Chapter 8
Bending Problems

8.1 Mathematical Modeling

Bending describes the deformation of thin objects under small forces. Typically, the object is neither stretched nor sheared, but large deformations occur. A simple example is the deformation of a sheet of paper that is clamped on part of its boundary and subject to a force such as gravity. Since curvatures are important to describe such a behavior, the related mathematical models involve higher-order derivatives. We discuss the derivation of such models and their properties. For further details we refer to the textbooks [5, 6] and the seminal paper [10].

8.1.1 Bending Models

We consider a Lipschitz domain $\omega \subset \mathbb{R}^2$ representing the region occupied by a thin plate, a body force $f = (f_1, f_2, f_3)^\top : \omega \to \mathbb{R}^3$ acting on it, and clamped boundary conditions on the nonempty closed subset $\gamma_D \subset \partial\omega$ that prescribe the displacement by a function u_D and the rotation by a mapping Φ_D on γ_D.

Definition 8.1 The *nonlinear Kirchhoff model* seeks a deformation $u : \omega \to \mathbb{R}^3$ that minimizes the functional

$$I^{\mathrm{Ki}}(u) = \frac{1}{2}\int_\omega |D^2 u|^2 \,\mathrm{d}x - \int_\omega f \cdot u \,\mathrm{d}x,$$

subject to the *isometry constraint* $(\nabla u)^\top \nabla u = I_2$ and the boundary conditions $u|_{\gamma_D} = u_D$ and $\nabla u|_{\gamma_D} = \Phi_D$.

The isometry constraint reflects the fact that pure bending theories do not allow for a shearing or stretching of the plate. This limits the class of boundary conditions that lead to nonempty sets of admissible deformations. In particular, the function Φ_D

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prescribes the normal, of the deformed surface on γ_D. The model sets no limitations on the size of the deformation, but does not prohibit self-penetrations, i.e., it does not enforce the surface parametrized by u be embedded. We will show below that the isometry constraint allows us to replace the Frobenius norm of the Hessian by the Euclidean norm of the Laplacian, i.e., $|D^2u| = |\Delta u|$, and that these expressions coincide with the modulus of the mean curvature. For small displacements

$$\phi = u - [\mathrm{id}_2, 0]^\top,$$

i.e., if $|\nabla\phi| \ll 1$, the isometry constraint can be omitted and it suffices to consider the vertical component $w = u_3$ of the deformation. Typical large deformation and small displacement situations are depicted in Fig. 8.1.

Definition 8.2 The *linear Kichhoff model* seeks a vertical displacement $w : \omega \to \mathbb{R}$ that minimizes the functional

$$I^{\mathrm{Ki}'}(w) = \frac{1}{2}\int_\omega |D^2w|^2 \,\mathrm{d}x - \int_\omega f_3 w \,\mathrm{d}x$$

subject to the boundary conditions $w|_{\gamma_D} = 0$ and $\nabla w|_{\gamma_D} = 0$, i.e., w belongs to the set $H^2_D(\omega) = \{v \in H^2(\Omega) : v|_{\gamma_D} = 0,\ \nabla v|_{\gamma_D} = 0\}$.

The linear Kirchhoff model is closely related to a model in which no second-order derivatives occur. It may be regarded as an approximation of the linear Kirchhoff model in which small shearing effects may occur. Mathematically, the second order derivatives are replaced by an additional variable and the difference is penalized with a penalty parameter, which may be regarded as a small artificial plate thickness. Notice that the symmetric gradient of a gradient is the Hessian, i.e., $\varepsilon(\nabla w) = D^2 w$.

Definition 8.3 The *linear Reissner–Mindlin model* seeks for given $t > 0$ a vertical displacement $w : \omega \to \mathbb{R}$ and a rotation $\theta : \omega \to \mathbb{R}^3$ that minimize the functional

$$I^{\mathrm{RM}}(w, \theta) = \frac{t^{-2}}{2}\int_\omega |\theta - \nabla w|^2 \,\mathrm{d}x + \frac{1}{2}\int_\omega |\varepsilon(\theta)|^2 \,\mathrm{d}x - \int_\omega f_3 w \,\mathrm{d}x,$$

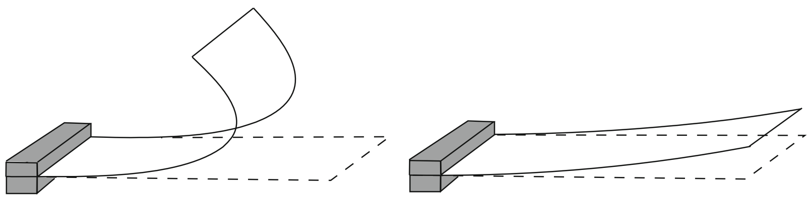

Fig. 8.1 Large isometric deformation of a thin clamped plate (*left*) and small displacement described by a linear model (*right*)

where $\varepsilon(\theta) = [(\nabla\theta)^\top + (\nabla\theta)]/2$, subject to the boundary conditions $w|_{\gamma_D} = 0$ and $\theta|_{\gamma_D} = 0$.

A solution u of the nonlinear Kirchhoff model defines an open surface in $\mathbb{R}^3$ that is parametrized by the deformation u. Since this surface is isometric to ω, we have that the Gaussian curvature K vanishes, i.e., that the local length and angle relations are preserved under the deformation. The mean curvature is given by $H^2 = |D^2u|^2$ and this identity establishes a relation to a bending model that is used to describe the deformation of fluid membranes such as cell surfaces. Here, the considered surfaces are closed. The justification of the model is less clear than in the case of solids. In particular, fluid membranes can undergo large shearing effects that are not seen by its description as a surface.

Definition 8.4 The *Willmore model* seeks a closed surface $\mathscr{M} \subset \mathbb{R}^3$ that minimizes the functional

$$I^{\mathrm{Wi}}(\mathscr{M}) = \frac{1}{2}\int_{\mathscr{M}} H^2 \,\mathrm{d}s - \int_{\mathscr{M}} K \,\mathrm{d}s,$$

subject to constraints that the surface area of $\mathscr{M}$ or that the volume enclosed by $\mathscr{M}$ be prescribed.

The integral over the Gaussian curvature is a topological invariant and can be neglected if a minimizer is sought in a fixed topology class. If the surface area and the enclosed volume are prescribed, then the model is referred to as the *Helfrich model*.

8.1.2 Relations to Hyperelasticity

In three-dimensional hyperelasticity, pure bending is characterized by a cubic scaling of the energy with respect to the plate thickness t, i.e., that

$$I_t(u_t) = \int_{\Omega_t} W(\nabla u_t)\,\mathrm{d}x - \int_{\Omega_t} f_t \cdot u_t \,\mathrm{d}x \sim t^3$$

for the optimal deformations $u_t \in H^1(\Omega_t;\mathbb{R}^3)$ as $t \to 0$ for $\Omega_t = \omega \times (-t/2, t/2) \subset \mathbb{R}^3$, such that $u_t|_{\Gamma_D} = \mathrm{id}$ on $\Gamma_D = \gamma_D \times (-t/2, t/2)$. This motivates considering the rescaled energy functionals $\widehat{I}_t = t^{-3} I_t$ and investigating the limiting behavior for $t \to 0$ in the framework of Γ-convergence. We let ∇' denote the gradient with respect to the first two variables $x' = (x_1, x_2)$. The corresponding three-dimensional objects are denoted $\nabla = (\nabla', \partial_3)$ and $x = (x', x_3)$.

Theorem 8.1 (Dimension reduction [10]) *Let*

$$W(F) = \mathrm{dist}^2\big(F, SO(3)\big)$$

for all $F \in \mathbb{R}^{3\times 3}$ *and* $SO(3) = \{F \in \mathbb{R}^{3\times 3} : F^\top F = I_3, \det F = 1\}$. *Set* $\widehat{f}_t(x', \widehat{x}_3) = t^{-2} f_t(x', t\widehat{x}_3)$ *and assume* $\widehat{f}_t \to f$ *in* $L^2(\Omega_1; \mathbb{R}^3)$ *and that* f *is independent of* $\widehat{x}_3 \in (-1, 1)$. *Let* $(u_t)_{t>0}$ *be a sequence of minimizers for the sequence of functionals* $(I_t)_{t>0}$, *i.e.,* $u_t \in H^1(\Omega_t; \mathbb{R}^3)$ *with* $u_t|_{\Gamma_\mathrm{D}} = \mathrm{id}_{\Gamma_\mathrm{D}}$. *Then the rescaled functions* $\widehat{u}(x', \widehat{x}_3) = u(x', t\widehat{x}_3)$ *converge in* $H^1(\Omega_1; \mathbb{R}^3)$ *to a function* $u \in H^1(\Omega_1; \mathbb{R}^3)$. *This function is independent of* $\widehat{x}_3$, *defines a parametrized surface with the first fundamental form* $g = (\nabla' u)^\top (\nabla' u) = I_2$ *in* Ω_1, *and satisfies* $u \in H^2(\Omega_1; \mathbb{R}^3)$. *Moreover, it has the boundary values* $u|_{\gamma_\mathrm{D}} = [\mathrm{id}, 0]^\top$ *and* $\nabla' u|_{\gamma_\mathrm{D}} = [I_2, 0]^\top$ *and minimizes*

$$I^{\mathrm{Ki}}(u) = \frac{1}{12} \int_\omega |h|^2 \,\mathrm{d}x' - \int_\omega f \cdot u \,\mathrm{d}x',$$

with the normal $b = \partial_1 u \times \partial_2 u$ *and the second fundamental form* $h = -(\nabla' b)^\top (\nabla' u)$, *in functions* $v \in H^1(\Omega_1; \mathbb{R}^3)$, *that are independent of* $\widehat{x}_3$, *satisfy* $(\nabla' v)^\top (\nabla' v) = I_2$ *in* Ω_1, *and have the same boundary conditions as* u. *Conversely, every such minimizer* u *of* I^{Ki} *is the limit of a sequence of rescaled minimizers of* I_t *and the minimal energies converge to* $I^{\mathrm{Ki}}(u)$.

Remarks 8.1 (i) We will show below that $|h| = |D^2 u|$ for the Frobenius norms of the second fundamental form and the Hessian of u.
(ii) The result also holds for isotropic, frame-indifferent energy densities $W \in C^2(\mathbb{R}^{n\times n})$ with $W(I_3) = 0$, and $W(F) \ge \mathrm{dist}^2\big(F, SO(3)\big)$, cf. [10].

For a heuristic justification of the result, we follow [7] and consider the rescaled energy functional

$$\widehat{I}_t(u) = t^{-3} \int_{\Omega_t} W(\nabla u) \,\mathrm{d}x$$

with W given by

$$W(F) = \mathrm{dist}^2\big(F, SO(3)\big) = \min_{Q \in SO(3)} |F - Q|^2.$$

We assume that the optimal deformation $u_t = u$ is of the form

$$u(x', x_3) = v(x') + x_3 b(x')$$

with t-independent vector fields $v, b : \omega \to \mathbb{R}^3$ and b is normal to the surface parametrized by v, i.e., $\partial_\ell v(x') \cdot b(x') = 0$ for $\ell = 1, 2$. This means that v is the deformation of the middle surface ω and the segments normal to ω are mapped to straight lines that are normal to the deformed surface, cf. the right plot of Fig. 8.2. We have

$$\nabla u = [\nabla' v, b] + [x_3 \nabla' b, 0].$$

For matrices $F \in \mathbb{R}^{3\times 3}$ in a neighborhood of $SO(3)$, we use the approximation

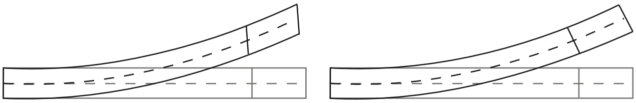

Fig. 8.2 Normal segments are mapped to straight line segments under the Reissner–Mindlin hypotheses (*left*); the Kirchhoff–Love hypotheses require that the deformed segments be normal to the deformed middle surface (*right*)

$$W(F) = \operatorname{dist}^2\big(F, SO(3)\big) \approx \frac{1}{4}|F^\top F - I_3|^2.$$

For a proof of this relation consider $F = P + \varepsilon G$, where $P = \pi_{SO(3)}(F)$ is the nearest-neighbor projection of F onto $SO(3)$ and G is normal to $SO(3)$ at P. We may assume that $P = I_3$, which implies that G is symmetric. Then $\operatorname{dist}^2\big(F, SO(3)\big) = \varepsilon^2|G|^2$ and $|F^\top F - I_3|^2 = \varepsilon^2|G + G^\top|^2 + \mathscr{O}(\varepsilon^3) = 4\varepsilon^2|G|^2 + \mathscr{O}(\varepsilon^3)$. Since $\widehat{I}_t(u) = t^{-3}I_t(u) \le C$ and t is small, we expect that $W(\nabla u)$ is small, i.e., that ∇u is close to $SO(3)$ so that

$$\widehat{I}_t(u) \approx \frac{t^{-3}}{4}\int_{\Omega_t} \big|(\nabla u)^\top \nabla u - I_3\big|^2 \,\mathrm{d}x.$$

Noting $(\nabla' v)^\top \nabla' b = (\nabla' b)^\top \nabla' v$, we have

$$(\nabla u)^\top \nabla u = \begin{bmatrix} (\nabla' v)^\top \nabla' v & 0 \\ 0 & |b|^2 \end{bmatrix} + x_3 \begin{bmatrix} 2(\nabla' b)^\top \nabla' v & (\nabla' b)^\top b \\ b^\top \nabla' b & 0 \end{bmatrix} + x_3^2 \begin{bmatrix} (\nabla' b)^\top \nabla' b & 0 \\ 0 & 0 \end{bmatrix}.$$

With the abbreviations

$$\widehat{g}_t = t^{-1}\big((\nabla' v)^\top \nabla' v - I_2\big), \quad h = -(\nabla' v)^\top \nabla' b, \quad k = (\nabla' b)^\top b,$$

we obtain

$$\begin{aligned} \widehat{I}_t(u) &\approx \frac{t^{-3}}{4}\int_{\Omega_t} \left| \begin{bmatrix} t\widehat{g}_t & 0 \\ 0 & |b|^2 - 1 \end{bmatrix} + x_3 \begin{bmatrix} -2h & (\nabla' b)^\top b \\ b^\top (\nabla' b) & 0 \end{bmatrix} + x_3^2 \begin{bmatrix} k & 0 \\ 0 & 0 \end{bmatrix} \right|^2 \mathrm{d}x \\ &= \frac{t^{-3}}{4}\int_{\Omega_t} \left| \begin{bmatrix} t\widehat{g}_t - 2x_3 h + x_3^2 k & (\nabla' b)^\top b \\ b^\top (\nabla' b) & |b|^2 - 1 \end{bmatrix} \right|^2 \mathrm{d}x. \end{aligned}$$

To guarantee that this expression is bounded t-independently, we need to impose the condition $|b|^2 = 1$, and with the resulting identity $b^\top \nabla' b = 0$, we deduce that

$$\widehat{I}_t(u) \approx \frac{t^{-3}}{4}\int_{\Omega_t} \big|t\widehat{g}_t - 2x_3 h + x_3^2 k\big|^2 \,\mathrm{d}x.$$

By carrying out the integration with respect to x_3, we obtain

$$\widehat{I}_t(u) \approx \frac{1}{4} \int_\omega |\widehat{g}_t|^2 + \frac{1}{3}|h|^2 + \frac{t^2}{5 \cdot 2^4}|k|^2 + \frac{t}{6}\widehat{g}_t : k \, \mathrm{d}x'.$$

Again, to obtain a t-independent limit, we need that $\widehat{g}_t = 0$. Neglecting the term involving the factor t^2, this leads to the reduced, t-independent functional

$$\widehat{I}_t(u) = \frac{1}{12} \int_\omega |h|^2 \, \mathrm{d}x',$$

subject to the pointwise constraint $(\nabla' v)^\top \nabla' v = I_2$. We finally remark that for forces described by functions f_t that are independent of x_3 and such that $t^{-2} f_t \to f$ in $L^2(\omega; \mathbb{R}^3)$ as $t \to 0$, we find with the assumed expansion $u(x) = v(x') + x_3 b(x')$ that

$$\begin{aligned} t^{-3} \int_{\Omega_t} f_t \cdot u \, \mathrm{d}x &= t^{-3} \int_{\Omega_t} f_t \cdot v \, \mathrm{d}x + t^{-3} \int_{-t/2}^{t/2} \int_\omega x_3 b \cdot f_t \, \mathrm{d}x' \, \mathrm{d}x_3 \\ &= t^{-2} \int_{\Omega_t} f_t \cdot v \, \mathrm{d}x \to \int_\omega f \cdot v \, \mathrm{d}x' \end{aligned}$$

as $t \to 0$.

8.1.3 Relations to Linear Elasticity

Linear elasticity employs a *geometric linearization* defined through the symmetric gradient

$$\varepsilon(\phi) = \frac{1}{2}\big((\nabla \phi)^\top + \nabla \phi\big) \approx \frac{1}{2}\big((\nabla u)^\top \nabla u - I_3\big)$$

for small displacements $\phi = u - \mathrm{id}_3 : \Omega \to \mathbb{R}^3$ with $\Omega \subset \mathbb{R}^3$. The energy density W is approximated by the quadratic expression

$$W(\nabla u) \approx \frac{1}{2} D^2 W(I_3)[\nabla \phi, \nabla \phi] = \frac{1}{2} D^2 W(I_3)[\varepsilon(\phi), \varepsilon(\phi)],$$

provided W is isotropic and frame-indifferent, using that $W(I_3) = 0$, and $D\widetilde{W}(I_3) = 0$. For homogeneous materials it follows that with the Lamé constants λ, μ we have for every symmetric matrix $E \in \mathbb{R}^{3\times 3}$ with $\mathbb{C} = D^2 W(I_3)$ that

$$\mathbb{C}E = 2\mu E + \lambda(\operatorname{tr} E)I_3.$$

The related minimization problem looks for $\phi : \Omega \to \mathbb{R}^3$ to be minimal for the *Navier–Lamé functional*

$$I^{\mathrm{NL}}(\phi) = \frac{1}{2}\int_\Omega \mathbb{C}\varepsilon(\phi) : \varepsilon(\phi)\,\mathrm{d}x - \int_\Omega \widehat{f} \cdot \phi\,\mathrm{d}x,$$

subject to $\phi|_{\Gamma_\mathrm{D}} = 0$. For thin plates $\Omega_t = \omega \times (-t/2, t/2)$ with Dirichlet boundary $\Gamma_\mathrm{D} = \gamma_\mathrm{D} \times (-t/2, t/2)$ for $\gamma_\mathrm{D} \subset \partial\omega$, often the following assumptions are made to obtain a dimensionally reduced model. The different assumptions are illustrated in Fig. 8.2.

Assumption 8.1 (*Reissner–Mindlin hypotheses*) (1) Points on the middle surface are only displaced in the vertical direction, i.e., $\phi_1(x', 0) = \phi_2(x', 0) = 0$ for all $x' \in \omega$.
(2) The vertical displacement does not depend on x_3, i.e., $\phi_3(x', x_3) = w(x')$.
(3) Segments that are normal to the middle surface are linearly deformed, i.e., $\phi(x', x_3) = \phi(x', 0) - x_3\widehat{\theta}(x')$ for all $(x', x_3) \in \Omega_t$.

The assumption implies that the minimizer for I^{NL} is given by

$$\phi(x', x_3) = \begin{bmatrix} -x_3\theta(x') \\ w(x') \end{bmatrix}$$

with the rotation $\theta : \omega \to \mathbb{R}^2$ and the vertical displacement $w : \omega \to \mathbb{R}$.

Assumption 8.2 (*Kirchhoff–Love hypotheses*) In addition to the Reissner–Mindlin hypotheses, assume that segments that are normal to the middle surface are mapped linearly and isometrically to segments that are normal to the deformed middle surface, i.e., $\phi(x', x_3) = \phi(x', 0) - x_3\widehat{\theta}(x')$ for all $(x', x_3) \in \Omega_t$ with

$$\widehat{\theta}(x', 0) = (1 + |\nabla' w|^2)^{-1/2} \begin{bmatrix} \nabla' w \\ 0 \end{bmatrix} \approx \begin{bmatrix} \nabla' w \\ 0 \end{bmatrix}.$$

Note that ϕ is the displacement, so that the third component of the normal vector $\widehat{\theta}$ disappears. The additional assumption implies that the solution of the linearly elastic problem is given by

$$\phi(x', x_3) = \begin{bmatrix} -x_3\nabla' w(x') \\ w(x') \end{bmatrix}$$

for the vertical displacement $w : \omega \to \mathbb{R}$.

Proposition 8.1 (Linear bending) *Assume that f_t is independent of x_3 and set $f_3 = t^{-2}f_{t,3}$. Suppose that $\mathbb{C}E = E$ for all symmetric matrices $E \in \mathbb{R}^{3\times 3}$. Let $\phi \in H^1_\mathrm{D}(\Omega_t; \mathbb{R}^3)$ be the minimizer of the three-dimensional elasticity functional I^{NL} with*

$\Omega = \Omega_t$ and $\widehat{f} = f_t$. Up to a change of constants we have:
(i) Under the Reissner–Mindlin hypotheses the pair $(w, \theta) \in H^1_{\mathrm{D}}(\omega) \times H^1_{\mathrm{D}}(\omega; \mathbb{R}^2)$ that specifies ϕ solves the linear Reissner–Mindlin model.
(ii) Under the Kirchhoff–Love hypotheses the function $w \in H^2_{\mathrm{D}}(\omega)$ that specifies ϕ solves the linear Kirchhoff model.

Proof In the case of the Reissner–Mindlin hypotheses we have

$$\varepsilon'(\phi) = \frac{1}{2}\begin{bmatrix} -x_3\nabla'\theta & -\theta \\ (\nabla' w)^\top & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -x_3(\nabla'\theta)^\top & \nabla' w \\ -\theta^\top & 0 \end{bmatrix} = \begin{bmatrix} -x_3\varepsilon'(\theta) & (\nabla' w - \theta)/2 \\ (\nabla' w - \theta)^\top/2 & 0 \end{bmatrix}.$$

Therefore, due to the assumption $\mathbb{C}E = E$,

$$\mathbb{C}\varepsilon'(\phi) : \varepsilon'(\phi) = x_3^2|\varepsilon'(\theta)|^2 + \frac{1}{2}|\nabla' w - \theta|^2.$$

An integration over $\Omega_t = \omega \times (-t/2, t/2)$ shows that

$$\frac{1}{2}\int_{\Omega_t} \mathbb{C}\varepsilon'(\varphi) : \varepsilon'(\varphi)\,\mathrm{d}x = \frac{t^3}{24}\int_\omega |\varepsilon'(\theta)|^2\,\mathrm{d}x' + \frac{t}{4}\int_\omega |\nabla' w - \theta|^2\,\mathrm{d}x'.$$

Since f_t is independent of x_3, we have

$$\int_{\Omega_t} f_t \cdot \varphi\,\mathrm{d}x = \int_\omega \int_{-t/2}^{t/2} (-x_3)\theta \cdot f_{t,12}\,\mathrm{d}x_3\,\mathrm{d}x' + \int_\omega \int_{-t/2}^{t/2} w f_{t,3}\,\mathrm{d}x_3\,\mathrm{d}x' = t\int_\omega f_{t,3} w\,\mathrm{d}x'.$$

Hence,

$$t^{-3} I^{\mathrm{NL}}(\varphi) = \frac{1}{24}\int_\omega |\varepsilon(\theta)|^2\,\mathrm{d}x' + \frac{t^{-2}}{4}\int_\omega |\nabla w - \theta|^2\,\mathrm{d}x' - \int_\omega f_3 w\,\mathrm{d}x'.$$

For the Kirchhoff hypothesis, this simplifies to $I^{\mathrm{Ki}'}$ due to the identities $\nabla' w = \theta$ and $\varepsilon'(\nabla' w) = \nabla'\nabla' w$. □

Remark 8.2 If $\mathbb{C}E = 2\mu E + \lambda(\operatorname{tr} E)I_3$ is considered then the assumption that for $\sigma = \mathbb{C}\varepsilon(\phi)$ we have $\sigma_{33} = 0$ has to be included.

8.1.4 Properties of Isometries

Given a surface $\mathscr{M}$ parametrized by $u : \omega \to \mathbb{R}^3$ the *first and second fundamental forms* $g, h : \omega \to \mathbb{R}^{2\times 2}$ are given by

$$g = (\partial_i u \cdot \partial_j u)_{1 \le i,j \le 2} = (\nabla u)^\top \nabla u,$$
$$h = -(\partial_i b \cdot \partial_j u)_{1 \le i,j \le 2} = -(\nabla b)^\top \nabla u = b^\top D^2 u,$$

where $b = \partial_1 u \times \partial_2 u / |\partial_1 u \times \partial_2 u|$ is a unit normal to $\mathscr{M}$. The parametrization is assumed to be an immersion, so that the tangent vectors $\partial_1 u$ and $\partial_2 u$ are linearly independent everywhere in ω. The first and second fundamental form are interpreted as bilinear forms on the tangent space $T\mathscr{M}$ in terms of the coefficients of the family of bases $\big(\partial_1 u(x), \partial_2 u(x)\big)_{x\in\omega}$. It follows that g is a symmetric and positive definite matrix for every $x \in \omega$ that defines a metric on the tangent space of $\mathscr{M}$. The *Gauss and mean curvature* are the determinant and the trace of the *Weingarten map*

$$s = -hg^{-1}$$

and given by

$$K = \det s = \frac{\det h}{\det g}, \quad H = \operatorname{tr} s = -\frac{h : \det' g}{\det g},$$

respectively. The Weingarten map measures variations of the normal b and is interpreted as a linear mapping on the tangent space. The second fundamental form is the bilinear form associated with s. We refer the reader to Sect. 8.4 for a detailed discussion.

Definition 8.5 The parametrization $u : \omega \to \mathbb{R}^3$ is called *isometry* if $g(x) = I_2$ for every $x \in \omega$.

Proposition 8.2 *Suppose that $u : \omega \to \mathbb{R}^3$ is a C^2-isometry. Then $\partial_i \partial_j u \cdot \partial_k u = 0$, $K = 0$, and*

$$|D^2 u| = |\Delta u| = |h| = |H|,$$

where $|\cdot|$ denotes the Frobenius norm on the respective spaces.

Proof We first note that for $1 \le i, j \le 2$, we have $0 = \partial_i(\partial_j u \cdot \partial_j u) = 2\partial_i \partial_j u \cdot \partial_j u$. To show that we also have $\partial_i^2 u \cdot \partial_j u = 0$ for $i \ne j$, we note $0 = \partial_i(\partial_i \cdot \partial_j u) = \partial_i^2 u \cdot \partial_j u + \partial_i u \cdot \partial_i \partial_j u$, i.e., $\partial_i^2 u \cdot \partial_j u = -\partial_i u \cdot \partial_i \partial_j u = 0$. Hence, we have

$$\partial_i \partial_j u \cdot \partial_k u = 0$$

for $i, j, k = 1, 2$, i.e., the Christoffel symbols of the second kind vanish. As a consequence of Gauss' theorem, cf. Lemma 8.3, we have $K = 0$. Moreover, we deduce that $-\Delta u = \beta b$ and since $(-\Delta u) \cdot b = \operatorname{tr}(-h) = H$, we have $\beta = H$. The vectors $(\partial_1 u, \partial_2 u, b)$ form an orthonormal basis of $\mathbb{R}^3$ for every $x \in \omega$, so that $|\partial_i \partial_j u| = |\partial_i \partial_j u \cdot b|$ and hence

$$|D^2 u|^2 = \sum_{i,j=1}^{2} |\partial_i \partial_j u \cdot b|^2 = |h|^2.$$

Moreover, we have

$$|h|^2 = |s|^2 = (\mathrm{tr}\, s)^2 - 2 \det s = H^2 - 2K = H^2,$$

which proves the assertion. □

Remark 8.3 Since isometries in $H^2(\omega; \mathbb{R}^3)$ can be approximated by isometries in $C^2(\overline{\omega}; \mathbb{R}^3)$ in the norm of $H^2(\omega; \mathbb{R}^3)$, the results of the proposition also hold for isometries $u \in H^2(\omega; \mathbb{R}^3)$, cf. [12].

8.2 Approximaton of Linear Bending Models

We discuss in this section numerical methods for the approximation of the linear Kirchhoff and the linear Reissner–Mindlin model. Finite element methods for dimensionally reduced models have to be carefully developed to avoid so-called *locking effects*. This describes the phenomenon that deformations obtained by numerical computation are too small in comparison to the true deformation. In particular, *membrane locking* is the inability of a finite element method to capture bending effects without stretching while *shear locking* refers to the problem that a finite element method is too stiff to describe certain in-plane deformations due to the occurrence of a small parameter. Another effect that occurs in the description of thin elastic structures is the *Babuška paradox* that states that if a domain is approximated by polygons, then the numerical solutions may fail to converge to the correct solution. We follow closely the presentation of [5] and refer the reader to [4] for further aspects.

8.2.1 Discrete Kirchhoff Triangles

To avoid an H^2-conforming finite element method for the linear Kirchhoff model, we employ a nonconforming discretization that is based on the construction of a discrete gradient operator

$$\nabla_h : W_h \to \Theta_h$$

with H^1-conforming finite element spaces $W_h \subset H^1(\omega)$ and $\Theta_h \subset H^1(\omega; \mathbb{R}^2)$. These are for a regular triangulation $\mathscr{T}_h$ of ω defined as

$$\begin{aligned} W_h = \{w_h \in C(\overline{\omega}) : w_h|_T \in P_3^{\mathrm{red}}(T) \text{ for all } T \in \mathscr{T}_h, \\ \nabla w_h \text{ continuous at all } z \in \mathscr{N}_h\}, \\ \Theta_h = \{\theta_h \in C(\overline{\omega}) : \theta_h|_T \in P_2(T) \text{ for all } T \in \mathscr{T}_h\}. \end{aligned}$$

Here, $P_k(T)$ for every $T \in \mathscr{T}_h$ denotes the set of polynomials of total degree less or equal to $k \geq 0$ restricted to T. The superscript in P_3^{red} means that one degree of

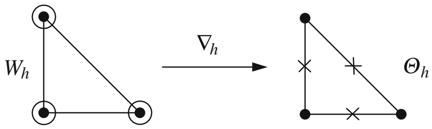

Fig. 8.3 Schematic description of the elementwise reduced cubic finite element space W_h (*left*) and the space of elementwise quadratic vector fields Θ_h (*right*)

freedom is eliminated, i.e., with the center of mass $x_T = (1/3)\sum_{z\in\mathcal{N}_h\cap T} z$ of T,

$$P_3^{\rm red}(T) = \Big\{p \in P_3(T) : p(x_T) = \frac{1}{3}\sum_{z\in\mathcal{N}_h\cap T}\big[p(z) + \nabla p(z)\cdot(x_T - z)\big]\Big\}.$$

The degrees of freedom in W_h are the function values and the derivatives at the vertices of the elements, cf. Fig. 8.3. For $w \in H^3(\omega)$, we define the nodal interpolant $\widetilde{\mathscr{I}}_h^3 w \in W_h$ by the conditions $\widetilde{\mathscr{I}}_h^3 w(z) = w(z)$ and $\nabla\widetilde{\mathscr{I}}_h^3 w(z) = \nabla w(z)$ for all $z \in \mathcal{N}_h$.

Definition 8.6 The *discrete gradient operator* $\nabla_h : W_h \to \Theta_h$ is for $w_h \in W_h$ the uniquely defined function $\theta_h = \nabla_h w_h \in \Theta_h$ with

$$\begin{aligned} \theta_h(z) &= \nabla w_h(z) && \text{for all } z \in \mathcal{N}_h,\\ \theta_h(z_S)\cdot n_S &= \frac{1}{2}\big(\nabla w_h(z_S^1) + \nabla w_h(z_S^2)\big)\cdot n_S && \text{for all } S \in \mathscr{S}_h,\\ \theta_h(z_S)\cdot t_S &= \nabla w_h(z_S)\cdot t_S && \text{for all } S \in \mathscr{S}_h, \end{aligned}$$

where, for all sides $S \in \mathscr{S}_h$, the orthonormal vectors $n_S, t_S \in \mathbb{R}^2$ are chosen such that n_S is normal to S, $z_S^1, z_S^2 \in \mathcal{N}_h$ are the endpoints of S, and $z_S = (z_S^1 + z_S^2)/2$ is the midpoint of S. For $w \in H^3(\Omega)$, we set $\nabla_h w = \nabla_h \widetilde{\mathscr{I}}_h^3 w$.

Remark 8.4 For every $S \in \mathscr{S}_h$ we have

$$\nabla_h w_h(z_S) = \frac{1}{2}\big[\big(\nabla w_h(z_S^1) + \nabla w_h(z_S^2)\big)\cdot n_S\big]n_S + \big[\nabla w_h(z_S)\cdot t_S\big]t_S.$$

The following lemma shows that ∇_h may be regarded as an interpolation operator on the space of gradients of functions in $H^3(\omega)$. We let $\gamma_{\rm D} \subset \partial\omega$ be closed and of positive surface measure and define $\gamma_{\rm N} = \partial\omega \setminus \gamma_{\rm D}$.

Lemma 8.1 (Properties of ∇_h [5]) (i) *There exists* $c_1 > 0$ *such that for all* $w_h \in W_h$ *and* $T \in \mathscr{T}_h$, *we have for* $\ell = 0, 1$ *that*

$$c_1^{-1}\|\nabla^{\ell+1} w_h\|_{L^2(T)} \le \|\nabla^\ell \nabla_h w_h\|_{L^2(T)} \le c_1\|\nabla^{\ell+1} w_h\|_{L^2(T)},$$

where $\nabla^1 = \nabla$ *and* $\nabla^0 = I$.
(ii) *There exists* $c_2 > 0$ *such that for all* $w \in H^3(\omega)$ *and* $T \in \mathcal{T}_h$, *we have*

$$\|\nabla_h w - \nabla w\|_{L^2(T)} + h_T \|\nabla\nabla_h w - D^2 w\|_{L^2(T)} \le c_2 h_T^2 \|D^3 w\|_{L^(T)}.$$

(iii) *There exists* $c_3 > 0$ *such that for all* $w_h \in W_h$ *and* $T \in \mathcal{T}_h$, *we have*

$$\|\nabla_h w_h - \nabla w_h\|_{L^2(T)} \le c_3 h_T \|D^2 w_h\|_{L^2(T)}.$$

(iv) *The mapping* $w_h \mapsto \|\nabla\nabla_h w_h\|$ *defines a norm on*

$$W_{h,\mathrm{D}} = \big\{w_h \in W_h : w_h(z) = 0,\ \nabla w_h(z) = 0 \text{ for all } z \in \mathcal{N}_h \cap \gamma_\mathrm{D}\big\},$$

and we have $w_h|_{\gamma_\mathrm{D}} = 0$ *and* $\nabla w_h|_{\gamma_\mathrm{D}} = 0$ *for all* $w_h \in W_{h,\mathrm{D}}$.

Proof (i) Both expressions define semi-norms and we show that $\nabla^{\ell+1} w_h = 0$ if and only if $\nabla^\ell \nabla_h w_h = 0$ for all $w_h \in W_h$. Assume that $\nabla_h w_h|_T = c_T$ for some $c_T \in \mathbb{R}^2$. Then $\nabla w_h(z) = c_T$ for all $z \in \mathcal{N}_h \cap T$ and $\nabla w_h(z_S) = c_T$ for all $S \in \mathcal{S}_h \cap T$. Thus, the cubic polynomials $w_h|_S$ are affine for all $S \in \mathcal{S}_h \cap \partial T$, and also the function $w_h|_{\partial T}$ is affine. Due to the elementwise constraint in the definition of W_h, it follows that $w_h|_T$ is affine and thus $\nabla w_h = c_T$. If conversely $\nabla w_h|_T = c_T$, then also $\nabla_h w_h|_T = c_T$. Hence, the expressions $\|\nabla^{\ell+1} w_h\|_{L^2(T)}$ and $\|\nabla^\ell \nabla_h w_h\|_{L^2(T)}$ are equivalent semi-norms on $W_h|_T$ and a scaling argument proves the first assertion.
(ii) Since $\nabla_h w|_T$ is affine if $\nabla w|_T$ is affine, the Bramble–Hilbert lemma yields the interpolation estimate

$$\|\theta - \theta_h\|_{L^2(T)} + h_T \|\nabla(\theta - \theta_h)\|_{L^2(T)} \le c h_T^2 \|D^2\theta\|_{L^2(T)}$$

for $\theta = \nabla w \in H^2(\omega)$ and $\theta_h = \nabla_h w$.
(iii) The estimate is a consequence of (ii) and the inverse estimate $\|D^3 w_h\|_{L^2(T)} \le c h_T^{-1} \|D^2 w_h\|_{L^2(T)}$.
(iv) If $w_h(z) = 0$ and $\nabla_h w_h(z) = 0$ for all $z \in \mathcal{N}_h \cap \gamma_\mathrm{D}$ then, since $w_h|_S$ is a cubic polynomial for every $S \in \mathcal{S}_h$, it follows that $w_h|_{\gamma_\mathrm{D}} = 0$ and $\nabla_h w_h|_{\gamma_\mathrm{D}} = 0$. Assume that $\|\nabla\nabla_h w_h\| = 0$. Then, since $\nabla_h w_h|_{\gamma_\mathrm{D}} = 0$ we deduce by Poincaré inequality that $\nabla_h w_h = 0$ in ω. With (i) and $w_h|_{\gamma_\mathrm{D}} = 0$ we find $w_h = 0$ in ω. □

The interpolation estimates allow us to prove the following error estimate.

Theorem 8.2 (Error estimate) *Assume that* $w \in H^2_\mathrm{D}(\omega) \cap H^3(\omega)$ *is the solution of the linear Kirchhoff model, i.e.,*

$$(D^2 w, D^2 v) = (f, v)$$

for all $v \in H^2_\mathrm{D}(\omega)$ *and let* $w_h \in W_{h,\mathrm{D}}$ *solve*

$$(\nabla\nabla_h w_h, \nabla\nabla_h v_h) = (f, v_h)$$

for all $v_h \in W_{h,\mathrm{D}}$. Then we have

$$\|D^2 w - \nabla\nabla_h w_h\| \le ch\|w\|_{H^3(\omega)}.$$

Proof The Lax–Milgram lemma and Lemma 8.1(iv) imply the existence of unique solutions $w \in H^2_{\mathrm{D}}(\omega)$ and $w_h \in W_{h,\mathrm{D}}$. The assumption $w \in H^3(\omega)$, the boundary condition $(D^2 w)n|_{\gamma_{\mathrm{N}}} = 0$, an integration by parts, and the identities div $D^2 = \Delta\nabla = \nabla\Delta$ show, that for all $v \in H^2_{\mathrm{D}}(\omega)$, we have

$$(f, v) = (D^2 w, D^2 v) = -(\nabla\Delta w, \nabla v)$$

and this identity holds for all $v \in H^1_{\mathrm{D}}(\omega)$. Therefore, for $v_h \in W_{h,\mathrm{D}}$ it follows that

$$\begin{aligned}
(\nabla\nabla_h w, \nabla\nabla_h v_h) &= (D^2 w, \nabla\nabla_h v_h) + (\nabla[\nabla_h w - \nabla w], \nabla\nabla_h v_h) \\
&= -(\nabla\Delta w, \nabla_h v_h) + (\nabla[\nabla_h w - \nabla w], \nabla\nabla_h v_h) \\
&= -(\nabla\Delta w, \nabla v_h) - (\nabla\Delta w, [\nabla_h v_h - \nabla v_h]) \\
&\quad + (\nabla[\nabla_h w - \nabla w], \nabla\nabla_h v_h).
\end{aligned}$$

Recalling that $\nabla_h w = \nabla_h \widetilde{\mathscr{I}}_h^3 w$ and incorporating the discrete and continuous formulations, this yields that

$$\begin{aligned}
\|\nabla\nabla_h[w - w_h]\|^2 &= (\nabla\nabla_h w, \nabla\nabla_h[w - w_h]) - (\nabla\nabla_h w_h, \nabla\nabla_h[w - w_h]) \\
&= (f, \widetilde{\mathscr{I}}_h^3 w - w_h) + (\nabla\Delta w, \nabla_h[w - w_h] - \nabla[\widetilde{\mathscr{I}}_h^3 w - w_h]) \\
&\quad + (\nabla[\nabla_h w - \nabla w], \nabla\nabla_h[w - w_h]) - (f, \widetilde{\mathscr{I}}_h^3 w - w_h) \\
&= (\nabla\Delta w, \nabla_h[w - w_h] - \nabla[\widetilde{\mathscr{I}}_h^3 w - w_h]) \\
&\quad + (\nabla[\nabla_h w - \nabla w], \nabla\nabla_h[w - w_h]).
\end{aligned}$$

For the first term on the right-hand side we have by Lemma 8.1(i) and (iii) that

$$(\nabla\Delta w, \nabla_h[w - w_h] - \nabla[\widetilde{\mathscr{I}}_h^3 w - w_h]) \le ch\|\nabla\Delta w\|\,\|\nabla\nabla_h[w - w_h]\|.$$

The second term is estimated with the help of Lemma 8.1(ii), i.e.,

$$(\nabla[\nabla_h w - \nabla w], \nabla\nabla_h[w - w_h]) \le ch\|D^3 w\|\,\|\nabla\nabla_h[w - w_h]\|$$

The combination of the last three estimates, the triangle inequality, and the bound $\|D^2 w - \nabla\nabla_h w\| \le ch\|D^3 w\|$ of Lemma 8.1(ii) prove the assertion. $\square$

8.2.2 Realization

For the implementation of the discrete Kirchhoff triangle, we identify functions $w_h \in W_h$ and $\theta_h \in \Theta_h$ with vectors $W \in \mathbb{R}^{3L}$ and $\Theta \in \mathbb{R}^{2(L+M)}$, where $L = n_C = \#\mathscr{N}_h$ and $M = n_S = \#\mathscr{S}_h$, defined by

$$W = \begin{bmatrix} w_h(z_1) \\ \nabla w_h(z_1) \\ w_h(z_2) \\ \nabla w_h(z_2) \\ \vdots \\ w_h(z_L) \\ \nabla w_h(z_L) \end{bmatrix} = \begin{bmatrix} w_{z_1} \\ \delta w_{z_1} \\ w_{z_2} \\ \delta w_{z_2} \\ \vdots \\ w_{z_L} \\ \delta w_{z_L} \end{bmatrix}, \quad \Theta = \begin{bmatrix} \theta_h(z_1) \\ \theta_h(z_2) \\ \vdots \\ \theta_h(z_L) \\ \theta_h(z_{S_1}) - \big(\theta_h(z_{S_1}^1) + \theta_h(z_{S_1}^2)\big)/2 \\ \theta_h(z_{S_2}) - \big(\theta_h(z_{S_2}^1) + \theta_h(z_{S_2}^2)\big)/2 \\ \vdots \\ \theta_h(z_{S_M}) - \big(\theta_h(z_{S_M}^1) + \theta_h(z_{S_M}^2)\big)/2 \end{bmatrix} = \begin{bmatrix} \theta_{z_1} \\ \theta_{z_2} \\ \vdots \\ \theta_{z_L} \\ \theta_{S_1} \\ \theta_{S_2} \\ \vdots \\ \theta_{S_M} \end{bmatrix}$$

with $\mathscr{N}_h = \{z_1, z_2, \dots, z_L\}$ and $\mathscr{S}_h = \{S_1, S_2, \dots, S_M\}$. For the coefficient of θ_h related to a side $S \in \mathscr{S}_h$, we subtract half of the values of θ_h at the corresponding endpoints z_S^1 and z_S^2 since we use the hierarchical basis

$$\big(\varphi_{z_1}, \varphi_{z_2}, \dots, \varphi_{z_L}, \varphi_{S_1}, \varphi_{S_2}, \dots \varphi_{S_M}\big)$$

of the space $\mathscr{S}^2(\mathscr{T}_h) = \{v_h \in C(\overline{\omega}) : v_h|_T \in P_2(T) \text{ for all } T \in \mathscr{T}_h\}$ given by the nodal basis $(\varphi_{z_1}, \varphi_{z_2}, \dots, \varphi_{z_L})$ of $\mathscr{S}^1(\mathscr{T}_h)$ and the functions $\varphi_S = 4\varphi_{z_S^1}\varphi_{z_S^2}$ for all $S \in \mathscr{S}_h$. A straightforward calculation shows that, for a function $w_h \in P_3^{\mathrm{red}}(T)$, we have that $w_h|_S$ is cubic for every side $S \subset \partial T$ with

$$\big(\nabla w_h(z_S)\big) \cdot t_S = \frac{3}{2|S|}\big(w_h(z_S^2) - w_h(z_S^1)\big) - \frac{1}{4}\big(\nabla w_h(z_S^1) + \nabla w_h(z_S^2)\big) \cdot t_S$$

with $|S| = |z_S^2 - z_S^1|$ and $z_S^2 - z_S^1 = |S| t_S$. Since (n_S, t_S) are orthonormal vectors it follows for $\theta_h = \nabla_h w_h$ that

$$\begin{aligned} \theta_h(z_S) &= \big(\nabla w_h(z_S) \cdot t_S\big) t_S + \Big[\frac{1}{2}\big(\nabla w_h(z_S^1) + \nabla w_h(z_S^2)\big) \cdot n_S\Big] n_S \\ &= \big(\nabla w_h(z_S) \cdot t_S\big) t_S + \frac{1}{2}\big(\nabla w_h(z_S^1) + \nabla w_h(z_S^2)\big) \\ &\quad - \Big[\frac{1}{2}\big(\nabla w_h(z_S^1) + \nabla w_h(z_S^2)\big) \cdot t_S\Big] t_S \\ &= \frac{3}{2|S|}\big(w_h(z_S^2) - w_h(z_S^1)\big) t_S - \frac{3}{4}\Big[\big(\nabla w_h(z_S^1) + \nabla w_h(z_S^2)\big) \cdot t_S\Big] t_S \\ &\quad + \frac{1}{2}\big(\nabla w_h(z_S^1) + \nabla w_h(z_S^2)\big). \end{aligned}$$

Since $\theta_h(z_S^j) = \nabla w_h(z_S^j)$, $j = 1, 2$, the corresponding coefficient is given by

$$\begin{aligned}\theta_S &= \theta_h(z_S) - \big(\theta_h(z_S^1) + \theta_h(z_S^2)\big)/2 \\ &= \frac{3}{2|S|}\big(w_h(z_S^2) - w_h(z_S^1)\big)t_S - \frac{3}{4}\big[\big(\nabla w_h(z_S^1) + \nabla w_h(z_S^2)\big)\cdot t_S\big]t_S.\end{aligned}$$

With these identifications, the discrete gradient operator can be represented by a matrix $D_h \in \mathbb{R}^{2(L+M)\times 3L}$. For a single element $T = \operatorname{conv}\{z_1, z_2, z_3\}$ with sides $S_1 = \operatorname{conv}\{z_2, z_3\}$, $S_2 = \operatorname{conv}\{z_3, z_1\}$, and $S_3 = \operatorname{conv}\{z_1, z_2\}$, we have

$$\begin{bmatrix}\theta_{z_1}\\ \theta_{z_2}\\ \theta_{z_3}\\ \theta_{S_1}\\ \theta_{S_2}\\ \theta_{S_3}\end{bmatrix} = \begin{bmatrix}0 & I_2 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & I_2 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & I_2\\ 0 & 0 & \widetilde{t}_{S_1} & \widetilde{T}_{S_1} & -\widetilde{t}_{S_1} & \widetilde{T}_{S_1}\\ \widetilde{t}_{S_2} & \widetilde{T}_{S_2} & 0 & 0 & -\widetilde{t}_{S_2} & \widetilde{T}_{S_2}\\ \widetilde{t}_{S_3} & \widetilde{T}_{S_3} & -\widetilde{t}_{S_3} & \widetilde{T}_{S_3} & 0 & 0\end{bmatrix}\begin{bmatrix}w_{z_1}\\ \delta w_{z_1}\\ w_{z_2}\\ \delta w_{z_2}\\ w_{z_3}\\ \delta w_{z_3}\end{bmatrix}$$

where $\widetilde{T}_{S_\ell} = -(3/4)t_{S_\ell}t_{S_\ell}^\top$ and $\widetilde{t}_{S_\ell} = -(3/(2|S_\ell|))t_{S_\ell}$. For a simpler implementation we approximated the right-hand side using numerical integration, i.e.,

$$\int_\omega f_3 w_h \,\mathrm{d}x \approx \int_\omega \mathscr{I}_h[f_3 w_h]\,\mathrm{d}x$$

which is computed with the lumped mass matrix. Figure 8.5 displays an implementation of the approximation of the linear Kirchhoff model with the discrete Kirchhoff triangle. The $M \times 2$ field `n4s` provides an enumeration of the edges and defines their endpoints. The field `s4e` has dimension $n_E \times 3$, $n_E = \#\mathscr{T}_h$, and contains the global numbers of the sides of the elements in $\mathscr{T}_h$, where the convention that the jth edge of T is opposite to the jth node of T is used, cf. Fig. 8.4. These arrays are provided by the subroutine `sides`. The stiffness matrix of the $P2$ finite element space with respect to the hierarchical basis defined above is provided by the routine `fe_matrix_p2.m`.

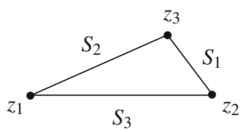

Fig. 8.4 Local enumeration of the sides of a *triangle* every side is associated to the opposite node

```
function kirchhoff_linear(red)
[c4n,n4e,Db,Nb] = triang_cube(2);
for j = 1:red
    [c4n,n4e,Db,Nb] = red_refine(c4n,n4e,Db,Nb);
end
[n4s,s4e] = sides(n4e);
nC = size(c4n,1); nS = size(n4s,1);
dNodes = unique(Db);
FNodes = setdiff(1:3*nC,[3*dNodes-2;3*dNodes-1;3*dNodes-0]);
u = zeros(3*nC,1); b = zeros(3*nC,1);
D = sparse(2*(nC+nS),3*nC);
for j = 1:nC
    D(2*j-[1,0],3*j-[1,0]) = eye(2);
end
for j = 1:nS
    t_S = (c4n(n4s(j,2),:)-c4n(n4s(j,1),:))';
    length_S = norm(t_S); t_S = t_S/length_S;
    D(2*nC+2*j-[1,0],3*n4s(j,1)-2) = -3/(2*length_S)*t_S;
    D(2*nC+2*j-[1,0],3*n4s(j,2)-2) = 3/(2*length_S)*t_S;
    D(2*nC+2*j-[1,0],3*n4s(j,1)-[1,0]) = -(3/4)*(t_S*t_S');
    D(2*nC+2*j-[1,0],3*n4s(j,2)-[1,0]) = -(3/4)*(t_S*t_S');
end
[s_p1,¬,m_lumped,vol_T] = fe_matrices(c4n,n4e);
s_p2 = fe_matrix_p2(c4n,n4e,n4s,s4e,s_p1,vol_T);
S = sparse(2*(nC+nS),2*(nC+nS));
S(1:2:2*(nC+nS),1:2:2*(nC+nS)) = s_p2;
S(2:2:2*(nC+nS),2:2:2*(nC+nS)) = s_p2;
S_dkt = D'*S*D;
b(3*(1:nC)-2) = m_lumped*f(c4n);
u(FNodes) = S_dkt(FNodes,FNodes)\b(FNodes);
show_p1(c4n,n4e,Db,Nb,u(1:3:3*nC))

function [n4s,s4e] = sides(n4e)
sides = reshape(n4e(:,[2,3,3,1,1,2])',2,[])';
[n4s,¬,sideNrs] = unique(sort(sides,2),'rows','first');
s4e = reshape(sideNrs(1:3*size(n4e,1)),3,[])';

function val = f(x)
val = ones(size(x,1),1);
```

Fig. 8.5 MATLAB routine for the approximation of the linear Kirchhoff model with Kirchhoff triangles

8.2.3 Reissner–Mindlin Plate

The linear Reissner–Mindlin model seeks a pair $(w, \theta) \in H^1_{\mathrm{D}}(\omega) \times H^1_{\mathrm{D}}(\omega; \mathbb{R}^2)$ such that

$$\big(\varepsilon(\theta), \varepsilon(\psi)\big) + t^{-2}(\theta - \nabla w, \psi - \nabla \eta) = (f, \eta)$$

for all $(\psi, \eta) \in H^1_{\mathrm{D}}(\omega; \mathbb{R}^2) \times H^1_{\mathrm{D}}(\omega)$. The corresponding strong form of the problem reads as

$$
\begin{aligned}
-\operatorname{div}\varepsilon(\theta) + t^{-2}(\theta - \nabla w) = 0 \text{ in } \omega, \quad \theta|_{\gamma_{\mathrm{D}}} = 0, \quad \partial_n\theta|_{\gamma_{\mathrm{N}}} = 0,\\
t^{-2}\operatorname{div}(\theta - \nabla w) = f \text{ in } \omega, \quad w|_{\gamma_{\mathrm{D}}} = 0, \quad (\theta - \nabla w)\cdot n|_{\gamma_{\mathrm{N}}} = 0
\end{aligned}
$$

with $\gamma_{\mathrm{N}} = \partial\omega \setminus \gamma_{\mathrm{D}}$. The problem can be simplified by employing a Helmholtz decomposition of $\theta - \nabla w$. For a function $p \in H^1(\omega)$ we write

$$
\operatorname{Curl} p = (\nabla p)^{\perp} = [-\partial_2 p, \partial_1 p]^{\top}.
$$

Proposition 8.3 (Equivalent formulation) *Assume that ω is simply connected. There exist uniquely defined functions $r \in H^1_{\mathrm{D}}(\omega)$ and $p \in H^1(\omega)$ with $\int_\omega p \,\mathrm{d}x = 0$ and* Curl $p \cdot n|_{\gamma_{\mathrm{N}}} = 0$, *such that $t^{-2}(\theta - \nabla w) = -\nabla r -$* Curl p. *The function $r \in H^1_{\mathrm{D}}(\omega)$ satisfies*

$$
(\nabla r, \nabla \eta) = (f, \eta)
$$

for all $\eta \in H^1_{\mathrm{D}}(\omega)$. The pair (θ, p) is uniquely defined by the equations

$$
\begin{aligned}
\big(\varepsilon(\theta), \varepsilon(\psi)\big) - \quad (\operatorname{Curl} p, \psi) \quad &= (\nabla r, \psi),\\
(\theta, \operatorname{Curl} q) - t^2(\operatorname{Curl} p, \operatorname{Curl} q) &= 0
\end{aligned}
$$

for all $(\psi, q) \in H^1_{\mathrm{D}}(\omega; \mathbb{R}^2) \times H^1(\omega)$ with Curl $q \cdot n|_{\gamma_{\mathrm{N}}} = 0$. *The function $w \in H^1_{\mathrm{D}}(\omega)$ satisfies*

$$
(\nabla w, \nabla v) = (\theta, \nabla v) + t^2(\nabla r, \nabla v)
$$

for all $v \in H^1_{\mathrm{D}}(\omega)$.

Proof Let $r \in H^1_{\mathrm{D}}(\omega)$ be the unique solution of

$$
(\nabla r, \nabla \eta) = (f, \eta) = -t^{-2}(\theta - \nabla w, \nabla \eta)
$$

for all $\eta \in H^1_{\mathrm{D}}(\omega)$. Since $F = t^{-2}(\theta - \nabla w) + \nabla r$ satisfies div $F = 0$ in ω and since $F \cdot n|_{\gamma_{\mathrm{N}}} = 0$, there exists a uniquely defined function $p \in H^1(\omega)$ with $\int_\omega p\,\mathrm{d}x = 0$, Curl $p \cdot n = 0$ on γ_{N}, and $F = -$ Curl p, cf., e.g., [11]. For all $\eta \in H^1_{\mathrm{D}}(\omega)$, we then have

$$
(\operatorname{Curl} p, \nabla \eta) = \int_{\partial\omega} \eta \operatorname{Curl} p \cdot n \,\mathrm{d}s = 0.
$$

The equations now follow from the weak formulation of the linear Reissner–Mindlin model and the identity that defines Curl p. □

The equations derived in the proposition show that the solution of the linear Reissner-Mindlin model can be computed by successively solving three problems. The first and the third formulations that define r and w are Poisson problems, while the second one defines the pair (θ, p) through a saddle-point problem with a penalty term that is qualitatively equivalent to the Stokes problem. In particular, the inf-sup condition is satisfied and the solution operator is bounded t-independently. This implies the robust solvability of the Reissner–Mindlin model, provided that the finite element spaces used for the approximation of (θ, p) satisfy a discrete inf-sup condition. A possible choice is the so-called *mini-element*, which is the lowest order conforming polynomial element for the Stokes problem. To guarantee that a discrete Helmholtz decomposition is available, the variables r and w then need to be approximated in the nonconforming Crouzeix–Raviart finite element space, cf. [1] for related details and optimal, t-independent error estimates.

8.3 Approximation of the Nonlinear Kirchhoff Model

The linear Kirchhoff model may be regarded as a simplification of the nonlinear Kirchhoff model in the case of small displacements. We generalize in this section the finite element method based on discrete Kirchhoff triangles for the linear model to the nonlinear one that describes large bending deformations. The proposed method uses techniques developed in [3].

8.3.1 Discretization

We employ the spaces W_h and Θ_h introduced for the approximation of the linear Kirchhoff model. The fact that the gradient of a function in W_h is continuous at vertices of elements allows us to impose the isometry constraint at those points. We thus consider the minimization problem defined by

$$
\begin{aligned}
&I_h^{\mathrm{Ki}}(u_h) = \frac{1}{2}\int_\omega |\nabla\nabla_h u_h|^2 \,\mathrm{d}x - \int_\omega f \cdot u_h \,\mathrm{d}x \\
&\text{subject to } u_h \in \mathscr{A}_h = \big\{ v_h \in W_h^3, \quad [\nabla v_h(z)]^\top \nabla v_h(z) = I_2 \text{ for all } z \in \mathscr{N}_h, \\
&\qquad\qquad\qquad\qquad v_h(z) = u_{\mathrm{D}}(z),\ \nabla v_h(z) = \Phi_{\mathrm{D}}(z) \text{ for all } z \in \mathscr{N}_h \cap \gamma_{\mathrm{D}} \big\}.
\end{aligned}
$$

For the vector field $u_h \in W_h^3$, the approximate gradient $\nabla_h u_h$ is obtained by applying ∇_h to each component of u_h. We suppose that the boundary data u_{D} and Φ_{D} are compatible in the sense that for a function $\widetilde{u}_{\mathrm{D}} \in H^2(\omega; \mathbb{R}^3)$ with $(\nabla\widetilde{u}_{\mathrm{D}})^\top \nabla\widetilde{u}_{\mathrm{D}} = I_2$ in ω, we have $u_{\mathrm{D}} = \widetilde{u}_{\mathrm{D}}|_{\gamma_{\mathrm{D}}}$ and $\Phi_{\mathrm{D}} = \nabla\widetilde{u}_{\mathrm{D}}|_{\gamma_{\mathrm{D}}}$. We also assume that u_{D} and Φ_{D} can be approximated with arbitrary accuracy by nodal interpolation on γ_{D}, i.e.,

$$
\big\| u_{\mathrm{D}} - \mathscr{I}_h \widetilde{u}_{\mathrm{D}}|_{\gamma_{\mathrm{D}}} \big\|_{L^2(\gamma_{\mathrm{D}})} + \big\| \Phi_{\mathrm{D}} - \mathscr{I}_h \nabla\widetilde{u}_{\mathrm{D}}|_{\gamma_{\mathrm{D}}} \big\|_{L^2(\gamma_{\mathrm{D}})} \to 0
$$

as $h \to 0$. For analyzing convergence of the numerical scheme, we assume that there exists a solution of the nonlinear Kirchhoff model that is smooth or which can be approximated by smooth isometries. This assumption is not a restriction because of corresponding density results in [12].

Theorem 8.3 (Approximation) *Assume that there exists a minimizer $u \in \mathscr{A}$ with*

$$\mathscr{A} = \{v \in H^2(\omega;\mathbb{R}^3) : (\nabla v)^\top \nabla v = I_2,\ v|_{\gamma_\mathrm{D}} = u_\mathrm{D},\ \nabla v|_{\gamma_\mathrm{D}} = \Phi_\mathrm{D}\}$$

for the nonlinear Kirchhoff model which can be approximated in $H^2(\omega;\mathbb{R}^3)$ by functions $v \in \mathscr{A} \cap H^3(\omega;\mathbb{R}^3)$. For every $h > 0$ there exists a minimizer $u_h \in W_h^3$ of I_h^{Ki}. If $(u_h)_{h>0}$ is a sequence of minimizers, then $\|\nabla u_h\| \le C$, for all $h > 0$, and every accumulation point $u \in H^1(\omega;\mathbb{R}^3)$ of the sequence is a strong accumulation point, belongs to $H^2(\omega;\mathbb{R}^3)$, satisfies $(\nabla u)^\top \nabla u = I_2$ almost everywhere in ω, $u|_{\gamma_\mathrm{D}} = u_\mathrm{D}$, and $\nabla u|_{\gamma_\mathrm{D}} = \Phi_\mathrm{D}$, and is a minimizer for I^{Ki}.

Proof By Lemma 8.1 (iii) we have that $\|\nabla\nabla_h u_h\|$ is a norm and this implies that I_h^{Ki} has a minimizer. Because of the assumptions on the boundary data, it follows by Poincaré inequality and Lemma 8.1 (i) that $\|\nabla u_h\| \le C$ and $\|\nabla\nabla_h u_h\| \le C$ for all $h > 0$. Let $u \in H^1(\omega;\mathbb{R}^3)$ and $z \in H^1(\omega;\mathbb{R}^{3\times 2})$ be such that for a subsequence (which is not relabeled), we have $u_h \rightharpoonup u$ in $H^1(\omega;\mathbb{R}^3)$ and $\nabla_h u_h \rightharpoonup z$ in $H^1(\omega;\mathbb{R}^{3\times 2})$. With Lemma 8.1 we verify that $\|\nabla_h u_h - \nabla u_h\| \le ch\|\nabla\nabla_h u_h\|$ and this yields $\nabla u = z$, in particular $u \in H^2(\omega;\mathbb{R}^3)$. The attainment of the boundary data follows from continuity properties of the trace operators and the fact that

$$\|u_h - \mathscr{I}_h u_h\| + \|\nabla_h u_h - \mathscr{I}_h \nabla_h u_h\| \to 0$$

as $h \to 0$. A nodal interpolation estimate and an inverse estimate yield that for every $T \in \mathscr{T}_h$, we have

$$\begin{aligned}
\big\|(\nabla u_h)^\top \nabla u_h - I_2\big\|_{L^1(T)} &\le c h_T^2 \big\|D^2\big[(\nabla u_h)^\top \nabla u_h\big]\big\|_{L^1(T)} \\
&\le c h_T^2 \big(\|D^3 u_h\|_{L^2(T)}\|\nabla u_h\|_{L^2(T)} + \|D^2 u_h\|_{L^2(T)}^2\big) \\
&\le c h_T \big(\|D^2 u_h\|_{L^2(T)}\|\nabla u_h\|_{L^2(T)} + \|D^2 u_h\|_{L^2(T)}^2\big).
\end{aligned}$$

A summation over all $T \in \mathscr{T}_h$ together with the fact that ∇u_h converges strongly to ∇u implies that $(\nabla u)^\top \nabla u = I_2$ almost everywhere in ω. To verify that u minimizes I^{Ki}, we first note that by weak lower semicontinuity of the L^2 norm, we have

$$\|D^2 u\| = \|\nabla z\| \le \liminf_{h\to 0} \|\nabla\nabla_h u_h\|$$

and

$$\int_\omega u_h \cdot f \,\mathrm{d}x \to \int_\omega u \cdot f \,\mathrm{d}x.$$

This proves that

$$I^{\mathrm{Ki}}(u) \le \liminf_{h\to 0} I_h^{\mathrm{Ki}}(u_h).$$

To show that the minimal energy is attained let $\widetilde{u} \in \mathscr{A}$ be a minimizing isometry for I^{Ki}. Due to the assumed approximability of $\widetilde{u}$ by smooth isometries, we may assume that $\widetilde{u} \in H^3(\omega; \mathbb{R}^3)$. We define $\widetilde{u}_h = \widetilde{\mathscr{I}}_h^3 \widetilde{u} \in \mathscr{A}_h$ and note with Lemma 8.1(ii) that

$$\|\nabla_h \widetilde{u}_h - \nabla \widetilde{u}\| + h\|\nabla\nabla_h \widetilde{u}_h - D^2\widetilde{u}\| \le ch^2\|\widetilde{u}\|_{H^3(\omega)}$$

which implies the attainment of the minimal energy. □

8.3.2 Iterative Minimization

Our iterative scheme for the practical solution of the discretized minimization problem realizes a discrete H^2-gradient flow of the energy functional with a linearization of the nodal isometry constraint about the current iterate. For this, it is important to realize that for the employed finite element space W_h, the nodal values of the discrete deformation $\big(u_h(z) : z \in \mathscr{N}_h\big)$ and its gradient $\big(\nabla u_h(z) : z \in \mathscr{N}_h\big)$ are mutually independent variables in the minimization problem.

Algorithm 8.1 (Discrete H^2-isometry-flow) *Let $\tau > 0$ and $u_h^0 \in W_h^3$ be such that*

$$\big[\nabla u_h^0(z)\big]^\top \nabla u_h^0(z) = I_2$$

for all $z \in \mathscr{N}_h$ and $u_h^0(z) = u_{\mathrm{D}}(z)$ and $\nabla_h u_h^0(z) = \Phi_{\mathrm{D}}(z)$ for all $z \in \mathscr{N}_h \cap \gamma_{\mathrm{D}}$. For $k = 1, 2, \ldots$, define

$$\begin{aligned}&\mathscr{F}_h[u_h^{k-1}]\\ &\quad = \big\{w_h \in W_{h,\mathrm{D}}^3 : [\nabla w_h(z)]^\top \nabla u_h^{k-1}(z) + [\nabla u_h^{k-1}(z)]^\top \nabla w_h(z) = 0 \textit{ f.a. } z \in \mathscr{N}_h\big\}\end{aligned}$$

and compute $u_h^k = u_h^{k-1} + \tau d_t u_h^k$ with $d_t u_h^k \in \mathscr{F}_h[u_h^{k-1}]$ satisfying

$$\big(\nabla\nabla_h d_t u_h^k, \nabla\nabla_h w_h\big) + \alpha\big(\nabla\nabla_h(u_h^{k-1} + \tau d_t u_h^k), \nabla\nabla_h w_h\big) = \big(f, w_h\big)$$

for all $w_h \in \mathscr{F}_h[u_h^{k-1}]$. Stop the iteration if $\|\nabla\nabla_h d_t u_h^k\| \le \varepsilon_{\mathrm{stop}}$.

The iterates $(u_h^k)_{k=0,1,\ldots}$ will in general not satisfy the nodal isometry constraint exactly, but the violation is independent of the number of iterations and controlled by the step size τ.

Theorem 8.4 (Iteration) *The iterates $(u_h^k)_{k=0,1,\ldots}$ of Algorithm 8.1 are well defined and satisfy*

$$I_h^{\mathrm{Ki}}(u_h^k) + \frac{\tau}{2}\|\nabla\nabla_h d_t u_h^k\|^2 \le I_h^{\mathrm{Ki}}(u_h^{k-1}).$$

Moreover, we have

$$\|\mathscr{I}_h\big[(\nabla u_h^k)^\top \nabla u_h^k\big] - I_2\|_{L^1(\omega)} \le C\tau I_h^{\mathrm{Ki}}(u_h^0).$$

Proof The existence of a unique $d_t u_h^k \in F_h[u_h^{k-1}]$ in every step of the iteration follows from the fact that the bilinear form $(v_h, w_h) \mapsto \big(\nabla\nabla_h v_h, \nabla\nabla_h w_h\big)$ defines a coercive and continuous bilinear form on $\mathscr{F}_h[u_h^{k-1}]$, cf. Lemma 8.1(iv). Upon choosing $w_h = d_t u_h^k$, we find that

$$\|\nabla\nabla_h d_t u_h^k\|^2 + \frac{1}{2} d_t \|\nabla\nabla_h u_h^k\|^2 + \frac{\tau}{2}\|\nabla\nabla_h d_t u_h^k\|^2 = \big(f, d_t u_h^k\big)$$

and this proves the energy decreasing property. Using $u_h^k = u_h^{k-1} + \tau d_t u_h^k$, we have

$$\begin{aligned}\big(\nabla u_h^k\big)^\top \nabla u_h^k = \big(\nabla u_h^{k-1}\big)^\top \nabla u_h^{k-1} + \tau\big(\nabla d_t u_h^k\big)^\top \nabla u_h^{k-1} \\ + \tau\big(\nabla u_h^{k-1}\big)^\top \nabla d_t u_h^k + \tau^2\big(\nabla d_t u_h^k\big)^\top \nabla d_t u_h^k.\end{aligned}$$

Since $d_t u_h^k \in F_h[u_h^{k-1}]$, the sum of the second and third term on the right-hand side vanishes at every $z \in \mathscr{N}_h$ and an inductive argument, together with the assumptions on u_h^0, leads to

$$\big|\big[\nabla u_h^L(z)\big]^\top \nabla u_h^L(z) - I_2\big| \le \tau^2 \sum_{k=1}^{L} \big|\nabla d_t u_h^k(z)\big|^2.$$

A discrete norm equivalence and a local inverse inequality imply the assertion. □

8.3.3 Realization

The implementation of Algorithm 8.1 is based on the realization of the discrete Kirchhoff triangle for the linear problem. We also employ quadrature to discretize the forcing term which we assume to act only in the vertical direction. This implies that only the nodal values $\big(u_h(z) : z \in \mathscr{N}_h\big)$ and $\big(\nabla u_h(z) : z \in \mathscr{N}_h\big)$ are needed for the implementation, in particular, no evaluation of u_h in the interior of elements in $\mathscr{T}_h$ is required. If S_2 is the stiffness matrix related to piecewise quadratic vector fields with six components, D realizes the operator $\nabla_h : W_h^3 \to \Theta_h^3$, and B_{k-1} encodes the constraints and boundary conditions defined in the space $\mathscr{F}_h[u_h^{k-1}]$, then one step of the discrete gradient flow leads to the linear system of equations

$$\begin{bmatrix}(1+\alpha\tau)D^\top S_2 D & B_{k-1}^\top \\ B_{k-1} & 0\end{bmatrix}\begin{bmatrix}d_t U^k \\ \Lambda\end{bmatrix} = \begin{bmatrix}-\alpha D^\top S_2 D\, U^{k-1} + \tau F \\ 0\end{bmatrix}.$$

```
function kirchhoff_nonlinear(red)
[c4n,n4e,Db,Nb] = triang_strip(10);
alpha = 1; tau = 2^(-red)/10;
for j = 1:red
    [c4n,n4e,Db,Nb] = red_refine(c4n,n4e,Db,Nb);
end
nC = size(c4n,1);
dNodes = unique(Db); DNodes = [3*dNodes-2;3*dNodes-1;3*dNodes-0];
FNodes = setdiff(1:9*nC,[0*nC+DNodes;3*nC+DNodes;6*nC+DNodes]);
S_dkt = fe_matrix_dkt(c4n,n4e);
[¬,¬,m_lumped] = fe_matrices(c4n,n4e);
Z = sparse(3*nC,3*nC);
SSS = [S_dkt,Z,Z;Z,S_dkt,Z;Z,Z,S_dkt];
SSS_free = SSS(FNodes,FNodes);
u = u_moebius(c4n);
dt_u = zeros(9*nC,1);
bbb = zeros(9*nC,1);
bbb(6*nC+(1:3:3*nC)) = m_lumped*f3(c4n);
corr = 1; eps_stop = 1e-2;
while corr > eps_stop
    B = sparse(3*nC,9*nC);
    for j = 1:nC
        for k = 1:3
            idx_j = 3*(j-1); idx_jk = (k-1)*3*nC+3*(j-1);
            B(idx_j+1,idx_jk+2) = u(idx_jk+2);
            B(idx_j+2,idx_jk+3) = u(idx_jk+3);
            B(idx_j+3,idx_jk+2) = u(idx_jk+3);
            B(idx_j+3,idx_jk+3) = u(idx_jk+2);
        end
    end
    B(DNodes,:) = [];
    ZZZ = sparse(size(B,1),size(B,1));
    AAA = [(1+tau*alpha)*SSS_free,B(:,FNodes)';B(:,FNodes),ZZZ];
    rhs = -alpha*SSS*u+bbb;
    ddd = [rhs(FNodes);zeros(size(B,1),1)];
    xxx = AAA\ddd;
    dt_u(FNodes) = xxx(1:size(SSS_free,1));
    corr = sqrt(dt_u'*SSS*dt_u)
    u = u+tau*dt_u; show_p1_para(c4n,n4e,u);
end

function val = f3(x)
val = 0*ones(size(x,1),1);

function u = u_moebius(x)
L = max(x(:,1)); nX = size(x,1); u = zeros(9*nX,1);
u(0*nX+(1:3:3*nX)) = sin(2*pi*x(:,1)/L);
u(3*nX+(1:3:3*nX)) = x(:,2)+(1-2*x(:,2)).*sin(pi*x(:,1)/(2*L));
u(6*nX+(1:3:3*nX)) = sin(pi*x(:,1)/L);
u(0*nX+(2:3:3*nX)) = ones(nX,1);
u(3*nX+(3:3:3*nX)) = ones(nX,1)-2*(x(:,1)>L/2).*ones(nX,1);
```

Fig. 8.6 Approximation of the nonlinear Kirchhoff model with discrete Kirchhoff triangles

The matrix $D^\top S_2 D$ is generated as in the case of the linear model and provided by the routine `dkt_matrix.m`. The initial deformation is assumed to satisfy the boundary conditions which may be inhomogeneous. We refer to the implementation displayed in Fig. 8.6 for details.

8.4 Willmore Flow

We discuss in this section numerical methods for approximating the Willmore flow. This is the L^2-gradient flow of the Willmore energy which is defined on closed surfaces in $\mathbb{R}^3$. To compute the evolution equation, we review concepts from differential geometry to differentiate quantities on surfaces and to measure variations of surfaces. The reader is referred to the textbooks [13, 14] for further details. The numerical schemes are based on results in [2, 8, 9].

8.4.1 *Tangential Differentiation and Curvature*

Let $\mathscr{M} \subset \mathbb{R}^3$ be a surface, i.e., an orientable two-dimensional C^2-submanifold $\mathscr{M}$ in $\mathbb{R}^3$, with continuous unit normal $n : \mathscr{M} \to \mathbb{R}^3$. For scalar functions $f : \mathscr{M} \to \mathbb{R}$ and vector fields $F : \mathscr{M} \to \mathbb{R}^3$ on $\mathscr{M}$ that admit continuously differentiable extensions $\widetilde{f} : \mathscr{U}(\mathscr{M}) \to \mathbb{R}$ and $\widetilde{F} : \mathscr{U}(\mathscr{M}) \to \mathbb{R}^3$ to an open neighborhood of $\mathscr{M}$, we define the *tangential gradient* and the *tangential divergence* by

$$\nabla_{\mathscr{M}} f = \nabla \widetilde{f} - (n \cdot \nabla \widetilde{f})n, \quad \operatorname{div}_{\mathscr{M}} F = \operatorname{div} \widetilde{F} - n^\top D\widetilde{F} n.$$

The operators satisfy the product rule

$$\operatorname{div}_{\mathscr{M}}(fF) = \nabla_{\mathscr{M}} f \cdot F + f \operatorname{div}_{\mathscr{M}} F.$$

The tangential gradient $\nabla_{\mathscr{M}} F$ of a vector field F is the matrix whose i-th row coincides with the transpose of the tangential gradient of the i-th component of F. The *Laplace–Beltrami operator* is defined as

$$\Delta_{\mathscr{M}} f = \operatorname{div}_{\mathscr{M}} \nabla_{\mathscr{M}} f.$$

For a local parametrization $u : \omega \to \mathbb{R}^3$ of $\mathscr{M}$, the tangent vectors $\partial_\ell u$, $\ell = 1, 2$, are linearly independent and define a unit normal $b = \pm \partial_1 u \times \partial_2 u / |\partial_1 u \times \partial_2 u|$, cf. Fig. 8.7. We assume in the following that the sign is chosen so that $b = n \circ u$. The *first fundamental form* is the matrix g with entries

$$g_{ij} = \partial_i u \cdot \partial_j u.$$

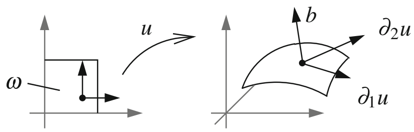

Fig. 8.7 Local parametrization of a surface by a mapping $u : \omega \to \mathbb{R}^3$; the partial derivatives $\partial_1 u$ and $\partial_2 u$ of u define a basis of the tangent space for every point on the image of u; their normalized cross product defines a unit normal b to the surface

It defines a metric on the tangent space of $\mathscr{M}$, e.g., the length of a tangent vector $\alpha_1\partial_1 u + \alpha_2\partial_2 u$ is given by the square root of $\alpha \cdot (g\alpha)$. The matrix g is symmetric and positive definite everywhere in ω; and we let $g^{-1} = (g^{ij})$ be its inverse and $g^{-1/2} = (g_{ij}^{(-1/2)})$ the symmetric and positive definite square root of g^{-1}.

Proposition 8.4 (Differential operators on $\mathscr{M}$) *We have*

$$(\nabla_{\mathscr{M}} f) \circ u = \sum_{i,j=1}^{2} g^{ij}\partial_j(f \circ u)\partial_i u, \quad (\operatorname{div}_{\mathscr{M}} F) \circ u = \sum_{i,j=1}^{2} g^{ij}\partial_j(F \circ u) \cdot \partial_i u.$$

If $F = \sum_{i=1}^2 F_i\partial_i u$ is tangential or $F = \nabla_{\mathscr{M}} f$, then

$$(\operatorname{div}_{\mathscr{M}} F) \circ u = (\det g)^{-1/2} \sum_{i=1}^{2} \partial_i\big(F_i \circ u(\det g)^{1/2}\big),$$
$$(\Delta_{\mathscr{M}} f) \circ u = (\det g)^{-1/2} \sum_{i,j=1}^{2} \partial_i\big((\det g)^{1/2} g^{ij}\partial_j(f \circ u)\big).$$

In particular, the operators are independent of the extensions.

Proof We occasionally omit the composition with u, e.g., we write $\nabla_{\mathscr{M}} f$ for $(\nabla_{\mathscr{M}} f) \circ u$. For $k = 1, 2$ we have

$$(\nabla_{\mathscr{M}} f) \cdot \partial_k u = \nabla\widetilde{f} \cdot \partial_k u = \partial_k(\widetilde{f} \circ u) = \partial_k(f \circ u)$$

and $(\nabla_{\mathscr{M}} f) \cdot n = 0$. Since

$$\Big(\sum_{i,j=1}^{2} g^{ij}\partial_j(f \circ u)\partial_i u\Big) \cdot \partial_k u = \sum_{i,j=1}^{2} g^{ij} g_{ik}\partial_j(f \circ u) = \sum_{j=1}^{2} \delta_{jk}\partial_j(f \circ u) = \partial_k(f \circ u)$$

and since the sum on the right-hand side of the first asserted identity is orthogonal to n, we deduce the formula for $\nabla_{\mathscr{M}} f$. With $V_i = \sum_{j=1}^2 g_{ij}^{(-1/2)}\partial_j u$ for $i = 1, 2$, the vectors (V_1, V_2, b) define an orthonormal basis in $\mathbb{R}^3$, i.e.,

$$V_i \cdot V_k = \sum_{j,\ell=1}^{2} g_{ij}^{(-1/2)} g_{k\ell}^{(-1/2)} \partial_j u \cdot \partial_\ell u = \sum_{j,\ell=1}^{2} g_{ij}^{(-1/2)} g_{k\ell}^{(-1/2)} g_{j\ell} = \delta_{ik}$$

and $V_i \cdot b = 0$ for $i = 1, 2$. With this we have

$$\operatorname{div} \widetilde{F} = \operatorname{tr} D\widetilde{F} = \sum_{i=1}^{2} V_i^\top D\widetilde{F} V_i + b^\top D\widetilde{F} b,$$

and hence by definition of $\operatorname{div}_{\mathscr{M}}$

$$\operatorname{div}_{\mathscr{M}} F = \sum_{i,j,k=1}^{2} g_{ij}^{(-1/2)} g_{ik}^{(-1/2)} (\partial_j u)^\top D\widetilde{F} \partial_k u = \sum_{j,k=1}^{2} g^{jk} \partial_j (F \circ u) \cdot \partial_k u$$

which is the second identity. Assume now that F is tangential so that $F \circ u = \sum_{i=1}^{2} F_i \partial_i u$ with uniquely defined functions $F_i : \omega \to \mathbb{R}$. It then follows that

$$\begin{aligned}
\operatorname{div}_{\mathscr{M}} F &= \sum_{i,j,k=1}^{2} g^{ij} (\partial_j F_k \partial_k u + F_k \partial_j \partial_k u) \cdot \partial_i u \\
&= \sum_{i,j,k=1}^{2} g^{ij} (\partial_j F_k g_{ik} + F_k \partial_j \partial_k u \cdot \partial_i u) \\
&= \sum_{k=1}^{2} \Big(\partial_k F_k + \sum_{i,j=1}^{2} g^{ij} F_k (\partial_k \partial_j u \cdot \partial_i u) \Big).
\end{aligned}$$

Since g^{-1} is symmetric, $g^{-1} = (\det g)^{-1} \det{}' g$, and $2\partial_k (\det g)^{1/2} = (\det g)^{-1/2} \det{}' g : \partial_k g$, we have for $k = 1, 2$ that

$$\sum_{i,j=1}^{2} g^{ij} (\partial_k \partial_j u \cdot \partial_i u) = \frac{1}{2} \sum_{i,j=1}^{2} g^{ij} \partial_k g_{ij} = (\det g)^{-1/2} \partial_k (\det g)^{1/2}.$$

The combination of the last two equations shows that

$$\operatorname{div}_{\mathscr{M}} F = \sum_{k=1}^{2} \big(\partial_k F_k + F_k (\det g)^{-1/2} \partial_k (\det g)^{1/2} \big),$$

which is the asserted identity. The identity for the Laplace–Beltrami operator now follows from the characterization of $\nabla_{\mathscr{M}}$. □

Example 8.1 For the parametrization $u(\theta, \phi) = r(\sin\theta \sin\phi, \sin\theta\cos\phi, \cos\theta)$ of the sphere $S_r \subset \mathbb{R}^3$ with radius $r > 0$, we have $\det g(\theta, \phi) = r^4 \sin^2\theta$ and $\Delta_{S_r} f = (r^2 \sin\theta)^{-1}\big[\partial_\theta(\sin\theta\,\partial_\theta f) + (\sin\theta)^{-1}\partial_\phi^2 f\big]$.

Remark 8.5 The representation $F = \sum_{i=1}^2 (V_i, F)V_i = \sum_{i,j=1}^2 g^{ij}(F \cdot \partial_i u)\partial_j u$ of a tangential vector field F with the orthonormal vectors (V_1, V_2) constructed in the proof of Proposition 8.4 yields the *Weingarten equation* $\partial_k b = -\sum_{i,j=1}^2 g^{ij} h_{ki} \partial_j u$ with the coefficients h_{ki} of the second fundamental form defined below.

To define a measure of curvature, we let $c : (-\varepsilon, \varepsilon) \to \mathscr{M}$ be a C^2 curve in $\mathscr{M}$ with $|c'(t)| = 1$ for all $t \in (-\varepsilon, \varepsilon)$ and consider the quantity $\kappa = c'' \cdot (n \circ c)$. Since $c' \cdot (n \circ c) = 0$ we have

$$\kappa = -c' \cdot (n \circ c)' = -c' \cdot \big(\nabla_{\mathscr{M}} n\, c'\big).$$

We call $\nabla_{\mathscr{M}} n$ the *shape operator* which is closely related to the *second fundamental form* defined through the symmetric matrix

$$h_{ij} = -\partial_i b \cdot \partial_j u = b \cdot \partial_i \partial_j u.$$

The mapping induced by $\nabla_{\mathscr{M}} n$ is also called the *Weingarten map*.

Proposition 8.5 (Shape operator) *The matrix $\nabla_{\mathscr{M}} n$ is symmetric and defines a self-adjoint linear operator on the tangent space of $\mathscr{M}$ into itself and is in the basis $(\partial_1 u, \partial_2 u)$ given by the generally nonsymmetric matrix $s = -hg^{-1}$.*

Proof For $i = 1, 2, 3$ we have $(\nabla_{\mathscr{M}} n_i) \cdot n = 0$ and hence $(\nabla_{\mathscr{M}} n)n = 0$. The identity $|n|^2 = 1$ implies that $n^\top(\nabla_{\mathscr{M}} n) = 0$. Therefore, $\nabla_{\mathscr{M}} n$ defines an endomorphism on the tangent space of $\mathscr{M}$; and for $i = 1, 2$ there exist s_{ij}, $j = 1, 2$, such that $(\nabla_{\mathscr{M}} n)\partial_i u = \sum_{j=1}^2 s_{ij}\partial_j u$, i.e.,

$$\sum_{j=1}^{2} s_{ij}\partial_j u \cdot \partial_k u = (\nabla_{\mathscr{M}} n \partial_i u) \cdot \partial_k u = \partial_i(n \circ u) \cdot \partial_k u = \partial_i b \cdot \partial_k u = -h_{ik}$$

and hence with $\partial_j u \cdot \partial_k u = g_{jk}$ we deduce $sg = -h$. The identity also implies the symmetry of $\nabla_{\mathscr{M}} n$. □

The *principal curvatures* of $\mathscr{M}$ are the eigenvalues of the self-adjoint symmetric operator $\nabla_{\mathscr{M}} n$ restricted to the tangent space of $\mathscr{M}$ and are denoted by κ_1 and κ_2. The eigenvectors corresponding to κ_1 and κ_2 are called *directions of principal curvature*. The possibly nonsymmetric matrix s has the eigenvalues κ_1 and κ_2 and the *mean* and *Gauss curvature* are defined as

$$H = \operatorname{tr} s = \kappa_1 + \kappa_2, \quad K = \det s = \kappa_1\kappa_2,$$

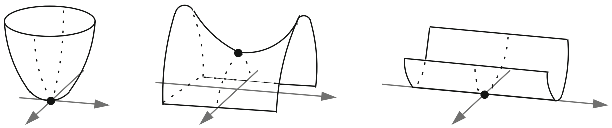

Fig. 8.8 Ellipsoidal surface with $\kappa_1 < 0$, $\kappa_2 < 0$ (*left*), hyperbolic surface with $\kappa_1 < 0$, $\kappa_2 > 0$ (middle), and parabolic surface with $\kappa_1 = 0$, $\kappa_2 > 0$ (*right*) relative to the unit normal $n = e_3$

respectively. We have that $|\nabla_{\mathscr{M}} n|^2 = s^\top : s = \operatorname{tr}(s^2) = \kappa_1^2 + \kappa_2^2 = (\operatorname{tr} s)^2 - 2\det s = H^2 - 2K$. We also note the identities $H = -h : g^{-1} = \operatorname{tr}(-hg^{-1})$.

Remark 8.6 The sign of H depends on the choice of the unit normal, whereas K is independent of the sign of $\pm n$. The definition implies $\kappa_1, \kappa_2 \geq 0$ if $\mathscr{M}$ is locally convex with respect to the chosen unit normal. The mean curvature H is often defined as $(1/2)\operatorname{tr} s = (\kappa_1 + \kappa_2)/2$.

Typical local shapes of two-dimensional surfaces are given in the following example and are shown in Fig. 8.8.

Example 8.2 Consider a local parametrization of a surface that is given by the graph of the function $f : \omega \to \mathbb{R}$, i.e., $u(x) = \big(x, f(x)\big)$. Also assume that $0 \in \omega$ with $\nabla f(0) = 0$. Noting $\partial_i u = e_i$ for $i = 1, 2$, and $b = e_3$, $g = I$, and $h = b \cdot \partial_i \partial_j u = D^2 f$, we find that $s = -hg^{-1} = -D^2 f$ at $x = 0$.

Proposition 8.6 (Mean curvature) *We have*

$$\operatorname{div}_{\mathscr{M}} n = H, \quad -\Delta_{\mathscr{M}} \mathrm{id}_{\mathscr{M}} = Hn,$$

where $\mathrm{id}_{\mathscr{M}} : \mathscr{M} \to \mathbb{R}^3$ *denotes the identity on* $\mathscr{M}$*, i.e.,* $\mathrm{id}_{\mathscr{M}}(p) = p$ *for all* $p \in \mathscr{M}$ *and* $\Delta_{\mathscr{M}}$ *is applied to every component of* $\mathrm{id}_{\mathscr{M}}$.

Proof With the characterization of $\operatorname{div}_{\mathscr{M}}$ of Proposition 8.4, we have

$$\operatorname{div}_{\mathscr{M}} n = \sum_{i,j=1}^{m} g^{ij} \partial_j (n \circ u) \cdot \partial_i u = -\sum_{i,j=1}^{2} g^{ij} h_{ij} = -\operatorname{tr}(hg^{-1}) = \operatorname{tr} s.$$

We have $\nabla_{\mathscr{M}} \mathrm{id}_{\mathscr{M}} = I - nn^\top$ and thus $-\Delta_{\mathscr{M}} \mathrm{id}^i_{\mathscr{M}} = \operatorname{div}_{\mathscr{M}}(n^i n) = n^i H$. □

We have the following generalized integration-by-parts formula.

Proposition 8.7 (Integration-by-parts) *For a vector field* $F : \mathscr{M} \to \mathbb{R}^3$ *and a compactly supported function* $\varphi : \mathscr{M} \to \mathbb{R}$*, we have*

$$\int_{\mathscr{M}} \nabla_{\mathscr{M}} \varphi \cdot F \, \mathrm{d}s = -\int_{\mathscr{M}} \varphi \operatorname{div}_{\mathscr{M}} F \, \mathrm{d}s + \int_{\mathscr{M}} H(F \cdot n)\varphi \, \mathrm{d}s.$$

Proof We assume that φ belongs to a coordinate chart parametrized by u and consider the vector field $G = \varphi F$ on $\mathscr{M}$. We set $G = G_{\tan} + G_{\mathrm{nor}}$ with $G_{\mathrm{nor}} = \gamma n$ for $\gamma = G \cdot n$. Then $G_{\tan} = \sum_{i=1}^{2} G_i \partial_i u$ and Proposition 8.4 and an integration-by-parts in $\mathbb{R}^2$ yield

$$\int_{\mathscr{M}} \operatorname{div}_{\mathscr{M}} G_{\tan} \, \mathrm{d}s = \sum_{i=1}^{2} \int_{\omega} \partial_i (G_i (\det g)^{1/2}) \, \mathrm{d}x$$
$$= \int_{\omega} \operatorname{div} \left((\det g)^{1/2} [G_1, G_2] \right) \mathrm{d}x = 0.$$

The product rule and $(\nabla_{\mathscr{M}} \gamma) \cdot n = 0$ show that

$$\int_{\mathscr{M}} \operatorname{div}_{\mathscr{M}} G_{\mathrm{nor}} \, \mathrm{d}s = \int_{\mathscr{M}} \gamma \, \operatorname{div}_{\mathscr{M}} n \, \mathrm{d}s = \int_{\mathscr{M}} \gamma H \, \mathrm{d}s = \int_{\mathscr{M}} (G \cdot n) H \, \mathrm{d}s.$$

The combination of the identities and an application of the product rule prove the asserted formula. □

Remark 8.7 If φ does not vanish on the boundary of $\mathscr{M}$, then the boundary term $\int_{\partial \mathscr{M}} \varphi F \cdot \mu \, \mathrm{d}t$ with the conormal $\mu = \tau \times n$, where τ is the tangent on $\partial \omega$, has to be included on the right-hand side.

8.4.2 Normal Variations

For a surface $\mathscr{M} \subset \mathbb{R}^3$ with unit normal n and a function $\phi : \mathscr{M} \to \mathbb{R}$, we consider for $-\varepsilon < t < \varepsilon$ the normal variations of $\mathscr{M}$ defined by

$$\mathscr{M}_t = \{q \in \mathbb{R}^3 : q = p + t\varphi(p) n(p), \ p \in \mathscr{M}\},$$

cf. Fig. 8.9. Then $\mathscr{M}_0 = \mathscr{M}$ and for sufficiently small $\varepsilon > 0$, the sets $\mathscr{M}_t$ are surfaces in $\mathbb{R}^3$. If $u : \omega \to \mathbb{R}^3$ is a local parametrization of $\mathscr{M}$, then

$$u_t = u + t(\phi \circ u)(n \circ u)$$

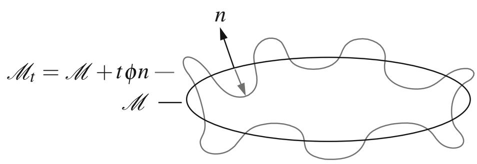

Fig. 8.9 Normal variation of a surface defined by a scalar function ϕ

is a local parametrization of $\mathscr{M}_t$. For a function $f_t : \mathscr{M}_t \to \mathbb{R}$ we denote $f = f_0$ and define

$$\delta f(p) = \lim_{t\to 0} t^{-1}\big(f_t(p) - f_0(p)\big)$$

for $p \in \mathscr{M}$. The proposition below studies the changes of geometric quantities on the surfaces $\mathscr{M}_t$ and employs Gauss' equation and an equivalent characterization of the Laplace–Beltrami operator stated in the following lemma.

Lemma 8.2 (Christoffel symbols) *With the* Christoffel symbols *of the first kind* $\Gamma_{ij,m} = \partial_i\partial_j u \cdot \partial_m u$ *and of the second kind* $\Gamma_{ij}^k = \sum_{m=1}^2 g^{km}\Gamma_{ij,m}$, *we have* Gauss' equation *and a representation of the Laplace–Beltrami operator, i.e.,*

$$\partial_i\partial_j u = \sum_{k=1}^2 \Gamma_{ij}^k \partial_k u + h_{ij} b, \quad \Delta_{\mathscr{M}}\phi = \sum_{i,j}^2 g^{ij}\Big(\partial_i\partial_j\phi - \sum_{k=1}^2 \Gamma_{ij}^k \partial_k\phi\Big).$$

Proof We have $\partial_i\partial_j u \cdot n = h_{ij}$ and hence there exist α_{ij}^k with

$$\partial_i\partial_j u \cdot \partial_\ell u = \sum_{k=1}^2 \alpha_{ij}^k \partial_k u \cdot \partial_\ell u = \sum_{k=1}^2 \alpha_{ij}^k g_{k\ell},$$

i.e., $\alpha_{ij}^m = \sum_{\ell=1}^2 g^{\ell m}(\partial_i\partial_j u)\cdot\partial_\ell u$. This implies the representation of $\partial_i\partial_j u$. According to Proposition 8.4 we have

$$\begin{aligned}
\Delta_{\mathscr{M}}\phi &= \sum_{i,j,\ell,m=1}^2 g^{ij}\partial_j\big(g^{\ell m}\partial_m\phi\partial_\ell u\big)\cdot\partial_i u \\
&= \sum_{i,j,\ell,m=1}^2 g^{ij}\big[\partial_j g^{\ell m}\partial_m\phi\partial_\ell u + g^{\ell m}(\partial_j\partial_m\phi)\partial_\ell u + g^{\ell m}\partial_m\phi(\partial_j\partial_\ell u)\big]\cdot\partial_i u \\
&= \sum_{i,j,\ell,m=1}^2 g^{ij}\big[\partial_j g^{\ell m}\partial_m\phi g_{\ell i} + g^{\ell m}(\partial_j\partial_m\phi)g_{\ell i} + g^{\ell m}\partial_m\phi\Gamma_{j\ell,i}\big].
\end{aligned}$$

Using $0 = \partial_j\sum_{r=1}^2(g^{\ell r}g_{rm}) = \sum_{r=1}^2(\partial_j g^{\ell r}g_{rm} + g^{\ell r}\partial_j g_{rm})$, we find that $\partial_j g^{\ell m} = -\sum_{r,k=1}^2 g^{\ell r}\partial_j g_{rk}g^{km}$ and noting $\partial_j g_{rk} = \Gamma_{jr,k} + \Gamma_{jk,r}$, i.e.,

$$\partial_j g^{\ell m} = -\sum_{r,k=1}^2 g^{\ell r}(\Gamma_{jr,k} + \Gamma_{jk,r})g^{km},$$

shows that $\Delta_{\mathscr{M}}\phi$ equals

$$\sum_{i,j,\ell,m=1}^{2} g^{ij}\big[-\sum_{r,k=1}^{2} g^{\ell r}(\Gamma_{jr,k}+\Gamma_{jk,r})g^{km}\partial_m\phi g_{\ell i}+g^{\ell m}(\partial_j\partial_m\phi)g_{\ell i}+g^{\ell m}\partial_m\phi\Gamma_{j\ell,i}\big]$$
$$=\sum_{i,j=1}^{2} g^{ij}\big[-\sum_{k,m=1}^{2}(\Gamma_{ji,k}+\Gamma_{jk,i})g^{km}\partial_m\phi+\partial_j\partial_i\phi+\sum_{\ell,m=1}^{2} g^{\ell m}\partial_m\phi\Gamma_{j\ell,i}\big]$$
$$=\sum_{i,j=1}^{2} g^{ij}\big[\partial_j\partial_i\phi-\sum_{k,m=1}^{2} g^{km}\Gamma_{ij,k}\partial_m\phi\big].$$

This implies the asserted formula for $\Delta_{\mathscr{M}}\phi$. □

A consequence of this is Gauss' *theorema egregium* which is stated below for isometric parametrizations, cf. Proposition 8.2.

Lemma 8.3 (Gauss curvature for isometries) *Assume that* $\Gamma_{ij,k}=\partial_i\partial_j u\cdot\partial_k u=0$ *for all* $1\le i,j,k\le 2$. *Then* $K=0$.

Proof Using $\partial_2(\partial_1^2 u)=\partial_1(\partial_1\partial_2 u)$ and the identities $\partial_i\partial_j u=h_{ij}b$, Lemma 8.2 shows that

$$0=\partial_2(h_{11}b)-\partial_1(h_{12}b)=(\partial_2 h_{11}-\partial_1 h_{12})b+h_{11}\partial_2 n-h_{12}\partial_1 n.$$

The Weingarten equations $\partial_k b=-\sum_{i,j=1}^{2} g^{ij}h_{ki}\partial_j u$, cf. Remark 8.5, imply that for the tangential part of the identity, we have

$$0=-h_{11}\sum_{i,j=1}^{2} g^{ij}h_{2i}\partial_j u+h_{12}\sum_{i,j=1}^{2} g^{ij}h_{1i}\partial_j u=-\sum_{i,j=1}^{2} g^{ij}(h_{11}h_{2i}-h_{12}h_{1i})\partial_j u.$$

The contributions to the sum vanish for $i=1$ and hence

$$0=-(\det h)\sum_{j=1}^{2} g^{2j}\partial_j u.$$

Since $\partial_1 u$ and $\partial_2 u$ are linearly independent, this implies $\det h=0$ and $K=0$. □

Proposition 8.8 (Normal variations of geometric quantities) *For* $1\le i,j\le 2$ *we have*

$$\delta g_{ij}=-2\phi h_{ij},\quad \delta g_{ij}^{-1}=2\phi\sum_{k,\ell=1}^{2} g^{ik}h_{k\ell}g^{\ell j},\quad \delta(\det g)^{1/2}=\phi H(\det g)^{1/2}$$

and

$$\delta n=-\nabla_{\mathscr{M}}\phi,\quad \delta H=-\Delta_{\mathscr{M}}\phi-|s|^2.$$

Proof We identify ϕ with the function $\phi \circ u$ and write $b = n \circ u$. We also omit the dependence on t in the following. Noting $\partial_i b \cdot b = 0$, we have

$$g_{ij}^t = \partial_i u_t \cdot \partial_j u_t = g_{ij} + t\phi\big(\partial_i u \cdot \partial_j b + \partial_j u \cdot \partial_i b\big) + t^2 \partial_i \phi \partial_j \phi + t^2 \phi^2 \partial_i b \cdot \partial_j b,$$

which implies $\delta g_{ij} = -2\phi h_{ij}$. With $g^{-1} g = I_2$ we find that $\delta g^{-1} = -g^{-1}(\delta g) g^{-1}$ and hence

$$\delta g^{ij} = -\sum_{k,\ell=1}^{2} g^{ik}(\delta g_{k\ell}) g^{\ell j} = 2\phi \sum_{k,\ell=1}^{2} g^{ik} h_{k\ell} g^{\ell j}.$$

The relations $(\det g)^{-1} \det{}' g = g^{-1}$ and $g^{-1} : h = -H$ imply

$$\begin{aligned}\delta(\det g)^{1/2} &= \frac{1}{2}(\det g)^{-1/2}(\det{}' g) : \delta g = \frac{1}{2}(\det g)^{1/2} g^{-1} : \delta g \\ &= -\phi(\det g)^{1/2} g^{-1} : h = \phi(\det g)^{1/2} H.\end{aligned}$$

Using $b \cdot \partial_i u = 0$, we deduce $\delta b \cdot \partial_i u + b \cdot \delta \partial_i u = 0$ and with $\delta \partial_i u = \phi \partial_i b + (\partial_i \phi) b$ and $b \cdot \partial_i b = 0$, it follows that $\delta b \cdot \partial_i u = -\partial_i \phi$. Since $0 = \delta |b|^2 = 2\delta b \cdot b$, we have that there exist α_1, α_2 with $\delta b = \alpha_1 \partial_1 u + \alpha_2 \partial_2 u$. Noting

$$\sum_{i=1}^{2} \alpha_i \partial_i u \cdot \partial_k u = \delta b \cdot \partial_k u = -\partial_k \phi$$

we find that $\alpha_i = -\sum_{j=1}^{2} g^{ij} \partial_j \phi$ which implies

$$\delta b = -\sum_{i,j=1}^{2} g^{ij} \partial_j \phi \partial_i u,$$

and this expression coincides with $-\nabla_{\mathscr{M}} \phi$. It remains to compute δH. For this we first compute δh_{ij}. Noting

$$\delta \partial_i \partial_j u = (\partial_i \partial_j \phi) b + \partial_i \phi \partial_j b + \partial_j \phi \partial_i b + \phi \partial_i \partial_j b,$$

and using $b \cdot \partial_i \partial_j b = -\partial_i b \cdot \partial_j b$, we have

$$b \cdot (\delta \partial_i \partial_j u) = \partial_i \partial_j \phi - \phi \partial_i b \cdot \partial_j b.$$

The Weingarten equation $\partial_k b = \sum_{i,j=1}^{2} g^{ij} h_{ki} \partial_j u$ leads to

$$\partial_i b \cdot \partial_j b = \sum_{\ell,m,r,s=1}^{2} g^{\ell m} h_{i\ell} g^{rs} h_{jr} \partial_m u \cdot \partial_s u = \sum_{r,s=1}^{2} g^{rs} h_{is} h_{rj}.$$

The formula for δb and Gauss' equation show that

$$\delta b \cdot (\partial_i \partial_j u) = -\Big(\sum_{k,\ell=1}^{2} g^{k\ell} \partial_\ell \phi \partial_k u \Big) \cdot \Big(\sum_{m=1}^{2} \Gamma_{ij}^{m} \partial_m u \Big) = -\sum_{\ell=1}^{2} \Gamma_{ij}^{\ell} \partial_\ell \phi.$$

We thus have

$$\delta h_{ij} = (\delta b) \cdot \partial_i \partial_j u + b \cdot (\delta \partial_i \partial_j u) = -\sum_{\ell=1}^{2} \Gamma_{ij}^{\ell} \partial_\ell \phi + \partial_i \partial_j \phi - \phi \sum_{k,\ell=1}^{2} g^{k\ell} h_{i\ell} h_{kj}$$

and

$$\begin{aligned} \sum_{i,j=1}^{2} g^{ij} \delta h_{ij} &= \sum_{i,j=1}^{2} g^{ij} \Big(\partial_i \partial_j \phi - \sum_{\ell=1}^{2} \Gamma_{ij}^{\ell} \partial_\ell \phi \Big) - \phi \sum_{i,j,k,\ell=1}^{2} g^{ij} g^{k\ell} h_{i\ell} h_{kj} \\ &= \Delta_{\mathscr{M}} \phi - \phi |s|^2. \end{aligned}$$

For the mean curvature we find that

$$\begin{aligned} \delta H &= -\delta \sum_{i,j=1}^{2} g^{ij} h_{ij} \\ &= -\sum_{i,j=1}^{2} \big((\delta g^{ij}) h_{ij} + g^{ij} (\delta h_{ij}) \big) \\ &= -2\phi \sum_{i,j,k,\ell=1}^{2} g^{ik} h_{k\ell} g^{\ell j} h_{ij} - \Delta_{\mathscr{M}} \phi + \phi |s|^2 \\ &= -2\phi |s|^2 - \Delta_{\mathscr{M}} \phi + \phi |s|^2. \end{aligned}$$

This proves the proposition. □

We finally derive variations for functionals measuring the surface area and the enclosed volume by a surface. The variation of a functional $\mathscr{G}$ defined on C^2-surfaces is the limit

$$\delta \mathscr{G}(\mathscr{M})[\phi] = \lim_{t \to 0} t^{-1} \big(\mathscr{G}(\mathscr{M}_t) - \mathscr{G}(\mathscr{M}_0) \big)$$

for a surface $\mathscr{M}$ that is perturbed in the normal direction with a function ϕ as above.

Proposition 8.9 (Variations of area and volume functional) *For $\mathscr{M} = \partial \Omega$ define*

$$\mathscr{A}(\mathscr{M}) = \int_{\mathscr{M}} 1 \, \mathrm{d}s, \quad \mathscr{V}(\mathscr{M}) = \int_{\Omega} 1 \, \mathrm{d}\xi = \frac{1}{3} \int_{\mathscr{M}} s \cdot n \, \mathrm{d}s.$$

We have

$$\delta\mathscr{A}(\mathscr{M})[\phi] = \int_{\mathscr{M}} H\phi \,\mathrm{d}s, \quad \delta\mathscr{V}(\mathscr{M})[\phi] = \frac{1}{3}\int_{\mathscr{M}} (1+H)\phi \,\mathrm{d}s.$$

Proof The first identity is a direct consequence of Proposition 8.8. The second identity follows from $\mathrm{id}_{\mathscr{M}_t} \cdot n = t\phi$. $\square$

8.4.3 Variation of the Willmore Functional

The normal variations of geometric quantities allow us to characterize stationary surfaces for the Willmore functional and to define related evolution problems. For a closed surface $\mathscr{M} \subset \mathbb{R}^3$, the bending energy is given by the *Willmore functional*

$$W(\mathscr{M}) = \frac{1}{2}\int_{\mathscr{M}} H^2 \,\mathrm{d}s.$$

The following theorem characterizes critical points of the functional.

Theorem 8.5 (Euler–Lagrange equations) *For a normal variation of $\mathscr{M}$ defined by a function $\phi : \mathscr{M} \to \mathbb{R}$, we have*

$$\delta W(\mathscr{M})[\phi] = \int_{\mathscr{M}} (-\Delta_{\mathscr{M}} H)\phi - |\nabla_{\mathscr{M}} n|^2 H\phi + \frac{1}{2}H^3\phi \,\mathrm{d}s,$$

where $|\nabla_{\mathscr{M}} n|^2 = H^2 - 2K$.

Proof We assume that ϕ is supported in a coordinate chart. We then have

$$\begin{aligned}
\delta\frac{1}{2}\int_{\mathscr{M}} H^2 \,\mathrm{d}s &= \frac{1}{2}\delta\int_{\omega} H^2(\det g)^{1/2} \,\mathrm{d}x \\
&= \int_{\omega} H(\delta H)(\det g)^{1/2} + \frac{1}{2}H^2\delta(\det g)^{1/2} \,\mathrm{d}x \\
&= \int_{\omega} H(-\Delta_{\mathscr{M}}\phi - \phi|s|^2)(\det g)^{1/2} + \frac{1}{2}\phi H^3(\det g)^{1/2} \,\mathrm{d}x \\
&= \int_{\mathscr{M}} H(-\Delta_{\mathscr{M}}\phi) - \phi H|s|^2 + \frac{1}{2}\phi H^3 \,\mathrm{d}s.
\end{aligned}$$

Noting $|s|^2 = |\nabla_{\mathscr{M}} n|^2 = H^2 - 2K$ and integrating-by-parts proves the theorem. $\square$

Definition 8.7 For a family of surfaces $(\mathscr{M}_t)_{t\in[0,T]}$ and a family of points on the surfaces given by a differentiable function $c : [0, T] \to \mathbb{R}^3$ with $c(t) \in \mathscr{M}_t$ for all $t \in [0, T]$ we define the *normal velocity* of $\mathscr{M}_t$ at $q_0 = c(t_0)$ by

$$V(q_0, t_0) = c'(t_0) \cdot n(q_0).$$

We let

$$(\phi, \psi)_{\mathscr{M}_t} = \int_{\mathscr{M}_t} \phi\psi \, \mathrm{d}s$$

denote the L^2 inner product on $\mathscr{M}_t$.

Definition 8.8 (i) A family of surfaces $(\mathscr{M}_t)_{t\in[0,T]}$ evolves according to the *Willmore flow* if

$$(V(t), \phi)_{\mathscr{M}_t} = -\delta W(\mathscr{M}_t)[\phi]$$

for all $t \in [0, T]$ and all $\phi \in C^\infty(\mathscr{M}_t)$.
(ii) A family of surfaces $(\mathscr{M}_t)_{t\in[0,T]}$ evolves according to the *Helfrich flow* if there exist $\lambda, \mu : [0, T] \to \mathbb{R}$ such that

$$(V(t), \phi)_{\mathscr{M}_t} = -\delta W(\mathscr{M}_t)[\phi] + \lambda(t)\delta\mathscr{A}(\mathscr{M}_t)[\phi] + \mu(t)\delta\mathscr{V}(\mathscr{M}_t)[\phi]$$

for all $t \in [0, T]$ and all $\phi \in C^\infty(\mathscr{M}_t)$ and the mappings $t \mapsto \mathscr{A}(\mathscr{M}_t)$ and $t \mapsto \mathscr{V}(\mathscr{M}_t)$ are constant.

Remark 8.8 The existence of solutions for the Willmore and Helfrich flow is only understood in special situations, e.g., when the initial surface $\mathscr{M}_0$ is a small perturbation of a sphere.

8.4.4 Discretization of the Laplace–Beltrami Operator

For a surface $\mathscr{M} \subset \mathbb{R}^3$, let $\mathscr{M}_h$ be an approximate surface that is the union of flat triangles in the triangulation $\mathscr{T}_h$ with vertices $\mathscr{N}_h \subset \mathbb{R}^3$, cf. Fig. 8.10. The elementwise constant unit normal n_h on $\mathscr{M}_h$ defines the tangential gradient of a function $v_h \in \mathscr{S}^1(\mathscr{T}_h)$ via

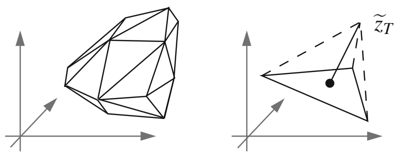

Fig. 8.10 Triangulated surface (*left*) and construction of an auxiliary tetrahedron with the auxiliary node $\widetilde{z}_T = x_T + |T|^{1/2} n_T$ (*right*)

$$\nabla_{\mathcal{M}_h} v_h = P_h \nabla \widetilde{v}_h = \big(I - n_h \otimes n_h\big) \nabla \widetilde{v}_h,$$

where $\widetilde{v}_h$ is an arbitrary extension of v_h to $\mathbb{R}^3$, e.g., by introducing for each triangle $T \in \mathcal{T}_h$ the auxiliary node $\widetilde{z}_T = x_T + |T|^{1/2} n_h|_T$, cf. Fig. 8.10, and setting $\widetilde{v}_h(\widetilde{z}_T) = 0$. The Laplace–Beltrami operator on a surface $\mathcal{M}$ leads to a Poisson problem on $\mathcal{M}$ of the form

$$-\Delta_{\mathcal{M}} u = f \text{ on } \mathcal{M}, \quad u = u_{\mathrm{D}} \text{ on } \gamma_{\mathrm{D},h}, \quad \nabla_{\mathcal{M}_h} u \cdot \mu_h = g \text{ on } \gamma_{\mathrm{N},h},$$

where μ_h is the conormal on $\Gamma_{\mathrm{N},h} \subset \partial \mathcal{M}_h$. A discrete approximation seeks $u_h \in \mathcal{S}^1(\mathcal{T}_h)$ such that $u_h|_{\gamma_{\mathrm{D},h}} = u_{\mathrm{D},h}$

$$\int_{\mathcal{M}_h} \nabla_{\mathcal{M}_h} u_h \cdot \nabla_{\mathcal{M}_h} v_h \,\mathrm{d}s = \int_{\mathcal{M}_h} f v_h \,\mathrm{d}s + \int_{\gamma_{\mathrm{N},h}} g_h v_h \,\mathrm{d}t$$

for all $v_h \in \mathcal{S}^1(\mathcal{T}_h)$ with $v_h|_{\gamma_{\mathrm{D},h}} = 0$. If $\gamma_{\mathrm{D},h} = \emptyset$, then the condition $\int_{\mathcal{M}_h} u_h \,\mathrm{d}s = 0$ is imposed. The MATLAB code displayed in Fig. 8.11 realizes the numerical scheme for the Laplace–Beltrami operator.

8.4.5 A Numerical Scheme for the Willmore Flow

We recall that the Willmore flow for a given initial surface $\mathcal{M}_0 \subset \mathbb{R}^3$ seeks a family of surfaces $(\mathcal{M}_t)_{t\in[0,T]}$ that solve the equation

$$V = \Delta_{\mathcal{M}} H + H|\nabla_{\mathcal{M}} n|^2 - \frac{1}{2} H^3,$$

where V is the normal velocity of $(\mathcal{M}_t)_{t\in[0,T]}$, n a unit normal on $\mathcal{M}_t$, and H the mean curvature of $\mathcal{M}_t$. For the position vector $X : \mathcal{M}_t \to \mathbb{R}^3$ on $\mathcal{M}$, we have $V = (\partial_t X) \cdot n$ and $Hn = -\Delta_{\mathcal{M}} \mathrm{id}_{\mathcal{M}}$. To discretize the evolution equation we consider a time step $t_k \in [0, T]$ and assume that we are given a triangulation $\mathcal{T}_h^k$ that defines the closed polyhedral surface $\mathcal{M}_h^k$ with unit normal $n_h^k \in \mathcal{L}^0(\mathcal{T}_h)^3$. We also suppose that $\widetilde{n}_h^k \in \mathcal{S}^1(\mathcal{T}_h^k)^3$ and $H_h^k \in \mathcal{S}^1(\mathcal{T}_h^k)$ approximate the unit normal n and the mean curvature of a smooth approximation of $\mathcal{M}_h^k$. To define the new surface $\mathcal{M}_h^{k+1}$, we compute a mapping

$$X_h^{k+1} : \mathcal{M}_h^k \to \mathbb{R}^3$$

```
function laplace_beltrami(red)
[c4n,n4e,Db,Nb] = triang_torus(.5,1,red);
nE = size(n4e,1); nC = size(c4n,1);
nNb = size(Nb,1); nDb = size(Db,1);
dNodes = unique(Db); fNodes = setdiff(1:nC,dNodes);
max_ctr = 9*nE; ctr = 0;
I = zeros(max_ctr,1); J = zeros(max_ctr,1);
X_s = zeros(max_ctr,1);
b = zeros(nC,1); c = zeros(nC,1); u = zeros(nC,1);
for j = 1:nE
    n_T = cross(c4n(n4e(j,2),:)-c4n(n4e(j,1),:),...
        c4n(n4e(j,3),:)-c4n(n4e(j,2),:));
    area_T = norm(n_T)/2;
    n_T = n_T/norm(n_T);
    mp_T = sum(c4n(n4e(j,:),:))/3;
    aux_tetra = [c4n(n4e(j,:),:);mp_T+sqrt(area_T)*n_T];
    grads3_T = [1,1,1,1;aux_tetra']\[0,0,0;eye(3)];
    P_T = eye(3)-n_T'*n_T;
    for k = 1:3
        b(n4e(j,k)) = b(n4e(j,k))+(1/3)*area_T*f(mp_T);
        c(n4e(j,k)) = c(n4e(j,k))+(1/3)*area_T;
        for ell = 1:3
            ctr = ctr+1;
            I(ctr) = n4e(j,k); J(ctr) = n4e(j,ell);
            X_s(ctr) = area_T*(P_T*grads3_T(k,:)')'...
                *(P_T*grads3_T(ell,:)');
        end
    end
end
s = sparse(I,J,X_s,nC,nC);
for j = 1:nNb
    length_E = norm(c4n(Nb(j,1),:)-c4n(Nb(j,2),:));
    mp_E = (c4n(Nb(j,1),:)-c4n(Nb(j,2),:))/2;
    b(Nb(j,1)) = b(Nb(j,1))+(1/2)*length_E*g(mp_E);
    b(Nb(j,2)) = b(Nb(j,2))+(1/2)*length_E*g(mp_E);
end
if isempty(dNodes)
    s = [s,c;c',0]; b = [b;0];
else
    for j = 1:nDb
        u(dNodes(j)) = u_D(c4n(dNodes(j),:));
    end
    b = b-s*u;
end
u(fNodes) = s(fNodes,fNodes)\b(fNodes);
show_p1_surf(c4n,n4e,u);

function val = f(X); val = X(2);
function val = u_D(X); val = 0;
function val = g(X); val = 0;
```

Fig. 8.11 MATLAB routine for the approximation of the Poisson problem on a surface

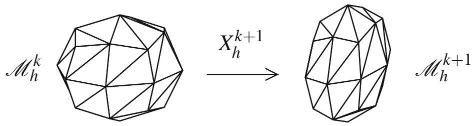

Fig. 8.12 Deformation $X_h^{k+1} : \mathscr{M}_h^k \to \mathbb{R}^3$ of a surface $\mathscr{M}_h^k$ that defines the new surface $\mathscr{M}_h^{k+1}$

that defines $\mathscr{M}_h^{k+1} = X_h^{k+1}(\mathscr{M}_h^k)$, cf. Fig. 8.12. A function or vector field on $\mathscr{M}_h^k$ is identified with a function on $\mathscr{M}_h^{k+1}$ via the parametrization X_h^{k+1}. The vector field $X_h^{k+1} \in \mathscr{S}^1(\mathscr{T}_h^k)^3$ is obtained by the following semi-implicit discretization of the Willmore flow from [2].

Algorithm 8.2 (Discrete Willmore flow) *For a discrete surface $\mathscr{M}_h^0$, functions $\widetilde{n}_h^0 \in \mathscr{S}^1(\mathscr{T}_h^0)^3$ and $H_h^0 = \mathscr{A}_h^0 \operatorname{div}_{\mathscr{M}_h^0} \widetilde{n}_h^0$. and a step size $\tau > 0$, compute the sequence $(\mathscr{M}_h^k)_{k=0,\dots,K}$ via $\mathscr{M}_h^{k+1} = X_h^{k+1}(\mathscr{M}_h^k)$, where $X_h^{k+1} \in \mathscr{S}^1(\mathscr{T}_h^k)^3$ and $H_h^{k+1} \in \mathscr{S}^1(\mathscr{T}_h^k)$ solve*

$$
\begin{aligned}
\frac{1}{\tau}\big(X_h^{k+1} - \mathrm{id}_{\mathscr{M}_h^k}, v_h \widetilde{n}_h^k\big)_{k,h} + \big(\nabla_{\mathscr{M}_h^k} H_h^{k+1}, \nabla_{\mathscr{M}_h^k} v_h\big)_k + \frac{1}{2}\big(|H_h^k|^2 H_h^{k+1}, v_h\big)_{k,h} & \\
= \big(H_h^k \mathscr{A}_h^k |\nabla_{\mathscr{M}_h^k} \widetilde{n}_h^k|^2, v_h\big)_{k,h}, & \\
\big(H_h^{k+1} \widetilde{n}_h^k, Y_h\big)_{k,h} - \big(\nabla_{\mathscr{M}_h^k} X_h^{k+1}, \nabla_{\mathscr{M}_h^k} Y_h\big)_k = 0 &
\end{aligned}
$$

for all $v_h \in \mathscr{S}^1(\mathscr{T}_h^k)$ and $Y_h \in \mathscr{S}^1(\mathscr{T}_h^k)^3$, and set $\widetilde{n}_h^{k+1} = \mathscr{A}_h^{k+1} n_h^{k+1}$. Stop the iteration if $\|v_h^{k+1}\|_{h,k} \le \varepsilon_{\mathrm{stop}}$ for $V_h^{k+1} = (X_h^{k+1} - \mathrm{id}_{\mathscr{M}_h^k})/\tau$ and $v_h^{k+1} = V_h^{k+1} \cdot \widetilde{n}_h^k$.

The averaging operator $\mathscr{A}_h^k : L^1(\mathscr{M}_h^k) \to \mathscr{S}^1(\mathscr{T}_h^k)$ is defined through

$$
\mathscr{A}_h^k v(z) = \frac{1}{|\omega_z|} \sum_{T \in \mathscr{T}_h^k,\, z \in T} |T|\, v|_T, \qquad |\omega_z| = \sum_{T \in \mathscr{T}_h^k,\, z \in T} |T|,
$$

and the inner product $(\cdot,\cdot)_{k,h}$ is for $v, w \in C(\mathscr{M}_h^k)$ defined by

$$
(v, w)_{k,h} = \int_{\mathscr{M}_h^k} \mathscr{I}_h^k[vw] \,\mathrm{d}x.
$$

Remark 8.9 The precise stability and convergence properties of Algorithm 8.2 are not known. The algorithm has an equidistribution property in the sense that it equidistributes the nodes of the discrete surface which avoids mesh irregularities. Details are discussed in [2].

According to Proposition 8.9 it suffices to impose that

$$\int_{\mathscr{M}} V \,\mathrm{d}s = \int_{\mathscr{M}} VH \,\mathrm{d}s = 0$$

to guarantee that the surface area and the enclosed volume are preserved. This leads to an identity for the associated Lagrange multipliers in the evolution equation, i.e.,

$$V = \Delta_{\mathscr{M}} H + H|\nabla_{\mathscr{M}} n|^2 - \frac{1}{2}H^3 + \lambda H + \mu.$$

Testing the equation with a constant function and with $H - \overline{H}$, where $\overline{H}$ is the integral mean of H, leads to

$$\mu = \frac{1}{|\mathscr{M}|} \int_{\mathscr{M}} -H|\nabla_{\mathscr{M}} n|^2 + \frac{1}{2}H^3 - \lambda H \,\mathrm{d}s,$$

$$\lambda = \frac{\int_{\mathscr{M}} \big(- H|\nabla_{\mathscr{M}} n|^2 + \frac{1}{2}H^3\big)(H - \overline{H}) + |\nabla_{\mathscr{M}} H|^2 \,\mathrm{d}s}{\int_{\mathscr{M}} (H - \overline{H})^2 \,\mathrm{d}s}.$$

To incorporate the constraints in Algorithm 8.2, the term λH is discretized implicitly if $\lambda \geq 0$ and explicitly otherwise. The MATLAB implementation displayed in Fig. 8.14 requires the bilinear forms

$$(\varphi_z^\ell, \varphi_y)_{k,h}, \quad (\nabla\varphi_z, \nabla\varphi_y)_k, \quad (\nabla\varphi_z^\ell, \nabla\varphi_y^m)_k,$$
$$(\varphi_z^\ell, n\varphi_y)_{k,h}, \quad (\varphi_z \mathscr{A}_h^k |\nabla_{\mathscr{M}_h^k} \widetilde{n}_h^k|^2, \varphi_y)_{k,h}, \quad (|H_h^k|^2 \varphi_z, \varphi_y)_{k,h},$$

for pairs of nodes $z, y \in \mathscr{N}_h^k$ and associated scalar nodal basis functions $\varphi_z, \varphi_y \in \mathscr{S}^1(\mathscr{T}_h)^k$ and vectorial nodal basis functions $\varphi_z^\ell = \varphi_z e_\ell$ and $\varphi_y^m = \varphi_y e_m$ with the canonical basis vectors $e_\ell, e_m \in \mathbb{R}^3$. The representing matrices are encoded in the arrays `m`, `s`, `S`, `M_n`, `m_w` provided by the routine shown in Fig. 8.13 while the last one is directly computed and stored in the array `m_H`. The routine `willmore_matrices.m` also computes an approximation of the mean curvature through $H_h^k = \mathscr{A}_h^k(\mathrm{div}_{\mathscr{M}_h^k} \widetilde{n}_h^k)$.

```
function [m,s,S,M_n,m_w,H] = willmore_matrices(c4n,n4e,w)
nC = size(c4n,1); nE = size(n4e,1);
max_ctr = 9*nE; ctr = 0;
I = zeros(max_ctr,1); J = zeros(max_ctr,1);
X_s = zeros(max_ctr,1);
diag_m = zeros(nC,1);
diag_m_w = zeros(nC,1);
diag_M_n = zeros(nC,3);
tr_nabla_w = zeros(nC,1);
for j = 1:nE
    n_T = cross(c4n(n4e(j,2),:)-c4n(n4e(j,1),:),...
        c4n(n4e(j,3),:)-c4n(n4e(j,2),:));
    area_T = norm(n_T)/2;
    n_T = n_T/norm(n_T);
    mp_T = sum(c4n(n4e(j,:),:))/3;
    tmp_tetra = [c4n(n4e(j,:),:);mp_T+sqrt(area_T)*n_T];
    grads3_T = [1,1,1,1;tmp_tetra']\[0,0,0;eye(3)];
    P_T = eye(3)-n_T'*n_T;
    P_Dphi_T = grads3_T(1:3,:)*P_T;
    nabla_T_w = w(n4e(j,:),:)'*P_Dphi_T;
    tr_nabla_w(j) = trace(nabla_T_w);
    W_sq = sum(sum(nabla_T_w.^2));
    for k = 1:3
        diag_m(n4e(j,k)) = diag_m(n4e(j,k))+area_T/3;
        diag_m_w(n4e(j,k)) = diag_m_w(n4e(j,k))+area_T*W_sq/3;
        diag_M_n(n4e(j,k),:) = diag_M_n(n4e(j,k),:)...
            +(area_T/3)*n_T;
        for ell = 1:3
            ctr = ctr+1;
            I(ctr) = n4e(j,k); J(ctr) = n4e(j,ell);
            X_s(ctr) = area_T...
                *(P_T*grads3_T(k,:)')'*(P_T*grads3_T(ell,:)');
        end
    end
end
m = spdiags(diag_m,0,nC,nC); m_w = spdiags(diag_m_w,0,nC,nC);
II = [3*I-2;3*I-1;3*I]; JJ = [3*J-2;3*J-1;3*J];
s = sparse(I,J,X_s); S = sparse(II,JJ,repmat(X_s,3,1));
I = [1:3:3*nC,2:3:3*nC,3:3:3*nC]'; J = [1:nC,1:nC,1:nC]';
M_n = sparse(I,J,diag_M_n(:));
H = average_quant_surf(c4n,n4e,tr_nabla_w);
```

Fig. 8.13 Matrices required in the implementation of the Willmore and the Helfrich flow

```
function willmore_helfrich_flow(red)
[n4e,c4n,¬,¬] = triang_sphere(red);
c4n(:,3) = .4*c4n(:,3);
tau = 2^(-red)/200;
nC = size(c4n,1);
w = averaged_normal(c4n,n4e);
[¬,¬,¬,¬,¬,H] = willmore_matrices(c4n,n4e,w);
X = reshape(c4n',3*nC,1);
corr = 1; eps_stop = 1e-1;
while corr > eps_stop
    w = averaged_normal(c4n,n4e);
    [m,s,S,M_n,m_w,¬] = willmore_matrices(c4n,n4e,w);
    m_H = spdiags(diag(m).*H.^2,0,nC,nC);
    [lambda,mu] = helfrich_constraints(c4n,H,s,m,m_w,m_H);
    A = [M_n',tau*(s+m_H/2-max(lambda,0)*m);-S,M_n];
    b = [tau*m_w*H+M_n'*X+tau*(mu*m*ones(nC,1)...
        +min(lambda,0)*m*H);zeros(3*nC,1)];
    xx = A\b;
    V = (xx(1:3*nC)-X)/tau;
    v = sum(reshape(V',3,nC)'.*w,2);
    corr = sqrt(v'*m*v)
    H = xx(3*nC+(1:nC)); X = X+tau*V; c4n = reshape(X',3,nC)';
    show_p1_surf(c4n,n4e,H);
end

function [lambda,mu] = helfrich_constraints(c4n,H,s,m,m_w,m_H)
nC = size(c4n,1); I = ones(nC,1);
mean_H = I'*m*H/(I'*m*I);
q = (H-mean_H)'*m*(H-mean_H);
lambda = 0;
if q > 0
    lambda = (-H'*m_w*H+H'*m_H/2*H...
        -(-H'*m_w*I+H'*m_H/2*I)*mean_H+H'*s*H)/q;
end
mu = (-H'*m_w*I+I'*m_H/2*H-lambda*I'*m*H)/(I'*m*I);

function w = averaged_normal(c4n,n4e)
nC = size(c4n,1); nE = size(n4e,1);
n = zeros(nE,3); w = zeros(nC,3);
for j = 1:nE
    n_T = cross(c4n(n4e(j,2),:)-c4n(n4e(j,1),:),...
        c4n(n4e(j,3),:)-c4n(n4e(j,2),:));
    n(j,:) = n_T/norm(n_T);
end
for k = 1:3
    w(:,k) = average_quant_surf(c4n,n4e,n(:,k));
end
norm_w = sqrt(sum(w.^2,2));
w = w./(norm_w*ones(1,3));
```

Fig. 8.14 Numerical approximation of the Willmore and the Helfrich flow

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