Singular Limits for Models of Compressible, Viscous, Heat Conducting, and/or Rotating Fluids

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Abstract

The complete set of equations describing the motion of a general compressible, viscous, heat-conducting, and possibly rotating fluid arises as a mathematical model in a large variety of real world applications. The scale analysis aims at two different objectives: Rigorous derivation of a simplified asymptotic set of equations and understanding the passage from the original primitive system to the simplified target system. These issues are discussed in the context of compressible, viscous, heat conducting, and/or rotating fluids.

1 Introduction

The complete set of equations describing the motion of a general compressible, viscous, heat-conducting, and possibly rotating fluid arises as a mathematical model in a large variety of real world applications. The scale analysis aims at two different objectives:

- Rigorous derivation of a simplified asymptotic set of equations which can be solved analytically or, in practice, with less numerical effort than the original system.
- Understanding the passage from the original primitive system to the simplified target system; identifying the essential part of the solution that persist and its residual component that is suppressed in the asymptotic limit.

Throughout the chapter, the fluid motion is considered in the framework of classical *continuum mechanics* described in terms of certain observable macroscopic quantities: the mass density ϱ, the (absolute) temperature ϑ, and the (bulk) velocity $\mathbf{u}$. The numerical values of these quantities – fields – depend on the time t and the spatial position x. Throughout the whole text, the *Eulerian description* is used, where the coordinate frame is attached to the physical domain occupied by the fluid or to the rotating frame as the case may be. The fields are interrelated through a system of *field equations* – balance laws – reflecting the underlying physical principles of conservation of mass, momentum, and energy.

1.1 Scaling and Dimensionless Equations

To begin, the basic equations of fluid dynamics will be rewritten in their *dimensionless form*. To this end, all physical quantities X will be replaced by their dimensionless counterpart

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$$X \approx \frac{X}{X_{\rm char}},$$

where $X_{\rm char}$ denotes the characteristic value of the quantity X in the regime of interest. Here and hereafter, the same symbol X is used for both the quantity and its rescaled value. The list of physical quantities considered in this chapter includes:

Symbol	Name
$\varrho = \varrho(t,x)$	mass density
$\mathbf{u} = \mathbf{u}(t,x)$	velocity field (in the rotating frame)
$\vartheta = \vartheta(t,x)$	(absolute) temperature
ω	rotation vector
$p = p(\varrho,\vartheta)$	pressure
$\mathbb{S} = \mathbb{S}(\varrho,\vartheta,\nabla_x\mathbf{u})$	viscous stress
$G = \lvert x \times \omega \rvert^2$	centrifugal force
$\mathbf{f} = \mathbf{f}(t,x)$	external force density
$e = e(\varrho,\vartheta)$	(specific) internal energy
$\mathbf{q} = \mathbf{q}(\varrho,\vartheta,\nabla_x\vartheta)$	heat flux
$\mathcal{Q} = \mathcal{Q}(t,x)$	external heat source

In accordance with the second law of thermodynamics, the pressure $p = (\varrho,\vartheta)$ and the specific internal energy $e = e(\varrho,\vartheta)$ are interrelated through *Gibbs' equation*

$$\vartheta Ds(\varrho,\vartheta) = De(\varrho,\vartheta) + p(\varrho,\vartheta)D\left(\frac{1}{\varrho}\right), \tag{1}$$

where $s = s(\varrho,\vartheta)$ is the specific entropy; see [43]. In addition, the *thermodynamic stability hypothesis*

$$\frac{\partial p(\varrho,\vartheta)}{\partial\varrho} > 0, \ \frac{\partial e(\varrho,\vartheta)}{\partial\vartheta} > 0 \tag{2}$$

is assumed for all $\varrho > 0$, $\vartheta > 0$; see [8].

In this chapter, we restrict ourselves to *linearly viscous* (Newtonian) fluids, for which the viscous stress is determined by Newton's rheological law

$$\mathbb{S}(\varrho,\vartheta,\nabla_x\mathbf{u}) = \mu(\varrho,\vartheta)\left(\nabla_x\mathbf{u} + \nabla_x^t\mathbf{u} - \frac{2}{3}{\rm div}_x\mathbf{u}\mathbb{I}\right) + \eta(\varrho,\vartheta){\rm div}_x\mathbf{u}\mathbb{I}, \tag{3}$$

and the heat flux is given by Fourier's law

$$\mathbf{q}(\varrho,\vartheta,\nabla_x\vartheta) = -\kappa(\varrho,\vartheta)\nabla_x\vartheta, \tag{4}$$

where the so-called *transport coefficients* $\mu(\varrho, \vartheta)$, $\eta(\varrho, \vartheta)$ and $\kappa(\varrho, \vartheta)$ may depend on the scalar state variables ϱ and ϑ; see, e.g., [15, 44].

1.1.1 Field Equations, Primitive System

A system of equations describing a general compressible, viscous, and heat-conducting fluid rotating along the axis $\omega/|\omega|$, where $|\omega|$ is the angular velocity, will be considered. The motion in the rotational reference frame can be described by a general system of partial differential equations written in their dimensionless form:

$$[\mathrm{Sr}]\partial_t \varrho + \mathrm{div}_x (\varrho \mathbf{u}) = 0, \tag{5}$$

$$\begin{aligned} &[\mathrm{Sr}]\partial_t (\varrho \mathbf{u}) + \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{[\mathrm{Ro}]}(\omega \times \varrho \mathbf{u}) + \frac{1}{[\mathrm{Ma}]^2} \nabla_x p(\varrho, \vartheta) \\ &= \frac{1}{[\mathrm{Re}]} \mathrm{div}_x \mathbb{S}(\varrho, \vartheta, \nabla_x \mathbf{u}) + \frac{1}{[\mathrm{Ro}]^2} \varrho \nabla_x G + \frac{1}{[\mathrm{Fr}]^2} \varrho \mathbf{f}, \end{aligned} \tag{6}$$

$$\begin{aligned} &[\mathrm{Sr}]\partial_t (\varrho e(\varrho, \vartheta)) + \mathrm{div}_x (\varrho e(\varrho, \vartheta) \mathbf{u}) + \frac{1}{[\mathrm{Pe}]} \mathrm{div}_x \mathbf{q}(\varrho, \vartheta, \nabla_x \vartheta) \\ &= \frac{[\mathrm{Ma}]^2}{[\mathrm{Re}]} \mathbb{S}(\varrho, \vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - p(\varrho, \vartheta) \mathrm{div}_x \mathbf{u} + [\mathrm{Hr}] \varrho \mathcal{Q}, \end{aligned} \tag{7}$$

see [59, 80, 91, 92].

The system of Eqs. (5), (6), and (7) is called *primitive system*. It contains dimensionless quantities [X] – characteristic numbers – specified in the next section.

1.1.2 Characteristic Numbers

Denote L_{ref} the characteristic length and T_{ref} the characteristic time. The characteristic numbers are defined as follows:

Symbol	Definition	Name
Sr	$L_{\mathrm{ref}}/(T_{\mathrm{ref}} U_{\mathrm{ref}})$	Strouhal number
Ma	$U_{\mathrm{ref}}/\sqrt{p_{\mathrm{ref}}/\varrho_{\mathrm{ref}}}$	Mach number
Re	$\varrho_{\mathrm{ref}} U_{\mathrm{ref}} L_{\mathrm{ref}}/\mu_{\mathrm{ref}}$	Reynolds number
Fr	$U_{\mathrm{ref}}/\sqrt{L_{\mathrm{ref}} f_{\mathrm{ref}}}$	Froude number
Pe	$p_{\mathrm{ref}} L_{\mathrm{ref}} U_{\mathrm{ref}}/(\vartheta_{\mathrm{ref}} \kappa_{\mathrm{ref}})$	Péclet number
Ro	$U_{\mathrm{ref}}/\omega_{\mathrm{ref}} L_{\mathrm{ref}}$	Rossby number
Hr	$\varrho_{\mathrm{ref}} Q_{\mathrm{ref}} L_{\mathrm{ref}}/(p_{\mathrm{ref}} U_{\mathrm{ref}})$	Heat release parameter

The scale analysis reviewed in this chapter is devoted to the study of the asymptotic behavior of solutions to primitive system (5), (6), and (7) in the regime where one or several characteristic numbers become singular – either vanish or become infinitely large. The complete fluid systems, for which the first and second laws of thermodynamics are included in the equations, represent the main target. Accordingly, there is a lot of material omitted in what follows:

- Singular limits of reduced systems, in particular the incompressible ones and the inviscid ones. The reader may consult the surveys [18, 40, 75, 76, 82, 84], and/or the seminal paper [57] among others.
- The results in one-space dimension; see, e.g., [49, 50].
- Other results related to singular limits. The interested reader may consult [14, 17, 41, 42, 68, 69], among many others.

2 Scale Analysis and Singular Limits

The problems addressed in this text feature the following general structure:

Given

- A set of *primitive equations* – typically system (5), (6), and (7) or its suitable simplification already neglecting certain phenomena as compressibility, heat conductivity, viscosity, and others as the case may be,
- A set of *singular parameters* – a suitable subset of characteristic numbers supposed to be small or excessively large in the limit regime of interest,

the goal is to identify

- The *target system* describing the behavior of the fluid in the asymptotic regime,
- The way how the solutions of the primitive system approach their asymptotic limit.

Problems of this type are termed *singular limits* in the theory of partial differential equations. There are several methods available in the literature how to attack these problems in the context of fluid dynamics.

2.1 Framework of Classical Solutions

The most natural way is to study the singular limits in the framework of *classical solutions* for both the primitive and the target system as proposed in the pioneering work [22, 57]; see also [69, 83, 91, 92] among many other more recent studies. As the primitive system usually contains nonlinear equations, a rigorous theory based on classical solutions reduces to possibly short time lapse on which the classical solutions are known to exist. Note that global-in-time regularity of solutions to systems in fluid dynamics remains largely open even in seemingly simple and well-studied cases including the famous *millennium problem* concerning the incompressible Navier-Stokes system; see [24].

Besides the fact that a (hypothetically) short life span of classical solutions may become a nuisance in certain applications, the mathematical treatment faces another substantial difficulty. Specifically, one has to show that the length of the time interval on which the solutions of the primitive system exist remains positive *independently* of the value of singular parameters.

2.2 Weak Solutions

Global-in-time weak (distributional) solutions are known to exist for a relatively vast class of systems arising in fluid mechanics. The following are the most prominent examples:

- Incompressible Navier-Stokes system and related problems in the 3D geometry; see [51, 60], and, more recently, [11] and the references therein;
- Compressible barotropic Navier-Stokes system and related problems in the 2D and 3D geometries; see [25, 64], and also related results [87];
- Complete Navier-Stokes-Fourier system and related problems in the 2D and 3D geometries; see [25, 27], and an alternative approach [9].

Note that the theory of global-in-time solutions for the above problems in the simple 1D geometry is well established both in the strong and weak solutions framework; see, e.g., [4].

The apparent advantage of working with weak solutions in the singular limit problems is that their life span is typically *independent* of the singular parameters. The weak point of such results is convergence to the target systems in relatively poor topologies. A seminal work in this direction is [65] discussed below.

2.3 Weak → Strong Approach

Combining the weak and strong solutions framework was effective in the *inviscid limit* and related problems, where the global estimates independent of the scaling parameters do not allow the passage to the limit system by means of the classical compactness arguments. Instead methods based on evaluation of the “distance” between solutions of the primitive and target system by means of relative entropy or energy functionals have been developed; see, e.g., [29, 70, 72, 81, 86, 89] among others.

2.4 Well- vs. Ill-Prepared Data

A singular limit problem concerns not only the equations but also the data – the initial state and/or boundary conditions as the case may be. The data are called *well prepared* if they approach those for the target problem in the asymptotic limit. If this is not the case, the data are *ill prepared*. For the well-prepared data, one expects the primitive system to enjoy the same level of regularity as for the target system and a relatively strong convergence to the limit problem although proving this could be rather involved. In the ill-prepared case, the solutions of the primitive system typically decompose in two components: a regular one that approaches the limit system and an "oscillatory" one supposed to disappear in the asymptotic limit. Obviously, it is the oscillatory component that makes the singular limit problem mathematically delicate. Examples of these phenomena will be given below.

3 High Reynolds/Péclet Number Regime: Inviscid Limits

One of the central problems in mathematical fluid mechanics is the asymptotic limit when the dissipative terms in (6) and (7) vanish. Specifically,

$$\mathrm{Re} \to \infty, \ \mathrm{Pe} \to \infty,$$

while the remaining characteristic numbers are of order one. A particularly interesting case is the fluid contained in a domain with boundary, where the incompatibility between the boundary conditions for the primitive and target system may result in a boundary layer separation. At the purely mathematical level, these phenomena are still poorly understood, in particular a rigorous justification of the boundary layer theory.

A standard example of the inviscid limit is that for the *incompressible* Navier-Stokes system

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla_x)\mathbf{u} + \nabla_x \Pi = \varepsilon \Delta \mathbf{u}, \ \mathrm{div}_x \mathbf{u} = 0 \text{ in } (0,T) \times \Omega, \tag{8}$$

$$\mathbf{u}|_{\partial\Omega} = 0, \tag{9}$$

$$\mathbf{u}(0,\cdot) = \mathbf{u}_0 \text{ in } \Omega, \tag{10}$$

where $\mathbf{u}$ is the velocity and Π the pressure in a viscous incompressible fluid occupying a domain $\Omega \subset R^3$. Obviously $\mathrm{Re} = \frac{1}{\varepsilon}$ and the limit $\varepsilon \to 0$ can be examined.

In the absence of a physical boundary, say $\Omega = R^3$, the following result of [53, Section 1,Theorem] is classical:

Theorem 1. *Let* $\Omega = R^3$ *and*

$$\mathbf{u}_0 \in W^{k,2}(R^3; R^3),\ \mathrm{div}_x \mathbf{u}_0 = 0,$$

be given, $k \geq 3$.

Then there exists $T > 0$ *that may depend on* $\|\mathbf{u}_0\|_{W^{k,2}(R^3;R^3)}$ *but is* independent *of* ε *such that the initial-value problem (8), (10) admits a solution* $\mathbf{u}_\varepsilon$ *unique in the class*

$$\mathbf{u}_\varepsilon \in C([0,T]; W^{k,2}(R^3; R^3)) \cap C^1([0,T]; W^{k-1}(R^3; R^3)).$$

Furthermore, the family $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ *is bounded in* $C([0,T]; W^{2,k}(R^3; R^3))$ *and*

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ in } C([0,T]; W^{2,k-1}(R^3; R^3)),$$

where $\mathbf{u}$ *is the unique solution of the (inviscid) Euler system*

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla_x)\mathbf{u} + \nabla_x \tilde{\Pi} = 0,\ \mathrm{div}_x \mathbf{u} = 0 \text{ in } (0,T) \times R^3, \tag{11}$$

$$\mathbf{u}(0,\cdot) = \mathbf{u}_0 \text{ in } R^3. \tag{12}$$

If $\Omega \subset R^3$ is a general *bounded* domain, the convergence result in Theorem 1 is conditional; see [55]:

Theorem 2. *Let* $\Omega \subset R^3$ *be a regular bounded domain. Suppose that the Navier-Stokes system* (8), (9)*, and* (10) *admits a family of solutions*

$$\mathbf{u}_\varepsilon \in C([0,T]; W^{k,2}(\Omega; R^3) \cap C^1([0,T]; W^{k-1,2}(\Omega; R^3),\ k \geq 3, \tag{13}$$

and that the limit Euler system (11)*,* (12)*, supplemented with the impermeability boundary condition*

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0 \tag{14}$$

possesses a solution $\mathbf{u}$ *in the class specified in* (13)*.*

Then the following statements are equivalent:

- $$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ in } C([0,T]; L^2(\Omega; R^3));$$
- $$\varepsilon \int_0^T \int_{\Gamma_\varepsilon} |\nabla_x \mathbf{u}_\varepsilon|^2 \, \mathrm{d}x \, \mathrm{d}t \to 0, \text{ where } \Gamma_\varepsilon = \left\{ x \in \Omega \ \middle|\ \mathrm{dist}[x, \partial\Omega] < \varepsilon \right\};$$
- $$\varepsilon \int_0^T \int_\Omega |\nabla_x \mathbf{u}_\varepsilon|^2 \, \mathrm{d}x \, \mathrm{d}t \to 0.$$

In general, the solutions $\mathbf{u}_\varepsilon$ of the Navier-Stokes system on domains with boundaries are expected to take the form

$$\mathbf{u}_\varepsilon = \mathbf{u} + \mathbf{u}_{\mathrm{BL}}, \tag{15}$$

where $\mathbf{u}$ is the solution of the limit inviscid problem and $\mathbf{u}_{\mathrm{BL}}$ is small except at a small neighborhood of the boundary. The behavior of $\mathbf{u}_{\mathrm{BL}}$ is determined by *Prandtl's equation*; however, rigorous results concerning validity of (15) are in a short supply; see the survey papers [48, 90], or [73, Chapter 2].

3.1 A Model Problem: Inviscid Limits of Compressible Fluids

A very particular case of the system (5), (6), and (7) will be considered, with constant viscosity coefficients $\mu > 0$, $\eta \geq 0$ and with the pressure $p = p(\varrho)$ independent of the temperature. Accordingly, Eq. (7) becomes irrelevant for the fluid motion and the problem reduces to the *barotropic Navier-Stokes system*. Taking $\mathrm{Sr} = \mathrm{Ma} = 1$, $\omega = 0$, $\mathbf{f} = 0$, and $\mathrm{Re} = 1/\varepsilon$ one obtains:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{16}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \varepsilon \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}), \tag{17}$$

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}. \tag{18}$$

The problem is completed by imposing the no-slip boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0, \tag{19}$$

and the initial state

$$\varrho(0,\cdot) = \varrho_{0,\varepsilon}, \ \mathbf{u}(0,\cdot) = \mathbf{u}_{0,\varepsilon}. \tag{20}$$

The inviscid limit $\varepsilon \to 0$ will be examined by means of the weak $\to$ strong approach. To this end, the concept of a *finite energy weak solution* ϱ, $\mathbf{u}$ to problem (16), (17), (18), (19), and (20), is needed to be defined through a family of integral identities:

$$\left[\int_\Omega \varrho(t,\cdot)\varphi(t,\cdot)\, \mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left(\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi \right)\, \mathrm{d}x\, \mathrm{d}t \tag{21}$$

for any $\tau \in [0,T]$ and any $\varphi \in C_c^\infty([0,T] \times \overline{\Omega})$;

$$\left[\int_\Omega \varrho(t,\cdot)\mathbf{u}(t,\cdot) \cdot \varphi(t,\cdot)\, \mathrm{d}x \right]_{t=0}^{t=\tau} \tag{22}$$

$$= \int_0^\tau \int_\Omega \left(\varrho \mathbf{u} \cdot \partial_t \varphi + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \varphi + p \mathrm{div}_x \varphi - \varepsilon \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \varphi \right) \, \mathrm{d}x$$

for any $\tau \in [0, T]$ and any $\varphi \in C_c^\infty([0, T] \times \Omega; R^3)$. In addition, the energy inequality is required in the form

$$\int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + H(\varrho) \right) (\tau, \cdot) \, \mathrm{d}x + \varepsilon \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \, \mathrm{d}x \, \mathrm{d}t \tag{23}$$

$$\leq \int_\Omega \left(\frac{1}{2} \varrho_{0,\varepsilon} |\mathbf{u}_{0,\varepsilon}|^2 + H(\varrho_{0,\varepsilon}) \right) \, \mathrm{d}x, \ H(\varrho) = \varrho \int_1^\varrho \frac{p(z)}{z^2} \, \mathrm{d}z,$$

to be satisfied for a.a. $\tau \in [0, T]$.

An obvious candidate for the limit system of equations is the compressible Euler system

$$\partial_t \varrho + \mathrm{div}_x (\varrho \mathbf{u}) = 0, \tag{24}$$

$$\partial_t (\varrho \mathbf{u}) + \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = 0, \tag{25}$$

supplemented with the impermeability boundary condition

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{26}$$

Although solutions of (24), (25), and (26) are known to develop singularities (shock waves) in a finite time (see, e.g., [85, Chapter 15]), there exists a possible short time interval $[0, T]$ on which they remain smooth as long as:

- $\Omega \subset R^3$ is a bounded domain with a sufficiently smooth boundary, say $\partial\Omega$ of class C^∞;
- the pressure $p \in C^5(0, \infty) \cap C^1[0, \infty)$;
- the initial data $[\varrho_0, \mathbf{u}_0]$ belong to the class

$$\varrho_0 \in W^{3,2}(\Omega), \ \mathbf{u}_0 \in W^{3,2}(\Omega; R^3), \ \varrho_0 > 0 \text{ in } \overline{\Omega}; \tag{27}$$

- the compatibility conditions

$$\partial_t^k \mathbf{u}_0 \cdot \mathbf{n}|_{\partial\Omega} = 0 \tag{28}$$

  hold for $k = 0, 1, 2$,

see [82, Theorem 1].

3.1.1 Inviscid Limit with Boundary Layer

Denote

$$d_{\partial\Omega}(x) = \mathrm{dist}[x, \partial\Omega], \ \Gamma_a = \left\{ x \in \Omega \ \middle| \ \mathrm{dist}[x, \partial\Omega] < a \right\}.$$

The following result may be seen as a "compressible" counterpart of Theorem 2; see [86, Theorem 3]:

Theorem 3. *Let* $\lambda > 0$ *be given. Suppose that* $\Omega \subset R^3$ *is a smooth bounded domain,*

$$p(\varrho) = a\varrho^{\gamma},\ a > 0,\ \gamma > \frac{3}{2}, \tag{29}$$

and $[\varrho_0, \mathbf{u}_0]$ *comply with hypotheses* (27) *and* (28). *Let* $[\varrho, \mathbf{u}]$ *be the unique classical solution of the Euler problem* (24), (25), *and* (26) *in* $(0,T) \times \Omega$ *emanating from the initial data* $[\varrho_0, \mathbf{u}_0]$, *and let* $\{[\varrho_\varepsilon, \mathbf{u}_\varepsilon]\}_{\varepsilon>0}$ *be a family of finite energy weak solutions to the Navier-Stokes system* (21), (22), *and* (23).

Assume that

$$\varrho_{0,\varepsilon} \geq 0,\ \|\varrho_{0,\varepsilon} - \varrho_0\|_{L^{\gamma}(\Omega)} + \int_{\Omega} \varrho_{0,\varepsilon}|\mathbf{u}_{0,\varepsilon} - \mathbf{u}_0|^2 \, \mathrm{d}x \to 0$$

and suppose, in addition, that

$$\varepsilon \int_0^T \int_{\Gamma_{\lambda\varepsilon}} \left(\frac{\varrho_\varepsilon |\mathbf{u}_\varepsilon|^2}{d_{\partial\Omega}^2} + \frac{\varrho_\varepsilon^2 (\mathbf{u}_\varepsilon \cdot \mathbf{n})^2}{d_{\partial\Omega}^2} + |\mathbb{S}(\nabla_x \mathbf{u}_\varepsilon)|^2 \right) \mathrm{d}x \, \mathrm{d}t \to 0$$

as $\varepsilon \to 0$.

Then

$$\|\varrho_\varepsilon(t,\cdot) - \varrho(t,\cdot)\|_{L^{\gamma}(\Omega)} + \int_{\Omega} \varrho_\varepsilon(t,\cdot)|\mathbf{u}_\varepsilon(t,\cdot) - \mathbf{u}(t,\cdot)|^2 \, \mathrm{d}x \to 0$$

as $\varepsilon \to 0$ *uniformly for* $t \in [0,T]$.

In Theorem 3, the extension of the normal vector field to Ω

$$\mathbf{n} = \nabla_x d_{\partial\Omega}$$

has been used. Note that *existence* of global-in-time finite energy weak solutions for the Navier-Stokes system under hypothesis (29) was proved in [33, Theorem 1.1].

3.1.2 Inviscid Limit Without Boundary Layer

The disturbing effect of the boundary layer in the inviscid limit may be eliminated by imposing proper (well-prepared) boundary conditions for the primitive system. One of plausible solutions is replacing the no-slip boundary condition (19) by *Navier's condition*

$$\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega}=0,\ [\mathbb{S}(\nabla_x\mathbf{u})\cdot\mathbf{n}]_{\tan}+\beta[\mathbf{u}]_{\tan}|_{\partial\Omega}=0, \tag{30}$$

where $\beta\geq 0$ plays a role of a friction coefficient,
cf. [10]. Accordingly, the integral identity (22) as well as the energy inequality (23) in the weak formulation of the Navier-Stokes system must be modified:

$$\left[\int_\Omega \varrho(t,\cdot)\mathbf{u}(t,\cdot)\cdot\varphi(t,\cdot)\,\mathrm{d}x\right]_{t=0}^{t=\tau} \tag{31}$$

$$=\int_0^\tau\int_\Omega\left(\varrho\mathbf{u}\cdot\partial_t\varphi+\varrho\mathbf{u}\otimes\mathbf{u}:\nabla_x\varphi+p\mathrm{div}_x\varphi-\varepsilon\mathbb{S}(\nabla_x\mathbf{u}):\nabla_x\varphi\right)\,\mathrm{d}x$$
$$-\int_0^\tau\int_{\partial\Omega}\beta\mathbf{u}\cdot\varphi\,\mathrm{dS}_x\,\mathrm{d}t$$

for any $\tau\in[0,T]$ and any $\varphi\in C_c^\infty([0,T]\times\overline{\Omega};R^3)$, $\varphi\cdot\mathbf{n}|_{\partial\Omega}=0$;

$$\int_\Omega\left(\frac{1}{2}\varrho|\mathbf{u}|^2+H(\varrho)\right)(\tau,\cdot)\,\mathrm{d}x \tag{32}$$

$$+\varepsilon\int_0^\tau\int_\Omega\mathbb{S}(\nabla_x\mathbf{u}):\nabla_x\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t+\int_0^\tau\int_{\partial\Omega}\beta|\mathbf{u}|^2\,\mathrm{dS}_x\,\mathrm{d}t$$
$$\leq\int_\Omega\left(\frac{1}{2}\varrho_{0,\varepsilon}|\mathbf{u}_{0,\varepsilon}|^2+H(\varrho_{0,\varepsilon})\right)\,\mathrm{d}x$$

for a.a. $\tau\in[0,T]$.

The following result was proved in [86, Theorem 4] (cf. also [7]):

Theorem 4. *Let $T>0$, p, and the classical solution $[\varrho,\mathbf{u}]$ of the Euler system be the same as in Theorem (3). Assume that $\{[\varrho_\varepsilon,\mathbf{u}_\varepsilon]\}_{\varepsilon>0}$ is a family of finite energy weak solutions to the Navier-Stokes system supplemented with Navier's boundary condition (30), meaning satisfying (21), (31), (32), where*

$$\beta=\beta_\varepsilon\to 0 \text{ as } \varepsilon\to 0.$$

Let

$$\varrho_{0,\varepsilon}\geq 0,\ \|\varrho_{0,\varepsilon}-\varrho_0\|_{L^\gamma(\Omega)}+\int_\Omega\varrho_{0,\varepsilon}|\mathbf{u}_{0,\varepsilon}-\mathbf{u}_0|^2\,\mathrm{d}x\to 0 \text{ as } \varepsilon\to 0.$$

Then

$$\|\varrho_\varepsilon(t,\cdot) - \varrho(t,\cdot)\|_{L^\gamma(\Omega)} + \int_\Omega \varrho_\varepsilon(t,\cdot)|\mathbf{u}_\varepsilon(t,\cdot) - \mathbf{u}(t,\cdot)|^2 \,\mathrm{d}x \to 0$$

as $\varepsilon \to 0$ *uniformly for* $t \in [0,T]$.

Note that *existence* of global-in-time weak solutions to the Navier-Stokes system has been shown in [37, Theorem 2.1].

The results claimed in Theorems 3, 4 are based on the concept of *relative energy/entropy* introduced in the context of hyperbolic conservation laws by [16]. Specifically, consider the functional

$$\mathcal{E}\left(\varrho,\mathbf{u}\,\middle|\, r,\mathbf{U}\right) = \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}-\mathbf{U}|^2 + H(\varrho) - H'(r)(\varrho - r) - H(r)\right)\,\mathrm{d}x.$$

As shown in [34, 36], any finite energy weak solution $[\varrho,\mathbf{u}]$ of (21), (22), and (23) satisfies the relative energy inequality

$$\mathcal{E}\left(\varrho,\mathbf{u}\middle| r,\mathbf{U}\right)(\tau) + \varepsilon\int_0^\tau\int_\Omega \Big(\mathbb{S}(\nabla_x\mathbf{u}) - \mathbb{S}(\nabla_x\mathbf{U})\Big) : \Big(\nabla_x\mathbf{u} - \nabla_x\mathbf{U}\Big)\,\mathrm{d}x\,\mathrm{d}t \tag{33}$$

$$\leq \mathcal{E}\Big(\varrho_0,\mathbf{u}_0\Big| r(0,\cdot),\mathbf{U}(0,\cdot)\Big) + \int_0^\tau \mathcal{R}(\varrho,\mathbf{u},r,\mathbf{U})\,\mathrm{d}t$$

holds for a.a. $\tau > 0$, where

$$\mathcal{R}(\varrho,\mathbf{u},r,\mathbf{U}) \equiv \int_\Omega \varrho\Big(\partial_t\mathbf{U} + \mathbf{u}\nabla_x\mathbf{U}\Big)\cdot(\mathbf{U}-\mathbf{u})\,\mathrm{d}x$$
$$+\varepsilon\int_\Omega \mathbb{S}(\nabla_x\mathbf{U}) : \nabla_x(\mathbf{U}-\mathbf{u})\,\mathrm{d}x$$
$$+\int_\Omega \big((r-\varrho)\partial_t H'(r) + \nabla_x H'(r)\cdot(r\mathbf{U} - \varrho\mathbf{u})\big)\,\mathrm{d}x - \int_\Omega \mathrm{div}_x\mathbf{U}\Big(p(\varrho) - p(r)\Big)\,\mathrm{d}x,$$

where $r > 0$, $\mathbf{U}$ are smooth "test functions", $\mathbf{U}$ satisfying the relevant boundary conditions. In the proof of Theorems 3 and 4, the functions r and $\mathbf{U}$ correspond to the density and velocity of the limit problem.

3.2 Vanishing Dissipation Limit for the Full Navier-Stokes-Fourier System

The discussion on vanishing dissipation limit will be concluded by discussing the full system (5), (6), and (7) in the regime of large Reynolds and Péclet numbers.

Similarly to the previous section, the weak → strong approach requiring a proper concept of weak solution to the primitive system will be applied.

3.2.1 Weak Solutions to the Full System

Taking

$$\mathbf{f} = \omega = \mathcal{Q} = 0, \ \mathrm{Ma} = \mathrm{Sr} = 1$$

and using Gibbs' relation (1), one may rewrite the system (5), (6), and (7) in the *entropy formulation*:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{34}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) - \lambda \mathbf{u} \tag{35}$$

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) = \sigma, \tag{36}$$

with the entropy production rate

$$\sigma \equiv \frac{1}{\vartheta} \left(\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right), \tag{37}$$

where

$$\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) = \nu \left[\mu(\vartheta) \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta(\vartheta) \mathrm{div}_x \mathbf{u} \mathbb{I} \right], \ \nu > 0, \tag{38}$$

and

$$\mathbf{q} = -\omega \kappa(\vartheta) \nabla_x \vartheta, \ \omega > 0. \tag{39}$$

The parameters $\nu = \frac{1}{\mathrm{Re}}$ and $\omega = \frac{1}{\mathrm{Pe}}$ will be small in the asymptotic limit. Note that a "friction" term $\lambda \mathbf{u}$ has been added in the momentum Eq. (35).

To avoid the aforementioned and still unsurmountable difficulties connected with the presence of a boundary layer, the complete slip boundary conditions (Navier's slip with $\beta = 0$) for the velocity will be imposed

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ [\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) \cdot \mathbf{n}] \times \mathbf{n}|_{\partial\Omega} = 0, \tag{40}$$

accompanied with the no-flux condition

$$\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{41}$$

The system of Eqs. (34), (35), (36), and (37), supplemented with the constitutive relations (38) and (39) is called *Navier-Stokes-Fourier system*.

The concept of weak solution to problem (34), (35), (36), (37), (38), (39), (40), and (41) introduced in [27, Chapter 2] is adopted:

- the Eqs. (34) and (35), together with the boundary conditions (40), are understood in the sense of distributions, exactly as in (21) and (31);
- the entropy balance (36), (37), with (41), is relaxed to an *inequality*

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) + \mathrm{div}_x\left(\frac{\mathbf{q}}{\vartheta}\right) \geq \sigma$$

  satisfied in the sense of distributions:

$$\begin{aligned}
&\left[\int_\Omega \varrho(t,\cdot)s(\varrho,\vartheta)(t,\cdot)\,\mathrm{d}x\right]_{t=0}^{t=\tau} \\
&\geq \int_0^T \int_\Omega \left(\varrho s(\varrho,\vartheta)\partial_t\varphi + \varrho s(\varrho,\vartheta)\mathbf{u}\cdot\nabla_x\varphi + \frac{\mathbf{q}\cdot\nabla_x\varphi}{\vartheta}\right)\mathrm{d}x\,\mathrm{d}t \\
&+ \int_0^T \int_\Omega \frac{1}{\vartheta}\left(\mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\right)\varphi\,\mathrm{d}x\,\mathrm{d}t
\end{aligned} \tag{42}$$

  for a.a. $\tau \in (0, T)$ and any $\varphi \in C_c^\infty([0, T] \times \overline{\Omega})$, $\varphi \geq 0$;
- the system is augmented by the total energy balance

$$\begin{aligned}
&\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\right](\tau,\cdot)\,\mathrm{d}x + \lambda\int_0^\tau\int_\Omega |\mathbf{u}|^2\,\mathrm{d}x \\
&\leq \int_\Omega \left[\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + \varrho_0 e(\varrho_0,\vartheta_0)\right]\mathrm{d}x
\end{aligned} \tag{43}$$

  for a.a. $\tau \in (0, T)$.

The weak solutions enjoy the important *compatibility property*, namely, any weak solution that is smooth satisfies the classical formulation of the Navier-Stokes-Fourier system; see [27, Chapter 2].

The mathematical theory developed in [27, Chapter 3] requires certain structural restrictions to be imposed on the constitutive relations, in particular the pressure and the associated components of the internal energy and entropy are augmented by a radiative component providing certain regularizing effect on the (hypothetical) vacuum zones.

- The pressure p takes the form

$$p(\varrho,\vartheta) = p_M(\varrho,\vartheta) + \frac{a}{3}\vartheta^4,\ a > 0,\ \text{with } p_M(\varrho,\vartheta) = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right), \tag{44}$$

where $P \in C^1[0,\infty) \cap C^5(0,\infty)$ satisfies

$$P(0) = 0,\ P'(Z) > 0 \text{ for all } Z \geq 0, \tag{45}$$

$$0 < \frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z} < c \text{ for all } Z > 0,\ \lim_{Z\to\infty} \frac{P(Z)}{Z^{5/3}} = P_\infty > 0. \tag{46}$$

- In agreement with Gibbs' relation (1), we take

$$e(\varrho,\vartheta) = e_M(\varrho,\vartheta) + a\frac{\vartheta^4}{\varrho},\ \text{with } e_M(\varrho,\vartheta) = \frac{3}{2}\vartheta\left(\frac{\vartheta^{3/2}}{\varrho}\right) P\left(\frac{\varrho}{\vartheta^{3/2}}\right), \tag{47}$$

and

$$s(\varrho,\vartheta) = s_M(\varrho,\vartheta) + \frac{4a}{3\varrho}\vartheta^3,\ s_M(\varrho,\vartheta) = S\left(\frac{\varrho}{\vartheta^{3/2}}\right), \tag{48}$$

where

$$S'(Z) = -\frac{3}{2}\frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z^2} < 0. \tag{49}$$

In addition, it is required that

$$\lim_{Z\to\infty} S(Z) = 0. \tag{50}$$

- The viscosity coefficients in (38) are continuously differentiable functions of $\vartheta \in [0,\infty)$ satisfying

$$|\mu'(\vartheta)| \leq c,\ \underline{\mu}(1+\vartheta) \leq \mu(\vartheta),\ 0 \leq \eta(\vartheta) \leq \overline{\eta}(1+\vartheta) \text{ for all } \vartheta \geq 0 \tag{51}$$

for certain constants $\underline{\mu} > 0$, $\overline{\eta} > 0$.
- The heat conductivity coefficient in Fourier's law (39) satisfies

$$\kappa \in C^1[0,\infty),\ \underline{\kappa}(1+\vartheta^3) \leq \kappa(\vartheta) \leq \overline{\kappa}(1+\vartheta^3) \text{ for all } \vartheta \geq 0 \tag{52}$$

for certain constants $\underline{\kappa} > 0$, $\overline{\kappa} > 0$.

As pointed out, the a-dependent terms in (44), (47), and (48) correspond to radiation effects. Under hypotheses (44), (45), (46), (47), (48), (49), (50), (51), and

(52), the Navier-Stokes-Fourier system admits a global-in-time weak solution for any finite energy initial data; see [27, Chapter 3, Theorem 3.1].

3.2.2 Target Problem

An obvious candidate for the target problem in the vanishing dissipation limit is the full Euler system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{53}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p_M(\varrho, \vartheta) = 0, \tag{54}$$

$$\partial_t \left(\frac{1}{2}\varrho |\mathbf{u}|^2 + \varrho e_M(\varrho, \vartheta) \right) + \mathrm{div}_x \left[\left(\frac{1}{2}\varrho |\mathbf{u}|^2 + \varrho e_M(\varrho, \vartheta) \right) \mathbf{u} + p_M(\varrho, \vartheta)\mathbf{u} \right] = 0, \tag{55}$$

with the boundary condition

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{56}$$

and the initial condition

$$\varrho(0, \cdot) = \varrho_0, \ \vartheta(0, \cdot) = \vartheta_0, \ \mathbf{u}(0, \cdot) = \mathbf{u}_0. \tag{57}$$

Note that the entropy balance has been replaced by the energy conservation (55). In view of Gibbs' relation (1), the entropy and energy balance equations are *equivalent* at least in the framework of smooth solutions of the Euler system (53), (54), and (55). In addition, the radiation terms have been deliberately omitted as the plan is to let $a \to 0$ in the asymptotic limit.

Similarly to its barotropic version studied in Sect. 3.1, the Euler system (53), (54), (55), (56), and (57) admits local-in-time classical solutions defined on a certain time interval $[0, T]$ if

- $\Omega \subset R^3$ is a bounded domain with a boundary $\partial\Omega$ of class C^∞;
- the initial data $[\varrho_0, \vartheta_0, \mathbf{u}_0]$ belong to the class

$$\varrho_0, \ \vartheta_0 \in W^{3,2}(\Omega), \ \mathbf{u}_0 \in W^{3,2}(\Omega; R^3), \ \varrho_0 > 0, \ \vartheta_0 > 0 \text{ in } \overline{\Omega}; \tag{58}$$

- the compatibility conditions

$$\partial_t^k \mathbf{u}_0 \cdot \mathbf{n}|_{\partial\Omega} = 0 \tag{59}$$

  hold for $k = 0, 1, 2,$

see [82, Theorem 1].

3.2.3 Vanishing Dissipation Limit

Similarly to [28], the ballistic free energy is introduced

$$H_\Theta(\varrho,\vartheta) = \varrho\Big(e(\varrho,\vartheta) - \Theta s(\varrho,\vartheta)\Big),$$

along with the *relative energy functional*

$$\mathcal{E}\left(\varrho,\vartheta,\mathbf{u}\Big|r,\Theta,\mathbf{U}\right) \tag{60}$$

$$= \int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}-\mathbf{U}|^2 + H_\Theta(\varrho,\vartheta) - \frac{\partial H_\Theta(r,\Theta)}{\partial\varrho}(\varrho - r) - H_\Theta(r,\Theta)\right]\,\mathrm{d}x.$$

As shown in [27, Chapter 5, Lemma 5.1], the functional $\mathcal{E}$ represents a "distance" between the quantities $[\varrho,\vartheta,\mathbf{u}]$ and $[r,\Theta,\mathbf{U}]$. More precisely, for any compact set $K \subset (0,\infty)^2$, there exists a positive constant $c(K)$, depending solely on the structural properties of the thermodynamic functions stated in (44), (45), (46), (47), (48), (49), and (50) such that

$$\mathcal{E}\left(\varrho,\vartheta,\mathbf{u}\Big|r,\Theta,\mathbf{U}\right) \tag{61}$$

$$\geq c(K)\begin{cases} |\varrho - r|^2 + |\vartheta-\Theta|^2 + |\mathbf{u}-\mathbf{U}|^2 \text{ if } [\varrho,\vartheta]\in K,\ [r,\Theta]\in K \\ 1 + \varrho|\mathbf{u}-\mathbf{U}|^2 + \varrho e(\varrho,\vartheta) + \varrho|s(\varrho,\vartheta)| \text{ if } [\varrho,\vartheta]\in(0,\infty)^2\setminus K,\ [r,\Theta]\in K. \end{cases}$$

As shown in [28], the relative energy $\mathcal{E}$ satisfies a Gronwall type inequality similar to (33). On the basis of this observation, the following result was proved in [26, Theorem 3.1]:

Theorem 5. *Let $\Omega \subset R^3$ be a bounded domain with smooth boundary. Suppose that the functions p, e, and s e_M satisfy (44), (45), (46), (47), (48), (49), and (50) and the transport coefficients μ, η and λ obey (51) and (52). Let $[\varrho_E,\vartheta_E,\mathbf{u}_E]$ be the classical solution of the Euler system (53), (54), (55), (56), and (57) defined on a time interval $(0,T)$, originating from the initial data $[\varrho_{0,E},\vartheta_{0,E},\mathbf{u}_{0,E}]$ satisfying (58) and (59). Let $[\varrho,\vartheta,\mathbf{u}]$ be a weak solution of the Navier-Stokes-Fourier system (34), (35), (36), (37), (38), (39), (40), and (41) in the sense specified in (42) and (43), and with the initial data $[\varrho_0,\vartheta_0,\mathbf{u}_0]$ satisfying*

$$\varrho_0,\vartheta_0 > 0 \text{ a.a. in } \Omega,\ \int_\Omega \varrho_0\,\mathrm{d}x \geq M,\ \|\varrho_0\|_{L^\infty(\Omega)} + \|\vartheta_0\|_{L^\infty(\Omega)} + \|\mathbf{u}_0\|_{L^\infty(\Omega;R^3)} \leq D.$$

Finally, suppose that the scaling parameters a, ν, ω, and λ are positive numbers.

Then

$$\mathcal{E}\left(\varrho,\vartheta,\mathbf{u}\Big|\varrho_E,\vartheta_E,\mathbf{u}_E\right)(\tau) \stackrel{<}{\sim} c(T,M,D)\Bigg(\mathcal{E}\left(\varrho_0,\vartheta_0,\mathbf{u}_0\Big|\varrho_{0,E},\vartheta_{0,E},\mathbf{u}_{0,E}\right)$$
$$+\max\left\{a,\nu,\omega,\lambda,\frac{\nu}{\sqrt{a}},\frac{\omega}{a},\left(\frac{a}{\sqrt{\nu^3\lambda}}\right)^{1/3}\right\}\Bigg)$$

for a.a. $\tau \in (0,T)$.

As a direct consequence of (61) the following holds true:

Corollary 1. *Under the hypotheses of Theorem* (5) *suppose that*

$$a,\nu,\omega,\lambda \to 0, \text{ and } \frac{\omega}{a} \to 0,\ \frac{\nu}{\sqrt{a}} \to 0,\ \frac{a}{\sqrt{\nu^3\lambda}} \to 0. \tag{62}$$

Then

$$\operatorname*{ess\,sup}_{\tau\in(0,T)} \int_\Omega \left[\varrho|\mathbf{u}-\mathbf{u}_E|^2 + |\varrho-\varrho_E|^{5/3} + \varrho|\vartheta-\vartheta_E|\right]\,\mathrm{d}x$$
$$\leq c(T,D,M)\Lambda\Big(a,\nu,\omega,\lambda,\|\varrho_0-\varrho_{0,E}\|_{L^\infty(\Omega)},\|\vartheta_0$$
$$-\vartheta_{0,E}\|_{L^\infty(\Omega)},\|\mathbf{u}_0-\mathbf{u}_{0,E}\|_{L^\infty(\Omega;R^3)}\Big),$$

where Λ *is an explicitly computable function of its arguments,*

$$\Lambda\left(a,\nu,\omega,\lambda,\|\varrho_0-\varrho_{0,E}\|_{L^\infty(\Omega)},\|\vartheta_0-\vartheta_{0,E}\|_{L^\infty(\Omega)},\|\mathbf{u}_0-\mathbf{u}_{0,E}\|_{L^\infty(\Omega;R^3)}\right)\to 0$$

provided a,ν,ω,λ *satisfy* (62)*, and*

$$\|\varrho_0-\varrho_{0,E}\|_{L^\infty(\Omega)},\|\vartheta_0-\vartheta_{0,E}\|_{L^\infty(\Omega)},\|\mathbf{u}_0-\mathbf{u}_{0,E}\|_{L^\infty(\Omega;R^3)}\to 0.$$

The convergence stated in Corollary 1 is *path dependent*, the parameters a,ν,ω,λ are interrelated through (62). It is easy to see that (62) holds provided, for instance,

$$a\to 0,\ \nu = a^\alpha,\ \omega = a^\beta,\ \lambda = a^\gamma,$$

where

$$\beta > 1,\ \frac{1}{2} < \alpha < \frac{2}{3},\ 0 < \gamma < 1-\frac{3}{2}\alpha.$$

4 Low Mach Number Regime: Incompressible Limits

Another interesting class of singular limit problems appears in the regime of low Mach number Ma $\to$ 0. Physically, this represents the asymptotic limit of small flow velocity or infinitely fast speed of sound propagation; the elastic features of the fluid become negligible and sound-wave propagation insignificant. When the Mach number approaches zero, the pressure is almost constant, while the speed of sound tends to be infinite. If, simultaneously, the temperature tends to a constant, the fluid is driven to incompressibility. Other interesting phenomena appear for a general nonconstant temperature.

4.1 A Model Problem: Incompressible Limit for Barotropic Fluid Flows

The mathematical theory of weak solutions for the compressible Navier-Stokes system developed in [64] initiated a new direction in the study of incompressible limits based on the weak $\to$ weak approach. Consider the barotropic system (16), (17), and18) in the low Mach number regime:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{63}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2}\nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}), \tag{64}$$

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}, \tag{65}$$

supplemented with the space-periodic boundary conditions, meaning the underlying spatial domain can be identified with the "flat" torus

$$\Omega = \left([0,1]|_{\{0,1\}} \right)^N, \ N = 2, 3. \tag{66}$$

To reveal the principal subtleties of the low Mach number limit, it is convenient to rewrite (63), (64) in the form of *Lighthill's acoustic analogy*

$$\varepsilon \partial_t \left(\frac{\varrho - \overline{\varrho}}{\varepsilon} \right) + \mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon) = 0, \tag{67}$$

$$\varepsilon \partial_t(\varrho_\varepsilon \mathbf{u}_\varepsilon) + p'(\overline{\varrho}) \nabla_x \left(\frac{\varrho - \overline{\varrho}}{\varepsilon} \right) = \varepsilon \mathrm{div}_x \mathbb{L}, \tag{68}$$

where

$$\mathbb{L} = \left[\mathbb{S}(\nabla_x \mathbf{u}) - \varrho \mathbf{u} \otimes \mathbf{u} - \frac{1}{\varepsilon^2} \left(p(\varrho) - p'(\overline{\varrho})(\varrho - \overline{\varrho}) - p(\overline{\varrho}) \right) \mathbb{I} \right]$$

is Lighthill's tensor; see [61, 62].

If the momentum $\varrho\mathbf{u}$ is written in terms of its *Helmholtz projection*

$$\varrho\mathbf{u} = H[\varrho\mathbf{u}] + \nabla_x\Phi,\ \Phi = \Delta^{-1}\mathrm{div}_x(\varrho\mathbf{u}),$$

it is possible to deduce from (67), (68) that the asymptotic limit is determined by the solenoidal component $H[\varrho\mathbf{u}]$ while the potential Φ satisfies the *acoustic wave equation*

$$\varepsilon\partial_t\left(\frac{\varrho-\overline{\varrho}}{\varepsilon}\right) + \Delta\Phi = 0, \tag{69}$$

$$\varepsilon\partial_t\nabla_x\Phi + p'(\overline{\varrho})\nabla_x\left(\frac{\varrho-\overline{\varrho}}{\varepsilon}\right) = \varepsilon(I-H)[\mathrm{div}_x\mathbb{L}]. \tag{70}$$

Note that the speed of propagation of acoustic waves governed by (69), (70) is $\sqrt{p'(\overline{\varrho})}/\varepsilon$ becoming infinite in the asymptotic limit $\varepsilon \to 0$. Accordingly, the acoustic waves, when confined to a bounded domain with "acoustically hard boundary" like the periodic box Ω, develop fast oscillations responsible for mere *weak* convergence (in time) to the limit problem. The following result was proved in [65, Theorem II.1]:

Theorem 6. *Let $\Omega \subset R^N$ be the flat torus* (66)*, $N = 2, 3$. Suppose that*

$$p(\varrho) = a\varrho^\gamma,\ \gamma > \frac{N}{2}, a > 0,$$

and that the initial data for the Navier-Stokes system (63)*,* (64)*, and* (65) *take the form*

$$\varrho(0,\cdot) = \overline{\varrho} + \varepsilon\varrho^{(1)}_{0,\varepsilon},\ \mathbf{u}(0,\cdot) = \mathbf{u}_{0,\varepsilon}, \tag{71}$$

where

$$\|\varrho^{(1)}_{0,\varepsilon}\|_{L^\infty(\Omega)} + \|\mathbf{u}_{0,\varepsilon}\|_{L^\infty(\Omega;R^N)} \leq c \text{ uniformly for } \varepsilon \to 0.$$

Let $[\varrho_\varepsilon, \mathbf{u}_\varepsilon]$ be a family of finite energy weak solutions of (63)*,* (64)*, and* (65) *in $(0,T)\times\Omega$ in the sense specified in* (21)*,* (22)*, and* (23)*.*

Then

$$\sup_{t\in(0,T)} \|\varrho_\varepsilon - \overline{\varrho}\|_{L^\gamma} \leq \varepsilon c,$$

and, passing to a subsequence as the case may be,

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ weakly in } L^2(0,T;W^{1,2}(\Omega;R^N)), \tag{72}$$

where $\mathbf{u}$ *is a weak solution to the incompressible Navier-Stokes system*

$$\mathrm{div}_x \mathbf{u} = 0,$$

$$\overline{\varrho}\Big[\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla_x)\mathbf{u}\Big] + \nabla_x \Pi = \mu \Delta \mathbf{u}$$

in $(0, T) \times \Omega$ *with the initial data*

$$\mathbf{u}(0, \cdot) = \mathbf{v}_0 = H[\mathbf{u}_0],$$

where H *denotes the Helmholtz projection and* $\mathbf{u}_0$ *is a weak-(*) limit of* $\mathbf{u}_{0,\varepsilon}$ *in* $L^\infty(\Omega; R^N)$.

The initial data (71) are *ill prepared* as they generate nontrivial solutions of acoustic Eq. (69) and (70) responsible for the weak convergence in (72). The *well-prepared* initial data in this context satisfy

$$\varrho^{(1)}_{0,\varepsilon} \to 0, \ \mathbf{u}_{0,\varepsilon} - P[\mathbf{u}_{0,\varepsilon}] \to 0 \text{ in, say, } L^2(\Omega)$$

eliminating a priori the acoustic component.

There are other possibilities how to get rid of the acoustic effects:

- dispersion of acoustic waves on large or unbounded spatial domains;
- damping mechanism induced by the presence of physical boundaries.

Both possibilities will be discussed in forthcoming sections.

4.2 Incompressible Limits for the Navier-Stokes-Fourier System

Going back to the original problem (5), (6), and (7), it is possible to write the scaled Navier-Stokes-Fourier system in the form

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{73}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2} \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) + \frac{1}{\varepsilon} \varrho \nabla_x F \tag{74}$$

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) + \mathrm{div}_x\left(\frac{\mathbf{q}}{\vartheta}\right)$$
$$= \frac{1}{\vartheta}\left(\varepsilon^2 \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta}\right), \tag{75}$$

with

$$\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) = \left[\mu(\vartheta)\left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I}\right) + \eta(\vartheta)\mathrm{div}_x \mathbf{u}\mathbb{I}\right], \tag{76}$$

and

$$\mathbf{q} = -\kappa(\vartheta)\nabla_x \vartheta. \tag{77}$$

The fluid is confined to a bounded domain $\Omega \subset R^3$, on the boundary of which we impose the complete slip conditions

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ [\mathbb{S} \cdot \mathbf{n}] \times \mathbf{n}|_{\partial\Omega} = 0, \tag{78}$$

and no-flux condition

$$\mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{79}$$

Similarly to Sect. (4.1 missing reference), the boundary conditions (78) enable the separation of the acoustic component satisfying an analogue of (69) and (70).

4.2.1 Incompressible Limit on Bounded Domains

In addition to the scaling Ma $\approx \varepsilon$, we have taken Fr $\approx \sqrt{\varepsilon}$ in (74). As a result, the asymptotic limit $\varepsilon \to 0$ gives rise to the *Oberbeck-Boussinesq approximation*:

$$\mathrm{div}_x \mathbf{U} = 0, \tag{80}$$

$$\overline{\varrho}\left[\partial_t \mathbf{U} + (\mathbf{U} \cdot \nabla_x)\mathbf{U}\right] + \nabla_x \Pi = \mu \Delta \mathbf{U} + r \nabla_x F, \tag{81}$$

$$\overline{\varrho} c_p \left[\partial_t \Theta + \mathbf{U} \cdot \nabla_x \Theta\right] - \kappa(\overline{\vartheta})\Delta\Theta - \overline{\varrho}\, \overline{\vartheta} \alpha \mathrm{div}_x (F\mathbf{U}) = 0, \tag{82}$$

$$r + \overline{\varrho}\alpha\Theta = 0, \tag{83}$$

see [93]. Here, $\overline{\varrho}$ and $\overline{\vartheta}$ are the reference values of the density and temperature, and

$$\alpha = \frac{1}{\overline{\varrho}} \frac{\partial_\vartheta p}{\partial_\varrho p}(\overline{\varrho}, \overline{\vartheta}), \ c_p = \partial_\vartheta e(\overline{\varrho}, \overline{\vartheta}) + \alpha \frac{\overline{\vartheta}}{\overline{\varrho}} \partial_\vartheta p(\overline{\varrho}, \overline{\vartheta}). \tag{84}$$

The asymptotic limit passage is specified in the following statement, see [27, Chapter 5, Theorem 5.2]:

Theorem 7. *Let $\Omega \subset R^3$ be a bounded domain of class $C^{2+\nu}$. Assume that the functions p, e, and s satisfy hypotheses* (44), (45), (46), (47), (48), (49), *and* (50), *the transport coefficients μ, η, and κ comply the growth restrictions* (51), (52), *and the driving force is determined by a scalar potential $F = F(x)$ such that*

$$F \in W^{1,\infty}(\Omega), \ \int_\Omega F \, \mathrm{d}x = 0.$$

Let $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$ *be a family of weak solutions to the scaled Navier-Stokes-Fourier system* (73)*,* (74)*,* (75)*,* (76)*, and* (77) *in the set* $(0,T) \times \Omega$ *in the sense specified in Sect. (*3.2.1 *missing reference), supplemented with the boundary conditions* (78)*,* (79)*, and the initial data*

$$\varrho_\varepsilon(0,\cdot) = \overline{\varrho} + \varepsilon \varrho^{(1)}_{0,\varepsilon},\ \mathbf{u}_\varepsilon(0,\cdot) = \mathbf{u}_{0,\varepsilon},\ \vartheta_\varepsilon(0,\cdot) = \vartheta_{0,\varepsilon} = \overline{\vartheta} + \varepsilon \vartheta^{(1)}_{0,\varepsilon}, \tag{85}$$

where

$$\overline{\varrho} > 0,\ \overline{\vartheta} > 0$$

are constant, and

$$\int_\Omega \varrho^{(1)}_{0,\varepsilon}\,\mathrm{d}x = \int_\Omega \vartheta^{(1)}_{0,\varepsilon}\,\mathrm{d}x = 0 \textit{ for all } \varepsilon > 0.$$

Moreover, assume that

$$\left\{ \begin{array}{c} \varrho^{(1)}_{0,\varepsilon} \to \varrho^{(1)}_0 \textit{ weakly-(*) in } L^\infty(\Omega), \\ \mathbf{u}_{0,\varepsilon} \to \mathbf{U}_0 \textit{ weakly-(*) in } L^\infty(\Omega;\mathbb{R}^3), \\ \vartheta^{(1)}_{0,\varepsilon} \to \vartheta^{(1)}_0 \textit{ weakly-(*) in } L^\infty(\Omega). \end{array} \right\} \tag{86}$$

Then

$$\operatorname*{ess\ sup}_{t\in(0,T)} \|\varrho_\varepsilon(t) - \overline{\varrho}\|_{L^{\frac{5}{3}}(\Omega)} \le \varepsilon c,$$

and, at least for a suitable subsequence,

$$\mathbf{u}_\varepsilon \to \mathbf{U} \textit{ weakly in } L^2(0,T;W^{1,2}(\Omega;R^3)),$$
$$\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \to \Theta \textit{ weakly in } L^2(0,T;W^{1,2}(\Omega;R^3)),$$

where $\mathbf{U}$ *and* Θ *are weak solutions to the Oberbeck-Boussinesq approximation* (80)*,* (81)*,* (82)*, and* (83)*, with the boundary conditions*

$$\mathbf{U}\cdot\mathbf{n}|_{\partial\Omega} = 0,\ \left[\left(\nabla_x \mathbf{U} + \nabla_x^t \mathbf{U}\right)\cdot\mathbf{n}\right]\times\mathbf{n}|_{\partial\Omega} = 0,\ \nabla_x\Theta\cdot\mathbf{n}|_{\partial\Omega} = 0,$$

and the initial conditions

$$\mathbf{U}(0,\cdot) = \mathbf{U}_0,\ \Theta(0,\cdot) = \Theta_0,$$

$$\Theta_0 \equiv \frac{\overline{\vartheta}}{c_p(\overline{\varrho}, \overline{\vartheta})} \Big(\frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} \varrho_0^{(1)} + \frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} \vartheta_0^{(1)} + \alpha(\overline{\varrho}, \overline{\vartheta}) F \Big). \tag{87}$$

The fact that the initial temperature is given through (87) reflects the well-known *data adjustment problem* discussed in detail in [27, Section 5.5.3].

Next, the same situation as in Theorem 7 will be discussed in the context of the no-slip boundary condition

$$\mathbf{u}|_{\partial\Omega} = 0 \tag{88}$$

for the velocity field in the primitive system. As observed in [21], the presence of viscosity in (74) creates a boundary layer that may provide an effective damping mechanism to eliminate acoustic waves. However, such a scenario occurs only on *admissible domains* Ω enjoying the following property:

Property (S). *The overdetermined eigenvalue problem*

$$\Delta w = \lambda w \text{ in } \Omega, \ \nabla_x w|_{\partial\Omega} = 0$$

admits only the trivial solution $w = \text{const}, \ \lambda = 0$ *in* Ω.

Remark 1. Validity of (S) is intimately related to Schiffer conjecture, See, e.g., [12], asserting that the only NON-admissible domain among all simply connected domains in R^3 is a ball.

To simplify analysis, a slightly different geometry will be used, namely, an infinite periodic slab

$$\Omega = \Big\{ (x_1, x_2, x_3) \ \Big| \ (x_1, x_2) \in \mathcal{T}^2, \ 0 < x_3 < h(x_1, x_2) \Big\}, \tag{89}$$

where

$$\mathcal{T}^2 = \big([0,1]|_{\{0,1\}} \big)^2$$

is the flat torus in R^2.

The application of the method developed in [21] yields the following result; see [27, Chapter 7, Theorem 7.1]:

Theorem 8. *Under the hypotheses of Theorem 7, let* $\Omega \subset R^3$ *be the infinite slab specified in* (89)*, with*

$$h \in C^3(\mathcal{T}^2), \ h > 0 \text{ on } \mathcal{T}^2, \ h \not\equiv \text{const}, \tag{90}$$

on the boundary of which the velocity in the primitive Navier-Stokes-Fourier system satisfies the no-slip boundary condition (88).

Then the conclusion of Theorem 7 *remains valid, and, in addition,*

$$\mathbf{u}_\varepsilon \to \mathbf{U} \text{ (strongly) in } L^2((0,T)\times\Omega). \tag{91}$$

Thus the strong convergence claimed in (91) results from the presence of a boundary layer generated (88) imposed on the non-flat part of $\partial\Omega$.

4.2.2 Incompressible Limit on Large/Unbounded Domains

In meteorology and similar real world applications, the acoustic waves have a little influence on the behavior of the fluid because of the *dispersive effect* of typically large physical domains. This was used in the seminal paper by [20], where Strichartz estimates are applied to the acoustic system arising in the low Mach number limit of a barotropic Navier-Stokes system.

The Navier-Stokes-Fourier system (73), (74), (75), (76),and (77) will be considered on a family of domains

$$\Omega_\varepsilon = \Omega \cap \left\{ x \in R^3 \ \middle|\ |x| < \frac{1}{\varepsilon^r} \right\},\ r > 1, \tag{92}$$

where Ω is an unbounded (exterior) domain with smooth and compact boundary, together with slip boundary conditions

$$\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega_\varepsilon} = 0,\ [\mathbb{S}(\nabla_x\mathbf{u})\cdot\mathbf{n}]\times\mathbf{n}|_{\partial\Omega_\varepsilon} = 0,\ \mathbf{q}\cdot\mathbf{n}|_{\partial\Omega_\varepsilon} = 0. \tag{93}$$

Next, the class of admissible potentials F will be restricted, specifically

$$-\Delta F = m \text{ in } R^3,\ \nabla_x F \in L^2(R^3;R^3),\ \text{supp}[m] \subset R^3 \setminus \overline{\Omega}. \tag{94}$$

Such a choice of F corresponds to a gravitational force acting on the fluid generated by an object with mass distribution m placed outside Ω.

The following results is an analogue of Theorem 7; see [32, Theorem 2.2]:

Theorem 9. *Let $\Omega \subset R^3$ be an unbounded (exterior) domain with a compact boundary of class $C^{2+\nu}$. Assume that the functions p, e, and s satisfy hypotheses* (44)*,* (45)*,* (46)*,* (47)*,* (48)*,* (49)*, and* (50) *and the transport coefficients μ, η, and κ comply the growth restrictions* (51) *and* (52)*. Let $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$ be a family of weak solutions to the scaled Navier-Stokes-Fourier system* (73)*,* (74)*,* (75)*,* (76)*, and* (77) *in the set $(0,T)\times\Omega_\varepsilon$ in the sense specified in Sect. (*3.2.1 *missing reference), driven by a potential F satisfying* (94)*, and supplemented with the boundary conditions* (93)*. Let the initial data satisfy*

$$\varrho_\varepsilon(0,\cdot) = \tilde{\varrho}_\varepsilon + \varepsilon\varrho^{(1)}_{0,\varepsilon},\ \mathbf{u}_\varepsilon(0,\cdot) = \mathbf{u}_{0,\varepsilon},\ \vartheta_\varepsilon(0,\cdot) = \vartheta_{0,\varepsilon} = \overline{\vartheta} + \varepsilon\vartheta^{(1)}_{0,\varepsilon},$$

where $\overline{\vartheta} > 0$ *is constant,* $\tilde{\varrho}_\varepsilon$ *is the unique solution of the problem*

$$\nabla_x p(\tilde{\varrho}_\varepsilon, \overline{\vartheta}) = \varepsilon \tilde{\varrho}_\varepsilon \nabla_x F \text{ in } R^3, \ \tilde{\varrho}_\varepsilon \to \overline{\varrho} > 0 \text{ as } |x| \to \infty,$$

and

$$\int_{\Omega_\varepsilon} \varrho_{0,\varepsilon}^{(1)} \, \mathrm{d}x = \int_{\Omega_\varepsilon} \vartheta_{0,\varepsilon}^{(1)} \, \mathrm{d}x = 0 \text{ for all } \varepsilon > 0.$$

Moreover, assume that

$$\left\{ \begin{array}{c} \varrho_{0,\varepsilon}^{(1)} \to \varrho_0^{(1)} \text{ weakly-(*) in } L^\infty \cap L^2(\Omega), \\ \mathbf{u}_{0,\varepsilon} \to \mathbf{U}_0 \text{ weakly-(*) in } L^\infty \cap L^2(\Omega; \mathbb{R}^3), \\ \vartheta_{0,\varepsilon}^{(1)} \to \vartheta_0^{(1)} \text{ weakly-(*) in } L^\infty \cap L^2(\Omega). \end{array} \right\}$$

Then

$$\varrho_\varepsilon \to \overline{\varrho} \text{ in } L^\infty(0,T; L^{5/3}(K)),$$

and, at least for a suitable subsequence,

$$\mathbf{u}_\varepsilon \to \mathbf{U} \text{ weakly in } L^2(0,T; W^{1,2}(K; R^3)) \text{ and strongly in } L^2((0,T) \times K; R^3),$$
$$\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \to \Theta \text{ weakly in } L^2(0,T; W^{1,2}(K; R^3)),$$

for any compact $K \subset \Omega$*, where* $\mathbf{U}$ *and* Θ *is a weak solutions to the Oberbeck-Boussinesq approximation (80), (81), (82), and (83), with the boundary conditions*

$$\mathbf{U} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ \left[\left(\nabla_x \mathbf{U} + \nabla_x^t \mathbf{U} \right) \cdot \mathbf{n} \right] \times \mathbf{n}|_{\partial\Omega} = 0, \ \nabla_x \Theta \cdot \mathbf{n}|_{\partial\Omega} = 0,$$

and the initial conditions

$$\mathbf{U}(0,\cdot) = \mathbf{U}_0, \ \Theta(0,\cdot) = \Theta_0,$$
$$\Theta_0 \equiv \frac{\overline{\vartheta}}{c_p(\overline{\varrho}, \overline{\vartheta})} \Big(\frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} \varrho_0^{(1)} + \frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} \vartheta_0^{(1)} + \alpha(\overline{\varrho}, \overline{\vartheta}) F \Big).$$

4.3 Low Mach Number Limit for the Navier-Stokes-Fourier System

Unlike in the purely barotropic case in which constant pressure implies constant density, the low Mach number regime for the full Navier-Stokes-Fourier system offers richer variety of possible scenarios depending on the initial data. The system (73), (74), (75), (76), and (77) will be considered for a specific choice of constitutive equations, namely,

- $$p(\varrho,\vartheta) = R\varrho\vartheta,\ e(\varrho,\vartheta) = c_v\vartheta,\ R, c_v > 0 \text{ positive constants;} \tag{95}$$

- constant transport coefficients

$$\mathbb{S}(\vartheta,\nabla_x\mathbf{u}) = \nu\left[\mu(\vartheta)\left(\nabla_x\mathbf{u} + \nabla_x^t\mathbf{u} - \frac{2}{3}\mathrm{div}_x\mathbf{u}\mathbb{I}\right) + \eta(\vartheta)\mathrm{div}_x\mathbf{u}\mathbb{I}\right],$$
$$\mathbf{q}(\vartheta,\nabla_x\vartheta) = -k\kappa(\vartheta)\nabla_x\vartheta,\ 0 \le \nu,\ k \le 1; \tag{96}$$

- $$F = 0.$$

Accordingly, system (73), (74), (75), (76), and (77) can be rewritten in terms of new state variables $[P = p(\varrho,\vartheta),\ \mathbf{u},\ \vartheta]$ as

$$\begin{aligned}\partial_t P + \mathbf{u}\cdot\nabla_x P + \gamma P\mathrm{div}_x\mathbf{u} &= (\gamma-1)k\mathrm{div}_x(\kappa(\vartheta)\nabla_x\vartheta)\\ &+ (\gamma-1)\varepsilon^2\nu\mathbb{S}(\vartheta,\nabla_x\mathbf{u}) : \nabla_x\mathbf{u},\end{aligned} \tag{97}$$

$$\varrho\left(\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla_x)\mathbf{u}\right) + \frac{1}{\varepsilon^2}P = \nu\mathrm{div}_x\mathbb{S}(\vartheta,\nabla_x\mathbf{u}), \tag{98}$$

$$\varrho c_V\left(\partial_t\vartheta + \mathbf{u}\cdot\nabla_x\vartheta\right) + P\mathrm{div}_x\mathbf{u} = k\mathrm{div}_x(\kappa(\vartheta)\nabla_x\vartheta) + \varepsilon^2\nu\mathbb{S}(\vartheta,\nabla_x\mathbf{u}) : \nabla_x\mathbf{u}, \tag{99}$$

where we have set

$$\varrho = \frac{P}{R\vartheta},\ \gamma = 1 + \frac{R}{c_v}.$$

4.3.1 Existence of Smooth Solutions

In contrast with the previous part of this section, we use strong $\to$ strong approach. The necessary local-in-time existence result is proved in [2, Theorem 1.2]:

Theorem 10. *Denote*

$$\mathcal{A} = \left\{ a = [\varepsilon, \nu, k] \;\middle|\; 0 < \varepsilon \le 1,\ 0 \le k, \nu \le 1 \right\}.$$

Let

$$\textit{either } \Omega = \left([0,1]|_{\{0,1\}} \right)^N \textit{ or } \Omega = R^N,\ N = 1, 2, 3,$$

and let $n > 1 + \frac{N}{2}$ *be given. Suppose that* μ, η *and* κ *are functions of class* $C^\infty(0,\infty)$ *such that*

$$\mu(\vartheta) > 0,\ \eta(\vartheta) \ge 0,\ \kappa(\vartheta) > 0 \textit{ for any } \vartheta > 0.$$

Then for all

$$\overline{P} > 0,\ \overline{\vartheta} > 0,\ M_0 > 0$$

there exists $T > 0$ *such that for all* $a = (\varepsilon, \nu, k) \in \mathcal{A}$ *and any initial data*

$$P(0,\cdot) = P_0 > 0,\ \vartheta(0,\cdot) = \vartheta_0 > 0,\ \mathbf{u}(0,\cdot) = \mathbf{u}_0 \tag{100}$$

satisfying

$$\left\| \frac{P_0 - \overline{P}}{\varepsilon} \right\|_{W^{n+1,2}(\Omega)} + \left\| \vartheta_0 - \overline{\vartheta} \right\|_{W^{n+1,2}(\Omega)} + \|\mathbf{u}_0\|_{W^{n+1,2}(\Omega;R^N)} \le M_0,$$

the initial-value problem (97), (98), and (99), (100) admits a solution $[P, \vartheta, \mathbf{u}]$ *unique in the class*

$$P - \overline{P} \in C([0,T]; W^{n+1,2}(\Omega)),\ \vartheta - \overline{\vartheta} \in C([0,T]; W^{n+1,2}(\Omega)),$$
$$\mathbf{u} \in C([0,T]; W^{n+1,2}(\Omega; R^N))$$

and with P, ϑ *strictly positive. In addition, there exists* M *depending only on* M_0, $\overline{P}$, *and* $\overline{\vartheta}$ *such that*

$$\sup_{t\in[0,T], a\in\mathcal{A}} \left\| \frac{P(t,\cdot) - \overline{P}}{\varepsilon} \right\|_{W^{n,2}(\Omega)} + \left\| \vartheta(t,\cdot) - \overline{\vartheta} \right\|_{W^{n,2}(\Omega)} + \|\mathbf{u}(t,\cdot)\|_{W^{n,2}(\Omega;R^N)} \le M. \tag{101}$$

As typical in the analysis of the low Mach number limit, the estimate (101) provides space but not time regularity of solutions. Similar results for systems involving general state equations and also large source terms in (97), (99) were obtained by [1].

4.3.2 The Low Mach Number Limit

Formally, the asymptotic limit for $\varepsilon \to 0$ can be identified as

$$\gamma \overline{P} \mathrm{div}_x \mathbf{u} = (\gamma - 1) k \mathrm{div}_x \left(\kappa(\vartheta) \nabla_x \vartheta \right), \tag{102}$$

$$\varrho \left(\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla_x) \mathbf{u} \right) + \nabla_x \Pi = \nu \mathrm{div}_x \mathbb{S}(\vartheta, \nabla_x \mathbf{u}), \tag{103}$$

$$\varrho c_p \left(\partial_t \vartheta + \mathbf{u} \cdot \nabla_x \vartheta \right) = k \mathrm{div}_x \left(\kappa(\vartheta) \nabla_x \vartheta \right), \ \varrho = \frac{\overline{P}}{R\vartheta}. \tag{104}$$

System (102) is highly nonstandard, in particular in view of the constraint (102), and its properties are of independent interest; see [19]. As for the asymptotic limit, we quote the following result, see [2, Theorem 1.5]:

Theorem 11. *Under the hypotheses of Theorem 10, suppose that*

$$\Omega = R^N, \ N = 2, 3,$$

and fix the parameters $\nu \in [0, 1]$, $k \in [0, 1]$. *Let* $[P_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon]$ *be a family of solutions of problem (97), (98), and (99) such that*

$$\sup_{t \in [0,T], 0 < \varepsilon \leq 1} \left\| \frac{P_\varepsilon(t, \cdot) - \overline{P}}{\varepsilon} \right\|_{W^{n,2}(\Omega)} + \left\| \vartheta_\varepsilon(t, \cdot) - \overline{\vartheta} \right\|_{W^{n,2}(\Omega)} + \| \mathbf{u}_\varepsilon(t, \cdot) \|_{W^{n,2}(\Omega; R^N)} < \infty$$

on a time interval $[0, T]$, *where* n *is sufficiently large. Suppose, in addition, that* $\vartheta_\varepsilon(0, \cdot) - \overline{\vartheta}$ *is compactly supported.*

Then

$$\frac{P_\varepsilon(t, \cdot) - \overline{P}}{\varepsilon} \to 0 \ in \ L^2(0, T; W^{s,2}(K)),$$

$$\vartheta_\varepsilon \to \vartheta \ in \ L^2(0, T; W^{s,2}(K)), \ \mathbf{u}_\varepsilon \to \mathbf{u} \ in \ L^2(0, T; W^{s,2}(K; R^N))$$

for any compact $K \subset \Omega$ *and any* $s < n$, *where* $[\vartheta, \mathbf{u}]$ *solves system (102), (103), and (104).*

Note that convergence of the family $\{\vartheta_\varepsilon, \mathbf{u}_\varepsilon\}_{\varepsilon > 0}$ is stated in the Lebesgue space L^2 in time; the initial values are "lost" in the limit due to rapid oscillations.

5 Incompressible–Inviscid Limits

In this section, a more complex situation will be examined when

$$\text{Ma} \to 0 \text{ and, simultaneously Re, Pe} \to \infty,$$

meaning the two regimes discussed in the previous part occur and interact at the same time. We use the weak → strong approach to preclude the so far unsurmountable difficulties related to the low regularity of solutions to the target Euler system. Accordingly, all results presented below are local-in-time, where the life span is that of the limit problem.

5.1 A Model Problem: Incompressible, Inviscid Limit for Barotropic Fluid Flows in Large Domains

The *barotropic Navier-Stokes system* will be considered:

$$\partial_t \varrho + \text{div}_x(\varrho \mathbf{u}) = 0, \tag{105}$$

$$\partial_t(\varrho \mathbf{u}) + \text{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2} \nabla_x p(\varrho) = \nu \text{div}_x \mathbb{S}(\nabla_x \mathbf{u}), \tag{106}$$

$$\mathbb{S}(\nabla_x \mathbf{u}) = \nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \text{div}_x \mathbf{u} \mathbb{I}, \tag{107}$$

in a spatial domain $\Omega_M \subset R^3$, supplemented with the no-slip boundary conditions

$$\mathbf{u}|_{\partial \Omega_M} = 0. \tag{108}$$

Next, a family $\{\Omega_M\}_{M>0}$ of domains is supposed to enjoy the following properties:

- $\Omega_M \subset R^3$ are simply connected, bounded uniformly(with respect to M)-C^2 domains;
- there exists $\omega > 0$ such that

$$\left\{ x \in R^3 \;\middle|\; |x| < \omega M \right\} \subset \Omega_M; \tag{109}$$

- there exists $\beta > 0$ such that

$$|\partial \Omega_M|_2 \leq \beta M^2, \tag{110}$$

where $|\cdot|_2$ denotes the standard two-dimensional Hausdorff measure.

The work [23] may be consulted for the concept of uniformly C^k domains. The goal is to identify the triple limit

$$\varepsilon \to 0,\ \nu \to 0,\ M \to \infty.$$

5.1.1 Target System

By analogy with Sects. 3.1 and 4.1, it is to be expected that

$$\varrho \to \overline{\varrho},$$

while an obvious candidate to describe the asymptotic behavior of the velocity field is the *incompressible Euler system*:

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v} + \nabla_x \Pi = 0,\ \mathrm{div}_x \mathbf{v} = 0 \text{ in } R^3. \tag{111}$$

The initial data

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0 = H[\mathbf{u}_0], \tag{112}$$

are considered, where H denotes the Helmholtz projection and

$$\mathbf{u}_0 \in C^m(R^3; R^3) \text{ for a certain } m > 4,\ \mathrm{supp}[\mathbf{u}_0] \text{ compact in } R^3$$

is a given function. As shown by [56], system (111) endowed with the initial condition (112) admits a unique classical solution

$$\mathbf{v} \in C^k([0, T_{\max}); W^{m-k,2}(R^3; R^3)),\ k = 1, .., m-1$$

defined on a maximal time interval $[0, T_{\max})$, $T_{\max} > 0$.

5.1.2 Acoustic Equation

Similarly to Sect. (4.1 missing reference), the behavior of the gradient component of the velocity field is governed by the acoustic system

$$\varepsilon \partial_t s + \Delta \Psi = 0,\ \varepsilon \partial_t \nabla_x \Psi + a \nabla_x s = 0,\ a = p'(1) > 0, \tag{113}$$

$$s(0, \cdot) = \varrho_0^{(1)},\ \nabla_x \Psi(0, \cdot) = \nabla_x \Psi_0 = \mathbf{u}_0 - H[\mathbf{u}_0]. \tag{114}$$

5.1.3 Inviscid Incompressible Limit

The following result can be obtained applying a scaled variant of the relative energy inequality (33) for

$$r = \frac{1}{\varepsilon} \overline{\varrho} + s,\ \mathbf{U} = \mathbf{v} + \nabla_x \Psi,$$

see [38, Theorem 2.1]:

Theorem 12. *Let the pressure p satisfy*

$$p \in C[0,\infty) \cap C^3(0,\infty),\ p(0) = 0,\ p'(\varrho) > 0 \, \textit{for} \, \varrho > 0,$$
$$\lim_{\varrho\to\infty} \frac{p'(\varrho)}{\varrho^{\gamma-1}} = p_\infty,\ \gamma > 3/2.$$

Let $\{\Omega_M\}_{M>0}$ be a family of uniformly C^2-domains in R^3 such that (109), (110) hold for $M = M(\varepsilon)$,

$$\varepsilon M(\varepsilon) \to \infty \, \textit{as} \, \varepsilon \to 0.$$

Let the initial data for the compressible Navier-Stokes system (105), (106), (107), and (108) be of the form

$$\varrho(0,\cdot) = \varrho_{0,\varepsilon} = 1 + \varepsilon\varrho^{(1)}_{0,\varepsilon},\ \mathbf{u}(0,\cdot) = \mathbf{u}_{0,\varepsilon},\ \|\varrho^{(1)}_{0,\varepsilon}\|_{L^2\cap L^\infty(R^3)} + \|\mathbf{u}_{0,\varepsilon}\|_{L^2(R^3;R^3)} \le D.$$

In addition, suppose we are given functions $\mathbf{u}_0$, $\varrho^{(1)}_0$ such that

$$\mathbf{u}_0 \in C^m(R^3;R^3),\ \varrho_0^{(1)} \in C^m(R^3),\ \|\mathbf{u}_0\|_{C^m(R^3;R^3)} + \|\varrho_0^{(1)}\|_{C^m(R^3)} \le D,\ m > 4,$$
$$\mathrm{supp}[\mathbf{u}_0],\ \mathrm{supp}[\varrho_0^{(1)}] \, \textit{compact in} \, R^3.$$

Let $T_{\max} > 0$ be the life span of the smooth solution $\mathbf{v}$ of the Euler system (111), endowed with the initial datum $\mathbf{v}_0 = H[\mathbf{u}_0]$, and let $0 < T < T_{\max}$. Let $[s,\Psi]$ be the solution of the acoustic system (113), with the initial data (114).

Then there exists $\varepsilon_0 > 0$ such that

$$\begin{aligned}
&\left\| \sqrt{\varrho}\Big(\mathbf{u} - \nabla_x\Psi - \mathbf{v}\Big)(\tau,\cdot)\right\|_{L^2(\Omega_M;R^3)} + \left\| \left(\frac{\varrho-1}{\varepsilon}\right)(\tau,\cdot) - s(\tau,\cdot)\right\|_{L^2+L^\gamma(\Omega_M)} \\
&\le c(D,T,\alpha)\left[\|\mathbf{u}_{0,\varepsilon} - \mathbf{u}_0\|_{L^2(\Omega_M;R^3)} + \left\|\varrho^{(1)}_{0,\varepsilon} - \varrho^{(1)}_0\right\|_{L^2(\Omega_M)} \right. \\
&\left. + \left(\nu + \varepsilon^\alpha + \frac{1}{\varepsilon M(\varepsilon)}\right)^{1/2}\right],\ \tau \in [0,T],\ 0 < \alpha < 1,\ \textit{and}\ 0 < \varepsilon \le \varepsilon_0,
\end{aligned} \tag{115}$$

for any weak solution $[\varrho, \mathbf{u}]$ of the compressible Navier-Stokes system (105–108).

Formula (115) provides a qualitative piece of information concerning the *convergence rate* in terms of the initial data and the scaling parameters ε, ν, and M. Although the proof in [38] leans on dispersive estimates for the acoustic system (113) and (114), the convergence can be established also in the periodic setting $\Omega = \mathcal{T}^N$; see Masmoudi [72, 89].

5.2 Inviscid Incompressible Limit for the Navier-Stokes-Fourier System

In accordance with the main topic of this chapter, the scaled Navier-Stokes-Fourier system is considered:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{116}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2} \nabla_x p(\varrho, \vartheta) = \varepsilon^a \mathrm{div}_x \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) + \frac{1}{\varepsilon} \varrho \nabla_x F, \tag{117}$$

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) + \varepsilon^b \mathrm{div}_x \left(\frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \right)$$
$$= \frac{1}{\vartheta} \left(\varepsilon^{2+a} \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \varepsilon^b \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right), \tag{118}$$

where, similarly to the above,

$$\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) = \mu(\vartheta) \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \right) + \eta(\vartheta) \mathrm{div}_x \mathbf{u} \mathbb{I}, \tag{119}$$

$$\mathbf{q}(\vartheta, \nabla_x \vartheta) = -\kappa(\vartheta) \nabla_x \vartheta, \tag{120}$$

where the quantities μ, η, and κ are temperature-dependent transport coefficients. The value of the characteristic numbers in (116), (117), and (118) are:

$$\mathrm{Ma} = \varepsilon, \ \mathrm{Fr} = \sqrt{\varepsilon}, \ \mathrm{Re} = \frac{1}{\varepsilon^a}, \ \mathrm{Pe} = \frac{1}{\varepsilon^b}.$$

5.2.1 Physical Space, Boundary Conditions

The fluid is supposed to occupy an exterior domain $\Omega \subset R^3$, with impermeable, thermally insulating, and frictionless boundary, specifically,

$$\mathbf{u} \cdot \mathbf{n}|_{\partial \Omega} = [\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) \cdot \mathbf{n}] \times \mathbf{n}|_{\partial \Omega} = 0, \ \nabla_x \vartheta \cdot \mathbf{n}|_{\partial \Omega} = 0. \tag{121}$$

As Ω is unbounded, the *far-field* conditions

$$\varrho \to \overline{\varrho}, \ \vartheta \to \overline{\vartheta}, \ \mathbf{u} \to 0 \text{ as } |x| \to \infty, \tag{122}$$

are prescribed, where $\overline{\varrho}$, $\overline{\vartheta}$ are positive constants.

Similarly to Sect. 4.2.2, it is supposed that

$$-\Delta F = m \text{ in } R^3, \ \nabla_x F \in L^2(R^3; R^3), \ \mathrm{supp}[m] \subset R^3 \setminus \overline{\Omega}, \tag{123}$$

meaning the motion is driven by the gravitational force of objects lying outside the fluid domain. We fix the equilibrium solution $\tilde{\varrho}_\varepsilon$ associate to the far-field conditions,

$$\nabla_x p(\tilde{\varrho}_\varepsilon, \overline{\vartheta}) = \varepsilon \tilde{\varrho}_\varepsilon \nabla_x F, \quad \tilde{\varrho}_\varepsilon \to \overline{\varrho} \quad \text{as } |x| \to \infty, \tag{124}$$

and take the initial data in the form

$$\varrho(0,\cdot) = \varrho_{0,\varepsilon} = \tilde{\varrho}_\varepsilon + \varepsilon \varrho^{(1)}_{0,\varepsilon},\ \vartheta(0,\cdot) = \vartheta_{0,\varepsilon} = \overline{\vartheta} + \varepsilon \vartheta^{(1)}_{0,\varepsilon},\ \mathbf{u}(0,\cdot) = \mathbf{u}_{0,\varepsilon}. \tag{125}$$

5.2.2 Target Problem

The target problem can be formally identified as the incompressible *Euler-Boussinesq system*:

$$\operatorname{div}_x \mathbf{U} = 0, \tag{126}$$

$$\partial_t \mathbf{U} + (\mathbf{U} \cdot \nabla_x)\mathbf{U} + \nabla_x \Pi = -\alpha \Theta \nabla_x F, \tag{127}$$

$$c_p \left(\partial_t \Theta + \mathbf{U} \cdot \nabla_x \Theta\right) - \overline{\vartheta} \alpha \mathbf{U} \cdot \nabla_x F = 0, \tag{128}$$

where c_p and α are the same as in (84).

In agreement with the nowadays standard theory of well-posedness for hyperbolic systems, see, e.g. [54], system (126), (127), and (128), endowed with the initial data

$$\Theta(0,\cdot) = \Theta_0,\ \Theta_0 \in W^{k,2}(\Omega),\ \|\Theta_0\|_{W^{k,2}(\Omega)} \le D,$$

$$\mathbf{U}(0,\cdot) = \mathbf{U}_0,\ \mathbf{U}_0 \in W^{k,2}(\Omega; R^3),\ \|\mathbf{U}_0\|_{W^{k,2}(\Omega;R^3)} \le D,\ \operatorname{div}_x \mathbf{U}_0 = 0,\ \mathbf{U}_0 \cdot \mathbf{n}|_{\partial\Omega} = 0,$$

with $k > \frac{5}{2}$ possesses a regular solution $[\Theta, \mathbf{U}]$,

$$\Theta \in C([0, T_{\max}); W^{k,2}(\Omega)),\ \mathbf{U} \in C([0, T_{\max}); W^{k,2}(\Omega; R^3)),$$
$$\partial_t \mathbf{U},\ \nabla_x \Pi \in C([0, T_{\max}); W^{k-1,2}(\Omega; R^3)),$$

defined on a maximal time interval $[0, T_{\max})$, $T_{\max} = T_{\max}(D)$.

5.2.3 Asymptotic Limit

The following result was proved in [30, Theorem 3.1]:

Theorem 13. *Let the thermodynamic functions p, e, s, and the transport coefficients μ, η, and κ satisfy the hypotheses (44), (45), (46), (47), (48), (49), (50), (51), and (52), Let the potential force F be given by (123). Let the exponents a, b, determining the Reynold and Péclet number scales, satisfy*

$$b > 0,\ 0 < a < \frac{5}{3}. \tag{129}$$

Let the initial data (125) be chosen in such a way that

$$\{\varrho^{(1)}_{0,\varepsilon}\}_{\varepsilon>0},\ \{\vartheta^{(1)}_{0,\varepsilon}\}_{\varepsilon>0} \text{ are bounded in } L^2 \cap L^\infty(\Omega),$$
$$\varrho^{(1)}_{0,\varepsilon} \to \varrho^{(1)}_0,\ \vartheta^{(1)}_{0,\varepsilon} \to \vartheta^{(1)}_0 \text{ in } L^2(\Omega),$$
$$\{\mathbf{u}_{0,\varepsilon}\}_{\varepsilon>0} \text{ is bounded in } L^2(\Omega; R^3),\ \mathbf{u}_{0,\varepsilon} \to \mathbf{u}_0 \text{ in } L^2(\Omega; R^3),$$

where

$$\varrho^{(1)}_0,\ \vartheta^{(1)}_0 \in W^{1,2} \cap W^{1,\infty}(\Omega),\ H[\mathbf{u}_0] = \mathbf{v}_0 \in W^{k,2}(\Omega; R^3) \text{ for a certain } k > \frac{5}{2}.$$

Suppose that the Euler-Boussinesq system (126), (127), and (128), endowed with the initial data

$$\mathbf{U}_0 = H[\mathbf{u}_0],\ \Theta_0 = \frac{\overline{\vartheta}}{c_p}\left(\frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial \varrho}\varrho^{(1)}_0 + \frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial \vartheta}\vartheta^{(1)}_0\right),$$

admits a regular solution $[\mathbf{U}, \Theta]$ *defined on a maximal time interval* $[0, T_{\max})$.

Finally, let $\{\varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon\}$ *be a dissipative weak solution of the Navier-Stokes-Fourier system (116), (117), (118), (119), (120), (121), and (122) in* $(0, T) \times R^3$, $T < T_{\max}$.

Then

$$\operatorname{ess}\sup_{t\in(0,T)} \|\varrho_\varepsilon(t,\cdot) - \overline{\varrho}\|_{L^{5/3}_{\rm loc}(\overline{\Omega})} \le \varepsilon c,\ \sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon \to \sqrt{\overline{\varrho}}\,\mathbf{U} \text{ in } L^\infty_{\rm loc}((0,T];$$
$$L^2_{\rm loc}(\overline{\Omega}; R^3)) \text{ and weakly-(*) in } L^\infty(0,T; L^2(\Omega; R^3)),$$

and

$$\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \to \Theta \text{ in } L^\infty_{\rm loc}((0,T]; L^2_{\rm loc}(\overline{\Omega})),\ \text{and weakly-(*) in } L^\infty(0,T; L^2(\Omega)).$$

In view of (129), the result is *path dependent*, it depends on the way the asymptotic limit is achieved.

6 Stratified Fluids

The low Mach number limit in *strongly stratified fluids* is accompanied with a comparable scaling of the Froude number:

$$\mathrm{Ma} \approx \mathrm{Fr}.$$

In the simplified barotropic case, the limit distribution of the mass density $\tilde{\varrho}$ is no longer constant but satisfies

$$\nabla_x p(\tilde{\varrho}) = \tilde{\varrho}\nabla_x F. \tag{130}$$

More interesting situations occur if the changes of the temperature are taken into account.

6.1 A Model Problem: Anelastic Limit

As an avatar of stratified fluids consider the barotropic Navier-Stokes system

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0, \tag{131}$$

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2}\nabla_x \varrho^\gamma = \operatorname{div}_x \mathbb{S}(\nabla_x \mathbf{u}) + \frac{1}{\varepsilon^2}\nabla_x F, \tag{132}$$

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu\left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\operatorname{div}_x \mathbf{u}\mathbb{I}\right) + \eta \operatorname{div}_x \mathbf{u}\mathbb{I},\ \mu > 0,\ \eta \geq 0 \tag{133}$$

in a spatial domain $\Omega \subset R^N$, $N = 2, 3$, supplemented with the no-slip boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0. \tag{134}$$

The goal is to examine the asymptotic limit $\varepsilon \to 0$.

6.1.1 Acoustic Waves, Target Problem

To formulate the target problem, a weighted analogue of Helmholtz decomposition is needed

$$\mathbf{u} = H_{\tilde{\varrho}}[\mathbf{u}] + \tilde{\varrho}\nabla_x \Phi,\ \operatorname{div}_x H_{\tilde{\varrho}}[\mathbf{u}] = 0,\ \nabla_x \Phi \cdot \mathbf{n}|_{\partial\Omega} = 0,$$

see [27, Section 6.3.3] and [63, 74].

An analogue of the acoustic equations (69) and (70) reads

$$\varepsilon \partial_t \left(\frac{\varrho - \tilde{\varrho}}{\varepsilon}\right) + \operatorname{div}_x(\tilde{\varrho}\nabla_x \Phi) = 0, \tag{135}$$

$$\varepsilon \tilde{\partial}_t \nabla_x \Phi + \tilde{\varrho}\nabla_x\left(\gamma \tilde{\varrho}^{\gamma-2}\frac{\varrho - \tilde{\varrho}}{\varepsilon}\right) = \varepsilon(I - H_{\tilde{\varrho}})[\operatorname{div}_x \tilde{\mathbb{L}}], \tag{136}$$

while the expected target system takes the form of the so-called *anelastic approximation*; see [58, 67, 78]:

$$\operatorname{div}_x(\tilde{\varrho}\mathbf{U}) = 0, \tag{137}$$

$$\partial_t(\tilde{\varrho}\mathbf{U}) + \operatorname{div}_x(\tilde{\varrho}\mathbf{U} \otimes \mathbf{U}) + \tilde{\varrho}\nabla_x \Pi = \operatorname{div}_x \mathbb{S}(\nabla_x \mathbf{U}). \tag{138}$$

6.1.2 Asymptotic Limit

The following result was proved with the help of the so-called "local method" developed in [66]; see [74, Theorem 1.1]:

Theorem 14. *Let $\Omega \subset R^N$, $N = 2, 3$ be a bounded regular domain. Suppose that $\gamma > N$ and $F \in W^{1,\infty}(\Omega)$. Let $[\varrho_\varepsilon, \mathbf{u}_\varepsilon]_{\varepsilon>0}$ be a family of weak solutions to the Navier-Stokes system (131), (132), (133), and (134) in $(0,T) \times \Omega$,*

$$\varrho_\varepsilon(0,\cdot) = \tilde{\varrho} + \varepsilon \varrho^{(1)}_{0,\varepsilon}, \ \mathbf{u}_\varepsilon(0,\cdot) = \mathbf{u}_{0,\varepsilon},$$

$$\|\varrho^{(1)}_{0,\varepsilon}\|_{L^\infty} \leq c \text{ uniformly}, \ \mathbf{u}_{0,\varepsilon} \to \mathbf{u}_0 \text{ weakly-(*) in } L^\infty(\Omega; R^N)$$

as $\varepsilon \to 0$.

Then

$$\sup_{t \in (0,T)} \|\varrho_\varepsilon - \tilde{\varrho}\|_{L^2(\Omega)} \leq \varepsilon c,$$

and, at least for a suitable subsequence,

$$\mathbf{u}_\varepsilon \to \mathbf{U} \text{ weakly in } L^2(0,T; W^{1,2}(\Omega; R^N)),$$

where $\mathbf{U}$ is a weak solution of the anelastic system (137) and (138),

$$\tilde{\varrho}\mathbf{U}(0,\cdot) = H_{\tilde{\varrho}}[\tilde{\varrho}\mathbf{u}_0].$$

6.2 Stratified Fluid Flows in Meteorology and Astrophysics

Certain singular limits arising in the analysis of stratified fluid flows in meteorology and astrophysics will be discussed [3, 58]. In addition to the situation examined in [74], we include variations of a kind of entropy variable – the *potential temperature*, Θ, at second order in the Mach number. The resulting system of equations reads:

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0, \tag{139}$$

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2}\nabla_x(\varrho\Theta)^\gamma = \operatorname{div}_x \mathbb{S}(\nabla_x \mathbf{u}) + \frac{1}{\varepsilon^2}\varrho\nabla_x F \tag{140}$$

$$\partial_t(\varrho\Theta) + \operatorname{div}_x(\varrho\Theta\mathbf{u}) = 0, \tag{141}$$

with

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}, \ \mu > 0, \ \eta \geq 0. \tag{142}$$

The fluid occupies a slab

$$\Omega = R^2 \times (0, 1), \tag{143}$$

The velocity is supposed to satisfy the complete slip (closed lid in the present context) boundary conditions,

$$\mathbf{u} \cdot \mathbf{n} = u_3|_{\partial\Omega} = 0, \ (\mathbb{S}(\nabla_x \mathbf{u}) \cdot \mathbf{n}) \times \mathbf{n}|_{\partial\Omega} = 0. \tag{144}$$

6.2.1 Target System

The goal is to identify the asymptotic limit for $\varepsilon \to 0$. It is possible to take $F = -g x_3$ and fix a static density distribution

$$\nabla_x \tilde{\varrho}^\gamma = \tilde{\varrho} \nabla_x F \text{ in } \Omega.$$

Accordingly, the far-field conditions must be given

$$\varrho \to \tilde{\varrho}, \ \Theta \to 1 \text{ as } |x| \to \infty. \tag{145}$$

Similarly to Sect. (6.1 missing reference), the asymptotic behavior will be captured by a variant of the anelastic system:

$$\mathrm{div}_x (\tilde{\varrho} \mathbf{U}) = 0, \tag{146}$$

$$\tilde{\varrho} \partial_t \mathbf{U} + \tilde{\varrho} (\mathbf{U} \cdot \nabla_x) \mathbf{U} + \tilde{\varrho} \nabla_x \Pi = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U}) - \tilde{\varrho} \mathcal{T} \nabla_x F, \tag{147}$$

and

$$\partial_t \mathcal{T} + \mathbf{U} \cdot \nabla_x \mathcal{T} = 0, \tag{148}$$

where

$$\frac{\Theta - 1}{\varepsilon^2} \approx \mathcal{T} \text{ for } \varepsilon \to 0,$$

supplemented with the complete slip boundary conditions

$$\mathbf{U} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ (\mathbb{S}(\nabla_x \mathbf{U}) \cdot \mathbf{n}) \times \mathbf{n}|_{\partial\Omega} = 0. \tag{149}$$

6.2.2 Asymptotic Limit

Consider the following distribution of the initial data

$$\left\{ \begin{array}{c} \varrho(0,\cdot) = \varrho_{0,\varepsilon} = \tilde{\varrho} + \varepsilon \varrho^{(1)}_{0,\varepsilon}, \\ \mathbf{u}(0,\cdot) = \mathbf{u}_{0,\varepsilon}, \\ \Theta(0,\varepsilon) = \Theta_{0,\varepsilon} = 1 + \varepsilon^2 \Theta^{(2)}_{0,\varepsilon}. \end{array} \right\} \tag{150}$$

The asymptotic limit is characterized in the following assertion, see [39, Theorem 2.2]:

Theorem 15. *Let* $\Omega = \mathbb{R}^2 \times (0,1)$ *be an infinite slab and* $\gamma > 3$*. Let* $[\varrho_\varepsilon, \mathbf{u}_\varepsilon, \Theta_\varepsilon]_{\varepsilon>0}$ *be family of weak solutions of the primitive system (139), (140), (141), (142), (144), on a time interval* $(0,T)$*, with the initial data (150), where*

$$\|\varrho^{(1)}_{0,\varepsilon}\|_{L^1\cap L^\infty(\Omega)} \le c \text{ uniformly for } \varepsilon \to 0,$$

$$\Theta^{(2)}_{0,\varepsilon} \to \mathcal{T}_0 \text{ weakly-(*) in } L^1 \cap L^\infty(\Omega),$$

$$\mathbf{u}_{0,\varepsilon} \to \mathbf{u}_0 \text{ weakly-(*) in } L^2 \cap L^\infty(\Omega; R^3)$$

as $\varepsilon \to 0$.

Then, passing to a suitable subsequence as the case may be, we have

$$\sup_{t\in(0,T)} \|\varrho_\varepsilon - \tilde{\varrho}\|_{L^2(\Omega)} \to 0,$$

$$\mathbf{u}_\varepsilon \to \mathbf{U} \text{ weakly in } L^2(0,T; W^{1,2}(\Omega; R^3)) \text{ and (strongly) in } L^2_{\mathrm{loc}}([0,T] \times \overline{\Omega}; R^3),$$

$$\frac{\Theta_\varepsilon - 1}{\varepsilon^2} \to \mathcal{T} \text{ weakly-(*) in } L^\infty((0,T)\times\Omega),$$

where $[\mathbf{U}, \mathcal{T}]$ *is a weak solution to the target system (146), (147), (148), and (149), with the initial data*

$$\tilde{\varrho}\mathbf{U}(0,\cdot) = H_{\tilde{\varrho}}[\tilde{\varrho}\mathbf{u}_0],\ \mathcal{T}(0,\cdot) = \mathcal{T}_0.$$

The weak → weak approach had been used here. Note that the *existence* of weak solutions for the primitive system was shown in [64]; see also [77]. In contrast with Theorem 14, the velocity field converges strongly (a.a. in $(0,T)\times\Omega$) in Theorem 15. This is because of the dispersion of acoustic waves on the *unbounded* strip Ω.

6.2.3 Multiple Singular Limits for Well-Prepared Data

The restriction $\gamma > 3$ in Theorems 14 and 15 is purely technical and a nuisance from the point of view of possible applications. It can be relaxed at least in the case of well-prepared data. To see this, we consider a slightly modified primitive system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{151}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2} \nabla_x (\varrho \Theta)^\gamma = \nu \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) + \frac{1}{\varepsilon^2} \varrho \nabla_x F \tag{152}$$

$$\partial_t(\varrho \Theta) + \mathrm{div}_x(\varrho \Theta \mathbf{u}) = 0 \tag{153}$$

in a "periodic" slab

$$\Omega = \left([0,1]|_{\{0,1\}}\right)^2 \times (0,1).$$

The goal is to perform the *triple* limit

$$\mathrm{Ma} = \mathrm{Fr} = \varepsilon \to 0, \ \mathrm{Re} = \frac{1}{\nu} \to \infty.$$

Accordingly, the target system reads

$$\mathrm{div}_x(\tilde{\varrho} \mathbf{U}) = 0, \tag{154}$$

$$\partial_t \mathbf{U} + (\mathbf{U} \cdot \nabla_x) \mathbf{U} + \nabla_x \Pi = -\mathcal{T} \nabla_x F, \tag{155}$$

$$\partial_t \mathcal{T} + \mathbf{U} \cdot \nabla_x \mathcal{T} = 0. \tag{156}$$

In view of the global existence result by Oliver [79, Theorem 3] for the 2D system, it is plausible to anticipate the existence of local-in-time strong solutions $[\mathbf{U}, \mathcal{T}, \Pi]$ defined on a maximal time interval $[0, T_{\max})$ also in the 3D case:

$$\mathbf{U} \in C([0, T_{\max}); W^{m,2}(\Omega; R^3)), \ \mathcal{T} \in C([0, T_{\max}); W^{m,2}(\Omega)), \ m \geq 3 \tag{157}$$

provided

$$\mathbf{U}(0,\cdot) = \mathbf{U}_0 \in W^{m,2}(\Omega; R^3), \ \mathrm{div}_x(\tilde{\varrho} \mathbf{U}_0) = 0,$$
$$\mathcal{T}(0,\cdot) = \mathcal{T}_0 \in W^{m,2}(\Omega).$$

The following result is of the type weak $\to$ strong, its proof leans on a variant of the relative energy inequality similar to (33), see [39, Theorem 2.1]:

Theorem 16. *Let* $\Omega = \left([0,1]|_{\{0,1\}}\right)^2 \times (0,1)$ *and* $\gamma > \frac{3}{2}$. *Let* $[\varrho_\varepsilon, \mathbf{u}_\varepsilon, \Theta_\varepsilon]$ *be a weak solution of system (151), (152), (153), (144) on a time interval* $(0, T)$*, with the initial data (150) such that*

$$\|\varrho^{(1)}_{0,\varepsilon}\|_{L^\infty(\Omega)} + \|\Theta^{(2)}_{0,\varepsilon}\|_{L^\infty(\Omega)} + \|\mathbf{u}_{0,\varepsilon}\|_{L^\infty(\Omega;R^3)} \leq D.$$

Suppose that the target system (154), (155), and (156) admits a smooth solution $[\mathbf{U}, \mathcal{T}]$ *on the same time interval* $[0, T]$ *emanating from the initial data*

$$\mathbf{U}(0, \cdot) = \mathbf{U}_0, \ \mathcal{T}(0, \cdot) = \mathcal{T}_0.$$

Then

$$\sup_{t \in [0,T]} \int_\Omega \left[\varrho_\varepsilon |\mathbf{u}_\varepsilon - \mathbf{U}|^2 + \left| \frac{\varrho_\varepsilon - \tilde{\varrho}}{\varepsilon} \right|^\gamma + \varrho_\varepsilon \left| \frac{\Theta_\varepsilon - 1}{\varepsilon^2} - \mathcal{T} \right|^2 \right] \mathrm{d}x \tag{158}$$

$$\leq c(T, D) \left[\varepsilon + \nu + \int_\Omega |\mathbf{u}_{0,\varepsilon} - \mathbf{U}_0|^2 + \left| \varrho_{0,\varepsilon}^{(1)} \right|^2 + \left| \Theta_{0,\varepsilon}^{(2)} - \mathcal{T}_0 \right|^2 \mathrm{d}x \right],$$

where the constant $c = c(T, D)$ *depends on the norm of the limit solution* $[\mathbf{U}, \mathcal{T}]$ *and on the size* D *of the initial data perturbation.*

Theorem 16 implies convergence toward the target system provided the right-hand side of (158) vanishes in the asymptotic limit, in particular, the initial data for the primitive system must be *well prepared*.

6.3 Another Singular Limit Arising in Astrophysics

An example of the flow dynamics in stellar radiative zones representing a major challenge of the current theory of stellar interiors is presented. Under these circumstances, the fluid is a plasma characterized by the following specific features:

- A strong *radiative transport* predominates the molecular one. This is due to extremely hot and energetic radiation fields prevailing the plasma. The Péclet number is therefore expected to be extremely *small*.
- Strong *stratification effects* because of the enormous gravitational potential of gaseous celestial bodies determine many of the properties of the fluid in the large.
- The convective motions are much slower than the speed of sound yielding the Mach number small. The fluid is therefore almost *incompressible*, whereas the density variations can be simulated via the anelastic approximation (see also Gough [47], Gilman and Glatzmaier [45, 46]).

6.3.1 Scaling and the Primitive System

A conveniently scaled Navier-Stokes-Fourier system (73)–(77) as the primitive system is considered. In addition to the *geometric scaling* considered so far in this chapter, we perform also *constitutive scaling* reflecting the material properties of the fluid. The fluid will occupy an infinite slab bounded above and below by two parallel plates:

$$\Omega = \mathcal{T}^2 \times (0, 1),$$

where

$$\mathcal{T}^2 = \Big([0,1]|_{\{0,1\}}\Big)^2 \text{ is the two-dimensional flat torus,}$$

on the boundary of which the fluid velocity satisfies the complete slip boundary conditions

$$\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = 0,\ (\mathbb{S}(\vartheta, \nabla_x \mathbf{u})\cdot\mathbf{n})\times\mathbf{n}|_{\partial\Omega} = 0. \tag{159}$$

The bottom part of the boundary is thermally insulated, meaning,

$$\mathbf{q}(\vartheta, \nabla_x\vartheta)\cdot\mathbf{n}|_{\{x_3=0\}} = 0, \tag{160}$$

while a radiative condition

$$\mathbf{q}(\vartheta, \nabla_x\vartheta)\cdot\mathbf{n} = \beta\vartheta(\vartheta - \overline{\vartheta})|_{\{x_3=1\}} \tag{161}$$

will be imposed on the upper part of the boundary.

Starting from the general constitutive relations (44)–(52) and keeping in mind the characteristic features of the underlying physical system discussed above, we suppose:

- the characteristic temperature of the system is large, specifically of order $\varepsilon^{-2\alpha/3}$, where ε is a small positive parameter, and $2 < \alpha < 3$;
- the radiative constant satisfy $a \approx \varepsilon^{2\alpha+1}$;
- the characteristic velocity is of order $\varepsilon^{1-\alpha/3}$, the characteristic length of order $\varepsilon^{-1-\alpha/3}$, the reference time is of order ε^{-2} so that the Strouhal number Sr equals 1;
- the gravitational constant g is of order $\varepsilon^{1-\alpha/3}$;
- $\beta \approx \varepsilon^{\alpha/3}$.

Under these circumstances, the primitive Navier-Stokes-Fourier system reads:

$$\partial_t\varrho + \mathrm{div}_x(\varrho\mathbf{u}) = 0; \tag{162}$$

$$\partial_t(\varrho\mathbf{u}) + \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) + \frac{1}{\varepsilon^2}\nabla_x p_\varepsilon(\varrho,\vartheta) = \mathrm{div}_x\mathbb{S}_\varepsilon(\vartheta,\nabla_x\mathbf{u}) - \frac{1}{\varepsilon^2}\varrho g\mathbf{j},\ \mathbf{j} = [0,0,1]; \tag{163}$$

$$\partial_t(\varrho s_\varepsilon(\varrho,\vartheta)) + \mathrm{div}_x(\varrho s_\varepsilon(\varrho,\vartheta)\mathbf{u}) + \frac{1}{\varepsilon^2}\mathrm{div}_x\Big(\frac{\mathbf{q}_\varepsilon(\vartheta,\nabla_x\vartheta)}{\vartheta}\Big) = \sigma_\varepsilon, \tag{164}$$

with

$$\sigma_\varepsilon = (\geq)\frac{1}{\vartheta}\left(\varepsilon^2\mathbb{S}_\varepsilon(\vartheta,\nabla_x\mathbf{u}) : \nabla_x\mathbf{u} - \frac{1}{\varepsilon^2}\frac{\mathbf{q}_\varepsilon(\vartheta,\nabla_x\vartheta)\cdot\nabla_x\vartheta}{\vartheta}\right),$$

where the inequality sign is being used in the weak formulation (42) and (43).

The relevant constitutive equations are:

$$\mathbb{S}_\varepsilon(\vartheta,\nabla_x\mathbf{u}) = (\varepsilon^{2\alpha/3}\mu_0 + \mu_1\vartheta)\left(\nabla_x\mathbf{u} + \nabla_x^\perp\mathbf{u} - \frac{2}{3}\mathrm{div}_x\mathbf{u}\mathbb{I}\right),\ \mu_0,\ \mu_1 > 0, \tag{165}$$

$$\mathbf{q}_\varepsilon(\vartheta,\nabla_x\vartheta) = -\left(\varepsilon^{2+2\alpha/3}\kappa_0 + \varepsilon^2\kappa_1\vartheta + d\vartheta^3\right)\nabla_x\vartheta,\ \kappa_0,\ \kappa_1 > 0, \tag{166}$$

together with

$$p_\varepsilon(\varrho,\vartheta) = \frac{\vartheta^{\frac{5}{2}}}{\varepsilon^\alpha}P\left(\varepsilon^\alpha\frac{\varrho}{\vartheta^{\frac{3}{2}}}\right) + \varepsilon\frac{a}{3}\vartheta^4, \tag{167}$$

$$e_\varepsilon(\varrho,\vartheta) = \frac{3}{2\varrho}\frac{\vartheta^{\frac{5}{2}}}{\varepsilon^\alpha}P\left(\varepsilon^\alpha\frac{\varrho}{\vartheta^{\frac{3}{2}}}\right) + \varepsilon a\frac{\vartheta^4}{\varrho}, \tag{168}$$

$$s_\varepsilon(\varrho,\vartheta) = S\left(\varepsilon^\alpha\frac{\varrho}{\vartheta^{\frac{3}{2}}}\right) - S(\varepsilon^\alpha) + \varepsilon\frac{4a}{3}\frac{\vartheta^3}{\varrho}, \tag{169}$$

where $a > 0$ is fixed, and where P and S are the same as in (44)–(50).

Under the scaling imposed through (167),

$$p_\varepsilon(\varrho,\vartheta) \to p_0\varrho\vartheta \text{ as } \varepsilon \to 0;$$

whence the limit static density $\tilde{\varrho}$ obeys

$$p_0\overline{\vartheta}\nabla_x\tilde{\varrho} + \tilde{\varrho}g\mathbf{j} = 0 \text{ in } \Omega,\ \text{with a prescribed total mass } \int_\Omega \tilde{\varrho}\ \mathrm{d}x = M_0. \tag{170}$$

The initial data are therefore taken in the form

$$\varrho(0,\cdot) = \varrho_{0,\varepsilon} = \tilde{\varrho} + \varepsilon\varrho^{(1)}_{0,\varepsilon},\ \mathbf{u}(0,\cdot) = \mathbf{u}_{0,\varepsilon},\ \vartheta(0,\cdot) = \overline{\vartheta} + \varepsilon\vartheta^{(1)}_{0,\varepsilon}. \tag{171}$$

6.3.2 Target System

The target system reads:

$$\mathrm{div}_x(\tilde{\varrho}\mathbf{U}) = 0; \tag{172}$$

$$\partial_t(\tilde{\varrho}\mathbf{U}) + \mathrm{div}_x(\tilde{\varrho}\mathbf{U}\otimes\mathbf{U}) + \tilde{\varrho}\nabla_x\Pi = \mu_1\overline{\vartheta}\Delta\mathbf{U} + \frac{1}{3}\mu_1\overline{\vartheta}\nabla_x\mathrm{div}_x\mathbf{U} + \frac{\vartheta^{(2)}}{\overline{\vartheta}}\tilde{\varrho}g\mathbf{j}, \tag{173}$$

where $\mathbf{U}$ satisfies the complete slip boundary conditions

$$\mathbf{U} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ \left[\mu_1 \overline{\vartheta}\left(\nabla_x \mathbf{U} + \nabla_x^T \mathbf{U}\right)\mathbf{n}\right] \times \mathbf{n}|_{\partial\Omega} = 0, \tag{174}$$

and $\vartheta^{(2)}$ is related to the third component of the velocity through

$$\tilde{\varrho} g U_3 = d\overline{\vartheta}^3 \Delta\vartheta^{(2)} \text{ in } \Omega, \ \nabla_x \vartheta^{(2)} \cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{175}$$

6.3.3 Asymptotic Limit

The following results is proved in [27, Chapter 6, Theorem 6.1]:

Theorem 17. *Let* $\Omega = \mathcal{T}^2 \times (0,1)$. *Suppose that* P *and* S *are as in hypotheses (44)–(50). Let* $[\varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon]_{\varepsilon>0}$ *be a family of weak solutions to the Navier-Stokes-Fourier system (162)–(169) on* $(0,T) \times \Omega$, *with the parameter* $2 < \alpha < 3$, *supplemented with the boundary conditions (159)–(161), and the initial conditions (171), where,*

$$\int_\Omega \varrho_{0,\varepsilon}^{(1)} \, \mathrm{d}x = \int_\Omega \vartheta_{0,\varepsilon}^{(1)} \, \mathrm{d}x = 0,$$

and

$$\left\{ \begin{array}{c} \{\varrho_{0,\varepsilon}^{(1)}\}_{\varepsilon>0}, \ \{\vartheta_{0,\varepsilon}^{(1)}\}_{\varepsilon>0} \textit{ are bounded in } L^\infty(\Omega), \\ \mathbf{u}_{0,\varepsilon} \to \mathbf{u}_0 \textit{ weakly-(*) in } L^\infty(\Omega; R^3). \end{array} \right\}$$

Then, at least for suitable subsequences, we have

$$\varrho_\varepsilon \to \tilde{\varrho} \textit{ in } C([0,T]; L^q(\Omega)) \textit{ for any } 1 \le q < \frac{5}{3},$$

$$\mathbf{u}_\varepsilon \to \mathbf{U} \textit{ weakly in } L^2(0,T; W^{1,2}(\Omega; R^3)),$$

$$\vartheta_\varepsilon \to \overline{\vartheta} \textit{ in } L^2(0,T; W^{1,2}(\Omega)),$$

and

$$\nabla_x \left(\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon^2} \right) \to \nabla_x \vartheta^{(2)} \textit{ weakly in } L^1(0,T; L^1(\Omega; R^3)),$$

where $\tilde{\varrho}$, $\overline{\vartheta}$, $\mathbf{U}$, $\vartheta^{(2)}$ *is a weak solution to problem (172)–(175), supplemented with the initial condition*

$$\tilde{\varrho}\mathbf{U}_0 = H_{\tilde{\varrho}}[\tilde{\varrho}\mathbf{u}_0].$$

7 Rotating Fluids

Rotating fluid systems appear in many applications of fluid mechanics, in particular in models of atmospheric and geophysical flows, see the monograph [13]. Earth's rotation, together with the influence of gravity and the fact that atmospheric Mach number is typically very small, give rise to a large variety of singular limit problems, where some of these characteristic numbers become large or tend to zero, see Klein [58]. Here, a simple situation of a compressible fluid, where the Rossby number Ro is proportional to small parameter ε.

For the sake of simplicity, the effects of the boundary will be ignored, in particular the Ekman layers created by a particular choice of boundary conditions, see, e.g., [71]. Accordingly, the spatial domain Ω will be always the infinite slab

$$\Omega = R^2 \times (0,1), \tag{176}$$

the rotation axis ω |parallel to $[0,0,1]$, and the (relative) velocity field $\mathbf{u}$ will satisfy the complete slip boundary conditions

$$\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = 0,\ (\mathbb{S}(\nabla_x\mathbf{u})\cdot\mathbf{n})\times\mathbf{n}|_{\partial\Omega} = 0. \tag{177}$$

7.1 A Model Problem: Rapidly Rotating Compressible Barotropic Fluid

The influence of the temperature being neglected, the scaled Navier-Stokes system is considered

$$\partial_t\varrho + \mathrm{div}_x(\varrho\mathbf{u}) = 0, \tag{178}$$

$$\partial_t(\varrho\mathbf{u}) + \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) + \frac{1}{\varepsilon}(\omega\times\varrho\mathbf{u}) + \frac{1}{\varepsilon^{2m}}\nabla_x p(\varrho) = \mathrm{div}_x\mathbb{S}(\nabla_x\mathbf{u}) + \frac{1}{\varepsilon^2}\varrho\nabla_x G, \tag{179}$$

with

$$\mathbb{S}(\nabla_x\mathbf{u}) = \mu\Big(\nabla_x\mathbf{u} + \nabla_x^t\mathbf{u} - \frac{2}{3}\mathrm{div}_x\mathbf{u}\mathbb{I}\Big),\ \mu > 0, \tag{180}$$

where a possible influence of the bulk viscosity component is ignored. Unlike its incompressible counterpart, where the effect of the centrifugal force can be included in the pressure, the compressible system should contain a gradient-like driving term corresponding to the centrifugal force, here represented by $\frac{1}{\varepsilon^2}\varrho\nabla_x G$,

$$G = |x\times\omega|^2 \approx |x_h|^2,\ x_h = [x_1, x_2].$$

In particular, the centrifugal force becomes exceedingly large for $|x| \to \infty$ unless its strength is moderated by an adequate decay of the density ϱ.

As shown in [13], *incompressible* rotating fluids stabilize to a 2D motion described by the vertical averages of the velocity provided the Rossby number ε is small enough. Note that the stabilizing effect of rotation has been exploited by several authors; see, e.g., [5, 6]. On the other hand, as observed in Theorem 6, compressible fluid flows in the low Mach number regime behave like the incompressible ones. Thus, (at least for $m \gg 1$), solutions of the scaled system (178) and (179) are first rapidly driven to incompressibility and then stabilize to a purely horizontal motion as $\varepsilon \to 0$. On the other hand, the abovementioned scenario changes completely in the critical case $m = 1$.

7.1.1 Dominating Incompressibility $m \gg 1$

Suppose that

$$p \in C^1[0,\infty) \cap C^2(0,\infty),\ p'(\varrho) > 0 \text{ for all } \varrho > 0,\ \lim_{\varrho\to\infty} \frac{p'(\varrho)}{\varrho^{\gamma-1}} = p_\infty > 0 \tag{181}$$

for a certain $\gamma > 1$ specified below, and that

$$\left\{ \begin{array}{c} G \in W^{1,\infty}(\Omega),\ G(x) \geq 0, \\ |\nabla_x G(x)| \leq c(1 + |x_h|) \text{ for all } x \in \Omega,\ x_h = [x_1, x_2]. \end{array} \right\} \tag{182}$$

The following result holds for $m >> 1$, see [35, Theorem 1]:

Theorem 18. *Let the pressure p and the potential of the driving force G satisfy hypotheses (181), (182), with $\gamma > 3/2$. Let $[\varrho_\varepsilon, \mathbf{u}_\varepsilon]$ be a finite energy weak solution of the Navier-Stokes system (178)–(180), (177) in $(0,T) \times \Omega$, emanating from the initial data*

$$\varrho_\varepsilon(0,\cdot) = \tilde{\varrho}_\varepsilon + \varepsilon^m \varrho_\varepsilon^{(1)},\ \mathbf{u}_\varepsilon(0,\cdot) = \mathbf{u}_{0,\varepsilon}, \tag{183}$$

where

$$\nabla_x p(\tilde{\varrho}_\varepsilon) = \varepsilon^{2(m-1)} \tilde{\varrho}_\varepsilon \nabla_x G,\ \tilde{\varrho}_\varepsilon \to 1 \text{ as } |x| \to \infty, \tag{184}$$

and

$$\begin{aligned} &\|\varrho_{0,\varepsilon}^{(1)}\|_{L^2 \cap L^\infty(\Omega)} + \|\sqrt{\tilde{\varrho}_\varepsilon} \mathbf{u}_{0,\varepsilon}\|_{L^2(\Omega;R^3)} \leq c,\ \left\| \tilde{\varrho}_\varepsilon^{\frac{\gamma-2}{2}} \varrho_{0,\varepsilon}^{(1)} \right\|_{L^2(\Omega)} \leq c \text{ if } \gamma > 2, \\ &\mathbf{u}_{0,\varepsilon} \to \mathbf{U}_0 \text{ weakly in } L^2(\Omega; R^3). \end{aligned} \tag{185}$$

Finally, suppose that

$$m > 10$$

Then

$$\text{ess}\sup_{t\in(0,T)} \|\varrho_\varepsilon - 1\|_{(L^2+L^\gamma)(K)} \leq \varepsilon^m c(K) \text{ for any compact } K \subset \Omega,$$

$$\mathbf{u}_\varepsilon \to \mathbf{U} \text{ weakly in } L^2(0,T; W^{1,2}(\Omega; R^3)),$$

where $\mathbf{U} = [\mathbf{U}_h(x_h), 0]$ *is the unique solution to the 2D incompressible Navier-Stokes system*

$$\text{div}_h \mathbf{U}_h = 0,$$

$$\partial_t \mathbf{U}_h + \text{div}_h(\mathbf{U}_h \otimes \mathbf{U}_h) + \nabla_h \Pi = \mu \Delta_h \mathbf{U}_h,$$

with the initial data

$$\mathbf{U}_h(0,\cdot) = \left[H\left[[\langle \mathbf{U}_0 \rangle_h, 0] \right] \right]_h, \text{ where } \langle \mathbf{U}_0 \rangle_h (x_h) = \int_0^1 \mathbf{U}_0(x_h, x_3)\, \mathrm{d}x_3.$$

The subscript D_h indicates that the differential operator D acts only on the horizontal variable $[x_1, x_2]$.

7.1.2 Critical Case $m = 1$

The critical case $m = 1$ is qualitatively very different from $m >> 1$, see [35, Theorem 2]:

Theorem 19. *Let the pressure* p *satisfy hypotheses (181), with* $\gamma > 3$ *and let* $G(x_h) = |x_h|^2$. *Let* ϱ_ε, $\mathbf{u}_\varepsilon$ *be a finite energy weak solution of the Navier-Stokes system in* $(0,T) \times \Omega$, *emanating from the initial data (183)–(185), with*

$$m = 1.$$

In addition, suppose that

$$\varrho^{(1)}_{0,\varepsilon} \to r_0 \text{ weakly in } L^2(\Omega),\ \mathbf{u}_{0,\varepsilon} \to \mathbf{U}_0 \text{ weakly in } L^2(\Omega; R^3).$$

Then

$$r_\varepsilon \equiv \frac{\varrho_\varepsilon - \tilde{\varrho}}{\varepsilon} \to r \text{ weakly-(*) in } L^\infty(0,T; L^2(K)) \text{ for any compact } K \subset \Omega,$$

$$\mathbf{u}_\varepsilon \to \mathbf{U} \text{ weakly in } L^2(0,T; W^{1,2}(\Omega; R^3)),$$

with

$$r = r(t, x_h) \text{ radially symmetric }, \quad \mathbf{U} = [\mathbf{U}_h(t, x_h), 0],$$

and

$$\nabla_h(P'(\tilde{\varrho})r) + \mathbf{U}_h^{\perp} = 0.$$

Moreover, the function r satisfies

$$\partial_t\Big(r - \mathrm{div}_h(\tilde{\varrho}\nabla_h(P'(\tilde{\varrho})r))\Big) + \Delta_h^2\Big(P'(\tilde{\varrho})r\Big) = 0.$$

In addition, the initial value $r(0)$ *is the unique radially symmetric function satisfying the integral identity*

$$\int_{R^2}\Big(\tilde{\varrho}\nabla_h\left(P'(\tilde{\varrho})r(0)\right)\cdot\nabla_h\psi \ + \ r(0)\,\psi\Big)\mathrm{d}x_h \ = \ \int_{R^2}\Big(\langle\tilde{\varrho}\mathbf{U}_{0,h}\rangle\cdot\nabla_h^{\perp}\psi + \langle r_0\rangle\,\psi\Big)\mathrm{d}x_h$$

for all radially symmetric $\psi = \psi(x_h) \in C_c^{\infty}(R^2)$.

7.2 Scale Interactions for Rotating Fluids

The singular limit of a rotating compressible fluid described by a scaled barotropic Navier-Stokes system will be considered, where the Rossby number $\mathrm{Ro} = \varepsilon$, the Mach number $\mathrm{Ma} = \varepsilon^m$, the Reynolds number $\mathrm{Re} = \varepsilon^{-\alpha}$, and the *Froude number* $\mathrm{Fr} = \varepsilon^n$ are proportional to a small parameter $\varepsilon \to 0$. The system of equations describes the time evolution of the mass density $\varrho = \varrho(t,x)$ and the (relative) velocity $\mathbf{u} = \mathbf{u}(t,x)$ of a compressible, rotating fluid:

$$\partial_t\varrho + \mathrm{div}_x(\varrho\mathbf{u}) = 0, \tag{186}$$

$$\partial_t(\varrho\mathbf{u}) + \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) + \frac{1}{\varepsilon}\varrho(\omega\times\mathbf{u}) + \frac{1}{\varepsilon^{2m}}\nabla_x p(\varrho) \tag{187}$$

$$= \varepsilon^{\alpha}\mathrm{div}_x\mathbb{S}(\nabla_x\mathbf{u}) + \frac{1}{\varepsilon^{2n}}\varrho\nabla_x G,$$

$$\mathbb{S}(\nabla_x\mathbf{u}) = \mu\left(\nabla_x\mathbf{u} + \nabla_x^t\mathbf{u} - \frac{2}{3}\mathrm{div}_x\mathbf{u}\mathbb{I}\right) + \eta\mathrm{div}_x\mathbf{u}\mathbb{I},\ \mu > 0,\ \eta \geq 0. \tag{188}$$

The fluid is confined to an infinite slab

$$\Omega = R^2 \times (0,1), \tag{189}$$

where it satisfies the *slip condition*

$$\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = (\mathbb{S}(\nabla_x\mathbf{u})\cdot\mathbf{n})\times\mathbf{n}|_{\partial\Omega} = 0. \tag{190}$$

The model may be viewed as a crude approximation (f-plane model) of the Earth's atmosphere in a plane tangent to the Earth at a certain latitude; see [88, Chapter 2, Section 2.3]. Accordingly, the gravitational force is taken parallel to the vertical projection of the rotation axis:

$$\omega/|\omega| = [0,0,1],\ \nabla_x G = [0,0,-1].$$

We consider the singular limit problem for $\varepsilon \to 0$ in the multiscale regime:

$$\frac{m}{2} > n \geq 1,\ \alpha > 0 \tag{191}$$

for the ill-prepared initial data

$$\varrho(0,\cdot) = \varrho_{0,\varepsilon} = \tilde{\varrho}_\varepsilon + \varepsilon^m \varrho^{(1)}_{0,\varepsilon},\ \mathbf{u}(0,\cdot) = \mathbf{u}_0, \tag{192}$$

where $\tilde{\varrho}_\varepsilon$ is a solution to the static problem

$$\nabla_x p(\tilde{\varrho}_\varepsilon) = \varepsilon^{2(m-n)} \tilde{\varrho}_\varepsilon \nabla_x G \text{ in } \Omega. \tag{193}$$

7.2.1 Target Problem

Formally, it is not difficult to identify the limit problem. Indeed fast rotation is expected to eliminate the vertical motion, the vanishing viscosity (high Reynolds number) makes the limit system *inviscid* (hyperbolic), while the low Mach number regime drives the fluid to *incompressibility*. The limit problem is therefore expected to be the incompressible Euler system for the planar velocity field $\mathbf{v} = [v_1, v_2]$,

$$\partial_t \mathbf{v} + \mathrm{div}_x(\mathbf{v} \otimes \mathbf{v}) + \nabla_x \Pi = 0,\ \mathrm{div}_x \mathbf{v} = 0 \text{ in } (0,T) \times R^2. \tag{194}$$

As is well known, see, e.g., [52], the target problem admits *global-in-time* solutions provided the initial data are smooth enough.

7.2.2 Asymptotic Limit

Let $p \in C[0,\infty) \cap C^3(0,\infty)$ be a given function of the density such that

$$p(0) = 0,\ p'(\varrho) > 0 \text{ for all } \varrho > 0,\ \lim_{\varrho \to \infty} \frac{p'(\varrho)}{\varrho^{\gamma-1}} = p_\infty > 0,\ \gamma > \frac{3}{2}. \tag{195}$$

In addition, without loss of generality, the pressure p can be "normalized" by setting

$$p'(1) = 1. \tag{196}$$

The following result was proved in [31, Theorem 3.1]:

Theorem 20. *Let the pressure $p = p(\varrho)$ satisfy the hypotheses (195) and (196). Suppose that the exponents α, m, and n are given such that*

$$\alpha > 0, \ \frac{m}{2} > n \geq 1.$$

Let the initial data $[\varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon}]$ be given by (192), where the stationary states $\tilde{\varrho}_\varepsilon$ satisfy (193),

$$\|\varrho^{(1)}_{0,\varepsilon}\|_{L^2 \cap L^\infty(\Omega)} \leq c, \ \varrho^{(1)}_{0,\varepsilon} \to \varrho^{(1)}_0 \textit{ in } L^2(\Omega), \ \mathbf{u}_{0,\varepsilon} \to \mathbf{u}_0 \textit{ in } L^2(\Omega; R^3),$$

with

$$\varrho^{(1)}_0 \in W^{k-1,2}(\Omega), \ \mathbf{u}_0 \in W^{k,2}(\Omega; R^3) \textit{ for a certain } k \geq 3.$$

Let $[\varrho_\varepsilon, \mathbf{u}_\varepsilon]$ be a finite energy weak solution of the problem (186)–(190) in the space-time cylinder $(0, T) \times \Omega$.

Then

$$\operatorname*{ess\ sup}_{t \in (0,T)} \|\varrho_\varepsilon(t, \cdot) - \tilde{\varrho}_\varepsilon\|_{(L^2 + L^\gamma)(\Omega)} \leq \varepsilon^m c$$

$$\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon \to \mathbf{v} \begin{cases} \textit{weakly-(*) in } L^\infty(0, T; L^2(\Omega; R^3)), \\ \textit{strongly in } L^1_{\mathrm{loc}}((0, T) \times \Omega; R^3), \end{cases}$$

where $\mathbf{v} = [\mathbf{v}_h, 0]$ is the unique solution of the Euler system (194), with the initial data

$$\mathbf{v}(0, \cdot) = H_h \left[\int_0^1 \mathbf{u}_0(x_h, x_3) \, \mathrm{d}x_3 \right].$$

8 Conclusion

The material collected in this section is limited to problems involving the complete fluid systems, meaning those incorporating both the first and second laws of thermodynamics. There is a vast amount of literature devoted to simplified or augmented models, where more aspects of the motion like the effect of a magnetic field and/or chemical reactions are taken into account. The reader should consult the bibliography appended to this chapter as well as the references cited therein.

9 Cross-References

- ▸ Concepts of Solutions in the Thermodynamics of Compressible Fluids
- ▸ Low Mach Number Limits and Acoustic Waves
- ▸ Scale Analysis of Compressible Flows from an Application Perspective
- ▸ Weak Solutions for the Compressible Navier-Stokes Equations: Existence, Stability, and Longtime Behavior

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