Chapter 7
Harmonic Maps

7.1 Analytical Properties

Harmonic maps are stationary points of the Dirichlet energy in the set of vector fields that attain their values in a given target manifold, e.g., the unit sphere. Related problems arise in various applications and the problem of computing harmonic maps serves as a mathematical model problem for constrained minimization problems on infinite-dimensional spaces. We will consider the case of computing harmonic maps into the unit sphere $S^{m-1} = \{s \in \mathbb{R}^m : |s| = 1\}$, i.e., unit-length vector fields, but notice that a large class of target manifolds can be treated with the same ideas. We thus aim at approximating minimizers $u \in \mathscr{A}$ for

$$I(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \,\mathrm{d}x$$

with the set of admissible vector fields

$$\mathscr{A} = \{v \in H^1(\Omega; \mathbb{R}^m) : |v(x)| = 1 \text{ for a.e. } x \in \Omega,\ v|_{\Gamma_\mathrm{D}} = u_\mathrm{D}\}.$$

The function $u_\mathrm{D} \in L^2(\Gamma_\mathrm{D}; \mathbb{R}^m)$ on the nonempty set $\Gamma_\mathrm{D} \subset \partial\Omega$ is assumed to admit an extension $\widetilde{u}_\mathrm{D} \in H^1(\Omega; \mathbb{R}^m)$ with $|\widetilde{u}_\mathrm{D}(x)| = 1$ for almost every $x \in \Omega$. We briefly summarize the main properties of harmonic maps and refer the reader to the textbooks [9, 12] for more details.

7.1.1 Existence and Nonuniqueness

The existence of minimizers is established by the direct method in the calculus of variations.

S. Bartels, *Numerical Methods for Nonlinear Partial Differential Equations*,
Springer Series in Computational Mathematics 47,
DOI 10.1007/978-3-319-13797-1_7

Theorem 7.1 (Existence) *There exists a minimizer* $u \in \mathscr{A}$.

Proof Since u_D admits an extension to a unit-length vector field field $\widetilde{u}_D \in \mathscr{A}$ there exists an infimizing sequence $(u_j)_{j\in\mathbb{N}} \subset \mathscr{A}$ with $\lim_{j\to\infty} I(u_j) = \inf_{v\in\mathscr{A}} I(v)$. Since $u_j - \widetilde{u}_D \in H^1_D(\Omega;\mathbb{R}^m)$, we have that $(u_j)_{j\in\mathbb{N}}$ is bounded in $H^1(\Omega;\mathbb{R}^m)$. A subsequence converges weakly to a vector field $u \in H^1(\Omega;\mathbb{R}^m)$ with $u|_{\Gamma_D} = u_D$. To show that $u \in \mathscr{A}$ we notice that the subsequence converges strongly in $L^2(\Omega;\mathbb{R}^m)$, and hence there exists a further subsequence that converges pointwise almost everywhere to u. Therefore, $|u| = 1$ almost everywhere in Ω, i.e., $u \in \mathscr{A}$. The weak lower semicontinuity of I implies that u is a minimizer. □

Remark 7.1 The proof shows that the set $\mathscr{A}$ is weakly closed.

The essential condition that $\mathscr{A} \neq \emptyset$ may be difficult to verify in practice even if $u_D \in L^2(\Gamma_D;\mathbb{R}^m)$ is smooth and satisfies $|u_D(x)| = 1$ for almost every $x \in \partial\Omega$.

Example 7.1 (Nonexistence) For $\Omega = B_1(0) \subset \mathbb{R}^2$ and $u_D(x) = x$ there is no function $\widetilde{u}_D \in H^1(\Omega;\mathbb{R}^2)$ with $\widetilde{u}_D|_{\partial\Omega} = u_D$ and $|\widetilde{u}_D(x)| = 1$ for almost every $x \in \Omega$. This is a consequence of the Hopf–Poincaré formula and Brouwer's fixed point theorem.

Due to the invariance of the Dirichlet energy under rotations, we cannot expect harmonic maps to be unique.

Example 7.2 (Nonuniqueness) Let $\Omega = (0,1)$, $\Gamma_D = \partial\Omega = \{0,1\}$, $m = 3$, and let $u : (0,1) \to S^2$ be minimal for

$$I(u) = \frac{1}{2}\int_0^1 |u'|^2\,\mathrm{d}x$$

in the set of functions $v \in \mathscr{A}$ with $v(0) = e$ and $v(1) = -e$ for some $e \in S^2$. Then for every rotation $Q \in SO(3) = \{R \in \mathbb{R}^{3\times 3} : R^\top R = I_3,\ \det R = 1\}$ with $Qe = e$, we have that $\widetilde{u} = Qu$ is another minimizer. The harmonic maps $u_1(x) = [\cos(\pi x), 0, \sin(\pi x)]^\top$, $x \in (0,1)$, and $u_2 = Qu_1$, where $Q \in \mathbb{R}^{3\times 3}$ realizes a rotation by π about the first coordinate axis, with identical Dirichlet energy are shown in Fig. 7.1.

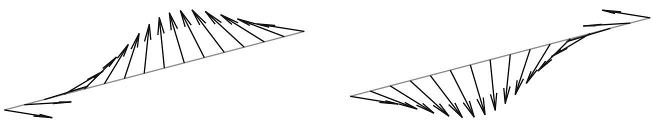

Fig. 7.1 Two harmonic maps on $\Omega = (0,1)$ with the same boundary values and identical Dirichlet energy; the length of the *arrows* is scaled for graphical purposes

Remarks 7.2 (i) Harmonic maps can be approximated by penalizing the pointwise constraint, e.g., considering for $\varepsilon > 0$ the Ginzburg–Landau regularization

$$I_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x + \frac{\varepsilon^{-2}}{4}\int_\Omega (|u|^2-1)^2 \,\mathrm{d}x$$

and investigating the limiting behavior of minimizers $(u_\varepsilon)_{\varepsilon>0}$ as $\varepsilon \to 0$.
(ii) Formally, a harmonic map u and the Lagrange multiplier λ associated to the length constraint define a saddle-point for the functional

$$L(u,\lambda) = \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x + \int_\Omega \lambda(|u|^2-1)\,\mathrm{d}x.$$

7.1.2 Euler–Lagrange Equations and Nonregularity

The Euler–Lagrange equations define a nonlinear partial differential equation.

Theorem 7.2 (Euler–Lagrange equations) *Let $u \in \mathscr{A}$ be stationary for the Dirichlet energy. Then we have*

$$(\nabla u, \nabla w) = (|\nabla u|^2 u, w)$$

for all $w \in H^1_\mathrm{D}(\Omega;\mathbb{R}^m) \cap L^\infty(\Omega;\mathbb{R}^m)$.

Proof Let $w \in H^1(\Omega;\mathbb{R}^m) \cap L^\infty(\Omega;\mathbb{R}^m)$ and $\varepsilon > 0$ be such that $\varepsilon\|w\|_{L^\infty(\Omega)} \le 1/2$. We then have that $|u(x)+rw(x)| \ge 1/2$ for almost every $x \in \Omega$ and every $r \in \mathbb{R}$ with $|r| \le \varepsilon$. It follows that the map

$$u^r(x) = \frac{u(x)+rw(x)}{|u(x)+rw(x)|}$$

belongs to $H^1(\Omega;\mathbb{R}^m)$ and satisfies $|u^r| = 1$ in Ω and $u^r|_{\Gamma_\mathrm{D}} = u_\mathrm{D}$. Since $u^0 = u$, we have that the function $t \mapsto I(u^r)$ is minimal at $r = 0$. We note that

$$\frac{d}{dr}\Big|_{r=0} u^r = w - u(u\cdot w).$$

A differentiation shows that

$$0 = \frac{d}{dr}\Big|_{r=0} I(u^r) = \sum_{\ell=1}^d \int_\Omega \partial_\ell u \cdot \partial_\ell\big[w - u(u\cdot w)\big]\,\mathrm{d}x$$

and the orthogonality $(\partial_\ell u)\cdot u = 0$ for $\ell = 1, 2, \dots, d$ implies the assertion. □

Definition 7.1 Solutions $u \in \mathscr{A}$ of the Euler–Lagrange equation are called *harmonic maps* (*into the sphere*).

Remark 7.3 The function $\lambda = |\nabla u|^2 \in L^1(\Omega)$ is the Lagrange multiplier associated to the pointwise constraint $|u(x)|^2 = 1$.

Solutions of the Euler–Lagrange equations are in general neither energy minimizing nor regular.

Example 7.3 (Nonregularity) Let $\Omega = (-1, 1)^3$ and $u_D(x) = x/|x|$ for $x \in \Gamma_D = \partial\Omega$. Then $u(x) = x/|x|$ for $x \in \Omega$ satisfies $u \in \mathscr{A}$ and is a harmonic map. Moreover u is minimal for I in the set of vector fields in $\mathscr{A}$.

Remarks 7.4 (i) For $d = 2$, harmonic maps are smooth.
(ii) If $d = 3$, then energy minimizing harmonic maps u are partially regular in the sense that u is smooth in $\Omega \setminus S$ for a set S with $\mathscr{H}^1(S) = 0$, e.g., a set of points. Harmonic maps that are not globally energy minimizing can be discontinuous everywhere.

7.1.3 Compactness

The lack of uniqueness and regularity of harmonic maps makes it difficult to quantify stability properties. The weaker concept of compactness shows that accumulation points of (almost) harmonic maps are again harmonic maps, i.e., that bounded subsets of the set of harmonic maps are weakly compact. The key to this property is the following equivalent characterization of harmonic maps. We restrict ourselves to the case $m = 3$ for ease of presentation.

Lemma 7.1 (Equivalent characterization) *Let $m = 3$. The function $u \in \mathscr{A}$ is a harmonic map if and only if*

$$(\nabla u, \nabla[u \times \phi]) = \sum_{\ell=1}^{d} (\partial_\ell u, u \times \partial_\ell \phi) = 0$$

for all $\phi \in H^1_D(\Omega; \mathbb{R}^3) \cap L^\infty(\Omega; \mathbb{R}^3)$. This is the case if and only if

$$(\nabla u, \nabla w) = 0$$

for all $w \in H^1_D(\Omega; \mathbb{R}^3)$ satisfying $u \cdot w = 0$ almost everywhere in Ω.

Proof (i) Let $u \in \mathscr{A}$ be a harmonic map. Then the choice $w = u \times \phi$ in the Euler–Lagrange equations, the fact that $u \cdot (u \times \phi) = 0$, and the identity

$$(\partial_\ell u, \partial_\ell[u \times \phi]) = (\partial_\ell u, u \times \partial_\ell \phi)$$

for $\ell = 1, 2, \ldots, d$ imply the first characterization. The second one is an immediate consequence of the Euler–Lagrange equations if $w \in H^1_{\mathrm{D}}(\Omega; \mathbb{R}^3) \cap L^\infty(\Omega; \mathbb{R}^3)$ with $w \cdot u = 0$ in Ω. A truncation argument shows that this is satisfied for all $w \in H^1_{\mathrm{D}}(\Omega; \mathbb{R}^3)$ with $w \cdot u = 0$ almost everywhere in Ω.

(ii) Assume that the first equation of the lemma is satisfied and let $w \in H^1_{\mathrm{D}}(\Omega; \mathbb{R}^3) \cap L^\infty(\Omega; \mathbb{R}^3)$. For $\phi = u \times w$ we have, due to the formula $a \times (b \times c) = b(a \cdot c) - c(a \cdot b)$ that

$$u \times \phi = u \times (u \times w) = (u \cdot w)u - |u|^2 w = (u \cdot w)u - w.$$

Moreover, we have for $\ell = 1, 2, \ldots, d$ that

$$\partial_\ell[(u \cdot w)u] = (\partial_\ell u \cdot w)u + (u \cdot \partial_\ell w)u + (u \cdot w)\partial_\ell u.$$

With $\partial_\ell u \cdot u = 0$ this implies that

$$0 = \sum_{\ell=1}^{d} \big[(\partial_\ell u, (u \cdot w)\partial_\ell u) - (\partial_\ell u, \partial_\ell w)\big] = (|\nabla u|^2 u, w) - (\nabla u, \nabla w)$$

which proves that u is a harmonic map.

(iii) Suppose that the second characterization is satisfied and let $\phi \in H^1_{\mathrm{D}}(\Omega; \mathbb{R}^3) \cap L^\infty(\Omega; \mathbb{R}^3)$. The function $w = u \times \phi$ satisfies $u \cdot w = 0$ so that the first characterization holds. □

Remark 7.5 The condition that $(\nabla u, \nabla w) = 0$ for all $w \in H^1_{\mathrm{D}}(\Omega; \mathbb{R}^3)$ satisfying $u \cdot w = 0$ shows that u is stationary with respect to tangential perturbations.

The equivalent characterizations imply the following weak compactness result which will serve as a guideline to prove convergence of numerical approximations.

Theorem 7.3 (Weak compactness) *Let $(\mathscr{R}_j)_{j \in \mathbb{N}} \subset H^1_{\mathrm{D}}(\Omega; \mathbb{R}^3)'$ be a sequence of functionals with $\|\mathscr{R}_j\|_{H^1_{\mathrm{D}}(\Omega)'} \to 0$ as $j \to \infty$, and assume that $(u_j)_{j \in \mathbb{N}} \subset \mathscr{A}$ is such that*

$$(\nabla u_j, \nabla w) = (|\nabla u_j|^2 u_j, w) + \mathscr{R}_j(w)$$

for every $j \in \mathbb{N}$ and all $w \in H^1_{\mathrm{D}}(\Omega; \mathbb{R}^3) \cap L^\infty(\Omega; \mathbb{R}^3)$. If $u \in H^1(\Omega; \mathbb{R}^3)$ is such that $u_j \rightharpoonup u$ in $H^1(\Omega; \mathbb{R}^3)$ as $j \to \infty$, then we have $u \in \mathscr{A}$ and u is a harmonic map.

Proof The weak closedness of $\mathscr{A}$ implies that $u \in \mathscr{A}$. For every $\phi \in H^1_{\mathrm{D}}(\Omega; \mathbb{R}^3) \cap C^\infty(\overline{\Omega}; \mathbb{R}^3)$ and $j \in \mathbb{N}$, the choice of $w = u_j \times \phi$ yields, using $\partial_\ell u_j \cdot (\partial_\ell u_j \times \phi) = 0$,

$$\sum_{\ell=1}^{d} (\partial_\ell u_j, u_j \times \partial_\ell \phi) = \mathscr{R}_j(u_j \times \phi).$$

Since $u_j \to u$ in $L^2(\Omega; \mathbb{R}^3)$ and $\partial_\ell u_j \rightharpoonup \partial_\ell u$ in $L^2(\Omega; \mathbb{R}^3)$, we have

$$(\partial_\ell u_j, u_j \times \partial_\ell \phi) = (\partial_\ell u_j, u \times \partial_\ell \phi) + (\partial_\ell u_j, [u_j - u] \times \partial_\ell \phi)$$
$$\to (\partial_\ell u, u \times \partial_\ell \phi)$$

as $j \to \infty$ for $\ell = 1, 2, \ldots, d$. Employing $\mathscr{R}_j \to 0$ in $H^1_{\mathrm{D}}(\Omega; \mathbb{R}^3)'$ and that $u_j \times \phi$ is bounded in $H^1_{\mathrm{D}}(\Omega; \mathbb{R}^3)$, we also have

$$\mathscr{R}_j(u_j \times \phi) \to 0$$

as $j \to \infty$. Altogether we find that u satisfies

$$\sum_{\ell=1}^{d} (\partial_\ell u, u \times \partial_\ell \phi) = 0$$

for all $\phi \in H^1_{\mathrm{D}}(\Omega; \mathbb{R}^3) \cap C^\infty(\overline{\Omega}; \mathbb{R}^3)$. A density argument shows that this identity holds for all $\phi \in H^1_{\mathrm{D}}(\Omega; \mathbb{R}^3) \cap L^\infty(\Omega; \mathbb{R}^3)$ so that Lemma 7.1 implies that u is a harmonic map. □

Remarks 7.6 (i) The equivalent characterization of harmonic maps involving the cross product allowed us to use that the product of a weakly and a strongly convergent sequence is weakly convergent. We remark that the identification of the limit of the square of a weakly convergent sequence is difficult in general and a passage to a limit in the Euler–Lagrange equations for harmonic maps does not imply that the limit is a harmonic map.
(ii) While the existence of harmonic maps into general target manifolds other than the unit sphere can be established analogously, related compactness results are false in general. For $d = 2$ and sufficiently smooth target manifolds, regularity and compactness can be proved, cf. [11].

7.1.4 Harmonic Map Heat Flow

The harmonic map heat flow is the L^2-gradient flow of the Dirichlet energy subject to the unit length constraint and is given by

$$\partial_t u - \Delta u = |\nabla u|^2 u, \ |u(t, \cdot)| = 1, \ u(0) = u_0, \ u|_{\Gamma_{\mathrm{D}}} = u_{\mathrm{D}}, \ \partial_n u|_{\Gamma_{\mathrm{N}}} = 0$$

for almost every $t \in [0, T]$. To avoid very irregular solutions, it is important to construct solutions that satisfy an energy law.

Theorem 7.4 (Existence) *Given $u_0 \in H^1(\Omega; \mathbb{R}^m)$ with $|u_0(x)| = 1$ for almost every $x \in \Omega$, there exists $u \in H^1([0, T]; L^2(\Omega; \mathbb{R}^m)) \cap L^\infty([0, T]; H^1(\Omega; \mathbb{R}^m))$ such that $|u(t, x)| = 1$ for almost every $(t, x) \in [0, T] \times \Omega$, $u(0) = u_0$,*

$$(\partial_t u, w) + (\nabla u, \nabla w) = (|\nabla u|^2 u, w)$$

for almost every $t \in [0, T]$ *and all* $w \in H^1_{\mathrm{D}}(\Omega; \mathbb{R}^m) \cap L^\infty(\Omega; \mathbb{R}^m)$*, and*

$$I(u(T')) + \int_0^{T'} \|\partial_t u\|^2 \, \mathrm{d}t \le I(u_0)$$

for almost every $T' \in [0, T]$.

Proof The result follows from the convergence of numerical approximations proved below. □

Remark 7.7 Uniqueness of solutions is known within the class of energy decreasing solutions if $d = 2$.

Solutions of the harmonic map heat flow can develop singularities in finite time.

Example 7.4 (Finite-time blowup [8]) Let $\Omega = B_1(0) \subset \mathbb{R}^2$, $\Gamma_{\mathrm{D}} = \partial\Omega$, and $u_{\mathrm{D}} = u_0|_{\Gamma_{\mathrm{D}}}$ for u_0 defined for $b > 0$ by

$$u_0(x) = \frac{1}{|x|}\big(x_1 \sin h(|x|), x_2 \sin h(|x|), |x| \cos h(|x|)\big)$$

for $x \in \Omega \setminus \{0\}$ and $h(r) = br^2$. If and only if $b \ge \pi$, the corresponding solution of the harmonic map heat flow is singular in the sense that there exists $T_c > 0$ with $\lim_{t \to T_c} \|\nabla u(t)\|_{L^\infty(\Omega)} = \infty$.

7.2 Numerical Approximation

We discuss in this section the approximation of harmonic maps and employ arguments from [1, 3, 5, 6, 10].

7.2.1 Discrete Harmonic Maps

It is straightforward to verify that the only polynomial vector fields that are pointwise of unit length are constant vector fields. Therefore, the constraint cannot be imposed almost everywhere on polynomial finite element functions. The following proposition shows that it is sufficient to impose the constraint at the nodes of a triangulation, cf. Fig. 7.2.

Proposition 7.1 (Nodal constraint) *Let* $(\mathcal{T}_h)_{h>0}$ *be a family of regular triangulations of* $\Omega \subset \mathbb{R}^d$ *and let* $(u_h)_{h>0} \subset H^1(\Omega; \mathbb{R}^m)$ *be such that* $u_h \in \mathcal{S}^1(\mathcal{T}_h)^m$ *and* $|u_h(z)| = 1$ *for all* $z \in \mathcal{N}_h$ *and every* $h > 0$*. If* $u_h \rightharpoonup u$ *in* $H^1(\Omega; \mathbb{R}^m)$ *for some* $u \in H^1(\Omega; \mathbb{R}^m)$*, then we have* $|u(x)| = 1$ *for almost every* $x \in \Omega$.

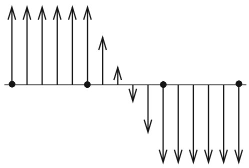

Fig. 7.2 The unit-length constraint is only imposed at the nodes of the triangulation; the linearly interpolated vector field may violate the constraint between two nodes

Proof We have $\mathcal{I}_h |u_h|^2 = 1$ for every $h > 0$ and hence by nodal interpolation estimates and $D^2 u_h|_T = 0$ for every $T \in \mathcal{T}_h$ that

$$\big\| |u_h|^2 - 1 \big\|_{L^2(T)} = \big\| |u_h|^2 - \mathcal{I}_h |u_h|^2 \big\|_{L^2(T)} \le c h_T^2 \| D^2 |u_h|^2 \|_{L^2(T)} \\ = c h_T^2 \| |\nabla u_h|^2 \|_{L^2(T)} = c h_T^2 \| \nabla u_h \|_{L^\infty(T)} \| \nabla u_h \|_{L^2(T)}.$$

The inverse estimate $\|\nabla u_h\|_{L^\infty(T)} \le c h_T^{-1} \|u_h\|_{L^\infty(T)} = c h_T^{-1}$ and a summation over $T \in \mathcal{T}_h$ imply

$$\big\| |u_h|^2 - 1 \big\| \le c h \|\nabla u_h\|$$

and prove that $|u_h| \to 1$ in $L^2(\Omega)$ as $h \to 0$. Since also $|u_{h'}| \to |u|$ as $h' \to 0$ almost everywhere in Ω for an appropriate subsequence $h' > 0$, we deduce $|u| = 1$ in Ω. □

The proposition motivates minimizing the Dirichlet energy restricted to finite element functions that satisfy the boundary conditions and the unit-length constraint at the nodes of the underlying triangulation.

Theorem 7.5 (Discrete harmonic maps) *Assume that* $\widetilde{u}_{D,h} \in \mathcal{S}^1(\mathcal{T}_h)^m$ *satisfies* $|\widetilde{u}_{D,h}(z)| = 1$ *for all* $z \in \mathcal{N}_h$ *and* $u_{D,h} = \widetilde{u}_{D,h}|_{\Gamma_D}$. *There exists a minimizer* $u_h \in \mathcal{A}_h$ *for* I *in the set of discrete admissible vector fields*

$$\mathcal{A}_h = \{ v_h \in \mathcal{S}^1(\mathcal{T}_h)^m : |v_h(z)| = 1 \text{ for all } z \in \mathcal{N}_h,\ v_h|_{\Gamma_D} = u_{D,h} \}.$$

The function $u_h \in \mathcal{A}_h$ *is stationary for* I *in the set of functions in* $\mathcal{A}_h$ *if and only if*

$$(\nabla u_h, \nabla w_h) = 0$$

for all $w_h \in \mathcal{F}_h[u_h]$ *with*

$$\mathcal{F}_h[u_h] = \big\{ w_h \in \mathcal{S}^1_D(\mathcal{T}_h)^m : w_h(z) \cdot u_h(z) = 0 \text{ for all } z \in \mathcal{N}_h \big\}.$$

Proof The functional I is coercive and continuous on $\mathcal{A}_h$, and this implies the existence of a minimizer. To verify the second statement, let $u_h \in \mathcal{A}_h$ be stationary for I and let $w_h \in \mathcal{F}_h[u_h]$. For every $r \in \mathbb{R}$, we have that $|u_h(z) + r w_h(z)|^2 = |u_h(z)|^2 + r^2 |w_h(z)|^2 \ge 1$ for all $z \in \mathcal{N}_h$ and we may define

$$u_h^r = \mathscr{I}_h\Big(\frac{u_h + rw_h}{|u_h + rw_h|}\Big) = \sum_{z\in\mathscr{N}_h} \frac{u_h(z) + rw_h(z)}{|u_h(z) + rw_h(z)|}\varphi_z.$$

For every $z \in \mathscr{N}_h$ a Taylor expansion at $r = 0$ shows that

$$u_h^r(z) = u_h(z) + rw_h(z) + r^2\xi_h(z)$$

for a function $\xi_h \in \mathscr{S}^1_{\mathrm{D}}(\mathscr{T}_h)^m$. Therefore, if u_h is stationary for I, we have

$$0 = \lim_{r\to 0} \frac{1}{r}\big(I(u_h^r) - I(u_h)\big) = (\nabla u_h, \nabla w_h).$$

Conversely, assume that $(\nabla u_h, \nabla w_h) = 0$ for all $w_h \in \mathscr{F}_h[u_h]$. If $(u_h^r)_{r\in(-\varepsilon,\varepsilon)}$ is a continuously differentiable path in $\mathscr{A}_h$ with $w_h^0 = u_h$, then we have

$$u_h^r = u_h + rw_h + \phi(r)\xi_h$$

with a vector field $\xi_h \in \mathscr{S}^1_{\mathrm{D}}(\mathscr{T}_h)^m$, a function ϕ such that $\phi(r)/r \to 0$ as $r \to 0$, and $w_h \in \mathscr{S}^1_{\mathrm{D}}(\mathscr{T}_h)^m$ defined by

$$w_h(z) = \frac{d}{dr}\Big|_{r=0} w_h^r(z).$$

Since $|u_h^r(z)|^2 = 1$ for every $z \in \mathscr{N}_h$ and $r \in (-\varepsilon, \varepsilon)$, we have $w_h(z)\cdot u_h(z) = 0$ for all $z \in \mathscr{N}_h$, i.e., $w_h \in \mathscr{F}_h[u_h]$. This implies

$$I(u_h^r) = I(u_h) + r(\nabla u_h, \nabla w_h) + \phi(r)(\nabla u_h, \nabla\xi_h) + I(rw_h + \phi(r)\xi_h)$$

and thus, using $(\nabla u_h, \nabla w_h) = 0$, we have $\big(I(u_h^r) - I(u_h)\big)/r \to 0$ as $r \to 0$, i.e., $r \mapsto I(u_h^r)$ is stationary at $r = 0$. □

The theorem motivates the following definition.

Definition 7.2 A function $u_h \in \mathscr{A}_h$ is called a *discrete harmonic map* if

$$(\nabla u_h, \nabla w_h) = 0$$

for all $w_h \in \mathscr{F}_h[u_h]$.

Remark 7.8 The space of admissible test functions $\mathscr{F}_h[u_h]$ may be regarded as the tangent space of $\mathscr{A}_h$ at u_h. In particular, a discrete harmonic map is stable with respect to discrete tangential perturbations.

The compactness result of Theorem 7.3 implies the convergence of discrete harmonic maps as $h \to 0$. For ease of presentation we again restrict to the case $m = 3$. The perturbation functionals $\mathscr{R}_h$ in the following theorem model an inexact solution of the discrete problems.

Theorem 7.6 (Discrete compactness) *Let $(u_h)_{h>0} \subset H^1(\Omega;\mathbb{R}^3)$ be a bounded sequence of almost discrete harmonic maps associated to the sequence $(\mathcal{T}_h)_{h>0}$, i.e., for every $h > 0$, we have $u_h \in \mathcal{A}_h$ and there exists $\mathcal{R}_h \in H^1_{\rm D}(\Omega;\mathbb{R}^3)'$ with*

$$(\nabla u_h, \nabla w_h) = \mathcal{R}_h(w_h)$$

for all $w_h \in \mathcal{F}_h[u_h]$. If $\mathcal{R}_h \to 0$ in $H^1_{\rm D}(\Omega;\mathbb{R}^m)'$ and $u_{{\rm D},h} \to u_{\rm D}$ in $L^2(\Gamma_{\rm D})$ as $h \to 0$, then every weak accumulation point of $(u_h)_{h>0}$ is a harmonic map.

Proof Let $u \in H^1(\Omega;\mathbb{R}^3)$ be a weak accumulation point of the sequence $(u_h)_{h>0}$ and without loss of generality, assume that the entire sequence converges weakly to u, i.e., $u_h \rightharpoonup u$ in $H^1(\Omega;\mathbb{R}^3)$ as $h \to 0$. Proposition 7.1 shows that $|u| = 1$ almost everywhere in Ω. Moreover, the weak continuity of the trace operator implies that $u|_{\Gamma_{\rm D}} = u_{\rm D}$. Given $\phi \in C^\infty(\overline{\Omega};\mathbb{R}^3) \cap H^1_{\rm D}(\Omega;\mathbb{R}^3)$, set $w_h = \mathcal{I}_h(u_h \times \phi)$. Then $w_h \in \mathcal{S}^1_{\rm D}(\mathcal{T}_h)^3$ with $w_h(z) \cdot u_h(z) = 0$ for all $z \in \mathcal{N}_h$. An element-wise nodal interpolation estimate and $D^2 u_h|_T = 0$ for every $T \in \mathcal{T}_h$ show that

$$\begin{aligned}\|\nabla(w_h - u_h \times \phi)\|_{L^2(T)} &\le c h_T \|D^2(u_h \times \phi)\|_{L^2(T)} \\ &\le c h_T \big(\|\nabla u_h\|_{L^2(T)} \|\nabla \phi\|_{L^\infty(T)} + \|u_h\|_{L^\infty(T)} \|\nabla \phi\|_{L^2(T)}\big).\end{aligned}$$

This implies that $\|\nabla w_h\| \le c$ and $w_h - u_h \times w \to 0$ in $H^1(\Omega;\mathbb{R}^3)$ as $h \to 0$. Therefore, we have

$$\mathcal{R}_h(w_h) = (\nabla u_h, \nabla w_h) = (\nabla u_h, \nabla[u_h \times \phi]) + (\nabla u_h, \nabla[w_h - u_h \times \phi])$$

with

$$(\nabla u_h, \nabla[w_h - u_h \times \phi]) \to 0$$

as $h \to 0$. For the other term on the right-hand side, we have

$$\sum_{\ell=1}^d (\partial_\ell u_h, \partial_\ell[u_h \times \phi]) = \sum_{\ell=1}^d (\partial_\ell u_h, u_h \times \partial_\ell \phi)$$

and since $u_h \to u$ in $L^2(\Omega;\mathbb{R}^3)$ and $\nabla u_h \rightharpoonup \nabla u$ in $L^2(\Omega;\mathbb{R}^{3\times 3})$ as $h \to 0$, we deduce that

$$0 = \lim_{h\to 0} (\nabla u_h, \nabla[u_h \times \phi]) = \sum_{\ell=1}^d (\partial_\ell u, u \times \partial_\ell \phi).$$

This proves that u is a harmonic map. □

7.2.2 Iterative Computation

The iterative computation of discrete harmonic maps is based on the computation of tangential corrections that define a new approximation after a node-wise projection onto the unit sphere. The following algorithm may be regarded as a discrete version of the H^1-flow for harmonic maps which is formally defined as

$$(\nabla \partial_t u, \nabla w) = -(\nabla u, \nabla w) + (|\nabla u|^2 u, w).$$

For w with $w \cdot u = 0$, the second term on the right-hand side disappears. Moreover, we have $\partial_t u \cdot u = 0$ if $|u(t,x)| = 1$ for almost every $(t,x) \in [0,T] \times \Omega$. We employ a semi-implicit discretization of this problem to compute approximations v_h^k of $\partial_t u(t_k)$ to find discrete harmonic maps with bounded energy. In particular, the linearized constraint will be treated explicitly, which leads to linear systems of equations in every time-step. The approach is illustrated in Fig. 7.3.

Algorithm 7.1 (*Discrete H^1-flow* [1]) Let $u_h^0 \in \mathscr{A}_h, \theta \in [0,1]$, and $\tau > 0$ and define the sequence $(u_h^k)_{k=0,1,\ldots} \subset \mathscr{A}_h$ by computing $v_h^k \in \mathscr{F}_h[u_h^{k-1}]$ such that

$$(\nabla v_h^k, \nabla w_h) + (\nabla [u_h^{k-1} + \theta \tau v_h^k], \nabla w_h) = 0$$

for all $w_h \in \mathscr{F}_h[u_h^{k-1}]$ and setting

$$u_h^k = \sum_{z \in \mathscr{N}_h} \frac{u_h^{k-1}(z) + \tau v_h^k(z)}{|u_h^{k-1}(z) + \tau v_h^k(z)|} \varphi_z$$

until $\|\nabla v_h^k\| \le \varepsilon_{\text{stop}}$.

Proposition 7.2 (Termination I) *Assume that $\mathscr{T}_h$ is weakly acute. The iterates $(u_h^k)_{k=0,1,\ldots} \subset \mathscr{A}_h$ of Algorithm 7.1 are well defined and satisfy*

$$\frac{1}{2}\|\nabla u_h^L\|^2 + (2 + 2\tau\theta - \tau)\frac{\tau}{2}\sum_{k=1}^{L}\|\nabla v_h^k\|^2 \le \frac{1}{2}\|\nabla u_h^0\|^2$$

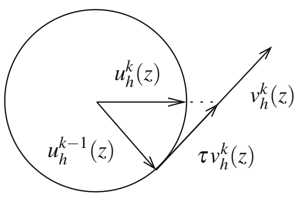

Fig. 7.3 The iteration of Algorithm 7.1 computes corrections v_h^k in the tangent space of the unit sphere at the current iterate u_h^{k-1} and then employs a projection onto the unit sphere to define the update u_h^k

for every $L \geq 1$. In particular, if $\tau(1-2\theta) \leq 2$, then the iteration terminates and the output $u_h^* \in \mathscr{A}_h$ satisfies

$$(\nabla u_h^*, \nabla w_h) = \mathscr{R}_h(w_h)$$

for all $w_h \in \mathscr{F}_h[u_h^*]$ and $\|\mathscr{R}_h\|_{H^1_D(\Omega;\mathbb{R}^m)'} \leq (1+\theta\tau)\varepsilon_{\rm stop}$.

Proof Given $u_h^{k-1} \in \mathscr{A}_h$, the space $\mathscr{F}_h[u_h^{k-1}]$ is a closed subspace of $\mathscr{S}^1_D(\mathscr{T}_h)^m$ and the Lax–Milgram lemma implies the existence of a uniquely defined $v_h^k \in \mathscr{F}_h[u_h^{k-1}]$ with

$$(\nabla v_h^k, \nabla w_h) + (\nabla[u_h^{k-1} + \theta\tau v_h^k], \nabla w_h) = 0$$

for all $w_h \in \mathscr{F}_h[u_h^{k-1}]$. Since $|u_h^{k-1}(z)| = 1$ and $v_h^k(z) \cdot u_h^{k-1}(z) = 0$ for all $z \in \mathscr{N}_h$, we have $|u_h^{k-1}(z) + \tau v_h^k(z)| \geq 1$ and $u_h^k \in \mathscr{A}_h$ is well defined. The mapping

$$F : s \mapsto \begin{cases} s/|s| & \text{if } |s| \geq 1, \\ s & \text{if } |s| \leq 1 \end{cases}$$

is Lipschitz continuous with $\|DF\|_{L^\infty(\mathbb{R}^m)} = 1$ so that Proposition 3.2 implies

$$\|\nabla u_h^k\| \leq \|\nabla(u_h^{k-1} + \tau v_h^k)\|.$$

The choice of $w_h = v_h^k$ in the equation of Algorithm 7.1 and the formula $2\tau(a + \theta\tau b)b = (a+\tau b)^2 - a^2 + \tau^2(2\theta-1)b^2$ show that

$$\|\nabla v_h^k\|^2 + \frac{1}{2\tau}\|\nabla(u_h^{k-1} + \tau v_h^k)\|^2 - \frac{1}{2\tau}\|\nabla u_h^{k-1}\|^2 + \frac{\tau}{2}(2\theta-1)\|\nabla v_h^k\|^2 = 0.$$

A combination with the bound for $\|\nabla u_h^k\|$ and a multiplication by τ, together with a summation over $k = 1, 2, \ldots, L$, imply

$$\frac{1}{2}\|\nabla u_h^L\|^2 + (2 + 2\tau\theta - \tau)\frac{\tau}{2}\sum_{k=1}^{L}\|\nabla v_h^k\|^2 \leq \frac{1}{2}\|\nabla u_h^0\|^2.$$

This yields that $\|\nabla v_h^K\| \leq \varepsilon$ for $K \geq 0$ sufficiently large and the functions $u_h^* = u_h^{K-1}$ and $v_h^* = v_h^K$ satisfy

$$(\nabla u_h^*, \nabla w_h) = -(1+\theta\tau)(\nabla v_h^*, \nabla w_h)$$

for all $w_h \in \mathscr{F}_h[u_h^*]$. Setting $\mathscr{R}_h(w) = -(1+\theta\tau)(\nabla v_h^*, \nabla w)$ for $w \in H^1_D(\Omega;\mathbb{R}^m)$ proves the assertion. □

Remarks 7.9 (i) The proof of the proposition shows that we have the local energy decay property $\|\nabla u_h^k\| \leq \|\nabla u_h^{k-1}\|$ for all $k \geq 1$.

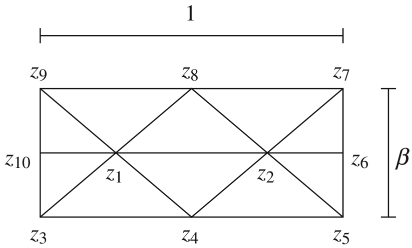

Fig. 7.4 A triangulation $\mathcal{T}_h$ that is weakly acute if and only if $\beta \geq 1/2$

(ii) Note that for all choices of θ the large step size $\tau = 1$ leads to a stable and convergent iterative scheme.

The acuteness property is necessary in general to guarantee that the projection step is stable in the sense that $\|\nabla u_h^k\| \leq \|\nabla[u_h^{k-1} + \tau v_h^k]\|$.

Proposition 7.3 (Necessity of acuteness) *For $\beta > 0$, let $\mathcal{T}_h$ be the triangulation of $\Omega = (0,1) \times (0,\beta)$ shown in Fig. 7.4, and let $\tau > 0$. Let $u_h \in \mathcal{S}^1(\mathcal{T}_h)^m$ and $v_h \in \mathcal{F}_h[u_h]$, be defined by $u_h(z_j) = e_1$ and $v_h(z_j) = 0$ for $j = 3, 4, \ldots, 10$, and*

$$\begin{aligned} u_h(z_1) &= e_1, & u_h(z_2) &= -e_1, \\ v_h(z_1) &= -(s/\tau)e_2, & v_h(z_2) &= 0, \end{aligned}$$

where $s = 1/2 - \beta$ and e_ℓ denotes the ℓ-th canonical basis vector in $\mathbb{R}^m$. Then for $P\widetilde{u}_h \in \mathcal{S}^1(\mathcal{T}_h)^m$ defined with $\widetilde{u}_h = u_h + \tau v_h$ by

$$P\widetilde{u}_h(z) = \frac{\widetilde{u}_h(z)}{|\widetilde{u}_h(z)|}$$

for all $z \in \mathcal{N}_h$, we have $\|\nabla P\widetilde{u}_h\| \leq \|\nabla \widetilde{u}_h\|$ if and only if $\mathcal{T}_h$ is weakly acute, i.e., if and only if $\beta \geq 1/2$.

Proof Since $|\widetilde{u}_h(z)| \geq 1$ for all $z \in \mathcal{N}_h$, Proposition 3.2 implies that $\|\nabla P\widetilde{u}_h\| \leq \|\nabla \widetilde{u}_h\|$ if $\mathcal{T}_h$ is weakly acute and this is the case if and only if $\beta \geq 1/2$. Suppose that $\beta < 1/2$. Then with the entries A_{jk}, $j, k = 1, 2, \ldots, 10$, of the stiffness matrix and the identity $\widetilde{u}_h(z_j) = P\widetilde{u}_h(z_j)$ for $j = 2, 3, 4, \ldots, 10$, the representation of $\|\nabla w_h\|^2$ in terms of the nodal values of w_h and the entries of A, cf. the proof of Proposition 3.2, we have that

$$\begin{aligned} \delta^2 = \|\nabla \widetilde{u}_h\|^2 - \|\nabla P\widetilde{u}_h\|^2 &= -\frac{1}{2}\sum_{j,k=1}^{10} A_{jk}\big(|\widetilde{u}_h(z_j) - \widetilde{u}_h(z_k)|^2 \\ &\qquad - |P\widetilde{u}_h(z_j) - P\widetilde{u}_h(z_k)|^2\big) \\ &= -\sum_{j=2}^{10} A_{1j}\big(|\widetilde{u}_h(z_j) - \widetilde{u}_h(z_1)|^2 \end{aligned}$$

$$- |P\widetilde{u}_h(z_j) - P\widetilde{u}_h(z_1)|^2\big).$$

We have $|\widetilde{u}_h(z_1) - \widetilde{u}_h(z_2)|^2 = 4 + s^2$ and $|\widetilde{u}_h(z_j) - \widetilde{u}_h(z_1)|^2 = s^2$ and

$$t_1^2 = |P\widetilde{u}_h(z_1) - P\widetilde{u}_h(z_2)|^2 = 2 + 2/(1+s^2)^{1/2},$$
$$t_2^2 = |P\widetilde{u}_h(z_j) - P\widetilde{u}_h(z_2)|^2 = 2 - 2/(1+s^2)^{1/2}$$

for $j = 3, 4, \ldots, 10$. Since $\sum_{j=1}^{10} A_{1j} = 0$ we have $\sum_{j=3}^{10} A_{1j} = -A_{11} - A_{22}$ and hence

$$\begin{aligned}\delta^2 &= (s^2 - t_2^2)(A_{11} + A_{12}) - A_{12}(4 + s^2 - t_1^2)\\ &= A_{11}(s^2 - t_2^2) - A_{12}(4 + t_2^2 - t_1^2).\end{aligned}$$

Direct calculations show that

$$A_{11} = (12\beta^2 + 5)/(4\beta), \quad A_{12} = (1 - 4\beta^2)/(4\beta).$$

With $\phi(s) = (1+s^2)^{1/2} - 1 - s^2/2$ and $\beta^2 = 1/4 - s + s^2$ we verify that

$$\begin{aligned}4\beta(1+s^2)^{1/2}\delta^2 &= \big(12\beta^2 + 5\big)\big(s^4/2 + s^2\phi(s) - 2\phi(s)\big) - \big(1 - 4\beta^2\big)\big(2s^2 + 4\phi(s)\big)\\ &= \big(8 - 12s + 12s^2\big)\big(s^4/2 + s^2\phi(s) - 2\phi(s)\big)\\ &\quad - 16(s - s^2)\big(s^2/2 + \phi(s)\big)\\ &= -8s^3 + 12s^4 - 6s^5 + 6s^6 + \phi(s)\big(-16s - 12s^3 + 12s^4\big)\\ &= -6s^3(1 - 2s) - 6s^5(1 - s) + 4s\phi(s)\big(2 - 3s^2 + 3s^3\big)\\ &\quad - 2\big(s^3 + 8\phi(s)\big).\end{aligned}$$

Since $0 < s < 1/2$ and $\phi(s) < 0$, the first three terms on the right-hand side are negative. The estimate $-s^4/8 \leq \phi(s)$ implies that the last term on the right-hand side is nonpositive. This shows $\delta < 0$ if $\beta < 1/2$ and proves the assertion. □

7.2.3 *Projection-Free Iteration*

The acuteness condition of Proposition 7.2 is restrictive if $d = 3$ but allows for large step sizes. In the continuous situation we have that the identity $u \cdot \partial_t u = 0$ implies that the initial length is preserved. In the discrete setting a semi-implicit discretization of this orthogonality leads to approximations that violate the constraint when the projection step is omitted, cf. Fig. 7.5.

Algorithm 7.2 (H^1*-flow without projection*) Let $u_h^0 \in \mathscr{A}_h$, $\tau > 0$, and define the sequence $(u_h^k)_{k=0,1,\ldots} \subset \mathscr{S}^1(\mathscr{T}_h)^m$ by computing $v_h^k \in \mathscr{F}_h[u_h^{k-1}]$ such that

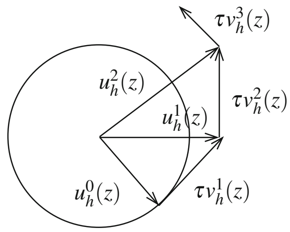

Fig. 7.5 Omitting the projection step in the semi-implicit H^1-flow leads to approximations that violate the unit-length constraint; the corresponding error in $L^1(\Omega)$ is independent of the number of iterations and controlled by the step size

$$(\nabla v_h^k, \nabla w_h) + (\nabla[u_h^{k-1} + \tau v_h^k], \nabla w_h) = 0$$

for all $w_h \in \mathscr{F}_h[u_h^{k-1}]$ and setting

$$u_h^k = u_h^{k-1} + \tau v_h^k$$

until $\|\nabla v_h^k\| \leq \varepsilon_{\mathrm{stop}}$.

The following proposition shows that the violation of the constraint is independent of the number of iterations and controlled by the step size.

Proposition 7.4 (Termination II) *The iterates* $(u_h^k)_{k=0,1,\ldots} \subset \mathscr{S}^1(\mathscr{T}_h)^m$ *of Algorithm 7.2 satisfy* $u_h^k|_{\Gamma_\mathrm{D}} = u_{\mathrm{D},h}$ *for* $k = 0, 1, \ldots$ *and*

$$\frac{1}{2}\|\nabla u_h^L\|^2 + (2+\tau)\frac{\tau}{2}\sum_{k=1}^{L}\|\nabla v_h^k\|^2 = \frac{1}{2}\|\nabla u_h^0\|^2$$

for every $L \geq 1$. *Moreover, we have every* $L \geq 1$ *that*

$$\left\|\mathscr{I}_h\big[|u_h^L|^2\big] - 1\right\|_{L^1(\Omega)} \leq c\tau\|\nabla u_h^0\|^2.$$

Proof Due to the Lax–Milgram lemma the iteration is well-defined and the choice $w_h^k = v_h^{k+1}$ shows, using the formula $2\tau(a+\tau b)b = (a+\tau b)^2 - a^2 + \tau^2 b^2$, that

$$\frac{2+\tau}{2}\|\nabla v_h^k\|^2 + \frac{1}{2\tau}\|\nabla u_h^k\|^2 - \frac{1}{2\tau}\|\nabla u_h^{k-1}\|^2 = 0$$

which implies the first asserted estimate. For every $z \in \mathscr{N}_h$, we have

$$|u_h^k(z)|^2 - 1 = |u_h^{k-1}(z)|^2 + \tau^2|v_h^k(z)|^2 - 1$$

and inductively with $|u_h^0(z)| = 1$, we find that

$$|u_h^L(z)|^2 - 1 = \tau^2\sum_{k=1}^{L}|v_h^k(z)|^2.$$

The discrete norm equivalences of Lemma 3.4 yield

$$(1/c)\big\|\mathscr{I}_h\big[|u_h^L|^2\big]-1\big\|_{L^1(\Omega)} \le \sum_{z\in\mathcal{N}_h} h_z^d\big||u_h^L(z)|^2-1\big| \\ \le \tau^2\sum_{k=1}^L\sum_{z\in\mathcal{N}_h} h_z^d|v_h^k(z)|^2 \le c\tau^2\sum_{k=1}^L\|v_h^k\|^2.$$

Poincaré's inequality and the first estimate of the proposition imply

$$\big\|\mathscr{I}_h\big[|u_h^L|^2\big]-1\big\|_{L^1(\Omega)} \le c\tau^2\sum_{k=1}^L\|\nabla v_h^k\|^2 \le c\tau\|\nabla u_h^0\|^2,$$

which proves the proposition. □

We conclude the discussion with a lemma which shows that the approximate treatment of the constraint at the nodes implies that it is satisfied by accumulation points in the limit $(h,\tau)\to 0$.

Lemma 7.2 (Constraint approximation) *If $(u_h)_{h>0}$ is a bounded sequence in $H^1(\Omega;\mathbb{R}^m)$ such that $u_h\in\mathscr{S}^1(\mathscr{T}_h)^m$ for all $h>0$, $u_h\to u$ in $L^2(\Omega;\mathbb{R}^m)$ for some $u\in H^1(\Omega;\mathbb{R}^m)$ as $h\to 0$, and*

$$\|\mathscr{I}_h\big[|u_h|^2\big]-1\|_{L^1(\Omega)}\to 0$$

as $h\to 0$, then we have $|u|^2=1$ almost everywhere in Ω.

Proof Two applications of the triangle inequality show that

$$\||u|^2-1\|_{L^1(\Omega)} \\ \le \||u|^2-|u_h|^2\|_{L^1(\Omega)}+\||u_h|^2-\mathscr{I}_h\big[|u_h|^2\big]\|_{L^1(\Omega)}+\|\mathscr{I}_h\big[|u_h|^2\big]-1\|_{L^1(\Omega)}.$$

Due to the assumptions of the lemma we have that the third term on the right-hand side tends to zero as $h\to 0$. Since

$$\||u|^2-|u_h|^2\|_{L^1(\Omega)}\le\|u-u_h\|\|u+u_h\|$$

we have that also the first term on the right-hand side vanishes as $h\to 0$. We use Hölder's inequality and a nodal interpolation estimate to verify that for every $T\in\mathscr{T}_h$, we have

$$\||u_h|^2-\mathscr{I}_h\big[|u_h|^2\big]\|_{L^1(T)} \le ch_T^{d/2}\||u_h|^2-\mathscr{I}_h\big[|u_h|\big]\|_{L^2(T)} \\ \le ch_T^{d/2}h_T^2\|D^2|u_h|^2\|_{L^2(T)} \le ch_T^2\|\nabla u_h\|_{L^2(T)}^2.$$

With a summation over $T \in \mathscr{T}_h$ we deduce for the second term that

$$\| |u_h|^2 - \mathscr{I}_h\big[|u_h|^2\big]\|_{L^1(\Omega)} \le ch^2 \|\nabla u_h\|^2.$$

Since the upper bound vanishes as $h \to 0$, this implies that $|u|^2 = 1$. □

7.2.4 Other Target Manifolds

The ideas outlined above can be generalized to approximate harmonic maps into target manifolds other than the unit sphere. We let $\mathscr{M} \subset \mathbb{R}^m$ be an $(m-1)$-dimensional C^2-submanifold and let $T_p\mathscr{M}$ denote the tangent space at $p \in \mathscr{M}$. Moreover, we let $\pi_{\mathscr{M}} : U_\delta(\mathscr{M}) \to \mathscr{M}$ be the nearest neighbor projection onto $\mathscr{M}$ which is uniquely defined in a neighborhood $U_\delta(\mathscr{M}) = \{q \in \mathbb{R}^m : \operatorname{dist}(p, \mathscr{M}) < \delta\}$ of $\mathscr{M}$ for some $\delta > 0$. The function $\pi_{\mathscr{M}}$ satisfies $|\pi_{\mathscr{M}}(q) - q| = \inf_{p\in\mathscr{M}} |p-q|$ for all $q \in U_\delta(\mathscr{M})$. If $\mathscr{M} = \partial\mathscr{C}$ for a convex set $\mathscr{C} \subset \mathbb{R}^m$, then $\pi_{\mathscr{M}}$ is well defined in $\mathbb{R}^m \setminus \mathscr{C}$.

Definition 7.3 Given $\widetilde{u}_{\mathrm{D},h} \in \mathscr{S}^1(\mathscr{T}_h)^m$ with $\widetilde{u}_{\mathrm{D},h}(z) \in \mathscr{M}$ for all $z \in \mathscr{N}_h$ set

$$\mathscr{A}_h = \{u_h \in \mathscr{S}^1(\mathscr{T}_h)^m : u_h|_{\Gamma_\mathrm{D}} = \widetilde{u}_{\mathrm{D},h}|_{\Gamma_\mathrm{D}} \text{ and } u_h(z) \in \mathscr{M} \text{ for all } z \in \mathscr{N}_h\big\}$$

and for $u_h \in \mathscr{A}_h$, let

$$\mathscr{F}_h[u_h] = \big\{v_h \in \mathscr{S}^1_\mathrm{D}(\mathscr{T}_h)^m : v_h(z) \in T_{u_h(z)}\mathscr{M} \text{ for all } z \in \mathscr{N}_h\big\}.$$

With these definitions we can define the following generalization of Algorithm 7.1.

Algorithm 7.3 (H^1*-flow for general target manifolds*) Let $u_h^0 \in \mathscr{A}_h$ and $\tau > 0$ and define the sequence $(u_h^k)_{k=0,1,\ldots} \in \mathscr{A}_h$ by computing $v_h^k \in \mathscr{F}_h[u_h^{k-1}]$ such that

$$(\nabla v_h^k, \nabla w_h) + (\nabla[u_h^{k-1} + \tau v_h^k], \nabla w_h) = 0$$

for all $w_h \in \mathscr{F}_h[u_h^{k-1}]$ and setting

$$u_h^k = \sum_{z\in\mathscr{N}_h} \pi_{\mathscr{M}}\big(u_h^{k-1}(z) + \tau v_h^k(z)\big)\varphi_z$$

until $\|\nabla v_h^k\| \le \varepsilon_{\mathrm{stop}}$.

Remarks 7.10 (i) Well-posedness of the algorithm requires that τ be sufficiently small so that $u_h^{k-1}(z) + \tau v_h^k(z) \in U_\delta(\mathscr{M})$ for all $z \in \mathscr{N}_h$, cf. Fig. 7.6. If $\mathscr{M} = \partial\mathscr{C}$ for a convex set $\mathscr{C}$, then this is always satisfied.
(ii) A stability proof employs an expansion of $\pi_{\mathscr{M}}$ and the fact that $D\pi_{\mathscr{M}}(s)|_{T_s\mathscr{M}} = \mathrm{id}_{T_s\mathscr{M}}$ provided that $\mathscr{M}$ is a C^3-submanifold.

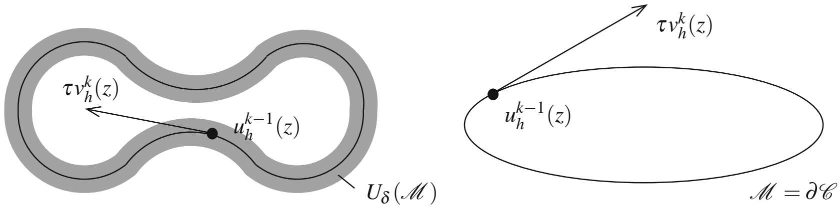

Fig. 7.6 The projection of $u_h^{k-1}(z) + \tau v_h^k(z)$ onto the target manifold is in general only well defined within a tubular neighborhood of $\mathscr{M}$ in the case of nonconvex manifolds and a step-size restriction needs to be imposed (*left*); for boundaries of convex sets no restriction on the step size is required (*right*)

(iii) The projection step can be omitted if an appropriate version of a shifted tangent space is available, e.g., if $\mathscr{M} = g^{-1}(\{0\})$ for an appropriate function $g : \mathbb{R}^m \to \mathbb{R}$.

7.2.5 Practical Realization

The implementation of Algorithm 7.1 requires working with discrete vector fields $u_h \in \mathscr{S}^1(\mathscr{T}_h)^m$ which are given by

$$u_h = \sum_{z \in \mathscr{N}_h} u_z \varphi_z$$

with coefficients $u_z = u_h(z) \in \mathbb{R}^m$ for all $z \in \mathscr{N}_h$. The function u_h will be identified with the vector $U \in \mathbb{R}^{mL}$ defined by

$$U = \begin{bmatrix} u_{z_1} \\ u_{z_2} \\ \vdots \\ u_{z_L} \end{bmatrix} \in \mathbb{R}^{mL}$$

with $L = \#\mathscr{N}_h$. The constraint $u_h(z) \cdot v_h(z) = 0$ for all $z \in \mathscr{N}_h$ for a vector field $v_h \in \mathscr{S}^1(\mathscr{T}_h)^m$ is then equivalently imposed by $B_U V = 0$ with the matrix $B_U \in \mathbb{R}^{L \times L}$ defined through

$$B_U = \begin{bmatrix} u_{z_1}^\top & 0 & & \\ 0 & u_{z_2}^\top & 0 & \\ & 0 & \ddots & 0 \\ & & 0 & u_{z_L}^\top \end{bmatrix}$$

so that $B_U V = [u_{z_1} \cdot v_{z_1}, u_{z_2} \cdot v_{z_2}, \ldots, u_{z_L} \cdot v_{z_L}]^\top$. The solution of the linearly constrained linear problems is based on the fact that we have

$$B_U V = 0, \quad W^\top S_m V = W^\top b \text{ for all } W \in \ker B$$

if and only if there exists $\Lambda \in \mathbb{R}^L$ such that

$$\begin{bmatrix} S_m & B_U^\top \\ B_U & 0 \end{bmatrix} \begin{bmatrix} V \\ \Lambda \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix},$$

where S_m is the $P1$ finite element stiffness matrix for vector fields with m components. A MATLAB implementation is shown in Fig. 7.7.

7.3 Approximation of Constrained Evolution Problems

The iterative schemes discussed above are discrete H^1-gradient flows for harmonic maps and can be modified to provide approximations of the L^2-gradient flow of harmonic maps. We show that this leads to convergent approximations of the harmonic map heat flow. In addition to this we analyze discretizations that preserve the constraint without an explicit correction of the iterates. We also discuss the application of the developed techniques to a hyperbolic problem. The presentation is based on results from [2, 4, 7].

7.3.1 Harmonic Map Heat Flow

The harmonic map heat flow is the L^2-gradient flow for the Dirichlet energy that is constrained to unit-length vector fields. In the strong form it seeks a function $u : [0, T] \times \Omega \to \mathbb{R}^m$ such that $|u| = 1$ in $[0, T] \times \Omega$ and

$$\partial_t u - \Delta u = |\nabla u|^2 u, \quad u|_{\Gamma_D} = u_D, \quad \partial_n u|_{\Gamma_N} = 0, \quad u(0) = u_0,$$

where Γ_D may be empty. The following proposition provides useful equivalent characterizations for the practically relevant case $m = 3$.

Proposition 7.5 (Equivalent formulations) *The following formulations are equivalent for a function $u \in H^1([0, T]; L^2(\Omega; \mathbb{R}^3)) \cap L^\infty([0, T]; H^1(\Omega; \mathbb{R}^3))$ satisfying $|u(t, x)| = 1$ for almost every $(t, x) \in [0, T] \times \Omega$:*
(i) *For almost every $t \in [0, T]$ and every $w \in H^1_D(\Omega; \mathbb{R}^3) \cap L^\infty(\Omega; \mathbb{R}^3)$, we have*

$$(\partial_t u, w) + (\nabla u, \nabla w) = (|\nabla u|^2 u, w).$$

```
function h1_flow_hm(d,red)
[c4n,n4e,Db,Nb] = triang_cube(d); Db = [Db;Nb]; c4n = c4n-.5;
for j = 1:red
    [c4n,n4e,Db,Nb,¬,¬] = red_refine(c4n,n4e,Db,Nb);
end
theta = 1/2; tau = 2; eps_stop = 1e-4; nC = size(c4n,1);
dNodes = unique(Db); fNodes = setdiff(1:nC,dNodes);
FNodes = [3*fNodes-2,3*fNodes-1,3*fNodes-0];
nDb = size(dNodes,1); nF = size(fNodes,2);
[s,¬,¬,¬] = fe_matrices(c4n,n4e); SSS = sparse(3*nC,3*nC);
for k = 1:3
    idx = k:3:3*nC; SSS(idx,idx) = s;
end
u = zeros(3*nC,1);
for j = 1:nC
    u(3*j-[2,1,0]) = u_0(c4n(j,:));
end
for j = 1:nDb
    u(3*dNodes(j)-[2,1,0]) = u_D(c4n(dNodes(j),:));
end
Flist = [FNodes,3*nC+fNodes];
norm_corr = 1;
while norm_corr > eps_stop
    B = sparse(nC,3*nC);
    for j = 1:nC
        B(j,3*j-[2,1,0]) = u(3*j-[2,1,0])';
    end
    X = [SSS,B';B,sparse(nC,nC)];
    b = [-(1+theta*tau)^(-1)*SSS*u;zeros(nC,1)];
    x = X(Flist,Flist)\b(Flist);
    v = zeros(3*nC,1);
    v(FNodes) = x(1:3*nF); tu = u+tau*v;
    norm_corr = sqrt(v'*SSS*v);
    for j = 1:nC
        u(3*j-[2,1,0]) = tu(3*j-[2,1,0])/norm(tu(3*j-[2,1,0]));
    end
    % u = tu;
    show_p1_field(c4n,u); axis square; view(30,30); drawnow;
end

function val = u_D(x)
val = [x/norm(x),zeros(1,3-size(x,2))];

function val = u_0(x)
val_tmp = rand(1,3)-.5;
val = val_tmp/norm(val_tmp);

function show_p1_field(c4n,u)
[nC,d] = size(c4n); X = [c4n,zeros(nC,3-d)];
quiver3(X(:,1),X(:,2),X(:,3),u(1:3:3*nC),u(2:3:3*nC),u(3:3:3*nC));
```

Fig. 7.7 Iterative approximation of harmonic maps into the sphere S^2 incorporating a projection step which can be deactivated by uncommenting the command `u = tu;`

(ii) *For almost every $t \in [0, T]$ and every $w \in H^1_{\rm D}(\Omega; \mathbb{R}^3)$ with $w(x) \cdot u(t, x) = 0$ for almost every $x \in \Omega$, we have*

$$(\partial_t u, w) + (\nabla u, \nabla w) = 0.$$

(iii) *For almost every $t \in [0, T]$ and every $\phi \in H^1_{\rm D}(\Omega; \mathbb{R}^3) \cap L^\infty(\Omega; \mathbb{R}^3)$, we have*

$$(\partial_t u, \phi) - (\nabla u, \nabla[u \times (u \times \phi)]) = 0.$$

Proof The proof is similar to the proof of Lemma 7.1. Assume that formulation (i) is satisfied. If $w(x) \cdot u(x, t) = 0$, then the right-hand side vanishes and a truncation argument shows that formulation (ii) holds. Using the identity $w = u \times (u \times \phi) = u(u \cdot \phi) - \phi$ implies the equivalence of (i) and (iii). Finally, (iii) follows from choosing $w = u \times (u \times \phi)$ in (ii) and noting that $\partial_t u \cdot u = 0$. □

Remark 7.11 The equivalence of (i) and (ii) can also be established for functions with values in $\mathbb{R}^m$ with $m \neq 3$.

7.3.2 Semi-implicit, Linear Schemes

The L^2-flow of harmonic maps can be approximated by replacing the H^1-inner product in Algorithm 7.1 by the L^2-inner product. As in that algorithm, the projection step can be omitted leading to a violation of the unit length constraint that is controlled by the step size independently of the number of iterations or time steps. As above we denote

$$\mathscr{A}_h = \big\{ v_h \in \mathscr{S}^1(\mathscr{T}_h)^m : |v_h(z)| = 1 \text{ for all } z \in \mathscr{N}_h,\ v_h|_{\Gamma_{\rm D}} = u_{{\rm D},h} \big\}$$

and given any $u_h \in \mathscr{S}^1(\mathscr{T}_h)^m$, we denote

$$\mathscr{F}_h[u_h] = \big\{ v_h \in \mathscr{S}^1_{\rm D}(\mathscr{T}_h)^m : v_h(z) \cdot u_h(z) = 0 \text{ for all } z \in \mathscr{N}_h \big\}.$$

Here and throughout the following the set $\Gamma_{\rm D}$ may be empty.

Algorithm 7.4 (*Discrete L^2-flow with optional projection*) Let $u_h^0 \in \mathscr{A}_h$, $\theta \in [0,1]$, and $\tau > 0$ and define the sequence $(u_h^k)_{k=0,\dots,K} \subset \mathscr{S}^1(\mathscr{T}_h)^m$ for $K = \lceil T/\tau \rceil$ by computing for $k = 1, 2, \dots, K$ the function $v_h^k \in \mathscr{F}_h[u_h^{k-1}]$ such that

$$(v_h^k, w_h) + (\nabla[u_h^{k-1} + \theta\tau v_h^k], \nabla w_h) = 0$$

for all $w_h \in \mathscr{F}_h[u_h^{k-1}]$ and setting $\widetilde{u}_h^k = u_h^{k-1} + \tau v_h^k$ and

$$u_h^k = \widetilde{u}_h^k \qquad \text{or} \qquad u_h^k = \sum_{z\in\mathscr{N}_h} \frac{\widetilde{u}_h^k(z)}{|\widetilde{u}_h^k(z)|}\varphi_z.$$

We discuss the stability properties of the algorithm for the case $\theta = 1$.

Proposition 7.6 (Stability) *Let* $(u_h^k)_{k=0,\dots,K} \subset \mathscr{S}^1(\mathscr{T}_h)^m$ *be the iterates of Algorithm 7.4 for* $\theta = 1$.
(i) *If the projection is omitted, then we have* $v_h^k = d_t u_h^k$ *for* $k = 1, 2, \dots, K$ *and for* $L = 1, 2, \dots, K$

$$\frac{1}{2}\|\nabla u_h^L\|^2 + \tau \sum_{k=1}^{L} \Big(\frac{\tau}{2}\|\nabla d_t u_h^k\|^2 + \|d_t u_h^k\|^2\Big) = \frac{1}{2}\|\nabla u_h^0\|^2,$$
$$\big\|\mathscr{I}_h\big[|u_h^L|^2\big] - 1\big\|_{L^1(\Omega)} \le c_0\tau.$$

(ii) *If the projection step is included and if* $\mathscr{T}_h$ *is weakly acute, then* $u_h^k \in \mathscr{A}_h$ *for* $k = 0, 1, \dots, K$ *and for every* $L = 1, 2, \dots, K$, *we have*

$$\frac{1}{2}\|\nabla u_h^L\|^2 + \tau \sum_{k=1}^{L} \Big(\frac{\tau}{2}\|\nabla v_h^k\|^2 + \|v_h^k\|^2\Big) \le \frac{1}{2}\|\nabla u_h^0\|^2,$$
$$\tau \sum_{k=1}^{L} \|v_h^k - d_t u_h^k\|_{L^1(\Omega)} \le c_0\tau.$$

Proof The well-posedness of Algorithm 7.4 follows as in the case of Algorithm 7.1 with the help of the Lax–Milgram lemma and the fact that $|\widetilde{u}_h^k(z)| \ge 1$ for all $k \ge 1$ and $z \in \mathscr{N}_h$.
(i) Assume that the projection step in Algorithm 7.4 is omitted. We then have $v_h^k = d_t u_h^k$ and the choice of $w_h = d_t u_h^k$ yields

$$\|d_t u_h^k\|^2 + \frac{d_t}{2}\|\nabla u_h^k\|^2 + \frac{\tau}{2}\|\nabla d_t u_h^k\|^2 = 0.$$

A summation over $k = 1, 2, \dots, L$ and multiplication by τ prove the stability estimate. For all $z \in \mathscr{N}_h$ and $k = 1, 2, \dots, L$, we have

$$|u_h^k(z)|^2 = |u_h^{k-1}(z) + \tau d_t u_h^k(z)|^2 = |u_h^{k-1}(z)|^2 + \tau^2 |d_t u_h^k(z)|^2$$

and inductively it follows with $|u_h^0(z)| = 1$ that

$$|u_h^L(z)|^2 - 1 = \tau^2 \sum_{k=1}^{L} |d_t u_h^k(z)|^2.$$

Multiplication by h_z^d the norm equivalences of Lemma 3.4, and the stability estimate imply, as in the proof of Proposition 7.4, that

$$\big\| \mathscr{I}_h\big[|u_h^L|^2\big] - 1 \big\|_{L^1(\Omega)} \le c\tau^2 \sum_{k=1}^{L} \|d_t u_h^k\|^2 \le c\tau \|\nabla u_h^0\|^2.$$

(ii) If the projection step is included, then the choice of $w_h = v_h^k$ shows that

$$\|v_h^k\|^2 + \frac{1}{2\tau}\big(\|\nabla(u_h^{k-1} + \tau v_h^k)\|^2 - \|\nabla u_h^{k-1}\|^2\big) + \frac{\tau}{2}\|v_h^k\|^2 = 0.$$

Since $\mathscr{T}_h$ is weakly acute and $u_h^k(z) = F\big(\widetilde{u}_h^k(z)\big)$ for all $z \in \mathscr{N}_h$ with the Lipschitz continuous mapping $F(s) = s/|s|$ for $|s| \ge 1$ and $F(s) = s$ otherwise, Proposition 3.2 implies as in the proof of Proposition 7.2 that

$$\|\nabla u_h^k\| \le \|\nabla[u_h^{k-1} + \tau v_h^k]\|.$$

With this, a summation over $k = 1, 2, \dots, L$, and a multiplication by τ, the previous identity implies the asserted stability estimate. To prove the estimate for the difference $v_h^k - d_t u_h^k$, let $z \in \mathscr{N}_h$. Then

$$\tau\big(d_t u_h^k(z) - v_h^k(z)\big) = u_h^k(z) - \big(u_h^{k-1}(z) + \tau v_h^k(z)\big) = \frac{\widetilde{u}_h^k(z)}{|\widetilde{u}_h^k(z)|} - \widetilde{u}_h^k(z).$$

With the identity

$$\Big|s - \frac{s}{|s|}\Big| = \Big|\frac{s}{|s|}\Big|\,\big||s| - 1\big| = \big||s| - 1\big|$$

for every $s \in \mathbb{R}^m$, it follows that

$$\tau\big|d_t u_h^k(z) - v_h^k(z)\big| = \big||\widetilde{u}_h^k(z)| - 1\big| = \big||u_h^{k-1}(z) + \tau v_h^k(z)| - 1\big|.$$

The relations $u_h^{k-1}(z) \cdot v_h^k(z) = 0$ and $|u_h^{k-1}(z)| = 1$ and the estimate $(1 + s^2)^{1/2} \le 1 + s^2/2$ imply that

$$|u_h^{k-1}(z) + \tau v_h^k(z)| = \big(1 + \tau^2 |v_h^k(z)|^2\big)^{1/2} \le 1 + \tau^2 |v_h^k(z)|^2/2.$$

A combination of the estimates and a summation over $z \in \mathscr{N}_h$ yield

$$\sum_{z \in \mathscr{N}_h} h_z^d |d_t u_h^k(z) - v_h^k(z)| \le \tau \sum_{z \in \mathscr{N}_h} h_z^d |v_h^k(z)|^2.$$

Norm equivalences and the stability result imply the asserted estimate. □

Remark 7.12 Under the conditions of Proposition 7.6 we have the local energy decay property $\|\nabla u_h^k\| \le \|\nabla u_h^{k-1}\|$ for all $k \ge 1$.

The stability estimates provide a priori bounds for the numerical approximations which allow us to pass to the limits for appropriate interpolants. Given the iterates $(u_h^k)_{k=0,\dots,K}$ of Algorithm 7.4 we define the interpolants $\widehat{u}_{h,\tau} : [0,T] \times \Omega \to \mathbb{R}^m$, $u_{h,\tau}^{\pm} : [0,T] \times \Omega \to \mathbb{R}^m$ and $v_{h,\tau}^{-} : [0,T] \times \Omega \to \mathbb{R}^m$ for $t \in (t_{k-1}, t_k)$ with $t_k = k\tau$ and $x \in \Omega$ by

$$\begin{aligned}
\widehat{u}_{h,\tau}(t,x) &= \frac{t_k - t}{\tau} u_h^{k-1}(x) + \frac{t - t_{k-1}}{\tau} u_h^k(x), \\
u_{h,\tau}^{-}(t,x) &= u_h^{k-1}(x), \quad u_{h,\tau}^{+}(t,x) = u_h^k(x), \\
v_{h,\tau}^{+}(t,x) &= v_h^k(x).
\end{aligned}$$

For ease of presentation, we again restrict the presentation to the case $m = 3$.

Theorem 7.7 (Convergence) *Suppose that $\Gamma_{\mathrm{D}} = \emptyset$, $u_h^0 \to u_0$ in $H^1(\Omega;\mathbb{R}^3)$ as $h \to 0$, and that $\mathscr{T}_h$ is weakly acute for every $h > 0$ if the projection step is carried out. Then every accumulation point of the sequence $(u_{h,\tau}^{+})_{h,\tau>0}$ in $L^\infty([0,T]; H^1(\Omega;\mathbb{R}^3))$ as $(h,\tau) \to 0$ is a weak solution of the harmonic map heat flow.*

Proof Step 1: *Selection of a weak limit.* The stability bounds of Proposition 7.6 imply that the sequences $(u_{h,\tau}^{+})_{h,\tau>0}$ and $(v_{h,\tau}^{+})_{h,\tau>0}$ are uniformly bounded in the spaces $L^\infty([0,T]; H^1(\Omega;\mathbb{R}^3))$ and $L^2([0,T]; L^2(\Omega;\mathbb{R}^3))$, respectively, so that after the extraction of a subsequence which is not relabeled, we have the existence of $u \in L^\infty([0,T]; H^1(\Omega;\mathbb{R}^3))$ and $v \in L^2([0,T]; L^2(\Omega;\mathbb{R}^3))$ with

$$\begin{aligned}
u_{h,\tau}^{\pm} &\rightharpoonup^* u \quad \text{in } L^\infty([0,T]; H^1(\Omega;\mathbb{R}^3)), \\
v_{h,\tau}^{+} &\rightharpoonup v \quad \text{in } L^2([0,T]; L^2(\Omega;\mathbb{R}^3))
\end{aligned}$$

as $(h,\tau) \to 0$. Since $v_{h,\tau}^{+} - \partial_t \widehat{u}_{h,\tau} \to 0$ in $L^2([0,T]; L^1(\Omega;\mathbb{R}^3))$ as $\tau \to 0$ we deduce that $u \in H^1([0,T]; L^2(\Omega;\mathbb{R}^3))$ and $v = \partial_t u$.

Step 2: *Verification of the energy law.* From the stability bounds we have for almost every $T' \in [0,T]$ up to a subsequence that $\nabla u_{h,\tau}^{+}(T',\cdot) \rightharpoonup \nabla u(T',\cdot)$. The weak lower semicontinuity of norms induced by inner products shows that

$$\frac{1}{2}\|\nabla u(T')\|^2 + \int_0^{T'} \|\partial_t u\|^2 \,\mathrm{d}t \le \frac{1}{2}\|\nabla u_0\|^2$$

for almost every $T' \in [0,T]$.

Step 3: *Unit-length constraint.* An interpolation estimate and $D^2 u_{h,\tau}|_R = 0$ for all elements $R \in \mathscr{T}_h$ yield for every $t \in [0,T]$ that

$$\begin{aligned}\big\|\mathscr{I}_h\big[|u_{h,\tau}^+|^2\big] - |u_{h,\tau}^+|^2\big\|_{L^1(R)} &\le c|R|^{1/2}h_R^2\big\|D^2|u_{h,\tau}^+|^2\big\|_{L^2(R)}\\ &\le c|R|^{1/2}h_R^2\|\nabla u_{h,\tau}^+\|_{L^4(R)}^2\\ &= ch_R^2|R|\big|\nabla u_{h,\tau}|_R\big|^2 \le ch_R^2\|\nabla u_{h,\tau}^+\|_{L^2(R)}^2.\end{aligned}$$

In the case of no projection we have

$$\||\mathscr{I}_h[u_{h,\tau}^+(t,\cdot)|^2] - 1\|_{L^1(\Omega)} \le c\tau,$$

while for the scheme including the projection step we have, cf. the proof of Proposition 7.1,

$$\||u_{h,\tau}^+(t,\cdot)|^2 - 1\| \le ch\|\nabla u_h^+(t,\cdot)\|.$$

The triangle inequality yields that $|u_{h,\tau}^+| \to 1$ in $L^1([0,T]\times\Omega)$ in both cases, i.e., that $|u(t,x)| = 1$ for almost every $(t,x)\in[0,T]\times\Omega$.
Step 4: *Attainment of initial data.* The weak continuity of the trace operator and $u_h^0 \to u^0$ in $L^2(\Omega;\mathbb{R}^3)$ as $h\to 0$ prove $u(0,\cdot) = u_0$.
Step 5: *Passage to the limit in the equation.* It remains to show that the function u solves the partial differential equation. For this, we choose $\varphi \in L^2([0,T];C^\infty(\overline{\Omega};\mathbb{R}^3))$ and define $w^{(h,\tau)} = u_{h,\tau}^- \times \varphi$ and

$$w_{h,\tau} = \mathscr{I}_h\big[u_{h,\tau}^- \times \varphi\big].$$

For this function we have $u_{h,\tau}^-(t,z)\cdot w_{h,\tau}(t,z) = 0$ for almost every $t\in[0,T]$ and every $z\in\mathscr{N}_h$. Moreover, we have using $D^2 u_{h,\tau}|_R = 0$ for all elements $R\in\mathscr{T}_h$ that

$$\begin{aligned}\|\nabla(w^{(h,\tau)} - w_{h,\tau})\|_{L^2(R)} &\le ch_R\|D^2[u_{h,\tau}^- \times\varphi]\|_{L^2(R)}\\ &\le h_R\big(\|\nabla u_{h,\tau}\|_{L^2(R)}\|\nabla\varphi\|_{L^2(R)} + \|u_{h,\tau}\|_{L^2(R)}\|D^2\varphi\|_{L^2(R)}\big).\end{aligned}$$

A summation over $R\in\mathscr{T}_h$ shows that $w_{h,\tau} - w^{h,\tau} \to 0$ in $L^\infty([0,T];H^1(\Omega;\mathbb{R}^3))$ as $(h,\tau)\to 0$. The equation of Algorithm 7.4 yields

$$(v_{h,\tau}^+, w_{h,\tau}) + (\nabla[u_{h,\tau}^- + \tau v_{h,\tau}^+], \nabla w_{h,\tau}) = 0$$

for almost every $t\in[0,T]$. Due to Lemma 7.6 we have that $\tau^{1/2}v_{h,\tau}^+$ is uniformly bounded in $L^2([0,T];H^1(\Omega;\mathbb{R}^3))$ and hence the term

$$\int_0^T (\tau(\nabla v_{h,\tau}^+, \nabla w_{h,\tau})\,\mathrm{d}t \le \tau^{1/2}\Big(\int_0^T \tau\|\nabla v_{h,\tau}^+\|^2\,\mathrm{d}t\Big)^{1/2}\Big(\int_0^T \|\nabla w_{h,\tau}^+\|^2\,\mathrm{d}t\Big)^{1/2}$$

converges to 0 as $(h,\tau)\to 0$. We write

$$\int_0^T (\nabla u_{h,\tau}^-, \nabla w_{h,\tau})\,\mathrm{d}t = \int_0^T (\nabla u_{h,\tau}^-, \nabla w^{(h,\tau)})\,\mathrm{d}t + \int_0^T (\nabla u_{h,\tau}^-, \nabla [w_{h,\tau} - w^{(h,\tau)}])\,\mathrm{d}t$$

and note that the second term on the right-hand side converges to 0 as $(h, \tau) \to 0$, while for the first term on the right-hand side we have

$$\begin{aligned}\int_0^T (\nabla u_{h,\tau}^-, \nabla w^{(h,\tau)})\,\mathrm{d}t &= \int_0^T \sum_{\ell=1}^d (\partial_\ell u_{h,\tau}^-, \partial_\ell [u_{h,\tau}^- \times \varphi])\,\mathrm{d}t \\ &= \int_0^T \sum_{\ell=1}^d (\partial_\ell u_{h,\tau}^-, u \times \partial_\ell \varphi)\,\mathrm{d}t \\ &\quad + \int_0^T \sum_{\ell=1}^d (\partial_\ell u_{h,\tau}^-, [u_{h,\tau}^- - u] \times \partial_\ell \varphi)\,\mathrm{d}t.\end{aligned}$$

This implies that for $(h, \tau) \to 0$, we have

$$\int_0^T (\nabla [u_{h,\tau}^- + \tau v_{h,\tau}^+], \nabla w_{h,\tau})\,\mathrm{d}t \to \int_0^T (\nabla u, \nabla [u \times \varphi])\,\mathrm{d}t.$$

Finally, we verify that

$$\begin{aligned}\int_0^T (v_h^+, w_{h,\tau})\,\mathrm{d}t &= \int_0^T (v_h^+, u \times \varphi) + (v_h^+, [u_h - u] \times \varphi) + (v_h^+, w_{h,\tau} - w^{(h,\tau)}\,\mathrm{d}t \\ &\to \int_0^T (\partial_t u, u \times \varphi)\,\mathrm{d}t\end{aligned}$$

as $(h, \tau) \to 0$. Altogether we have proved that u satisfies

$$\int_0^T (\partial_t u, u \times \varphi) + (\nabla u, \nabla [u \times \varphi])\,\mathrm{d}t = 0$$

for all $\varphi \in L^2([0, T]; C^\infty(\overline{\Omega}; \mathbb{R}^3))$. Choosing $\varphi(t, x) = \rho(t) w(x)$ we deduce that

$$(\partial_t u, u \times w) + (\nabla u, \nabla [u \times w])\,\mathrm{d}t = 0$$

for all $w \in C^\infty(\overline{\Omega}; \mathbb{R}^3)$. A density argument proves that this is satisfied for every $w \in H^1(\Omega; \mathbb{R}^3) \cap L^\infty(\Omega; \mathbb{R}^3)$. For $\phi \in H^1(\Omega; \mathbb{R}^3) \cap L^\infty(\Omega; \mathbb{R}^3)$ and $w = u \times \phi$, we verify with the identities $u \times (u \times \phi) = (u \cdot \phi)u - \phi$ and $\partial_t u \cdot u = 0$ that

$$-(\partial_t u, \phi) + (\nabla u, \nabla[u \times (u \times \phi)]) = 0$$

for almost every $t \in [0, T]$. According to Proposition 7.5 this implies that u is a weak solution of the harmonic map heat flow. □

7.3.3 Constraint Preservation

The third characterization of solutions of the harmonic map heat flow in Proposition 7.5 reads in the strong form that

$$\partial_t u + u \times (u \times \Delta u) = 0;$$

this reveals a symplectic structure and implies that the L^2-flow of harmonic maps is constraint preserving, i.e., if $|u_0(x)| = 1$ for almost every $x \in \Omega$, then we have $|u(t, x)| = 1$ for almost every $(t, x) \in [0, T] \times \Omega$. We consider the case $\Gamma_{\mathrm{D}} = \emptyset$ for ease of presentation.

Lemma 7.3 (Constraint preservation) *Let* $u \in L^\infty([0, T]; H^1(\Omega; \mathbb{R}^3))$ *satisfy* $\partial_t u \in L^2([0, T]; L^2(\Omega; \mathbb{R}^3))$ *and* $u(0, \cdot) = u_0$ *with* u_0 *such that* $|u_0(x)| = 1$ *for almost every* $x \in \Omega$. *Assume that*

$$(\partial_t u, \phi) + (\nabla u, \nabla[u \times (u \times \phi)]) = 0$$

for almost every $t \in [0, T]$ *and every* $\phi \in H^1(\Omega; \mathbb{R}^3) \cap L^\infty(\Omega; \mathbb{R}^3)$. *Then we have* $|u(t, x)| = 1$ *for almost every* $(t, x) \in [0, T] \times \Omega$.

Proof Let $\rho \in C^\infty(\mathbb{R}^n)$ be a nonnegative function with $\|\rho\|_{L^1(B_1(0))} = 1$ and $\operatorname{supp} \rho \subset B_1(0)$. Given $\varepsilon > 0$, set $\rho_\varepsilon(x) = \rho(x/\varepsilon)$ for $x \in \Omega$. For $x_0 \in \Omega$ the choice of $\phi = \rho_\varepsilon(\cdot - x_0)u$ implies that

$$\frac{d}{dt}\frac{1}{2}(|u(t, \cdot)|^2 * \rho_\varepsilon)(x_0) = (\partial_t u, \rho_\varepsilon u) = 0,$$

i.e., $(|u(T', \cdot)|^2 * \rho_\varepsilon)(x_0) = (|u_0(\cdot)|^2 * \rho_\varepsilon)(x_0)$ for every $T' \in [0, T]$. Noting that $(|u(t, \cdot)|^2 * \rho_\varepsilon)(x_0) \to |u(t, x_0)|$ as $\varepsilon \to 0$ implies the assertion. □

The lemma motivates the development of numerical schemes that preserve the length-constraint in a discrete sense. For the Crank–Nicolson type discretization of the strong form

$$d_t u^k + u^{k-1/2} \times (u^{k-1/2} \times \Delta u^{k-1/2}) = 0$$

we observe that testing with $u^{k-1/2} = (u^k + u^{k-1})/2$ formally yields the length-preservation property

$$d_t |u^k|^2 = 0.$$

To obtain this property for a fully discrete scheme, reduced integration has to be incorporated. We define the discrete Laplacian $\widetilde{\Delta}_h u_h \in \mathscr{S}^1(\mathscr{T}_h)^3$ of a function $u_h \in \mathscr{S}^1(\mathscr{T}_h)^3$ by

$$(\widetilde{\Delta}_h u_h, v_h)_h = -(\nabla u_h, \nabla v_h)$$

for all $v_h \in \mathscr{S}^1(\mathscr{T}_h)^3$.

Algorithm 7.5 (*Constraint-preserving iteration*) Let $u_h^0 \in \mathscr{S}^1(\mathscr{T}_h)^3$ with $|u_h^0(z)| = 1$ for all $z \in \mathscr{N}_h$ and $\tau > 0$ and define the sequence $(u_h^k)_{k=0,\dots,K} \subset \mathscr{S}^1(\mathscr{T}_h)^3$ such that

$$(d_t u_h^k, \phi_h)_h + (u_h^{k-1/2} \times [u_h^{k-1/2} \times \widetilde{\Delta}_h u_h^{k-1/2}], \phi_h)_h = 0$$

for all $\phi_h \in \mathscr{S}^1(\mathscr{T}_h)^3$.

To establish the well-posedness of the algorithm, we note that a corollary of Brouwer's fixed-point theorem states that if $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is continuous with $\Phi(s) \cdot s \geq 0$ for all $s \in \mathbb{R}^n$ with $|s| \geq R$ for some $R \in \mathbb{R}$, then there exists $s^* \in \mathbb{R}^n$ with $|s^*| \leq R$ and $\Phi(s^*) = 0$.

Proposition 7.7 (Stability and constraint preservation) *There exists a sequence* $(u_h^k)_{k=0,\dots,K} \subset \mathscr{S}^1(\mathscr{T}_h)^3$ *that solves the scheme of Algorithm*7.5. *We have* $|u_h^k(z)| = 1$ *for* $k = 0, 1, \dots, K$ *and*

$$\frac{1}{2}\|\nabla u_h^L\|^2 + \tau \sum_{k=1}^{L} \|d_t u_h^k\|^2 \leq \frac{1}{2}\|\nabla u_h^0\|^2.$$

Proof Let $k \geq 1$ and define $\Phi_h : \mathscr{S}^1(\mathscr{T}_h)^3 \to \mathscr{S}^1(\mathscr{T}_h)^3$ by

$$\Phi_h(v_h) = \frac{2}{\tau}(v_h - u_h^{k-1}) + \mathscr{I}_h\big[v_h \times (v_h \times \widetilde{\Delta}_h v_h)\big].$$

The function Φ_h is continuous and the Cauchy–Schwarz inequality, employing that $(\mathscr{I}_h w_h, v_h)_h = (w_h, v_h)_h$ for all $v_h, w_h \in \mathscr{S}^1(\mathscr{T}_h)^3$, proves that

$$(\Phi_h(v_h), v_h)_h = \frac{2}{\tau}(v_h - u_h^{k-1}, v_h)_h \geq \frac{1}{\tau}\|v_h\|_h^2 - \frac{1}{\tau}\|u_h^{k-1}\|_h^2,$$

i.e., $(\Phi_h(v_h), v_h)_h \geq 0$ for all $v_h \in \mathscr{S}^1(\mathscr{T}_h)^3$ with $\|v_h\|_h \geq \|u_h^{k-1}\|_h$. Brouwer's fixed-point theorem thus implies that there exists $r_h^k \in \mathscr{S}^1(\mathscr{T}_h)^3$ with $\Phi_h(r_h^k) = 0$ or equivalently that $u_h^k = 2r_h^k - u_h^{k-1}$ solves

$$0 = d_t u_h^k + \mathscr{I}_h\big[u_h^{k-1/2} \times (u_h^{k-1/2} \times \widetilde{\Delta}_h u_h^{k-1/2})\big],$$

i.e.,

$$(d_t u_h^k, w_h)_h + (u_h^{k-1/2} \times [u_h^{k-1/2} \times \widetilde{\Delta}_h u_h^{k-1/2}], w_h)_h = 0$$

for all $w_h \in \mathscr{S}^1(\mathscr{T}_h)^3$. For $z \in \mathscr{N}_h$ and the function $w_h = [u_h^{k-1/2}(z)]\varphi_z$, the properties of the discrete inner product imply that

$$\begin{aligned}
\beta_z d_t |u_h^k(z)|^2 &= \beta_z d_t u_h^k(z) \cdot u_h^{k-1/2}(z) = (d_t u_h^k, \varphi_z u_h^{k-1/2})_h \\
&= (u_h^{k-1/2} \times [u_h^{k-1/2} \times \widetilde{\Delta}_h u_h^{k-1/2}], \varphi_z u_h^{k-1/2})_h \\
&= \beta_z \big(u_h^{k-1/2}(z) \times [u_h^{k-1/2}(z) \times \widetilde{\Delta}_h u_h^{k-1/2}(z)]\big) \cdot u_h^{k-1/2}(z) = 0,
\end{aligned}$$

i.e., $|u_h^k(z)| = |u_h^{k-1}(z)|$, and inductively the assumption $|u_h^0(z)| = 1$ implies $|u_h^k(z)| = 1$. For $w_h = \widetilde{\Delta}_h u_h^{k-1/2}$, we obtain

$$\begin{aligned}
&d_t \|\nabla u_h^k\|_h^2 + \big\|u_h^{k-1/2} \times \widetilde{\Delta}_h u_h^{k-1/2}\big\|_h^2 \\
&\quad = (\nabla d_t u_h^k, \nabla u_h^{k-1/2}) - (u_h^{k-1/2} \times [u_h^{k-1/2} \times \widetilde{\Delta}_h u_h^{k-1/2}], \widetilde{\Delta} u_h^{k-1/2})_h \\
&\quad = -(d_t u_h^k, \widetilde{\Delta}_h u_h^{k-1/2}) - (u_h^{k-1/2} \times [u_h^{k-1/2} \times \widetilde{\Delta}_h u_h^{k-1/2}], \widetilde{\Delta} u_h^{k-1/2})_h = 0.
\end{aligned}$$

The choice of $w_h = d_t u_h^k$ shows that

$$\begin{aligned}
\|d_t u_h^k\|_h^2 &= -(u_h^{k-1/2} \times \widetilde{\Delta}_h u_h^{k-1/2}, u_h^{k-1/2} \times d_t u_h^k)_h \\
&\le \|u_h^{k-1/2} \times \widetilde{\Delta}_h u_h^{k-1/2}\|_h \|u_h^{k-1/2} \times d_t u_h^k\|_h
\end{aligned}$$

and with $|u_h^{k-1/2}(z)| \le 1$ for every $z \in \mathscr{N}_h$, we deduce $\|d_t u_h^k\| \le \|u_h^{k-1/2} \times \widetilde{\Delta}_h u_h^{k-1/2}\|_h$. A combination of the last two estimates, multiplication by τ, and a summation over $k = 1, 2, \ldots, L$ thus prove the asserted bound. □

Remarks 7.13 (i) The stability bound implies unconditional convergence to a weak solution of the harmonic map heat flow.
(ii) The existence of the iterates in Algorithm 7.5 was established by Brouwer's fixed point theorem which is nonconstructive and in fact the iterates may not be uniquely defined. If $\tau \le c h_{\min}^2$, the following linear iteration is constraint-preserving and converges to the uniquely defined function $u_h^{k-1/2}$. Set $r_h^0 = u_h^{k-1}$ and define the sequence $(r_h^\ell)_{\ell=0,1,\ldots} \subset \mathscr{S}^1(\mathscr{T}_h)^3$ via

$$\frac{2}{\tau}(r_h^\ell, \phi_h)_h + (r_h^\ell \times [r_h^{\ell-1} \times \widetilde{\Delta}_h r_h^{\ell-1}], \phi_h)_h = \frac{2}{\tau}(u_h^{k-1}, \phi_h)_h$$

for all $\phi_h \in \mathscr{S}^1(\mathscr{T}_h)^3$.

7.3.4 Approximation of Wave Maps

Wave maps are solutions of the wave equation subject to a pointwise unit-length constraint. They solve the partial differential equation

$$\partial_t^2 u - \Delta u = \lambda u$$

in $[0, T] \times \Omega$ with a Lagrange multiplier $\lambda : [0, T] \times \Omega \to \mathbb{R}$ associated to the constraint $|u(t, x)| = 1$ for almost every $(t, x) \in [0, T] \times \Omega$ and subject to the boundary condition $\partial_n u = 0$ on $[0, T] \times \partial\Omega$ and the initial conditions $u(0, \cdot) = u_0$ and $\partial_t u(0, \cdot) = u_1$. Qualitatively, similar partial differential equations arise in general relativity and particle physics. Wave maps may also be regarded as harmonic maps on $[0, T] \times \Omega$ for the Dirichlet energy defined with the Minkowski metric on the $\mathbb{R}^{1+d}$ time-space domain, i.e., they are stationary for

$$I_g(u) = \frac{1}{2} \int_0^T \int_\Omega |Du|_g^2 \,\mathrm{d}t \,\mathrm{d}x$$

with $Du = (\partial_t u, \nabla u)$ and $|v|_g^2 = -v_0^2 + v_1^2 + \cdots v_d^2$ for $v \in \mathbb{R}^{d+1}$. An important feature of solutions for the wave map equation is the energy conservation property that the mapping

$$t \mapsto I\big(u(t, \cdot), \partial_t u(t, \cdot)\big) = \frac{1}{2} \int_\Omega |\partial_t u(t, \cdot)|^2 \,\mathrm{d}x + \frac{1}{2} \int_\Omega |\nabla u(t, \cdot)|^2 \,\mathrm{d}x$$

is constant as a function of $t \in [0, T]$.

Definition 7.4 Given $u_0 \in H^1(\Omega; \mathbb{R}^m)$ and $u_1 \in L^2(\Omega; \mathbb{R}^m)$, a *wave map* is a function $u : [0, T] \times \Omega \to \mathbb{R}^m$ such that
(a) $u \in H^1([0, T]; L^2(\Omega; \mathbb{R}^m)) \cap L^2([0, T]; H^1(\Omega; \mathbb{R}^m))$,
(b) $|u(t, x)| = 1$ for almost every $(t, x) \in [0, T] \times \Omega$,
(c) for all $w \in C_c^\infty([0, T); C^\infty(\overline{\Omega}; \mathbb{R}^m))$ with $u(t, x) \cdot w(t, x) = 0$ for almost every $(t, x) \in [0, T] \times \Omega$, we have

$$-\int_0^T (\partial_t u, \partial_t w) + (\nabla u, \nabla w) \,\mathrm{d}t = (u_1, w(0)),$$

(d) the initial data u_0 is attained continuously by u as $t \to 0$ in $H^1(\Omega; \mathbb{R}^m)$,
(e) for almost every $T' \in [0, T]$, we have

$$I\big(u(T', \cdot), \partial_t u(T', \cdot)\big) \le I(u_0, u_1).$$

The algorithm for approximating wave maps is a modification of Algorithm 7.4 for the approximation of the harmonic map heat flow. The sets $\mathscr{A}_h$ and $\mathscr{F}_h[u_h^{k-1}]$ are defined as above.

Algorithm 7.6 (*Wave map approximation*) Let $u_h^0, v_h^0 \in \mathscr{S}^1(\mathscr{T}_h)^m$ with $|u_h^0(z)| = 1$ for all $z \in \mathscr{N}_h$ and $\tau > 0$ and define the sequence $(u_h^k)_{k=0,\dots,K} \subset \mathscr{S}^1(\mathscr{T}_h)^m$ for $K = \lceil T/\tau \rceil$ by computing $v_h^k \in \mathscr{F}_h[u_h^{k-1}]$ such that

$$(d_t v_h^k, w_h) + (\nabla[u_h^{k-1} + \tau v_h^k], \nabla w_h) = 0$$

for all $w_h \in \mathscr{F}_h[u_h^{k-1}]$ and setting with $\widetilde{u}_h = u_h^{k-1} + \tau v_h^k$

$$u_h^k = \widetilde{u}_h^k \qquad \text{or} \qquad u_h^k = \sum_{z \in \mathscr{N}_h} \frac{\widetilde{u}_h^k(z)}{|\widetilde{u}_h^k(z)|} \varphi_z.$$

We have the following stability result.

Proposition 7.8 (Stability) (i) *If no projection is carried out, then* $v_h^k = d_t u_h^k$ *for* $k = 1, 2, \dots, K$ *and for* $L = 1, 2, \dots, K$*, we have*

$$\frac{1}{2}\|v_h^L\|^2 + \frac{1}{2}\|\nabla u_h^L\|^2 + \frac{\tau^2}{2}\sum_{k=1}^{L}\big(\|d_t v_h^k\|^2 + \|\nabla v_h^k\|^2\big) = \frac{1}{2}\|v_h^0\|^2 + \frac{1}{2}\|\nabla u_h^0\|^2,$$
$$\big\|\mathscr{I}_h\big[|u_h^L|^2\big] - 1\big\|_{L^1(\Omega)} \le c_0 \tau.$$

(ii) *If a projection is carried out in every step of the algorithm and if* $\mathscr{T}_h$ *is weakly acute, then we have* $|u_h^k(z)| = 1$ *for* $k = 0, 1, \dots, K$ *and all* $z \in \mathscr{N}_h$*, and for* $L = 1, 2, \dots, K$ *that*

$$\frac{1}{2}\|v_h^L\|^2 + \frac{1}{2}\|\nabla u_h^L\|^2 + \frac{\tau^2}{2}\sum_{k=1}^{L}\big(\|d_t v_h^k\|^2 + \|\nabla v_h^k\|^2\big) \le \frac{1}{2}\|v_h^0\|^2 + \frac{1}{2}\|\nabla u_h^0\|^2,$$
$$\|v_h^L - d_t u_h^L\|_{L^1(\Omega)} \le c_0 \tau.$$

```
function wave_maps(d,red)
[c4n,n4e,Db,Nb] = triang_cube(d); c4n = c4n-.5;
T = 10;
tau = 2^(-red)/4; K = ceil(T/tau);
for j = 1:red
    [c4n,n4e,Db,Nb] = red_refine(c4n,n4e,Db,Nb);
end
nC = size(c4n,1);
[s,m,¬,¬] = fe_matrices(c4n,n4e);
SSS = sparse(3*nC,3*nC); MMM = sparse(3*nC,3*nC);
for k = 1 : 3
    idx = k:3:3*nC; SSS(idx,idx) = s; MMM(idx,idx) = m;
end
u = zeros(3*nC,1); v = zeros(3*nC,1);
for j = 1:nC
    u(3*j-[2,1,0]) = u_0(c4n(j,:));
    v(3*j-[2,1,0]) = v_0(c4n(j,:));
end
for k = 1:K
    B = sparse(nC,3*nC);
    for j = 1:nC
        B(j,3*j-[2,1,0]) = u(3*j-[2,1,0]);
    end
    X = [MMM+tau^2*SSS,B';B,sparse(nC,nC)];
    b = [MMM*v-tau*SSS*u;zeros(nC,1)];
    x = X\b;
    v = x(1:3*nC);
    tu = u+tau*v;
    for j = 1:nC
        u(3*j-[2,1,0]) = tu(3*j-[2,1,0])/norm(tu(3*j-[2,1,0]));
    end
    show_p1_field(c4n,u); axis(.5*[-1,1,-1,1,-1,1]);
    view(30,18); drawnow; pause(.05)
end

function val = u_0(x)
d = size(x,2);
x = [x,zeros(1,3-d)];
r = norm(x); a = max(0,1-2*r)^4;
val = [2*a*x(1:2),a^2-r^2 ]/(a^2+r^2);

function val = v_0(x)
val = [0,0,0];
```

Fig. 7.8 MATLAB realization of Algorithm 7.6 for the approximation of wave maps

Proof The choice of $w_h = v_h^k$ yields

$$\frac{d_t}{2}\|v_h^k\|^2 + \frac{\tau}{2}\|d_t v_h^k\|^2 + \frac{1}{2\tau}\big(\|\nabla[u_h^{k-1} + \tau v_h^k]\|^2 - \|\nabla u_h^{k-1}\|^2\big) + \frac{\tau}{2}\|\nabla v_h^k\|^2 = 0.$$

In the case of no projection, we have $u_h^{k-1} + \tau v_h^k = u_h^k$, and a summation over $k = 1, 2, \ldots, L$ implies the stability bound. If u_h^k is obtained through a projection, then it follows as in the proof of Proposition 7.6 that $\|\nabla u_h^k\| \le \|\nabla[u_h^{k-1} + \tau v_h^k]\|$, and again a summation over $k = 1, 2, \ldots, L$ implies the stability bound. The other estimates follow as in the proof of Proposition 7.6. □

Remark 7.14 The stability bounds imply the convergence of approximations to a wave map.

Figure 7.8 displays a MATLAB realization of Algorithm 7.6 that is based on the implementation of Algorithm 7.1.

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