Chapter 10
Free Discontinuities

10.1 Functions of Bounded Variation

Many important phenomena require the description of physical quantities with discontinuous functions. Although Sobolev functions are not continuous in general, they are too restrictive to admit functions with jumps across lower-dimensional subsets. We introduce in this section the space of functions of bounded variations and discuss its properties. The reader is referred to the textbooks [2, 4, 9] for details.

10.1.1 Derivatives of Discontinuous Functions

Functions in $L^1(\Omega)$ define regular distributions and can be differentiated in the distributional sense, i.e., given $u \in L^1(\Omega)$, its *distributional derivative* is the linear functional $Du : C_c^\infty(\Omega; \mathbb{R}^d) \to \mathbb{R}$ defined by

$$\langle Du, \phi \rangle = -\int_\Omega u \operatorname{div} \phi \, \mathrm{d}x$$

for every $\phi \in C_c^\infty(\Omega; \mathbb{R}^d)$.

Remark 10.1 For $u \in L^1(\Omega)$ we have $u \in W^{1,1}(\Omega)$ if $Du \in L^1(\Omega; \mathbb{R}^d)$, i.e., if there exists $g \in L^1(\Omega; \mathbb{R}^d)$ such that for all $\phi \in C_c^\infty(\Omega; \mathbb{R}^d)$, we have

$$\langle Du, \phi \rangle = \int_\Omega g \cdot \phi \, \mathrm{d}x.$$

The space $C_0(\Omega; \mathbb{R}^m)$ denotes the completion of the space $C_c^\infty(\Omega; \mathbb{R}^m)$ with respect to the norm $\|v\|_{L^\infty(\Omega)} = \sup_{x \in \Omega} |v(x)|$ for $v \in C_c^\infty(\Omega; \mathbb{R}^m)$, defined through

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the Euclidean norm on $\mathbb{R}^m$. It is a separable Banach space and its dual is denoted by $\mathscr{M}(\Omega;\mathbb{R}^m)$. The elements in $\mathscr{M}(\Omega;\mathbb{R}^m)$ are through Riesz's representation theorem identified with (*vectorial*) *Radon measures*; and the application of $\mu \in \mathscr{M}(\Omega;\mathbb{R}^m)$ to $v \in C_0(\Omega;\mathbb{R}^m)$ is denoted by

$$\langle \mu, \phi\rangle = \int_\Omega \phi \, \mathrm{d}\mu = \int_\Omega \phi(x)\, \mathrm{d}\mu(x).$$

If $m = 1$, we call μ a scalar Radon measure and write $\mathscr{M}(\Omega)$ for $\mathscr{M}(\Omega;\mathbb{R}^m)$.

Examples 10.1 (i) Every $f \in L^1(\Omega;\mathbb{R}^m)$ defines a Radon measure $\mu_f = f \otimes \mathrm{d}x \in \mathscr{M}(\Omega;\mathbb{R}^m)$ through the Lebesgue integral

$$\langle \mu_f, \phi\rangle = \int_\Omega \phi \cdot f \, \mathrm{d}x$$

for all $\phi \in C_c^\infty(\Omega;\mathbb{R}^m)$. This is a bounded linear functional on $C_0(\Omega;\mathbb{R}^m)$ since

$$\langle \mu_f, \phi\rangle \le \|f\|_{L^1(\Omega)} \|\phi\|_{L^\infty(\Omega)}.$$

(ii) The *Dirac distribution* δ_{x_0} for $x_0 \in \overline{\Omega}$ defines a Radon measure in $\mathscr{M}(\Omega)$ which, for all $\phi \in C_0(\Omega)$ is given by

$$\langle \delta_{x_0}, \phi\rangle = \phi(x_0).$$

(iii) Given a union $C = \cup_{i=1}^{\ell} \Gamma_i$ of Lipschitz continuous curves $\Gamma_i \subset \overline{\Omega}$, $i = 1, 2, \ldots, n$, and a function $f \in L^1(C;\mathbb{R}^m)$, we define the Radon measure $\mu_{fC} = f \otimes \mathrm{d}s\lfloor_C$ by setting for $\phi \in C_0(\Omega)$

$$\langle \mu_{fC}, \phi\rangle = \int_C \phi f \, \mathrm{d}s.$$

Definition 10.1 A function $u \in L^1(\Omega)$ is said to be of *bounded variation* if its distributional derivative defines a Radon measure, i.e., if there exists $c \ge 0$ such that

$$\langle Du, \phi\rangle = -\int_\Omega u \ \mathrm{div}\ \phi \, \mathrm{d}x \le c\|\phi\|_{L^\infty(\Omega)}$$

for all $\phi \in C_c^1(\Omega;\mathbb{R}^d)$. The minimal constant $c \ge 0$ with this property is called *total variation* of Du and is given by

$$|Du|(\Omega) = \sup\Big\{ -\int_\Omega u \ \mathrm{div}\ \phi \, \mathrm{d}x : \phi \in C_c^1(\Omega;\mathbb{R}^n),\ \|\phi\|_{L^\infty(\Omega)} \le 1\Big\}.$$

The space of all such functions is denoted $BV(\Omega)$ and called the *space of functions of bounded variation*. It is equipped with the norm

$$\|u\|_{BV(\Omega)} = \|u\|_{L^1(\Omega)} + |Du|(\Omega)$$

for all $u \in BV(\Omega)$.

We summarize some basic facts about the space $BV(\Omega)$.

Remarks 10.2 (i) The space $BV(\Omega)$ is a nonseparable Banach space.
(ii) We have that $|Du|(\Omega)$ is the operator norm of $Du : C_c^\infty(\Omega; \mathbb{R}^d) \to \mathbb{R}$.
(iii) We have $W^{1,1}(\Omega) \subset BV(\Omega)$ with $\|u\|_{BV(\Omega)} = \|u\|_{W^{1,1}(\Omega)}$ for all $u \in W^{1,1}(\Omega)$.
(vi) We have that $u \in BV(\Omega)$ if and only if there exists $\mu \in \mathscr{M}(\Omega; \mathbb{R}^d)$ such that

$$\int_\Omega u \operatorname{div} \phi \, \mathrm{d}x = -\int_\Omega \phi \, \mathrm{d}\mu$$

for all $\phi \in C_c^1(\Omega; \mathbb{R}^d)$.
(v) If $u \in BV(\Omega)$ and $Du = 0$, then u is constant on every connected component of Ω. Moreover, $u \mapsto |Du|(\Omega)$ is a seminorm on $BV(\Omega)$.
(vi) If $u \in BV(\Omega)$ and $\psi : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous with constant L, then $\psi \circ u \in BV(\Omega)$ with $|D(\psi \circ u)|(\Omega) \le L|Du|(\Omega)$.
(vii) If $\Omega = (a, b) \subset \mathbb{R}^1$ and $u \in BV(\Omega)$, then there exists $\widetilde{u} \in BV(\Omega)$ with $u = \widetilde{u}$ almost everywhere in Ω and

$$|Du|(\Omega) = \sup_{a<x_0<x_1<\cdots<x_n<b} \sum_{j=1}^n |\widetilde{u}(x_j) - \widetilde{u}(x_{j-1})|,$$

where the supremum is over all partitions $a < x_0 < x_1 < \cdots < x_n < b$ with $n \ge 1$.

Typical examples of functions in $BV(\Omega)$ that do not belong to $W^{1,1}(\Omega)$ are functions that are piecewise weakly differentiable and jump across lower-dimensional subsets.

Examples 10.2 (i) For $\Omega = (-1, 1)$ and $u(x) = \operatorname{sign}(x)$, we have

$$\langle Du, \phi\rangle = -\int_{(-1,1)} u\phi' \, \mathrm{d}x = \int_{(0,1)} \phi' \, \mathrm{d}x - \int_{(0,1)} \phi' \, \mathrm{d}x = 2\phi(0)$$

for all $\phi \in C_0^1(\Omega)$, i.e., $Du = 2\delta_0$ and $u \in BV(\Omega)$ with $|Du|(\Omega) = 2$.
(ii) For $\Omega \subset \mathbb{R}^d$ and a Lipschitz domain $E \subset \Omega$, the characteristic function $u = \chi_E$ satisfies

$$\langle D\chi_E, \phi\rangle = -\int_\Omega \chi_E \operatorname{div} \phi \, \mathrm{d}x = -\int_E \operatorname{div} \phi \, \mathrm{d}x = -\int_{\partial E} \phi \cdot n_E \, \mathrm{d}s$$

for all $\phi \in C_0^1(\Omega; \mathbb{R}^d)$ with the outer unit normal n_E on ∂E, i.e., we have $D\chi_E = -n_E \otimes \mathrm{d}s\lfloor_{\partial E}$. This implies that $|D\chi_E|(\Omega) = \mathscr{H}^{d-1}(\partial E)$ is the length or surface area of ∂E for $d = 2$ and $d = 3$, respectively.

Remarks 10.3 (i) If $E \subset \mathbb{R}^d$, then E is said to be of finite perimeter in Ω if $\chi_E \in BV(\Omega)$, and in this case $|D\chi_E|(\Omega)$ is called the *perimeter* of E in Ω. The perimeter generalizes the length or surface area of the boundary of a measurable set $E \cap \Omega$.
(ii) The coarea formula states that the total variation coincides with the integral of the perimeters of the level sets of a function of bounded variation, i.e., we have that

$$|Du|(\Omega) = \int_{-\infty}^{+\infty} |D\chi_{\{u>t\}}|(\Omega)\,\mathrm{d}t.$$

10.1.2 Properties of BV(Ω)

The space $BV(\Omega)$ is an extension of $W^{1,1}(\Omega)$ in the sense that $W^{1,1}(\Omega) \subset BV(\Omega)$ and $\|u\|_{W^{1,1}(\Omega)} = \|u\|_{BV(\Omega)}$ for all $u \in W^{1,1}(\Omega)$. Since the set $C^\infty(\overline{\Omega})$ is dense in $W^{1,1}(\Omega)$, we have that $BV(\Omega)$ is not the closure of $C^\infty(\overline{\Omega})$ with respect to the norm in $BV(\Omega)$. In particular, convergence with respect to the norm in $BV(\Omega)$ or equivalently strong convergence in $BV(\Omega)$ is a notion of convergence that is too restrictive in applications.

Definition 10.2 (i) We say that the sequence $(u_n)_{n\in\mathbb{N}} \subset BV(\Omega)$ *converges intermediately* or *strictly* to $u \in BV(\Omega)$ if $u_n \to u$ in $L^1(\Omega)$ and $|Du_n|(\Omega) \to |Du|(\Omega)$ as $n \to \infty$.
(ii) We say that $(u_n)_{n\in\mathbb{N}} \subset BV(\Omega)$ *converges weakly* to $u \in BV(\Omega)$ if $u_n \to u$ in $L^1(\Omega)$ and $Du_n \rightharpoonup^* Du$ in $\mathscr{M}(\Omega; \mathbb{R}^d)$, i.e., $\langle Du_n, \phi\rangle \to \langle Du, \phi\rangle$ as $n \to \infty$ for every $\phi \in C_0(\Omega; \mathbb{R}^d)$.

Remarks 10.4 (i) The space $BV(\Omega)$ is the dual of a separable Banach space and therefore a natural weak* topology on $BV(\Omega)$ exists. It coincides with the notion of weak convergence introduced in the definition.
(ii) The weak topology in $BV(\Omega)$ in the sense of Banach spaces is difficult to characterize due to the lack of an efficient characterization of $BV(\Omega)'$.
(iii) For $(u_n)_{n\in\mathbb{N}} \subset W^{1,p}(\Omega)$ and $1 < p < \infty$, we have that $u_n \to u$ in $W^{1,p}(\Omega)$ for some $u \in W^{1,p}(\Omega)$ if and only if $u_n \rightharpoonup u$ and $\|u_n\|_{W^{1,p}(\Omega)} \to \|u\|_{W^{1,p}(\Omega)}$ as $n \to \infty$.

Examples 10.3 (i) For $\Omega = (-1, 1)$, let $(u_n)_{n\in\mathbb{N}} \subset BV(\Omega)$ be defined by $u_n(x) = nx$ if $|x| \le 1/n$ and $u_n(x) = \operatorname{sign}(x)$ for $|x| \ge 1/n$, cf. Fig. 10.1. We have that $u_n \to u$ in $L^1(\Omega)$ as $n \to \infty$ for $u(x) = \operatorname{sign}(x)$ for all $x \in \Omega$. Since $|Du_n|(\Omega) = \|\nabla u_n\|_{L^1(\Omega)} = 2$ for all $n \in \mathbb{N}$ and $Du = 2\delta_0$, we have $|Du_n|(\Omega) \to |Du|(\Omega)$; that is, the sequence $(u_n)_{n\in\mathbb{N}}$ converges intermediately to u as $n \to \infty$. Since $(u_n)_{n\in\mathbb{N}} \subset$

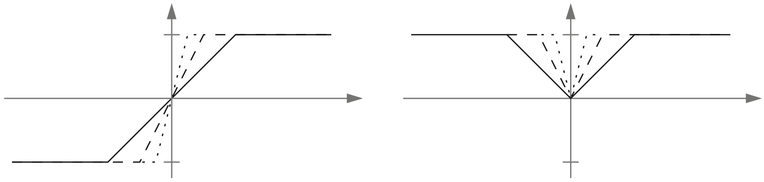

Fig. 10.1 Sequence of functions converging intermediately to $u = \text{sign}$ but not strongly (*left*); sequence of functions converging weakly to $u = 1$ but not intermediately (*right*)

$W^{1,1}(\Omega)$ but $u \notin W^{1,1}(\Omega)$, the sequence does not converge strongly to u.
(ii) For $\Omega = (-1, 1)$ let $(u_n)_{n\in\mathbb{N}}$ be defined by $u_n(x) = n|x|$ if $|x| \le 1/n$ and $u_n(x) = 1$ for $|x| \ge 1/n$, cf. Fig. 10.1. We have that $(u_n)_{n\in\mathbb{N}}$ converges in $L^1(\Omega)$ to the constant function $u = 1$, but $|Du_n|(\Omega) = 2$ and $|Du|(\Omega) = 0$ so that $(u_n)_{n\in\mathbb{N}}$ does not converge intermediately to u. Since $\langle Du_n, \chi_{\{|x|\le 1/m\}}\rangle = 0$ for $m \le n$, it follows that the sequence converges weakly to u.

An important property of the total variation is that it is lower semicontinuous with respect to strong convergence in $L^1(\Omega)$. The following theorem shows that this is equivalent to weak lower semicontinuity in $BV(\Omega)$.

Theorem 10.1 (Weak lower semicontinuity) *If $(u_n)_{n\in\mathbb{N}} \subset BV(\Omega)$ and $u \in L^1(\Omega)$ such that $|Du_n|(\Omega) \le c$ for all $n \in \mathbb{N}$ and $u_n \to u$ in $L^1(\Omega)$, then $u \in BV(\Omega)$ with $|Du|(\Omega) \le \liminf_{n\to\infty} |Du_n|(\Omega)$. Moreover, we have $u_n \rightharpoonup u$ in $BV(\Omega)$ as $n \to \infty$.*

Smooth functions are not dense in $BV(\Omega)$ with respect to strong convergence but with respect to intermediate convergence.

Theorem 10.2 (Approximation by smooth functions) *The spaces $C^\infty(\overline{\Omega})$ and $C^\infty(\Omega) \cap BV(\Omega)$ are dense in $BV(\Omega)$ with respect to intermediate convergence.*

The following compactness result allows us to extract weakly convergent subsequences from bounded sequences in $BV(\Omega)$. This is the crucial difference between the spaces $BV(\Omega)$ and $W^{1,1}(\Omega)$.

Theorem 10.3 (Compactness) *Let $(u_n)_{n\in\mathbb{N}} \subset BV(\Omega)$ be a bounded sequence. Then there exists a subsequence $(u_{n_j})_{j\in\mathbb{N}}$ and $u \in BV(\Omega)$ such that $u_{n_j} \rightharpoonup u$ in $BV(\Omega)$ as $j \to \infty$.*

The most important examples of functions in $BV(\Omega)$ are piecewise regular functions that jump across an interface.

Proposition 10.1 (Piecewise regular functions) *If $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$ and Ω_1, Ω_2 are such that $\Omega_1 \cap \Omega_2 = \emptyset$ and $\Sigma = \partial\Omega_1 \cap \partial\Omega_2$ and $u \in L^1(\Omega)$ such that $u|_{\Omega_j} \in W^{1,1}(\Omega_j)$ for $j = 1, 2$, then $u \in BV(\Omega)$ with*

$$Du = \nabla u \otimes \mathrm{d}x - [\![un]\!] \otimes \mathrm{d}s\lfloor_{\Sigma}$$

with the piecewise defined weak gradient $\nabla u|_{\Omega_j} = \nabla(u|_{\Omega_j})$ *and the jump* $[\![un]\!] = u_{\Omega_1} n_{\Omega_1}\lfloor_{\Sigma} + u_{\Omega_2} n_{\Omega_2}\lfloor_{\Sigma}$ *across* Σ *with the outer unit normals* n_{Ω_j} *to* Ω_j *for* $j = 1, 2$.

Proof For $\phi \in C_c^\infty(\Omega; \mathbb{R}^d)$ a piecewise integration by parts with $\phi|_{\partial\Omega_j \setminus \Sigma} = 0$ for $j = 1, 2$ shows that

$$\begin{aligned}\int_{\Omega} u \ \mathrm{div}\ \phi \,\mathrm{d}x &= \int_{\Omega_1} u \ \mathrm{div}\ \phi \,\mathrm{d}x + \int_{\Omega_2} u \ \mathrm{div}, \phi \,\mathrm{d}x \\ &= -\int_{\Omega_1} (\nabla u) \cdot \phi \,\mathrm{d}x - \int_{\Omega_2} (\nabla u) \cdot \phi \,\mathrm{d}x + \int_{\partial\Omega_1} u\phi \cdot n_{\Omega_1} \,\mathrm{d}s \\ &\quad + \int_{\partial\Omega_2} u\phi \cdot n_{\Omega_2} \,\mathrm{d}s \\ &= -\int_{\Omega} (\nabla u) \cdot \phi \,\mathrm{d}x + \int_{\Sigma} \phi \cdot [\![un]\!] \,\mathrm{d}s,\end{aligned}$$

which proves the assertion. □

The proposition can be generalized which leads to the following characterization of functions in $BV(\Omega)$.

Theorem 10.4 (Decomposition of Du) *For every* $u \in BV(\Omega)$ *we have*

$$Du = \nabla u \otimes \mathrm{d}x - [\![un]\!] \otimes \mathrm{d}s|_{S_u} + C_u,$$

where S_u *is a* $(d-1)$*-dimensional jump set,* $\nabla u \in L^1(\Omega)$ *is the weak gradient in the set* $\Omega \setminus S_u$*, and* C_u *either vanishes or is a measure supported on a Cantor set of vanishing* d*-dimensional Hausdorff measure that is zero for sets of finite* $(d-1)$*-dimensional Hausdorff measure. A point* $x \in \Omega$ *belongs to* S_u *if there exists a unit vector* $n \in \mathbb{R}^d$ *and distinct numbers* $a^\pm \in \mathbb{R}$ *such that*

$$\lim_{\varepsilon \to 0} |B_\varepsilon^\pm(x, n) \cap \Omega|^{-1} \int_{B_\varepsilon^\pm(x,n)\cap\Omega} u(y) \,\mathrm{d}y = a^\pm,$$

where $B_\varepsilon^\pm(x, n) = \{y \in B_\varepsilon(x) : \pm(y - x) \cdot n > 0\}$*, cf. Fig.* 10.2.

Some further important properties of $BV(\Omega)$ are listed below.

Remarks 10.5 (i) The embedding $BV(\Omega) \to L^p(\Omega)$ is continuous for $1 \le p \le d/(d-1)$ and compact for $1 \le p < d/(d-1)$.
(ii) We have $\|u\|_{L^p(\Omega)} \le c \ \mathrm{diam}(\Omega)|Du|(\Omega)$ if $u \in BV(\Omega)$ with $\int_\Omega u \,\mathrm{d}x = 0$ and

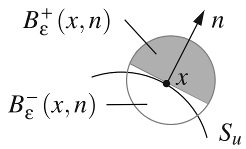

Fig. 10.2 Sets $B_\varepsilon^\pm(x,n)$ for a point $x \in S_u$ where the function u jumps from the value a^- to the value a^+ in the direction of n

$1 \le p \le d/(d-1)$.
(iii) There exists a linear operator $\text{tr} : BV(\Omega) \to L^1(\partial\Omega)$ such that $\text{tr}(u) = u|_{\partial\Omega}$ for all $u \in BV(\Omega) \cap C(\overline{\Omega})$. Moreover, we have the integration by parts formula

$$\int_\Omega \phi Du = -\int_\Omega u \operatorname{div} \phi \,\mathrm{d}x + \int_{\partial\Omega} \text{tr}(u)\phi \cdot n \,\mathrm{d}s$$

for all $u \in BV(\Omega)$ and all $\phi \in C^1(\overline{\Omega};\mathbb{R}^d)$. The operator tr is continuous with respect to intermediate convergence in $BV(\Omega)$. It is not continuous with respect to weak convergence in $BV(\Omega)$; for example for $(u_n)_{n\in\mathbb{N}} \subset BV(0,1)$ defined through $u_n(x) = nx$ for $x \le 1/n$ and $u(x) = 1$ for $x \ge 1/n$, we have $u_n \rightharpoonup u$ with $u \equiv 1$ but $u_n(0) = 0$ for all $n \in \mathbb{N}$.

10.1.3 A Variational Model Problem on $BV(\Omega)$

To understand the finite element approximation of variational problems involving total variation, we consider, for given $g \in L^2(\Omega)$ and $\alpha > 0$, minimizing the functional

$$I(u) = |Du|(\Omega) + \frac{\alpha}{2}\int_\Omega (u-g)^2 \,\mathrm{d}x$$

as defined for $u \in BV(\Omega) \cap L^2(\Omega)$. By the density of smooth functions we may choose a bounded infimizing sequence $(u_n)_{n\in\mathbb{N}} \subset W^{1,1}(\Omega) \cap L^2(\Omega)$. Due to the lack of reflexivity or more generally an existing separable predual space, we cannot extract a weakly convergent subsequence in $W^{1,1}(\Omega)$. A weak limit of a subsequence exists in the space $BV(\Omega) \cap L^2(\Omega)$.

Theorem 10.5 (Existence) *There exists a minimizer $u \in BV(\Omega) \cap L^2(\Omega)$ for I.*

Proof The functional I is bounded from below and the set of admissible functions is nonempty, and hence there exists a bounded infimizing sequence $(u_n)_{n\in\mathbb{N}} \subset BV(\Omega) \cap L^2(\Omega)$. Theorem 10.3 guarantees the existence of a weakly convergent subsequence $(u_{n_j})_{j\in\mathbb{N}}$ with weak limit $u \in BV(\Omega)$ and Theorem 10.1 implies $I(u) \le \liminf_{j\to\infty} I(u_{n_j})$, i.e., u is a minimizer for I. □

Remark 10.6 The existence of solutions subject to Dirichlet boundary conditions $u|_{\partial\Omega} = u_{\rm D}$ for $u_{\rm D} \in L^1(\partial\Omega)$ is difficult to establish due to the lack of weak continuity of the trace operator.

The following stability result implies the uniqueness of minimizers.

Theorem 10.6 (Stability and uniqueness) *For $g_1, g_2 \in L^2(\Omega)$ let the functions $u_1, u_2 \in BV(\Omega) \cap L^2(\Omega)$ be minimizers of I with g replaced by g_1 and g_2, respectively. We then have*

$$\|u_1 - u_2\| \le \|g_1 - g_2\|.$$

In particular, minimizers are uniquely defined.

Proof We define the convex functionals $F : BV(\Omega) \to \mathbb{R}$ and $G_\ell : L^2(\Omega) \to \mathbb{R}$, $\ell = 1, 2$, by

$$F(u) = |Du|(\Omega), \quad G_\ell(u) = (\alpha/2)\|u - g_\ell\|^2$$

and set $I_\ell = F + G_\ell$. We extend F to $L^2(\Omega)$ with the value $+\infty$, and note that G_ℓ is Fréchet differentiable with

$$\delta G_\ell(u)[v] = \alpha(u - g_\ell, v)$$

for all $v \in L^2(\Omega)$. Since F is convex, we have that its subdifferential is monotone, i.e., for $\mu_\ell \in \partial F(u_\ell)$, $\ell = 1, 2$, we have

$$(\mu_2 - \mu_1, u_2 - u_1) \ge 0.$$

Noting that $0 \in \partial I_\ell(u_\ell)$ we deduce that $-\delta G_\ell(u_\ell) \in \partial F(u_\ell)$ for $\ell = 1, 2$, and therefore

$$\big(-\alpha(u_2 - g_2) + \alpha(u_1 - g_1), u_2 - u_1\big) \ge 0.$$

This implies that

$$\|u_2 - u_1\|^2 \le (u_2 - u_1, g_2 - g_1)$$

and an application of Hölder's inequality proves the asserted bound. □

Due to a monotonicity property of the total variation, a maximum principle holds for the minimization problem.

Proposition 10.2 (Maximum principle) *If $g \in L^\infty(\Omega)$, then the minimizer $u \in BV(\Omega) \cap L^2(\Omega)$ for I satisfies $u \in L^\infty(\Omega)$ with $\|u\|_{L^\infty(\Omega)} \le \|g\|_{L^\infty(\Omega)}$.*

Proof Assume that $g(x) \le \overline{g}$ for almost every $x \in \Omega$ and given the minimizer $u \in BV(\Omega) \cap L^2(\Omega)$ for I, define $\widetilde{u}(x) = \min\{u(x), \overline{g}\}$ for $x \in \Omega$. According to Remark 10.2 we have $\widetilde{u} \in BV(\Omega)$ with $|D\widetilde{u}|(\Omega) \le |Du|(\Omega)$. Since also $\|\widetilde{u} - g\| \le \|u - g\|$, we deduce that $I(\widetilde{u}) \le I(u)$. This implies $u = \widetilde{u}$ and $u \le \overline{g}$. The same argument shows that $u \ge \underline{g}$ if $g(x) \ge \underline{g}$ for almost every $x \in \Omega$. Therefore $u \in L^\infty(\Omega)$ with the asserted bound. □

Useful information about the minimization of I is contained in the related dual problem. To identify it, we note that by a completion of $C_c^\infty(\Omega;\mathbb{R}^d)$ with respect to the norm $\|p\|_{H(\mathrm{div};\Omega)} = \|p\| + \|\operatorname{div} p\|$, the total variation $|Du|(\Omega)$ of a function $u \in BV(\Omega) \cap L^2(\Omega)$ can equivalently be characterized as

$$|Du|(\Omega) = \sup\Big\{ -\int_\Omega u \operatorname{div} p \,\mathrm{d}x : p \in H_N(\mathrm{div};\Omega),\ |p| \le 1 \text{ in } \Omega\Big\},$$

where

$$H_N(\mathrm{div};\Omega) = \{p \in L^2(\Omega;\mathbb{R}^d) : \operatorname{div} p \in L^2(\Omega),\ p\cdot n|_{\partial\Omega} = 0\}.$$

For the minimization problem defined through I, we thus have with the indicator functional $I_{K_1(0)}$ of the set

$$K_1(0) = \{p \in L^2(\Omega;\mathbb{R}^d) : |p| \le 1 \text{ almost everywhere in } \Omega\}$$

that

$$\begin{aligned}\inf_{u\in BV\cap L^2} I(u) &= \inf_{u\in BV\cap L^2} |Du|(\Omega) + \frac{\alpha}{2}\|u-g\|^2 \\ &= \inf_{u\in BV\cap L^2}\ \sup_{p\in H_N(\mathrm{div})} \Big(-\int_\Omega u \operatorname{div} p \,\mathrm{d}x + \frac{\alpha}{2}\|u-g\|^2 - I_{K_1(0)}(p)\Big).\end{aligned}$$

This defines a saddle point problem with unknowns u and p. The dual problem is obtained by eliminating u. For this we assume that the order of the infimum and supremum can be interchanged, i.e.,

$$\inf_{u\in BV\cap L^2} I(u) = \sup_{p\in H_N(\mathrm{div})}\ \inf_{u\in BV\cap L^2} \Big(-\int_\Omega u \operatorname{div} p \,\mathrm{d}x + \tfrac{\alpha}{2}\|u-g\|^2 - I_{K_1(0)}(p)\Big).$$

A direct calculation shows that the solution u of the inner minimization problem is for $p \in H_N(\mathrm{div};\Omega)$ given by

$$u = g + \alpha^{-1} \operatorname{div} p,$$

and thus

$$\inf_{u\in BV\cap L^2} I(u) = \sup_{p\in H_N(\mathrm{div})} -\frac{1}{2\alpha}\|\operatorname{div} p + \alpha g\|^2 + \frac{1}{2\alpha}\|\alpha g\|^2 - I_{K_1(0)}(p).$$

The maximization problem defined by the right-hand side is the dual problem. The heuristic interchange of the infimum and the supremum can be rigorously justified and leads to the following result.

Proposition 10.3 (Strong duality) *For $p \in H_N(\mathrm{div};\Omega)$ define*

$$D(p) = -\frac{1}{2\alpha}\|\ \mathrm{div}\ p + \alpha g\|^2 + \frac{\alpha}{2}\|g\|^2 - I_{K_1(0)}(p).$$

We have

$$\inf_{u\in BV(\Omega)\cap L^2(\Omega)} I(u) = \sup_{p\in H_N(\mathrm{div};\Omega)} D(p).$$

Moreover, there exists a solution $p \in H_N(\mathrm{div};\Omega)$ that maximizes the functional D.

Proof The reader is referred to [12] for a proof of the result which is established by showing that I is the Fenchel dual of D in the sense of [11]. □

Remark 10.7 Exchanging the order of the infimum and supremum always leads to the weak duality principle $\inf_u I(u) \geq \sup_p D(p)$.

Proposition 10.4 *The unique solution $u \in BV(\Omega) \cap L^2(\Omega)$ of the minimization problem defined by I and every solution $p \in H_N(\mathrm{div};\Omega)$ of the maximization problem defined by D correspond to a saddle point for the functional*

$$L(u,p) = -\int_\Omega u\ \mathrm{div}\ p\,\mathrm{d}x + \frac{\alpha}{2}\|u-g\|^2 - I_{K_1(0)}(p)$$

and are related by

$$\mathrm{div}\ p = \alpha(u-g), \quad Du \in \partial I_{K_1(0)}(p),$$

where the inclusion is understood as

$$-\big(u,\ \mathrm{div}\,(q-p)\big) \leq 0$$

for all $q \in H_N(\mathrm{div};\Omega) \cap K_1(0)$.

Proof The proof follows from standard arguments in convex optimization, cf., e.g., [11]. □

Remarks 10.8 (i) The inclusion $Du \in \partial I_{K_1(0)}(p)$ is formally equivalent to $p \in \partial|Du|$. In particular, we have $p = \nabla u/|\nabla u|$ in regions where $\nabla u \neq 0$.
(ii) In the case of Dirichlet boundary conditions on $\partial\Omega$, the space $H_N(\mathrm{div};\Omega)$ is replaced by $H(\mathrm{div};\Omega) = \{p \in L^2(\Omega;\mathbb{R}^d) :\ \mathrm{div}\ p \in L^2(\Omega)\}$.

An explicit solution can be constructed in the case of Dirichlet boundary conditions.

Example 10.4 Let $r > 0$ be such that $B_r(0) \subset \Omega$ and define $g = \chi_{B_r(0)}$. Then

$$u = \max\{0, 1 - d/(\alpha r)\}\chi_{B_r}(0)$$

is the minimizer for I subject to $u|_{\partial\Omega} = 0$.

Proof Assume that $d \leq \alpha r$ and define

$$p(x) = \begin{cases} -r^{-1}x & \text{for} \quad |x| \leq r, \\ -rx/|x|^2 & \text{for} \quad |x| \geq r. \end{cases}$$

Then $p \in H(\text{div};\Omega)$ with $\text{div}\; p = -(d/r)\chi_{B_r(0)}$ and $|p| \leq 1$. Moreover, we have $u = (1/\alpha)\;\text{div}\; p + g$. Since $p = -n$ on $\partial B_r(0)$ we have for every $q \in H(\text{div};\Omega)$ with $|q| \leq 1$ that

$$-(u, \text{div}(q-p)) = -\big(1 - d/(\alpha r)\big)\int_{\partial B_r(0)} (q-p)\cdot n\,\mathrm{d}s \leq 0.$$

If $d \geq \alpha r$, we define

$$p(x) = \begin{cases} -(\alpha/d)x & \text{for} \quad |x| \leq r, \\ -(\alpha/d)r^2x/|x|^2 & \text{for} \quad |x| \geq r \end{cases}$$

and verify $\text{div}\; p = -\alpha\chi_{B_r(0)} = -\alpha g$, i.e., $u = (1/\alpha)\;\text{div}\; p + g = 0$, and $|p| \leq \alpha r/d \leq 1$. Since $u = 0$ the variational inclusion $Du \in \partial I_{K_1(0)}(p)$ is satisfied. □

10.2 Numerical Approximation

We discuss in this section the numerical approximation and iterative solution of the minimization problem defined through the functional I, which for every function $u \in BV(\Omega) \cap L^2(\Omega)$ is given by

$$I(u) = |Du|(\Omega) + \frac{\alpha}{2}\|u - g\|^2$$

for $\alpha > 0$ and $g \in L^2(\Omega)$. The subsequent discussion is based on results in [6–8, 10].

10.2.1 $W^{1,1}$ Conforming Approximation

The finite element space $\mathscr{S}^1(\mathscr{T}_h)$ defines a subspace of $BV(\Omega) \cap W^{1,1}(\Omega)$. Due to the density of smooth functions in $BV(\Omega)$ with respect to intermediate convergence, we can approximate functions in $BV(\Omega)$ by functions in $\mathscr{S}^1(\mathscr{T}_h)$. The following lemma provides bounds on the approximation error. For ease of presentation we restrict to the case $d = 2$.

Lemma 10.1 (Approximation of BV functions) *Assume that $\Omega \subset \mathbb{R}^2$ is star-shaped and let $\varepsilon > 0$. For every $u \in BV(\Omega)$ there exists $u_{\varepsilon,h} \in \mathscr{S}^1(\mathscr{T}_h)$ such that*

$$\|\nabla u_{\varepsilon,h}\|_{L^1(\Omega)} \le (1 + ch\varepsilon^{-1} + c\varepsilon)|Du|(\Omega),$$

and

$$\|u_{\varepsilon,h} - u\|_{L^1(\Omega)} \le c(h^2\varepsilon^{-1} + \varepsilon)|Du|(\Omega).$$

If $u \in L^\infty(\Omega)$, then $\|u_{\varepsilon,h}\|_{L^\infty(\Omega)} \le \|u\|_{L^\infty(\Omega)}$.

Proof Since $C^\infty(\overline{\Omega})$ is dense in $BV(\Omega)$ with respect to intermediate convergence we may choose a function $\widetilde{u} \in C^1(\overline{\Omega})$, such that $\|\widetilde{u} - u\|_{L^1(\Omega)} \le c\varepsilon|Du|(\Omega)$ and $\|\nabla\widetilde{u}\|_{L^1(\Omega)} \le (1 + \varepsilon)|Du|(\Omega)$. Moreover, if $u \in L^\infty(\Omega)$, then we have that $\|\widetilde{u}\|_{L^\infty(\Omega)} \le \|u\|_{L^\infty(\Omega)}$. This allows us to assume $u \in C^1(\overline{\Omega})$ in the following. We suppose that Ω is star-shaped with respect to 0 and define the set $\widehat{\Omega}_\varepsilon = (1 + \varepsilon)\Omega$ and the linear transformation $\phi : \widehat{\Omega}_\varepsilon \to \Omega$, $\widehat{x} \mapsto \widehat{x}/(1 + \varepsilon)$. We set $\widehat{u}_\varepsilon = u \circ \phi$. and with a nonnegative convolution kernel $\rho_\varepsilon \in C^\infty(\mathbb{R}^2)$, we let $u_\varepsilon = (\widehat{u}_\varepsilon * \rho_\varepsilon)|_\Omega$ and define $u_{\varepsilon,h} = \mathscr{I}_h u_\varepsilon$. To prove the estimates we first note that nodal interpolation estimates guarantee

$$\|u_{\varepsilon,h} - u_\varepsilon\|_{L^1(\Omega)} + h\|\nabla(u_{\varepsilon,h} - u_\varepsilon)\| \le ch^2\|D^2 u_\varepsilon\|_{L^1(\Omega)}.$$

Standard mollification arguments show that

$$\begin{aligned} \|u_\varepsilon - \widehat{u}_\varepsilon\|_{L^1(\Omega)} &\le c\varepsilon\|\nabla\widehat{u}_\varepsilon\|_{L^1(\widehat{\Omega}_\varepsilon)}, \\ \varepsilon\|D^2 u_\varepsilon\|_{L^1(\Omega)} + \|\nabla u_\varepsilon\|_{L^1(\Omega)} &\le \|\nabla\widehat{u}_\varepsilon\|_{L^1(\widehat{\Omega}_\varepsilon)}. \end{aligned}$$

A transformation argument and a direct calculation imply that

$$\begin{aligned} \|\widehat{u}_\varepsilon - u\|_{L^1(\Omega)} &\le c\varepsilon\|\nabla u\|_{L^1(\Omega)}, \\ \|\nabla\widehat{u}_\varepsilon\|_{L^1(\Omega)} &\le (1 + \varepsilon)\|\nabla u\|_{L^1(\Omega)}. \end{aligned}$$

The combination of the estimates proves the asserted bounds for the case $u \in C^1(\overline{\Omega})$. The estimate $\|u_{\varepsilon,h}\|_{L^\infty(\Omega)} \le \|u\|_{L^\infty(\Omega)}$ is a direct consequence of the construction. □

Remarks 10.9 (i) For $d \ge 3$ the same result can be proved by employing a quasi-interpolation operator instead of the nodal interpolation operator.
(ii) The estimate of the lemma and Hölder's inequality imply that for functions $u \in BV(\Omega) \cap L^\infty(\Omega)$ we have $\inf_{v_h \in \mathscr{S}^1(\mathscr{T}_h)} \|u - v_h\|_{L^p(\Omega)} \le ch^{1/p}$ for $1 \le p < \infty$.

(iii) Optimizing the convergence rates of the estimates in the lemma simultaneously for intermediate convergence leads to the choice $\varepsilon = h^{1/2}$ and the suboptimal estimate $\|u - u_{\varepsilon,h}\|_{L^1(\Omega)} \leq ch^{1/2}$.

Since the functional I is strongly convex, the distance of any function to the minimum is controlled by the difference of the values of the functional.

Lemma 10.2 (Convexity) *If $u \in BV(\Omega) \cap L^2(\Omega)$ is the minimizer for I, then we have*

$$\frac{\alpha}{2}\|u - v\|^2 \leq I(v) - I(u)$$

for every $v \in BV(\Omega) \cap L^2(\Omega)$.

Proof We define $F : BV(\Omega) \to \mathbb{R}$ and $G : L^2(\Omega) \to \mathbb{R}$ by

$$F(u) = |Du|(\Omega), \quad G(u) = \frac{\alpha}{2}\|u - g\|^2$$

and extend F by $+\infty$ to $L^2(\Omega)$. Then F is convex and G is strongly convex and Fréchet differentiable with $\delta G(u)[w] = \alpha(u - g, w)$, i.e., we have

$$\delta G(u)[v - u] + \frac{\alpha}{2}\|u - v\|^2 + G(u) = G(v)$$

for all $u, v \in L^2(\Omega)$. Since $u \in BV(\Omega) \cap L^2(\Omega)$ is a minimizer, we have

$$0 \in \partial I(u) = \partial F(u) + \delta G(u),$$

or equivalently $-\delta G(u) \in \partial F(u)$, i.e.,

$$-\delta G(u)[v - u] + F(u) \leq F(v).$$

The strong convexity of G yields

$$\frac{\alpha}{2}\|u - v\|^2 + G(u) - G(v) + F(u) \leq F(v)$$

which proves the assertion. □

Theorem 10.7 (Error estimate) *Assume that $\Omega \subset \mathbb{R}^2$ is star-shaped and $g \in L^\infty(\Omega)$. Let $u \in BV(\Omega) \cap L^2(\Omega)$ and $u_h \in \mathcal{S}^1(\mathcal{T}_h)$ be the minimizers for I in the respective spaces. We then have*

$$\frac{\alpha}{2}\|u - u_h\|^2 \leq ch^{1/2}.$$

Proof Lemma 10.2 and the fact that $I(u_h) \leq I(v_h)$ for all $v_h \in \mathcal{S}^1(\mathcal{T}_h)$ imply that

$$
\begin{aligned}
\frac{\alpha}{2}\|u-u_h\|^2 &\le I(u_h)-I(u) \le I(v_h)-I(u)\\
&= \|\nabla v_h\|_{L^1(\Omega)} - |Du|(\Omega)\\
&\quad + \frac{\alpha}{2}\int_\Omega \big((v_h-g)-(u-g)\big)\big((v_h-g)+(u-g)\big)\,\mathrm{d}x\\
&\le \|\nabla v_h\|_{L^1(\Omega)} - |Du|(\Omega) + \frac{\alpha}{2}\|v_h-u\|_{L^1(\Omega)}\|v_h+u-2g\|_{L^\infty(\Omega)}.
\end{aligned}
$$

For $\varepsilon > 0$ we let $v_h = u_{\varepsilon,h} \in \mathscr{S}^1(\mathscr{T}_h)$ be an approximation of u as in Lemma 10.1 and deduce that

$$
\frac{\alpha}{2}\|u-u_h\|^2 \le c(h\varepsilon^{-1}+\varepsilon)|Du|(\Omega) + c(h^2\varepsilon^{-1}+\varepsilon)|Du|(\Omega).
$$

With $\varepsilon = h^{1/2}$ we find the asserted bound. □

Remarks 10.10 (i) Since for $u \in BV(\Omega)\cap L^2(\Omega)$ the best approximation in $\mathscr{S}^1(\mathscr{T}_h)$ satisfies $\inf_{v_h\in\mathscr{S}^1(\mathscr{T}_h)}\|u-v_h\| \le h^{1/2}$, the convergence rate of the theorem is sub-optimal. Numerical experiments indicate that the optimal convergence rate $\mathscr{O}(h^{1/2})$ in $L^2(\Omega)$ is in general not attained.
(ii) If $\Omega = (a,b) \subset \mathbb{R}$ and the minimizer $u \in BV(\Omega)\cap L^2(\Omega)$ is piecewise continuous, then we can employ the nodal interpolant $v_h = \mathscr{I}_h u$ in the proof of the theorem and noting that $\|\nabla \mathscr{I}_h u\| \le |Du|(\Omega)$ and $\|u-\mathscr{I}_h u\|_{L^1(\Omega)} \le ch|Du|(\Omega)$, we obtain the quasi-optimal estimate $\|u-u_h\| \le ch^{1/2}$.

The best approximation result $\inf_{v_h\in\mathscr{S}^1(\mathscr{T}_h)}\|u-v_h\|_{L^p(\Omega)} \le ch^{1/p}$ for functions $u \in BV(\Omega)\cap L^\infty(\Omega)$ can in general not be improved as the following example shows.

Example 10.5 Let $\Omega = (-1,1)$, $\mathscr{T}_h$ a uniform triangulation of Ω with mesh-size $h > 0$ such that $z = 0$ is a node of $\mathscr{T}_h$. For $u = \mathrm{sign}$, we then have

$$
\inf_{v_h\in\mathscr{S}^1(\mathscr{T}_h)}\|u-v_h\|_{L^p(\Omega)} \ge ch^{1/p}.
$$

To prove this we show that the entire approximation error is concentrated at the discontinuity at $x = 0$. We assume that there exists a minimal $w_h \in \mathscr{S}^1(\mathscr{T}_h)$ which is antisymmetric, i.e., we have $w_h(-x) = -w_h(x)$ for $x \in (0,1)$ and $w_h(0) = 0$. Then the function w_h is affine on $(-h,h)$ with slope $a/h \in \mathbb{R}$, cf. Fig. 10.3, for some $a \in \mathbb{R}$, and we have with the transformation $y = x/h$ that

$$
\int_{(-h,h)} |u-w_h|^p\,\mathrm{d}x = 2\int_{(0,h)} |1-ax/h|^p\,\mathrm{d}x = 2h\int_{(0,1)} |1-ay|^p\,\mathrm{d}y.
$$

The value of the integral related to the minimizing choice of a is positive and independent of h which implies that $\|u-w_h\|_{L^p(\Omega)} \ge ch^{1/p}$.

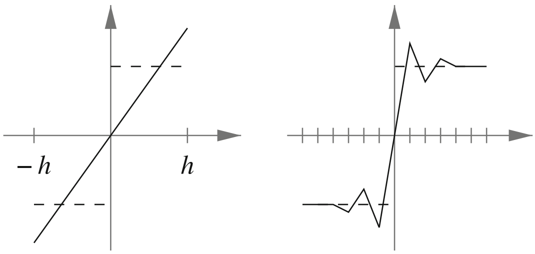

Fig. 10.3 The approximation of a discontinuous function with continuous, piecewise affine functions leads to an error $\|u - w_h\|_{L^p(\Omega)} \geq ch^{1/p}$ (*left*); for the best approximation of $u = \text{sign}$ in $\mathscr{S}^1(\mathscr{T}_h)$ with respect to the L^2 norm, the Gibb's phenomenon occurs at the discontinuity (*right*)

10.2.2 Piecewise Constant Approximation

The set of piecewise constant finite element functions $\mathscr{L}^0(\mathscr{T}_h)$ is a subset of $BV(\Omega)$. It is straightforward to check that for a sequence of triangulations their union defines a dense subset with respect to weak convergence. We will show that density with respect to intermediate convergence fails and hence that the discretization of the model problem with piecewise constant finite elements may not approximate the right minimum.

Proposition 10.5 (Piecewise constant functions) *For every $u_h \in \mathscr{L}^0(\mathscr{T}_h)$ we have*

$$|Du_h|(\Omega) = \sum_{S \in \mathscr{S}_h \cap \Omega} \|[\![u_h]\!]\|_{L^1(S)}.$$

Proof The identity follows directly from an elementwise integration by parts. □

Proposition 10.6 (Nonapproximation) *Let $\Omega = (-1/2, 1/2) \times (0, 1)$ and let $u \in BV(\Omega) \cap L^\infty(\Omega)$ be, for $x = (x_1, x_2) \in \Omega$, defined by $u(x_1, x_2) = \chi_{\{x_1 < 0\}}$. For each $n \geq 1$ let $\mathscr{T}_n$ be the triangulation of Ω with maximal mesh-size $h_n = 1/n$, as shown in Fig. 10.4. Then there is no sequence $(u_n)_{n \in \mathbb{N}} \subset L^1(\Omega)$ with $u_n \in \mathscr{L}^0(\mathscr{T}_n)$ for all $n \in \mathbb{N}$ such that $u_n \to u$ in $L^1(\Omega)$ and $|Du_n|(\Omega) \to |Du|(\Omega) = 1$ as $n \to \infty$.*

Proof Let $(u_n)_{n \in \mathbb{N}}$ be a sequence with $u_n \in \mathscr{L}^0(\mathscr{T}_n)$ such that $\|u_n - u\|_{L^1(\Omega)} \to 0$ and $|Du_n|(\Omega) \leq c$ for all $n \in \mathbb{N}$. Given $n \in \mathbb{N}$ we define the sets R_j^n for $j = 1, 2, \ldots, n$ by

$$R_j^n = \{(x_1, x_2) \in \Omega : (j-1)/n < x_2 < j/n\}$$

and set $R^n = R_1^n$. Let $\overline{u}_n \in L^1(R^n)$ be the average of u_n over all strips, i.e., for $(x_1, x_2) \in R^n$ set

$$\overline{u}_n(x_1, x_2) = \frac{1}{n} \sum_{j=1}^{n} u_n(x_1, x_2 + j/n),$$

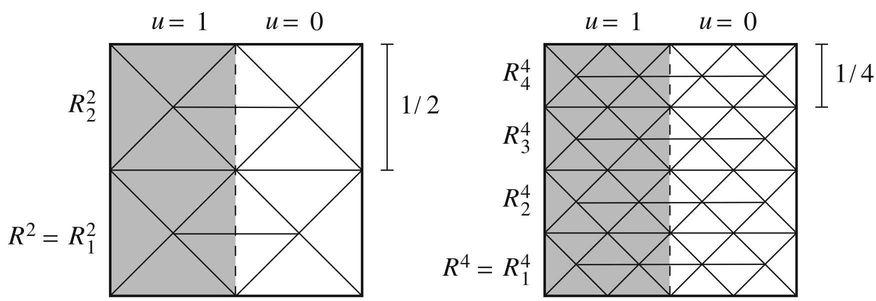

Fig. 10.4 Construction of triangulations $\mathscr{T}_n$, $n \in \mathbb{N}$, of $\Omega = (-1/2, 1/2) \times (0, 1)$ on which piecewise constant finite element functions are not dense in $BV(\Omega)$ with respect to intermediate convergence; the jump set of the function $u = \chi_{\{x_1<0\}}$ is not resolved by the triangulations

and reflect $\overline{u}_n$ across the x_1-axis, i.e., $\overline{u}_n(x_1, -x_2) = \overline{u}_n(x_1, x_2)$ for $(x_1, x_2) \in R^n$. We then define $\widetilde{u}_n \in L^1(\Omega)$ by periodically extending $\overline{u}_n$ with period $2/n$ in the x_2-direction. Then $\widetilde{u}_n \in L^1(\Omega)$ is continuous across the interfaces $\overline{R}_j^n \cap \overline{R}_{j+1}^n$ for $j = 1, 2, \ldots, n-1$ and we have $\|\widetilde{u}_n - u\|_{L^1(R_j^n)} = \|\overline{u}_n - u\|_{L^1(R^n)}$ and $|D\widetilde{u}_n|(R_j^n) = |D\overline{u}_n|(R^n)$ for $j = 1, 2, \ldots, n$, where $|D\overline{u}_n|(R^n)$ denotes the total variation of $D\overline{u}_n$ on R^n. With the triangle inequality we verify that

$$
\begin{aligned}
|D\widetilde{u}_n|(\Omega) &= n|D\overline{u}_n|(R^n) \le |Du_n|(\Omega), \\
\|\widetilde{u}_n - u\|_{L^1(\Omega)} &= n\|\overline{u}_n - u\|_{L^1(R^n)} \le \|u_n - u\|_{L^1(\Omega)}.
\end{aligned}
$$

For every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\|u_n - u\|_{L^1(\Omega)} < \varepsilon$ for all $n \ge N$, i.e.,

$$
\|\overline{u}_n - u\|_{L^1(R^n)} < \varepsilon/n.
$$

For each $n \ge N$ there exist distinct triangles $T_+^1, T_+^2, T_-^1, T_-^2 \in \mathscr{T}_n \cap R^n$ with $\overline{u}_n|_{T_+^1 \cup T_+^2} \ge 1 - 4\varepsilon$ and $\overline{u}_n|_{T_-^1 \cup T_-^2} \le 4\varepsilon$ since otherwise $\|\overline{u}_n - u\|_{L^1(R^n)} \ge \varepsilon/n$. The triangle inequality along disjoint paths of neighboring elements connecting T_-^j and T_+^j for $j = 1, 2$, respectively, yields that

$$
\begin{aligned}
(1 - 8\varepsilon)\sqrt{2}/n &\le (h_n/\sqrt{2})\Big(\big|\overline{u}_n|_{T_-^1} - \overline{u}_n|_{T_+^1}\big| + \big|\overline{u}_n|_{T_-^2} - \overline{u}_n|_{T_+^2}\big|\Big) \\
&\le \sum_{S \in \mathscr{S}_h \cap R^n} \|[\![\overline{u}_n]\!]\|_{L^1(S)} = |D\overline{u}_n|(R^n)
\end{aligned}
$$

and hence $|Du_n|(\Omega) \ge |D\widetilde{u}_n|(\Omega) \ge (1 - 8\varepsilon)\sqrt{2}$ for all $n \ge N$, i.e., we have that $|Du_n|(\Omega) \not\to 1 = |Du|(\Omega)$ as $n \to \infty$. □

10.2.3 Iterative Solution

To develop an iterative solution method for the nondifferentiable minimization problem, we first state optimality conditions for the minimization of I in $\mathscr{S}^1(\mathscr{T}_h)$. For this we note that the minimization of I can be equivalently expressed as a saddle-point problem; that is, due to the fact that ∇u_h is elementwise constant for $u_h \in \mathscr{S}^1(\mathscr{T}_h)$ we have

$$\begin{aligned}\inf_{u_h \in \mathscr{S}^1(\mathscr{T}_h)} \int_\Omega |\nabla u_h| \,\mathrm{d}x + \frac{\alpha}{2}\|u_h - g\|^2 &= \inf_{u_h \in \mathscr{S}^1(\mathscr{T}_h)} \sup_{p_h \in \mathscr{L}^0(\mathscr{T}_h)^d} \int_\Omega p_h \cdot \nabla u_h \,\mathrm{d}x \\ &\quad + \frac{\alpha}{2}\|u_h - g\|^2 - I_{K_1(0)}(p_h) \\ &= \inf_{u_h \in \mathscr{S}^1(\mathscr{T}_h)} \sup_{p_h \in \mathscr{L}^0(\mathscr{T}_h)^d} L_h(u_h, p_h),\end{aligned}$$

where $I_{K_1(0)}$ is the indicator functional of the set $K_1(0) = \{p \in L^\infty(\Omega; \mathbb{R}^d) : |p| \le 1 \text{ a.e. in } \Omega\}$.

Lemma 10.3 (Optimality) *The function $u_h \in \mathscr{S}^1(\mathscr{T}_h)$ minimizes I in $\mathscr{S}^1(\mathscr{T}_h)$ if and only if there exists $p_h \in \mathscr{L}^0(\mathscr{T}_h)^d$ with $|p_h| \le 1$ in Ω such that*

$$(p_h, \nabla v_h) = -\alpha(u_h - g, v_h), \quad (\nabla u_h, q_h - p_h) \le 0$$

for all $(v_h, q_h) \in \mathscr{S}^1(\mathscr{T}_h) \times \mathscr{L}^0(\mathscr{T}_h)^d$ with $|q_h| \le 1$ in Ω.

Proof The existence of a saddle point $(u_h, p_h) \in \mathscr{S}^1(\mathscr{T}_h) \times \mathscr{L}^0(\mathscr{T}_h)^d$ follows from the fact that the Lagrangian function L_h is a lower-semicontinuous, proper, convex-concave function, cf., e.g., [14] for details. The equations are the corresponding Kuhn–Tucker optimality conditions, i.e.,

$$0 = \delta_{u_h} L_h(u_h, p_h), \quad 0 \in \partial_{p_h} L_h(u_h, p_h),$$

where we note that $\xi_h \in \partial I_{K_1(0)}(p_h)$ for $\xi_h \in \mathscr{L}^0(\mathscr{T}_h)^d$ and $p_h \in \mathscr{L}^0(\mathscr{T}_h)^d \cap K_1(0)$, i.e.,

$$(\xi_h, q_h - p_h) + I_{K_1(0)}(p_h) \le I_{K_1(0)}(q_h)$$

for all $q_h \in \mathscr{L}^0(\mathscr{T}_h)^d$, if and only if

$$(\xi_h, q_h - p_h) \le 0$$

for all $q_h \in \mathscr{L}^0(\mathscr{T}_h)^d \cap K_1(0)$. □

To find a saddle point for L_h we use a descent flow with respect to u_h and an ascent flow with respect to p_h, i.e.,

$$\partial_t u_h = -\delta_{u_h} L_h(u_h, p_h), \quad \partial_t p_h \in \partial_{p_h} L_h(u_h, p_h).$$

With an appropriate time-discretization and a discrete inner product $(\cdot,\cdot)_{h,s}$ on $\mathscr{S}^1(\mathscr{T}_h)$ that may differ from the L^2 inner product, this motivates the following iteration which specifies the abstract primal-dual iteration of Algorithm 4.5.

Algorithm 10.1 (*Primal-dual iteration*) Let $(\cdot,\cdot)_{h,s}$ be an inner product on $\mathscr{S}^1(\mathscr{T}_h)$, $\tau > 0$, $(u_h^0, p_h^0) \in \mathscr{S}^1(\mathscr{T}_h) \times \mathscr{L}^0(\mathscr{T}_h)^d$, set $d_t u_h^0 = 0$, and for $k = 0, 1, \dots$ with $\widetilde{u}_h^k = u_h^{k-1} + \tau d_t u_h^{k-1}$ solve the equations

$$(-d_t p_h^k + \nabla \widetilde{u}_h^k, q_h - p_h^k) \le 0,$$
$$(d_t u_h^k, v_h)_{h,s} + (p_h^k, \nabla v_h) + \alpha(u_h^k - g, v_h) = 0$$

subject to $|p_h^k| \le 1$ in Ω for all $(v_h, q_h) \in \mathscr{S}^1(\mathscr{T}_h) \times \mathscr{L}^0(\mathscr{T}_h)^d$ with $|q_h| \le 1$ in Ω. Stop the iteration if $\|d_t u_h^k\|_{h,s} \le \varepsilon_{\text{stop}}$.

Remark 10.11 Notice that p_h^k is the unique minimizer of the mapping

$$q_h \mapsto \frac{1}{2\tau}\|q_h - p_h^{k-1}\|^2 - (q_h, \nabla \widetilde{u}_h^k) + I_{K_1(0)}(q_h)$$

and given by the truncation operation

$$p_h^k = \big(p_h^{k-1} + \tau \nabla \widetilde{u}_h^k\big) / \max\{1, |p_h^{k-1} + \tau \nabla \widetilde{u}_h^k|\}$$

which can be computed elementwise.

The iterates of Algorithm 10.1 converge to a stationary point if τ is sufficiently small.

Proposition 10.7 (Convergence) *Let $u_h \in \mathscr{S}^1(\mathscr{T}_h)$ be minimal for I in $\mathscr{S}^1(\mathscr{T}_h)$ and define*

$$\theta = \sup_{v_h \in \mathscr{S}^1(\mathscr{T}_h)\setminus\{0\}} \frac{\|\nabla v_h\|}{\|v_h\|_{h,s}}.$$

If $\tau\theta \le 1$, then the iterates of Algorithm 10.1 converge to u_h in the sense that they satisfy for every $N \ge 1$

$$\tau \sum_{k=1}^N \Big((1-\tau^2\theta^2)\frac{\tau}{2}\|d_t u_h^k\|_{h,s}^2 + \alpha\|u_h - u_h^k\|^2\Big) \le \frac{1}{2}\big(\|u_h - u_h^0\|_{h,s}^2 + \|p_h - p_h^0\|^2\big).$$

Proof Let $p_h \in \mathscr{L}^0(\mathscr{T}_h)^d$ be as in Lemma 10.3. Upon choosing $v_h = u_h - u_h^k$ and $q_h = p_h$ in Algorithm 10.1, we find that

$$\frac{d_t}{2}\big(\|u_h - u_h^k\|_{h,s}^2 + \|p_h - p_h^k\|^2\big) + \frac{\tau}{2}\big(\|d_t u_h^k\|_{h,s}^2 + \|d_t p_h^k\|^2\big) + \alpha\|u_h - u_h^k\|^2$$
$$= -(d_t u_h^k, u_h - u_h^k)_{h,s} - (d_t p_h^k, p_h - p_h^k) + \alpha\|u_h - u_h^k\|^2$$

$$\leq (p_h^k, \nabla(u_h - u_h^k)) + \alpha(u_h^k - g, u_h - u_h^k) - (p_h - p_h^k, \nabla\widetilde{u}_h^k) + \alpha\|u_h - u_h^k\|^2.$$

Using that

$$(u_h^k - g, u_h - u_h^k) + \|u_h - u_h^k\|^2 = (u_h - g, u_h - u_h^k)$$

and choosing $q_h = p_h^k$ in Lemma 10.3, we deduce that

$$\begin{aligned}
&\frac{d_t}{2}\big(\|u_h - u_h^k\|_{h,s}^2 + \|p_h - p_h^k\|^2\big) + \frac{\tau}{2}\big(\|d_t u_h^k\|_{h,s}^2 + \|d_t p_h^k\|^2\big) + \alpha\|u_h - u_h^k\|^2 \\
&\quad = (p_h^k, \nabla(u_h - u_h^k)) - (p_h - p_h^k, \nabla\widetilde{u}_h^k) + \alpha(u_h - g, u_h - u_h^k) \\
&\quad = (p_h^k, \nabla(u_h - u_h^k)) - (p_h - p_h^k, \nabla\widetilde{u}_h^k) - (p_h, \nabla(u_h - u_h^k)) \\
&\quad = (p_h - p_h^k, \nabla(u_h^k - \widetilde{u}_h^k)) + (p_h^k - p_h, \nabla u_h) \\
&\quad \leq (p_h - p_h^k, \nabla(u_h^k - \widetilde{u}_h^k)) = \tau^2(p_h - p_h^k, \nabla d_t^2 u_h^k),
\end{aligned}$$

where we used $u_h^k - \widetilde{u}_h^k = \tau^2 d_t^2 u_h^k$ in the last identity. Multiplication by τ, summation over $k = 1, 2, \ldots, K$, discrete integration by parts, Young's inequality, and $d_t u_h^0 = 0$ show that for the right-hand side we have

$$\begin{aligned}
\tau^3 \sum_{k=1}^K (p_h - p_h^k, \nabla d_t^2 u_h^k) &= \tau^3 \sum_{k=1}^K (d_t p_h^k, \nabla d_t u_h^{k-1}) + \tau^2(p_h - p_h^k, \nabla d_t u_h^k)\Big|_{k=0}^K \\
&\leq \frac{\tau^2}{2}\Big(\sum_{k=1}^K \tau^2\|\nabla d_t u_h^{k-1}\|^2 + \|d_t p_h^k\|^2\Big) \\
&\quad + \frac{1}{2}\|p_h - p_h^K\|^2 + \frac{\tau^4}{2}\|\nabla d_t u_h^K\|^2 \\
&\leq \frac{\tau^2}{2}\Big(\sum_{k=1}^K \tau^2\theta^2\|d_t u_h^{k-1}\|_{h,s}^2 + \|d_t p_h^k\|^2\Big) \\
&\quad + \frac{1}{2}\|p_h - p_h^K\|^2 + \frac{\tau^4\theta^2}{2}\|d_t u_h^K\|_{h,s}^2.
\end{aligned}$$

Due to the assumption $\tau\theta \leq 1$ we may absorb the terms of the right-hand side and conclude that

$$\begin{aligned}
&\frac{1}{2}\|u_h - u_h^K\|_{h,s}^2 + \tau\sum_{k=1}^K \frac{\tau}{2}(1 - \tau\theta^2)\|d_t u_h^k\|^2 + \tau\sum_{k=1}^K \alpha\|u_h - u_h^k\|^2 \\
&\quad \leq \frac{1}{2}\big(\|u_h - u_h^0\|_{h,s}^2 + \|p_h - p_h^0\|^2\big).
\end{aligned}$$

This proves the theorem. □

Remark 10.12 Notice that we cannot expect convergence $p_h^n \to p_h$ since p_h is not unique in general, e.g., if $\nabla u_h|_T = 0$ for some $T \in \mathscr{T}_h$.

Useful choices of the inner product $(\cdot,\cdot)_{h,s}$ are weighted combinations of the inner product in $L^2(\Omega)$ and the semi-inner product in $H^1(\Omega)$.

Proposition 10.8 (Discrete inner products) *For $s \in [0,1]$ and $v_h, w_h \in \mathscr{S}^1(\mathscr{T}_h)$ define*

$$(v_h, w_h)_{h,s} = (v_h, w_h) + h_{\min}^{(1-s)/s}(\nabla v_h, \nabla w_h),$$

where $h_{\min}^{(1-s)/s} = 0$ for $s = 0$. We then have $\|\nabla v_h\| \le c h_{\min}^{-\min\{1,(1-s)/(2s)\}} \|v_h\|_{h,s}$ for all $v_h \in \mathscr{S}^1(\mathscr{T}_h)$ with $c = 1$ if $s > 0$.

Proof If $s > 0$, then we have by definition of $\|v_h\|_{h,s}^2 = (v_h, v_h)_{h,s}$ that

$$\|\nabla v_h\|^2 \le h_{\min}^{-(1-s)/s} \|v_h\|_{h,s}^2$$

for all $v_h \in \mathscr{S}^1(\mathscr{T}_h)$. For $s \ge 0$ the inverse estimate $\|\nabla v_h\| \le c h_{\min}^{-1} \|v_h\|$, valid for all $v_h \in \mathscr{S}^1(\mathscr{T}_h)$, implies the assertion. □

To fully justify the choice of the scalar products $(\cdot,\cdot)_{h,s}$ for $s > 0$, we have to show that the right-hand side in the estimate of Proposition 10.7 is bounded h-independently. For $s \le 1/2$ this is guaranteed by the following lemma if the sequence $(u_h)_{h>0}$ of finite element approximations is uniformly bounded in the set $W^{1,1}(\Omega) \cap L^\infty(\Omega)$.

Lemma 10.4 (Discrete interpolation estimate) *For every $v_h \in \mathscr{S}^1(\mathscr{T}_h)$ we have*

$$h_{\min} \|\nabla v_h\|_{L^2(\Omega)}^2 \le c \|v_h\|_{L^\infty(\Omega)} \|\nabla v_h\|_{L^1(\Omega)}.$$

Proof For $T \in \mathscr{T}_h$, an integration by parts on T together with the fact that $\Delta v_h|_T = 0$, implies that

$$h_T \int_T |\nabla v_h|^2 \,\mathrm{d}x = h_T \int_{\partial T} v_h \nabla v_h \cdot n_T \,\mathrm{d}s \le h_T |\partial T| \|v_h\|_{L^\infty(T)} |T|^{-1} \|\nabla v_h\|_{L^1(T)}.$$

Noting $h_T |\partial T| \le c|T|$, a summation over $T \in \mathscr{T}_h$ implies the assertion. □

Remark 10.13 To obtain approximations with residuals that are bounded independently of the parameter s, the stopping criterion

$$\sup_{v_h \in \mathscr{S}^1(\mathscr{T}_h)} \frac{(d_t u_h^k, v_h)_{h,s}}{\|v_h\|} \le \varepsilon_{\mathrm{stop}}$$

should be used.

10.2.4 Realization

The MATLAB code displayed in Fig. 10.5 is an implementation of the primal dual method of Algorithm 10.1 with the scalar product $(\cdot,\cdot)_{h,1/2}$ defined in Proposition 10.8 and the corresponding choice $\tau = h^{1/2}/10$. It computes the update of p_h^{k-1} via the elementwise operation

$$p_h^k = \frac{p_h^{k-1} + \tau \nabla \widetilde{u}_h^{k-1}}{\max\{1, |p_h^{k-1} + \tau \nabla \widetilde{u}_h^{k-1}|\}}$$

and the linear system of equations

$$(d_t u_h^k, v_h)_{h,s} + (p_h^k, \nabla v_h) = -\alpha(u_h^k - g, v_h)$$

for all $v_h \in \mathscr{S}^1(\mathscr{T}_h)$. The second term on the left-hand side is represented by the matrix with the entries

$$(\chi_T e^\ell, \nabla \varphi_z) = |T|\, \partial_\ell \varphi_z|_T$$

for all $T \in \mathscr{T}_h$, $\ell = 1, 2, \dots, d$, and $z \in \mathscr{N}_h$ which is assembled in the routine `mixed_matrix`.

10.2.5 A Posteriori Error Control

We apply the abstract framework for a posteriori error estimates for strongly convex minimization problems of Theorem 4.2 to control the approximation error in the numerical minimization of I. The estimate states that the distance of an arbitrary approximation to the minimizer is controlled by the primal-dual gap. The dual functional is for $p \in H_N(\mathrm{div}; \Omega)$ given by

$$D(p) = -\frac{1}{2\alpha}\| \operatorname{div} p + \alpha g\|^2 + \frac{\alpha}{2}\|g\|^2 - I_{K_1(0)}(p),$$

and we have $D(q) \le I(u)$ for every $q \in H_N(\mathrm{div}; \Omega)$ with equality for a solution of the dual problem.

Theorem 10.8 (A posteriori error estimate) *Let $u \in BV(\Omega) \cap L^2(\Omega)$ be the minimizer for I. Then for every $u_h \in \mathscr{S}^1(\mathscr{T}_h)$ and $\widehat{p}_h \in H_N(\mathrm{div}; \Omega)$ with $|\widehat{p}_h| \le 1$, we have*

$$\frac{\alpha}{2}\|u - u_h\|^2 \le \|\nabla u_h\|_{L^1(\Omega)} - \int_\Omega \nabla u_h \cdot \widehat{p}_h \,\mathrm{d}x + \frac{1}{2\alpha}\| \operatorname{div} \widehat{p}_h - \alpha(u_h - g)\|^2.$$

```
function tv_reg_primal_dual(d,red)
[c4n,n4e,Db,Nb] = triang_cube(d); c4n = c4n-.5;
for j = 1:red
    [c4n,n4e,Db,Nb] = red_refine(c4n,n4e,Db,Nb);
end
h = 2^(-red); alpha = 100; tau = h^(1/2)/10; noise = .4;
[s,m,¬,¬] = fe_matrices(c4n,n4e);
ms = mixed_matrix(c4n,n4e);
A = m+h*s;
[nC,d] = size(c4n); nE = size(n4e,1);
gg = g(c4n)+noise*(rand(nC,1)-.5);
u = zeros(nC,1); u_tilde = u; p = zeros(nE,d);
corr = 1; eps_stop = 1e-2;
while corr > eps_stop
    du_tilde = comp_gradient(c4n,n4e,u_tilde);
    p_tmp = p+tau*du_tilde;
    p = p_tmp./max(1,(sqrt(sum(p_tmp.^2,2))*ones(1,d)));
    P = reshape(p',d*nE,1);
    u_new = (A+tau*alpha*m)\(A*u-tau*ms*P+tau*alpha*m*gg);
    dt_u = (u-u_new)/tau;
    corr = sqrt(dt_u'*A*dt_u)
    u_tilde = 2*u_new-u;
    u = u_new;
    show_p1(c4n,n4e,Db,Nb,u);
end

function ms = mixed_matrix(c4n,n4e)
[nC,d] = size(c4n); nE = size(n4e,1);
ctr = 0; ctr_max = d*(d+1)*nE;
I = zeros(ctr_max,1); J = zeros(ctr_max,1); X = zeros(ctr_max,1);
for j = 1:nE
    X_T = [ones(1,d+1);c4n(n4e(j,:),:)'];
    grads_T = X_T\[zeros(1,d);eye(d)];
    vol_T = det(X_T)/factorial(d);
    for k = 1:d+1
        for ell = 1:d
            ctr = ctr+1;
            I(ctr) = n4e(j,k); J(ctr) = (j-1)*d+ell;
            X(ctr) = vol_T*grads_T(k,ell);
        end
    end
end
ms = sparse(I,J,X,nC,d*nE);

function val = g(x)
val = zeros(size(x,1),1);
val(sqrt(sum(x.^2,2))<.2) = 1;
```

Fig. 10.5 MATLAB realization of Algorithm 10.1 for the iterative minimization of the total variation regularization problem

Proof We recall from Lemma 10.2 that

$$\frac{\alpha}{2}\|u-u_h\|^2 \le I(u_h) - I(u).$$

Incorporating the duality principle $I(u) \ge D(\widehat{p}_h)$ for all $\widehat{p}_h \in H_N(\mathrm{div};\Omega)$, we deduce that

$$\frac{\alpha}{2}\|u-u_h\|^2 \le \|\nabla u_h\|_{L^1(\Omega)} + \frac{\alpha}{2}\|u_h-g\|^2 + \frac{1}{2\alpha}\|\operatorname{div}\widehat{p}_h + \alpha g\|^2 - \frac{\alpha}{2}\|g\|^2 + I_{K_1(0)}(\widehat{p}_h).$$

We assume that $|\widehat{p}_h| \le 1$ in Ω and with straightforward calculations deduce that

$$\begin{aligned}\frac{\alpha}{2}\|u-u_h\|^2 &\le \|\nabla u_h\|_{L^1(\Omega)} + \frac{1}{2\alpha}\|\operatorname{div}\widehat{p}_h - \alpha(u_h-g)\|^2 \\ &\quad + \int_\Omega u_h(\operatorname{div}\widehat{p}_h + \alpha g)\,\mathrm{d}x + \frac{\alpha}{2}\|u_h-g\|^2 - \frac{\alpha}{2}\|g\|^2 - \frac{\alpha}{2}\|u_h\|^2 \\ &= \|\nabla u_h\|_{L^1(\Omega)} + \frac{1}{2\alpha}\|\operatorname{div}\widehat{p}_h - \alpha(u_h-g)\|^2 + \int_\Omega u_h\operatorname{div}\widehat{p}_h\,\mathrm{d}x.\end{aligned}$$

An integration by parts proves the asserted estimate. □

Remarks 10.14 (i) The error estimate is sharp in the sense that if $u = u_h$ and $\widehat{p}_h = p$ solves the dual problem, then the right-hand side vanishes.
(ii) The practical application requires us to compute a conforming approximate solution of the dual problem. The piecewise constant approximation provided by Algorithm 10.1 in general does not satisfy $\widehat{p}_h \in H_N(\mathrm{div};\Omega)$.
(iii) The error estimate gives rise to the nonnegative refinement indicators

$$\eta_T(u_h,\widehat{p}_h) = \|\nabla u_h\|_{L^1(T)} - \int_T \nabla u_h\cdot\widehat{p}_h\,\mathrm{d}x + \frac{1}{2\alpha}\|\operatorname{div}\widehat{p}_h - \alpha(u_h-g)\|^2_{L^2(T)}$$

for $u_h \in \mathscr{S}^1(\mathscr{T}_h)$ and $\widehat{p}_h \in H_N(\mathrm{div};\Omega)$ with $|\widehat{p}_h| \le 1$. Noting the optimality condition $\operatorname{div} p = \alpha(u-g)$ and the duality relation

$$|Du|(\Omega) = -\int_\Omega u\operatorname{div} p\,\mathrm{d}x$$

for an exact solution $(u,p) \in \big(BV(\Omega)\cap L^2(\Omega)\big)\times H_N(\mathrm{div};\Omega)$ with $|p| \le 1$ in Ω, the refinement indicators have the interpretation of a residual.

10.2.6 Regularized Minimization

In some situations a regularized treatment of the functional I provides accurate approximations and in this case a semi-implicit discretization of the corresponding gradient flow defines a useful iterative scheme. We define the regularized functional I_δ for $\delta > 0$ by

$$I_\delta(u) = \int_\Omega |\nabla u|_\delta \,\mathrm{d}x + \frac{\alpha}{2}\|u - g\|^2$$

for $u \in W^{1,1}(\Omega) \cap L^2(\Omega)$ and with $|p|_\delta = (|p|^2 + \delta^2)^{1/2}$ for every $p \in \mathbb{R}^d$.

Algorithm 10.2 (*Semi-implicit, regularized L^2-flow*) Given $\delta > 0$, $\tau > 0$, and $u_h^0 \in \mathscr{S}^1(\mathscr{T}_h)$ compute the sequence $(u_h^k)_{k=0,1,\dots}$ by solving

$$(d_t u_h^k, v_h) + \big(|\nabla u_h^{k-1}|_\delta^{-1} \nabla u_h^k, \nabla v_h\big) = -\alpha(u_h^k - g, v_h)$$

for all $v_h \in \mathscr{S}^1(\mathscr{T}_h)$. Stop if $\|d_t u_h^k\| \le \varepsilon_{\text{stop}}$.

Remark 10.15 The choice $v_h = u_h^k$ shows that the iteration is unconditionally weakly stable in the sense that

$$\frac{d_t}{2}\|u_h^k\|^2 + \frac{\tau}{2}\|d_t u_h^k\|^2 + \big\||\nabla u_h^{k-1}|_\delta^{-1/2}\nabla u_h^k\big\|^2 + \frac{\alpha}{2}\|u_h^k\|^2 \le \frac{\alpha}{2}\|g\|^2$$

for all $k \ge 1$. In order to obtain accurate approximations, the step size should be chosen so that $\tau \le c h_{\min}$. This scaling leads to practically strongly stable approximation schemes for $\delta > 0$ in the sense that the regularized energy I_δ decreases.

If $\delta \le ch^{1/2}$, we have the same error estimates as for the unregularized approximation.

Proposition 10.9 (Regularized approximation) *Let $u \in BV(\Omega) \cap L^2(\Omega)$ be the minimizer for I and let $u_{\delta,h} \in \mathscr{S}^1(\mathscr{T}_h)$ be minimal for*

$$I_\delta(v_h) = \int_\Omega |\nabla v_h|_\delta \,\mathrm{d}x + \frac{\alpha}{2}\|v_h - g\|^2$$

in the set of functions $v_h \in \mathscr{S}^1(\mathscr{T}_h)$. If $\delta \le ch^{1/2}$, then we have

$$\frac{\alpha}{2}\|u - u_{\delta,h}\|^2 \le ch^{1/2}.$$

Proof We first note that for every $p \in \mathbb{R}^d$ we have

$$|p| \le |p|_\delta \le |p| + \delta.$$

With Lemma 10.2 and the fact that $u_{\delta,h}$ is minimal for I_δ in $\mathscr{S}^1(\mathscr{T}_h)$ it follows for every $v_h \in \mathscr{S}^1(\mathscr{T}_h)$ that

$$\frac{\alpha}{2}\|u - u_{\delta,h}\|^2 \le I(u_{\delta,h}) - I(u) \le I_\delta(u_{\delta,h}) - I(u) \le I_\delta(v_h) - I(u)$$
$$= I_\delta(v_h) - I(v_h) + I(v_h) - I(u) \le \delta|\Omega| + I(v_h) - I(u).$$

With $v_h = u_{\varepsilon,h}$, as in Lemma 10.1 for $\varepsilon = h^{1/2}$, we deduce the asserted bound. □

Remark 10.16 An alternative definition for $|p|_\delta$ is given by

$$|p|_\delta = \begin{cases} |p| & \text{if} \quad |p| \ge \delta, \\ (|p|^2 + \delta^2)/2 & \text{if} \quad |p| \le \delta. \end{cases}$$

Figure 10.6 displays an implementation of Algorithm 10.2. The weighted stiffness matrix is computed in the routine `fe_matrices_weighted` which provides for elementwise constant functions $a, b : \Omega \to \mathbb{R}$ the matrices with entries

$$s_{a,zy} = \int_\Omega a\,\nabla\varphi_z \cdot \nabla\varphi_y \,\mathrm{d}x, \quad m_{b,zy} = \int_\Omega b\,\varphi_z\varphi_y \,\mathrm{d}x$$

for $z, y \in \mathscr{N}_h$.

10.2.7 Total Variation Flow

The total variation arises in various mathematical models describing evolution problems by subdifferential flows. The evolution problems are also often the basis for numerical minimization algorithms. An implicit discretization leads to the following algorithm.

Algorithm 10.3 (*Subdifferential flow*) Given $u_h^0 \in \mathscr{S}^1(\mathscr{T}_h)$ and $\tau > 0$, compute the sequence $(u_h^k)_{k=0,\dots,K} \subset \mathscr{S}^1(\mathscr{T}_h)$ by minimizing for $k = 1, 2, \dots, K$ the functionals

$$I_{\tau,h}^k(w_h) = \frac{1}{2\tau}\|w_h - u_h^{k-1}\|^2 + I(w_h)$$

in the set of functions $w_h \in \mathscr{S}^1(\mathscr{T}_h)$.

The scheme may be regarded as an implicit Euler method and is unconditionally stable.

Proposition 10.10 (Stability) *Assume that $I : L^2(\Omega) \to \mathbb{R} \cup \{+\infty\}$ is convex and lower-semicontinuous. For $L = 1, 2, \dots, K$ we have*

```
function tv_reg_regularized(d,red)
[c4n,n4e,Db,Nb] = triang_cube(d); c4n = c4n-.5;
for j = 1:red
    [c4n,n4e,Db,Nb] = red_refine(c4n,n4e,Db,Nb);
end
h = 2^(-red); alpha = 100; tau = h/10;
noise = .4; delta = h^(1/2);
nC = size(c4n,1); nE = size(n4e,1);
[¬,m,¬,¬] = fe_matrices(c4n,n4e);
gg = g(c4n)+noise*(rand(nC,1)-.5);
u = zeros(nC,1);
corr = 1; eps_stop = 1e-5;
while corr > eps_stop
    du = comp_gradient(c4n,n4e,u);
    a_du_inv = 1./sqrt(sum(du.^2,2)+delta^2);
    [s_du,¬] = fe_matrices_weighted(c4n,n4e,a_du_inv,zeros(nE,1));
    X = (1+alpha*tau)*m+tau*s_du;
    b = m*u+tau*alpha*m*gg;
    u_new = X\b;
    dt_u = (u_new-u)/tau;
    corr = sqrt(dt_u'*m*dt_u);
    u = u_new;
    show_p1(c4n,n4e,Db,Nb,u);
end

function val = g(x)
val = zeros(size(x,1),1);
val(sqrt(sum(x.^2,2))<.2) = 1;

function [s_a,m_b] = fe_matrices_weighted(c4n,n4e,a,b)
[nC,d] = size(c4n); nE = size(n4e,1);
m_loc = (ones(d+1,d+1)+eye(d+1))/((d+1)*(d+2));
ctr = 0; ctr_max = (d+1)^2*nE;
I = zeros(ctr_max,1); J = zeros(ctr_max,1);
X_s_a = zeros(ctr_max,1); X_m_b = zeros(ctr_max,1);
for j = 1:nE
    X_T = [ones(1,d+1);c4n(n4e(j,:),:)'];
    grads_T = X_T\[zeros(1,d);eye(d)];
    vol_T = det(X_T)/factorial(d);
    for m = 1:d+1
        for n = 1:d+1
            ctr = ctr+1;
            I(ctr) = n4e(j,m); J(ctr) = n4e(j,n);
            X_s_a(ctr) = vol_T*a(j)*grads_T(m,:)*grads_T(n,:)';
            X_m_b(ctr) = vol_T*b(j)*m_loc(m,n);
        end
    end
end
s_a = sparse(I,J,X_s_a,nC,nC); m_b = sparse(I,J,X_m_b,nC,nC);
```

Fig. 10.6 MATLAB realization of the semi-implicit gradient flow discretization of the regularized total variation functional I_δ defined in Algorithm 10.2

$$I(u_h^L) + \tau \sum_{k=1}^{L} \|d_t u_h^k\|^2 \le I(u_h^0).$$

Proof The existence of the iterates follows from the direct method in the calculus of variations, and the strong convexity of $I_{\tau,h}^k$ implies their uniqueness. For $k = 1, 2, \ldots, K$ we have $0 \in \partial I_{\tau,h}^k(u_h^k)$, i.e., $-d_t u_h^k \in \partial I(u_h^k)$ and hence for all $v_h \in \mathscr{S}^1(\mathscr{T}_h)$

$$(-d_t u_h^k, v_h - u_h^k) + I(u_h^k) \le I(v_h).$$

The choice $v_h = u_h^{k-1}$ yields

$$\tau \|d_t u_h^k\|^2 + \tau d_t I(u_h^k) \le 0$$

and a summation over $k = 1, 2, \ldots, L$ implies the stability estimate. □

We next bound the difference between the fully discrete and semi-discrete approximations, i.e., we estimate the difference $u_h^k - u^k$, where $(u^k)_{k=0,1,\ldots,K}$ is the sequence of minimizers for the functionals

$$I_\tau^k(w) = \frac{1}{2\tau}\|w - u^{k-1}\|^2 + I(w)$$

with an initial $u^0 = u_0 \in L^2(\Omega)$. For ease of presentation we restrict to the case $I(u) = |Du|(\Omega)$.

Proposition 10.11 (Partial error estimate) *Let $I(u) = |Du|(\Omega)$ for $u \in BV(\Omega)$ and assume that $u_0 \in BV(\Omega) \cap L^\infty(\Omega)$. For $L = 1, 2, \ldots, K$ we have*

$$\|u_h^L - u^L\|^2 \le \|u_h^0 - u_0\|^2 + ch^{1/3}.$$

The constant $c \ge 0$ depends on T, $|Du^0|(\Omega)$, $\|\nabla u_h^0\|_{L^1(\Omega)}$, and $\|u^0\|_{L^\infty(\Omega)}$.

Proof We let $(u^k)_{k=0,\ldots,K} \subset BV(\Omega) \cap L^2(\Omega)$ be the solution of the semi-discrete scheme with initial value $u^0 = u_0$. Then, for $k = 1, 2, \ldots, K$ and all $v \in BV(\Omega) \cap L^2(\Omega)$ we have

$$(-d_t u^k, v - u^k) + I(u^k) \le I(v).$$

For $k = 1, 2, \ldots, K$, and all $v_h \in \mathscr{S}^1(\mathscr{T}_h)$ we have

$$(-d_t u_h^k, v_h - u_h^k) + I(u_h^k) \le I(v_h).$$

Choosing $v = u_h^k$ we deduce that

$$(d_t[u^k - u_h^k], u^k - u_h^k) + I(u^k) - I(v_h) \le (d_t u_h^k, v_h - u^k),$$

i.e.,

$$\frac{d_t}{2}\|u^k - u_h^k\|^2 + \frac{\tau}{2}\|d_t(u^k - u_h^k)\|^2 \le I(v_h) - I(u^k) + \|d_t u_h^k\|\|v_h - u^k\|.$$

For $\varepsilon > 0$ we let $v_h = u_{\varepsilon,h}^k$ be as in Lemma 10.1 so that

$$I(v_h) - I(u^k) \le c(\varepsilon + h\varepsilon^{-1})I(u^k)$$

and

$$\|v_h - u^k\|^2 \le \|v_h - u^k\|_{L^1(\Omega)}\|v_h - u^k\|_{L^\infty(\Omega)} \le c(h^2\varepsilon^{-1} + \varepsilon)|Du^k|(\Omega)\|u^k\|_{L^\infty(\Omega)}.$$

Arguing as in Proposition 10.2, we have $\|u^k\|_{L^\infty(\Omega)} \le \|u^0\|_{L^\infty(\Omega)}$ for $k = 1, 2, \dots, K$. The construction of $u_{\varepsilon,h}^k$ in Lemma 10.1 guarantees that $\|v_h\|_{L^\infty(\Omega)} \le \|u^k\|_{L^\infty(\Omega)}$. As in the proof of Proposition 10.10, we find that the semi-discrete scheme is energy-decreasing, i.e., we have $|Du^k|(\Omega) \le |Du^0|(\Omega)$ for $k = 1, 2, \dots, K$, and hence

$$|Du^k|(\Omega) + \tau\sum_{k=1}^{L}\|d_t u^k\|^2 \le |Du^0|(\Omega) = c_0.$$

Incorporating also the estimate from Proposition 10.10, it follows from a summation over $k = 1, 2, \dots, L$ that

$$\begin{aligned}
\frac{1}{2}\|u_h^L - u^L\|^2 &\le \frac{1}{2}\|u_h^0 - u^0\|^2 + \tau\sum_{k=1}^{L}\big(|Dv_h|(\Omega) - |Du^k|(\Omega)\big)\\
&\quad + \Big(\tau\sum_{k=1}^{L}\|d_t u_h^k\|^2\Big)^{1/2}\Big(\tau\sum_{k=1}^{L}\|v_h - u^k\|^2\Big)^{1/2}\\
&\le \frac{1}{2}\|u_h^0 - u^0\|^2 + cT(\varepsilon + h\varepsilon^{-1})c_0\\
&\quad + cT^{1/2}c_0^{1/2}\|u^0\|_{L^\infty(\Omega)}^{1/2}(h^2\varepsilon^{-1} + \varepsilon)^{1/2}.
\end{aligned}$$

Choosing $\varepsilon = h^{2/3}$ leads to the assertion. □

The combination of Proposition 10.11 with the abstract error estimate for implicit discretizations of subdifferential flows of Theorem 4.7 leads to the following error estimate.

Theorem 10.9 (Error estimate) *Assume that* $u_0 \in BV(\Omega) \cap L^\infty(\Omega)$ *and* $u_h^0 \in \mathcal{S}^1(\mathcal{T}_h)$ *is such that* $\|u_0 - u_h^0\| \le h^{1/6}$ *and* $|Du_h^0|(\Omega) \le c$ *for all* $h > 0$. *We then have*

$$\max_{k=1,\ldots,K} \|u(t_k) - u_h^k\| \le c(\tau^{1/2} + h^{1/6}).$$

Proof The assertion is a direct consequence of the abstract error estimate for implicit discretizations of subdifferential flows of Theorem 4.7 and Proposition 10.11. □

Remarks 10.17 (i) The upper bound can be improved to $\tau + h^{1/4}$ provided that $\partial I(u^0) \neq \emptyset$ and $\|d_t u_h^k\|_{L^\infty(\Omega)} \le c$ for $k = 1, 2, \ldots, K$.
(ii) In the case of Dirichlet boundary conditions and $d = 1$, any monotone function $u \in BV(\Omega)$ is stationary for I, whereas only the affine interpolant of the boundary data is stationary for the regularized functional I_δ.

10.3 Segmentation

We discuss in this section the numerical approximation of segmentation problems. The considered simple model problems detect edges in certain images and serve as bases for the development of models that describe damage and fracture in solid mechanics. We refer the reader to [5, 9] for further details.

10.3.1 The Mumford–Shah Functional

The *Mumford–Shah* functional detects certain edges in an image $g : \Omega \to \mathbb{R}$ by minimizing the functional

$$I(u, K) = \frac{\alpha}{2} \int_{\Omega\setminus K} |\nabla u|^2 \,\mathrm{d}x + \beta \mathscr{H}^{d-1}(K) + \frac{\gamma}{2} \int_{\Omega\setminus K} (u-g)^2 \,\mathrm{d}x$$

in closed sets $K \subset \overline{\Omega}$ and functions $u \in H^1(\Omega\setminus K)$ with given parameters $\alpha, \beta, \gamma > 0$. For a minimizing pair (u, K) the $(d-1)$-dimensional Hausdorff measure $\mathscr{H}^{d-1}(K)$ has to be finite, e.g., K is the union of curves or surfaces for $d = 2$ or $d = 3$, respectively, and $\mathscr{H}^{d-1}$ is the corresponding surface measure. The function u approximates the data g and may jump across the set K. Establishing the existence of minimizing pairs is a difficult task, since the unknowns u and K are different objects and the Hausdorff measure is not lower semicontinuous.

Example 10.6 For $k \in \mathbb{N}$ recursively define $S_k \subset [0, 1]$ through $S_0 = [0, 1/2]$ and

$$S_k = (1/2)S_{k-1} \cup (1/2)\big(S_{k-1} + 1/2\big) = \cup_{\ell=0}^{2^k-1} 2^{-(k+1)}[2\ell, 2\ell + 1]$$

e.g., $S_1 = [0, 1/4] \cup [2/4, 3/4]$. Then the sequence $(S_k)_{k\in\mathbb{N}}$ converges to $S = [0, 1]$ with respect to the Hausdorff metric

$$d_{\mathscr{H}}(K, L) = \inf\{\varepsilon > 0 : K \subset U_\varepsilon(L),\ L \subset U_\varepsilon(K)\},$$

where $U_\varepsilon(K) = \{x \in \mathbb{R}^d : \operatorname{dist}(x, K) < \varepsilon\}$. Since $\mathscr{H}^{d-1}(S) = 1$ and $\mathscr{H}^{d-1}(S_k) = 1/2$ for all $k \in \mathbb{N}$, we conclude that the mapping $K \mapsto \mathscr{H}^{d-1}(K)$ is not lower semicontinuous with respect to the Hausdorff metric.

The main idea to establish the existence of solutions is to consider functions of bounded variation and to identify K with the discontinuity set S_u of a function $u \in BV(\Omega)$. We recall that the distributional derivative of $u \in BV(\Omega)$ permits the decomposition

$$Du = \nabla u \otimes \mathrm{d}x - [\![un]\!] \otimes \mathrm{d}s|_{S_u} + C_u$$

with a vector field $\nabla u \in L^1(\Omega; \mathbb{R}^d)$ and the discontinuity set S_u of finite $(d-1)$-dimensional Hausdorff measure. The Cantor part C_u is in general supported on a set of infinite $(d-1)$-dimensional Hausdorff measure. If $C_u = 0$, it is natural to consider

$$I'(u) = \frac{\alpha}{2} \int_\Omega |\nabla u|^2 \,\mathrm{d}x + \beta \mathscr{H}^{d-1}(S_u) + \frac{\gamma}{2} \int_\Omega (u - g)^2 \,\mathrm{d}x.$$

The functions $u \in BV(\Omega)$ with $C_u = 0$ are called *special functions of bounded variation* and the set of all such functions is denoted $SBV(\Omega)$, i.e.,

$$SBV(\Omega) = \{u \in BV(\Omega) : C_u = 0\}.$$

Sequences $(u_j)_{j \in \mathbb{N}} \subset SBV(\Omega) \cap L^\infty(\Omega)$ that are uniformly bounded in $L^\infty(\Omega)$ and for which we have $\nabla u_j \in L^2(\Omega)$ for every $j \in \mathbb{N}$, such that the expression

$$\int_\Omega |\nabla u_j|^2 \,\mathrm{d}x + \mathscr{H}^{d-1}(S_{u_j})$$

is uniformly bounded, provide convergent subsequences $(u_{j_k})_{k \in \mathbb{N}}$ with limit $u \in SBV(\Omega)$, i.e., we have that $u_{j_k} \to u$ almost everywhere in Ω, $\nabla u_{j_k} \rightharpoonup \nabla u$ in $L^2(\Omega)$, and

$$\mathscr{H}^{d-1}(S_u) \le \liminf_{k \to \infty} \mathscr{H}^{d-1}(S_{u_{j_k}}).$$

This compactness property implies the following existence result.

Theorem 10.10 (Existence [1]) *If $g \in L^\infty(\Omega)$, then the functional I' has a minimizer $u \in SBV(\Omega) \cap L^\infty(\Omega)$. The pair (u, K) with $K = \overline{S_u} \cap \overline{\Omega}$ minimizes the Mumford–Shah functional in pairs (u, K) consisting of a closed set $K \subset \overline{\Omega}$ with $\mathscr{H}^{d-1}(K) < \infty$ and $u \in W^{1,2}(\Omega \setminus K)$.*

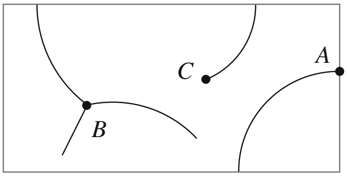

Fig. 10.7 Typical vertices of the singularity set K in the minimization of the Mumford–Shah functional; vertices are either points on the boundary where K intersects $\partial\Omega$ perpendicularly (A), triple points where three smooth segments intersect with equal angles (B), or endpoints of curves (C)

Precise characterizations of the singularity set K are available.

Remark 10.18 Assume $d = 2$ and a minimizing pair (u, K) is such that K is the finite union of $C^{1,1}$ curves. Then every vertex of K is either (a) A point on $\partial\Omega$ where K and $\partial\Omega$ intersect perpendicularly, (b) A point in Ω at which three $C^{1,1}$ curves intersect with angles $2\pi/3$, or (c) A point in Ω at which a $C^{1,1}$ curve ends, cf. Fig. 10.7. The technical results follow from contradictions and local modifications to lower the energy.

10.3.2 Regularization of $I'(u)$

It is difficult to approximate the Mumford–Shah functional directly with finite element methods since the singularity sets of discontinuous, piecewise polynomial finite element functions are subsets of the skeleton of the underlying triangulation which is in general too restrictive to approximate a given curve. An approach to regularizing the Mumford–Shah functional is to describe the set K by the zero level set $\Gamma_\phi = \phi^{-1}(\{0\})$ of a function $\phi : \Omega \to \mathbb{R}$ and noting that the Hausdorff measure of Γ_ϕ is approximated by the Modica–Mortola type length functional L_ε, i.e.,

$$\mathscr{H}^{d-1}(\Gamma_\phi) \approx L_\varepsilon(\Gamma_\phi) = \frac{\varepsilon}{2}\int_\Omega |\nabla\phi|^2 \,\mathrm{d}x + \frac{1}{2\varepsilon}\int_\Omega (\phi-1)^2 \,\mathrm{d}x.$$

This relation follows from Young's inequality together with the transformation $w = (\phi-1)^2$, i.e., $|\nabla w| = 2|\phi-1||\nabla\phi|$. We have

$$L_\varepsilon(\Gamma_\phi) = \frac{\varepsilon}{2}\int_\Omega |\nabla\phi|^2 \,\mathrm{d}x + \frac{1}{2\varepsilon}\int_\Omega (\phi-1)^2 \,\mathrm{d}x \geq \int_\Omega |\nabla\phi||\phi-1| \,\mathrm{d}x = \frac{1}{2}\int_\Omega |\nabla w| \,\mathrm{d}x.$$

We assume that Γ_ϕ is a smooth curve and, for every $r \in \Gamma_\phi$, denote by n_r the unit normal to Γ_ϕ at r. With the tubular neighborhood

$$\Gamma_{\phi,\varepsilon} = \{x \in \Omega : x = r + tn_r, |t| \le \varepsilon\}$$

of Γ_ϕ we have

$$L_\varepsilon(\Gamma_\phi) \ge \frac{1}{2} \int_{\Gamma_{\phi,\varepsilon}} |\nabla w| \,\mathrm{d}x \ge \frac{1}{2} \int_{\Gamma_\phi} \int_{-\varepsilon}^{\varepsilon} |\nabla w \cdot n_r| \,\mathrm{d}t \,\mathrm{d}r.$$

Assuming that $L_\varepsilon(\Gamma_\phi)$ remains bounded as $\varepsilon \to 0$, the function ϕ approaches the value 1 away from Γ_ϕ for ε sufficiently small, so that we may assume that $w = (\phi - 1)^2 \approx 0$ in $\Omega \setminus \Gamma_{\phi,\varepsilon}$. The integral of the modulus of the derivative of w in normal direction to Γ_ϕ is then approximately 2 and we obtain

$$L_\varepsilon(\Gamma_\phi) \ge \int_{\Gamma_\phi} 1 \,\mathrm{d}s = \mathscr{H}^{d-1}(\Gamma_\phi).$$

These observations motivate us to consider the Ambrosio–Tortorelli approximation of the Mumford–Shah functional in which L_ε approximates $\mathscr{H}^{d-1}(S_u)$ and enforces ϕ to be close to one, while a term $\phi^2|\nabla u|^2$ favors $\phi \approx 0$ to permit large, unbounded gradients of u.

Theorem 10.11 (Regularization [3]) *For* $(u, \phi) \in H^1(\Omega) \times H^1(\Omega)$ *and* $\varepsilon > 0$*, define the* Ambrosio–Tortorelli functional

$$AT_\varepsilon(u, \phi) = \frac{\alpha}{2} \int_\Omega (\phi^2 + \varepsilon^2)|\nabla u|^2 \,\mathrm{d}x + \beta\Big(\frac{\varepsilon}{2} \int_\Omega |\nabla \phi|^2 \,\mathrm{d}x + \frac{1}{2\varepsilon} \int_\Omega (\phi - 1)^2 \,\mathrm{d}x\Big) + \frac{\gamma}{2} \int_\Omega (u - g)^2 \,\mathrm{d}x$$

and extend AT_ε *with value* $+\infty$ *to* $L^1(\Omega) \times L^1(\Omega)$*. Then, as* $\varepsilon \to 0$*, we have that* $AT_\varepsilon \to^\Gamma I''$ *with respect to strong convergence in* $L^1(\Omega) \times L^1(\Omega)$*, and where* $I''(u, \phi) = I'(u)$ *if* $(u, \phi) \in SBV(\Omega) \times L^1(\Omega)$ *with* $\phi = 1$ *almost everywhere and* $I''(u, \phi) = +\infty$ *otherwise, i.e.,* $I'(u) = I''(u, 1)$ *for all* $u \in SBV(\Omega)$*.*

10.3.3 Numerical Approximation of AT_ε

The functional AT_ε can be directly discretized with H^1-conforming finite element methods; that is, given $\varepsilon > 0$ and a triangulation $\mathscr{T}_h$ of Ω, we consider the separately convex functional

$$AT_{\varepsilon,h}(u_h,\phi_h) = \frac{\alpha}{2}\int_\Omega (\phi_h^2+\varepsilon^2)|\nabla u_h|^2\,\mathrm{d}x + \beta\Big(\frac{\varepsilon}{2}\int_\Omega |\nabla\phi_h|^2\,\mathrm{d}x + \frac{1}{2\varepsilon}\int_\Omega (\phi_h-1)^2\,\mathrm{d}x\Big) + \frac{\gamma}{2}\int_\Omega (u_h-g)^2\,\mathrm{d}x$$

for $(u_h,\phi_h)\in\mathscr{S}^1(\mathscr{T}_h)$. Extending $AT_{\varepsilon,h}$ by $+\infty$ on $L^1(\Omega)^2\setminus\mathscr{S}^1(\mathscr{T}_h)^2$, the density of $\mathscr{S}^1(\mathscr{T}_h)$ in $L^1(\Omega)$ leads to a Γ-convergence result as in Theorem 10.11. The iterative solution of $AT_{\varepsilon,h}$ is based on a semi-implicit discretization of a gradient flow with respect to ϕ_h. This leads to two uncoupled equations in every step of the iteration. We let $P_0 v\in\mathscr{L}^0(\mathscr{T}_h)$ denote the elementwise average of a function $v\in L^1(\Omega)$.

Algorithm 10.4 (*Semi-implicit gradient flow for $AT_{\varepsilon,h}$*) Given $\tau>0$ and $\phi_h^0\in\mathscr{S}^1(\mathscr{T}_h)$, define the sequence $(u_h^k,\phi_h^k)_{k=1,2,\dots}$ by solving for $k=1,2,\dots$ the equations

$$\alpha((|P_0\phi_h^{k-1}|^2+\varepsilon^2)\nabla u_h^k,\nabla v_h)+\gamma(u_h^k-g,v_h)=0,$$
$$(d_t\phi_h^k,w_h)+\alpha(|\nabla u_h^k|^2\phi_h^k,w_h)+\beta\varepsilon(\nabla\phi_h^k,\nabla w_h)+\frac{\beta}{\varepsilon}(\phi_h^k-1,w_h)=0$$

for all $(v_h,w_h)\in\mathscr{S}^1(\mathscr{T}_h)\times\mathscr{S}^1(\mathscr{T}_h)$. Stop the iteration if $\|d_t\phi_h^k\|\le\varepsilon_{\mathrm{stop}}$.

In the implementation of the scheme shown in Fig. 10.8 we used the parameter $\beta=1$.

10.3.4 The Perona–Malik Equation

The *Perona–Malik* equation is a nonlinear parabolic partial differential equation that denoises an image g for a parameter $\lambda>0$ through

$$\partial_t u - \operatorname{div}\Big(\frac{\nabla u}{(1+|\nabla u|^2/\lambda^2)^2}\Big)=0,\quad \partial_n u(t,\cdot)=0,\quad u(0)=g.$$

The diffusion coefficient $a(|\nabla u|)=(1+|\nabla u|^2/\lambda^2)^{-2}$ is small in regions where $|\nabla u|$ is large and this leads to a preservation of edges in the images that are characterized by large gradients. In the remaining regions where $|\nabla u|\le c$, the diffusion coefficient $a(|\nabla u|)$ is larger and causes a smoothing of u away from the edges. This leads to a simultaneous denoising and steepening of edges, but analytically to the problem that the equation is of backward and forward parabolic type, so that the well-posedness of the initial boundary value problem is false in general. The equation has an interesting relation to the Mumford–Shah model, i.e., to its Ambrosio–Tortorelli regularization, described in [13]. An implicit discretization in time of the Perona–Malik equation leads to the problem of determining u^k such that

```
function ambrosio_tortorelli(d,red)
[c4n,n4e,Db,Nb] = triang_cube(d); c4n = 2*(c4n-.5);
for j = 1:red
    [c4n,n4e,Db,Nb] = red_refine(c4n,n4e,Db,Nb);
end
[nC,d] = size(c4n); nE = size(n4e,1); gg = g(c4n);
alpha = 1; gamma = 10; tau = 2^(-red)/10; eps = 1/10;
[s,m,¬] = fe_matrices(c4n,n4e);
a_0 = zeros(nE,1);
phi = zeros(nC,1); corr = 1; eps_stop = 1e-2;
while corr > eps_stop
    a_phi_sq = eps^2+(sum(phi(n4e),2)/(d+1)).^2;
    [s_phi,¬] = fe_matrices_weighted(c4n,n4e,a_phi_sq,a_0);
    X_u = gamma*m+alpha*s_phi;
    b_u = gamma*m*gg;
    u = X_u\b_u;
    du = comp_gradient(c4n,n4e,u);
    mod_du_sq = sum(du.^2,2);
    [¬,m_du] = fe_matrices_weighted(c4n,n4e,a_0,mod_du_sq);
    X_phi = m+eps*tau*s+tau*alpha*m_du+(1/(2*eps))*tau*m;
    b_phi = m*phi+(1/(2*eps))*tau*m*ones(nC,1);
    phi_new = X_phi\b_phi;
    dt_phi = (phi_new-phi)/tau;
    corr = sqrt(dt_phi'*m*dt_phi);
    phi = phi_new;
    figure(1); show_p1(c4n,n4e,Db,Nb,u);
    figure(2); show_p1(c4n,n4e,Db,Nb,phi);
end

function val = g(x)
val = tanh(100*(sum(x.^2,2)-1/2));
```

Fig. 10.8 MATLAB realization of Algorithm 10.4 for the iterative minimization of the Ambrosio–Tortorelli regularization of the Mumford–Shah functional

$$\operatorname{div}\left(\frac{\nabla u^k}{(1+|\nabla u^k|^2/\lambda^2)^2}\right) = \frac{1}{\tau}(u^k - u^{k-1}). \tag{10.1}$$

The Euler–Lagrange equations of the Ambrosio–Tortorelli functional AT_ε define the pair (u,ϕ) via

$$\alpha \operatorname{div}\big((\phi^2+\varepsilon^2)\nabla u\big) = \gamma(u-g),$$
$$\alpha\varepsilon|\nabla u|^2\phi - \beta\varepsilon^2\Delta\phi + \beta(\phi-1) = 0.$$

Neglecting terms with a factor ε^2, we find that

$$\phi = \frac{1}{1+(\alpha/\beta)\varepsilon|\nabla u|^2}$$

and

$$\operatorname{div}\left(\frac{\nabla u}{\left(1+(\alpha/\beta)\varepsilon|\nabla u|^2\right)^2}\right) = \frac{\gamma}{\alpha}(u-g). \tag{10.2}$$

For $k = 1$ and $u^0 = g$ in (10.1) and, e.g., $\alpha = \lambda^{-1/2}$, $\beta = \varepsilon$, and $\gamma = \alpha/\tau$ in (10.2), the partial differential equations coincide. The practical solution of the Perona–Malik equation is based on a semi-implicit discretization of the equation.

Algorithm 10.5 (*Semi-implicit Perona–Malik equation*) Given $\tau > 0$ and $g_h \in \mathscr{S}^1(\mathscr{T}_h)$, define the sequence $(u_h^k)_{k=0,1,\ldots}$ by setting $u_h^0 = g_h$ and solving for $k = 1, 2, \ldots$ the equations

$$(d_t u_h^k, v_h) + \left(\frac{\nabla u_h^k}{(1+|\nabla u_h^{k-1}|^2/\lambda^2)^2}, \nabla v_h\right) = 0$$

for all $v_h \in \mathscr{S}^1(\mathscr{T}_h)$. Stop the iteration if $\|d_t u_h^k\| \le \varepsilon_{\mathrm{stop}}$.

An implementation of the scheme is shown in Fig. 10.9.

```
function perona_malik(d,red)
[c4n,n4e,Db,Nb] = triang_cube(d); c4n = 2*(c4n-.5);
lambda = .5;
for j = 1:red
    [c4n,n4e,Db,Nb] = red_refine(c4n,n4e,Db,Nb);
end
nE = size(n4e,1);
tau = 2^(-red)/10;
[¬,m,¬] = fe_matrices(c4n,n4e);
u = g(c4n);
corr = 1; eps_stop = 1e-2;
while corr > eps_stop
    du = comp_gradient(c4n,n4e,u);
    a_du = (1+sum(du.^2,2)/lambda^2).^(-2);
    [s_du,¬] = fe_matrices_weighted(c4n,n4e,a_du,zeros(nE,1));
    X = m+tau*s_du;
    b = m*u;
    u_new = X\b;
    dt_u = (u_new-u)/tau;
    u = u_new;
    corr = sqrt(dt_u'*m*dt_u);
    show_p1(c4n,n4e,Db,Nb,u);
end

function val = g(x)
val = tanh(100*(sum(x.^2,2)-1/2))+.25*(rand(size(x,1),1)-.5);
```

Fig. 10.9 MATLAB realization of the semi-implicit discretization of the Perona–Malik equation specified in Algorithm 10.5

Remarks 10.19 (i) A stability proof for the iteration is expected to require restrictive conditions on the step size τ. Practically, the iteration provides satisfactory results for $\tau \leq ch$. Difficulties in the numerical analysis reflect the fact that no general existence theory for the Perona–Malik equation is available and in fact solutions may fail to exist due to occurring backward diffusion.
(ii) An alternative choice for the diffusion coefficient in the Perona–Malik equation is $a(s) = e^{-s^2/\lambda^2}$.

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