Chapter 23
Nonlinear System Identification Through Backbone Curves and Bayesian Inference

A. Cammarano, P.L. Green, T.L. Hill, and S.A. Neild

Abstract Nonlinear structures exhibit complex behaviors that can be predicted and analyzed once a mathematical model of the structure is available. Obtaining such a model is a challenge. Several works in the literature suggest different methods for the identification of nonlinear structures. Some of the methods only address the question of whether the system is linear or not, others are more suitable for localizing the source of nonlinearity in the structure, only a few suggest some quantification of the nonlinear terms. Despite the effort made in this field, there are several limits in the identification methods suggested so far, especially when the identification of a multi-degree of freedom (MDOF) nonlinear structure is required.

This work presents a novel method for the identification of nonlinear structures. The method is based on estimating backbone curves and the relation between backbone curves and the response of the system in the frequency domain. Using a Bayesian framework alongside Markov chain Monte Carlo (MCMC) methods, nonlinear model parameters were inferred from the backbone curves of the response and the Second Order Nonlinear Normal Forms which gives a relationship between the model and the backbone curve. The potential advantage of this method is that it is both efficient from a computation and from an experimental point of view.

Keywords Identification • Nonlinear vibrations • Markov Chain Monte Carlo • Bayesian inference • Nonlinear normal forms

23.1 Introduction

In the last decades the scientific community has shown a growing interest in the dynamics of nonlinear structures. This is mainly due to the increasing demand for lighter structures where the same level of safety is guaranteed and, eventually, the region of operation is extended. When the structures are particularly light, in order to assess the safety of the structure, large deflections have to be taken into account, but the theory commonly used to study the dynamics of structures relies on the assumption of linear behavior. This assumption is not true for larger deformations.

Many authors have developed numerical tools able to capture the nonlinear behavior in structures. Although the resulting numerical models give great insight into the physics and the mechanisms that govern the nonlinear behavior, a correct characterization of the nonlinear laws that characterize such behavior in a real structure is still challenging.

The problem of defining such a law is commonly referred to as “nonlinear identification”. Unlike linear identification, where the basic assumption is that the structure can be described by a set of linear equations and that only the coefficients of such linear equations are unknown, in nonlinear identification both the coefficients and the form of the equations are to be found.

Several nonlinear identification techniques attempted to linearize the behavior of the system in the neighborhood of a parameter set (see for example [1, 2]). This is equivalent to describing the system in terms of a set of linear oscillators, each able to reproduce the response of the system for a particular range of amplitude. Although this approach has the advantage of re-introducing some of the tools of linear analysis, it comes with many disadvantages. One of the main disadvantages is that these techniques are able to describe the system only when the nonlinear contribution to the response is extremely small. Also it is essential that the response does not go through any bifurcation. When these requirements are not met, other techniques are to be preferred.

A. Cammarano (✉) • T.L. Hill • S.A. Neild
Department of Mechanical Engineering, University of Bristol, Queen’s Building, University Walk, Bristol BS8 1TR, UK
e-mail: andrea.cammarano@bristol.ac.uk

P.L. Green
Department of Mechanical Engineering, University of Sheffield, Sir Frederick Mappin Building, Mappin Street, Sheffield S1 3JD, UK

G. Kerschen (ed.), *Nonlinear Dynamics, Volume 1*, Conference Proceedings of the Society for Experimental Mechanics Series, DOI 10.1007/978-3-319-15221-9_23

A set of techniques that overcome this limitation is based on the restoring force surface [3]. This technique, using some features of the surface that relate the restoring force to its state variables, provides a way to estimate the nonlinear parameters. This approach is incredibly powerful for single degrees of freedom systems where the equivalent parameters can be extracted by studying the intersection of the surface with planes characterized by either zero velocities or zero displacement—zero planes. When the number of degrees of freedom increases, the construction of these surfaces becomes more complicated and the interpretation of the intersection with the zero-planes is not trivial. Besides, building these surfaces requires the acquisition of a great number of time histories which can be incredibly time consuming.

It has long been established that probabilistic parameter estimates of nonlinear systems can be realized through the use of a Bayesian framework. This often involves inferring a set of parameters from a measured time history (see [4] for example). One drawback of the techniques is that the input data are compared with simulated data generated through the integration of differential equations. This process can be extremely time demanding.

In this work we present a different approach that uses the Bayesian framework in conjunction with the nonlinear normal forms [5, 6]. The advantage of this approach is that the normal forms provide a useful framework to reduce a complex nonlinear model to a set of algebraic equations. The method does not require forced responses but instead uses the resonant decay which provides an estimation of the backbone curves of the system. The equations provided by the second order nonlinear normal forms are then used to investigate the set of nonlinear parameters that minimizes the error between simulated and input data.

After a brief introduction to nonlinear normal forms and how they can be used for identification purposes, a brief overview of the application of the Bayesian inference for identification with the nonlinear normal forms is provided. Then, a simulated two-degree of freedom nonlinear system with known nonlinear parameters is used to show the procedure to assess the accuracy of the identification technique. Finally, after a brief discussion of the results, the conclusions are drawn.

23.2 Nonlinear Normal Forms and Backbone Curves

A description of the second order nonlinear normal forms can be found in [6, 7]. Here, for sake of simplicity, a brief description in general terms is provided.

The concept behind the nonlinear normal forms [5] is that a set of nonlinear differential equations can be simplified by applying a set of transformations that separates the effect of the nonlinear terms, retaining only those that are more influential for the description of the response. The second order nonlinear normal forms [6], a variation of the original formulation, is particularly convenient for describing the dynamics of structures. In fact, the equations of motion are always expressed in terms of second order differential equations. The second order normal forms avoid the need to rewrite the set of equation in terms of a Cauchy problem (i.e. to reduce the system to the first order).

In this work the second order nonlinear normal forms are used to find algebraic equations that describe the backbone curves of the system. To find the backbone curves, the system has to be considered unforced and undamped. Therefore, the damping and forcing terms must be removed from the equations, which can then be transformed using the following transformations:

- A linear modal transform—the system is projected onto the linear modes (i.e. the modes of the underlying linear system).
- A nonlinear near-identity transform—this removes any non-resonant terms from each equation of motion.

Once these transforms are applied, each mode is described by an equation of motion consisting only of the terms resonating at one frequency—the response frequency (ω_{ri} for the ith mode—close to the linear natural frequency ω_{ni}). This way, the harmonic components of the equation are balanced, and any time-dependence removed. We invite the reader to note that the frequency at which the ith mode responds is named here as ω_{ri}, whereas ω_{ni} is the ith underlying linear natural frequency.

Broadly adopting the notation used by [8], we write the resulting unforced modal dynamics as

$$\ddot{\mathbf{v}} + \mathbf{\Lambda v} + \mathbf{N}_v(\mathbf{v}, \dot{\mathbf{v}}) = 0, \tag{23.1}$$

where $\mathbf{v}$ is a vector of modal displacements, $\mathbf{\Lambda}$ is a diagonal matrix having the ith diagonal term equal to ω_{ni}^2 and $\mathbf{N}_v$ contains the nonlinear terms, which are assumed to be small. The final step in the technique is the near-identity nonlinear transform $\mathbf{v} \rightarrow \mathbf{u}$

$$\ddot{\mathbf{v}} + \mathbf{\Lambda v} + \mathbf{N}_v(\mathbf{v}, \dot{\mathbf{v}}) = 0 \quad \xrightarrow{\mathbf{v} = \mathbf{u} + \mathbf{H}(\mathbf{u}, \dot{\mathbf{u}})} \quad \ddot{\mathbf{u}} + \mathbf{\Lambda u} + \mathbf{N}_u(\mathbf{u}, \dot{\mathbf{u}}) = 0, \tag{23.2}$$

where $\mathbf{H}$ stores all the terms that do not contribute to the fundamental response. All nonlinear terms resonating at ω_{ri}, for the ith mode, are collected in $\mathbf{N}_u$. A general solution for the ith component of the fundamental response $\mathbf{u}$ can be written as $u_i = u_{ip} + u_{im} = U_i e^{j\omega_{ri}tc} + U_i e^{-j\omega_{ri}t+\phi_i}$, where U_i represents the amplitude of the sinusoidal response and ϕ_i its phase. Substituting this solution for $\mathbf{u}$ in Eq. (23.2), and writing $\mathbf{N_u}(\mathbf{u}, \dot{\mathbf{u}}) = [\mathbf{N_u}]\mathbf{u}^*$ and $\mathbf{H}(\mathbf{u}, \dot{\mathbf{u}}) = [\mathbf{H}]\mathbf{u}^*$ as a linear combination of all nonlinear terms, where in general the lth element in the vector $\mathbf{u}^*$ is

$$u_\ell^* = \prod_{i=1}^{I} \left\{ u_{ip}^{s_{\ell ip}} u_{im}^{s_{\ell im}} \right\}, \tag{23.3}$$

it is possible to eliminate the time dependency from Eq. (23.2) and write the response of the system in terms of algebraic equations. The coefficients of these linear combinations can be found following the procedure described in [7]. Since the response is now expressed in terms of algebraic equations, relating the response to the change in parameters becomes extremely simple and computationally efficient. This is one of the great advantage of performing the identification of the parameters of the system using these equations, rather than the original differential equations.

23.3 Identification with Bayesian Inference

In the previous section, the second order nonlinear normal forms were used to find algebraic expression for a generic backbone curve of the system. To use these expressions for identification purposes, the backbone curve of the system must be estimated with an experimental procedure. For this purpose the techniques introduced by Feldman in [9] is used.

This techniques requires that the system is excited with a step sine so that the maximum amplitude response on one resonant peak is reached (red star in Fig. 23.1). Once this response is achieved, the forcing is stopped and the free response of the system is measured.

Because the system was vibrating around a resonant peak, the initial condition for the free vibration is also compatible with that resonance, hence the system response decays along its backbone curve (for more details see [10]). A schematic representation of this method is shown in Fig. 23.1. Here the response to the swept sine is represented with a dotted line, the resonant peak, where the forcing is stopped, is indicated with a red star and the backbone curve followed by the decaying signal is represented by a thick solid black line. The complete response of the system is shown with a thin line. The response is shown in solid black if stable, dashed red if unstable.

Note that the maximum amplitude response is in the proximity of a fold bifurcation (after which the response becomes unstable). This means that it is impossible to reach the peak (the basin of attraction of that solution is almost null and any small perturbation makes the solution jump to the lower solution). For this reason, when conducting the experiment, the forcing has to be stopped before the resonant peak. From numerical simulation it seems this does not cause any problem in the estimation of the backbone curve, since the free response rapidly converges to it.

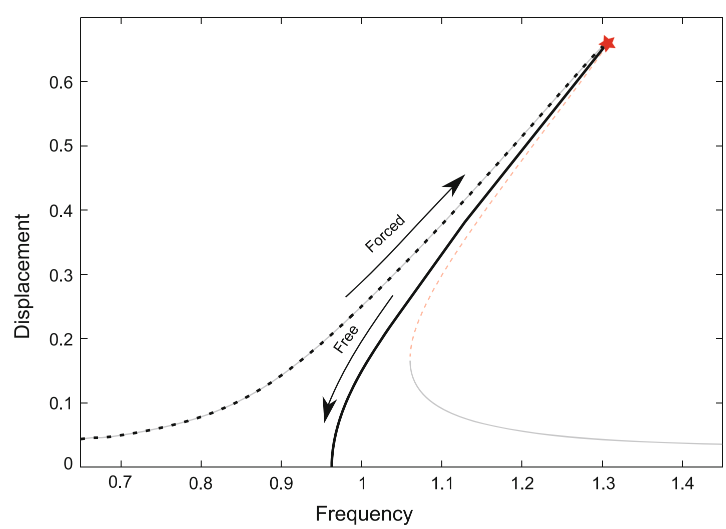

Fig. 23.1 Schematic of the backbone curve estimation with the free decay curve method. The *thin lines* represent the frequency response around resonance (*grey solid curves* show the stable responses whereas *red dashed lines* represent the unstable responses). The system is forced with a step sine until the forced response (*black dots*) reaches the resonant peak (*red star*). At this point the forcing is brought to zero and the damping in the system causes the response of the system to freely decay to zero (Color figure online)

Once the free vibration has been measured, the estimation of the backbone curve can be done by considering the period between too consecutive zeros and the amplitude of the signal at that instant of time (since the signal is decaying an average value on the interval is considered). For the backbone curves to be used with the nonlinear normal forms, they need to be expressed in terms of the underlying linear modal coordinates. This can be achieved applying the modal transform to the measured physical displacements. The modal transform can be obtained by running an initial test at low amplitude so that the nonlinear contribution is very small. This test is also needed to find the linear natural frequencies.

Once the backbone curves are evaluated, the Bayesian inference can be used. Here a brief description of the method is provided. For more details see [4].

Using Bayes' rule

$$P(\boldsymbol{\theta}|\mathcal{D},\mathcal{M}) = \frac{P(\mathcal{D}|\boldsymbol{\theta},\mathcal{M})P(\boldsymbol{\theta}|\mathcal{M})}{P(\mathcal{D}|\mathcal{M})} \tag{23.4}$$

it is possible to evaluate the probability that parameter vector assumes a value *boldsymbol*θ given data $\mathcal{D}$ and a model structure $\mathcal{M}$. In this case, the data consist of the measured values of $U_{1,n}$ and $U_{2,n}$ along the backbone curves together with the corresponding frequencies ω_{rn}. Here with the subscript n we indicate the resonance. So $U_{1,1}$ is the component U_1 of the first backbone curve. From now on the data are separated in two sets: the set of the $U_{1,n}$ values and their corresponding response frequency ω_{rn} is named $\mathcal{D}_1$ and the set of the $U_{2,n}$ values together with their response frequency $\mathcal{D}_2$.

The vector of parameters $\boldsymbol{\theta}$ contains all the parameters that must be identified. The model $\mathcal{M}$, in this case is the algebraic equation derived from the second order normal forms that maps values of ω_{rn} in values of $U_{1,n}$ and $U_{2,n}$.

To define the likelihood $P(\mathcal{D}|\boldsymbol{\theta},\mathcal{M})$, we assume that, for a given model $\mathcal{M}$ and a parameter set $\boldsymbol{\theta}$, the probability of witnessing $U_{1,n}$ and $U_{2,n}$ are mutually independent, so that

$$P(\mathcal{D}|\boldsymbol{\theta},\mathcal{M}) = P(\mathcal{D}_1|\boldsymbol{\theta},\mathcal{M})P(\mathcal{D}_2|\boldsymbol{\theta},\mathcal{M}). \tag{23.5}$$

Also it is assumed that the PDF describing the probability of witnessing a data point is given by a Gaussian distribution with mean equal to the model output,

$$P(U_1^{(i)}|\boldsymbol{\theta},\mathcal{M}) = \mathcal{N}(\hat{U}_1^{(i)}(\boldsymbol{\theta}),\sigma_1^2), \tag{23.6}$$

$$P(U_2^{(i)}|\boldsymbol{\theta},\mathcal{M}) = \mathcal{N}(\hat{U}_2^{(i)}(\boldsymbol{\theta}),\sigma_2^2). \tag{23.7}$$

Here the $\hat{U}_1$ and $\hat{U}_2$ are the output values of the model $\mathcal{M}$ for U_1 and U_2 (the hat indicates that this values are simulated) and therefore they are a function of the parameters $\boldsymbol{\theta}$. The parameters σ_1 and σ_2 can be included in the parameter vector and considered as unknown for the problem.

It is also assumed that there is mutual independence between the data points, that is

$$P(U_1^{(i)},U_1^{(j)}|\mathcal{D},\mathcal{M}) = P(U_1^{(i)}|\boldsymbol{\theta},\mathcal{M})P(U_1^{(j)}|\mathcal{D},\mathcal{M}), \tag{23.8}$$

$$P(U_2^{(i)},U_2^{(j)}|\mathcal{D},\mathcal{M}) = P(U_2^{(i)}|\boldsymbol{\theta},\mathcal{M})P(U_2^{(j)}|\mathcal{D},\mathcal{M}). \tag{23.9}$$

Under these assumptions the likelihood of the data set $\mathcal{D}$ given a parameter set $\boldsymbol{\theta}$ and a model $\mathcal{M}$ is

$$P(\mathcal{D}|\boldsymbol{\theta},\mathcal{M}) = \prod_{i=1}^{N}\mathcal{N}(\hat{U}_1^{(i)}(\boldsymbol{\theta}),\sigma_1^2)\mathcal{N}(\hat{U}_2^{(i)}(\boldsymbol{\theta}),\sigma_2^2). \tag{23.10}$$

In Eq. (23.4) the distribution $P(\boldsymbol{\theta}|\mathcal{M})$ is the prior distribution, that is a probability distribution representing the knowledge of $\boldsymbol{\theta}$ before the data are known. In this work the prior is assumed to have a uniform distribution. The denominator of Eq. (23.4) can be seen as a normalizing constant which ensure that the integral of the posterior distribution over the support of the prior is equal to 1.

To generate the samples from the posterior we use a Markov chain Monte Carlo (MCMC). In particular for this work we used the Simulated Annealing algorithm which involves targeting the sequence of distributions:

$$\pi_{\beta_j} \propto P(\mathcal{D}|\boldsymbol{\theta},\mathcal{M})^{\beta_j}P(\boldsymbol{\theta}|\mathcal{M}) \qquad j = 1,2,\ldots \tag{23.11}$$

where β_1, β_2, β_3, is a sequence which increases monotonically from 0 to 1—the annealing schedule. Using Eq. (23.4) an adaptive annealing schedule was chosen. This ensures that the information content of the data (measured using the Shannon entropy) is introduced into the target distribution at a constant rate [11]. Moreover, to avoid local traps, multiple Markov chain can be grown in parallel.

23.4 Nonlinear Identification of a 2-DOF Nonlinear Oscillator

In this section a two mass system featuring a nonlinear spring is presented. This system is used to produce a set of numerical data which can be used for the nonlinear identification. As the system is simulated the characteristics are known and so allows us to verify the accuracy of the identified parameters using the proposed technique.

A schematic of the system is provided in Fig. 23.2. Both masses are connected to ground and between each other with linear springs and viscous dampers. All the springs have the same stiffness value K of 1 N/m, the external dampers (labeled C)have a damping constant of 1×10^{-3} Ns/m and the central dampers (labeled C_2) have a damping constant of 5×10^{-4} Ns/m. The spring between ground and the first mass has an additional cubic term with a nonlinear coefficient κ of 0.5 N/m^3.

In this example, the system has not been excited as previously described. Since the system presented is simulated, it was possible to reverse the time flow and start the time decay from a solution in the close proximity of zero. The advantage of this method is that when the amplitude of vibration approaches zero, the system behaves linearly and the initial displacement can be chosen so that it is compatible with the modeshape corresponding to the nth mode of the underlying linear system. In this case, the eigenvectors of the underlying linear system are $[1, 1]$ and $[1, -1]$. A simulation starting from $[\delta, \delta]$ is used to generate a time history from which the first backbone curve can be estimated. The same approach, starting from an initial displacement of $[\delta, -\delta]$ gives the time history to estimate the second backbone curve. Here with δ we mean a generic small displacement.

The backbone curve, estimated evaluating the instantaneous frequency at the zero crossing, are then used as input data for the Bayesian technique illustrated in the previous section. For this test we hypothesized that the number of nonlinear springs and their position were known and that only the value of the nonlinearity was unknown. Using the nonlinear normal forms, analytical expressions for the backbone curves of a system featuring two masses and three linear springs are derived:

$$S1 : \left(\omega_{n1}^2 - \omega_{r1}^2\right) U_1 + \frac{3}{8m}\left[\kappa\left(U_1 + U_2\right)^3\right] = 0, \qquad (23.12)$$

$$S2 : \left(\omega_{n2}^2 - \omega_{r2}^2\right) U_2 + \frac{3}{8m}\left[\kappa\left(U_1 + U_2\right)^3\right] = 0. \qquad (23.13)$$

In the model, the number of nonlinear springs is one and its position is known: this simplified the derivation of the backbone curves. Also, only one nonlinear parameter has to be identified. For the estimation of the nonlinear parameter a uniform distribution for the prior has been used. The limit of the distribution for each parameter are listed in Table 23.1.

The parameter vector $\boldsymbol{\theta}$ in this case is made of the nonlinear parameter κ and the variance of the data around the value estimated by the model. To find the parameter vector 8,000 posterior samples were generated using MCMC.

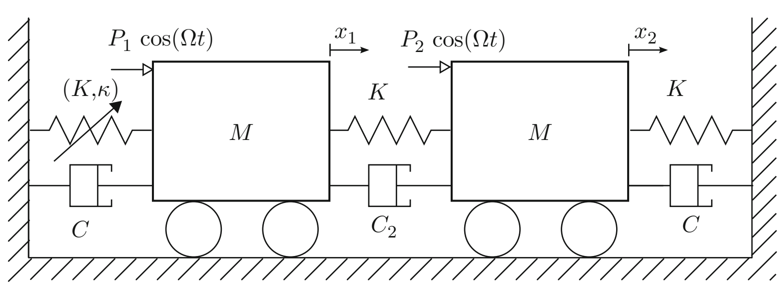

Fig. 23.2 A schematic diagram of the nonlinear two-mass oscillator

Table 23.1 Parameter limits for the prior distribution

Parameter	Lower limit	Upper limit
κ	0	1
σ_1	0	0.5
σ_2	0	0.5

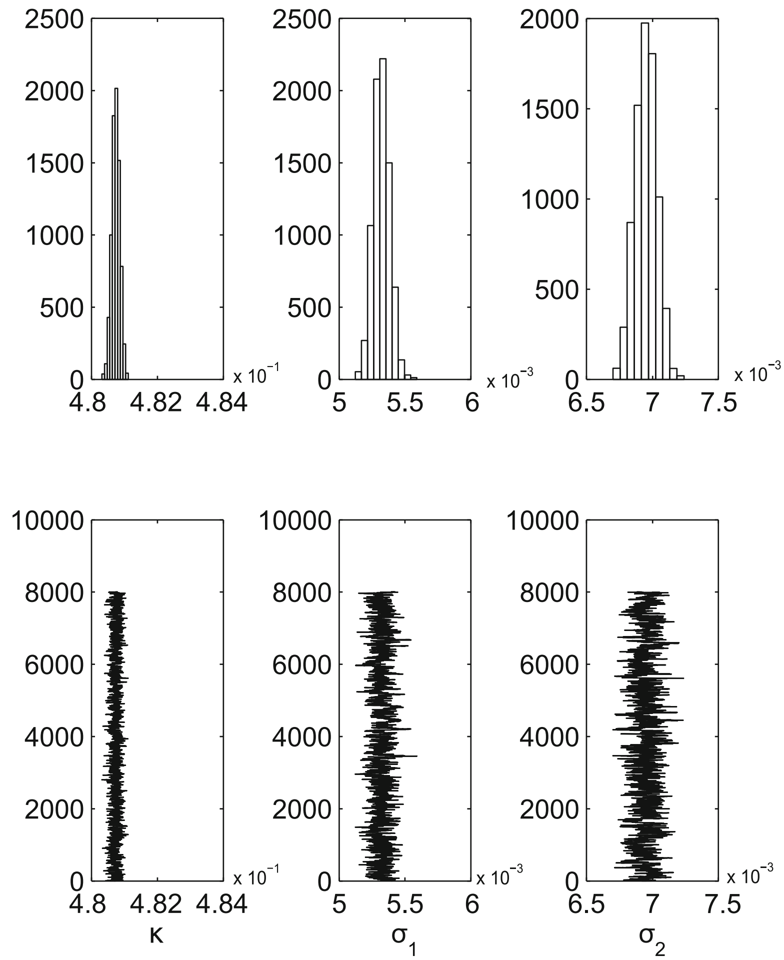

Fig. 23.3 Parameter distribution in the data generated by the MCMC. Each of the three columns of figures shows the evolution of one the parameters (*bottom*) and its histogram (*top*). The mean nonlinear coefficient κ is estimated to be 4.81 N/m^3. The *second* and the *third columns* show the values of σ_1 and σ_2 for the distribution of the input data around the values predicted by the model

The identification methods predicted a mean value for κ of 0.481 N/m^3 as shown in Fig. 23.3 whereas the real value was 0.5 N/m^3. Figures 23.4 and 23.5 show the measured backbone curves (black dots), the most probable backbone curves identified by the techniques using the estimated value of κ (red lines). Also the $\pm 3\ \sigma$ intervals (delimited by the blue curves) are shown. The estimated backbone curve, the predicted ones and the interval limits are very close to each other: from the figures they are barely distinguishable. The backbone curves in Figs. 23.4 and 23.5 are represented in terms of the amplitude of the fundamentals U_1 and U_2 respectively. This representation has been chosen to be consistent with the variable used in the identification methods. Other representations are possible (for example in terms of the physical coordinates).

23.5 Conclusions

A procedure for identifying nonlinear system has been described. The method, based on the second order nonlinear normal forms, uses a Bayesian framework to find probabilistic estimates of the parameters of a nonlinear system. The advantage of this technique over other techniques using a Bayesian framework is that the simulated data are obtained using algebraic equations rather than differential equations. This results in reduced computation time and in the possibility of comparing the input data with a higher number of nonlinear functions. After the introduction of the techniques, a numerical model of a 2DOF nonlinear oscillator has been used to provide input data for the identification procedure. In the previous section, the identification procedure was implemented on the simulated data and more insight into the technique provided. The parameter used for the simulation was correctly identified by the identification method shown in this work with an error of 5%. For this initial test, the position and the number of nonlinear springs were assumed as known. Future work will assess whether this technique is able to localize the nonlinearity and if the estimation error can be reduced.

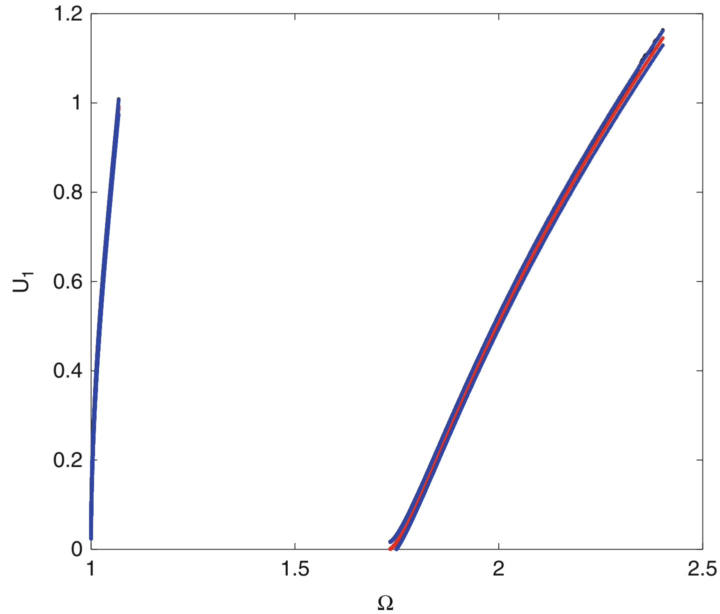

Fig. 23.4 Backbone curves (U_1 component): estimated backbone curve (*black dots*), identified backbone curve (*red dots*) and limit of the $\pm 3\ \sigma_1$ region (*blue dots*). Both the $S1$ and the $S2$ backbone curves are shown (Color figure online)

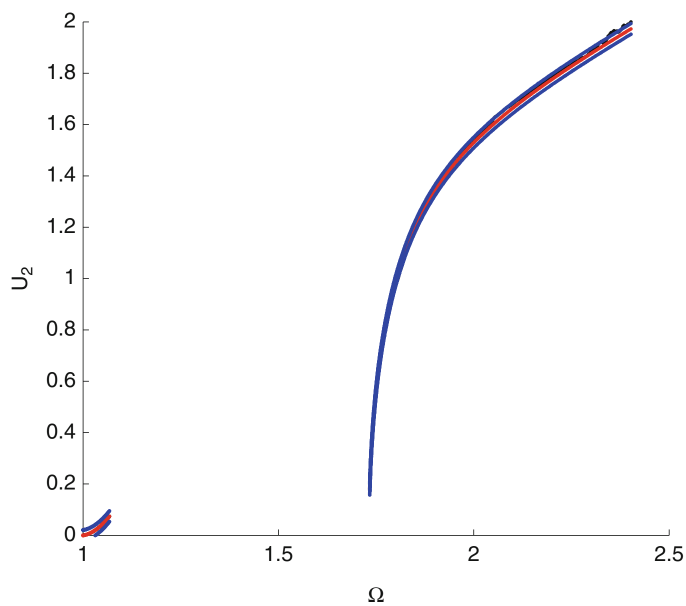

Fig. 23.5 Backbone curves (U_2 component): estimated backbone curves (*black dots*), identified backbone curves (*red dots*) and limit of the $\pm 3\ \sigma_2$ region (*blue dots*). Both the $S1$ and the $S2$ backbone curves are shown (Color figure online)

Acknowledgements The authors would like to acknowledge the support of the Engineering and Physical Sciences Research Council. A. Cammarano and P.L.Green are supported by EPSRC Programme Grant "Engineering Nonlinearity" EP/K003836/1. T.L. Hill is supported by an EPSRC studentship and S.A. Neild is supported by EPSRC Fellowship EP/K005375/1.

References

1. Carrella A, Ewins DJ (2011) Identifying and quantifying structural nonlinearities in engineering applications from measured frequency response functions. Mech Syst Signal Process 25(3):1011–1027
2. Josefsson A, Magnevall M, Ahlin K (2007) On nonlinear parameter estimation with random noise signals. In: Proceedings of IMAC XXV
3. Worden K, Tomlinson GR (2010) Nonlinearity in structural dynamics: detection, identification and modelling. CRC Press, Boca Raton, pp 286–376
4. Green PL (2014) Bayesian system identification of a nonlinear dynamical system using a novel variant of Simulated Annealing. Mech Syst Signal Process. doi:10.1016/j.ymssp.2014.07.010

5. Nayfeh AH (1993) Method of normal forms. Wiley, New York
6. Neild SA, Wagg DJ (2011) Applying the method of normal forms to second order nonlinear vibration problems. Proc R Soc Lond A 467: 1141–1163
7. Neild SA (2012) Approximate methods for analysing nonlinear structures. Exploiting nonlinear behavior in structural dynamics. Springer, New York, pp 53–109
8. Cammarano A, Hill TL, Neild SA, Wagg DJ (2013) Bifurcations of backbone curves for systems of coupled nonlinear two mass oscillator. Nonlinear Dyn 1–10. doi:10.1007/s11071-014-1295-3
9. Feldman M (1994) Non-linear system vibration analysis using Hilbert transform—I. Free vibration analysis method'Freevib'. Mech Syst Signal Process 8(2):119–127
10. Kerschen G, Peeters M, Golinval JC, Vakakis AF (2009) Nonlinear normal modes, Part I: a useful framework for the structural dynamicist. Mech Syst Signal Process 23:170–194
11. Green PL (2014) Bayesian system identification of nonlinear dynamical systems using a fast MCMC algorithm. In: Proceedings of ENOC 2014, European nonlinear dynamics conference