Maximizing Flows with Message-Passing: Computing Spatially Continuous Min-Cuts

Egil Bae[1], Xue-Cheng Tai[3], and Jing Yuan[2]

[1] Department of Mathematics, University of California, Los Angeles, USA
ebae@math.ucla.edu
[2] Department of Medical Biophysics, Schulich Medical School, Western University, Canada
cn.yuanjing@gmail.com
[3] Department of Mathematics, University of Bergen, Norway
tai@math.uib.no

Abstract. In this work, we study the problems of computing spatially continuous cuts, which has many important applications of image processing and computer vision. We focus on the convex relaxed formulations and investigate the corresponding flow-maximization based dual formulations. We propose a series of novel continuous max-flow models based on evaluating different constraints of flow excess, where the classical pre-flow and pseudo-flow models over graphs are re-discovered in the continuous setting and re-interpreted in a new variational manner. We propose a new generalized proximal method, which is based on a specific entropic distance function, to compute the maximum flow. This leads to new algorithms exploring flow-maximization and message-passing simultaneously. We show the proposed algorithms are superior to state of art methods in terms of efficiency.

1 Introduction

Many problems in image processing and computer vision can be modeled and formulated by the theory of Markov Random Fields (MRF) over graphs, in terms of computing a maximum a posteriori probability (MAP) estimate, see [23] for reference. Graph-cuts and message-passing, e.g. [5,4,30,31,19] are two main categories of efficient algorithms for the combinatorial optimization problem. However, graph-based methods suffer from visible grid bias, and reducing such bias requires either adding more neighbors locally or considering high-order cliques, which inevitably leads to a more intensive computation and memory cost.

On the other hand, variational methods can be applied to solve the same class of optimization problems in the spatially continuous setting, while avoiding the metrication errors generated by combinatorial algorithms. In particular, convex relaxation methods [21,7,15,34,24,9,2,20] were recently developed by relaxing the discrete constraint to some convex set, which leads great advantages both in theory and numerics: the convex optimization theory is well-established, efficient and reliable solvers are available with provable convergence properties, and also

X.-C. Tai et al. (Eds.): EMMCVPR 2015, LNCS 8932, pp. 15–28, 2015.

easy to handle large-scale computation and speed up by GPUs. In this regard, the proximal method plays the central element to build up a wide range of efficient first-order methods, see e.g. [11,10] for references.

1.1 Contributions

In this work, we propose a series of max-flow dual formulations, to compute minimum cuts in the continuous setting. In contrast to previous work on continuous max-flow [33,1], we formulate the flow excess constraints in different ways, which directly lead to new generalized proximal algorithms, where the Bregman divergence acts as the distance measurement for updating the labeling function. We propose primal-dual algorithmic schemes which combine both a flow-maximizing step and message-passing step in one unified numerical framework. This reveals close connections between the proposed flow-maximization methods and the classical methods, where 'cuts' over the graphs can be computed by maximizing flows or propagating messages. Finally, we compare the proposed algorithms with state-of-art continuous optimization methods: the Split-Bregman algorithm [15], the primal-dual algorithm [10] and the max-flow algorithm in [33] through experiments.

2 Revisit: Max-flow and Full-Flow Representation

Many discrete optimization problems in image processing and computer vision can be formulated as finding the minimum cut over appropriate graphs, as first observed by Greig et. al. [16]. The two most efficient combinatorial algorithms for computing the minimum cut solve the dual max-flow problem over the graph, and are called the Ford Fulkerson algorithm [13] and push-relabel algorithm [14]. More recently, continuous max-flow algorithms [33] have been proposed that are able to solve isotropic versions of the min-cut / max flow problem by convex optimization techniques. Both the continuous max-flow algorithm in [33] and the Ford Fulkerson algorithm solve a *full-flow representation* of the max-flow problem, in contrast to the pseudo-flow representation in the push-relabel algorithm and the algorithms in this paper.

2.1 Discrete Min-cut and Max-flow Models

A graph $\mathcal{G}$ is a pair $(\mathcal{V}, \mathcal{E})$ consisting of a vertex set $\mathcal{V}$ and an edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. We let $C(v, w) \geq 0$ denote the cost / weight / capacity on edge (v, w) and use the convention $C(v, w) = 0$ if there is no edge (v, w). In the min-cut and max-flow problems, there are two special vertices in addition to $\mathcal{V}$, a source vertix s and a sink vetrex t. The min-cut problem is to find a partition of $\mathcal{V} \cup s \cup t$ into two sets V_s and V_t, such that $s \in V_s$ and $t \in V_t$ with smallest cost possible, i.e. to solve

$$\min_{V_s, V_t} \sum_{v \in V_s, w \in V_t} C(v, w), \quad \text{s.t. } s \in V_s,\ t \in V_t, \quad V_s \cup V_t = V, \quad V_s \cap V_t = \emptyset \quad (1)$$

It is well known that the min-cut problem (1) is dual to the maximum flow problem over the same graph. We let $p_s(v)$ denote the flow on the edge (s, v) and $C_s(v)$ denote its capacity $C(s, v)$. Similarly, $p_t(v)$ and $C_t(v)$ are the flow and capacity on (v, t) and $p(v, w)$ the flow on (v, w). The maximum flow problem can be formulated as follows

$$\max_{p_s} \sum_{v \in \mathcal{V}} p_s(v) \tag{2}$$

$$\text{s.t. } |p(v,w)| \le C(v,w) \quad p_s(v) \le C_s(v) \quad p_t(v) \le C_t(v) \quad \forall v, w \in V \tag{3}$$

$$\sum_{(w,v) \,:\, w \in V} p((w,v)) - p_s(v) + p_t(v) = 0 \quad \forall v \in V \tag{4}$$

where the objective (2) is to push the maximum amount of flow from the source to the sink under flow capacity constraints (3). Additionally, the flow conservation constraint (4) should hold, which states that the total amount of incoming flow should be balanced by the amount of outgoing flow at each vertex.

The classical Ford-Fulkerson algorithm [13] solves the max-flow problem (2) by successively pushing flow from s to t along non-saturated paths, while maintaining the flow conservation constraint (4) each iteration. In this paper, we also call (2) subject to (3) and (4), the *full-flow representation* of max-flow.

2.2 Continuous Min-cut and Max-flow Models

In the spatially continuous setting, the min-cut problem (1), especially for image segmentation, can be similarly formulated in terms of finding the two segments $S, \Omega \backslash S \subset \Omega$ such that

$$\min_S \int_S C_s(x)\, dx + \int_{\Omega \backslash S} C_t \, dx + \int_{\partial S} C(s)\, ds \,, \tag{5}$$

where $C_s(x)$ and $C_t(x)$ are pointwise costs for assigning any x to the foreground S and background $\Omega \backslash S$ respectively. As proposed by [21,7], this problem can be solved globally and exactly by solving the *continuous min-cut* as follows

$$\min_{u(x) \in [0,1]} E(u) = \int_\Omega (1-u) C_s \, dx + \int_\Omega u C_t \, dx + \int_\Omega C(x) \left| \nabla u \right|_2 \, dx \,, \tag{6}$$

which results in a convex optimization problem. Further studies can be found in [22,15] etc.

Continuous Max-flow: Full-Flow Representation. An interesting study on the *continuous min-cut model* (6) was proposed in [32,33], which built up the duality connection between (6) to the so-called *continuous max-flow model*. It directly presents the analogue to the well-known duality beetween max-flow and min-cut [12] discussed above.

As the discrete graph configuration shown above, given the continuous image domain Ω and two terminals, link the source s and the sink t to each pixel $x \in \Omega$

respectively; define three flow fields around the pixel x: $p_s(x) \in \mathbb{R}$ directed from the source s to x, $p_t(x) \in \mathbb{R}$ directed from x to the sink t and the spatial flow field $p(x) \in \mathbb{R}^2$ around x within the image plain.

By the above spatially continuous setting, the *continuous max-flow model* tries to maximize the total flow passing from the source s:

$$\max_{p_s, p_t, p} \int_{\Omega} p_s \, dx \tag{7}$$

subject to the three flow capacity constraints:

$$p_s(x) \,\leq\, C_s(x)\,, \quad p_t(x) \,\leq\, C_t(x)\,, \quad |p(x)|_2 \,\leq\, C(x)\,, \quad \forall x \in \Omega\,. \tag{8}$$

and the flow conservation condition:

$$p_t(x) - p_s(x) + \operatorname{div} p(x) \,=\, 0\,, \quad \forall x \in \Omega\,. \tag{9}$$

The authors [32,33] proved that the continuous max-flow model (7) is equivalent to the continuous min-cut problem (6) in terms of primal and dual, where the labeling function $u(x)$ just works as a multiplier to the linear flow conservation condition (9). To see this, the equivalent primal-dual model

$$\min_u \max_{p_s, p_t, p} \int_{\Omega} p_s \, dx \,+\, \langle u, p_t - p_s + \operatorname{div} p \rangle \;, \tag{10}$$

subject to the flow capacity constraints (8) was considered. The flow conservation condition (9) played a central role in constructing the duality between the max-flow and min-cut models: (7) and (6).

We call (7) the *full-flow representation* of the continuous max-flow model in this paper. In the following sections, we will discuss the other two continuous max-flow models which are distinct from the full-flow representation model (7). We will see that different continuous max-flow models can be constructed through variants of flow preservation (9), while the full-flow representation model (7) just corresponds to the balance of in-flow and out-flow.

To compute a solution to (6) or (7), discretization of the domain Ω is necessary. One fundamental difference to the discrete max-flow and min-cut models is the rotationally invariant 2-norm in (6) and (8), which corresponds to the Euclidean perimeter in (5). In this paper we assume a general discretized image domain and differential operators when deriving the duality theory, but we keep the continuous notation ∇, div, $\int$ to ease readability. To derive rigorous existence proofs for infinite dimensional spaces is quite involved and out of the scope of this conference paper.

3 Continuous Max-flow Models Represented by Pre-flows and Pseudo-flows

In this section, we propose and study two other continuous max-flow models in terms of the representations of *pre-flows* and *pseudo-flows*. Both models are dual to the continuous min-cut model (6).

3.1 Continuous Max-flow: Pre-flow Representation

Now we partially optimize the max-flow model (7) by maximizing over the source flow $p_s(x) \leq C_s(x)$. By simple computation, we can prove that

Proposition 1. *The continuous max-flow model* (7) *is equivalent to the following flow-maximization problem:*

$$\max_{p_t, p} \int_\Omega p_t \, dx \tag{11}$$

$$s.t. \; C_s(x) - \operatorname{div} p(x) - p_t(x) \, \geq \, 0 \, , \quad \forall x \in \Omega \tag{12}$$

$$p_t(x) \, \leq \, C_t(x) \, , \quad |p(x)| \, \leq \, C(x) \, , \quad \forall x \in \Omega . \tag{13}$$

Proof. We first observe that the max-flow model (7) can be equivalently formulated as

$$\max_{p_t, p} \int_\Omega p_t \, dx \tag{14}$$

$$\text{s.t. } p_s(x) + \operatorname{div} p(x) - p_t(x) \, = \, 0 \, , \quad \forall x \in \Omega \tag{15}$$

$$p_s(x) \, \leq \, C_s(x) \, , \quad p_t(x) \, \leq \, C_t(x) \, , \quad |p(x)| \, \leq \, C(x) \, , \quad \forall x \in \Omega . \tag{16}$$

This just comes from the fact that the total source flow $\int p_s \, dx$ equals to the total sink flow $\int p_t \, dx$, due to the flow balance condition (9). Changing the positive direction of flows p_s and p_t in (7), we then have (14).

Therefore, by the same procedures as in [32], optimizing (14) over the constraint $p_s(x) \, \leq \, C_s(x)$, we see that (14) can be equivalently expressed as

$$\min_{u \geq 0} \max_{p_t, p} \int_\Omega p_t \, dx \, + \, \langle u, C_s + \operatorname{div} p - p_t \rangle \tag{17}$$

$$\text{s.t. } p_t(x) \, \leq \, C_t(x) \, , \quad |p(x)| \, \leq \, C(x) \quad \forall x \in \Omega .$$

where u is a Lagrange multiplier for $C_s + \operatorname{div} p - p_t \geq 0$. Clearly, (17) is just the primal-dual formulation of (11). Hence, we have:

$$(7) \iff (14) \iff (17) \iff (11) \, .$$

The equivalence between (7) and (11) is proved.

Obviously, (11) gives another continuous max-flow model which tries to maximize the total flow streaming out to the sink t while keeping the maximum source flow $p_s(x) = C_s(x)$. We see that the excess of flows at each pixel is no longer constrained to vanish, but to be non-negative (12), i.e. the flow conservation condition (9) is not kept.

Moreover, we will show that (11) results in a novel max-flow algorithm, in the continuous context, which has similar steps as the well-known *push-relabel* algorithm proposed in [14]. With this perspective, the constraint (12) recovers the *pre-flow* condition. We call (11) the *pre-flow representation* of the continuous max-flow model. In view of (17), we have that

Proposition 2. *The pre-flow based max-flow model* (11) *is dual to the continuous min-cut problem* (6)*, and also equivalent to its primal-dual model*

$$\min_{u \geq 0} \max_{p_t, p} \int_\Omega p_t \, dx \; + \; \langle u, C_s + \operatorname{div} p - p_t \rangle \tag{18}$$
$$s.t. \; p_t(x) \; \leq \; C_t(x) \, , \; |p(x)| \; \leq \; C(x) \quad \forall x \in \Omega .$$

The proof follows by (17).

3.2 Continuous Max-flow: Pseudo-flow Representation

By maximizing the continuous max-flow model (7) over the flows $p_s(x) \leq C_s(x)$ and $p_t(x) \leq C_t(x)$ simultaneously, we have that

Proposition 3. *The continuous max-flow model* (7) *is equivalent to the following flow-maximization problem:*

$$\max_{|p(x)| \, \leq \, C(x)} \int_\Omega \min(0, C_t + \operatorname{div} p - C_s) \, dx \, , \tag{19}$$

The flow excess at each point $(C_t + \operatorname{div} p - C_s)(x) \neq 0$ is neither balanced nor non-negative, i.e. the pseudo-flow condition. Problem (19) is also related to the dual formulation of multi-region partitions proposed in [2].

Proof. Following the same steps in [32], optimizing the continuous max-flow model (7) over $p_s(x) \leq C_s(x)$ and $p_t(x) \leq C_t(x)$ results in

$$\min_{u(x) \in [0,1]} \max_{|p(x)| \, \leq \, C(x)} \int_\Omega u \, (C_t + \operatorname{div} p - C_s) \, dx \tag{20}$$

The min and max operators are interchangeable, by the minimax theorem. Then, by minimizing the above functional over $u(x) \in [0,1]$ at each pixel $x \in \Omega$, we obtain the optimization problem (19).

The formulation (19) emphasizes: first, the flow excess at each pixel x is neither balanced nor non-negative (pre-flow condition); actually, the flow excess can be either positive or negative; second, the object is to find the spatial flow field $p(x)$ which maximizes the total negative flow excess, i.e. $(C_t + \operatorname{div} p - C_s)(x) \leq 0$. Observe that we find the third equivalent max-flow model in terms of the *pseudo-flow* condition, proposed in [17]. In this regard, we call (19) the *pseudo-flow representation* of the continuous max-flow model. In the following sections, we propose a new algorithm associated to the peudo-flow based max-flow model (19).

4 Entropic Proximal Max-flow Algorithms

In this section, we consider the generalized proximal method to solve the newly proposed continuous max-flow models: (11) and (19) which are dual to the continuous min-cut problem (6). We will see that such proximal method based on the generalized entropic distance functions leads to the generalized augmented Lagrangian method [28,18], and builds up a class of novel continuous max-flow algorithms which explores flow-maximization joint with message-passing simultaneously.

We first introduce the entropic proximal method using the generalized Bregman distance as its mapping kernel. Then we build up the new entropic-proximal based algorithms to the proposed continuous max-flow models (11) and (19). We also discuss their essential links to the *push-relabel* and *pseudoflow* algorithms over graphs.

4.1 Proximal Methods with Bregman Distance

Given the closed proper convex function $f(x)$, the proximal mapping of any point z is defined by [26]:

$$\mathrm{prox}_f(z) \,=\, (I + \lambda \partial f)^{-1}(z) \,=\, \arg\min_x \left\{ \frac{1}{2\lambda} \|x - z\|^2 + f(x) \right\}. \tag{21}$$

Then the classical proximal method [8] to minimize the function $f(x)$ can formulated as computing a sequence of proximal mappings iteratively:

$$x^{k+1} \,=\, (I + \lambda \partial f)^{-1}(x^k) = \arg\min_x \left\{ \frac{1}{2\lambda_k} \left\|x - x^k\right\|^2 + f(x) \right\}. \tag{22}$$

Convergence properties of the proximal method was studied in [27]. Its close connections to the augmented Lagrangian method were demonstrated in [25,28] by computing the iterative proximal mappings of the dual sequence.

The proximal method is one of important elements to design most the efficient first-order primal-dual algorithms [10]. One of its interesting extensions is to incorporate the generalized Bregman distance or divergence functions $D_g(x,y)$ [6] as the proximity measurement, which results in the entropic proximal method:

$$x^{k+1} \,=\, \arg\min_x \left\{ D_g(x, x^k) + f(x) \right\} \tag{23}$$

where

$$D_g(x,y) \,=\, g(x) - g(y) - \langle \partial g, x - y \rangle \,, \tag{24}$$

$g(x)$ is a differentiable and strictly convex function.

Clearly, the Bregman distance (24) provides a quite general conception on the proximity measurement: for example, the function $g(x) = \frac{1}{2}\|\cdot\|^2$ just gives the common squared Euclidean distance $\frac{1}{2}\|x - y\|^2$; the entropy function for the vector $x := (x_1, \ldots, x_n) \in (\mathbb{R}^+)^n$

$$g(x) = \sum_i (x_i \log x_i - x_i)$$

results in the generalized *Kullback-Leibler* divergence of two vectors $x, y \in (\mathbb{R}^+)^n$ such that

$$D_g(x,y) \,=\, \sum_{i=1}^n \Big(x_i \log(x_i/y_i) - x_i + y_i \Big) \,, \tag{25}$$

see also [3] for the definition of more Bregman distances.

Generalized Augmented Lagrangian Method. [28] showed the entropic proximal method (23) over the dual sequence just amounts to the *generalized augmented Lagrangian method*, which incorporates the classical augmented Lagrangian method as its special case with the quadratic Euclidean distance.

Now we consider the generalized optimization problem associated to the continuous max-flow models:

$$\min_{u \in C_u} \max_{p \in C_p} L(p, u) \, = \, f(p) + \langle u, G(p) \rangle \tag{26}$$

where C_u and C_p are the constraint sets on u and p respectively.

Let the dual function $D(u)$ be

$$D(u) \, := \, \max_{p \in C_p} L(p, u) \, .$$

As in [28], the entropic proximal method (23) to the dual function $D(u)$ gives the generalized augmented Lagrangian method

$$u^{k+1} \, = \, \arg \min_{u \in C_u} \, \left\{ c \, D_g(u, u^k) + D(u) \right\}$$

where c is some positive constant. Therefore, we have the corresponding augmented Lagrangian function as follows:

$$L_c(p, v) \, = \, \min_{u \in C_u} \left\{ L(p, u) + c \, D_g(u, v) \right\} .$$

The generalized augmented Lagrangian method contains the following two steps at each iteration:

$$p^{k+1} \, = \, \arg \max_{p \in C_p} \, L_{c^k}(p, u^k) \, , \tag{27}$$

$$u^{k+1} \, = \, \arg \min_{u \in C_u} \, \left\langle u, G(p^{k+1}) \right\rangle + c^k D_g(u, u^k) \, . \tag{28}$$

It is important to notice that the function $L_c(x, v)$ is the smoothed approximation to $L(x, u)$, hence better properties in numerics. In particular, when the quadratic L_2-norm is used as the distance function, then the classical augmented Lagrangian method is recovered.

In the following part, we propose and discuss a class of new continuous max-flow algorithms based on the entropic proximal method, especially the generalized augmented Lagrangian method. We will also show its close connections to the existing max-flow algorithms over graphs.

4.2 Entropic Proximal Max-flow Algorithm to (11)

For the pre-flow represented max-flow model (11), its corresponding primal-dual model (18) gives the common Lagrangian function:

$$\max_{p_t, p} \min_{u(x) \geq 0} \, L(p_t, p, u) \, := \, \int_\Omega p_t \, dx \, + \, \langle u, C_s + \operatorname{div} p - p_t \rangle \tag{29}$$

$$\text{s.t.} \;\; p_t(x) \, \leq \, C_t(x) \, , \; |p(x)| \, \leq \, C(x) \quad \forall x \in \Omega .$$

In this work, we consider the Kullback-Leibler distance (25) as the proximal function, i.e.

$$D(u,v) \;=\; \int_\Omega \Big\{ u(x)\log(u(x)/v(x)) - u(x) + v(x) \Big\}\, dx\,.$$

This results in the augmented Lagrangian function

$$L_c(p_t, p, v) \;=\; \min_{u(x)\geq 0} \; L(p_t, p, u) + c\, D(u,v) \tag{30}$$

From the first order optimality condition, we obtain the explicit expression for the minimizer $u = v\exp\big[-(C_s + \operatorname{div} p - p_t)/c\big]$ for $v \geq 0$, which leads to

$$L_c(p_t, p, v) \;=\; \int_\Omega \Big\{ p_t - c\,\big[v\, \exp\big\{ -\frac{C_s + \operatorname{div} p - p_t}{c} \big\} + 1\big] \Big\}\, dx\,, \quad v \geq 0. \tag{31}$$

In view of the step (27) of the generalized augmented Lagrangian method, the augmented Lagrangian function (30) can then be expressed, in terms of u^k at each iteration, as:

$$L_c(p_t, p, u^k) \;=\; \int_\Omega \Big\{ p_t - c\,\big[u^k\, \exp\big\{ -\frac{C_s + \operatorname{div} p - p_t}{c} \big\} + 1\big] \Big\}\, dx\, dx\,, \quad u^k \geq 0.$$

By means of (30), we have the new continuous max-flow algorithm corresponding to the pre-flow model (11):

Algorithm 4. *Initialize* $u^0(x) \in (0,1)$ $\forall x \in \Omega$, p_t^0, p^0. *For k=0,1,... until convergence, perform the following two steps (flow maximization and message passing):*

- *Maximize over the flows* p_t *and* p *by*

$$p_t^{k+1} \;:=\; \arg \max_{p_t(x) \leq C_t(x)} \; L_c(p_t, p^k, u^k)\,; \tag{32}$$

$$p^{k+1} \;:=\; \arg \max_{|p(x)| \leq C(x)} \; L_c(p_t^k, p, u^k)\,; \tag{33}$$

  where the step (32) *can be solved explicitly through simple variational computation and the step* (33) *can be solved iteratively, as shown below.*
- *Update the message function* u *by*

$$u^{k+1} \;:=\; u^k \exp\big\{ -\frac{C_s + \operatorname{div} p^{k+1} - p_t^{k+1}}{c} \big\} \tag{34}$$

For the flow-maximization step (32), it's easy to solve the given maximization problem explicitly by

$$p_t^{k+1} \;=\; \min \Big\{ C_s + \operatorname{div} p^k - c\log u^k,\; C_t \Big\}\,, \tag{35}$$

since $\frac{\partial L_c(p_t, p^k, u^k)}{\partial p_t} = 0$ if $p_t = C_s + \operatorname{div} p^k - c \log u^k$.

For the flow-maximization step (33), we apply one iteration of the projected-gradient method

$$p^{k+1} = \operatorname{Proj}_{|\,\cdot\,| \le C(x)} \left\{ p^k + \gamma \nabla \Big(u^k \exp\big\{ - \frac{C_s + \operatorname{div} p^k - p_t^k}{c} \big\} \Big) \right\}, \qquad (36)$$

where $\gamma > 0$ is the step-size.

4.3 Entropic Proximal Max-flow Algorithm to (19)

Likewise, for the pseudo-flow represented max-flow model (19), its corresponding primal-dual formulation (20) expresses the common Lagrangian function

$$\min_{u(x) \in [0,1]} \max_{|p(x)| \le C(x)} L(p, u) := \int_\Omega u \, (C_t + \operatorname{div} p - C_s) \, dx \, .$$

Consider the function $u(x) \in [0, 1]$, we apply the following Bregman distance as the proximal function

$$D(u, v) = \int_\Omega \left\{ u \log(\frac{u}{v}) + (1 - u) \log(\frac{1 - u}{1 - v}) \right\} dx \, .$$

The resulting augmented Lagrangian function is

$$L_c(p, v) = \min_{u(x) \in [0,1]} L(p, u) + c \, D(u, v) \qquad (37)$$

$$= -c \int_\Omega \log \left\{ (1 - v) + v \exp\big(- \frac{C_t + \operatorname{div} p - C_s}{c} \big) \right\} dx \, , \qquad (38)$$

where $c > 0$ works as the step-size.

Considering $\exp(0/c) = 1$, it is easy to see that

$$L_c(p, v) = -c \int_\Omega \log \left\{ (1 - v) \exp(\frac{0}{c}) + v \exp\big(- \frac{C_t + \operatorname{div} p - C_s}{c} \big) \right\} dx \, .$$

As $c \to 0^+$, we have the limit function [29]

$$\lim_{c \to 0^+} L_c(p, v) = \int_\Omega c \, \min(0, C_t + \operatorname{div} p - C_s) \, dx$$

which is just the original pseudo-flow represented max-flow model (19). To this end, we see that the augmented Lagrangian function (37) just works as the smoothed version of the energy function (19).

Following the step (27) of the generalized augmented Lagrangian method, the augmented Lagrangian function (37) can then be expressed, in terms of u^k at each iteration, as:

$$L_c(p, u^k) = -c \int_\Omega \log \left\{ (1 - u^k) + u^k \exp\big(- \frac{C_t + \operatorname{div} p - C_s}{c} \big) \right\} dx \, .$$

By means of (30), we have the new continuous max-flow algorithm to its pseudo-flow represented model (19):

Algorithm 5. *Initialize $u^0(x) \in (0,1)$ $\forall x \in \Omega$, p_t^0, p^0. For k=0,1,... until convergence, perform the following two steps (flow maximization and message passing):*

- *Maximize over the flows p by*

$$p^{k+1} := \arg \max_{|p(x)| \leq C(x)} L_c(p, u^k) \, ; \tag{39}$$

which can be solved approximately by one iteration of projected gradient.

- *Update the message function u by*

$$u^{k+1} := \frac{u^k \exp(-G^{k+1}/c)}{1 - u^k + u^k \exp(-G^{k+1}/c)} \, , \tag{40}$$

where for $\forall x \in \Omega$

$$G^{k+1}(x) = (C_t + \operatorname{div} p^{k+1} - C_s)(x) \, .$$

Algorithm 5 is similar to the smoothing dual algorithm proposed in [2] for multiphase partition problems. One crucial difference is that algorithm 5 solves the problem exactly without any smoothing approximation.

(a) (b) (c) (d)

Fig. 1. Segmentation with data term (41): (b) result on image (a) with $C(x) = \alpha = 0.5$, $c_1 = 0.15$ and $c_2 = 0.6$; (d) result on image (c) with $C(x) = \alpha = 0.25$, $c_1 = 0.16$ and $c_2 = 0.5$

5 Experiments

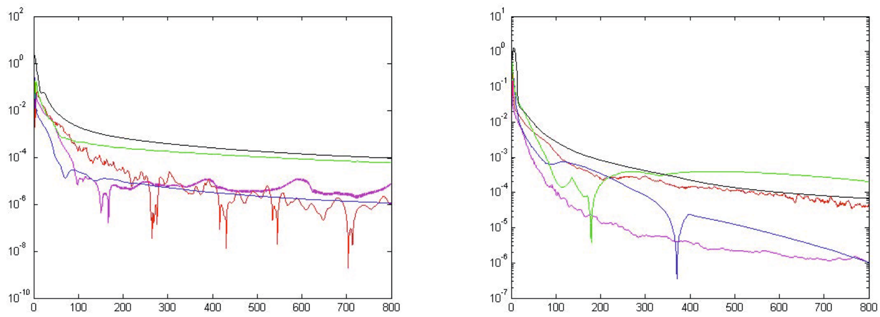
This section validates the convergence of the algorithms 4 and 5 on some image segmentation examples and comparisons are given to the previous max-flow algorithm [33], the Split-Bregman algorithm [15] and the primal-dual algorithm [10]. They are regarded as the state of the art algorithms for solving the convex partition problem. We choose the fidelity term

$$C_s(x) = |I(x) - c_1|^2, \quad C_t(x) = |I(x) - c_2|^2, \tag{41}$$

where I is the input image and c_1 and c_2 are two scalar gray values approximating the mean image intensities within each region. Results are shown in Figure 1. Figure 2 shows plots of the relative energy error

$$\frac{|E(u^k) - E(u^*)|}{E(u^*)}$$

where E is the energy (6), u^k is the solution at iteration k and u^* is the ground truth solution computed by 100000 iterations for each method. It can be observed that the two new variants of the max-flow algorithm converge at a similar rate as the old max-flow algorithm on example figure 1 (a), while on figure 1 (c) algorithm 5 is faster and algorithm 5 is slower. In both images all the max-flow algorithms converge considerably faster that the Split-Bregman and primal-dual algorithm. We speculate the reason for the faster convergence is that the max-flow algorithms avoids the projection step for incorporating the constraint $u(x) \in [0, 1], \forall x \in \Omega$. The CPU times are



(a) (b)

Fig. 2. Convergence of relative energy error $\frac{|E(u^k)-E(u^*)|}{E(u^*)}$ for iterations $k = 1, ..., 800$: (a) image 1(a); (b) image 1(c). The function u^* is the ground truth solutions computed by 100000 iterations of each method. Red is the new max-flow algorithm 5, magneta is new max-flow algorithm 4, blue is the old max-flow algorithm [33], green is Split-Bregman [15] and black is the primal-dual algorithm [10].

6 Conclusions

In this paper, we propose a series of novel flow-maximization models dual to the continuous min-cut problem by formulating the flow excess conditions in different ways. In theory, the proposed dual formulations discover and re-interpret the conventional pre-flow and pseudo-flow models over discrete graphs in the spatially continuous setting under a new variational perspective. In addition, the new dual formulations, i.e. the continuous max-flow models, directly lead to new generalized proximal dual optimization based algorithms, which embed both flow maximization and message-passing in a single algorithmic framework. Moreover, we show the proposed algorithms numerically outperform the state-of-art methods by experiments.

Acknowledgements. This research has been supported by the Norwegian Research Council eVita project 214889 and eVita project 166075 and Natural Sciences and Engineering Research Council of Canada (NSERC) Accelerator Grant R3584A04.

References

1. Appleton, B., Talbot, H.: Globally minimal surfaces by continuous maximal flows. IEEE Transactions on PAMI 28, 106–118 (2006)
2. Bae, E., Yuan, J., Tai, X.-C.: Global minimization for continuous multiphase partitioning problems using a dual approach. International Journal of Computer Vision 92(1), 112–129 (2011)
3. Banerjee, A., Merugu, S., Dhillon, I.S., Ghosh, J.: Clustering with bregman divergences. Journal of Machine Learning Research 6, 1705–1749 (2005)
4. Boykov, Y., Kolmogorov, V.: An experimental comparison of min-cut/max-flow algorithms for energy minimization in vision. IEEE Transactions on PAMI 26, 359–374 (2001)
5. Boykov, Y., Veksler, O., Zabih, R.: Fast approximate energy minimization via graph cuts. IEEE Transactions on PAMI 23, 1222 (2001)
6. Bregman, L.M.: The relaxation method of finding the common point of convex sets and its application to the solution of problems in convex programming. USSR Computational Mathematics and Mathematical Physics 7, 200–217 (1967)
7. Bresson, X., Esedoglu, S., Vandergheynst, P., Thiran, J.P., Osher, S.: Fast global minimization of the active contour/snake model. Journal of Mathematical Imaging and Vision 28(2), 151–167 (2007)
8. Censor, Y.A., Zenios, S.A.: Parallel Optimization: Theory, Algorithms and Applications. Oxford University Press (1997)
9. Chambolle, A., Cremers, D., Pock, T.: A convex approach for computing minimal partitions. Technical Report TR-2008-05, University of Bonn (November 2008)
10. Chambolle, A., Pock, T.: A first-order primal-dual algorithm for convex problems with applications to imaging. Journal of Mathematical Imaging and Vision 40(1), 120–145 (2011)
11. Esser, E., Zhang, X., Chan, T.F.: A general framework for a class of first order primal-dual algorithms for convex optimization in imaging science. SIAM J. Imaging Sciences 3(4), 1015–1046 (2010)
12. Ford, L.R., Fulkerson, D.R.: Flows in Networks. Princeton University Press, Princeton (1962)
13. Ford Jr., L.R., Fulkerson, D.R.: Maximal flow through a network. Canad. J. Math. 8, 399–404 (1956)
14. Goldberg, A.V., Tarjan, R.E.: A new approach to the maximum-flow problem. J. ACM 35(4), 921–940 (1988)
15. Goldstein, T., Bresson, X., Osher, S.: Geometric applications of the split bregman method: Segmentation and surface reconstruction. J. Sci. Comput. 45(1-3), 272–293 (2010)

16. Greig, D.M., Porteous, B.T., Seheult, A.H.: Exact maximum a posteriori estimation for binary images. J. Royal Stat. Soc., Series B, 271–279 (1989)
17. Hochbaum, D.S.: The pseudoflow algorithm: A new algorithm for the maximum-flow problem. Operations Research 56(4), 992–1009 (2008)
18. Iusem, A.N., Svaiter, B.F., Teboulle, M.: Entropy-like proximal methods in convex programming. Mathematics of Operations Research 19(4), 790–814 (1994)
19. Kolmogorov, V., Wainwright, M.J.: On the optimality of tree-reweighted max-product message-passing. In: UAI, pp. 316–323 (2005)
20. Lellmann, J., Kappes, J., Yuan, J., Becker, F., Schnörr, C.: Convex multi-class image labeling by simplex-constrained total variation. In: Tai, X.-C., Mørken, K., Lysaker, M., Lie, K.-A. (eds.) SSVM 2009. LNCS, vol. 5567, pp. 150–162. Springer, Heidelberg (2009)
21. Nikolova, M., Esedoglu, S., Chan, T.F.: Algorithms for finding global minimizers of image segmentation and denoising models. SIAM J. App. Math. 66(5), 1632–1648 (2006)
22. Olsson, C., Byröd, M., Overgaard, N.C., Kahl, F.: Extending continuous cuts: Anisotropic metrics and expansion moves. In: ICCV, pp. 405–412 (2009)
23. Paragios, N., Chen, Y., Faugeras, O.: Handbook of Mathematical Models in Computer Vision. Springer-Verlag New York, Inc., Secaucus (2005)
24. Pock, T., Schoenemann, T., Graber, G., Bischof, H., Cremers, D.: A convex formulation of continuous multi-label problems. In: Forsyth, D., Torr, P., Zisserman, A. (eds.) ECCV 2008, Part III. LNCS, vol. 5304, pp. 792–805. Springer, Heidelberg (2008)
25. Rockafellar, R.T.: Augmented Lagrangians and applications of the proximal point algorithm in convex programming. Math. Oper. Res. 1(2), 97–116 (1976)
26. Rockafellar, R.T.: Convex analysis. Princeton Mathematical Series, vol. 28. Princeton University Press, Princeton (1970)
27. Rockafellar, R.T.: Monotone operators and the proximal point algorithm. SIAM J. Control Optimization 14(5), 877–898 (1976)
28. Teboulle, M.: Entropic proximal mappings with applications to nonlinear programming. Math. Oper. Res. 17(3), 670–690 (1992)
29. Teboulle, M.: A unified continuous optimization framework for center-based clustering methods. J. Mach. Learn. Res. 8, 65–102 (2007)
30. Wainwright, M., Jaakkola, T., Willsky, A.: Map estimation via agreement on (hyper)trees: Message-passing and linear programming approaches. IEEE Transactions on Information Theory 51, 3697–3717 (2002)
31. Wainwright, M.J., Jaakkola, T., Willsky, A.S.: Map estimation via agreement on trees: message-passing and linear programming. IEEE Transactions on Information Theory 51(11), 3697–3717 (2005)
32. Yuan, J., Bae, E., Tai, X.C.: A study on continuous max-flow and min-cut approaches. In: CVPR, USA, San Francisco (2010)
33. Yuan, J., Bae, E., Tai, X.-C., Boykov, Y.: A spatially continuous max-flow and min-cut framework for binary labeling problems. Numerische Mathematik 126, 559–587 (2013)
34. Zach, C., Gallup, D., Frahm, J.-M., Niethammer, M.: Fast global labeling for real-time stereo using multiple plane sweeps. In: VMV 2008 (2008)