A Parametric Interval Approximation of Fuzzy Numbers

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Abstract In this paper we present a parametric formulation of interval approximation of fuzzy numbers. It is based on a more complex version of generalized Trutschnig et al. distance. General conclusions are showed and particular cases are studied in details.

Keywords Fuzzy sets · Fuzzy quantities · Interval type-2 fuzzy sets · Evaluation fuzzy numbers · Interval approximation · Weighting functions

1 Introduction

The problem to approximate a fuzzy set has been studied by several authors. Some of them use an interval approximation [1, 4–11], others use triangular or trapezoidal approximation. This idea is born with the aim of simplifying complicated calculations that appear in modeling and processing fuzzy optimization and control problems. In all these cases, even if we start with data described by fuzzy numbers of the easier forms like triangular and trapezoidal, the fuzzy outputs we meet are complicated and may have lost all the peculiarity of the starting sets. It is sufficient to look to a usual fuzzy control system output that may be neither normal nor convex.

In this paper we propose to substitute a given fuzzy number with an interval which has some properties like to be the nearest in some sense we describe. The results we present start from a paper of Grzegorzewski [10] in which he proposes the nearest interval to the original fuzzy number with respect to several distances. In particular we have focused on the distance introduced by Trutschnig et al. [13]. This distance depends on two parameters, a constant θ and a function $f(\alpha)$ that depends on the

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variable that individualizes the α-cuts of a fuzzy set. We propose to consider even θ as a function of α. This simple variation generates interesting results and connection with other important intervals connected with the initial fuzzy number defined in previous researches.

In Sect. 2 we give basic definitions and notations. In Sect. 3 we present an interval approximation for a fuzzy number obtained by minimizing a suitable functional. In Sects. 4 and 5 we study some properties of the approximation interval we have proposed. Finally, in Sect. 6 we apply our method to the interval approximation of trapezoidal fuzzy numbers.

2 Preliminaries and Notation

Let X denote a universe of discourse. A fuzzy set A in X is defined by a membership function $\mu_A : X \to [0, 1]$ which assigns to each element of X, a grade of membership to the set A. The height of A is $h_A = height\, A = \sup_{x \in X} \mu_A(x)$. The support and the core of A are defined, respectively, as the crisp sets $supp(A) = \{x \in X; \mu_A(x) > 0\}$ and $core(A) = \{x \in X; \mu_A(x) = 1\}$. A fuzzy set A is normal if its core is nonempty. A fuzzy number A is a fuzzy set of the real line $\mathbb{R}$ with a normal, fuzzy convex and upper-semicontinuous membership function of bounded support (see, e.g., [2]). In accordance with the definition given above there exist four numbers $a_1, a_2, a_3, a_4 \in \mathbb{R}$, with $a_1 \leq a_2 \leq a_3 \leq a_4$, and two functions $l_A, r_A : \mathbb{R} \to [0, 1]$ called the left side and the right side of A, respectively, where l_A is nondecreasing and right-continuous and r_A is nonincreasing and left-continuous, such that

$$\mu_A(x) = \begin{cases} 0 & x < a_1 \\ l_A(x) & a_1 \leq x < a_2 \\ 1 & a_2 \leq x \leq a_3 \\ r_A(x) & a_3 < x \leq a_4 \\ 0 & x > a_4 \,. \end{cases}$$

The α-cut of a fuzzy set A, $0 \leq \alpha \leq 1$, is defined as the crisp set $A_\alpha = \{x \in X; \mu_A(x) \geq \alpha\}$ if $0 < \alpha \leq 1$ and as the closure of the support if $\alpha = 0$. Every α-cut of a fuzzy number is a closed interval $A_\alpha = [a_L(\alpha), a_R(\alpha)]$, for $0 \leq \alpha \leq 1$, where $a_L(\alpha) = \inf A_\alpha$ and $a_R(\alpha) = \sup A_\alpha$.
In the following we will employ the mid-spread representation of intervals. The middle point and the spread of the interval $I = [a, b]$ will be denoted, respectively, by $mid(I) = (a + b)/2$ and $spr(I) = (b - a)/2$.

3 Interval Approximation of Fuzzy Numbers

Our proposal starts from the Grzegorzewski papers in which the author defines and finds an interval approximation of a fuzzy number. Starting from a distance between two fuzzy numbers and observing that any closed interval is a fuzzy number, the author defines the approximating interval of a fuzzy number as the interval of minimum distance. The distance he uses is based on the distance between two closed intervals I and J introduced by Trutschnig et al. [13] defined by

$$d_{\bar{\theta}}(I, J) = \sqrt{(mid(I) - mid(J))^2 + \bar{\theta}(spr(I) - spr(J))^2}$$

where the mid-spread representation of the involved intervals is employed. The parameter $\bar{\theta} \in]0, 1]$ indicates the relative importance of the spreads against the mids [10, 13]. The distance $d_{\bar{\theta}}$ is extended to the space of all fuzzy number $\mathbb{F}(\mathbb{R})$ by defining $D_{\bar{\theta}} : \mathbb{F}(\mathbb{R}) \times \mathbb{F}(\mathbb{R}) \rightarrow [0, +\infty[$ such that for two arbitrary fuzzy numbers A and B

$$D^2_{f,\bar{\theta}}(A, B) = \frac{1}{\int_0^1 f(\alpha)\, d\alpha} \int_0^1 d^2_{\bar{\theta}}(A_\alpha, B_\alpha)\, f(\alpha)\, d\alpha$$

where the weighting function $f :]0, 1] \rightarrow [0, +\infty[$ is such that $\int_0^1 f(\alpha)\, d\alpha > 0$.

In this paper we extend Grzegorzewski's idea by considering the parameter θ as a function of α having in mind that the relative importance of the spreads against the mids may depend on the level of uncertainty. This hypothesis leads to interesting results.

Definition 1 We say that $C^*(A) = [c^*_L, c^*_R]$ is an approximation interval of the fuzzy number A with respect to the pair (f, θ) if it minimizes the weighted mean of the squared distances

$$\begin{aligned}
\mathcal{D}^{(2)}_{f,\theta}(C; A) &= \frac{1}{\int_0^1 f(\alpha)\, d\alpha} \int_0^1 d^2_{\theta(\alpha)}(C, A_\alpha)\, f(\alpha)\, d\alpha \\
&= \frac{1}{\int_0^1 f(\alpha)\, d\alpha} \int_0^1 \left[(mid(C) - mid(A_\alpha))^2 + \theta(\alpha)\,(spr(C) - spr(A_\alpha))^2\right] \\
&\qquad \times f(\alpha) d\alpha
\end{aligned}$$

among all the intervals $C = [c_L, c_R]$, where the weight function $f : [0, 1] \rightarrow [0, +\infty[$ is such that $\int_0^1 f(\alpha)\, d\alpha > 0$ and $\theta : [0, 1] \rightarrow]0, 1]$ is a function that indicates the relative importance of the spreads against the mids [10, 13].

Theorem 1 *The approximation interval $C^*(A) = C^*(A; f, \theta) = [c_L^*, c_R^*]$ of the fuzzy number A with respect to (f, θ) is given by*

$$\begin{aligned} c_L^* &= \frac{\int_0^1 mid(A_\alpha)\, f(\alpha)\, d\alpha}{\int_0^1 f(\alpha)\, d\alpha} - \frac{\int_0^1 spr(A_\alpha)\, f(\alpha)\,\theta(\alpha)\, d\alpha}{\int_0^1 f(\alpha)\,\theta(\alpha)\, d\alpha} \\ c_R^* &= \frac{\int_0^1 mid(A_\alpha)\, f(\alpha)\, d\alpha}{\int_0^1 f(\alpha)\, d\alpha} + \frac{\int_0^1 spr(A_\alpha)\, f(\alpha)\,\theta(\alpha)\, d\alpha}{\int_0^1 f(\alpha)\,\theta(\alpha)\, d\alpha}\,. \end{aligned} \tag{1}$$

Proof We have to minimize the function

$$\begin{aligned} g(c_L, c_R) = &\int_0^1 \left(\frac{c_L + c_R}{2} - \frac{a_L(\alpha) + a_R(\alpha)}{2}\right)^2 f(\alpha) d\alpha \\ &+ \int_0^1 \theta(\alpha) \left(\frac{c_R - c_L}{2} - \frac{a_R(\alpha) - a_L(\alpha)}{2}\right)^2 f(\alpha) d\alpha \end{aligned}$$

with respect to c_L and c_R. We obtain

$$\begin{aligned} \frac{\partial g}{\partial c_L}(c_L, c_R) = &\int_0^1 (mid(C) - mid(A_\alpha))\; f(\alpha) d\alpha \\ &- \int_0^1 \theta(\alpha)\, (spr(C) - spr(A_\alpha))\; f(\alpha) d\alpha \\ \frac{\partial g}{\partial c_R}(c_L, c_R) = &\int_0^1 (mid(C) - mid(A_\alpha))\; f(\alpha) d\alpha \\ &+ \int_0^1 \theta(\alpha)\, (spr(C) - spr(A_\alpha))\; f(\alpha) d\alpha\,. \end{aligned}$$

By solving

$$\begin{cases} \frac{\partial g}{\partial c_L}(c_L, c_R) = 0 \\ \frac{\partial g}{\partial c_R}(c_L, c_R) = 0 \end{cases}$$

we obtain that the solution $C^* = C^*(A) = [c_L^*, c_R^*]$ satisfies

$$mid(C^*) = \frac{\int_0^1 mid(A_\alpha)\, f(\alpha)\, d\alpha}{\int_0^1 f(\alpha)\, d\alpha}\,, \qquad spr(C^*) = \frac{\int_0^1 spr(A_\alpha)\, f(\alpha)\,\theta(\alpha)\, d\alpha}{\int_0^1 f(\alpha)\,\theta(\alpha)\, d\alpha}$$

and thus, since $c_L^* = mid(C^*) - spr(C^*)$ and $c_R^* = mid(C^*) + spr(C^*)$, we obtain (1). Moreover, by calculation, we get

$$\frac{\partial^2 g}{\partial c_L^2}(c_L, c_R) = \frac{\partial^2 g}{\partial c_R^2}(c_L, c_R) = \frac{1}{2}\left(\int_0^1 f(\alpha)\, d\alpha + \int_0^1 f(\alpha)\,\theta(\alpha)\, d\alpha\right)$$

and

$$\frac{\partial^2 g}{\partial c_R \partial c_L}(c_L, c_R) = \frac{\partial^2 g}{\partial c_L \partial c_R}(c_L, c_R) = \frac{1}{2}\left(\int_0^1 f(\alpha)\, d\alpha - \int_0^1 f(\alpha)\,\theta(\alpha)\, d\alpha\right)$$

and thus

$$\det \begin{bmatrix} \frac{\partial^2 g}{\partial c_L^2}(c_L, c_R) & \frac{\partial^2 g}{\partial c_R \partial c_L}(c_L, c_R) \\ \frac{\partial^2 g}{\partial c_L \partial c_R}(c_L, c_R) & \frac{\partial^2 g}{\partial c_R^2}(c_L, c_R) \end{bmatrix} = \left(\int_0^1 f(\alpha)\, d\alpha\right)\left(\int_0^1 f(\alpha)\,\theta(\alpha)\, d\alpha\right) > 0$$

and $\frac{\partial^2 g}{\partial c_L^2}(c_L, c_R) > 0$. Then (c_L^*, c_R^*) minimizes $g(c_L, c_R)$.

Remark 1 The previous theorem still holds if $\theta > 0$ almost everywhere in [0, 1].

4 Properties

In this section we study some properties of the approximation interval.

Proposition 1 *The approximation interval*

$$C^*(A) = C^*(A;\, f, \theta) = [c_L^*(A;\, f, \theta), c_R^*(A;\, f, \theta)]$$

given by (1) *satisfies the following properties:*

(i) *invariance under translations, that is*

$$C^*(A + z;\, f, \theta) = C^*(A;\, f, \theta) + z \qquad \forall z \in \mathbb{R}\,;$$

(ii) *scale invariance, that is*

$$C^*(z \cdot A;\, f, \theta) = z \cdot C^*(A;\, f, \theta) \qquad \forall\, z \in \mathbb{R} \backslash \{0\}\,.$$

Proof Let us prove (i). Since

$$mid((A + z)_\alpha) = \frac{1}{2}(a_L(\alpha) + z + a_R(\alpha) + z) = mid(A_\alpha) + z$$

and

$$spr((A + z)_\alpha) = \frac{1}{2}(a_R(\alpha) + z - a_L(\alpha) - z) = spr(A_\alpha)$$

from (1) we obtain

$$
\begin{aligned}
c_L^*(A+z;\, f,\theta) &= \frac{\int_0^1 mid((A+z)_\alpha)\, f(\alpha)\, d\alpha}{\int_0^1 f(\alpha)\, d\alpha} - \frac{\int_0^1 spr((A+z)_\alpha)\, f(\alpha)\,\theta(\alpha)\, d\alpha}{\int_0^1 f(\alpha)\,\theta(\alpha)\, d\alpha} \\
&= \frac{\int_0^1 mid(A_i^\alpha)\, p_i(\alpha)\, f(\alpha)\, d\alpha}{\int_0^1 f(\alpha)\, d\alpha} - \frac{\int_0^1 spr(A_i^\alpha)\, f(\alpha)\,\theta(\alpha)\, d\alpha}{\int_0^1 f(\alpha)\,\theta(\alpha)\, d\alpha} + z \\
&= c_L^*(A;\, f,\theta) + z\,.
\end{aligned}
$$

In a similar way, we get $c_R^*(A+z;\, f,\theta) = c_R^*(A;\, f,\theta)+z$ and thus $C^*(A+z;\, f,\theta) = C^*(A;\, f,\theta) + z$.
Let us prove (ii). First, we consider the case $z > 0$. We have

$$(z \cdot A)_\alpha = [z \cdot a_L(\alpha), z \cdot a_R(\alpha)]$$

and thus

$$mid((z \cdot A)_\alpha) = \frac{1}{2}(z \cdot a_L(\alpha) + z \cdot a_R(\alpha)) = z \cdot mid(A_\alpha)$$

and

$$spr((z \cdot A)_\alpha) = \frac{1}{2}(z \cdot a_R(\alpha) - z \cdot a_L(\alpha)) = z \cdot spr(A_\alpha)\,.$$

So from (1) we get

$$c_L^*(z \cdot A;\, f,\theta) = z \cdot c_L^*(A;\, f,\theta)\,, \qquad c_R^*(z \cdot A;\, f,\theta) = z \cdot c_R^*(A;\, f,\theta)$$

and thus

$$C^*(z \cdot A;\, f,\theta) = z \cdot C^*(A;\, f,\theta)\,.$$

If $z < 0$ we have

$$(z \cdot A)_\alpha = [z \cdot a_R(\alpha), z \cdot a_L(\alpha)]$$

and thus

$$mid((z \cdot A)_\alpha) = z \cdot mid(A_\alpha)\,, \qquad spr((z \cdot A)_\alpha) = (-z) \cdot spr(A_\alpha)\,.$$

Then from (1) we obtain

$$
\begin{aligned}
c_L^*(z \cdot A;\, f,\theta) &= \frac{\int_0^1 mid((z \cdot A)_\alpha)\, f(\alpha)\, d\alpha}{\int_0^1 f(\alpha)\, d\alpha} - \frac{\int_0^1 spr((z \cdot A)_\alpha)\, f(\alpha)\,\theta(\alpha)\, d\alpha}{\int_0^1 f(\alpha)\,\theta(\alpha)\, d\alpha} \\
&= z\frac{\int_0^1 mid(A_\alpha)\, f(\alpha)\, d\alpha}{\int_0^1 f(\alpha)\, d\alpha} + z\frac{\int_0^1 spr(A_\alpha)\, f(\alpha)\,\theta(\alpha)\, d\alpha}{\int_0^1 f(\alpha)\,\theta(\alpha)\, d\alpha} \\
&= z\, c_R^*(A;\, f,\theta)\,.
\end{aligned}
$$

In the similar way, we get $c^*_R(z \cdot A;\, f, \theta) = z\, c^*_L(A;\, f, \theta)$. Then, taking into account that $z < 0$, we have

$$\begin{aligned} C^*(z \cdot A;\, f, \theta) &= [c^*_L(z \cdot A;\, f, \theta), c^*_R(z \cdot A;\, f, \theta)] = [z\, c^*_R(A;\, f, \theta), z\, c^*_L(A;\, f, \theta)] \\ &= z \cdot [c^*_L(A;\, f, \theta), c^*_R(A;\, f, \theta)] = z \cdot C^*(A;\, f, \theta)\,. \end{aligned}$$

5 Relation to Expected Interval and Interval-Valued Possibilistic Mean

Some important intervals connected with a fuzzy number are often utilized to have its view. We pertain to the *expected interval* $EI(A)$ of a fuzzy number A, introduced by Dubois and Prade [6] and Heilpern [11]

$$EI(A) = \left[\int_0^1 a_L(\alpha)\, d\alpha, \int_0^1 a_R(\alpha)\, d\alpha \right],$$

the *interval-valued possibilistic mean* introduced by Carlsson and Fullér [3]

$$M(A) = \left[2\int_0^1 a_L(\alpha)\alpha\, d\alpha, 2\int_0^1 a_R(\alpha)\alpha\, d\alpha \right],$$

and the *f-weighted interval-valued possibilistic mean* proposed by Fullér and Majlender [8] for monotonic increasing weighting functions and by Liu [12] without the monotonic increasing assumption

$$M_f(A) = \left[\frac{\int_0^1 a_L(\alpha) f(\alpha)\, d\alpha}{\int_0^1 f(\alpha)\, d\alpha}, \frac{\int_0^1 a_R(\alpha) f(\alpha)\, d\alpha}{\int_0^1 f(\alpha)\, d\alpha} \right]$$

which is a generalization of the previous ones. It is interesting to see that there is an important connection between $M_f(A)$ and the approximation interval $C^*_{f,\theta}(A) = C^*(A;\, f, \theta)$ we have introduced before. As

$$mid(M_f(A)) = \frac{\int_0^1 mid(A_\alpha)\, f(\alpha)\, d\alpha}{\int_0^1 f(\alpha)\, d\alpha}, \qquad spr(M_f(A)) = \frac{\int_0^1 spr(A_\alpha)\, f(\alpha)\, d\alpha}{\int_0^1 f(\alpha)\, d\alpha}, \tag{2}$$

and observed that from (1) we have

$$mid(C^*_{f,\theta}(A)) = \frac{\int_0^1 mid(A_\alpha)\, f(\alpha)\, d\alpha}{\int_0^1 f(\alpha)\, d\alpha},$$
$$spr(C^*_{f,\theta}(A)) = \frac{\int_0^1 spr(A_\alpha)\, f(\alpha)\, \theta(\alpha)\, d\alpha}{\int_0^1 f(\alpha)\, \theta(\alpha)\, d\alpha} \tag{3}$$

we get

$$mid(C^*_{f,\theta}(A)) = \frac{\int_0^1 mid(A_\alpha)\, f(\alpha)\, d\alpha}{\int_0^1 f(\alpha)\, d\alpha} = mid(M_f(A)) \tag{4}$$

and thus $M_f(A)$ and $C^*_{f,\theta}(A)$ have the same middle point independently of the choice of θ. Obviously, they may differ in their spreads. Thus, for a given weighting functions f, they are all intervals centered at the same point but with different sizes. May we say something about these sizes? The following considerations reply to some questions.

To this aim, we consider the preference index value of the weighting function f

$$e_f = \frac{\int_0^1 \alpha\, f(\alpha)\, d\alpha}{\int_0^1 f(\alpha)\, d\alpha} \tag{5}$$

introduced in [12]. Similarly, we define the preference index value of the function $\theta \cdot f$ as

$$k_{f,\theta} = \frac{\int_0^1 \alpha\, \theta(\alpha)\, f(\alpha)\, d\alpha}{\int_0^1 \theta(\alpha)\, f(\alpha)\, d\alpha} \tag{6}$$

and

$$\varepsilon_f(\theta) = e_f - k_{f,\theta}\,. \tag{7}$$

First, we prove the following lemma.

Lemma 1 *Let $\tilde{f}, \tilde{g} : [0,1] \to [0,+\infty[$ such that $\int_0^1 \tilde{f}(\alpha)\, d\alpha = 1$, $\int_0^1 \tilde{g}(\alpha)\, d\alpha = 1$ and*

$$\forall \alpha, \gamma \in [0,1] \qquad \alpha \geq \gamma \implies \tilde{f}(\alpha)\tilde{g}(\gamma) - \tilde{f}(\gamma)\tilde{g}(\alpha) \geq 0\,. \tag{8}$$

Then if h is an increasing function we have

$$\int_0^1 h(\alpha)\tilde{f}(\alpha)\, d\alpha - \int_0^1 h(\alpha)\tilde{g}(\alpha)\, d\alpha \geq 0;$$

if h is a decreasing function we have

$$\int_0^1 h(\alpha)\tilde{f}(\alpha)\, d\alpha - \int_0^1 h(\alpha)\tilde{g}(\alpha)\, d\alpha \leq 0\,.$$

Proof We have

$$
\begin{aligned}
&\int_0^1 h(\alpha)\tilde{f}(\alpha)\,d\alpha - \int_0^1 h(\alpha)\tilde{g}(\alpha)\,d\alpha \\
&= \int_0^1 h(\alpha)\tilde{f}(\alpha)\,d\alpha \int_0^1 \tilde{g}(\gamma)\,d\gamma - \int_0^1 h(\gamma)\tilde{g}(\gamma)\,d\alpha \int_0^1 \tilde{f}(\alpha)\,d\alpha \\
&= \int_0^1 \int_0^1 (h(\alpha) - h(\gamma))\tilde{f}(\alpha)\tilde{g}(\gamma)\,d\alpha\,d\gamma \\
&= \int_0^1 d\alpha \int_0^\alpha (h(\alpha) - h(\gamma))\tilde{f}(\alpha)\tilde{g}(\gamma)\,d\gamma + \int_0^1 d\alpha \int_\alpha^1 (h(\alpha) - h(\gamma))\tilde{f}(\alpha)\tilde{g}(\gamma)\,d\gamma \\
&= \int_0^1 d\alpha \int_0^\alpha (h(\alpha) - h(\gamma))\tilde{f}(\alpha)\tilde{g}(\gamma)\,d\gamma + \int_0^1 d\gamma \int_0^\gamma (h(\alpha) - h(\gamma))\tilde{f}(\alpha)\tilde{g}(\gamma)\,d\alpha \\
&= \int_0^1 d\alpha \int_0^\alpha (h(\alpha) - h(\gamma))\tilde{f}(\alpha)\tilde{g}(\gamma)\,d\gamma + \int_0^1 d\alpha \int_0^\alpha (h(\gamma) - h(\alpha))\tilde{f}(\gamma)\tilde{g}(\alpha)\,d\gamma \\
&= \int_0^1 d\alpha \int_0^\alpha (h(\alpha) - h(\gamma))[\tilde{f}(\alpha)\tilde{g}(\gamma) - \tilde{f}(\gamma)\tilde{g}(\alpha)]\,d\gamma\,.
\end{aligned}
$$

The assertions follow from (8).

Proposition 2 *Let* $\theta_1, \theta_2 : [0,1] \to]0,1]$ *such that*

$$
\forall\,\alpha, \gamma \in [0,1] \qquad \alpha \geq \gamma \implies \theta_1(\alpha)\theta_2(\gamma) - \theta_1(\gamma)\theta_2(\alpha) \geq 0\,. \tag{9}
$$

Then

$$
\varepsilon_f(\theta_2) \geq \varepsilon_f(\theta_1)
$$

and

$$
C^*_{f,\theta_2}(A) \supseteq C^*_{f,\theta_1}(A)\,.
$$

Proof If we choose

$$
\tilde{f}(\alpha) = \frac{\theta_1(\alpha) f(\alpha)}{\int_0^1 \theta_1(\alpha) f(\alpha)\,d\alpha} \qquad \tilde{g}(\alpha) = \frac{\theta_2(\alpha) f(\alpha)}{\int_0^1 \theta_2(\alpha) f(\alpha)\,d\alpha}
$$

we obtain

$$
\varepsilon_f(\theta_2) - \varepsilon_f(\theta_1) = \int_0^1 \alpha\tilde{f}(\alpha)\,d\alpha - \int_0^1 \alpha\tilde{g}(\alpha)\,d\alpha\,.
$$

It is trivial to see that as θ_1 and θ_2 verify (9) then even relation (8) is true. If in Lemma 1 we choose the increasing function $h(\alpha) = \alpha$ we obtain

$$
\varepsilon_f(\theta_2) - \varepsilon_f(\theta_1) \geq 0.
$$

Moreover, if we choose the decreasing function $h(\alpha) = spr(A_\alpha)$ from Lemma 1 we get

$$spr(C^*_{f,\theta_1}(A)) - spr(C^*_{f,\theta_2}(A)) = \int_0^1 spr(A_\alpha)\tilde{f}(\alpha)\, d\alpha - \int_0^1 spr(A_\alpha)\tilde{g}(\alpha)\, d\alpha \leq 0\,.$$

Since $mid(C^*_{f,\theta_1}(A)) = mid(C^*_{f,\theta_2}(A))$ we have $C^*_{f,\theta_1}(A) \subseteq C^*_{f,\theta_2}(A)$.

Property (9) means that for $\alpha \geq \gamma$ we have $\frac{\theta_1(\alpha)}{\theta_1(\gamma)} \geq \frac{\theta_2(\alpha)}{\theta_2(\gamma)}$. Thus, if θ_1 increases faster than θ_2, the approximation interval will have a smaller spread.

Corollary 1 (i) *If θ is constant then $\varepsilon_f(\theta) = 0$ and $C^*_{f,\theta}(A) = M_f(A)$;*
(ii) *if θ is a decreasing function then $\varepsilon_f(\theta) \geq 0$ and $C^*_{f,\theta}(A) \supseteq M_f(A)$;*
(iii) *if θ is an increasing function then $\varepsilon_f(\theta) \leq 0$ and $C^*_{f,\theta}(A) \subseteq M_f(A)$.*

Proof Assertion (i) follows by observing that if $\theta_1 = \bar{\theta}$ is constant, with $\bar{\theta} \in [0, 1]$, then $\varepsilon_f(\bar{\theta}) = 0$ and $C^*_{f,\bar{\theta}}(A) = M_f(A)$. To prove (ii), assume that θ is decreasing. The functions $\theta_1 = \bar{\theta}$ (constant) and $\theta_2 = \theta$ satisfy the property (8). Thus, from Proposition 2 we have

$$\varepsilon_f(\theta) \geq \varepsilon_f(\bar{\theta}) = 0$$

and

$$C^*_{f,\theta}(A) \supseteq C^*_{f,\bar{\theta}}(A) = M_f(A)\,.$$

Assertion (iii) follows in a similar way, assuming θ increasing and applying Proposition 2 to the functions $\theta_1 = \theta$ and $\theta_2 = \bar{\theta}$.

6 Trapezoidal Fuzzy Numbers

In this section we are interested in showing some results for a nonmonotonic θ to have a more evident connection between $\varepsilon_f(\theta)$ and the approximation interval size. Starting with the particular case of a trapezoidal fuzzy number to reach our aims we present a particular parametric representation of $\theta(\alpha)$ that includes the increasing, decreasing, and nonmonotonic case.
For a trapezoidal fuzzy number A the α-cuts are

$$A_\alpha = [a_1 + \alpha(a_2 - a_1), a_4 - \alpha(a_4 - a_3))] \qquad 0 \leq \alpha \leq 1\,.$$

Observing that

$$mid(A_\alpha) = \frac{a_1 + a_4}{2} + \frac{a_2 - a_1 + a_3 - a_4}{2}\alpha$$

$$spr(A_\alpha) = \frac{a_4 - a_1}{2} - \frac{a_4 - a_3 + a_2 - a_1}{2}\alpha$$

we obtain from (2)

$$mid(M_f(A)) = \frac{a_1 + a_4}{2} + \frac{a_2 - a_1 + a_3 - a_4}{2} e_f$$
$$spr(M_f(A)) = \frac{a_4 - a_1}{2} - \frac{a_4 - a_3 + a_2 - a_1}{2} e_f$$

where e_f is defined in (5). Note that the f-weighted interval-valued possibilistic mean of a trapezoidal fuzzy number is the α-cut at the level of the preference index value e_f, that is $M_f(A) = A_{e_f} = [a_L(e_f), a_R(e_f)]$. The approximation interval $C^*_{f,\theta}(A)$ is given by

$$mid(C^*_{f,\theta}(A)) = mid(M_f(A)) \tag{10}$$

and, from (3),

$$spr(C^*_{f,\theta}(A)) = \frac{a_4 - a_1}{2} - \frac{a_4 - a_3 + a_2 - a_1}{2} k_{f,\theta}$$

where $k_{f,\theta}$ is defined in (6). Observe that $mid(C^*_{f,\theta}(A)) = mid(A_{e_f})$ and $spr(C^*_{f,\theta}(A)) = spr(A_{k_{f,\theta}})$. Furthermore, the larger $k_{f,\theta}$ is, the smaller the spread of approximation will be. We have from (7)

$$spr(C^*_{f,\theta}(A)) - spr(M_f(A)) = \frac{a_4 - a_3 + a_2 - a_1}{2} \varepsilon_f(\theta)\,. \tag{11}$$

From (10) and (11) we obtain the following general result for trapezoidal fuzzy numbers

Proposition 3 (i) *If* $\varepsilon_f(\theta) = 0$ *then* $C^*_{f,\theta}(A) = M_f(A)$*;*
(ii) *if* $\varepsilon_f(\theta) > 0$ *then* $C^*_{f,\theta}(A) \supset M_f(A)$*;*
(iii) *if* $\varepsilon_f(\theta) < 0$ *then* $C^*_{f,\theta}(A) \subset M_f(A)$.

6.1 Example

Let us consider for $0 < \beta < 1, n > 0$

$$\theta(\alpha) = \begin{cases} \left(\frac{\beta-\alpha}{\beta}\right)^n & \alpha < \beta \\ \left(\frac{\alpha-\beta}{1-\beta}\right)^n & \alpha \geq \beta \end{cases} \qquad \alpha \in [0, 1]$$

Note that if $\beta = 1/2$ we have $\theta(\alpha) = |1 - 2\alpha|^n$.
Furthermore, we may consider the limit cases $\beta = 0$ corresponding to the increasing

function $\theta(\alpha) = \alpha^n$, and $\beta = 1$ corresponding to the decreasing function $\theta(\alpha) = (1-\alpha)^n$. In the following we will denote $\varepsilon_f(n, \beta) = \varepsilon_f(\theta)$.

6.2 Case $f(\alpha) = 1$

In the case $f(\alpha) = 1$ we have $M_f(A) = EI(A)$ where $EI(A)$ is the expected interval. We obtain

$$e_f = \frac{\int_0^1 \alpha \, d\alpha}{\int_0^1 d\alpha} = \frac{1}{2}, \qquad k_{f,\theta} = \frac{\int_0^1 \alpha \, \theta(\alpha) \, d\alpha}{\int_0^1 \theta(\alpha) \, d\alpha} = \frac{(1-\beta)n+1}{n+2}.$$

If $C^*(A) = C^*(A; f, n, \beta)$ is the approximation interval, when $n \to 0$ we have $C^*(A) \to EI(A)$. From Proposition 3 we obtain by computation

(i) $\beta = 1/2 \implies \varepsilon_f(n, \beta) = 0$ (for all $n > 0$) $\implies C^*(A) = EI(A)$,
(ii) $\beta > 1/2 \implies \varepsilon_f(n, \beta) > 0$ (for all $n > 0$) $\implies C^*(A) \supset EI(A)$,
(iii) $\beta < 1/2 \implies \varepsilon_f(n, \beta) < 0$ (for all $n > 0$) $\implies C^*(A) \subset EI(A)$.

Numerical example. To show how the interval approximation proposed works, we consider the symmetric trapezoidal fuzzy number $A = (1, 2, 4, 5)$ shown in Fig. 1 and compute the approximation interval of A for different values of parameters β and n when $f(\alpha) = 1$. By computation we obtain $EI(A) = [\frac{3}{2}, \frac{9}{2}]$, $C^*(A) = C^*(A; f, n, \beta) = [c_L^*, c_R^*] = \left[\frac{(2-\beta)n+3}{n+2}, \frac{(4+\beta)n+9}{n+2}\right]$ and that the smaller β is, the smaller $C^*(A)$ will be. Fig. 2a shows the approximation interval, in the case $n = 1$, for each level β. When $\beta = 0.5$ we have $C^*(A) = EI(A)$. In Fig. 2b we have represented the interval approximation if $n = 0$ (continuous line), $n = 1$ (dashed line), $n = 2$ (dotted line), $n = \infty$ (dashed–dotted line). Note that when $n = 0$ we have $C^*(A) = EI(A)$ and when $n \to +\infty$ we have $C^*(A) \to [2-\beta, 4+\beta]$. Furthermore, n can be interpreted as an intensification parameter because when $n \to +\infty$ we have $C^*(A) \searrow$ if $\beta < 1/2$ and $C^*(A) \nearrow$ if $\beta > 1/2$.

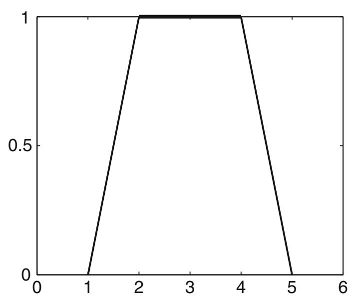

Fig. 1 Trapezoidal fuzzy number

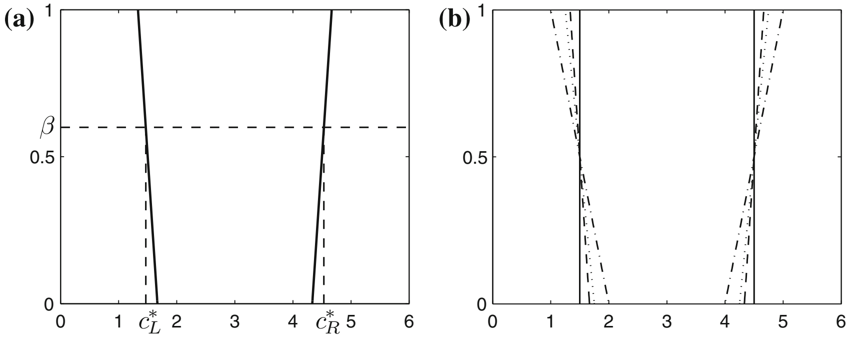

Fig. 2 Interval approximation

6.3 Case $f(\alpha) = \alpha$

In the case $f(\alpha) = \alpha$ we have $M_f(A) = M(A)$ where $M(A)$ is the interval-valued possibilistic mean. We get

$$e_f = \frac{\int_0^1 \alpha^2\, d\alpha}{\int_0^1 \alpha\, d\alpha} = \frac{2}{3}$$

$$k_{f,\theta} = \frac{\int_0^1 \alpha^2\, \theta(\alpha)\, d\alpha}{\int_0^1 \alpha\, \theta(\alpha)\, d\alpha} = \frac{(1-\beta)n^2 + (3-\beta-2\beta^2)n + 2}{(1-\beta)n^2 + (4-3\beta)n + 3}\,.$$

By computation we obtain that, for $n > 0$ fixed, the equation $\varepsilon_f(n,\beta) = 0$ has a unique solution in the interval (0, 1) given by

$$\beta_*(n) = \frac{1}{4} + \frac{\sqrt{n^2+18n+33} - n}{12}\,.$$

The solution $\beta_*(n)$ is increasing with respect to n, $\lim_{n\to+\infty} \beta_*(n) = 1$ and

$$\lim_{n\to 0} \beta_*(n) = \frac{1}{4} + \frac{\sqrt{33}}{12}\,.$$

From Proposition 3 we obtain

(i) $\beta = \beta_*(n) \implies \varepsilon_f(n,\beta) = 0 \implies C^*(A) = M(A)$,
(ii) $\beta > \beta_*(n) \implies \varepsilon_f(n,\beta) > 0 \implies C^*(A) \supset M(A)$,
(iii) $\beta < \beta_*(n) \implies \varepsilon_f(n,\beta) < 0 \implies C^*(A) \subset M(A)$.

Remark 2 Note that for $\beta = 1/2$ in the case $f(\alpha) = 1$ we have $C^*(A) = EI(A)$ and in the case $f(\alpha) = \alpha$ we have $C^*(A) \subset M(A)$.

7 Conclusion

In this paper we have seen a new generalization of the Trutschnig et al. distance to evaluate the nearest interval to a fuzzy number. We conclude that the classical interval connected with a fuzzy number as $M_f(A)$ has the same middle point of the approximation interval we propose. So if we want to have the average value of a fuzzy number we may use the two methods indifferently. But if we want to have information on its ambiguity or other quantities connected with its spread, we obtain a different evaluation by working with the method we propose.
We are working in the direction to apply this result to the triangular and trapezoidal approximation of a fuzzy number.

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