Chapter 18
Experimental Nonlinear Dynamics and Chaos of Post-buckled Plates

R. Wiebe and D. Ehrhardt

Abstract Panels and plates are an important structural element in many engineering applications, such as aircraft skin panels, ship hulls, and civil shell structures. These structures, particularly when their boundaries are in some way constrained, exhibit highly nonlinear behavior (e.g. spring hardening) even for relatively small deformations due to induced axial loading. An extreme, but highly important, example is dynamic snap-through buckling of curved or post-buckled thin panels. This phenomena is well represented in the literature, both for plates and for the simplified case of curved beams. The majority of the experimental studies, especially for panels, have been carried out using either wind tunnels or acoustic drivers to generate transverse loading. While this is directly applicable to real-world scenarios, say aircraft panel loading, it does not permit direct control of the loads that are applied. In this work, we instead apply loads to a thermally buckled panel using an electrodynamic shaker. This, along with the use of digital image correlation to capture the full field dynamic response allows for a complete picture of the complex characteristics of dynamic snap-through.

Keywords Nonlinear dynamics • Experimental mechanics • Snap-through • Chaos

18.1 Introduction

Given their common use as structural elements, plates and curved panels have been studied extensively in the past. The work that most closely matches what will be presented in this paper is [1], in which the nonlinear dynamics and snap-through of a thermally buckled plate under acoustic excitation is investigated experimentally. It is shown that regions of snap-through occur near linear resonance, and not surprising is more likely for larger loading magnitudes. Given the bistability of buckled plates, it is perhaps not surprising that, at least qualitatively, this mirrors the response of, in decreasing spatial complexity, buckled beams [2], and the double-well Duffing equation. In [3, 4] it is shown that the double-well Duffing equation yields surprising sensitivity to forcing parameters, with certain parameters yielding multiple co-existing response types.

The response of the Duffing equation hints that a buckled plate (which may be represented by a double-well Duffing type equation for first-mode response) may also present similar sensitivity in forcing parameter space, a characteristic which would certainly be of interest to engineers designing shell and plate type structures. Figure 18.1 shows the initially (nominally) flat, thermally-buckled aluminum plate with clamped boundary conditions all around that is used in this investigation (mounting visible in photo). The plate dimensions were $0.5 \times 165 \times 216$ mm. The plate was thermally buckled using two heat lamps. All tests were performed after the plate had reached a steady-state temperature. The rise of the center of the plate at the steady-state temperature was $h \approx 1$ mm. Forcing was applied inertially by mounting the plate in its frame to an MB Dynamics electrodynamic shaker (not visible in photo). The benefit of using a shaker, rather than standard wind-tunnel or acoustic testing, is that the harmonic forcing frequency, and amplitude could be easily and independently controlled. Random, or combined random and harmonic forcing could also be applied using this arrangement, however this paper is limited to harmonic forcing results. The displacement was measured using dynamic digital image correlation DIC. The use of DIC allowed for a vast improvement over more traditional vibrometers as this made it possible to obtain displacement time series at any location of interest.

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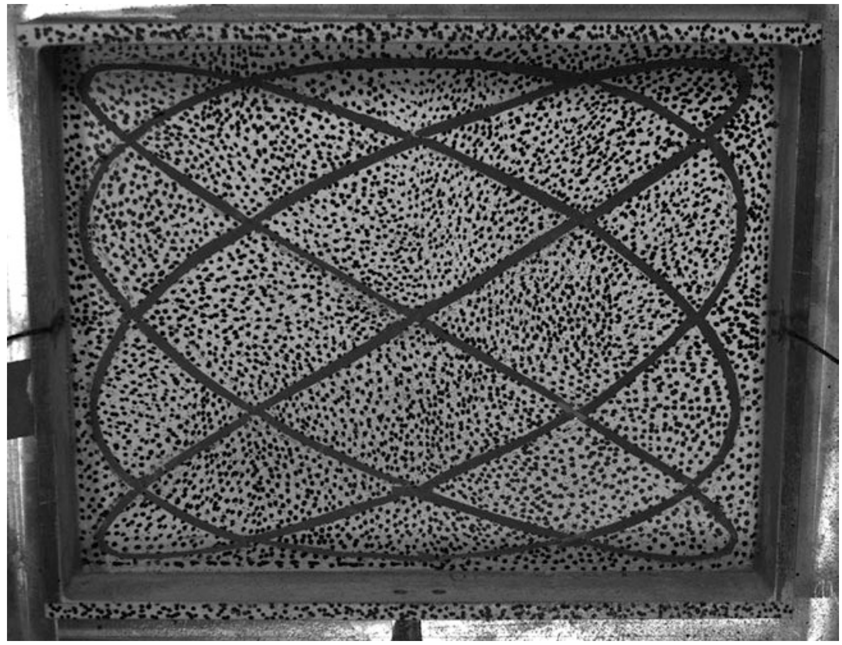

Fig. 18.1 Aluminum plate painted with speckle pattern for DIC data acquisition. The mounting frame provided clamped boundary conditions on all edges. The lissajous curves were applied to allow for scanning laser data acquisition (not discussed herein)

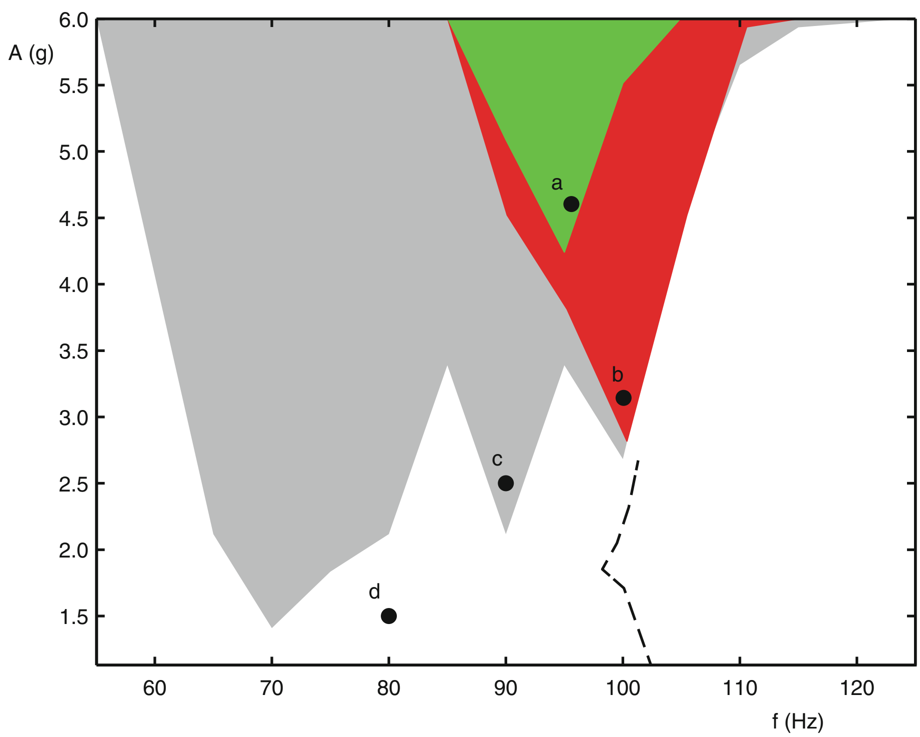

Fig. 18.2 Response-type plot of buckled plate under harmonic loading. The forcing amplitude is given in multiples of g (acceleration due to gravity), while the frequency is given in Hz. Four response types were observed, (*white*) non-snap, (*green*) P1 snap-through, (*red*) chaotic snap-through, (*gray*) co-existing chaotic snap-through and non-snap. The first nonlinear normal mode (*dashed black curve*) is also shown (Color figure online)

18.2 Experimental Results

The results of an experimental parametric study are presented as a response-type plot in Fig. 18.2. This figure divides the forcing parameter space into four response types. In the white regions of parameter space, the response was entirely non-snap-through, i.e. single-well. A typical time series of the plate measured at its center point (for parameters at point d) can be seen in Fig. 18.3d. In an engineering design context, this would be a preferable structural response. At the other extreme, the green region contains forcing parameters that resulted in period 1 snap-through. A time series of this response (for the parameters at point a) is shown in Fig. 18.3a. This is effectively the highest rate of snap-through that one could expect, and is thus likely to lead to the most rapid degradation through fatigue. The red shaded region instead resulted in chaotic

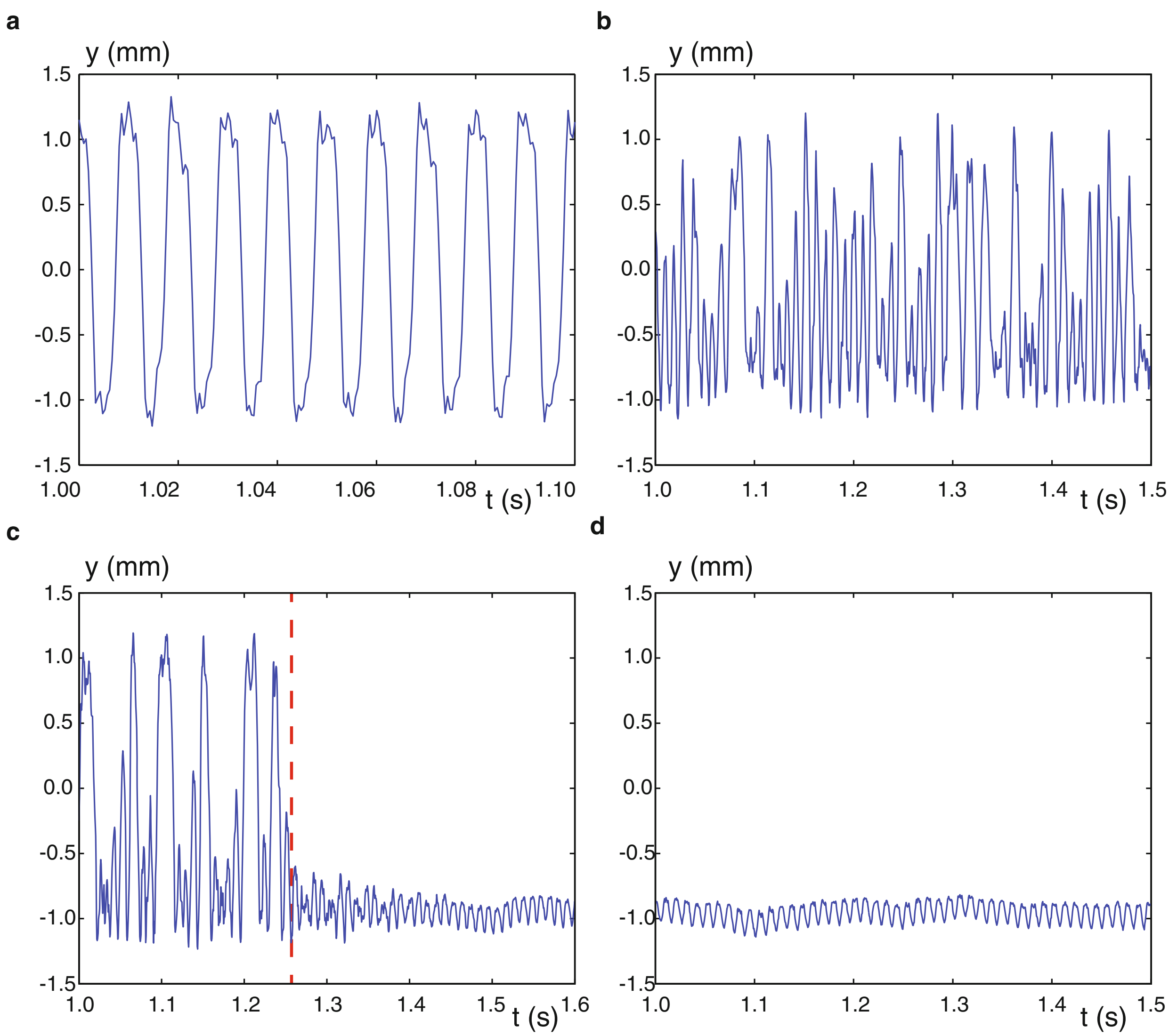

Fig. 18.3 Sample plate center-displacement time series. Parts (**a**) through (**d**) correspond to the labels (**a**) through (**d**) in Figure 18.2. The *vertical dashed line* in part (**c**) indicates an external perturbation

snap-through (also see point b and Fig. 18.3b), which yields a lower average snap-through rate than P1 snap-through. Finally, the gray shaded region denotes forcing parameters that lead to co-existing snap-through and non-snap-through response. As an example, the two response types that coexist for point c are shown in Fig. 18.3c. The gray region shows that this structure also exhibits sensitivity to initial conditions in terms of final response type, along with sensitivity to parameters. In part (c) an external perturbation (impacting the beam with a small hammer, vertical dashed line) was used to 'kick' the system between the competing response types. While the green region produces the most violent, highest-energy, response, the gray region is perhaps the most critical, as it could easily lull a designer into a false sense of security about the structural response.

Figure 18.2 was obtained using slow amplitude ramp-up and ramp-down tests at 15 different frequencies in 5 Hz increments. The boundaries between different response-type regions of parameter space were obtained by visual observation of the response, and are thus subject to some error. Tests were, however, repeated multiple times to ensure quality of the results. The boundary of the gray region was particularly difficult to observe. These boundaries were initially approximated by first ramping the amplitude up, then down at a given frequency. The amplitude at which chaotic (or periodic) snap-through occurred on the ramp-up was assumed to be in the vicinity of the gray-red (or gray-green) boundary. Then on the down-sweep, the amplitude at which snap-through ceased was assumed to be near the white-gray boundary. These approximate parameter locations were then examined further using multiple random perturbations (hitting the plate with a small hammer) to find at exactly which forcing amplitude the co-existing non-snap and snap-through behavior ceased to exist. The sliver of red-type response the divides the green and gray response types is suspect, as the transient behavior was quite long in this region, making difficult to distinguish chaotic response from long transience leading to periodic behavior. This sliver does not appear in the response-type plot of the double-well Duffing equation [3]. The seemingly ragged gray-white boundary is, at least qualitatively, similar to what is observed in the double-well Duffing equation, which is almost certainly fractal. The 5 Hz grid used experimentally, however, is much to coarse to prove or disprove the existence of fractal boundaries in this plot.

The first linear natural frequency of the plate about the buckled configuration was $f_0 \approx 108$ Hz. When compared with the lowest point of the red response type, which occurs at $f \approx 100$ Hz, it could be stated that this system presents a softening type behavior. An interesting method of characterizing the transition form linear to increasingly nonlinear behavior is given by the nonlinear normal modes (NNMs) of the response. These modes are extensions of the linear modes, in that they exhibit 'vibration in unison' (although other definitions exist), and present a promising approach for characterizing nonlinear vibrations, or as mode shapes in reduced order modeling [5]. The details of NNMs will not be discussed herein (see [6] for further details), however, as a point of interest the first NNM of the buckled plate (dashed black curve in Fig. 18.2) can be seen to approach the critical minimum snap-through point. If this is indeed the case, it would serve as another useful application of NNMs in detecting snap-through. Note that the NNM connects to the linear natural frequency of $f_0 \approx 108$ Hz, however, the plot is cut off at a forcing amplitude of 1 g.

18.3 Conclusions

Thermally buckled plates are shown experimentally to exhibit extreme sensitivity to initial conditions (chaos and co-existing solutions) and forcing parameters (sudden changes in the response-type plot) under harmonic loading. This has consequences in the structural design of curved plates and shells, as designers must be careful to ensure that they have captured the worst case, or limit state, of the system of interest.

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