Asymptotic Analysis of Closed Markov Retrial Queuing System with Collision

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Abstract. We consider a closed retrial queuing system M/M/1//N with collision of the customers. We assume that sources can be in two states: generating a primary customers and waiting for the end of successful service. Source which sends the customer for service, moves into the waiting state and stays in this state till the end of the service of this customer. This system is solved using the asymptotic method under conditions of infinitely increasing number of sources. We establish formulas for computing the prelimit distribution of the number of sources in "waiting" state. Also, we determine the range of applicability of the asymptotic results in prelimiting situation.

Keywords: closed queueing system, retrial queue, collision, asymptotic analysis.

1 Introduction

Retrial queue is a queuing system [1–3] characterized by the following feature: customers, who find server busy goes to the orbit and after random time repeat their demand. It is assumed that the orbit is infinitely large and every call retry its attempts until it is satisfied. The field of practical application of such system is very extensive. RQ-system can be applied for researching telecommunication and computer system, for engineering cellular mobile networks, computer networks, ets. For a detailed overviews of main results about retrial queues, we refer the reader to the excellent book of Falin and Templeton [4]. For an extensive bibliography, see [5]. As regards the closed retrial queuing with finite number of source, it is elaborately discussed by Almási B. et al [6, 7], Artalejo J.R. [8], and Dragieva V.I. [9, 10]. In this paper, we consider the M/M/1//N retrial queue with collision. In the papers Nazarov A.A., Lyubina T.V. are considered the various open retrial queuing systems with collision [11, 12].

2 Model Description

We consider a closed retrial queuing system of type M/M/1//N in Kendals notation with collision of the customers. This mean that the system has one

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server and N sources. Each one of them generated a primary customers according to a Poisson flow with rate λ/N. We assume that sources can be in two states: generating a primary customers and waiting for the end of successful service. Source which send the customer for service, moves into the "waiting" state and stays in this state till the end of the service of this customer. If a primary customer finds server idle, he enters into service immediately, during service time, which distributed exponentially with parameter μ. Otherwise, if server is busy, arriving customer involves into collision with servicing customer and they both moves into the orbit. Retrial customer repeat his demand for service with an exponential distribution with rate σ/N. We assume that primary customers, retrial customers and service time are mutually independent.

At time t let $i(t)$ be the number of sources locating in "waiting" state and $k(t)$ determines the server state

$$k(t) = \begin{cases} 0, & \text{if the server is free,} \\ 1, & \text{if the server is busy.} \end{cases}$$

Let us denote by $P\{k(t) = k, i(t) = i\} = P_k(i,t)$ the joint probability that at the time t there are i sources in "waiting" state and the server is in the "k" state. Under the above assumption the process $\{k(t), i(t)\}$ is a 2-dimentional Markov process with state space $\{0, 1, \ldots, N\} \times \{0, 1\}$

The differential Kolmogorov equations for probabilities $P_k(i,t)$ are

$$\begin{aligned}
\frac{\partial P_0(0,t)}{\partial t} &= -\lambda P_0(0,t) + \mu P_1(1,t) \ , \\
\frac{\partial P_1(1,t)}{\partial t} &= -\left(\lambda \frac{N-1}{N} + \mu\right) P_1(1,t) + \lambda P_0(0,t) + \frac{\sigma}{N} P_0(1,t) \ , \\
\frac{\partial P_0(i,t)}{\partial t} &= -\left(\lambda \frac{N-i}{N} + \sigma \frac{i}{N}\right) P_0(i,t) + \mu P_1(i+1,t) + \\
&\quad + \lambda \frac{N-i+1}{N} P_1(i-1,t) + \sigma \frac{i-1}{N} P_1(i,t) \ , \\
\frac{\partial P_1(i,t)}{\partial t} &= -\left(\lambda \frac{N-i}{N} + \sigma \frac{i-1}{N} + \mu\right) P_1(i,t) + \\
&\quad + \lambda \frac{N-i+1}{N} P_0(i-1,t) + \sigma \frac{i}{N} P_0(i,t) \ .
\end{aligned}$$

Note this system in steady state

$$\begin{aligned}
&-\lambda P_0(0) + \mu P_1(1) = 0 \ , \\
&-\left(\lambda \frac{N-1}{N} + \mu\right) P_1(1) + \lambda P_0(0) + \frac{\sigma}{N} P_0(1) = 0 \ , \\
&-\left(\lambda \frac{N-i}{N} + \sigma \frac{i}{N}\right) P_0(i) + \mu P_1(i+1) + \lambda \frac{N-i+1}{N} P_1(i-1) + \\
&\qquad\qquad + \sigma \frac{i-1}{N} P_1(i) = 0 \ , \\
&-\left(\lambda \frac{N-i}{N} + \sigma \frac{i-1}{N} + \mu\right) P_1(i) + \lambda \frac{N-i+1}{N} P_0(i-1) + \\
&\qquad\qquad + \sigma \frac{i}{N} P_0(i) = 0 \ .
\end{aligned} \tag{1}$$

The partial characteristic functions are denoted by

$$H_k(u) = \sum_{i=0}^{N} e^{jui} P_k(i) \ .$$

Then system (1) corresponds as

$$\begin{aligned}
&\frac{j}{N}(\sigma-\lambda)\frac{dH_0(u)}{du} + \frac{j}{N}\left(\lambda e^{ju}-\sigma\right)\frac{dH_1(u)}{du} - \lambda H_0(u) + \\
&\qquad\qquad + \left(\lambda e^{ju} + \mu e^{-ju} - \frac{\sigma}{N}\right) H_1(u) = 0 \ , \\
&\frac{j}{N}\left(\lambda e^{ju}-\sigma\right)\frac{dH_0(u)}{du} + \frac{j}{N}(\sigma-\lambda)\frac{dH_1(u)}{du} + \lambda e^{ju} H_0(u) + \\
&\qquad\qquad + \left(\frac{\sigma}{N} - \lambda - \mu\right) H_1(u) = 0 \ .
\end{aligned} \tag{2}$$

In order to solve this system, we use method of asymptotic analysis [13] under conditions of infinitely increasing number of sources ($N \to \infty$).

3 Asymptotic of the First Order

Let us denote $\frac{1}{N} = \varepsilon$.

Introducing following substitute

$$u = \varepsilon w, \qquad H_k(u) = F_k(w, \varepsilon) \ ,$$

we can transform system (2) to the form:

$$\begin{aligned}
&j(\sigma-\lambda)\frac{\partial F_0(w,\varepsilon)}{\partial w} + j\left(\lambda e^{j\varepsilon w}-\sigma\right)\frac{\partial F_1(w,\varepsilon)}{\partial w} - \lambda F_0(w,\varepsilon) + \\
&\qquad\qquad + \left(\lambda e^{j\varepsilon w} + \mu e^{-j\varepsilon w} - \varepsilon\sigma\right) F_1(w,\varepsilon) = 0, \\
&j\left(\lambda e^{j\varepsilon w}-\sigma\right)\frac{\partial F_0(w,\varepsilon)}{\partial w} + j(\sigma-\lambda)\frac{\partial F_1(w,\varepsilon)}{\partial w} + \lambda e^{j\varepsilon w} F_0(w,\varepsilon) + \\
&\qquad\qquad + (\varepsilon\sigma - \lambda - \mu) F_1(w,\varepsilon) = 0.
\end{aligned} \tag{3}$$

Theorem 1. *The limiting value $F_0(w), F_1(w)$ of function $F_0(w,\varepsilon), F_1(w,\varepsilon)$(the solutions of the system (3)), are given by the formulas*

$$F_0(w) = R_0 e^{jw\kappa_1}, \quad F_1(w) = R_1 e^{jw\kappa_1},$$

where R_k the stationary distributions of probabilities of the service state are defined as follows

$$\begin{aligned}
R_1 &= \frac{\sigma(2\lambda+\mu) - \sqrt{\sigma^2(2\lambda-\mu)^2 + 8\sigma\mu\lambda^2}}{4\mu(\sigma-\lambda)}, \\
R_0 &= 1 - \frac{\sigma(2\lambda+\mu) - \sqrt{\sigma^2(2\lambda-\mu)^2 + 8\sigma\mu\lambda^2}}{4\mu(\sigma-\lambda)},
\end{aligned} \tag{4}$$

and κ_1 *is*

$$\kappa_1 = \frac{2\mu R_1^2}{\sigma(1-2R_1)}.$$

4 Asymptotic of the Second Order

To find the asymptotic of the second order we must execute following substitute at system (2):

$$H_k(u) = H_k^{(2)}(u)\exp\{ju\kappa_1 N\}.$$

By putting $\dfrac{1}{N} = \varepsilon^2$, $u = \varepsilon w$, $H_k^{(2)}(u) = F_k^{(2)}(w,\varepsilon)$, we get

$$\begin{aligned}
&j\varepsilon(\sigma-\lambda)\frac{\partial F_0^{(2)}(w,\varepsilon)}{\partial w} + j\varepsilon\left(\lambda e^{j\varepsilon w}-\sigma\right)\frac{\partial F_1^{(2)}(w,\varepsilon)}{\partial w} - \\
&\qquad - \left[\lambda + (\sigma-\lambda)\kappa_1\right]F_0^{(2)}(w,\varepsilon) + \\
&\qquad + \left[\lambda e^{j\varepsilon w}(1-\kappa_1) + \mu e^{-j\varepsilon w} + \sigma\kappa_1 - \varepsilon^2\sigma\right]F_1^{(2)}(w,\varepsilon) = 0, \\
&j\varepsilon\left(\lambda e^{j\varepsilon w}-\sigma\right)\frac{\partial F_0^{(2)}(w,\varepsilon)}{\partial w} + j\varepsilon(\sigma-\lambda)\frac{\partial F_1^{(2)}(w,\varepsilon)}{\partial w} \\
&\qquad + \left[\lambda e^{j\varepsilon w}(1-\kappa_1) + \sigma\kappa_1\right]F_0^{(2)}(w,\varepsilon) - \\
&\qquad - \left[\lambda(1-\kappa_1) + \mu + \sigma\kappa_1 - \varepsilon^2\sigma\right]F_1^{(2)}(w,\varepsilon) = 0.
\end{aligned} \tag{5}$$

Theorem 2. *The limiting value* $F_0^{(2)}(w)$, $F_1^{(2)}(w)$ *of function* $F_0^{(2)}(w,\varepsilon)$, $F_1^{(2)}(w,\varepsilon)$ *(the solutions of the system (5)), are given by the formulas*

$$F_k^{(2)}(w) = R_k\Phi^{(2)}(w),$$

where

$$\Phi^{(2)}(w) = \exp\left\{\frac{(jw)^2}{2}\kappa_2\right\},$$

$$\kappa_2 = \mu R_1 \cdot \frac{1 + (R_1 - R_0)R_0}{\lambda - (\lambda-\sigma)(R_1-R_0)^2}.$$

5 Asymptotic of the Third Order

To find the asymptotic of the third order we must execute following substitute at system (2):

$$H_k(u) = H_k^{(3)}(u)\exp\left\{ju\kappa_1 N + \frac{(ju)^2}{2}\kappa_2 N\right\}. \tag{6}$$

At system (2) make substitutions

$$\frac{1}{N} = \varepsilon^3, \quad u = \varepsilon w, \quad H_k^{(3)}(u) = F_k^{(3)}(w, \varepsilon), \tag{7}$$

and we obtain

$$\begin{aligned}
& j\varepsilon^2 (\sigma - \lambda) \frac{\partial F_0^{(3)}(w,\varepsilon)}{\partial w} + j\varepsilon^2 \left(\lambda e^{j\varepsilon w} - \sigma\right) \frac{\partial F_1^{(3)}(w,\varepsilon)}{\partial w} + \\
& \quad + \left[(\lambda - \sigma)(\kappa_1 + j\varepsilon w \kappa_2) - \lambda\right] F_0^{(3)}(w,\varepsilon) + \\
& \quad + \left[\left(\sigma - \lambda e^{j\varepsilon w}\right)(\kappa_1 + j\varepsilon w \kappa_2) + \lambda e^{j\varepsilon w} + \mu e^{-j\varepsilon w} - \varepsilon^3 \sigma\right] F_1^{(3)}(w,\varepsilon) = 0, \\
& j\varepsilon^2 \left(\lambda e^{j\varepsilon w} - \sigma\right) \frac{\partial F_0^{(3)}(w,\varepsilon)}{\partial w} + j\varepsilon^2 (\sigma - \lambda) \frac{\partial F_1^{(3)}(w,\varepsilon)}{\partial w} + \\
& \quad + \left[\left(\sigma - \lambda e^{j\varepsilon w}\right)(\kappa_1 + j\varepsilon w \kappa_2) + \lambda e^{j\varepsilon w}\right] F_0^{(3)}(w,\varepsilon) + \\
& \quad + \left[(\lambda - \sigma)(\kappa_1 + j\varepsilon w \kappa_2) - \lambda - \mu + \varepsilon^3 \sigma\right] F_1^{(3)}(w,\varepsilon) = 0.
\end{aligned} \tag{8}$$

Theorem 3. *The limiting value* $F_0^{(3)}(w)$ *,* $F_1^{(3)}(w)$ *of function* $F_0^{(3)}(w,\varepsilon)$ *,* $F_1^{(3)}(w,\varepsilon)$ *(the solutions of the system (8)), are given by the formulas*

$$F_k^{(3)}(w) = R_k \Phi^{(3)}(w),$$

where

$$\Phi^{(3)}(w) = \exp\left\{\frac{(jw)^3}{3!}\kappa_3\right\},$$

$$\kappa_3 = 2 \cdot \left\{ \frac{\left[(\lambda - \sigma)(R_0 - R_1)^2 \kappa_2 + \mu a\right] \cdot \left[\frac{1}{2} - ((R_0 - R_1)(\sigma - \lambda) + \lambda) \frac{\kappa_2}{\mu R_1}\right]}{\lambda + (\sigma - \lambda)(R_0 - R_1)^2} + \right.$$
$$\left. + \frac{\lambda \kappa_2 \cdot \left[\frac{a}{R_0} + \frac{1}{2}\right] + \mu \left[\frac{1}{R_0} - \frac{a}{2}\right]}{\lambda + (\sigma - \lambda)(R_0 - R_1)^2} \right\},$$

$$a = R_0 R_1 (R_1 - R_0).$$

We can find the characteristic function $h(u)$ of the number of sources is in "waiting" state. Using a substitution, reversing to the (7), and, considering (6), we get

$$h(u) = \left(H_0^{(3)}(u) + H_1^{(3)}(u)\right) \exp\left\{ju\kappa_1 N + \frac{(ju)^2}{2}\kappa_2 N\right\},$$

where

$$H_0^{(3)}(u) = R_0 \exp\left\{\frac{(ju)^3}{3!}\kappa_3 N\right\}, \quad H_1^{(3)}(u) = R_1 \exp\left\{\frac{(ju)^3}{3!}\kappa_3 N\right\}.$$

Thus

$$h(u) = \exp\left\{ju\kappa_1 N + \frac{(ju)^2}{2}\kappa_2 N + \frac{(ju)^3}{3!}\kappa_3 N\right\}.$$

6 Computing Procedure

Consider now system M/M/1//N in prelimiting situation.

Theorem 4. *The joint distribution of the service and source state can be computed from the following steps:*

1. *Choose the model parameters* λ, μ, σ, N.
2. *Put* $P_1(0) = 0$.
3. *For* $i = 0$ *compute quantity* $\dfrac{P_1(1)}{P_0(0)}$ *from (1).*
4. *For* $1 \leq i \leq N-1$ *compute recursively* $\dfrac{P_0(i)}{P_0(0)}$ *and* $\dfrac{P_1(i)}{P_0(0)}$ *from the following formulas*

$$\frac{P_0(i)}{P_0(0)} = \frac{N}{i\sigma}\left\{\left(\lambda\frac{N-i}{N} + \sigma\frac{i-1}{N} + \mu\right)\frac{P_1(i)}{P_0(0)} - \lambda\left(\frac{N-i+1}{N}\right)\frac{P_0(i-1)}{P_0(0)}\right\},$$

$$\frac{P_1(i+1)}{P_0(0)} = \frac{1}{\mu}\left\{\lambda\frac{N-i}{N} + \sigma\frac{i}{N}\right\}\frac{P_0(i)}{P_0(0)} - \lambda\frac{(N-i+1)}{N}\frac{P_1(i-1)}{P_0(0)} - \sigma\frac{(i-1)}{N}\frac{P_1(i)}{P_0(0)},$$

which are obtain from (1).

5. *For* $i = N$ *compute quantity* $\dfrac{P_0(N)}{P_0(0)}$

$$\frac{P_0(N)}{P_0(0)} = \frac{1}{\sigma}\left\{\left(\sigma\frac{N-1}{N} + \mu\right)\frac{P_1(N)}{P_0(0)} - \frac{\lambda}{N}\cdot\frac{P_0(N-1)}{P_0(0)}\right\}.$$

6. *The quantity* $P_0(0)$ *may be found with the help of the normalizing conditions*

$$P_0(0) = 1 \Big/ \sum_{i=0}^{N}\left(\frac{P_0(i)}{P_0(0)} + \frac{P_1(i)}{P_0(0)}\right).$$

7. *Compute* $P_k(i)$ *from*

$$P_k(i) = \frac{P_k(i)}{P_0(0)} \cdot P_0(0).$$

Now we can get one-dimensional distribution of the number of sources in "waiting" state

$$P(i) = P_0(i) + P_1(i).$$

7 Numerical Results

We assume that prelimit distribution $P(i)$ can be approximated by asymptotic distribution $P_\nu(i)$ of the ν-th order ($\nu = 2, 3$) in some domains of the system parameters. In order to compare distributions, we use the Kolmogorov distance:

$$\Delta_\nu = \max_{0 \le k \le N} \left| \sum_{i=0}^{k} P_\nu(i) - \sum_{i=0}^{k} P(i) \right|.$$

In numerical computation the model parameters λ, μ, σ are fixed, $\lambda = 5$, $\mu = 10$, $\sigma = 20$ and the number of sources N takes values $5, 10, 18, 25, 100$.

The results are reported in Table 1.

Table 1. Kolmogorov distance between prelimit distribution $P(i)$ and asymptotic distribution $P_\nu(i)$ of the ν-th order ($\nu = 2, 3$)

	$\lambda = 5,\ \mu = 10,\ \sigma = 20$				
N	5	10	18	25	100
Δ_2	0,184	0,091	0,039	0,028	0,012
Δ_3	0,165	0,080	0,029	0,019	0,008

Table 1 show that asymptotic approximation of the second order with a good degree of accuracy approximates prelimit distribution $P(i)$ for $N \ge 25$. At the same time, the Kolmogorov distance between distributions $P(i)$ and $P_3(i)$ becomes less then $0,03$ for $N \ge 18$. Let us note, that accuracy of all approximations generally improves as N increases.

Our numerical experiment show, that accuracy increase with the growth of the order of approximation and determine the range of applicability of the asymptotic results in prelimiting situation.

8 Conclusion

In this paper, we research a closed retrial queuing system M/M/1//N with collision of the customers. Using the method of asymptotic analysis under conditions of infinitely increasing number of sources, we obtain a distribution of the number of sources in "waiting" state. Also, we obtain the probability distribution with help of numerical algorithm in prelimiting situation. Comparing there distributions we can conclude, that prelimit distribution can be approximated by asymptotic distribution for $N \ge 25$ (the second order approximation) and for $N \ge 18$ (the third order approximation). Therefore, those approximations have a high level of accuracy and can be used on practical engineering application where the relative error does not represent a severe constrain.

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