“Potential Interval of Root” of Nonlinear Equation: Labeling Algorithm

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Abstract. Novel Sequence Generating Algorithm (SGA) and Potential Interval Qualifier Algorithm (PIQA) are designed to classify potential interval estimates of a given nonlinear equation into intervals possessing roots and intervals containing extrema. Using trisection method, SGA is developed to generate conjugate pair of sequences that converge to a point in the interval. Further, PIQA qualifies each interval into interval enclosing a root or interval containing extrema. If the interval contains a root, the multiplicity of root is also obtained. The proposed methodologies have been implemented and demonstrated through a set of benchmark functions to illustrate the effectiveness.

Keywords: Convergence**,** extrema, Genetic Algorithms, interval based method, interval estimates, root, sequence.

1 Introduction

A number of real world problems in different areas of science and engineering are often reduced to the problem of finding all the real roots and/or the extrema of a function in a given interval. Most of the conventional numerical methods like Newton-Raphson method, secant method have some drawbacks such as sensitivity to initial guess and oscillatory behavior of root etc. Generally interval based root computation methods are known to be robust but they are slow. Finding roots through interval based bracketing root method called Bisection method is discussed in [1] by subdividing the given interval in to ‘n’ equal intervals. Most of the real world problems are likely to be complex and knowledge about the function is generally not known in advance and hence we do not know how many subdivisions of intervals are needed for a given problem in the region of interest. Various Metaheuristic optimization methods such as Genetic Algorithms (GA) [2, 3], Particle Swarm Optimization (PSO) [4] and Invasive Weed Optimization method (IWO) [5] have been proposed to address this problem of finding all the roots of given nonlinear equation or system of equations. In our earlier study, an interval based approach named as Novel GA has been proposed [6] to capture all the potential interval estimates for roots. In the present study, an “Intelligent Interval Labeling Algorithm” is proposed to identify these interval estimates into intervals containing roots and intervals containing extrema of given function.

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GA are simulation programs [7] that create an environment which would allow only the fittest population to survive. Thus, GA provides some possible solutions and these solutions are represented as chromosomes. Generally the parameter 'Length of Chromosome' (LC) in GA is arbitrarily chosen and this might sometimes effect the performance of GA. Novel GA methodology [6] proposed earlier has two algorithms, namely *Pre-processing algorithm* and *Post-processing algorithm.* The fitness function in GA is defined as the sum of all potential intervals in the region of interest satisfying the condition that the product of function values at the end points of each interval is less than zero. To establish the knowledge about the number of roots that a nonlinear equation possesses, *Pre-processing algorithm* is proposed to suggest a mechanism that adaptively fixes the parameter LC in GA. GA coupled with *Post-processing algorithm* is executed with this new LC as input. Thus, the methodology [6] produces significantly narrow and potential interval estimates facilitating root computation to be highly efficient. Additionally it is noticed that Novel GA [6] has the power of capturing all the interval estimates i.e., missing a root in the zone of interest has less probability.

To address the problem of finding roots with multiplicity, several researchers attempted to address multiple zeros of a nonlinear equation [8-12]. The approach suggested by [8] has been adapted in Novel GA method [6]. It has the following transformation that converts roots with multiplicity of the problem $f(x) = 0$ as a problem $g(x) = 0$ with simple roots where

$$g(x) = \begin{Bmatrix} \dfrac{f(x)}{f'(x)}, & f'(x) \neq 0 \\ 0, & f'(x) = 0 \end{Bmatrix} \tag{1}$$

Hence in Novel GA method [6], $g(x)$ is considered for the initial population in the place of $f(x)$ in *Pre-processing algorithm* and we obtain the potential interval estimates of g. It is observed that all these selected potential intervals need not possess a root of f. By the nature of the smoothness of the function f, in between two roots i.e., in between two crossings of X-axis, there should be a maxima or minima of f. This paper aims at classifying each of these interval estimates into interval containing a root or interval having extrema of f. King [12] suggested that multiplicity $'m'$ of a root is approximately the reciprocal of the divided difference for $'g'$ for successive iterates. In the present study, a schematic method named as SGA is proposed to produce two conjugate pair of sequences, one is increasing and the other is decreasing and these sequences converge to a point in the interval. PIQA is developed by considering the function values of these two sequences, their absolute maximum, absolute difference and absolute relative difference between each pair of numbers in the two sequences.

The paper is organized as follows. Section 2 describes about the trisection method based Sequence Generating Algorithm (SGA). Section 3 explains about the Potential Interval Qualifier Algorithm (PIQA). Section 4 provides few illustrations and Section 5 has numerical examples and Section 6 concludes the results.

2 Sequence Generating Algorithm (SGA)

Consider the function $f(x) = (x-1)^2 (\sin x)^2 + (x-1)^3 (\cos x)^3 + 5(x-1)$ that has 3 roots in [0,4]. Novel GA [6] is executed for the transformed function $g(x)$.

Table 1. The following 5 potential intervals are given as output from Novel GA [6]

S.No.	x-lower	x-upper
1	0.9776173	1.0271712
2	2.3490321	2.4026781
3	3.2538517	3.2985591
4	3.3498671	3.3588103
5	3.4281167	3.4696456

It is observed that 1st, 3rd and 5th intervals possess roots 1, 3.27577, 3.433599, whereas 2nd and 4th intervals do not possess any root of the given function f but these two intervals contain extrema of f. Thus, it is necessary to distinguish the intervals containing roots from the intervals containing extrema.

Let $[a,b]$ be a potential interval given as output from Novel GA [6] with respect to the function g such that $g(a) * g(b) < 0$. Generate two sequences $\{a_i\}, \{b_i\}, i = 0,1,..,n$ contained in $[a,b]$ with the only restriction that $\{a_i\}$ is an increasing sequence and $\{b_i\}$ is a decreasing sequence, $a_0 = a$ and $b_0 = b$. Thus, these two sequences have points that are uniformly densed. Now compute the absolute maximum, absolute difference and absolute relative difference defined by,

$$\text{Absolute maximum } h_i = Max\{|g(b_i)|, |g(a_i)|\},$$

$$\text{Absolute difference } d_i = |g(b_i) - g(a_i)|,$$

$$\text{Absolute relative difference } r_i = \frac{|g(b_i) - g(a_i)|}{b_i - a_i}, \text{for } i = 0,1,2..,n$$

Interesting results are obtained as depicted in Fig. 1 and Fig. 2 for the sequences in the 1st and 2nd intervals given in Table 1.

Two sequences $\{a_i\}, \{b_i\}$ that are increasing and decreasing respectively are generated in the interval $[0.97762, 1{,}0272]$ with uniformly densed points. It can be noticed from Fig.1 (subplot 1) that function values of both the sequences are

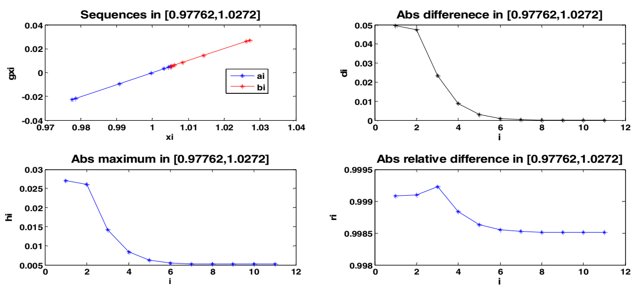

Fig. 1. Plot depicting the behavior of $\{a_i\}, \{b_i\}$ in 1st interval $[0.97762, 1{,}0272]$

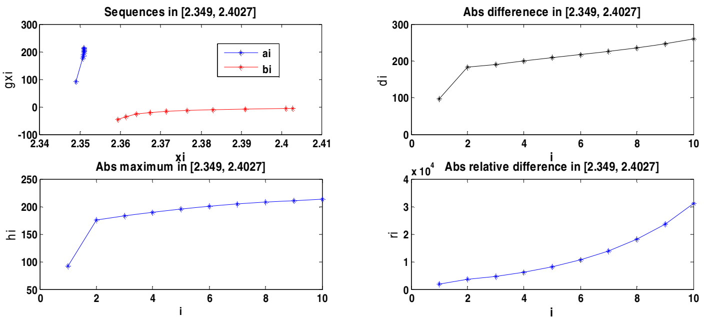

Fig. 2. Plot explaining the behavior of $\{a_i\},\{b_i\}$ in 2nd interval $[2.349, 2.4027]$

converging in this interval. Absolute maximum, absolute difference monotonically decrease and are converging. Further, absolute relative difference also converges.

It can be seen from Fig.2 that in the interval $[2.349, 2.4027]$, function values of both the sequences are diverging, absolute maximum and absolute difference are monotonically increasing and absolute relative difference exponentially increases. Due to the transformation (1), $g(x)$ will be zero when it has root or when $f'(x) = 0$, which corresponds to the maxima or minima of f. Therefore, it can be understood that this interval encloses a point at which $f'(x) = 0$.

The above Fig. 1 and Fig. 2 illustrating the behavior of sequences (having uniformly densed points) do not explicitly explain the nature of the function f and function g in the interval. Hence, it is required to develop a schematic method to generate conjugate pair of sequences $\{a_i\},\{b_i\}$ in $[a, b]$ such that always every pair $[a_i, b_i] \subseteq [a, b]\ \forall i = 0,1,2,\ldots,n$ maintains the potential interval property satisfying the condition $g(a_i) * g(b_i) < 0$ and every $[a_i, b_i]$ brackets a root. Motivated by the trisection method, the following SGA is developed for generating such conjugate pair of sequences.

Algorithm 1. ***Sequence Generating Algorithm - SGA***

//Generates two sequences $\{a_i\}$, $\{b_i\}, i = 0,1,2..n$ *in* $[a, b]$ *satisfying potential interval property//*

INPUT:

- Given transformed function $g(.)$
- Potential interval $[a, b]$ given as output from method [6] such that $ga * gb < 0$
- p // p indicates number of parts which is a user defined variable //
- $\in \leftarrow 10^{-6}$ // tolerance //

OUTPUT: $[\{a_i\},\{b_i\}, n]$ //$\{a_i\},\{b_i\}$ are increasing and decreasing sequences contained in $[a, b]$ respectively. $'n'$ corresponds to number of elements in the sequences $\{a_i\},\{b_i\}$ //

METHOD:

Step 1:

$i \leftarrow 0, a_i \leftarrow a, b_i \leftarrow b, \delta \leftarrow \frac{(b-a)}{p}$

Step 2:

while (ä >∈)

$i \leftarrow i + 1, anew \leftarrow a + \delta, bnew \leftarrow b - \delta$

compute $ganew \leftarrow g(anew), gbnew \leftarrow g(bnew)$
$Case$
$: ganew * gbnew < 0$
$a \leftarrow anew, b \leftarrow b_{new}, ga \leftarrow ganew, gb \leftarrow gbnew$
$Case$
$: ga * ganew < 0$
$b \leftarrow anew, gb \leftarrow ganew$
$Case$
$: else$
$a \leftarrow bnew, ga \leftarrow gbnew$
$end\ Case$
$a_i \leftarrow a, b_i \leftarrow b$
$\delta \leftarrow \frac{(b-a)}{p}$
$end\ while$
$n \leftarrow i$
End; *//SGA //*

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3 Potential Interval Qualifier Algorithm (PIQA)

The following result is given by [8, p5].

Lemma: If the given function $f(.)$ has a root $'\alpha'$ with multiplicity $'m'$, and $g(.)$ is the transformed function by (2), then

$$\lim_{x\to\alpha} \frac{g(x)}{(x-\alpha)} = \frac{1}{m}, \text{i.e., if } \delta = x-\alpha \ \Rightarrow x = \alpha+\delta \text{ and as } x \longrightarrow \alpha, \delta \longrightarrow 0, \text{then}$$

$$\lim_{x\to\alpha} \left\{\frac{1}{\delta}\frac{f(\alpha+\delta)}{f'(\alpha+\delta)}\right\} = \frac{1}{m}$$

King [9] proved that multiplicity $'m'$ of a root is approximately equal to the reciprocal of the divided difference for $'g'$ for successive iterates. We now state the following theorem.

Theorem: Given two sequences $\{a_i\}, \{b_i\}$ in $[a, b]$ generated by SGA such that both the sequences converge to root α with multiplicity $'m'$. Let $a_i = \alpha - \delta_i, b_i = \alpha + \delta_i$ for $i = 0,1,2..,n$. Then the absolute relative difference

$$\lim_{\delta_i\to 0} \left\{\frac{|g(b_i) - g(a_i)|}{b_i - a_i}\right\} = \frac{1}{m}.$$

Now, the following Potential Interval Qualifier Algorithm is proposed.

Notation:

- $[a^j, b^j]$ denotes j^{th} interval
- $\{a_i^j\}, \{b_i^j\}$ are monotonically increasing and decreasing sequences generated by SGA that confine to j^{th} interval.

Algorithm 2. ***PIQA***

//This algorithm qualifies the given potential interval estimates in to intervals enclosing root (with multiplicity) and intervals containing extrema //

INPUT:

- Given transformed function $g(.)$
- $d \leftarrow$ Number of potential intervals $[a^j, b^j]$, $j = 1,2..d$ given as output from Novel GA method [6] such that $g(a^j) * g(b^j) < 0$, for each j
- p // p indicates number of parts which is a user defined variable //
- $\in \leftarrow 10^{-6}$ // tolerance to generate sequences in SGA//
- $\in_1 \leftarrow 10^{-3}$// tolerance to qualify the interval whether it has a root or extrema //
- $[\{a_i^j\}, \{b_i^j\}, n^j], j = 12, \ldots d$ //output from SGA where n^j corresponds to number of terms in the pair of sequences $\{a_i^j\}, \{b_i^j\}$ generated in j^{th} interval//

OUTPUT: $[\{h_i^j\}, \{d_i^j\}, \{r_i^j\}, pi, npi, m]$ //$\{h_i^j\}, \{d_i^j\}, \{r_i^j\}$ correspond to sequences for absolute maximum, absolute difference, absolute relative difference respectively in j^{th} interval. $'pi'$ corresponds to number of potential intervals possessing root in the region of interest, $'npi'$ corresponds to number of intervals that have extrema but do not contain root and $'m'$ is an array representing multiplicity of each root in the corresponding interval //

METHOD:

Step 0: Initialization: $j \leftarrow 1, pi \leftarrow 0, npi \leftarrow 0$

Repeat Step 1 through Step 4 $while\ (j \le d)$

Step 1: Initialize j^{th} interval $[a^j, b^j]$

Step 2: $[\{a_i^j\}, \{b_i^j\}, n^j] \leftarrow \mathrm{SGA}(g(.), [a^j, b^j], p, \in)$

Step 3: Compute $for\ i \leftarrow 1\ to\ n^j$

$$h_i^j \leftarrow Max\{|g(a_i^j)|, |g(b_i^j)|\}$$

$$d_i^j \leftarrow |g(b_i^j) - g(a_i^j)|$$

$$r_i^j \leftarrow \frac{|g(b_i^j) - g(a_i^j)|}{b_i^j - a_i^j}$$

// sequences for absolute maximum, absolute difference, absolute relative difference in the interval $[a^j, b^j]$ //

Compute $Min\_hij \leftarrow min\ (h_i^j); Min\_dij \leftarrow min\ (d_i^j); Min\_rij \leftarrow min\ (r_i^j)$

$endfor$

Step 4: $if(\ (Min\_hij < \in_1\ )\ and\ (Min\_dij < \in_1)\ )$

$pi \leftarrow pi + 1$

$m(j) \leftarrow callmr(\{r_i^j\}, \in_1, n^j)$ //Returns multiplicity of each root//

$else$

$npi \leftarrow npi + 1$

$endif$

$endwhile$

End; *//PIQA //*

Algorithm to find multiplicity: $callmr(\{r_i^j\}, \in_1, n^j)$

$$l \leftarrow n^j$$

$$if((r_l^j - r_{l-1}^j) < \in_1)$$

$$m = \frac{1}{r_l^j}$$

$$endif$$

4 Illustration for SGA and PIQA

Now consider $f(x) = (x-1)^2\,(\sin x)^2 + (x-1)^3(\cos x)^3 + 5(x-1)$. While generating sequences by SGA, we come across three cases.

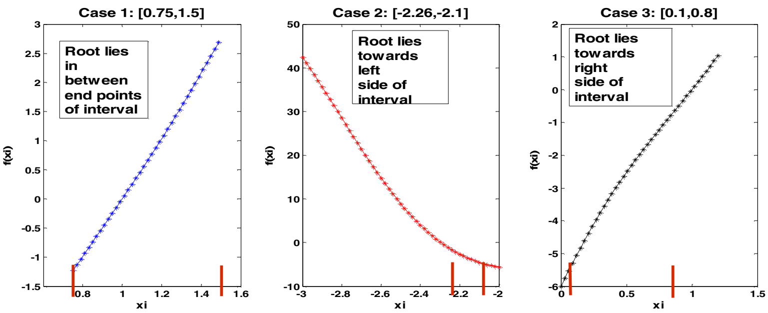

Fig. 3. Plot depicting the three possible cases in three different intervals

In the above Fig. 3, subplot 1 shows the behavior of function in the interval $[0.75,1.5]$, where root lies inside the interval satisfying the condition that product of function values at the end points of the interval is less than zero. Subplot 2 exhibits the case 2 where root lies outside the interval towards left side. Subplot 3 displays the case 3 where root lies outside the interval towards right side. Now, while generating two sequences inside a given interval by SGA, we always wish to bracket the root. Thus, the two sequences $\{a_i\}$, $\{b_i\}$ are generated in such a way that always every pair $[a_i, b_i] \subseteq [a, b]\ \forall i = 0,1,2,\ldots,n$ maintains the potential interval property satisfying the condition $g(a_i) * g(b_i) < 0$. The consecutive new values in each pair of sequences are altered accordingly to satisfy this condition as given in Algorithm 1. Hence this is a learning algorithm, a knowledge discovery approach by means of extracting the information in the given interval. Further, if the interval encloses a root, then function values of f and g for both the sequences converge to zero in the interval and hence both the sequences will converge to the root. If the potential interval does not contain a root, then the function values of f for both the sequences diverge whereas function values of g for both the sequences will have erratic behavior. As the transformed function g possesses quality information, the methodology is developed and analysis is made with focus on the sequences of function values of g, their absolute maximum, absolute difference and absolute relative difference.

It is understood that if δ is too small, we get refined sequences that are always falling into case 1 while generating them. In general, taking a smaller delta influences resource utilization. It is noticed that the potential interval estimates returned by Novel GA are very narrow and hence it is not required to have such smaller δ in SGA.

Now generate sequences $\{a_i\}$, $\{b_i\}$ by SGA in each of the potential intervals given in Table 1 and apply PIQA that qualifies each potential interval into interval enclosing root or interval containing extrema.

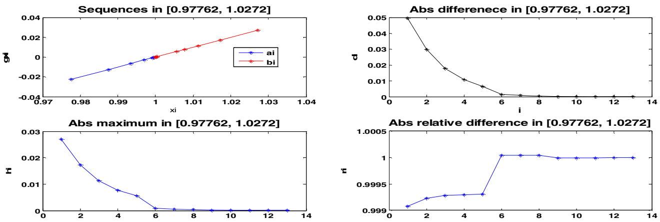

Fig. 4. Plot displaying the behavior $\{a_i\}, \{b_i\}$ generated by SGA in $[0.97762, 1{,}0272]$

It can be noticed from Fig.4 (subplot 1) that function values of both these sequences in $[0.97762, 1{,}0272]$ are converging to zero and both sequences are converging to '1' in this interval. Absolute maximum and absolute difference monotonically decrease and converge to zero. Thus by PIQA, this interval possesses a root and root is the converging point of two sequences. Further, absolute relative difference converges to 1, i.e., multiplicity $'m'$ of the root enclosed in this interval equals to 1. In comparison with Fig.1, it can be clearly understood from Fig. 4 that this algorithm qualifies whether a given interval possesses a root and if it has a root, its multiplicity is also obtained.

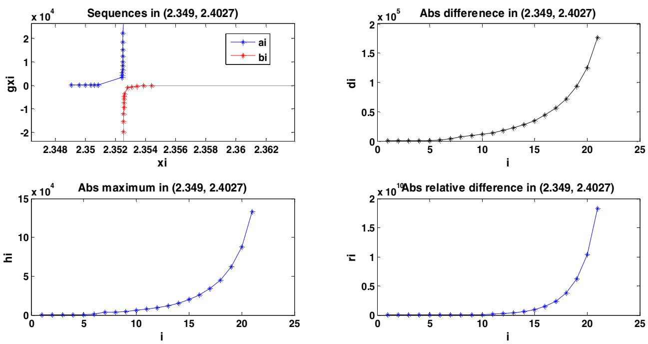

Fig. 5. Plot depicting the behavior of $\{a_i\}, \{b_i\}$ generated by SGA in $[2.349, 2.4027]$

In contrast to the Fig.2, it can be clearly seen from Fig.5 that in the interval $[2.349, 2.4027]$, function values of both the sequences are diverging, absolute maximum and absolute difference are monotonically increasing and absolute relative difference exponentially increases. Hence it can be concluded from PIQA that this interval does not possess root but it has extrema of f.

5 Numerical Experiments

In this section we present 5 examples to illustrate the efficiency of the proposed method.

Table 2. A set of bench mark functions and the results with the proposed method are listed below. Here in this table, NI represents number of intervals returned by Novel GA [6], NIR corresponds to number of intervals enclosing root and NIE corresponds to number of intervals having extrema. It can be noted that 'm' in 8th column denotes the multiplicity of root.

Function	**Zone**	**NI by [6]**	**NIR**	**NIE**	**No. of selected roots**		
					total	**simple**	**multiple**
$f_1 = (x(e^{x^2}) - (sinx)^2 + 3cosx + 5)^4$	[-3,4]	3	1	2	4	-	1 ($m = 4$)
$f_2 = \sin(0.2x)\cos(0.5x)$	[-50,50]	37	19	18	23	15	4 ($m = 2$)
$f_3 = (3x-2)^4(2x-3)^2 (96x^3 - 332x^2 + 325x - 75)$	[0.5,2]	7	4	3	8	2	2 ($m_1 = 4$)
$f_4 = \frac{2}{3} - -(0.1 - x^{11})e^{2-x^2}$	[-1,1]	5	3	2	3	3	-
$f_4 = \left(64x^4 - 16\pi x^3 - 3ð^2x^2 + \pi^3 x - \frac{ð^4}{16}\right)(sin5x + 0.5x + 2)$	[-1,1]	7	3	4	4	2	1 ($m = 2$)

Consider f_2 that has total 23 roots in $[-50,50]$ among which 19 are distinct roots.

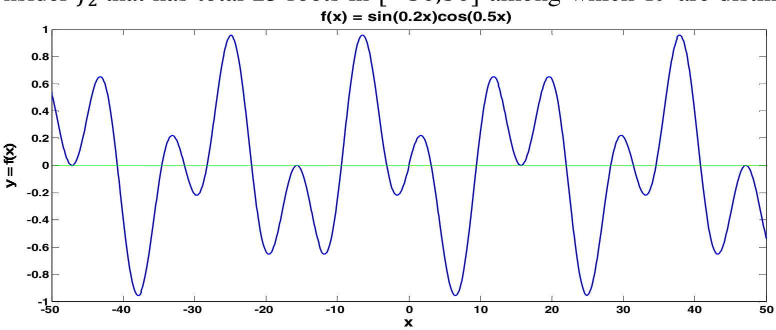

Fig. 6. Plot for $f_2 = sin(0.2x)cos(0.5x)$

As shown in Fig 6, f_2 has 15 simple roots and 4 roots are multiple roots, each with multiplicity 2. Novel GA [6] returned 37 narrow potential intervals and SGA along with PIQA qualifies these 37 intervals into 19 intervals enclosing a root of both g as well as f_2 and 18 intervals as intervals containing extrema of f_2. The following Table 3 displays the intervals enclosing roots and intervals containing extrema of the function f_2.

Table 3. Intervals possessing root or extrema of f_2

S. No.	x-lower	x-upper	PIQA output	S. No.	x-lower	x-upper	PIQA output
1	-47.3684606	-46.9983821	Root	20	1.6432410	1.7254402	Extrema
2	-43.2214511	-43.1660531	Extrema	21	3.125396	3.1685314	Root
3	-40.8441857	-40.8193238	Root	22	6.426091	6.5067159	Extrema
4	-37.9821880	-37.8580232	Extrema	23	9.323579	9.7045229	Root
5	-34.5748749	-34.4447411	Root	24	11.539284	11.915735	Extrema
6	-33.2650959	-33.0479707	Extrema	25	15.684648	15.772927	Root
7	-31.5959871	-31.3240455	Root	26	19.530681	19.656868	Extrema
8	-29.7383903	-29.6495947	Extrema	27	21.746960	22.132692	Root
9	-28.3528603	-28.2703011	Root	28	24.652562	25.024529	Extrema
10	-24.9694113	-24.8496210	Extrema	29	28.108451	28.296819	Root
11	-22.1215471	-21.9769910	Root	30	29.681578	29.814171	Extrema
12	-19.8794413	-19.5596421	Extrema	31	31.377958	31.623528	Root
13	-15.7578597	-15.6994188	Root	32	32.917715	33.118862	Extrema
14	-11.9152470	-11.7062010	Extrema	33	34.545802	34.775856	Root
15	-9.5616142	-9.3720057	Root	34	37.832769	37.948854	Extrema
16	-6.6704734	-6.3067219	Extrema	35	40.605173	40.843837	Root
17	-3.1601239	-2.8432123	Root	36	43.161613	43.511411	Extrema
18	-1.7326985	-1.2000641	Extrema	37	47.014463	47.584481	Root
19	-0.2161691	0.09399237	Root	-	-	-	-

It can be observed that from Table 3 and Fig.6 that the output given by PIQA matches with the roots and extrema of f_2. The following Fig 7 and Fig 8 depict the convergence criterion for the sequences generated in the 1^{st} and 4^{th} intervals from Table 3.

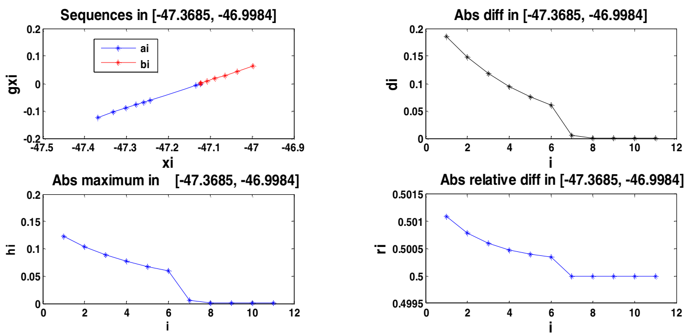

Fig. 7. Plot describing the behavior of $\{a_i\}, \{b_i\}$ generated by SGA in $[-47.3685, -46.9984]$

The function f_2 has a root in the interval $[-47.3685, -46.9984]$ with multiplicity 2. The behavior of the two sequences generated by SGA in this interval can be seen in Fig.7 that function values of both the sequences, absolute maximum and absolute difference converge to zero. Hence by PIQA, it can be concluded that this potential interval contains a root. It can also be observed from Fig.7 (subplot 4) that absolute relative difference converges to 0.5 and hence the multiplicity of the root enclosed in this interval is reciprocal of 0.5, which equals to 2.

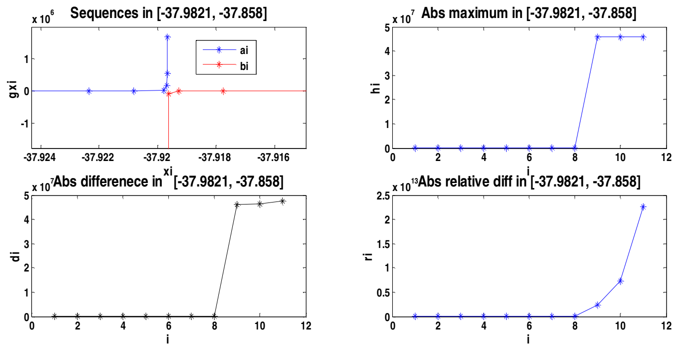

Fig. 8. Plot displays the behavior of $\{a_i\}, \{b_i\}$ generated by SGA in $[-37.9821, -37.858]$

The sequences for function values are diverging, absolute maximum and absolute difference are monotonically increasing. Further, sequence for absolute relative difference is exponentially increasing. Hence, it can be concluded that by PIQA that this interval contains extrema.

Now consider the function $f_3 = (3x-2)^4(2x-3)^2\,(96x^3 - 332x^2 + 325x - 75)$ that has 4 distinct roots in $[0.5, 2]$, among which 0.6666 is a root with multiplicity 4.

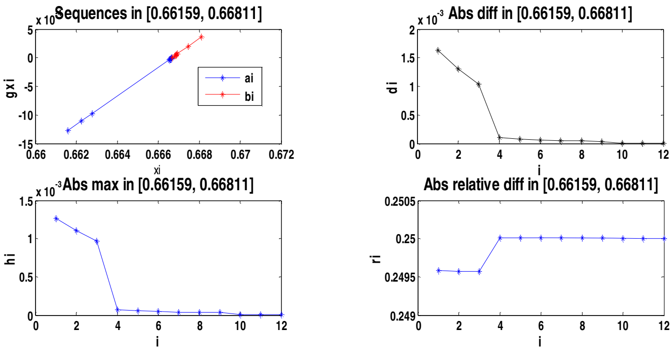

Fig. 9. Plot depicting the behavior of $\{a_i\}, \{b_i\}$ generated by SGA in $[0.66159, 0.66811]$

As the sequences for absolute maximum, absolute difference converge to zero, it can be stated that this potential interval encloses a root and subplot 4 explains that this root (0.6666) has multiplicity 4.

6 Conclusion

A schematic method is proposed to classify the given potential interval estimates into intervals possessing roots and intervals containing extrema through "SGA" coupled with "PIQA". SGA is developed to generate conjugate pair of sequences so that each pair $[a_i, b_i]$ of sequences $\{a_i\}, \{b_i\}, i = 0,1,2..n$ maintains the potential interval property. Analysis is made through PIQA by considering the function values of the sequences given by SGA, their absolute maximum, absolute difference and absolute relative difference to decide whether the potential interval possesses a root or it contains extrema. Based on [8,12], it can be stated that root of g with corresponding $m \geq 1$ is root of f with multiplicity $'m'$. PIQA qualifies remaining potential intervals of g as intervals containing extrema of f. The numerical experiments demonstrate the effectiveness of the proposed methodology.

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References

1. Numerical Recipes: The Art of Scientific Computing, 3rd edn. Cambridge Univ. Press
2. Holland, J.H.: Adaption in natural and artificial systems: An introductory analysis with applications to biology, control, and artificial intelligence. University of Michigan Press
3. Goldberg, D.E.: Genetic Algorithms in Search, Optimization and Machine Learning. Addison-Wesley Publishing
4. Castillo, O., Melin, P., Pedrycz, W.: Soft Computing for Hybrid Intelligent Systems
5. Pourjafari, E., Mojallali, H.: Solving nonlinear equations systems with a new approach based on Invasive weed optimization algorithm and clustering. Swarm and Evolutionary Computation 4, 33–43 (2012)
6. Nadimpalli, V.L.V., Wankar, R., Chillarige, R.R.: A Novel Genetic Algorithmic Approach for Computing Real Roots of a Nonlinear Equation. In: Proceedings of EVO Star 2014. LNCS (in press, 2014)
7. Deb, K.: Multi-objective Optimization Using Evolutionary Algorithms. Wiley (2001)
8. Traub, J.F.: Iterative Methods for the Solution of Equations. Prentice Hall, Englewood (1964)
9. Espelid, T.O.: On the behavior of the secant method near a multiple root. BIT I1, 112–115 (1972)
10. Parida, P.K., Gupta, D.K.: An improved method for finding multiple roots and it's multiplicity of nonlinear equations in R*. Applied Mathematics and Computation 202, 498–503 (2008)
11. Yun, B.I.: Iterative methods for solving nonlinear equations with finitely many zeros in an interval. Journal of Computational and Applied Mathematics 236, 3308–3318 (2012)
12. King, R.F.: A secant method for multiple roots. BIT 17, 321–328 (1977)