

Chapter 11

A Non-clamped Frictional Contact Problem with Normal Compliance

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Abstract In this chapter we study a dynamic frictional contact problem with normal compliance and non-clamped contact conditions, for thermo-viscoelastic materials. The weak formulation of the problem leads to a general system defined by a second order quasivariational evolution inequality coupled with a first order evolution equation. We state and prove an existence and uniqueness result, by using arguments on parabolic variational inequalities, monotone operators and fixed point. Then, we provide a numerical scheme of approximations and various numerical computations.

Keywords Thermo-viscoelasticity • Dynamic frictional process • Non-clamped condition • Normal compliance • Evolution inequality • Fixed point • Weak solution • Numerical simulations

AMS Classification. 74M15, 74M10, 74F05; 74S05, 74S20, 74H20, 74H25, 47J22

11.1 Introduction

Contact problems are omnipresent in mechanics, civil engineering, industry and everyday life, and represent a challenging topic, due to their important applications and various open questions they involve. In order to describe the behavior of deformable bodies subjected to various nonlinear and non-smooth solicitations such as contact, friction and thermal effects, mathematical models are necessary. They are useful in the study of a large number of problems related to impacts, cracks, packing, transport, process engineering and heat transfer. For this reason, the engineering and mathematical literature devoted to dynamic and quasistatic frictional contact

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problems, including their mathematical modeling, mathematical analysis, numerical analysis and numerical simulations, is continuously increasing.

An early study of contact problems within the mathematical analysis framework was done in the pioneering references [6, 11, 15]. For the error estimates analysis and numerical approximation, we refer the reader to [5, 7, 9, 19]. Functional nonlinear analysis results useful in the study of contact problems could be found in [2–4, 12, 20]. Mathematical models of frictional contact with viscoelastic and viscoplastic plastic materials have been studied in [10, 17, 18]. One of the purpose of these works was to show the cross-fertilization between various new and nonstandard models arising in contact mechanics and the abstract theory of variational inequalities. Further extensions to nonconvex contact conditions with nonmonotone and possible multivalued constitutive laws led to the recent domain of non-smooth mechanics within the framework of the so-called hemivariational inequalities. References in the field include [8, 13, 16].

This chapter is a companion work of our previous paper [1]. There, we studied a dynamic contact problem with friction, for thermo-viscoelastic materials with long memory and subdifferential boundary conditions. The model led to a system defined by a second order evolution inequality coupled with a first order evolution equation. An existence and uniqueness result for the displacement and the temperature fields has been established. Finally, a fully discrete scheme for numerical approximations was introduced and various numerical computations in dimension two have been provided.

In contrast, in this current work we investigate a dynamic contact problem with normal compliance and friction for thermo-viscoelastic materials with short memory. As in [1], the usual clamped condition has been deleted. This leads to a new and non-standard model of system defined by a second order quasi-variational inequality, coupled with a first order evolution equation. The main difficulties in the analysis of this model arise from the fact that Korn's inequality cannot be applied any more. Moreover, the model presents a strong nonlinearity due to the fact that the process is assumed to be frictional. Such kind of semi-coercive problems were first studied in [6] for Coulomb's friction models where the inertial term of the dynamic process has been used in order to compensate the loss of coerciveness in the a priori estimates. By a change of variable, we bring the coupled second order evolution inequality into a classical first order evolution inequality. Then, using a fixed point method frequently used in [10, 17], combined with monotonicity and convexity arguments, we prove the existence and uniqueness of the displacement and the temperature fields. Finally, to complete our study, we introduce a numerical scheme for the approximation of the solution and we perform numerical computations.

The chapter is organized as follows. In Sect. 11.2 we describe the mechanical problem, list the assumptions on the data, derive the variational formulation and then we state our main existence and uniqueness result, Theorem 11.1. In Sect. 11.3 we give the proof of the claimed result. In Sect. 11.4 we present several numerical simulations in the study of a two dimensional problem, which illustrate the evolution of the displacement and temperature fields.

11.2 The Contact Problem

The physical setting is as follows. A thermo-viscoelastic body occupies a bounded domain $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) with a Lipschitz boundary Γ that is partitioned into two disjoint measurable parts, Γ_F and Γ_C . Let $[0, T]$ be the time interval of interest, where $T > 0$. We assume that a volume force of density \mathbf{f}_0 acts in $\Omega \times (0, T)$ and that surface tractions of density \mathbf{f}_F apply on $\Gamma_F \times (0, T)$. The body may come in contact with an obstacle, the foundation, over the potential contact surface Γ_C . The contact is described with a normal compliance condition, with friction and heat exchange. Our aim is to study the dynamic evolution of the body, by using an appropriate mathematical model.

To this end, let us recall some classical notations, see e.g. [6, 10, 14] for further details. We denote by \mathbb{S}^d the space of second order symmetric tensors on \mathbb{R}^d , while “ \cdot ” and $\|\cdot\|$ will represent the inner product and the Euclidean norm on \mathbb{S}^d and \mathbb{R}^d . Let \mathbf{v} denote the unit outer normal on Γ . Everywhere in the sequel, the indices i, j, k, h run from 1 to d , summation over repeated indices is implied and the index that follows a comma represents the partial derivative with respect to the corresponding component of the independent variable. We use standard notation for continuous, L^p and Sobolev spaces of functions defined on Ω and Γ . In addition, we use the following notation:

$$H = L^2(\Omega)^d, \quad \mathcal{H} = \{ \boldsymbol{\sigma} = (\sigma_{ij}) \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega), 1 \leq i, j \leq d \},$$

$$H_1 = \{ \mathbf{u} \in H \mid \boldsymbol{\varepsilon}(\mathbf{u}) \in \mathcal{H} \}, \quad \mathcal{H}_1 = \{ \boldsymbol{\sigma} \in \mathcal{H} \mid \text{Div } \boldsymbol{\sigma} \in H \}.$$

Here $\boldsymbol{\varepsilon} : H_1 \rightarrow \mathcal{H}$ and $\text{Div} : \mathcal{H}_1 \rightarrow H$ are the deformation and the divergence operators, respectively, defined by

$$\boldsymbol{\varepsilon}(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})), \quad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \text{Div } \boldsymbol{\sigma} = (\sigma_{ij,j}).$$

The spaces H, \mathcal{H}, H_1 and \mathcal{H}_1 are real Hilbert spaces endowed with the canonical inner products given by

$$(\mathbf{u}, \mathbf{v})_H = \int_{\Omega} u_i v_i \, dx, \quad (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} = \int_{\Omega} \sigma_{ij} \tau_{ij} \, dx,$$

$$(\mathbf{u}, \mathbf{v})_{H_1} = (\mathbf{u}, \mathbf{v})_H + (\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}}, \quad (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}_1} = (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} + (\text{Div } \boldsymbol{\sigma}, \text{Div } \boldsymbol{\tau})_H.$$

The mechanical problem is then formulated as follows.

Problem \mathcal{P} . Find a displacement field $\mathbf{u} : \Omega \times [0, T] \rightarrow \mathbb{R}^d$, a stress field $\boldsymbol{\sigma} : \Omega \times [0, T] \rightarrow \mathbb{S}^d$ and a temperature field $\theta : \Omega \times [0, T] \rightarrow \mathbb{R}_+$ such that

$$\boldsymbol{\sigma}(t) = \boldsymbol{\varepsilon}(\mathbf{u}'(t)) + \mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}(t)) - \theta(t) C_e \quad \text{in } \Omega, \quad (11.1)$$

$$\mathbf{u}''(t) = \text{Div } \boldsymbol{\sigma}(t) + \mathbf{f}_0(t) \quad \text{in } \Omega, \tag{11.2}$$

$$\boldsymbol{\sigma}(t)\nu = \mathbf{f}_F(t) \quad \text{on } \Gamma_F, \tag{11.3}$$

$$-\sigma_\nu(t) = c_\nu (u_\nu(t) - g)_+ \quad \text{on } \Gamma_C, \tag{11.4}$$

$$\left. \begin{aligned} \|\boldsymbol{\sigma}_\tau(t)\| &\leq \mu_\tau c_\nu (u_\nu(t) - g)_+, \\ \|\boldsymbol{\sigma}_\tau(t)\| &< \mu_\tau c_\nu (u_\nu(t) - g)_+ \implies \mathbf{u}'_\tau(t) = \mathbf{0}, \\ \|\boldsymbol{\sigma}_\tau(t)\| &= \mu_\tau c_\nu (u_\nu(t) - g)_+ \\ &\implies \mathbf{u}'_\tau(t) = -\lambda \boldsymbol{\sigma}_\tau(t) \text{ for some } \lambda \geq 0, \end{aligned} \right\} \text{on } \Gamma_C, \tag{11.5}$$

$$\theta'(t) - \text{div}(K_c \nabla \theta(t)) = -c_{ij} \frac{\partial u'_i}{\partial x_j}(t) + q(t) \quad \text{on } \Omega, \tag{11.6}$$

$$-k_{ij} \frac{\partial \theta}{\partial x_j}(t) n_i = k_e (\theta(t) - \theta_R) \quad \text{on } \Gamma_C, \tag{11.7}$$

$$\theta(t) = 0 \quad \text{on } \Gamma_F, \tag{11.8}$$

for all $t \in [0, T]$ and, moreover,

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \mathbf{u}'(0) = \mathbf{v}_0, \quad \theta(0) = \theta_0 \quad \text{in } \Omega. \tag{11.9}$$

Here, (11.1) represents the thermo-visco-elastic constitutive law of the material in which \mathcal{A} is the viscosity tensor, \mathcal{B} is the elasticity operator and C_e denotes the thermal expansion tensor. Equation (11.2) represents the equation of motion in which we assume the mass density $\rho \equiv 1$. Condition (11.3) represents the traction boundary condition. Next, relation (11.4) represents the normal compliance contact condition in which σ_ν denotes the normal stress, c_ν is a positive constant related to the hardness of the foundation, u_ν represents the normal displacement and g is the initial gap between the foundation and the body. Here, the term $u_\nu(t) - g$ represents, when it is positive, the penetration of the surface asperities in those of the foundation. Conditions (11.5) represent a version of Coulomb’s dry friction law, where $\boldsymbol{\sigma}_\tau$ is the tangential stress, μ_τ represents the coefficient of friction and, finally, \mathbf{u}'_τ denotes tangential velocity. The differential equation (11.6) describes the evolution of the temperature field, where K_c represents the thermal conductivity tensor and q is the density of volume heat sources. The associated temperature boundary condition is given by (11.7) and (11.8), where θ_R is the temperature of the foundation, and k_e is the heat exchange coefficient between the body and the obstacle. Finally, the data $\mathbf{u}_0, \mathbf{v}_0, \theta_0$ in (11.9) represent the initial displacement, the initial velocity, and the initial temperature, respectively.

In order to derive the variational formulation of the mechanical problem (11.1)–(11.9), we need additional notation. Let $V = H_1$ be the space of admissible displacement fields, endowed with the inner product given by

$$(\mathbf{u}, \mathbf{v})_V = (\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} + (\mathbf{u}, \mathbf{v})_H \quad \forall \mathbf{u}, \mathbf{v} \in V,$$

and let $\|\cdot\|_V$ be the associated norm, i.e.

$$\|\mathbf{v}\|_V^2 = \|\boldsymbol{\varepsilon}(\mathbf{v})\|_{\mathcal{H}}^2 + \|\mathbf{v}\|_H^2 \quad \forall \mathbf{v} \in V.$$

It follows that $\|\cdot\|_{H_1}$ and $\|\cdot\|_V$ are equivalent norms on V and, therefore, $(V, \|\cdot\|_V)$ is a real Hilbert space. Moreover, by the Sobolev's trace theorem, there exists a constant $c_0 > 0$ depending on Ω such that

$$\|\mathbf{v}\|_{L^2(\Gamma_C)} \leq c_0 \|\mathbf{v}\|_V \quad \forall \mathbf{v} \in V. \tag{11.10}$$

Next, let

$$E = \{\eta \in H^1(\Omega) \mid \eta = 0 \text{ on } \Gamma_F\}$$

be the space of admissible temperature fields, endowed with the canonical inner product of $H^1(\Omega)$. We also need two Gelfand evolution triples (see e.g. [20] II/A, p. 416), given by

$$V \subset H \equiv H' \subset V', \quad E \subset L^2(\Omega) \equiv (L^2(\Omega))' \subset E',$$

where the inclusions are dense and continuous, and we denote by $\langle \cdot, \cdot \rangle_{V' \times V}$, $\langle \cdot, \cdot \rangle_{E' \times E}$ the corresponding duality pairing mappings.

In the study of the mechanical problem (11.1)–(11.9), we assume that the tensor $\mathcal{A} = (a_{ijkh}) : \Omega \times \mathbb{S}^d \rightarrow \mathbb{S}^d$ satisfies the usual properties of symmetry and ellipticity, i.e.

$$\left. \begin{aligned} \text{(i)} \quad & a_{ijkh} = a_{khij} = a_{ijhk} \in W^{1,\infty}(\Omega); \\ \text{(ii)} \quad & \text{there exists } m_{\mathcal{A}} > 0 \text{ such that} \\ & \mathcal{A}\boldsymbol{\tau} \cdot \boldsymbol{\tau} \geq m_{\mathcal{A}} \|\boldsymbol{\tau}\|^2 \quad \forall \boldsymbol{\tau} \in \mathbb{S}^d, \text{ a.e. in } \Omega. \end{aligned} \right\} \tag{11.11}$$

We also suppose that the elasticity operator $\mathcal{B} : \Omega \times \mathbb{S}^d \rightarrow \mathbb{S}^d$ and the thermal tensor $C_e = (c_{ij}) : \Omega \times \mathbb{S}^d \rightarrow \mathbb{S}^d$ satisfy the following conditions.

$$\left. \begin{aligned} \text{(i)} \quad & \text{there exists } L_B > 0 \text{ such that} \\ & \|\mathcal{B}(\mathbf{x}, \boldsymbol{\varepsilon}_1) - \mathcal{B}(\mathbf{x}, \boldsymbol{\varepsilon}_2)\| \leq L_B \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| \\ & \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \mathbf{x} \in \Omega; \\ \text{(ii)} \quad & \mathbf{x} \mapsto \mathcal{B}(\mathbf{x}, \boldsymbol{\varepsilon}) \text{ is Lebesgue measurable on } \Omega, \forall \boldsymbol{\varepsilon} \in \mathbb{S}^d; \\ \text{(iii)} \quad & \text{the mapping } \mathbf{x} \mapsto \mathcal{B}(\mathbf{x}, \mathbf{0}) \in \mathcal{H}. \end{aligned} \right\} \tag{11.12}$$

$$c_{ij} = c_{ji} \in L^\infty(\Omega). \tag{11.13}$$

In addition, the body forces, surface tractions and the heat sources density have the regularity

$$\mathbf{f}_0 \in L^2(0, T; H), \quad \mathbf{f}_F \in L^2(0, T; L^2(\Gamma_F)^d), \tag{11.14}$$

$$q \in L^2(0, T; L^2(\Omega)) \tag{11.15}$$

where, here and below, we use the standard notation for functions defined on $[0, T]$ with values in a Hilbert space.

The coefficients c_v and μ_τ verify

$$c_v \in L^\infty(\Gamma_C; \mathbb{R}^+), \quad \mu_\tau \in L^\infty(\Gamma_C; \mathbb{R}^+) \tag{11.16}$$

and, moreover, the boundary thermal data satisfy the regularity

$$k_e \in L^\infty(\Omega; \mathbb{R}^+), \quad \theta_R \in W^{1,2}(0, T; L^2(\Gamma_C)). \tag{11.17}$$

We also suppose that the thermal conductivity tensor $K_c = (k_{ij}) : \Omega \times \mathbb{R}^d \rightarrow \mathbb{S}^d$ verifies the usual properties of symmetry and ellipticity, i.e.

$$\left. \begin{aligned} \text{(i)} \quad & k_{ij} = k_{ji} \in L^\infty(\Omega); \\ \text{(ii)} \quad & \text{there exists } c_k > 0 \text{ such that} \\ & k_{ij} \xi_i \xi_j \geq c_k \xi_i \xi_i \quad \forall \xi = (\xi_{ij}) \in \mathbb{R}^d, \text{ a.e. in } \Omega. \end{aligned} \right\} \tag{11.18}$$

Finally, we assume that the initial data satisfy the conditions

$$\mathbf{u}_0 \in V, \quad \mathbf{v}_0 \in V, \quad \theta_0 \in E. \tag{11.19}$$

Next, using Green’s formula, we obtain the following weak formulation of the mechanical Problem \mathcal{P} .

Problem \mathcal{P}^V . Find a displacement field $\mathbf{u} : [0, T] \rightarrow V$ and a temperature field $\theta : [0, T] \rightarrow E$ such that

$$\begin{aligned} & \langle \mathbf{u}''(t) + A\mathbf{u}'(t) + B\mathbf{u}(t) + C\theta(t), \mathbf{w} - \mathbf{u}'(t) \rangle_{V' \times V} \\ & + j_v(\mathbf{u}(t), \mathbf{w} - \mathbf{u}'(t)) + j_\tau(\mathbf{u}(t), \mathbf{w}) - j_\tau(\mathbf{u}(t), \mathbf{u}'(t)) \\ & \geq \langle \mathbf{f}(t), \mathbf{w} - \mathbf{u}'(t) \rangle_{V' \times V} \quad \forall \mathbf{w} \in V, \\ & \theta'(t) + K\theta(t) = R\mathbf{u}'(t) + Q(t), \end{aligned}$$

a.e. $t \in (0, T)$ and, moreover,

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \mathbf{u}'(0) = \mathbf{v}_0, \quad \theta(0) = \theta_0. \tag{11.20}$$

Note that in the statement Problem \mathcal{P}^V we use various operators and functions, which are defined as follows: $A, B : V \rightarrow V', C : E \rightarrow V', j_v, j_\tau : V \times V \rightarrow \mathbb{R}, K : E \rightarrow E', R : V \rightarrow E', \mathbf{f} : [0, T] \rightarrow V'$ and $Q : [0, T] \rightarrow E'$,

$$\begin{aligned}
\langle \mathcal{A}\mathbf{v}, \mathbf{w} \rangle_{V' \times V} &= (\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{v}), \boldsymbol{\varepsilon}(\mathbf{w}))_{\mathcal{H}}, \\
\langle \mathcal{B}\mathbf{v}, \mathbf{w} \rangle_{V' \times V} &= (\mathcal{B}\boldsymbol{\varepsilon}(\mathbf{v}), \boldsymbol{\varepsilon}(\mathbf{w}))_{\mathcal{H}}, \\
\langle \mathcal{C}\zeta, \mathbf{w} \rangle_{V' \times V} &= -(\zeta C_e, \boldsymbol{\varepsilon}(\mathbf{w}))_{\mathcal{H}}, \\
j_v(\mathbf{v}, \mathbf{w}) &= \int_{\Gamma_C} c_v (v_v - g)_+ w_v \, da, \\
j_\tau(\mathbf{v}, \mathbf{w}) &= \int_{\Gamma_C} \mu_\tau c_v (v_v - g)_+ \|\mathbf{w}_\tau\| \, da, \\
\langle \mathbf{f}(t), \mathbf{w} \rangle_{V' \times V} &= (\mathbf{f}_0(t), \mathbf{w})_H + (\mathbf{f}_F(t), \mathbf{w})_{(L^2(\Gamma_F))^d}, \\
\langle K\zeta, \eta \rangle_{E' \times E} &= \sum_{i,j=1}^d \int_{\Omega} k_{ij} \frac{\partial \zeta}{\partial x_j} \frac{\partial \eta}{\partial x_i} \, dx + \int_{\Gamma_C} k_e \zeta \cdot \eta \, da, \\
\langle R\mathbf{v}, \eta \rangle_{E' \times E} &= - \int_{\Omega} c_{ij} \frac{\partial v_i}{\partial x_j} \eta \, dx, \\
\langle Q(t), \eta \rangle_{E' \times E} &= \int_{\Gamma_C} k_e \theta_R(t) \eta \, dx + \int_{\Omega} q(t) \eta \, dx,
\end{aligned}$$

$\forall \mathbf{v} \in V, \forall \mathbf{w} \in V, \forall \zeta \in E, \forall \eta \in E$, a.e. $t \in (0, T)$.

Our main existence and uniqueness result that we state here and prove in the next section is the following.

Theorem 11.1. *Assume that (11.11)–(11.19) hold. Then there exists a positive constant c_Ω depending on Ω such that there exists a unique solution $\{\mathbf{u}, \theta\}$ to Problem \mathcal{P}^V , if $\|\mu_\tau c_v\|_{L^\infty(\Gamma_C)} < c_\Omega$. Moreover, the solution has the regularity*

$$\left. \begin{aligned}
\mathbf{u} &\in C^1(0, T; H) \cap W^{1,2}(0, T; V) \cap W^{2,2}(0, T; V'); \\
\theta &\in C(0, T; L^2(\Omega)) \cap L^2(0, T; E) \cap W^{1,2}(0, T; E').
\end{aligned} \right\} \quad (11.21)$$

Note that Theorem 11.1 states the unique weak solvability of the thermo-mechanical Problem \mathcal{P} , under a smallness assumption on the coefficient of friction.

11.3 Proof of Theorem 11.1

The proof is based on monotonicity, convexity and fixed point arguments and will be carried out in several steps. Everywhere in this section we denote by $c > 0$ a generic constant which value may change from line to line. We start by introducing the velocity variable $\mathbf{v} = \mathbf{u}'$. Then, Problem \mathcal{P}^V leads to the following problem.

Problem \mathcal{Q}^V . Find a velocity field $\mathbf{v} : [0, T] \rightarrow V$ and a temperature field $\theta : [0, T] \rightarrow E$ such that

$$\begin{aligned} & \langle \mathbf{v}'(t) + A\mathbf{v}(t) + B\mathbf{u}(t) + C\theta(t), \mathbf{w} - \mathbf{v}(t) \rangle_{V' \times V} \\ & \quad + j_v(\mathbf{u}, \mathbf{w} - \mathbf{v}(t)) + j_\tau(\mathbf{u}, \mathbf{w}) - j_\tau(\mathbf{u}, \mathbf{v}(t)) \\ & \quad \geq \langle \mathbf{f}(t), \mathbf{w} - \mathbf{v}(t) \rangle_{V' \times V} \quad \forall \mathbf{w} \in V, \\ & \theta'(t) + K\theta(t) = R\mathbf{u}'(t) + Q(t), \end{aligned}$$

a.e. $t \in (0, T)$ and, moreover,

$$\mathbf{v}(0) = \mathbf{v}_0, \quad \theta(0) = \theta_0. \tag{11.22}$$

Here, $\mathbf{u} : [0, T] \rightarrow V$ is the function defined by

$$\mathbf{u}(t) = \mathbf{u}_0 + \int_0^t \mathbf{v}(s) \, ds \quad \forall t \in [0, T].$$

We start with the following result.

Lemma 11.2. For all $\eta \in L^2(0, T; V')$, there exists a unique function

$$\mathbf{v}_\eta \in C(0, T; H) \cap L^2(0, T; V) \cap W^{1,2}(0, T; V') \tag{11.23}$$

which satisfies

$$\left. \begin{aligned} & \langle \mathbf{v}'_\eta(t) + A\mathbf{v}_\eta(t), \mathbf{w} - \mathbf{v}_\eta(t) \rangle_{V' \times V} + \langle \eta(t), \mathbf{w} - \mathbf{v}_\eta(t) \rangle_{V' \times V} \\ & \quad + j_\tau(\mathbf{u}_\eta(t), \mathbf{w}) - j_\tau(\mathbf{u}_\eta(t), \mathbf{v}_\eta(t)) \geq \langle \mathbf{f}(t), \mathbf{w} - \mathbf{v}_\eta(t) \rangle_{V' \times V}, \\ & \quad \forall \mathbf{w} \in V, \quad \text{a.e. } t \in (0, T); \\ & \mathbf{v}_\eta(0) = \mathbf{v}_0, \end{aligned} \right\} \tag{11.24}$$

where

$$\mathbf{u}_\eta(t) = \mathbf{u}_0 + \int_0^t \mathbf{v}_\eta(s) \, ds.$$

Moreover, there exists a positive constant c_Ω , which depends on Ω , with the following property: if $\|\mu_\tau c_v\|_{L^\infty(\Gamma_C)} < c_\Omega$, then there exists $c > 0$ such that

$$\begin{aligned} & \|\mathbf{v}_{\eta_2}(t) - \mathbf{v}_{\eta_1}(t)\|_H^2 + \int_0^t \|\mathbf{v}_{\eta_2}(s) - \mathbf{v}_{\eta_1}(s)\|_V^2 \leq c \int_0^t \|\eta_1 - \eta_2\|_{V'}^2. \\ & \forall \eta_1, \eta_2 \in L^2(0, T; V'), \quad \forall t \in [0, T]. \end{aligned} \tag{11.25}$$

Proof. Given $\eta \in L^2(0, T; V')$ and $\xi \in C(0, T; V)$, by using a general result on parabolic variational inequalities (see e.g. [7, Chap. 3]), we obtain the existence of a unique function $\mathbf{v}_{\eta\xi} \in C(0, T; H) \cap L^2(0, T; V) \cap W^{1,2}(0, T; V')$ which satisfies

$$\left. \begin{aligned} & \langle \mathbf{v}'_{\eta\xi}(t) + A\mathbf{v}_{\eta\xi}(t), \mathbf{w} - \mathbf{v}_{\eta\xi}(t) \rangle_{V' \times V} + \langle \eta(t), \mathbf{w} - \mathbf{v}_{\eta\xi}(t) \rangle_{V' \times V} \\ & + j_\tau(\xi(t), \mathbf{w}) - j_\tau(\xi(t), \mathbf{v}_{\eta\xi}(t)) \geq \langle \mathbf{f}(t), \mathbf{w} - \mathbf{v}_{\eta\xi}(t) \rangle_{V' \times V}, \\ & \forall \mathbf{w} \in V, \quad \text{a.e. } t \in (0, T), \\ & \mathbf{v}_{\eta\xi}(0) = \mathbf{v}_0, \end{aligned} \right\} \quad (11.26)$$

Now let us fix $\eta \in L^2(0, T; V')$ and consider the operator $\Lambda_\eta : C(0, T; V) \rightarrow C(0, T; V)$ defined by

$$\forall \xi \in C(0, T; V), \quad \Lambda_\eta \xi(t) = \mathbf{u}_0 + \int_0^t \mathbf{v}_{\eta\xi}(s) ds.$$

We use some algebraic manipulation to see that

$$j_\tau(\mathbf{u}_1, \mathbf{w}_2) - j_\tau(\mathbf{u}_1, \mathbf{w}_1) + j_\tau(\mathbf{u}_2, \mathbf{w}_1) - j_\tau(\mathbf{u}_2, \mathbf{w}_2) \leq c \|\mathbf{u}_2 - \mathbf{u}_1\|_V \|\mathbf{w}_2 - \mathbf{w}_1\|_V,$$

for all $\mathbf{u}_1, \mathbf{u}_2, \mathbf{w}_1, \mathbf{w}_2 \in V$. Here $c > 0$ is a positive constant proportional to $c_0 \|\mu_\tau c_v\|_{L^\infty(\Gamma_C)}$ where c_0 is defined in (11.10).

Let $\xi_1, \xi_2 \in C(0, T; V)$ be given. We use inequality (11.26) with $\xi = \xi_1$ and $\mathbf{w} = \mathbf{v}_{\eta\xi_2}$, then with $\xi = \xi_2$ and $\mathbf{w} = \mathbf{v}_{\eta\xi_1}$, add the resulting inequalities and integrate the result over $[0, t]$, for all $t \in [0, T]$. In this way we obtain

$$\begin{aligned} & \|\mathbf{v}_{\eta\xi_2}(t) - \mathbf{v}_{\eta\xi_1}(t)\|_H^2 + \int_0^t \|\mathbf{v}_{\eta\xi_2}(s) - \mathbf{v}_{\eta\xi_1}(s)\|_V^2 ds \\ & \leq c \int_0^t \|\xi_2(s) - \xi_1(s)\|_V^2 ds + c \int_0^t \|\mathbf{v}_{\eta\xi_2}(s) - \mathbf{v}_{\eta\xi_1}(s)\|_H^2 ds \end{aligned}$$

for all $t \in [0, T]$. Next, using Gronwall's inequality, we deduce that

$$\|\Lambda_\eta(\xi_2)(t) - \Lambda_\eta(\xi_1)(t)\|_V^2 \leq c \int_0^t \|\xi_2(s) - \xi_1(s)\|_V^2 ds$$

for all $t \in [0, T]$. Thus, by Banach's fixed point principle we know that the operator Λ_η has a unique fixed point, denoted ξ_η . We then verify that

$$\mathbf{v}_\eta = \mathbf{v}_{\eta\xi_\eta}$$

is the unique solution of (11.24) with regularity (11.23).

Now let $\eta_1, \eta_2 \in L^2(0, T; V')$. We use (11.24) with $\eta = \eta_1$ and $w = v_{\eta_2}$, then with $\eta = \eta_2$ and $w = v_{\eta_1}$. We add the resulting inequalities and integrate their sum to obtain

$$\begin{aligned} & \|v_{\eta_2}(t) - v_{\eta_1}(t)\|_H^2 + \int_0^t \|v_{\eta_2}(s) - v_{\eta_1}(s)\|_V^2 ds \\ & \leq c \int_0^t \|\eta_2(s) - \eta_1(s)\|_{V'}^2 ds + c \int_0^t \|u_{\eta_2}(s) - u_{\eta_1}(s)\|_{V'}^2 ds \\ & \quad + c \int_0^t \|v_{\eta_2}(s) - v_{\eta_1}(s)\|_H^2 ds. \end{aligned}$$

for all $t \in [0, T]$. Here, again, $c > 0$ is a positive constant which is proportional to $c_0 \|\mu_\tau c_v\|_{L^\infty(\Gamma_C)}$. Let $\delta > 0$ be a given constant and let $c_\Omega = \frac{\delta}{c_0}$. It is clear that c_Ω depends on Ω and, moreover, if $\|\mu_\tau c_v\|_{L^\infty(\Gamma_C)} \leq c_\Omega$, then $c_0 \|\mu_\tau c_v\|_{L^\infty(\Gamma_C)} \leq \delta$. Therefore, choosing δ small enough we can assume that $2c < 1$. Then, using Gronwall's inequality we deduce (11.25), which concludes the proof. \square

We proceed with the following result.

Lemma 11.3. *For all $\eta \in L^2(0, T; V')$, there exists a function*

$$\theta_\eta \in C(0, T; L^2(\Omega)) \cap L^2(0, T; E) \cap W^{1,2}(0, T; E') \tag{11.27}$$

which satisfies

$$\left. \begin{aligned} & \theta'_\eta(t) + K \theta_\eta(t) = R v_\eta(t) + Q(t) \quad \text{in } E', \text{ a.e. } t \in (0, T), \\ & \theta_\eta(0) = \theta_0. \end{aligned} \right\} \tag{11.28}$$

Moreover, if $\|\mu_\tau c_v\|_{L^\infty(\Gamma_C)} < c_\Omega$, then there exists $c > 0$ such that for all $\eta_1, \eta_2 \in L^2(0, T; V')$ the following inequality holds:

$$\|\theta_{\eta_1}(t) - \theta_{\eta_2}(t)\|_{L^2(\Omega)}^2 \leq c \int_0^t \|\eta_1 - \eta_2\|_{V'}^2, \quad \forall t \in [0, T]. \tag{11.29}$$

Proof. We verify that the operator $K : E \rightarrow E'$ is linear continuous and strongly monotone, and from the expression of the operator R , we have

$$v_\eta \in L^2(0, T; V) \implies R v_\eta \in L^2(0, T; E').$$

Now, since $Q \in L^2(0, T; E')$ it follows that $R v_\eta + Q \in L^2(0, T; E')$. Therefore, the existence part of the lemma follows from a classical result on first order evolution equation.

Now, to provide the estimate (11.29), consider $\eta_1, \eta_2 \in L^2(0, T; V')$. We have

$$\begin{aligned}
& \langle \theta'_{\eta_1}(t) - \theta'_{\eta_2}(t), \theta_{\eta_1}(t) - \theta_{\eta_2}(t) \rangle_{E' \times E} \\
& + \langle K\theta_{\eta_1}(t) - K\theta_{\eta_2}(t), \theta_{\eta_1}(t) - \theta_{\eta_2}(t) \rangle_{E' \times E} \\
& = \langle R\mathbf{v}_{\eta_1}(t) - R\mathbf{v}_{\eta_2}(t), \theta_{\eta_1}(t) - \theta_{\eta_2}(t) \rangle_{E' \times E} \quad \text{a.e. } t \in (0; T).
\end{aligned}$$

We then integrate this inequality over $[0, t]$ and use the strong monotonicity of K and the Lipschitz continuity of $R : V \rightarrow E'$ to deduce that

$$\|\theta_{\eta_1}(t) - \theta_{\eta_2}(t)\|_{L^2(\Omega)}^2 \leq c \int_0^t \|\mathbf{v}_{\eta_1} - \mathbf{v}_{\eta_2}\|_V^2 \quad \forall t \in [0, T].$$

Inequality (11.29) follows then from Lemma 11.2. \square

Lemmas 11.2 and 11.3 allow to consider the operator $\Lambda : L^2(0, T; V') \rightarrow L^2(0, T; V')$ defined, for all $\boldsymbol{\eta} \in L^2(0, T; V')$, by the equality

$$\begin{aligned}
\langle \Lambda \boldsymbol{\eta}(t), \mathbf{w} \rangle_{V' \times V} &= \langle B\mathbf{u}_{\boldsymbol{\eta}}(t) + C\theta_{\boldsymbol{\eta}}(t), \mathbf{w} \rangle_{V' \times V} + j_v(\mathbf{u}_{\boldsymbol{\eta}}(t), \mathbf{w}), \\
\forall \mathbf{w} \in V, \text{ a.e. } t &\in (0, T).
\end{aligned}$$

Here

$$\mathbf{u}_{\boldsymbol{\eta}}(t) = \mathbf{u}_0 + \int_0^t \mathbf{v}_{\boldsymbol{\eta}}(s) ds \quad \forall t \in [0, T]$$

where $\mathbf{v}_{\boldsymbol{\eta}}$ and $\theta_{\boldsymbol{\eta}}$ are the functions defined in Lemmas 11.2 and 11.3.

We have the following result.

Lemma 11.4. *Assume that $\|\mu_{\tau} c_v\|_{L^\infty(\Gamma_C)} < c_\Omega$. Then Λ has a unique fixed point $\boldsymbol{\eta}^* \in L^2(0, T; V')$.*

Proof. Let $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2 \in L^2(0, T; V')$ be given. Then, it is easy to check that

$$\begin{aligned}
\|\Lambda \boldsymbol{\eta}_2(t) - \Lambda \boldsymbol{\eta}_1(t)\|_{V'} &\leq c \|B\mathbf{u}_{\boldsymbol{\eta}_2}(t) - B\mathbf{u}_{\boldsymbol{\eta}_1}(t)\|_{V'} + c \|\theta_{\boldsymbol{\eta}_2}(t) - \theta_{\boldsymbol{\eta}_1}(t)\|_{L^2(\Omega)} \\
&+ c \|\mathbf{u}_{\boldsymbol{\eta}_2}(t) - \mathbf{u}_{\boldsymbol{\eta}_1}(t)\|_V
\end{aligned}$$

a.e. $t \in (0, T)$. We combine (11.12), (11.25) and (11.29) to deduce that there exists $c > 0$ such that

$$\|\Lambda \boldsymbol{\eta}_2(t) - \Lambda \boldsymbol{\eta}_1(t)\|_{V'}^2 \leq c \int_0^t \|\boldsymbol{\eta}_2(s) - \boldsymbol{\eta}_1(s)\|_{V'}^2 ds \quad \forall t \in [0, T].$$

Lemma 11.4 is a consequence of the previous inequality combined with the Banach fixed point principle. \square

We now have all the ingredients to prove Theorem 11.1.

Proof of Theorem 11.1. Let \mathbf{u} , \mathbf{v} and θ the functions defined by

$$\mathbf{u}(t) := \mathbf{u}_0 + \int_0^t \mathbf{v}_{\eta^*}(s) ds, \quad \mathbf{v}(t) := \mathbf{v}_{\eta^*}(t), \quad \theta(t) := \theta_{\eta^*}(t) \quad \forall t \in [0, T].$$

Then, using (11.24) and (11.28), it is easy to see that the couple (\mathbf{v}, θ) is a solution to Problem \mathcal{Q}^V and, therefore (\mathbf{u}, θ) is a solution to Problem \mathcal{P}^V . Moreover, the regularity (11.21) follows from the regularity of the functions \mathbf{v}_η and θ_η in Lemmas 11.2 and 11.3, see (11.23) and (11.27), respectively. This proves the existence part of the theorem. The uniqueness part follows from the uniqueness of the solution in Lemmas 11.2 and 11.3. \square

11.4 Numerical Computations

In this section, we provide a fully-discrete numerical approximation scheme for the variational Problem \mathcal{P}^V , and the associated numerical simulations in the study of two dimensional tests by using MATLAB computation codes. To this end we denote by $\{\mathbf{u}, \theta\}$ the unique solution of the Problem \mathcal{P}^V and consider the velocity variable defined by

$$\mathbf{v}(t) = \mathbf{u}'(t) \quad \forall t \in [0, T].$$

We make the following additional assumptions on the solution and data:

$$\mathbf{f} \in C([0, T]; V'); \quad Q \in C([0, T]; E');$$

$$\mathbf{v} \in C(0, T; V); \quad \mathbf{v}' \in C(0, T; H);$$

$$\theta \in C(0, T; E); \quad \theta' \in C(0, T; L^2(\Omega)).$$

Now let $V^h \subset V$ and $E^h \subset E$ be a family of finite dimensional subspaces, defined by finite elements spaces of piecewise linear functions, where $h > 0$ is a discretization parameter which may be the maximal diameter of the elements. We divide the time interval $[0, T]$ into N equal parts: $t_n = nk, n = 0, 1, \dots, N$, with the uniform time step $k = T/N$. For a continuous function $\mathbf{w} \in C([0, T]; X)$ with values in a space X , we use the notation $\mathbf{w}_n = \mathbf{w}(t_n) \in X$.

Then, Problem \mathcal{Q}^V implies

$$\left. \begin{aligned} \langle \mathbf{v}'(t) + A\mathbf{v}(t) + B\mathbf{u}(t) + C\theta(t) + D\mathbf{u}(t), \mathbf{w} - \mathbf{v}(t) \rangle_{V' \times V} \\ + j_\tau(\mathbf{u}, \mathbf{w}) - j_\tau(\mathbf{u}, \mathbf{v}(t)) \geq \langle \mathbf{f}(t), \mathbf{w} - \mathbf{v}(t) \rangle_{V' \times V} \quad \forall \mathbf{w} \in V, \\ \langle \theta'(t) + K\theta(t) - R\mathbf{v}(t) - Q(t), \eta \rangle_{E' \times E} = 0 \quad \forall \eta \in E, \\ \mathbf{v}(0) = \mathbf{v}_0, \quad \theta(0) = \theta_0, \end{aligned} \right\}$$

for all $t \in [0, T]$, where

$$\begin{aligned}\langle D\mathbf{u}(t), \mathbf{w} \rangle_{V' \times V} &= j_v(\mathbf{u}(t), \mathbf{w}), \\ \langle \mathbf{v}'(t), \mathbf{w} \rangle_{V' \times V} &= (\mathbf{v}'(t), \mathbf{w})_H, \\ \langle \theta'(t), \eta \rangle_{E' \times E} &= (\theta'(t), \eta)_{L^2(\Omega)}.\end{aligned}$$

This suggests to introduce the following fully-discrete scheme.

Problem \mathcal{P}^{hk} . Find a discrete velocity field $\mathbf{v}^{hk} = \{\mathbf{v}_n^{hk}\}_{n=0}^N \subset V^h$ and a discrete temperature field $\theta^{hk} = \{\theta_n^{hk}\}_{n=0}^N \subset E^h$ such that

$$\mathbf{v}_0^{hk} = \mathbf{v}_0^h, \quad \theta_0^{hk} = \theta_0^h, \quad (11.30)$$

and for $n = 1, \dots, N$,

$$\begin{aligned}& \left\langle \frac{\mathbf{v}_n^{hk} - \mathbf{v}_{n-1}^{hk}}{k} + A\mathbf{v}_n^{hk}, \mathbf{w}^h - \mathbf{v}_n^{hk} \right\rangle_{V' \times V} \\ & + \langle B\mathbf{u}_{n-1}^{hk} + C\theta_{n-1}^{hk} + D\mathbf{u}_{n-1}^{hk}, \mathbf{w}^h - \mathbf{v}_n^{hk} \rangle_{V' \times V} \\ & + j_\tau(\mathbf{u}_{n-1}^{hk}, \mathbf{w}^h) - j_\tau(\mathbf{u}_{n-1}^{hk}, \mathbf{v}_n^{hk}) \\ & \geq \langle \mathbf{f}_n, \mathbf{w}^h - \mathbf{v}_n^{hk} \rangle_{V' \times V} \quad \forall \mathbf{w}^h \in V^h, \quad (11.31)\end{aligned}$$

$$\begin{aligned}& \left(\frac{\theta_n^{hk} - \theta_{n-1}^{hk}}{k}, \eta^h \right)_{L^2(\Omega)} + \langle K\theta_n^{hk}, \eta^h \rangle_{E' \times E} \\ & = \langle R\mathbf{v}_n^{hk}, \eta^h \rangle_{E' \times E} + \langle Q_n, \eta^h \rangle_{E' \times E}, \quad \forall \eta^h \in E^h. \quad (11.32)\end{aligned}$$

Here

$$\mathbf{u}_n^{hk} = \mathbf{u}_{n-1}^{hk} + k \mathbf{v}_n^{hk}, \quad \mathbf{u}_0^{hk} = \mathbf{u}_0^h. \quad (11.33)$$

Moreover, $\mathbf{u}_0^h \in V^h$, $\mathbf{v}_0^h \in V^h$ and $\theta_0^h \in E^h$ represent suitable approximations of the initial values \mathbf{u}_0 , \mathbf{v}_0 , θ_0 , respectively.

For $n = 1, \dots, N$, once \mathbf{u}_{n-1}^{hk} , \mathbf{v}_{n-1}^{hk} and θ_{n-1}^{hk} are known, we compute \mathbf{v}_n^{hk} , θ_n^{hk} and \mathbf{u}_n^{hk} by using (11.31)–(11.33) and classical result on variational inequality (see e.g. [10]). Therefore, the discrete scheme has a unique solution by starting with initial values on displacement, velocity and temperature fields. Moreover, under additional regularity of solution and using arguments similar as those used in [19], we can prove that the errors estimate order is proportional to the discretization parameters h and k .

In view of the numerical simulations, we consider the domain Ω , the partition of its boundary, the elasticity tensor and the viscosity operator as follows:

$$\Omega = (0, L_1) \times (0, L_2);$$

$$\Gamma_F = (\{0\} \times [0, L_2]) \cup ([0, L_1] \times \{L_2\}) \cup (\{L_1\} \times [0, L_2]); \Gamma_C = [0, L_1] \times \{0\};$$

$$(\mathcal{B}\boldsymbol{\tau})_{ij} = \frac{E\kappa}{1-\kappa^2}(\tau_{11} + \tau_{22})\delta_{ij} + \frac{E}{1+\kappa}\tau_{ij}, \quad 1 \leq i, j \leq 2, \boldsymbol{\tau} \in \mathbb{S}^2;$$

$$(\mathcal{A}\boldsymbol{\tau})_{ij} = \mu(\tau_{11} + \tau_{22})\delta_{ij} + \eta\tau_{ij}, \quad 1 \leq i, j \leq 2, \boldsymbol{\tau} \in \mathbb{S}^2.$$

Here E is the Young's modulus, κ is the Poisson's ratio of the material, δ_{ij} denotes the Kronecker symbol and μ and η are viscosity constants.

We refer to the previous numerical scheme, and use spaces of continuous piecewise affine functions $V^h \subset V$ and $E^h \subset E$ as families of approximating subspaces. For our computations, we consider also the following data (IS unity):

$$L_1 = L_2 = 1, \quad T = 1,$$

$$\mu = 10, \quad \eta = 10, \quad E = 2, \quad \kappa = 0.1,$$

$$c_{ij} = k_{ij} = k_e = 1, \quad 1 \leq i, j \leq 2,$$

$$\mathbf{f}_0(\mathbf{x}, t) = (0, -1.5), \quad q(\mathbf{x}, t) = 1 \quad \forall \mathbf{x} \in \Omega, t \in [0, T],$$

$$\mathbf{f}_F(\mathbf{x}, t) = (0, 0), \quad \forall \mathbf{x} \in \{0\} \times [0, L_2], t \in [0, T],$$

$$\mathbf{f}_F(\mathbf{x}, t) = (0.5, 0.4), \quad \forall \mathbf{x} \in ([0, L_1] \times \{L_2\}) \cup (\{L_1\} \times [0, L_2]), t \in [0, T],$$

$$\mathbf{u}_0(\mathbf{x}) = (0, 0), \quad \mathbf{v}_0(\mathbf{x}) = (0, 0), \quad \theta_0(\mathbf{x}) = 0 \quad \forall \mathbf{x} \in \Omega.$$

In Figs. 11.1 and 11.2 we show the deformed configurations at final time, for two different values of the normal compliance coefficient. We see that for a larger coefficient, penetration is less important. In Figs. 11.3 and 11.4, we show the deformed configurations at final time, for two different values of coefficients of friction. We note that for a smaller coefficient the slip phenomenon appears on the contact surface. In Figs. 11.5 and 11.6, we plot the deformed configurations at final time, for two values of the gap. In Figs. 11.7, 11.8, 11.9, 11.10, 11.11 and 11.12 we represent the Von Mises norm of the stress, corresponding to the numerical values in Figs. 11.1, 11.2, 11.3, 11.4, 11.5 and 11.6, respectively. These figures show that when penetration is more important then the norm of the stress on the contact surface is larger. In particular, the norm of the stress is minimal in the case where there is loss of contact with the foundation. Finally, in Figs. 11.13 and 11.14, we show the influence of the different temperatures of the foundation on the temperature field of the body. We observe that a high temperature of the foundation leads to a high temperature in the neighborhood of the contact surface.

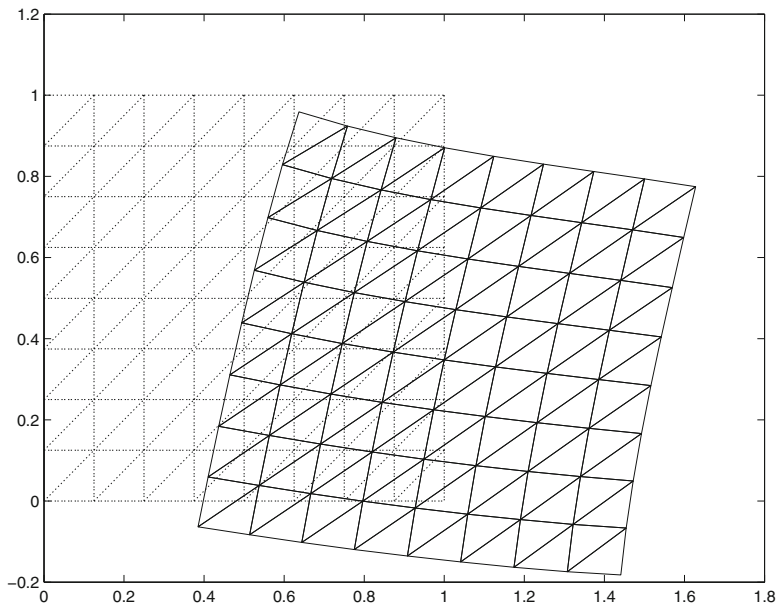


Fig. 11.1 Deformed configuration at final time, $\theta_R = 0, g = 0, \mu_\tau = 0.1, c_\nu = 10$

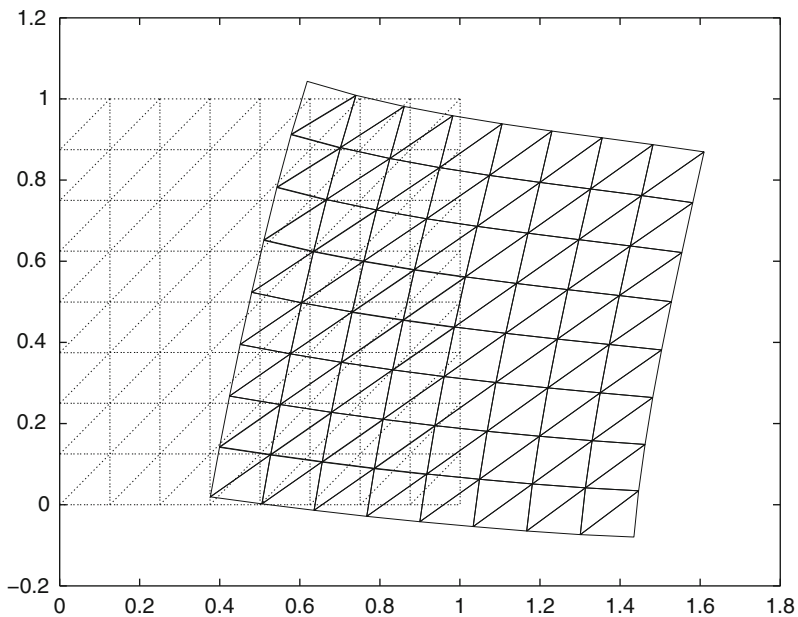


Fig. 11.2 Deformed configuration at final time, $\theta_R = 0, g = 0, \mu_\tau = 0.1, c_\nu = 20$

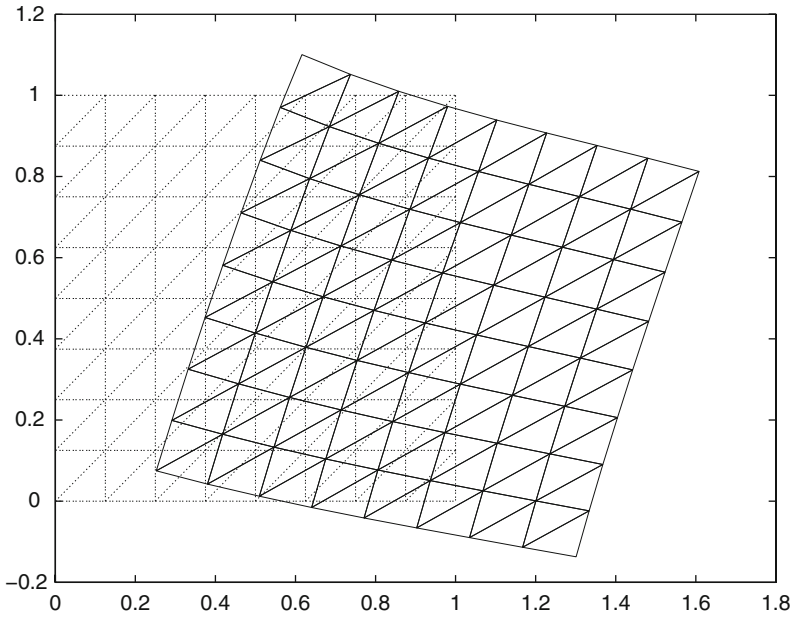


Fig. 11.3 Deformed configuration at final time, $\theta_R = 0$, $g = 0$, $c_v = 20$, $\mu_\tau = 0.30$

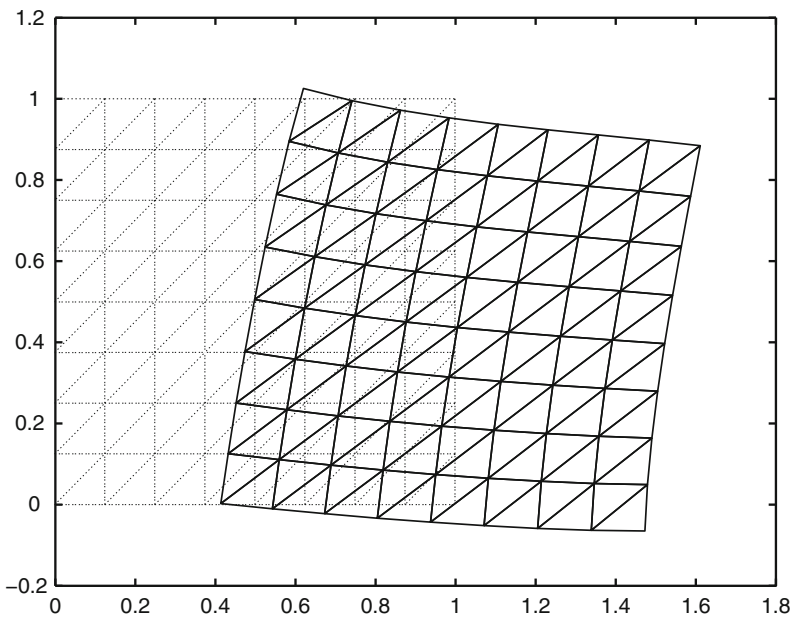


Fig. 11.4 Deformed configuration at final time, $\theta_R = 0$, $g = 0$, $c_v = 20$, $\mu_\tau = 0.05$

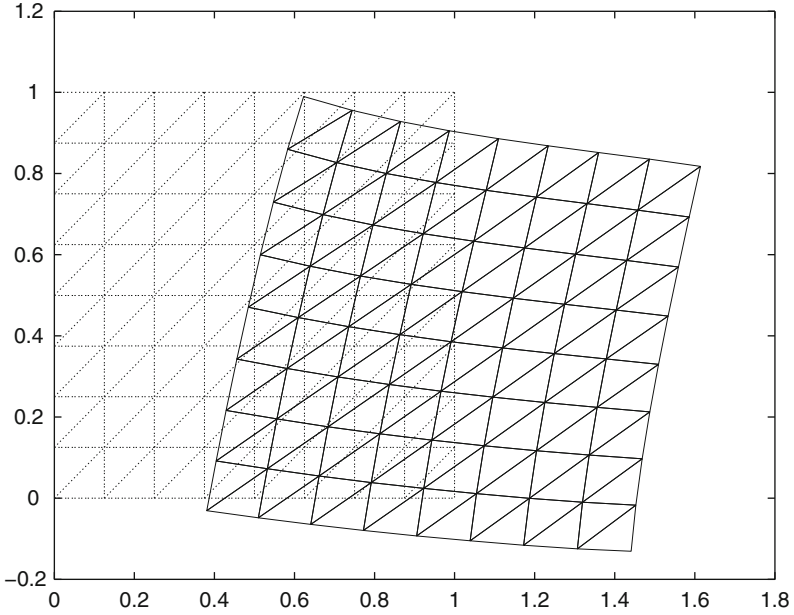


Fig. 11.5 Deformed configuration at final time, $\theta_R = 0$, $c_v = 20$, $\mu_\tau = 0.1$, $g = 0.03$

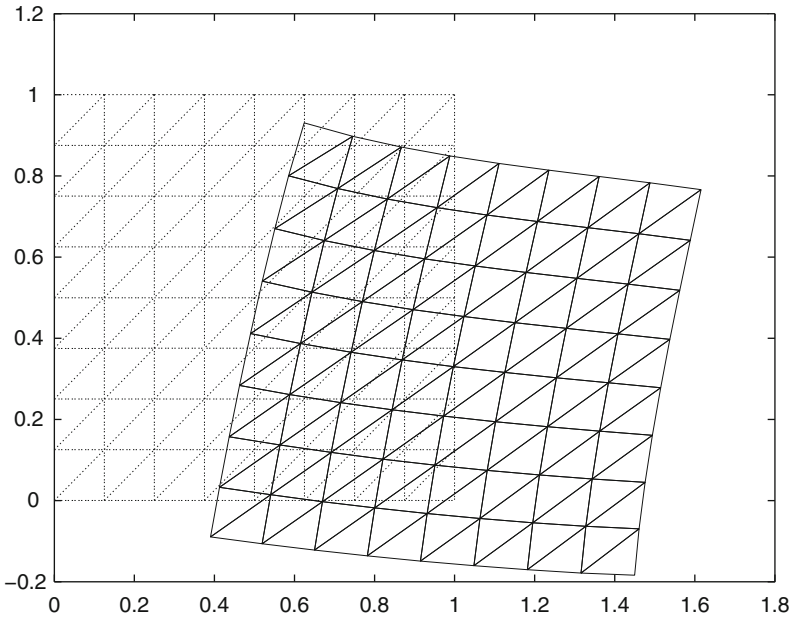


Fig. 11.6 Deformed configuration at final time, $\theta_R = 0$, $c_v = 20$, $\mu_\tau = 0.1$, $g = 0.06$

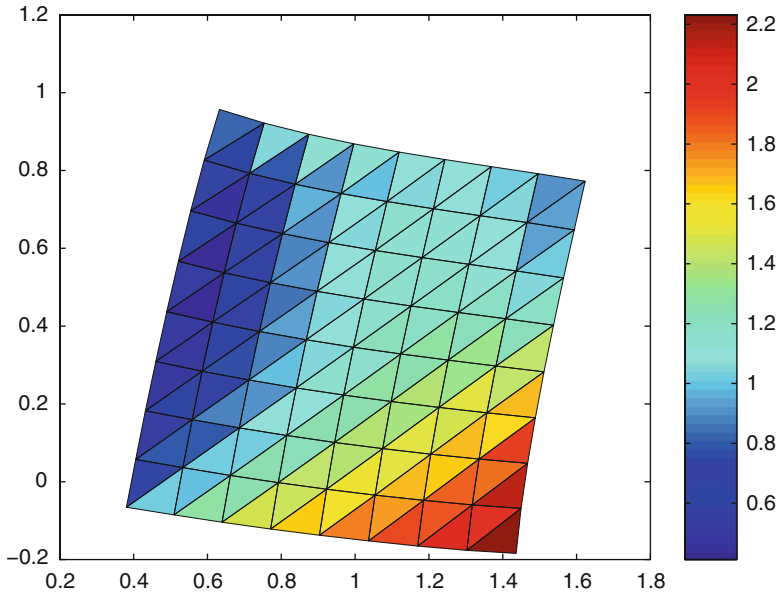


Fig. 11.7 Von Mises norm of the stress in deformed configurations, $\theta_R = 0, g = 0, \mu_\tau = 0.1, c_v = 10$

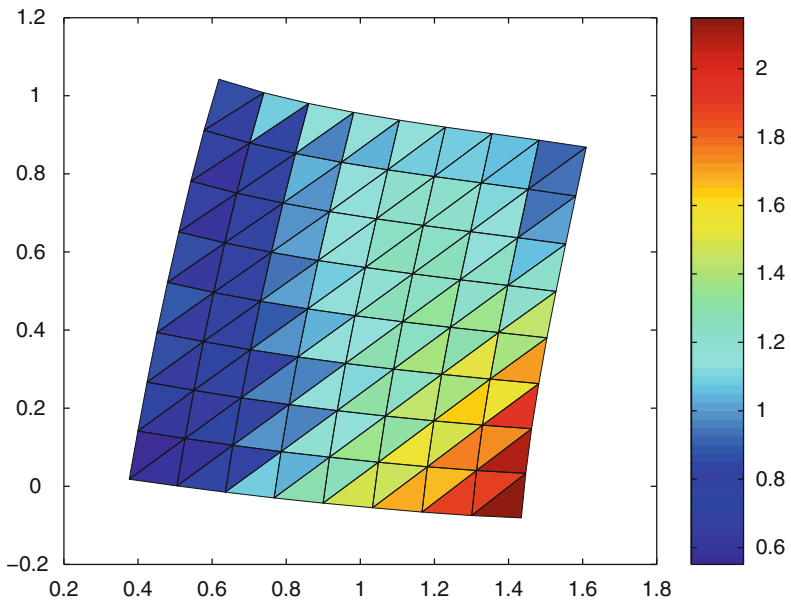


Fig. 11.8 Von Mises norm of the stress in deformed configuration, $\theta_R = 0, g = 0, \mu_\tau = 0.1, c_v = 20$

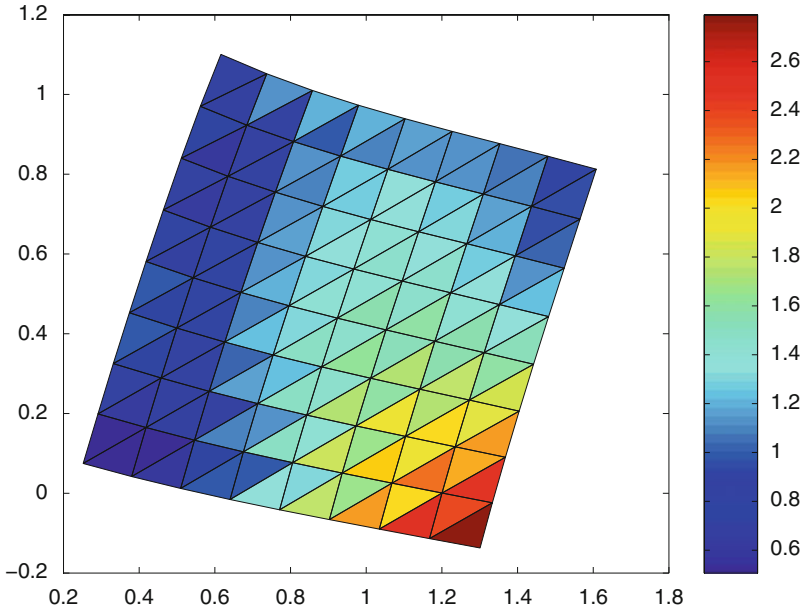


Fig. 11.9 Von Mises norm of the stress in deformed configuration, $\theta_R = 0$, $g = 0$, $c_v = 20$, $\mu_\tau = 0.3$

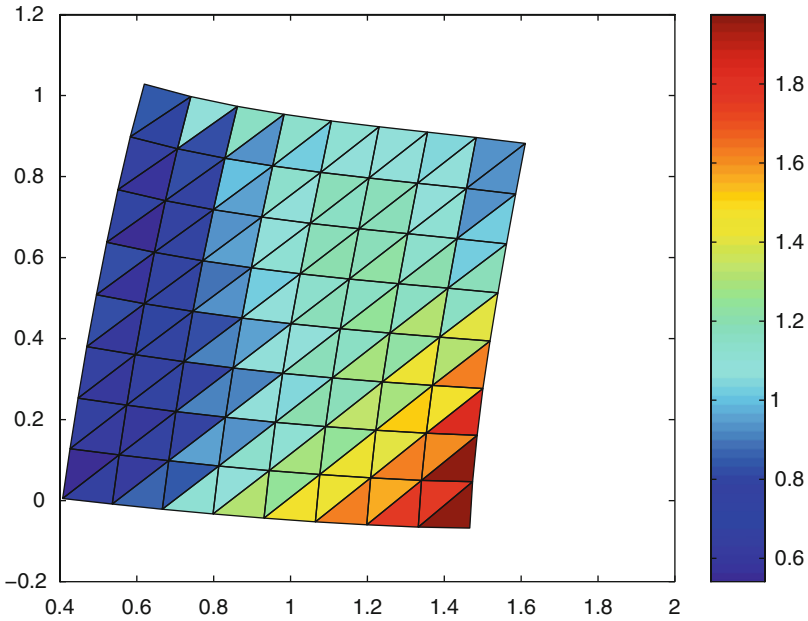


Fig. 11.10 Von Mises norm of the stress in deformed configuration, $\theta_R = 0$, $g = 0$, $c_v = 20$, $\mu_\tau = 0.05$

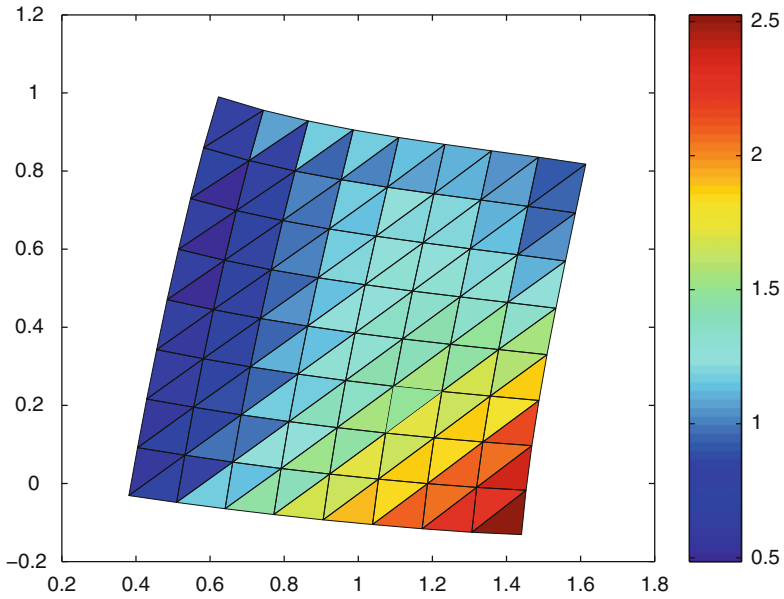


Fig. 11.11 Von Mises norm of the stress in deformed configuration, $\theta_R = 0$, $c_v = 20$, $\mu_\tau = 0.3$, $g = 0.03$

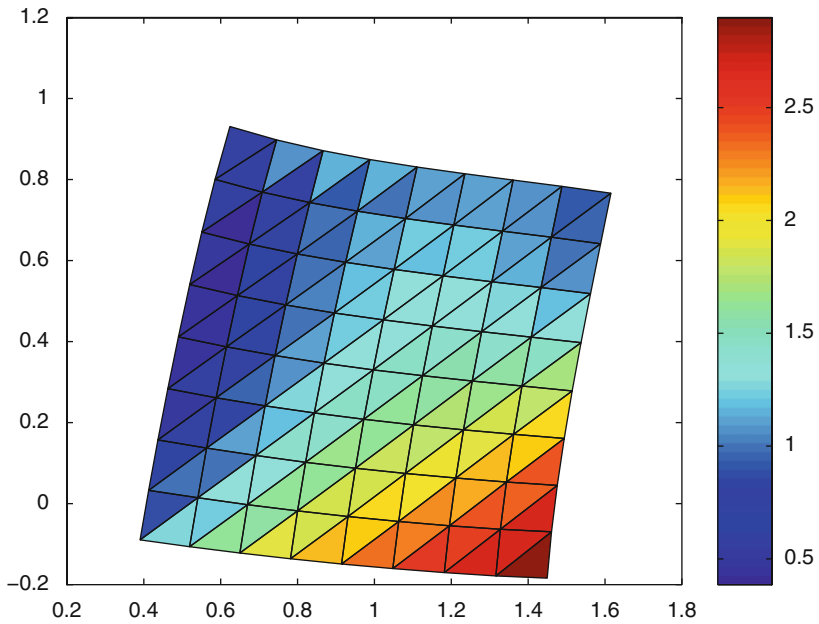


Fig. 11.12 Von Mises norm of the stress in deformed configuration, $\theta_R = 0$, $c_v = 20$, $\mu_\tau = 0.05$, $g = 0.06$

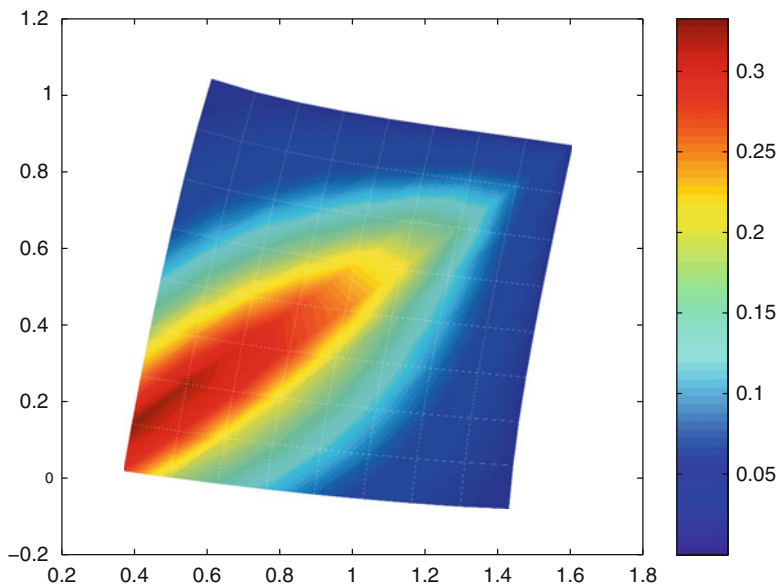


Fig. 11.13 Temperature field at final time, $c_v = 20$, $g = 0$, $\mu_\tau = 0.1$, $\theta_R = 0$

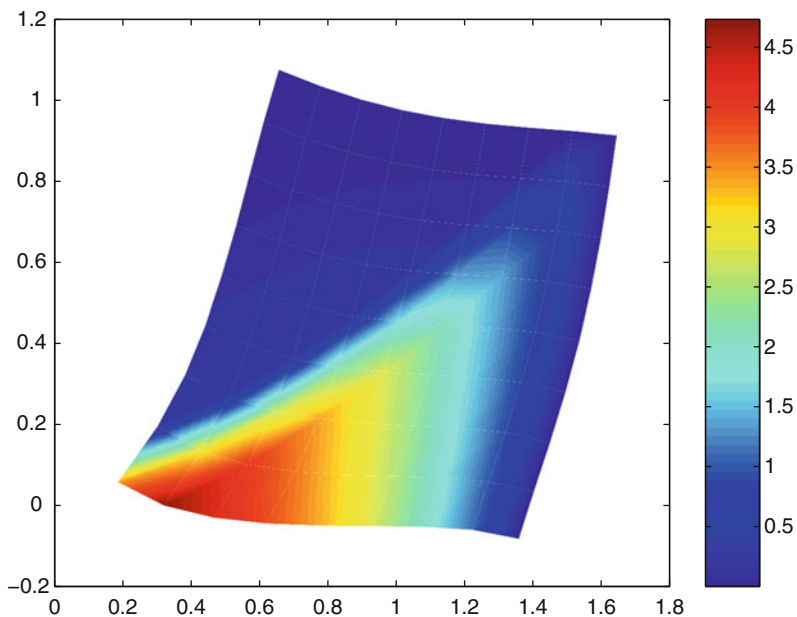


Fig. 11.14 Temperature field at final time, $c_v = 20$, $g = 0$, $\mu_\tau = 0.1$, $\theta_R = 10$

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