Stationary Navier-Stokes Flow in Exterior Domains and Landau Solutions

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Contents

Abstract

Consider the stationary Navier-Stokes flow in 3D exterior domains with zero velocity at infinity. What is of particular interest is the spatial behavior of the flow at infinity, especially optimal decay (summability) observed in general and the asymptotic structure. When the obstacle is translating, the answer is found in some classic literature by Finn; in fact, the optimal summability is L^q with $q > 2$ and the leading profile is the Oseen fundamental solution. This presentation is devoted to the other cases developed in the last decade, mainly the case where the obstacle is at rest, together with several remarks even on the challenging case where the obstacle is rotating. The optimal summability for those cases is $L^{3,\infty}$ (weak-L^3) and the leading term of small solutions being in this class is the homogeneous Navier-Stokes flow of degree (-1), which is called the Landau solution. In any case, the total net force is closely related to the asymptotic structure of the flow. An insight into the homogeneous Navier-Stokes flow of

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degree (-1), due to Šverák, plays an important role. It would be also worthwhile finding a class of the external force, as large as possible, which ensures the asymptotic expansion of the flow at infinity.

1 Introduction

Let Ω be an exterior domain in $\mathbb{R}^3$ occupied by a viscous incompressible fluid, where a compact set $\mathbb{R}^3 \setminus \Omega$ is identified with an obstacle (rigid body) and the boundary $\partial\Omega$ of Ω is assumed to be sufficiently smooth. Given external force $f = (f_1(x), f_2(x), f_3(x))^\top$, the stationary motion of the fluid is described by the velocity $u = (u_1(x), u_2(x), u_3(x))^\top$ and pressure $p = p(x)$ which obey the Navier-Stokes system

$$-\Delta u + \nabla p + (u - \eta - \omega \times x) \cdot \nabla u + \omega \times u = f, \qquad \operatorname{div} u = 0 \qquad \text{in } \Omega \tag{1}$$

subject to the boundary conditions

$$u = \eta + \omega \times x \qquad \text{on } \partial\Omega, \tag{2}$$

$$u \to 0 \qquad \text{as } |x| \to \infty, \tag{3}$$

where (2) is the usual no-slip condition in which the flow attains the rigid motion in the sense of trace, while (3) is understood from the class of solutions, mostly either pointwise or summability. All vectors are throughout column ones and $(\cdot)^\top$ denotes the transpose of vectors or matrices. To understand the Eq. (1) of momentum, one should start with nonstationary Navier-Stokes system in a time-dependent region exterior to a moving body and make a suitable change of variables to reduce the problem to an equivalent one in the reference frame attached to the moving body (see Galdi [26] for the details). The translational velocity η and angular velocity ω are then in general time dependent; however, both are assumed to be constant vectors, and the flow is assumed to be stationary in the reference frame. The main issue one would like to address here is the spatial behavior of solutions at infinity. It turns out that the rate of decay (3) is controlled by the total net force (momentum flux)

$$\begin{aligned} M &= M(\eta, \omega, f) \\ &= \int_{\partial\Omega} [T(u,p) - u \otimes (u - \eta - \omega \times y) - (\omega \times y) \otimes u]\, \nu \, d\sigma_y + \int_\Omega f(y)\, dy \end{aligned} \tag{4}$$

associated with (1), which is written as the divergence form

$$-\operatorname{div} [T(u,p) - u \otimes (u - \eta - \omega \times x) - (\omega \times x) \otimes u] = f,$$

where

$$T(u,p) = \nabla u + (\nabla u)^\top - p\mathbb{I} \tag{5}$$

is the Cauchy stress tensor, $\mathbb{I}$ is the 3×3 identity matrix, and ν stands for the outward unit normal to $\partial\Omega$. One has nonzero force $M \neq 0$ in general; however, if in particular u, p, and ∇u decay sufficiently fast at infinity, then integrating the equation above over $\{x \in \Omega; |x| < \rho\}$ and letting $\rho \to \infty$ yield $M = 0$. This simple observation suggests that M is related to the decay structure of the flow. One may also refer to [47, section 6] in the context of the self-propelled motion of a rigid body.

In his celebrated paper [61], Leray showed the existence of at least one solution in the class of finite Dirichlet integral $\nabla u \in L^2(\Omega)$ to (1), (2) and (3) (when $\eta = \omega = 0$) without any smallness condition on the data. The argument relies on compactness together with a priori estimate arising from structure of the Navier-Stokes system. Note that this structure is kept for the case $\eta + \omega \times x \neq 0$ as well. His theorem thus provides even large solutions, most of which would be unstable. From the viewpoint of stability, solutions of the Leray class do not give us enough information about the asymptotic behavior at infinity. In fact, the only thing one knows is $u \in L^6(\Omega)$; however, mathematical analysis developed so far requires better decay property such as $|u(x)| \leq C|x|^{-1}$ or $u \in L^{3,\infty}(\Omega)$ (as well as smallness) of the stationary flow u to show its stability, where $L^{3,\infty}$ denotes the weak-L^3 space (see [6, 7, 31, 38–40, 44, 46, 49, 55, 63, 65, 70], and the references therein). When $\eta = 0$, the summability $L^{3,\infty}$ of stationary flows observed in general is actually optimal unless assuming any specific condition such as symmetry. As compared with this case, better summability of stationary flows for the case $\eta \in \mathbb{R}^3 \setminus \{0\}$ mentioned in the next paragraph is helpful in the proof of stability of such flows (under smallness conditions); indeed, there is no need to analyze the full linearized operator; in other words, analysis of the Oseen semigroup is enough, while that is not the case when $\eta = 0$ (unless using an interpolation technique due to [76]).

The interest is focused on optimal decay/summability at infinity of the flow together with its asymptotic structure. This was addressed in a series of papers by Finn [22–24], in which the rigid body was assumed to be translating with velocity $\eta \in \mathbb{R}^3 \setminus \{0\}$; however, $\omega = 0$. In this case the essential step is to analyze the asymptotic behavior of solutions at infinity to the Oseen system

$$-\Delta u + \nabla p - \eta \cdot \nabla u = f, \qquad \operatorname{div} u = 0, \qquad \text{in } \Omega. \tag{6}$$

The Oseen fundamental solution possesses anisotropic decay structure with paraboloidal wake region behind the body. In fact, the flow decays faster outside wake than inside, and consequently the summability near infinity is better like $u \in L^q$ with $q > 2$ than the case $\eta = 0$. Finn [23], Farwig [15], and Shibata [70] proved the existence of small Navier-Stokes flow which exhibits the same decay structure with wake as mentioned above; actually, the leading profile of the flow is the Oseen fundamental solution, and its coefficient is given by the force $M(\eta, 0, f)$

(see (4)), provided f is of bounded support. Such a flow was called physically reasonable solution by Finn. Furthermore, Babenko [1], Galdi [25], and Farwig and Sohr [21] showed that any solution of the Leray class without restriction on the magnitude becomes a physically reasonable solution (see Galdi [28, Theorem X.8.1]). This is a contrast to the case $\eta = 0$; indeed, when the translation of the body is absent, one has no result on the asymptotic behavior of large solutions of the Leray class except for Choe and Jin [13], in which some pointwise decay rates of axisymmetric solutions of that class were deduced. Later on, Galdi and Silvestre [33], Galdi and Kyed [29], and Kyed [57–59] generalized the results mentioned above for purely translational regime to the case $\eta \cdot \omega \neq 0$. In fact, the presence of translation still implies fine decay/summability at infinity of the flow past a rotating obstacle except the case where $\omega \in \mathbb{R}^3 \setminus \{0\}$ is orthogonal to η, and, as a consequence, the leading profile of the Navier-Stokes flow is described in terms of the linear part, in which a remarkable role of axis of rotation can be also observed. Such a role was discovered first by Farwig and Hishida [18, 19] for the flow around a purely rotating obstacle, and it will be explained later.

This presentation studies the other case where the translation of the body is absent ($\eta = 0$). In this case the existence of solutions, which decay like

$$|u(x)| \leq C|x|^{-1}, \qquad |\nabla u(x)| \leq C|x|^{-2} \qquad (|x| \to \infty)$$

or

$$u \in L^{3,\infty}(\Omega), \qquad \nabla u \in L^{3/2,\infty}(\Omega),$$

for small external forces was proved by [7, 56, 69] ($\omega = 0$) and by [17, 27] ($\omega \neq 0$). For the case $\omega = 0$, Deuring and Galdi [14] clarified that the leading profile was no longer the Stokes fundamental solution. Indeed, the rate $|x|^{-1}$ of decay yields the balance between the linear part and nonlinearity since $\Delta u \sim u \cdot \nabla u \sim |x|^{-3}$ (formally). This observation would suggest that a sort of nonlinear effect is involved in the leading term of the flow. Nazarov and Pileckas [68] first derived asymptotic expansion under smallness conditions, but the leading term was less explicit. It was explicitly found much later by Korolev and Šverák [50] (case $\omega = 0$). When the nonlinearity is balanced with the linear part, it is reasonable to expect that the self-similar solution would be a candidate of the leading term. Since the stationary Navier-Stokes system ($\eta = \omega = 0$) is invariant under the scale transformation

$$u_\lambda(x) = \lambda u(\lambda x), \qquad p_\lambda(x) = \lambda^2 p(\lambda x), \qquad \lambda > 0, \tag{7}$$

a smooth solution $\{u, p\}$ to

$$-\Delta u + \nabla p + u \cdot \nabla u = 0, \qquad \operatorname{div} u = 0 \qquad \text{in } \mathbb{R}^3 \setminus \{0\} \tag{8}$$

is called (stationary) self-similar solution if

$$u_\lambda(x) = u(x), \qquad p_\lambda(x) = p(x), \qquad \forall \lambda > 0 \ \forall x \in \mathbb{R}^3 \setminus \{0\}$$

or, equivalently,

$$u(x) = \frac{1}{|x|} u\left(\frac{x}{|x|}\right), \qquad p(x) = \frac{1}{|x|^2} p\left(\frac{x}{|x|}\right), \qquad \forall x \in \mathbb{R}^3 \setminus \{0\}, \tag{9}$$

that is, u and p are homogeneous of degree (-1) and (-2), respectively. Landau [60] derived its exact form under the assumption of axisymmetry (see (78)), in order to describe jets from a thin pipe (see also Tian and Xin [75] and Cannone and Karch [9]). Finally, Šverák [74] has characterized completely the set $\mathcal{S}$ of all self-similar solutions as follows: $\mathcal{S}$ is parameterized by vectorial parameter as

$$\mathcal{S} = \left\{ \{U_b, P_b\};\ b \in \mathbb{R}^3 \right\} \tag{10}$$

whose member $\{U_b, P_b\}$ is symmetric about the axis $\mathbb{R}b$ and satisfies

$$-\Delta U_b + \nabla P_b + U_b \cdot \nabla U_b = b\delta, \qquad \mathrm{div}\, U_b = 0 \qquad \text{in } \mathcal{D}'(\mathbb{R}^3) \tag{11}$$

across the origin (see also [2, 9]), where δ denotes the Dirac measure. In other words, every self-similar solution must have its own axis of symmetry, and the set $\mathcal{S}$ eventually agrees with the family of solutions computed by Landau. This is the reason why the self-similar solution is often called the Landau solution. The proof of Šverák [74] is closely related to geometric properties of $\mathbb{S}^2$ (unit sphere). Based on his profound insight, he and Korolev [50] proved that the leading term of asymptotic expansion of solutions to (1) with $(\eta, \omega, f) = (0, 0, 0)$ (without assuming (2)), which decay like $|x|^{-1}$ at infinity, is given by the specific Landau solution U_M with label $M = M(0, 0, 0)$ (see (4)), provided $\limsup_{|x|\to\infty} |x||u(x)|$ is small enough, where the error term satisfies the pointwise estimate like $|x|^{-2+\varepsilon}$ for $\varepsilon > 0$ arbitrarily small (but the smallness of u depends on ε). The result was extended to small time-periodic solutions (case $\eta = \omega = 0$) with period $\mathcal{T} > 0$ by Kang, Miura, and Tsai [48], where the leading term is the Landau solution U_b with label $b = \frac{1}{\mathcal{T}} \int_0^{\mathcal{T}} M$, that is, the time average of the force (4). Note that the Landau solution must be useful to describe the local behavior related to singularity/regularity as well; indeed, it was proved by Miura and Tsai [64] that the leading term of point singularity like $|x|^{-1}$ at $x = 0$ of the Navier-Stokes flow is also given by a Landau solution provided it is small enough.

Later on, Farwig and Hishida [19] studied the case where the body is purely rotating with angular velocity $\omega \in \mathbb{R}^3 \setminus \{0\}$ and proved that the leading term of solutions to (1) (with $\eta = 0$, $f = 0$), which are small in $L^{3,\infty}(\Omega)$, is another Landau solution U_b with label $b = (\frac{\omega}{|\omega|} \cdot M)\frac{\omega}{|\omega|}$, whose axis of symmetry is parallel to the axis of rotation along which the flow is largely concentrated, where $M = M(0, \omega, 0)$ (see (4)). The solution enjoys better summability if and only if $\omega \cdot M = 0$, while so does the solution for the case where the body is at rest if and only if the full force M vanishes. Thus, one is able to find out the effect of rotation, which was not clear until the study of [18, 19]. They considered solutions in $L^{3,\infty}(\Omega)$ rather than pointwise decay properties, and the error term was estimated in terms of summability. Their result was then refined by Farwig, Galdi, and Kyed [16] in the sense that the asymptotic expansion with error term satisfying a pointwise estimate (as in [50]) was deduced even for solutions of the Leray class with the energy inequality under appropriate smallness of ω.

Most part of this presentation is devoted to the case $\omega = 0$, but key points for the purely rotating case $\omega \in \mathbb{R}^3 \setminus \{0\}$ and the remarkable difference between those cases are also explained. One specifies a class of the external force f, which ensures the asymptotic expansion of the Navier-Stokes flow $u \in L^{3,\infty}(\Omega)$ as long as it is small enough. Since the class of f is rather large, it seems difficult to deduce pointwise estimates of the error term; instead, it is estimated in terms of summability as in [19]. Before the analysis of the Navier-Stokes flow, it is also worthwhile showing the asymptotic expansion of the Stokes flow with general external force as above (see Theorem 1).

This presentation consists of six sections. After some preliminaries in the next section, asymptotic structure for the linearized system is studied in sect. 3. At the end of sect. 3, a few crucial facts which interpret why the axis of rotation is preferred for the case $\omega \in \mathbb{R}^3 \setminus \{0\}$ are mentioned. In sect. 4 the existence theorem (Theorem 2) for small solutions in $L^{3,\infty}(\Omega)$ is provided. It is based on the linear theory (see Theorem 3). The result is due to Kozono and Yamazaki [56], but one carries out linear analysis in a different way, which can be applied to the other cases $\eta + \omega \times x \neq 0$ (see [17, 71]). The asymptotic expansion of the Navier-Stokes flow obtained in Theorem 2 is studied in sect. 5 (see Theorem 4), in which one needs a bit more decay property of the external force than assumed in Theorem 2. The final section summarizes what is done and raises several open questions about the related issues. Although this is a survey article, the complete proof of Theorems 1, 3, and 4 will be presented.

2 Preliminaries

In this section some function spaces are introduced and notation is fixed. Let B_ρ be the open ball in $\mathbb{R}^3$ centered at the origin with radius $\rho > 0$. For sufficiently large $\rho > 0$, we set $\Omega_\rho = \Omega \cap B_\rho$, where Ω is the exterior domain under consideration.

Let D be a smooth domain in $\mathbb{R}^3$, such as the exterior domain Ω, whole space $\mathbb{R}^3$, or a bounded domain. By $C_0^\infty(D)$ one denotes the class of smooth functions with compact support in D. For $1 \leq q \leq \infty$ the usual Lebesgue spaces are denoted by $L^q(D)$ with norm $\|\cdot\|_{q,D}$. To introduce the Lorentz space (for details, see Bergh and Löfström [3]), given measurable function f on D, set

$$m_f(t) := |\{x \in D;\ |f(x)| > t\}|, \qquad t > 0,$$

where $|\cdot|$ stands for the Lebesgue measure. Then $m_f(\cdot)$ is monotonically non-increasing, right continuous, and measurable. It is well known that $f \in L^q(D)$, $1 \le q < \infty$, if and only if

$$\int_0^\infty \{t\, m_f(t)^{1/q}\}^q \frac{dt}{t} < \infty.$$

With this in mind, one denotes by $L^{q,r}(D)$ the vector space consisting of all measurable functions f on D which satisfy

$$\begin{aligned} &\left(\int_0^\infty \{t\, m_f(t)^{1/q}\}^r \frac{dt}{t}\right)^{1/r} < \infty \qquad \text{if } 1 \le r < \infty, \\ &\sup_{t>0} t\, m_f(t)^{1/q} < \infty \qquad \text{if } r = \infty. \end{aligned} \tag{12}$$

Note that

$$L^{q,r_0}(D) \subset L^{q,r_1}(D) \quad \text{if } r_0 \le r_1; \quad L^{q,q}(D) = L^q(D),$$

the latter of which is obvious as mentioned above. Each of finite quantities (12) is a quasi-norm; however, by the use of the average function, it is possible to introduce a norm $\|\cdot\|_{q,r,D}$, which is equivalent to that, unless $q = 1$ (see [3]). Then $L^{q,r}(D)$ equipped with $\|\cdot\|_{q,r,D}$ ($1 < q < \infty$, $1 \le r \le \infty$) is a Banach space, called the Lorentz space; in particular, $L^{q,\infty}(D)$ is well known as the weak-L^q space, in which $C_0^\infty(D)$ is not dense. As a typical function in this space, recall that $|x|^{-\alpha} \in L^{3/\alpha,\infty}(\mathbb{R}^3)$ as long as $0 < \alpha \le 3$. One also has the weak Hölder inequality ([7, Lemma 2.1]): let $1 < p \le \infty$, $1 < q < \infty$ and $1 < r < \infty$ satisfy $1/r = 1/p + 1/q$, and let $f \in L^{p,\infty}(D)$, $g \in L^{q,\infty}(D)$, then $fg \in L^{r,\infty}(D)$ with

$$\|fg\|_{r,\infty,D} \le \|f\|_{p,\infty,D} \|g\|_{q,\infty,D} \tag{13}$$

where $L^{\infty,\infty}(D) = L^\infty(D)$.

Let $1 < q < \infty$ and $1 \le r \le \infty$. The Lorentz spaces can be also constructed via real interpolation

$$L^{q,r}(D) = \big(L^1(D), L^\infty(D)\big)_{1-1/q,r}. \tag{14}$$

This together with the reiteration theorem in the interpolation theory ([3, 5.3.1]) implies that

$$L^{q,r}(D) = \big(L^{q_0,r_0}(D), L^{q_1,r_1}(D)\big)_{\theta,r} \tag{15}$$

provided

$$1 < q_0 < q < q_1 < \infty, \quad 1/q = (1-\theta)/q_0 + \theta/q_1, \quad 1 \le r_0, r_1, r \le \infty.$$

Then one knows

$$\|f\|_{q,r,D} \le C\|f\|_{q_0,r_0,D}^{1-\theta}\|f\|_{q_1,r_1,D}^{\theta} \tag{16}$$

for all $f \in L^{q_0,r_0}(D) \cap L^{q_1,r_1}(D) \subset L^{q,r}(D)$. For fixed $f \in L^{p,\infty}(D)$, the map $g \mapsto fg$ is bounded from $L^{q,\infty}(D)$ to $L^{r,\infty}(D)$ by (13), where $1 < p \le \infty$, $1 < q < \infty$, and $1 < r < \infty$ satisfy $1/r = 1/p + 1/q$. Hence, the interpolation (15) leads to

$$\|fg\|_{r,s,D} \le \|f\|_{p,\infty,D}\|g\|_{q,s,D} \tag{17}$$

for $f \in L^{p,\infty}(D)$ and $g \in L^{q,s}(D)$, where p, q, r are the same as above and $1 \le s \le \infty$. For

$$1 < q < \infty, \quad 1 < r \le \infty, \quad 1/q' + 1/q = 1, \quad 1/r' + 1/r = 1, \tag{18}$$

the duality relation

$$L^{q,r}(D) = L^{q',r'}(D)^*$$

holds, in particular, $L^{q,\infty}(D) = L^{q',1}(D)^*$. In what follows, the same symbols for vector and scalar function spaces are adopted as long as there is no confusion. The abbreviations $\|\cdot\|_q = \|\cdot\|_{q,\Omega}$ and $\|\cdot\|_{q,r} = \|\cdot\|_{q,r,\Omega}$ are used for the exterior domain Ω under consideration.

One needs the homogeneous Sobolev space. For $1 < q < \infty$, let $\dot{H}^1_q(D)$ be the completion of $C_0^\infty(D)$ with respect to the norm $\|\nabla(\cdot)\|_{q,D}$. For $D = \mathbb{R}^3$ one has

$$\dot{H}^1_q(\mathbb{R}^3) = \{u \in L^q_{\mathrm{loc}}(\mathbb{R}^3);\ \nabla u \in L^q(\mathbb{R}^3)\}/\mathbb{R}.$$

When $1 < q < 3$, one may take the canonical representative elements to adopt

$$\dot{H}^1_q(\mathbb{R}^3) = \{u \in L^{q_*}(\mathbb{R}^3);\ \nabla u \in L^q(\mathbb{R}^3)\},$$

where $1/q_* = 1/q - 1/3$, together with the embedding estimate

$$\|u\|_{q_*,\mathbb{R}^3} \le C\|\nabla u\|_{q,\mathbb{R}^3}.$$

Let

$$1 < q_0 < q < q_1 < \infty, \quad 1/q = (1-\theta)/q_0 + \theta/q_1, \quad 1 \le r \le \infty, \tag{19}$$

and define

$$\dot{H}^1_{q,r}(D) = \left(\dot{H}^1_{q_0}(D), \dot{H}^1_{q_1}(D)\right)_{\theta,r},$$

which is independent of the choice of $\{q_0, q_1\}$, with norm $\|\nabla(\cdot)\|_{q,r,D}$. Note that $C_0^\infty(D)$ is dense in $\dot{H}^1_{q,r}(D)$ unless $r = \infty$. For $D = \mathbb{R}^3$ the embedding relations ([56])

$$\begin{aligned} &\dot{H}^1_{q,r}(\mathbb{R}^3) \hookrightarrow L^{q*,r}(\mathbb{R}^3), \qquad \|u\|_{q*,r,\mathbb{R}^3} \le C\|\nabla u\|_{q,r,\mathbb{R}^3}, \\ &\dot{H}^1_{3,1}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3) \cap C(\mathbb{R}^3), \qquad \|u\|_{\infty,\mathbb{R}^3} \le C\|\nabla u\|_{3,1,\mathbb{R}^3}, \end{aligned} \tag{20}$$

hold provided $1 < q < 3$, $1/q_* = 1/q - 1/3$ and $1 \le r \le \infty$.

Let $1 < q < 3$, $1/q_* = 1/q - 1/3$ and $1 \le r \le \infty$. Let $\Omega \subset \mathbb{R}^3$ be the exterior domain. For every $u \in L^1_{\text{loc}}(\overline{\Omega})$ satisfying $\nabla u \in L^{q,r}(\Omega)$, there is a constant $k = k(u)$ such that $u + k \in L^{q*,r}(\Omega)$ with

$$\|u + k\|_{q*,r} \le C\|\nabla u\|_{q,r}$$

where $C > 0$ independent of u (see [7, Theorem 5.9]). By taking the canonical representative element of $u \in \dot{H}^1_{q,r}(\Omega)$, one has the characterization ([34, 52, 56])

$$\dot{H}^1_{q,r}(\Omega) = \{u \in L^{q*,r}(\Omega);\ \nabla u \in L^{q,r}(\Omega), u|_{\partial\Omega} = 0\}, \tag{21}$$

together with

$$\|u\|_{q*,r} \le C\|\nabla u\|_{q,r}. \tag{22}$$

One can also take the canonical representative element of $u \in \dot{H}^1_{3,1}(\Omega) \hookrightarrow L^\infty(\Omega) \cap C(\Omega)$, which goes to zero for $|x| \to \infty$ and satisfies $u|_{\partial\Omega} = 0$ as well as

$$\|u\|_\infty \le C\|\nabla u\|_{3,1}. \tag{23}$$

For $\{q, r\}$ satisfying (18), the space $\dot{H}^{-1}_{q,r}(D)$ is defined as the dual space of $\dot{H}^1_{q',r'}(D)$, and set $\dot{H}^{-1}_q(D) = \dot{H}^{-1}_{q,q}(D)$. The duality theorem for interpolation spaces ([3, 3.7.1]) implies that

$$\dot{H}^{-1}_{q,r}(D) = \left(\dot{H}^{-1}_{q_0}(D), \dot{H}^{-1}_{q_1}(D)\right)_{\theta,r} \tag{24}$$

for q, q_0, q_1, r satisfying (19) with $1 < r \le \infty$. For $r = 1$, one also defines $\dot{H}^{-1}_{q,1}(D)$, $1 < q < \infty$, as the dual space of the completion of $C_0^\infty(D)$ with respect to the norm $\|\nabla(\cdot)\|_{q',\infty,D}$, which is denoted by $\hat{H}^1_{q',\infty}(D) \left(\subsetneq \dot{H}^1_{q',\infty}(D)\right)$. Then (24) holds for $r = 1$ as well (see [3, p.55]). Let $1 < q < \infty$ and $1 \le r \le \infty$, then there exists a constant $C > 0$ such that for every $f \in \dot{H}^{-1}_{q,r}(D)$, one can take $F \in L^{q,r}(D)$ satisfying

$$\operatorname{div} F = f, \qquad \|F\|_{q,r,D} \le C\|f\|_{\dot{H}^{-1}_{q,r}(D)};$$

see [51, Lemma 2.2] and [56, Lemma 2.2].

Let $D \subset \mathbb{R}^3$ be a bounded domain. Then

$$L^{p,r}(D) \subset L^{q,s}(D) \quad \text{for all } 1 < q < p < \infty,\ r, s \in [1,\infty].$$

Both embeddings

$$\dot{H}^1_{q,r}(D) \hookrightarrow L^{q,r}(D) \hookrightarrow \dot{H}^{-1}_{q,r}(D) \tag{25}$$

are compact, where $1 < q < \infty$ and $1 \le r \le \infty$ (see [3, 3.14.8]). For the same $\{q, r\}$, one also has

$$\dot{H}^1_{q,r}(D) = \{u \in L^{q,r}(D);\ \nabla u \in L^{q,r}(D), u|_{\partial D} = 0\},$$

together with the Poincaré inequality

$$\|u\|_{q,r,D} \le C\|\nabla u\|_{q,r,D}. \tag{26}$$

Finally, consider the boundary value problem for the equation of continuity

$$\operatorname{div} w = f \quad \text{in } D, \qquad w = 0 \quad \text{on } \partial D,$$

where D is a bounded domain in $\mathbb{R}^3$ with Lipschitz boundary ∂D. Given f being in a suitable class, say $f \in L^q(D)$, with compatibility condition $\int_D f = 0$, there are a lot of solutions, some of which were found by many authors (see Galdi [28, Notes for Chapter III]). Among them a particular solution discovered by Bogovskii [4] is useful to recover the solenoidal condition in a cutoff procedure on account of some fine properties of his solution. By the following lemma, the operator $f \mapsto$ his solution w (called the Bogovskii operator) is well defined, and its properties are summarized. For the proof, see Borchers and Sohr [8], Galdi [28], as well as Bogovskii [4].

Lemma 1. *Let $D \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary. Then there is a linear operator $B : C_0^\infty(D) \to C_0^\infty(D)^3$ such that, for $1 < q < \infty$ and $k \ge 0$ integer,*

$$\|\nabla^{k+1} Bf\|_{q,D} \le C\|\nabla^k f\|_{q,D}$$

with some $C = C(D, q, k) > 0$ and that

$$\operatorname{div} Bf = f \qquad \text{if} \quad \int_D f(x)dx = 0,$$

where the constant C is invariant with respect to dilation of the domain D. By continuity, B is extended uniquely to a bounded operator from $\dot{H}^k_q(D)$ to $\dot{H}^{k+1}_q(D)^3$, where $\dot{H}^k_q(D)$ is the completion of $C_0^\infty(D)$ with respect to the norm $\|\nabla^k(\cdot)\|_{q,D}$. Furthermore, by real interpolation, it is extended uniquely to a bounded operator from $\dot{H}^k_{q,r}(D)$ to $\dot{H}^{k+1}_{q,r}(D)^3$, where $1 \le r \le \infty$ and

$$\dot{H}^k_{q,r}(D) = \left(\dot{H}^k_{q_0}(D), \dot{H}^k_{q_1}(D)\right)_{\theta,r}$$

with q_0, q_1 and θ satisfying (19).

3 Asymptotic Structure of the Stokes Flow

Let us start with asymptotic structure of the simplest case, that is, the exterior Stokes flow without external force

$$-\Delta u + \nabla p = 0, \qquad \operatorname{div} u = 0, \qquad \text{in } \Omega, \tag{27}$$

where nothing is imposed at the boundary $\partial\Omega$. Note that the result below does not depend on the boundary condition on $\partial\Omega$. Since (27) admits polynomial solutions, it is reasonable to impose a growth condition, for instance,

$$\begin{aligned} & u(x) = o(|x|) \qquad \text{as } |x| \to \infty, \\ \text{or} \quad & u/(1+|x|) \in L^{q,r}(\Omega) \qquad \text{for some } q \in (1,\infty),\ r \in [1,\infty], \\ \text{or} \quad & \nabla u \in L^{q,r}(\Omega) \qquad \text{for some } q \in (1,\infty),\ r \in [1,\infty], \end{aligned} \tag{28}$$

to exclude polynomials except constants. Note that the growth condition on the pressure is not needed here since it is controlled through the Eq. (27) by the velocity (but that is not the case in Theorem 1 below; see Remark 1). Then the asymptotic structure is described in terms of the Stokes fundamental solution

$$E(x) = \frac{1}{8\pi}\left(\frac{1}{|x|}\mathbb{I} + \frac{x \otimes x}{|x|^3}\right), \qquad Q(x) = \frac{x}{4\pi|x|^3}, \tag{29}$$

to be precise (Chang and Finn [11]), for every solution to (27) subject to (28), there are constants $u_\infty \in \mathbb{R}^3$ and $p_\infty \in \mathbb{R}$ such that

$$\begin{aligned} u(x) &= u_\infty + E(x)\int_{\partial\Omega} T(u,p)\nu\, d\sigma + O(|x|^{-2}), \\ p(x) &= p_\infty + Q(x)\cdot\int_{\partial\Omega} T(u,p)\nu\, d\sigma + O(|x|^{-3}), \end{aligned} \tag{30}$$

as $|x| \to \infty$. For the proof, there are two methods. One is to employ a potential representation formula (as in, for instance, [18]), and the other is a cutoff technique for reduction to the whole space problem. This section takes the latter way since it works for the Navier-Stokes system as well (see sect. 5). It will be also clarified which condition on the external force f ensures that the Stokes fundamental solution is still the leading profile at infinity for

$$-\Delta u + \nabla p = f, \qquad \operatorname{div} u = 0 \qquad \text{in } \Omega. \tag{31}$$

For simplicity, let us consider smooth solution to (31) for smooth external force. If $f(x) = O(|x|^{-3})$ or $f = \operatorname{div} F$ with $F(x) = O(|x|^{-2})$, it is formally balanced with the second derivative of $E(x)$ and thus the situation would be delicate (the equality $-\Delta(|x|^{-1}\log|x|) = |x|^{-3}$ suggests that one could not expect even the rate $|x|^{-1}$ of decay for the former case). In order to make this point clear, one needs the following lemma on asymptotic structure of the volume potentials $E * f$ and $Q * f$.

Lemma 2. *Let $f \in C^\infty(\mathbb{R}^3)$ and assume that there are constants $\alpha > 3$ and $C > 0$ such that*

$$|f(x)| \le \frac{C}{(1+|x|)^\alpha} \qquad \forall x \in \mathbb{R}^3.$$

Then

$$\nabla^j (E * f)(x) = \nabla^j E(x) \int_{\mathbb{R}^3} f(y)\,dy + \begin{cases} O(|x|^{-\alpha+2-j}), & 3<\alpha<4, \\ O(|x|^{-2-j}\log|x|), & \alpha = 4, \\ O(|x|^{-2-j}), & \alpha > 4, \end{cases}$$

$$(Q * f)(x) = Q(x) \cdot \int_{\mathbb{R}^3} f(y)\,dy + \begin{cases} O(|x|^{-\alpha+1}), & 3<\alpha<4, \\ O(|x|^{-3}\log|x|), & \alpha = 4, \\ O(|x|^{-3}), & \alpha > 4, \end{cases} \tag{32}$$

as $|x| \to \infty$, where $j = 0, 1$.

Proof. One can split the error term as

$$\begin{aligned} &\int_{\mathbb{R}^3} \{(\nabla^j E)(x-y) - \nabla^j E(x)\} f(y)\,dy \\ &= \int_{|y|<|x|/2} \{(\nabla^j E)(x-y) - \nabla^j E(x)\} f(y)\,dy - \nabla^j E(x) \int_{|y|\ge|x|/2} f(y)\,dy \\ &\quad + \int_{|x|/2\le|y|\le 2|x|} (\nabla^j E)(x-y) f(y)\,dy + \int_{|y|>2|x|} (\nabla^j E)(x-y) f(y)\,dy \\ &=: I_1 + I_2 + I_3 + I_4. \end{aligned}$$

It is easy to see that

$$|I_2| + |I_4| \le C|x|^{-\alpha+2-j}$$

and that

$$\begin{aligned} |I_3| &\le C|x|^{-\alpha} \int_{|y|\le 2|x|} \frac{dy}{|x-y|^{1+j}} \\ &\le C|x|^{-\alpha} \int_{|y-x|\le 3|x|} \frac{dy}{|x-y|^{1+j}} = C|x|^{-\alpha+2-j}. \end{aligned}$$

One also finds

$$\begin{aligned} |I_1| &\le C \int_{|y|<|x|/2} \int_0^1 \frac{d\tau}{|x-\tau y|^{2+j}} |y||f(y)|\, dy \\ &\le \frac{C}{|x|^{2+j}} \int_{|y|<|x|/2} \frac{|y|}{(1+|y|)^{\alpha}}\, dy \\ &\le \frac{C}{|x|^{2+j}} \int_0^{|x|/2} (1+r)^{-\alpha+3}\, dr \end{aligned}$$

which concludes (32) for $\nabla^j(E * f)$, $j = 0, 1$. The other one $Q * f$ can be discussed similarly. □

When the external force does not possess pointwise estimate, one needs the following existence result for the Stokes system in the whole space

$$-\Delta u + \nabla p = f, \qquad \text{div}\, u = 0 \qquad \text{in } \mathbb{R}^3. \tag{33}$$

Lemma 3. *Let* $1 < q < \infty$ *and* $1 \le r \le \infty$. *For every* $f \in \dot{H}^{-1}_{q,r}(\mathbb{R}^3)$, *there is a unique solution* $\{u, p\} \in \dot{H}^1_{q,r}(\mathbb{R}^3) \times L^{q,r}(\mathbb{R}^3)$ *to* (33) *with*

$$\|\{\nabla u, p\}\|_{q,r,\mathbb{R}^3} \le C\|f\|_{\dot{H}^{-1}_{q,r}(\mathbb{R}^3)} \tag{34}$$

in the sense that

$$\langle \nabla u, \nabla \varphi \rangle - \langle p, div\, \varphi \rangle = \langle f, \varphi \rangle$$

holds for all $\varphi \in C_0^\infty(\mathbb{R}^3)$, *where* $\langle \cdot, \cdot \rangle$ *stands for duality pairings. If in particular* $q < 3$, *then*

$$\|u\|_{q_*,r,\mathbb{R}^3} \le C\|f\|_{\dot{H}^{-1}_{q,r}(\mathbb{R}^3)} \tag{35}$$

and the solution $\{u, p\}$ *is unique within the class* $L^{q_*,r}(\mathbb{R}^3) \times L^{q,r}(\mathbb{R}^3)$, *where* $1/q_* = 1/q - 1/3$.

Proof. Suppose $\{u, p\}$ satisfies (33) with $f = 0$. One has $\nabla u \in \mathcal{S}'(\mathbb{R}^3)$, which implies $u \in \mathcal{S}'(\mathbb{R}^3)$ by [12, Proposition 1.2.1]. Since the Fourier transform of u is supported in $\{0\}$, the condition $\nabla u \in L^{q,r}(\mathbb{R}^3)$ concludes that u is a constant vector, which means zero element in $\dot{H}^1_{q,r}(\mathbb{R}^3)$. Then $\nabla p = 0$, yielding $p = 0$ in $L^{q,r}(\mathbb{R}^3)$, which proves the uniqueness. When $f = \text{div } F$ with $F \in C_0^\infty(\mathbb{R}^3)$, one has a solution $\{u, p\} = \{E * f, Q * f\}$. In terms of the Riesz transform $R = (R_j)$, we have

$$\nabla u = -(R \otimes R)p + R(R \cdot F), \qquad p = -(R \otimes R) : F,$$

yielding (34). The continuity argument then provides a solution for $f \in \dot{H}^{-1}_q(\mathbb{R}^3)$. Finally, by real interpolation, one gets a solution for $f \in \dot{H}^{-1}_{q,r}(\mathbb{R}^3)$ with desired estimate (34), which combined with (20) implies (35). □

From two lemmata above, one obtains the following theorem, in which (37) can be regarded as asymptotic expansion in terms partly of summability.

Theorem 1. *Let* $f = f_0 + \text{div } F$ *with*

$$F \in L^s(\Omega) \cap C^\infty(\Omega)$$

for some $s \in (1, 3/2]$. *Let* $f_0 \in C^\infty(\Omega)$ *and assume that there are constants* $\alpha > 3$ *and* $C > 0$ *such that*

$$|f_0(x)| \le \frac{C}{(1 + |x|)^\alpha} \qquad \forall x \in \Omega. \tag{36}$$

For every smooth solution of class $u, p, \nabla u \in L^s_{\text{loc}}(\overline{\Omega})$ *to* (31) *subject to* (28), *there is a constant* $u_\infty \in \mathbb{R}^3$ *such that*

$$u(x) = u_\infty + E(x)M_0 + v_0(x) + v_1(x), \tag{37}$$

for $|x| \ge 3R$ *with*

$$M_0 = \int_{\partial\Omega} \big(T(u, p) + F\big)\nu \, d\sigma + \int_\Omega f_0(y)\, dy,$$

$$\nabla^j v_0(x) = \begin{cases} O(|x|^{-\alpha+2-j}), & 3 < \alpha < 4, \\ O(|x|^{-2-j} \log |x|), & \alpha = 4, \\ O(|x|^{-2-j}), & \alpha > 4, \end{cases} \tag{38}$$

$(j = 0, 1)$ as $|x| \to \infty$ and

$$v_1 \in L^{s_*}(\Omega \setminus B_{3R}), \qquad \nabla v_1 \in L^s(\Omega \setminus B_{3R}),$$

where $s_* \in (3/2, 3]$ is defined by $1/s_* = 1/s - 1/3$. Here, $R > 0$ is taken such that $\mathbb{R}^3 \setminus \Omega \subset B_R$. If in particular $F = 0$, then v_1 is absent from (37).

Proof. First of all, note that the boundary integral of M_0 can be understood as $\langle (T(u,p) + F)\nu, 1\rangle_{\partial\Omega}$ in the sense of normal trace $(T(u,p) + F)\nu \in H_s^{-1/s}(\partial\Omega)$ since $T(u,p) + F \in L^s_{\mathrm{loc}}(\overline{\Omega})$ and div $(T(u,p) + F) = -f_0 \in L^\infty(\Omega) \subset L^s_{\mathrm{loc}}(\overline{\Omega})$. Let us reduce the problem to the one with vanishing flux at the boundary $\partial\Omega$. Without loss one may assume $0 \in \mathrm{int}\,(\mathbb{R}^3 \setminus \Omega)$. We introduce the flux carrier

$$z(x) = \beta\,\nabla \frac{1}{4\pi|x|}, \qquad \beta = \int_{\partial\Omega} \nu \cdot u\, d\sigma, \tag{39}$$

for given solution $\{u, p\}$. Since

$$\int_{\partial\Omega} \nu \cdot z\, d\sigma = \beta, \qquad \mathrm{div}\, z = 0, \qquad \Delta z = 0 \qquad \text{in } \mathbb{R}^3 \setminus \{0\}, \tag{40}$$

the pair $\{\tilde{u}, p\}$ with $\tilde{u} = u - z$ fulfills also (31) subject to

$$\int_{\partial\Omega} \nu \cdot \tilde{u}\, d\sigma = 0. \tag{41}$$

By using a cutoff function

$$\psi \in C_0^\infty(B_{3R}; [0,1]), \quad \psi(x) = 1 \quad (|x| \le 2R), \quad \|\nabla\psi\|_\infty \le \frac{C}{R} \tag{42}$$

and the Bogovskii operator B (Lemma 1) in the domain

$$A_R = \{x \in \mathbb{R}^3;\ R < |x| < 3R\}, \tag{43}$$

one sets

$$v = (1-\psi)\tilde{u} + B[\tilde{u} \cdot \nabla\psi], \qquad \theta = (1-\psi)p, \tag{44}$$

where the Bogovskii term is understood as its zero extension to the whole space $\mathbb{R}^3$. It should be noted that $\int_{A_R} \tilde{u} \cdot \nabla\psi\, dx = 0$ follows from (41). Then the pair $\{v, \theta\}$ obeys

$$-\Delta v + \nabla\theta = g + (1-\psi) f_0 + \mathrm{div}\,((1-\psi)F), \qquad \mathrm{div}\, v = 0 \qquad \text{in } \mathbb{R}^3 \tag{45}$$

for some function $g \in C_0^\infty(A_R)$. Here, one does not need any exact form of g and what is important is structure of the Eq. (45). When either $u(x) = o(|x|)$ or $u/(1+|x|) \in L^{q,r}(\Omega)$, it is obvious that $v \in \mathcal{S}'(\mathbb{R}^3)$. Under the alternative assumption $\nabla u \in L^{q,r}(\Omega)$ in (28), one has $\nabla v \in \mathcal{S}'(\mathbb{R}^3)$, which implies $v \in \mathcal{S}'(\mathbb{R}^3)$ ([12, Proposition 1.2.1]). Going back to the Eq. (45) yields $\nabla\theta \in \mathcal{S}'(\mathbb{R}^3)$ and, therefore, $\theta \in \mathcal{S}'(\mathbb{R}^3)$ by the same reasoning. Let

$$v_1 \in \dot{H}^1_s(\mathbb{R}^3) \hookrightarrow L^{s*}(\mathbb{R}^3), \qquad \theta_1 \in L^s(\mathbb{R}^3),$$

be the solution to (33) with the external force $\operatorname{div}((1-\psi)F) \in \dot{H}^{-1}_s(\mathbb{R}^3)$ obtained in Lemma 3. We then find

$$\begin{aligned} v(x) &= \big[E * \{g + (1-\psi)f_0\}\big](x) + v_1(x) + \mathcal{P}_v(x), \\ \theta(x) &= \big[Q * \{g + (1-\psi)f_0\}\big](x) + \theta_1(x) + \mathcal{P}_\theta(x), \end{aligned} \tag{46}$$

with some polynomials $\mathcal{P}_v$ and $\mathcal{P}_\theta$; however, from (28) it follows that $\mathcal{P}_v$ must be a constant vector, which is denoted by u_∞. Thus, one concludes from Lemma 2 that $u(x)$ can be represented as

$$u(x) = \tilde{u}(x) + z(x) = E(x)M_0 + v_0(x) + v_1(x) + u_\infty \qquad (|x| \geq 3R)$$

with

$$M_0 = \int_{\mathbb{R}^3} \{g + (1-\psi)f_0\}(y)\,dy,$$

where $\nabla^j v_0(x)$ behaves like the remainder of (32) since $\nabla^j z(x) = O(|x|^{-2-j})$ (see (39)). Let $\rho > 3R$. From

$$\int_{|y|=\rho} \big(\nabla z + (\nabla z)^\top\big)\frac{y}{\rho}\,d\sigma = \frac{\beta}{\pi\rho^4}\int_{|y|=\rho} y\,d\sigma = 0 \tag{47}$$

one can deduce

$$\begin{aligned} \int_{B_\rho} \{g + (1-\psi)f_0\}(y)\,dy &= -\int_{B_\rho} \operatorname{div}\{T(v,\theta) + (1-\psi)F\}\,dy \\ &= -\int_{|y|=\rho} (T(u,p) + F)\frac{y}{\rho}\,d\sigma \\ &= \int_{\partial\Omega} (T(u,p) + F)\nu\,d\sigma + \int_{\Omega_\rho} f_0(y)\,dy. \end{aligned}$$

Letting $\rho \to \infty$ leads to

$$M_0 = \int_{\partial\Omega} (T(u,p) + F)\nu\,d\sigma + \int_{\Omega} f_0(y)\,dy,$$

which concludes (37). This completes the proof. □

Remark 1. Because of less information about div F, one cannot say anything about the polynomial $\mathcal{P}_\theta$ in (46) unless assuming the behavior of p at infinity. If in particular $F = 0$ so that $\theta_1 = 0$, then $\mathcal{P}_\theta$ must be a constant. In fact, one knows that both $\Delta(E * h)$ and $\nabla(Q * h)$ belong to $L^r(\mathbb{R}^3)$ for every $r \in (1, \infty)$ because so does $h := g + (1 - \psi) f_0$. By going back to (31), one finds that $\mathcal{P}_\theta$ is a constant, which we denote by p_∞. As a consequence,

$$p(x) = p_\infty + Q(x) \cdot M_0 + \begin{cases} O(|x|^{-\alpha+1}), & 3 < \alpha < 4, \\ O(|x|^{-3} \log |x|), & \alpha = 4, \\ O(|x|^{-3}), & \alpha > 4, \end{cases} \tag{48}$$

as $|x| \to \infty$.

Theorem 1 immediately implies the following corollary.

Corollary 1. *In addition to the assumptions of Theorem 1 with $s = 3/2$, suppose either $u \in L^3(\Omega)$ or $\nabla u \in L^{3/2}(\Omega)$. Then $M_0 = 0$.*

Remark 2. For the Stokes system in n-dimensional exterior domains, either $u \in L^{n/(n-2)}(\Omega)$ or $\nabla u \in L^{n/(n-1)}(\Omega)$ yields $M_0 = 0$ (under suitable assumptions on the external force).

Consider (31) subject to $u|_{\partial\Omega} = 0$. Corollary 1 then tells us that the condition $u \in \dot{H}^1_{3/2,\infty}(\Omega)$ is an optimal class observed in general even if $f = \operatorname{div} F$ with $F \in C_0^\infty(\Omega)$. The following corollary claims the uniqueness of solutions in this class.

Corollary 2. *Let $\{u, p\} \in \dot{H}^1_{3/2,\infty}(\Omega) \times L^{3/2,\infty}(\Omega)$ be a solution to (27) subject to $u|_{\partial\Omega} = 0$. Then $\{u, p\} = \{0, 0\}$.*

Proof. Theorem 1 with $f = 0$ can be applied. Since $u \in L^{3,\infty}(\Omega)$, one has (37) with $u_\infty = 0$ as well as $v_1 = 0$. One also knows (48) with $p_\infty = 0$ (or even more directly, the same procedure as in the proof of Theorem 1 with use of $p \in L^{3/2,\infty}(\Omega)$ leads to the same expansion). As a consequence,

$$u(x) = O(|x|^{-1}), \qquad \{\nabla u(x), p(x)\} = O(|x|^{-2}) \tag{49}$$

as $|x| \to \infty$. Let $\phi \in C^\infty([0, \infty); [0, 1])$ satisfy $\phi(t) = 1$ for $0 \le t \le 1$ and $\phi(t) = 0$ for $t \ge 2$, and set $\phi_\rho(x) = \phi(|x|/\rho)$ for $\rho > 0$ large enough and $x \in \mathbb{R}^3$. Since the local regularity theory for the Stokes boundary value problem together with a bootstrap argument yields $\{u, p\} \in H^2(\Omega_{2\rho}) \times H^1(\Omega_{2\rho})$, one can multiply (27) by $\phi_\rho u$ to obtain

$$\int_{\Omega} |\nabla u|^2 \phi_\rho \, dx + \int_{\rho<|x|<2\rho} (\nabla u \cdot \nabla \phi_\rho) \cdot u \, dx - \int_{\rho<|x|<2\rho} (u \cdot \nabla \phi_\rho) p \, dx = 0. \tag{50}$$

Letting $\rho \to \infty$ by use of (49) together with $|\nabla \phi_\rho(x)| \leq C/\rho$ leads to $\int_\Omega |\nabla u|^2 \, dx = 0$, so that $u = 0$ by the boundary condition. Then $\nabla p = 0$, yielding $p = 0$ on account of (49). □

Remark 3. The key is that smallness at infinity in a weak sense implies the pointwise decay property (49). The argument works even for $\{u, p\} \in \dot{H}^1_{q,r}(\Omega) \times L^{q,r}(\Omega)$ provided $\{q, r\} \in (1, 3) \times [1, \infty]$ or $\{q, r\} = \{3, 1\}$; thus, the uniqueness of solutions within that class also holds true.

One turns to the case where the obstacle is purely rotating with angular velocity ω. The linearized system (the Stokes system with rotation) is given by

$$-\Delta u + \nabla p - (\omega \times x) \cdot \nabla u + \omega \times u = f, \qquad \operatorname{div} u = 0 \qquad \text{in } \Omega. \tag{51}$$

Without loss one may assume $\omega = ae_3$ with $a \in \mathbb{R} \setminus \{0\}$. Let us introduce

$$\Gamma(x, y) = \int_0^\infty O(at)^\top (G\mathbb{I} + H)(O(at)x - y, t) \, dt, \tag{52}$$

where

$$G(x, t) = (4\pi t)^{-3/2} e^{-|x|^2/4t}, \qquad H(x, t) = \int_t^\infty \nabla^2 G(x, s) \, ds$$

and

$$O(t) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $(G\mathbb{I} + H)(x, t)$ is a fundamental solution of the nonstationary Stokes system and that the fundamental solution (29) of the stationary Stokes system is represented as

$$E(x) = \int_0^\infty (G\mathbb{I} + H)(x, t) \, dt,$$

which can be compared with (52). By [18, 20] the pair $\{\Gamma(x, y), Q(x - y)\}$ with Q given by (29) is a fundamental solution to

$$-\Delta u + \nabla p - (\omega \times x) \cdot \nabla u + \omega \times u = f, \qquad \operatorname{div} u = 0 \qquad \text{in } \mathbb{R}^3 \tag{53}$$

with $\omega = ae_3$. The following lemma gives us the asymptotic behavior of $\Gamma(x, y)$; in particular, (57) plays a crucial role.

Lemma 4. *Let $\omega = ae_3$ with $a \in \mathbb{R} \setminus \{0\}$, and let $j = 0, 1$. Then*

$$|\nabla^j \Gamma(x, y)| \le C|x|^{-1-j} \qquad \textit{for } |x| > 2|y|, \tag{54}$$

$$|\nabla^j \Gamma(x, y)| \le C|y|^{-1-j} \qquad \textit{for } |y| > 2|x|, \tag{55}$$

$$\int_{|y|\le 2|x|} |\nabla^j \Gamma(x, y)|\, dy \le C|x|^{2-j} \qquad \textit{for } x \in \mathbb{R}^3, \tag{56}$$

where ∇ denotes either ∇_x or ∇_y. Set

$$\Phi(x) := \frac{1}{8\pi|x|^3} \begin{pmatrix} 0 & 0 & x_1x_3 \\ 0 & 0 & x_2x_3 \\ 0 & 0 & |x|^2 + x_3^2 \end{pmatrix},$$

then

$$|\Gamma(x, y) - \Phi(x)| \le C|y||x|^{-2} + C|a|^{-1}|x|^{-3} \qquad \textit{for } |x| > 2|y|. \tag{57}$$

Proof. A brief sketch will be presented here. Let us take the Taylor formula (with respect to y)

$$G(O(at)x - y, t) = G(x, t) + G(x, t)\frac{(O(at)x) \cdot y}{2t} + \text{(remainder)}$$

and consider each term multiplied by $O(at)^\top$. To get (57), the point is rapid decay due to oscillation

$$\left| \int_0^\infty \begin{pmatrix} \cos at \\ \sin at \end{pmatrix} G(x, t)\, dt \right| \le \frac{C}{|a||x|^3}$$

and a part of non-oscillating terms comes to the leading profile $\Phi(x)$. The term $H(O(at)x - y, t)$ can be discussed similarly although the computation is more complicated. The details are found in [18, section 4]. The others (54), (55) and (56) are much easier (see [18, (2.11)] and [47, (6.23)]). □

By the same splitting of the volume potential as in the proof of Lemma 2, one can make use of Lemma 4 to conclude the following asymptotic expansion. In the proof, the dominant term is I_1 for the region $|y| < |x|/2$, in which (57) is employed.

Lemma 5. *Let $\omega = ae_3$ with $a \in \mathbb{R} \setminus \{0\}$. Suppose f satisfies the same condition (with $\alpha > 3$) as in Lemma 2. Then*

$$u(x) = \int_{\mathbb{R}^3} \Gamma(x, y) f(y)\, dy$$

enjoys

$$u(x) = \left(e_3 \cdot \int_{\mathbb{R}^3} f(y)\, dy\right) E(x)e_3 + \begin{cases} O(|x|^{-\alpha+2}), & 3 < \alpha < 4, \\ O(|x|^{-2} \log |x|), & \alpha = 4, \\ O(|x|^{-2}), & \alpha > 4, \end{cases} \tag{58}$$

as $|x| \to \infty$, where $E(x)$ is the Stokes fundamental solution (29).

By means of harmonic analytic method developed by [20], it is possible to deduce exactly the same well-posedness for (53) as in Lemma 3, where the constants in (34) and (35) are independent of $a \in \mathbb{R} \setminus \{0\}$. It was done by [43, Theorem 2.1], [17, Proposition 3.2]. By using this together with Lemma 5, one can follow the proof of Theorem 1 (with a bit more care of the flux carrier, see [47, section 6]) under the same assumptions on the external force to obtain the asymptotic expansion of solutions to (51), in which the leading term is given by

$$V(x) := \left(\frac{\omega}{|\omega|} \cdot M_0\right) E(x)\frac{\omega}{|\omega|}, \qquad \frac{\omega}{|\omega|} = e_3, \tag{59}$$

where

$$M_0 = \int_{\partial\Omega} [T(u, p) + u \otimes (\omega \times y) - (\omega \times y) \otimes u + F]\nu\, d\sigma_y + \int_{\Omega} f_0(y)\, dy.$$

Since $e_3 \cdot (e_3 \times y) = 0$, the third term of the boundary integral does not contribute to the coefficient $e_3 \cdot M_0$. This was proved under the no-slip condition $u|_{\partial\Omega} = \omega \times x$ by Farwig and Hishida [18], who deduced not only the leading term but also the second one although they restricted their consideration to the simple case where $f = \operatorname{div} F$ with $F \in C_0^\infty(\overline{\Omega})$. The leading term (59) satisfies

$$-\Delta V + \nabla\Pi = (e_3 \cdot M_0)e_3\delta, \qquad \operatorname{div} V = 0$$

in $\mathcal{D}'(\mathbb{R}^3)$, where $\Pi(x) = (e_3 \cdot M_0)x_3/(4\pi|x|^3)$. But the pair $\{V, \Pi\}$ enjoys

$$-\Delta V + \nabla\Pi - (\omega \times x) \cdot \nabla V + \omega \times V = (e_3 \cdot M_0)e_3\delta, \qquad \operatorname{div} V = 0 \tag{60}$$

as well, since

$$(e_3 \times x) \cdot \nabla V - e_3 \times V = 0. \tag{61}$$

Note that (61) holds for all vector fields which are symmetric about $\mathbb{R}e_3$ (x_3-axis). In fact, because such vector fields must be of the form

$$V = (W(r, x_3)\cos\theta, W(r, x_3)\sin\theta, V_3(r, x_3))^\top$$

in cylindrical coordinates r, θ, x_3, one finds $(e_3 \times x) \cdot \nabla V = \partial_\theta V = e_3 \times V$. As long as $V(x) \sim 1/|x|$ near the origin, (61) holds also in $\mathcal{D}'(\mathbb{R}^3)$, that is,

$$\langle V \otimes (e_3 \times x) - (e_3 \times x) \otimes V, \nabla\phi\rangle = 0$$

for all $\phi \in C_0^\infty(\mathbb{R}^3)$. Hence, the leading term (59) satisfies (60) in $\mathcal{D}'(\mathbb{R}^3)$.

4 Existence of Flows in $L^{3,\infty}$

Consider the problem (1) and (2) with $\eta = \omega = 0$, that is,

$$-\Delta u + \nabla p + u \cdot \nabla u = f, \qquad \operatorname{div} u = 0 \qquad \text{in } \Omega, \tag{62}$$

$$u = 0 \qquad \text{on } \partial\Omega. \tag{63}$$

Before proceeding to study of asymptotic structure of the Navier-Stokes flow, one should establish the existence of (small) solutions having optimal asymptotic behavior at infinity. The existence of solutions decaying like $|x|^{-1}$ was proved by Finn [23], Galdi and Simader [35], Novotny and Padula [69], and Borchers and Miyakawa [7]. Indeed, such a pointwise estimate is fine, but one may take another way by the use of function spaces, so that the proof becomes easier. Among some function spaces which are able to catch homogeneous functions of degree (-1), the weak-L^3 space is probably the simplest one. Actually, Kozono and Yamazaki [56] succeeded in showing the existence of a unique Navier-Stokes flow in $L^{3,\infty}(\Omega)$ whenever the external force is small in a sense. Since the next section studies the asymptotic structure of solutions in $L^{3,\infty}(\Omega)$, one will provide the existence theorem due to [56].

It was known ([7, 53, 54]) that either $u \in L^3(\Omega)$ or $\nabla u \in L^{3/2}(\Omega)$ necessarily yields $M(0, 0, f) = 0$ (see (4)), although those Lebesgue spaces are invariant under the scale transformation (7). Hence, one has no chance to find the Navier-Stokes flow belonging to $L^3(\Omega)$ in generic situation. This is a nonlinear counter part of Corollary 1 (for the Stokes flow) and can be also interpreted by asymptotic expansion in the next section.

For the Stokes boundary value problem (31) subject to $u|_{\partial\Omega} = 0$, one has the well-posedness in the class $u \in \dot{H}^1_q(\Omega)$ if and only if

$$\left(\frac{n}{n-1} =\right) \frac{3}{2} < q < 3 \; (= n\text{: space dimension}), \tag{64}$$

see Borchers and Miyakawa [5], Galdi and Simader [34], and Kozono and Sohr [51, 52]. To be precise, the condition $q > 3/2$ is necessary for solvability and it is consistent with Corollary 1, while for uniqueness one needs $q < 3$; in fact, the proof of Corollary 2 does not work when $u \in \dot{H}^1_3(\Omega)$ because the constant u_∞ cannot be excluded in (37). Kozono and Yamazaki [56] clarified that all things for both Stokes and Navier-Stokes systems work well if replacing $L^3(\Omega)$ (resp. $L^{3/2}(\Omega)$) by $L^{3,\infty}(\Omega)$ (resp. $L^{3/2,\infty}(\Omega)$) for u (resp. ∇u). When $n \geq 4$, the L^q-theory is enough to construct (small) Navier-Stokes flow $u \in L^n(\Omega)$ with $\nabla u \in L^{n/2}(\Omega)$ because $\frac{n}{n-1} < \frac{n}{2} < n$ in this case.

The existence theorem due to [56] now reads as follows.

Theorem 2. *There is a constant $\kappa > 0$ such that for every $f \in \dot{H}^{-1}_{3/2,\infty}(\Omega)$ with $\|f\|_{\dot{H}^{-1}_{3/2,\infty}(\Omega)} < \kappa$, problem (62) and (63) admits a unique solution*

$$\begin{aligned} &u \in \dot{H}^1_{3/2,\infty}(\Omega) \hookrightarrow L^{3,\infty}(\Omega), \qquad p \in L^{3/2,\infty}(\Omega), \\ &\|\{\nabla u, p\}\|_{3/2,\infty} + \|u\|_{3,\infty} \leq C\|f\|_{\dot{H}^{-1}_{3/2,\infty}(\Omega)}, \end{aligned} \tag{65}$$

in the sense that

$$\langle \nabla u, \nabla \varphi \rangle - \langle p, div\, \varphi \rangle - \langle u \otimes u, \nabla \varphi \rangle = \langle f, \varphi \rangle$$

holds for all $\varphi \in C_0^\infty(\Omega)$ (and, therefore, all $\varphi \in \dot{H}^1_{3,1}(\Omega)$), where $\langle \cdot, \cdot \rangle$ stands for duality pairings.

Remark 4. It is an open question whether the small solution $\{u, p\}$ constructed in Theorem 2 is unique in the class $\dot{H}^1_{3/2,\infty}(\Omega) \times L^{3/2,\infty}(\Omega)$ without assuming smallness, in other words, whether $\{u, p\}$ coincides with other large solutions $\{v, q\}$ in this class. The difficulty seems to stem from unremovable singularity like $1/|x - x_0|$ of the velocity being in $L^{3,\infty}(\Omega)$. If such a singular behavior is ruled out for large solutions v by assuming additionally $v \in L^3(\Omega) + L^\infty(\Omega)$, then the answer is affirmative (provided $\|u\|_{3,\infty}$ is small enough that is accomplished by (65) when $\|f\|_{\dot{H}^{-1}_{3/2,\infty}(\Omega)}$ is still smaller). This interesting uniqueness criterion was proved by Nakatsuka [67] (see also [66]).

Once the following linear theory is established, it is straightforward by using a simple contraction argument with the aid of (13) to show Theorem 2 (whose proof may be omitted).

Theorem 3. *For every $f \in \dot{H}^{-1}_{3/2,\infty}(\Omega)$, problem (31) subject to (63) admits a unique solution*

$$\begin{aligned} &u \in \dot{H}^1_{3/2,\infty}(\Omega) \hookrightarrow L^{3,\infty}(\Omega), \qquad p \in L^{3/2,\infty}(\Omega), \\ &\|\{\nabla u, p\}\|_{3/2,\infty} + \|u\|_{3,\infty} \leq C\|f\|_{\dot{H}^{-1}_{3/2,\infty}(\Omega)}, \end{aligned} \tag{66}$$

in the sense that

$$\langle \nabla u, \nabla \varphi \rangle - \langle p, div\, \varphi \rangle = \langle f, \varphi \rangle$$

holds for all $\varphi \in C_0^\infty(\Omega)$ *(and, therefore, all* $\varphi \in \dot{H}^1_{3,1}(\Omega)$*), where* $\langle \cdot, \cdot \rangle$ *stands for duality pairings.*

The well-posedness in the class $\dot{H}^1_{q,r}(\Omega) \times L^{q,r}(\Omega)$ for every $f \in \dot{H}^{-1}_{q,r}(\Omega)$ was established first by Konozo and Yamazaki [56] when $\{q, r\}$ satisfies

$$\{q, r\} = \{3/2, \infty\}; \qquad \{q, r\} \in (3/2, 3) \times [1, \infty]; \qquad \{q, r\} = \{3, 1\},$$

which is a generalization of (64) (case $q = r$). Indeed Theorem 3 is just one of those cases, but it is the most important case to solve the nonlinear problem. Later on, Shibata and Yamazaki [71] proved the well-posedness not only in the class above but in the sum of function spaces

$$u \in \dot{H}^1_{q,r}(\Omega) + \dot{H}^1_{3/2,\infty}(\Omega), \qquad p \in L^{q,r}(\Omega) + L^{3/2,\infty}(\Omega)$$

even for the other cases

$$\{q, r\} \in (1, 3/2) \times [1, \infty]; \qquad q = 3/2 \text{ and } r \in [1, \infty).$$

This result suggests that ∇u and p do not decay faster than $|x|^{-2}$ in general. In [71] they discussed the Oseen system (6) as well as the Stokes system to study the relation between solutions to (1) with $\eta \neq 0$ and $\eta = 0$ (i.e., the behavior for the limit $\eta \to 0$). The well-posedness in the class above for (51) was proved by Farwig and Hishida [17] when the obstacle is purely rotating. It was generalized by Heck, Kim, and Kozono [37] when taking both translation and rotation of the obstacle into account. As a result, one has Theorem 2 for the Navier-Stokes boundary value problem (1) and (2) even if $\omega \neq 0$ provided the data ($\eta + \omega \times x$ in (2) as well as f) are small enough; in fact, the case $\eta \cdot \omega = 0$ is reduced to [17], while the other case $\eta \cdot \omega \neq 0$ is reduced to [37] by using the Mozzi-Chasles transform ([32], [28, Chapter VIII]). The pointwise estimate like $|u(x)| \leq C|x|^{-1}$ for (1) and (2) with $\omega \neq 0$ was successfully deduced by Galdi [27] and Galdi and Silvestre [32].

For the proof of Theorem 3, a sort of duality argument was employed in [56], but this way is not taken here; instead, a parametrix is constructed as in [71]. The latter method was also adopted in [17] and [37] since the argument of [56] does not seem to work because of lack of homogeneity of the equation with $\eta + \omega \times x \neq 0$. Also, one cannot use any continuity argument since $C_0^\infty(\Omega)$ is not dense in $L^{3/2,\infty}(\Omega)$. Given $f \in \dot{H}^{-1}_{3/2,\infty}(\Omega)$, one intends to construct directly a solution with the use of solutions in the whole space (Lemma 3) and in a bounded domain (Lemma 6 below).

Let D be a bounded domain in $\mathbb{R}^3$ with smooth boundary ∂D, and consider the boundary value problem

$$-\Delta u + \nabla p = f, \qquad \text{div}\, u = 0 \qquad \text{in } D, \tag{67}$$

$$u = 0 \qquad \text{on } \partial D. \tag{68}$$

The following lemma is due to Cattabriga [10], Solonnikov [73], Kozono and Sohr [51], and Kozono and Yamazaki [56]. In (69) below, the Poincaré inequality (26) is involved.

Lemma 6. *Let* $1 < q < \infty$ *and* $1 \le r \le \infty$. *For every* $f \in \dot{H}^{-1}_{q,r}(D)$, *problem* (67) *and* (68) *admits a solution*

$$\begin{aligned} &u \in \dot{H}^1_{q,r}(D), \qquad p \in L^{q,r}(D), \\ &\|\{\nabla u,\, u,\, p - \overline{p}\}\|_{q,r,D} \le C\|f\|_{\dot{H}^{-1}_{q,r(D)}}, \end{aligned} \tag{69}$$

with $\overline{p} := \frac{1}{|D|}\int_D p\, dx$, *in the sense that*

$$\langle \nabla u, \nabla\varphi\rangle - \langle p, div\, \varphi\rangle = \langle f, \varphi\rangle$$

holds for all $\varphi \in C_0^\infty(D)$, *where* $\langle\cdot,\cdot\rangle$ *stands for duality pairings. The solution is unique up to an additive constant for* p.

One is in a position to show Theorem 3.

Proof of Theorem 3. Since the uniqueness is already known by Corollary 2, one will show the existence part. Fix $R > 0$ so large that $\mathbb{R}^3 \setminus \Omega \subset B_{R-5}$. Take functions $\phi, \phi_1 \in C^\infty(\mathbb{R}^3; [0, 1])$ satisfying

$$\phi(x) = \begin{cases} 1, & |x| \le R - 3, \\ 0, & |x| \ge R - 2, \end{cases} \qquad \phi_1(x) = \begin{cases} 0, & |x| \le R - 5, \\ 1, & |x| \ge R - 4, \end{cases}$$

and set $A = \{x \in \mathbb{R}^3;\ R - 4 < |x| < R - 1\}$. Given $f \in \dot{H}^{-1}_{3/2,\infty}(\Omega)$, it is easily seen that

$$\begin{aligned} &f \in \dot{H}^{-1}_{3/2,\infty}(\Omega_R), \qquad \|f\|_{\dot{H}^{-1}_{3/2,\infty}(\Omega_R)} \le \|f\|_{\dot{H}^{-1}_{3/2,\infty}(\Omega)}, \\ &\phi_1 f \in \dot{H}^{-1}_{3/2,\infty}(\mathbb{R}^3), \qquad \|\phi_1 f\|_{\dot{H}^{-1}_{3/2,\infty}(\mathbb{R}^3)} \le C\|f\|_{\dot{H}^{-1}_{3/2,\infty}(\Omega)}, \end{aligned} \tag{70}$$

for the latter of which $(20)_2$ is used. Consider (67) and (68) with f in the bounded domain $D = \Omega_R$, and let $\{u_0, p_0\}$ be the solution obtained in Lemma 6 subject to $\int_{\Omega_R} p_0\, dx = 0$. Consider also (33) with f replaced by $\phi_1 f$, and let $\{u_1, p_1\}$ be the solution obtained in Lemma 3. Set

$$\begin{aligned} \Psi f &:= (1 - \phi)u_1 + \phi u_0 + B[(u_1 - u_0)\cdot\nabla\phi], \\ \Pi f &:= (1 - \phi)p_1 + \phi p_0, \end{aligned} \tag{71}$$

where B is the Bogovskii operator (see Lemma 1) in the bounded domain A. Since $\int_A (u_1 - u_0) \cdot \nabla\phi \, dx = 0$, one has div $\Psi f = 0$. Then it follows from (34), (35), (69), and (21), Lemma 1, and (70) that

$$\begin{aligned} &(\Psi f, \Pi f) \in \dot{H}^1_{3/2,\infty}(\Omega) \times L^{3/2,\infty}(\Omega), \\ &\|\{\nabla\Psi f, \Pi f\}\|_{3/2,\infty} + \|\Psi f\|_{3,\infty} \leq C\|f\|_{\dot{H}^{-1}_{3/2,\infty}(\Omega)} \end{aligned} \tag{72}$$

and that $(\Psi f, \Pi f)$ is a solution to

$$-\Delta\Psi f + \nabla\Pi f = f + \mathcal{R}f, \qquad \text{div } \Psi f = 0, \qquad \Psi f|_{\partial\Omega} = 0 \tag{73}$$

where

$$\mathcal{R}f = 2\nabla\phi \cdot \nabla(u_1 - u_0) + (\Delta\phi)(u_1 - u_0) - \Delta B[(u_1 - u_0) \cdot \nabla\phi] - (\nabla\phi)(p_1 - p_0),$$

which is in $L^{3/2,\infty}(\Omega_R) \hookrightarrow \dot{H}^{-1}_{3/2,\infty}(\Omega_R)$ and satisfies

$$\|\mathcal{R}f\|_{3/2,\infty,\Omega_R} \leq C\|f\|_{\dot{H}^{-1}_{3/2,\infty}(\Omega)}. \tag{74}$$

For every $\psi \in C_0^\infty(\Omega)$, one finds

$$|\langle \mathcal{R}f, \psi\rangle| \leq \|\mathcal{R}f\|_{3/2,\infty,\Omega_R}\|\psi\|_{3,1,\Omega_R}$$

which combined with $\|\psi\|_{3,1,\Omega_R} \leq C\|\psi\|_\infty \leq C\|\nabla\psi\|_{3,1}$ (see (23)) implies that $\mathcal{R}f \in \dot{H}^{-1}_{3/2,\infty}(\Omega)$ with

$$\|\mathcal{R}f\|_{\dot{H}^{-1}_{3/2,\infty}(\Omega)} \leq C\|\mathcal{R}f\|_{3/2,\infty,\Omega_R}.$$

Actually, one has even

$$\|\mathcal{R}f\|_{\dot{H}^{-1}_{3/2,\infty}(\Omega)} \leq C\|\mathcal{R}f\|_{\dot{H}^{-1}_{3/2,\infty}(\Omega_R)}. \tag{75}$$

In fact, with the use of a fixed function $\varphi \in C_0^\infty(\Omega_R)$ with $\varphi(x) = 1$ $(x \in \overline{A})$, one observes

$$\begin{aligned} |\langle \mathcal{R}f, \psi\rangle| = |\langle \varphi\mathcal{R}f, \psi\rangle| &\leq \|\mathcal{R}f\|_{\dot{H}^{-1}_{3/2,\infty}(\Omega_R)}\|\varphi\psi\|_{\dot{H}^1_{3,1}(\Omega_R)} \\ &\leq C\|\mathcal{R}f\|_{\dot{H}^{-1}_{3/2,\infty}(\Omega_R)}\|\nabla\psi\|_{3,1} \end{aligned}$$

for every $\psi \in C_0^\infty(\Omega)$, owing to (23) as well as $\mathcal{R}f = 0$ outside A. This implies (75).

Now it turns out that $\mathcal{R} : \dot{H}^{-1}_{3/2,\infty}(\Omega) \to \dot{H}^{-1}_{3/2,\infty}(\Omega)$ is a compact operator. In fact, suppose $\{f_j\}$ is a bounded sequence in $\dot{H}^{-1}_{3/2,\infty}(\Omega)$, and then by (74) the sequence $\{\mathcal{R} f_j\}$ is bounded in $L^{3/2,\infty}(\Omega_R)$ and, therefore, converges in $\dot{H}^{-1}_{3/2,\infty}(\Omega_R)$ along a subsequence on account of the compact embedding (25). Then it is also convergent in $\dot{H}^{-1}_{3/2,\infty}(\Omega)$ by virtue of (75).

One next shows that $1 + \mathcal{R}$ is injective. Suppose $f \in \dot{H}^{-1}_{3/2,\infty}(\Omega)$ fulfills $(1 + \mathcal{R}) f = 0$ in $\dot{H}^{-1}_{3/2,\infty}(\Omega)$. Since $f = -\mathcal{R} f \in L^{3/2,\infty}(\Omega_R)$ which vanishes outside A, one has $f = 0$ in $\Omega \setminus A$. It thus suffices to show that $f = 0$ in A. In view of (72) and (73), it follows from Corollary 2 that $\{\Psi f, \Pi f\} = \{0, 0\}$. Hence, by (71) one observes

$$\{u_1, p_1\} = \{0, 0\}, \quad |x| \geq R - 1; \qquad \{u_0, p_0\} = \{0, 0\}, \quad |x| \leq R - 4$$

which shows that both $\{u_1, p_1\}$ and $\{u_0, p_0\}$ can be regarded as solutions to

$$-\Delta v + \nabla \theta = f, \qquad \operatorname{div} v = 0, \qquad \text{in } B_R; \qquad v|_{\partial B_R} = 0$$

and belong to $\dot{H}^{1}_{3/2,\infty}(B_R) \times L^{3/2,\infty}(B_R)$. It follows from uniqueness assertion of Lemma 6 that $u_1 = u_0$ and that $p_1 = p_0 + c$ for some constant c. One goes back to (71) to see that $0 = \Pi f = (1 - \phi)(p_0 + c) + \phi p_0$; however, the side condition $\int_{\Omega_R} p_0 \, dx = 0$ yields $c = 0$, so that $p_1 = p_0$. After all, one finds $\{u_1, p_1\} = \{0, 0\}$, yielding $f = 0$ in A. By the Fredholm alternative, $1 + \mathcal{R}$ is bijective, and, therefore, the pair

$$u = \Psi(1 + \mathcal{R})^{-1} f, \qquad p = \Pi(1 + \mathcal{R})^{-1} f$$

provides the desired solution, which enjoys (66) on account of (72). The proof is complete. □

5 Asymptotic Structure of the Navier-Stokes Flow

This section is devoted to a precise look at the profile of solutions for $|x| \to \infty$ obtained in Theorem 2. In order to make essential points clear, it would be better to avoid things caused by less local regularity of solutions. In what follows, let us consider smooth solutions which are of class $\{u, p\} \in L^{3,\infty}(\Omega) \times L^{3/2,\infty}(\Omega)$. If the solution enjoyed slightly faster decay property than described in Theorem 2, such as $u \cdot \nabla u = O(|x|^{-\alpha})$ with $\alpha > 3$ or $u \otimes u \in L^s(\Omega)$ with $s \leq 3/2$, then one could regard the nonlinear term as the external force and use Theorem 1 to see that the leading profile would be still the Stokes fundamental solution. But that is not the case here. As mentioned in sect. 1, the balance between the linear part and nonlinearity implies that the leading term of asymptotic expansion would be a self-similar solution.

Given $b \in \mathbb{R}^3 \setminus \{0\}$, Landau [60] (see also [9, 75]) found a nontrivial exact solution to (8), which satisfies axial symmetry about $\mathbb{R}b$ as well as homogeneity (9). Set $x = |x|\sigma,\ \sigma = (\sigma_1, \sigma_2, \sigma_3)^T \in \mathbb{S}^2$ (unit sphere). When b is parallel to e_3, the Landau solution is of the form

$$\begin{aligned} U(x) &= \frac{2}{|x|}\left[\frac{c\sigma_3 - 1}{(c-\sigma_3)^2}\,\sigma + \frac{1}{c-\sigma_3}\,e_3\right], \\ P(x) &= \frac{4\,(c\sigma_3 - 1)}{|x|^2(c-\sigma_3)^2} \end{aligned} \tag{76}$$

with parameter $c \in (-\infty, -1) \cup (1, \infty)$, and it satisfies

$$-\Delta U + \nabla P + U \cdot \nabla U = k e_3 \delta, \qquad \operatorname{div} U = 0 \qquad \text{in } \mathcal{D}'(\mathbb{R}^3)$$

(see (11)), where k is given by

$$k = k(c) = \frac{8\pi c}{3(c^2-1)}\left(2 + 6c^2 - 3c(c^2-1)\log\frac{c+1}{c-1}\right). \tag{77}$$

This calculation was done by Cannone and Karch [9, Proposition 2.1] (see also Batchelor [2, p.209]). The function $k(\cdot)$ is monotonically decreasing on each of intervals $(-\infty, -1)$ and $(1, \infty)$ and fulfills

$$k(c) \to 0 \quad (|c| \to \infty); \quad k(c) \to -\infty \quad (c \to -1); \quad k(c) \to \infty \quad (c \to 1).$$

Hence, for every $b \in \mathbb{R}^3 \setminus \{0\}$ parallel to e_3, there is a unique $c \in (-\infty, -1) \cup (1, \infty)$ such that $k(c)e_3 = b$. Since the Navier-Stokes system (8) is rotationally invariant, the Landau solution $\{U_b, P_b\}$ for general $b \in \mathbb{R}^3 \setminus \{0\}$ is given by rotation of (76). Let $\mathcal{O} \in \mathbb{R}^{3\times 3}$ be an orthogonal matrix that fulfills $\mathcal{O}\frac{b}{|b|} = e_3$. Then one finds

$$\begin{aligned} U_b(x) &= \frac{2}{|x|}\left[\frac{c(\mathcal{O}\sigma)_3 - 1}{\{c-(\mathcal{O}\sigma)_3\}^2}\,\sigma + \frac{1}{c-(\mathcal{O}\sigma)_3}\,\frac{b}{|b|}\right], \\ P_b(x) &= \frac{4\,\{c(\mathcal{O}\sigma)_3 - 1\}}{|x|^2\{c-(\mathcal{O}\sigma)_3\}^2} \end{aligned} \tag{78}$$

for $x = |x|\sigma,\ \sigma \in \mathbb{S}^2$. Since

$$\|U_b\|_{\infty,\mathbb{S}^2} + \|P_b\|_{\infty,\mathbb{S}^2} = O(|c|^{-1})$$

for $|c| \to \infty$ or, equivalently, $|b| \to 0$, one observes

$$\|U_b\|_{3,\infty,\mathbb{R}^3} + \|P_b\|_{3/2,\infty,\mathbb{R}^3} \to 0 \qquad (b \to 0). \tag{79}$$

When $b = 0$, one may understand $\{U_0, P_0\} = \{0, 0\}$. As proved by Šverák [74], for each $b \in \mathbb{R}^3$ the Landau solution (78) is the only solution to (11) which is smooth in $\mathbb{R}^3 \setminus \{0\}$ and possesses the homogeneity (9) (however, without assuming axisymmetry), and the family of Landau solutions covers all of possible self-similar solutions to (8). This fact as well as the Eq. (11) itself is essential in the proof of Theorem 4 below, while the exact form (78) is not really needed except for (79).

Given Navier-Stokes flow $u \in L^{3,\infty}(\Omega)$, the aim is to clarify how a specific Landau solution is singled out from the set $\mathcal{S}$ (see (10)). One also takes care of the external force, to which less attention has been paid in previous literature except [48]. Concerning that, the situation is the same as in Theorem 1 for the Stokes flow, that is, the class $f \in \dot{H}^{-1}_{3/2,\infty}(\Omega)$ yields the balance between the Landau solution and the error term. Thus, one needs slightly more decay property of f. In this presentation the class of the external force is a bit larger than the one in [48], for pointwise decay of F is not assumed below.

The main result reads

Theorem 4. *Let $f = f_0 + div\ F$ with*

$$F \in L^{s,\infty}(\Omega) \cap L^{3/2,\infty}(\Omega) \cap C^\infty(\Omega) \tag{80}$$

for some $s \in (1, 3/2)$. Suppose $f_0 \in C^\infty(\Omega)$ satisfies (36) for some $\alpha > 3$. Let $\{u, p\}$ be a smooth solution of class

$$u \in L^{3,\infty}(\Omega), \qquad p \in L^{3/2,\infty}(\Omega), \qquad \nabla u \in L^{3/2,\infty}_{\mathrm{loc}}(\overline{\Omega}) \tag{81}$$

to (62). Set

$$M = \int_{\partial\Omega} [T(u, p) - u \otimes u + F]\, \nu\, d\sigma + \int_\Omega f_0(y)\, dy \tag{82}$$

and

$$q = \max\{3/(\alpha-1), s\} \in (1, 3/2), \qquad q_* = \max\{3/(\alpha-2), s_*\} \in (3/2, 3),$$

where $s_* \in (3/2, 3)$ is defined by $1/s_* = 1/s - 1/3$ (note that $1/q_* = 1/q - 1/3$). There is a constant $\gamma > 0$ such that if

$$\|u\|_{3,\infty} + |M| < \gamma, \tag{83}$$

then

$$u - U_M \in L^{r_*}(\Omega), \qquad \{\nabla u - \nabla U_M,\ p - P_M\} \in L^r(\Omega), \quad \forall r \in (q, 3/2), \tag{84}$$

where $\{U_M, P_M\} \in \mathcal{S}$ denotes the Landau solution with label given by (82), see (10), and $r_* \in (q_*, 3)$ is defined by $1/r_* = 1/r - 1/3$.

Even if F *satisfies*

$$F \in L^{3/2}(\Omega) \cap C^{\infty}(\Omega) \tag{85}$$

in place of (80)*, the conclusion above still holds true, in which* (84) *is replaced by*

$$u - U_M \in L^{3,3/2}(\Omega \setminus B_{3R}), \qquad \{\nabla u - \nabla U_M,\ p - P_M\} \in L^{3/2}(\Omega \setminus B_{3R}), \tag{86}$$

where $R > 0$ *is taken large enough.*

Proof. Since $T(u, p) - u \otimes u - F \in L^t_{\rm loc}(\overline{\Omega})$ for every $t \in (1, 3/2)$, the boundary integral of (82) makes sense by the same reasoning as in Theorem 1. Set $\beta = \int_{\partial\Omega} \nu \cdot u \, d\sigma$. One assumes $0 \in \mathrm{int}\,(\mathbb{R}^3 \setminus \Omega)$ without loss and uses the flux carrier $z(x)$ given by (39), which fulfills not only (40) but also $z \cdot \nabla z = \nabla \frac{|z|^2}{2}$. So the pair

$$\tilde{u} = u - z, \qquad \tilde{p} = p - \frac{|z|^2}{2}$$

belongs to the class (81) and obeys

$$-\Delta \tilde{u} + \nabla \tilde{p} + \tilde{u} \cdot \nabla \tilde{u} = f - u \cdot \nabla z - z \cdot \nabla u, \qquad \mathrm{div}\, \tilde{u} = 0 \qquad \text{in } \Omega$$

as well as vanishing flux condition (41). Fix $R_0 > 0$ such that $\mathbb{R}^3 \setminus \Omega \subset B_{R_0}$. Let $R \in [R_0, \infty)$ be the parameter to be determined later. One takes

$$v = (1 - \psi)\tilde{u} + B[\tilde{u} \cdot \nabla \psi], \qquad \theta = (1 - \psi)\tilde{p}, \tag{87}$$

by using the cutoff function (42) together with the Bogovskii operator B (Lemma 1) in the domain A_R (see (43)). One then finds

$$-\Delta v + \nabla \theta + v \cdot \nabla v = h, \qquad \mathrm{div}\, v = 0 \qquad \text{in } \mathbb{R}^3 \tag{88}$$

with

$$h := g + (1 - \psi) f_0 + \mathrm{div}\,\{(1 - \psi)(F - z \otimes u - u \otimes z)\}, \qquad g \in C_0^{\infty}(A_R),$$

where the exact form of g is not needed as in the proof of Theorem 1.

One is going to show that

$$\int_{\mathbb{R}^3} \{g + (1 - \psi) f_0\}(y)\, dy = M, \tag{89}$$

(see (82)) and that

$$\|v\|_{3,\infty,\mathbb{R}^3} \le C\|u\|_{3,\infty} + \frac{C|\beta|}{R}. \tag{90}$$

Let $\rho > 3R$. By taking

$$\int_{|y|=\rho} \left(\frac{|z|^2}{2} - z \otimes z\right) \frac{y}{\rho}\, d\sigma = \frac{-\beta^2}{32\pi^2\rho^5} \int_{|y|=\rho} y\, d\sigma = 0$$

as well as (47) into account, one finds

$$\begin{aligned}
&\int_{B_\rho} \{g + (1-\psi) f_0\}(y)\, dy \\
&= -\int_{B_\rho} \operatorname{div} \{T(v,\theta) - v \otimes v + (1-\psi)(F - z \otimes u - u \otimes z)\}\, dy \\
&= -\int_{|y|=\rho} [T(\tilde{u},\tilde{p}) - \tilde{u} \otimes \tilde{u} + F - z \otimes u - u \otimes z]\, \frac{y}{\rho}\, d\sigma \\
&= -\int_{|y|=\rho} [T(u,p) - u \otimes u + F]\, \frac{y}{\rho}\, d\sigma \\
&= \int_{\partial\Omega} [T(u,p) - u \otimes u + F]\, \nu\, d\sigma + \int_{\Omega_\rho} f_0(y)\, dy.
\end{aligned}$$

Letting $\rho \to \infty$ leads to (89). To show (90), set $v_u := (1-\psi)u + B[u \cdot \nabla\psi]$ and $v_z := (1-\psi)z + B[z \cdot \nabla\psi]$. Since the map $u \mapsto v_u$ is bounded from $L^q(\Omega)$ to $L^q(\mathbb{R}^3)$ for every $q \in (1,\infty)$, the real interpolation implies that

$$\|v_u\|_{3,\infty,\mathbb{R}^3} \le C\|u\|_{3,\infty}. \tag{91}$$

Fix $\tau \in (3,\infty)$ arbitrarily. It then follows from (42), the Gagliardo-Nirenberg and Poincaré inequalities together with dilation invariance of the estimate of the Bogovskii operator (due to Borchers and Sohr [8], see Lemma 1) that

$$\begin{aligned}
\|B[z \cdot \nabla\psi]\|_{\infty,A_R} &\le C\|B[z \cdot \nabla\psi]\|_{\tau,\mathbb{R}^3}^{1-3/\tau} \|\nabla B[z \cdot \nabla\psi]\|_{\tau,\mathbb{R}^3}^{3/\tau} \\
&\le CR^{1-3/\tau} \|\nabla B[z \cdot \nabla\psi]\|_{\tau,A_R} \\
&\le CR^{1-3/\tau} \|z \cdot \nabla\psi\|_{\tau,A_R} \\
&\le C\|z\|_{\infty,A_R} = \frac{C|\beta|}{R^2}.
\end{aligned}$$

Thus, $|x||v_z(x)| \le C|\beta|/R$ for all $x \in \mathbb{R}^3$ and thereby

$$\|v_z\|_{3,\infty,\mathbb{R}^3} \le \frac{C|\beta|}{R},$$

which together with (91) concludes (90).

Let $\{U, P\} = \{U_M, P_M\}$ be the Landau solution whose label is given by (82). To regularize $\{U, P\}$ around $x = 0$, one may follow the same cutoff procedure as in (87):

$$V = (1-\psi)U + B[U \cdot \nabla\psi], \qquad \Theta = (1-\psi)P.$$

One then observes $\int_{A_R} U \cdot \nabla\psi \, dx = 0$ because

$$\int_{A_R} \operatorname{div}(\psi U)\, dx = \int_{|x|=R} \frac{-x \cdot U}{R}\, d\sigma = \int_{|x|=\varepsilon} \frac{-x \cdot U}{\varepsilon}\, d\sigma = O(\varepsilon) \qquad (\varepsilon \to 0).$$

The same reasoning as in (91) implies that

$$\|V\|_{3,\infty,\mathbb{R}^3} \le C\|U\|_{3,\infty,\mathbb{R}^3}. \tag{92}$$

The pair $\{V, \Theta\}$ obeys

$$-\Delta V + \nabla\Theta + V \cdot \nabla V = H, \qquad \operatorname{div} V = 0 \qquad \text{in } \mathbb{R}^3 \tag{93}$$

for some function $H \in C_0^\infty(A_R)$ with

$$\int_{\mathbb{R}^3} H(y)\, dy = M. \tag{94}$$

In fact, by using a test function $\phi \in C^\infty(\mathbb{R}^3)$ satisfying $\phi(x) = 1$ ($|x| \le 3R$) and $\phi(x) = 0$ ($|x| \ge 4R$), one sees from (11) with $b = M$ that

$$\begin{aligned}
\int_{A_R} H(y)\, dy &= -\int_{|y|=3R} (T(U,P) - U \otimes U)\frac{y}{3R}\, d\sigma \\
&= \int_{3R<|y|<4R} (T(U,P) - U \otimes U)(\nabla\phi)\, dy \\
&= \langle M\delta, \phi\rangle
\end{aligned}$$

which leads to (94).

Set

$$\overline{v} := v - V \in L^{3,\infty}(\mathbb{R}^3), \qquad \overline{\theta} := \theta - \Theta \in L^{3/2,\infty}(\mathbb{R}^3),$$

and consider the auxiliary linear system

$$-\Delta w+\nabla\vartheta = (h-H)-\mathrm{div}\,(v\otimes w+w\otimes V), \qquad \mathrm{div}\, w=0 \qquad \text{in } \mathbb{R}^3 \tag{95}$$

with

$$\begin{aligned} h-H &= \{g+(1-\psi)f_0-H\}+\mathrm{div}\,\{(1-\psi)(F-z\otimes u-u\otimes z)\}, \\ &\int_{\mathbb{R}^3}\{g+(1-\psi)f_0-H\}(y)\,dy=0, \end{aligned} \tag{96}$$

where the latter property follows from (89) and (94). It is obvious that $\{\overline{v},\overline{\theta}\}$ satisfies (95). Furthermore, it is the only solution to (95) within the class $L^{3,\infty}(\mathbb{R}^3)\times L^{3/2,\infty}(\mathbb{R}^3)$ provided that

$$\|v\|_{3,\infty,\mathbb{R}^3}+\|V\|_{3,\infty,\mathbb{R}^3}<\gamma_1 \tag{97}$$

with a suitable small constant $\gamma_1 > 0$. In fact, let $\{w,\vartheta\}\in L^{3,\infty}(\mathbb{R}^3)\times L^{3/2,\infty}(\mathbb{R}^3)$ satisfy (95) in which $h-H$ is replaced by zero, then it follows from uniqueness for the homogeneous Stokes system within this class (see Lemma 3) that $\{w,\vartheta\}$ coincides with the solution to (33), where the external force is given by $-\mathrm{div}\,(v\otimes w+w\otimes V)\in \dot{H}^{-1}_{3/2,\infty}(\mathbb{R}^3)$, obtained in Lemma 3. By (35) together with

$$\|\mathrm{div}\,(v\otimes w+w\otimes V)\|_{\dot{H}^{-1}_{3/2,\infty}(\mathbb{R}^3)}\le C(\|v\|_{3,\infty,\mathbb{R}^3}+\|V\|_{3,\infty,\mathbb{R}^3})\|w\|_{3,\infty,\mathbb{R}^3} \tag{98}$$

which follows from (13), one gets $\{w,\vartheta\}=\{0,0\}$ under the smallness condition (97). Hence, the task is to find a solution

$$w\in L^{q_*,\infty}(\mathbb{R}^3)\cap L^{3,\infty}(\mathbb{R}^3), \qquad \{\nabla w,\ \vartheta\}\in L^{q,\infty}(\mathbb{R}^3)\cap L^{3/2,\infty}(\mathbb{R}^3) \tag{99}$$

to (95) under the condition (80). Once that is available, the only solution $\{\overline{v},\overline{\theta}\}$ must belong to the class (99), which combined with (16) (with $q=r$) and decay properties of (39) lead to

$$u-U\in L^{r_*}(\Omega\setminus B_{3R}), \qquad \{\nabla u-\nabla U,\ p-P\}\in L^r(\Omega\setminus B_{3R})$$

for every $r\in(q,3/2)$. By (81) one has also

$$u-U\in L^{r_*}(\Omega_{3R}), \qquad \{\nabla u-\nabla U,\ p-P\}\in L^r(\Omega_{3R})$$

for the same r as above. Thus, (84) is proved.

Set

$$w_1:=E*\{g+(1-\psi)f_0-H\}, \qquad \vartheta_1:=Q*\{g+(1-\psi)f_0-H\}. \tag{100}$$

It follows from Lemma 2 together $(96)_2$ that $\nabla^j w_1$ $(j = 0, 1)$ and ϑ_1 enjoy the same pointwise estimates at large distance as the error terms of (32) do. Therefore,

$$\begin{aligned} &w_1 \in L^{t_*,\infty}(\mathbb{R}^3 \setminus B_L) \cap L^\infty(\mathbb{R}^3 \setminus B_L), \\ &\{\nabla w_1,\ \vartheta_1\} \in L^{t,\infty}(\mathbb{R}^3 \setminus B_L) \cap L^\infty(\mathbb{R}^3 \setminus B_L) \end{aligned} \tag{101}$$

for some $L > 0$, where

$$\begin{array}{lll} t = 3/(\alpha-1), & t_* = 3/(\alpha-2) & \text{if } \alpha \in (3,4), \\ t \in (1, 3/2) \text{ is arbitrary}, & 1/t_* = 1/t - 1/3 & \text{if } \alpha \geq 4. \end{array}$$

Since $w_1,\ \nabla w_1,\ \vartheta_1 \in L^\tau(B_L)$ for every $\tau \in (1,\infty)$, which follows from $g + (1 - \psi) f_0 + H \in L^\tau(\mathbb{R}^3)$ for such τ, we obtain

$$w_1 \in L^{t_*,\infty}(\mathbb{R}^3) \cap L^{3,\infty}(\mathbb{R}^3), \qquad \{\nabla w_1,\ \vartheta_1\} \in L^{t,\infty}(\mathbb{R}^3) \cap L^{3/2,\infty}(\mathbb{R}^3). \tag{102}$$

One observes $\sqrt{1-\psi}|z| \in L^{3/2,\infty}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, which combined with $\sqrt{1-\psi}|u| \in L^{3,\infty}(\mathbb{R}^3)$ and (13) imply that

$$(1-\psi)(z \otimes u + u \otimes z) \in L^{\tau,\infty}(\mathbb{R}^3), \qquad \forall\, \tau \in (1, 3]. \tag{103}$$

It thus follows from (80) that

$$\operatorname{div}\left\{(1-\psi)(F - z \otimes u - u \otimes z)\right\} \in \dot{H}^{-1}_{s,\infty}(\mathbb{R}^3) \cap \dot{H}^{-1}_{3/2,\infty}(\mathbb{R}^3). \tag{104}$$

Let

$$\begin{aligned} &w_2 \in \dot{H}^1_{s,\infty}(\mathbb{R}^3) \cap \dot{H}^1_{3/2,\infty}(\mathbb{R}^3) \hookrightarrow L^{s_*,\infty}(\mathbb{R}^3) \cap L^{3,\infty}(\mathbb{R}^3), \\ &\vartheta_2 \in L^{s,\infty}(\mathbb{R}^3) \cap L^{3/2,\infty}(\mathbb{R}^3), \end{aligned} \tag{105}$$

be the solution to (33) with the external force (104) obtained in Lemma 3. Then one finds

$$\begin{aligned} &w_0 := w_1 + w_2 \in L^{q_*,\infty}(\mathbb{R}^3) \cap L^{3,\infty}(\mathbb{R}^3), \\ &\nabla w_0 \in L^{q,\infty}(\mathbb{R}^3) \cap L^{3/2,\infty}(\mathbb{R}^3), \\ &\vartheta_0 := \vartheta_1 + \vartheta_2 \in L^{q,\infty}(\mathbb{R}^3) \cap L^{3/2,\infty}(\mathbb{R}^3), \end{aligned} \tag{106}$$

with $q = \max\{\frac{3}{\alpha-1}, s\}$, where $1/q_* = 1/q - 1/3$.

Given $w \in L^{q_*,\infty}(\mathbb{R}^3) \cap L^{3,\infty}(\mathbb{R}^3)$ (see (99)), the velocity part of the unique solution to (33) with the external force

$$-\operatorname{div}(v \otimes w + w \otimes V) \in \dot{H}^{-1}_{q,\infty}(\mathbb{R}^3) \cap \dot{H}^{-1}_{3/2,\infty}(\mathbb{R}^3)$$

obtained in Lemma 3 is denoted by Tw. Then problem (95) is rewritten as

$$w = w_0 + Tw \tag{107}$$

and the right-hand side returns to the class (99) on account of (106). By (35) together with (98) (and the similar one in terms of $\|w\|_{q_*,\infty,\mathbb{R}^3}$), the map $w \mapsto w_0 + Tw$ is contractive from $L^{q_*,\infty}(\mathbb{R}^3) \cap L^{3,\infty}(\mathbb{R}^3)$ into itself provided that

$$\|v\|_{3,\infty,\mathbb{R}^3} + \|V\|_{3,\infty,\mathbb{R}^3} < \gamma_2 \tag{108}$$

with a suitable small constant $\gamma_2 = \gamma_2(q) > 0$. One thus gets a fixed point $w \in L^{q_*,\infty}(\mathbb{R}^3) \cap L^{3,\infty}(\mathbb{R}^3)$ with $\nabla w \in L^{q,\infty}(\mathbb{R}^3) \cap L^{3/2,\infty}(\mathbb{R}^3)$. Since the pressure associated with Tw belongs to $L^{q,\infty}(\mathbb{R}^3) \cap L^{3/2,\infty}(\mathbb{R}^3)$, so does the pressure associated with the fixed point because of (106).

In view of (90), (92) with $U = U_M$, and (79), the parameter $R \in [R_0, \infty)$ is first fixed so that $|\beta|/R$ is small enough, and then it is possible to take a suitable constant $\gamma > 0$ such that both (97) and (108) are accomplished under the condition (83).

Finally, consider the case when F satisfies (85) in place of (80), then

$$\operatorname{div}\left\{(1-\psi)(F - z \otimes u - u \otimes z)\right\} \in \dot{H}^{-1}_{3/2}(\mathbb{R}^3)$$

on account of (103). Hence, one has

$$w_2 \in \dot{H}^1_{3/2}(\mathbb{R}^3) \hookrightarrow L^{3,3/2}(\mathbb{R}^3), \qquad \vartheta_2 \in L^{3/2}(\mathbb{R}^3)$$

(see (20)), instead of (105). On the other hand, (101) yields

$$w_1 \in L^{3,3/2}(\mathbb{R}^3 \setminus B_L), \qquad \{\nabla w_1, \vartheta_1\} \in L^{3/2}(\mathbb{R}^3 \setminus B_L)$$

for some $L > 0$, which implies

$$w_1 \in L^{3,3/2}(\mathbb{R}^3), \qquad \{\nabla w_1, \vartheta_1\} \in L^{3/2}(\mathbb{R}^3) \tag{109}$$

by the same reasoning as in (102). One thus obtains

$$w_0 \in L^{3,3/2}(\mathbb{R}^3), \qquad \{\nabla w_0, \vartheta_0\} \in L^{3/2}(\mathbb{R}^3) \tag{110}$$

instead of (106). For the proof of (86), it suffices to find a solution

$$w \in L^{3,3/2}(\mathbb{R}^3), \qquad \{\nabla w, \vartheta\} \in L^{3/2}(\mathbb{R}^3) \tag{111}$$

to (95). Given $w \in L^{3,3/2}(\mathbb{R}^3)$, one observes

$$-\operatorname{div}(v \otimes w + w \otimes V) \in \dot{H}^{-1}_{3/2}(\mathbb{R}^3)$$

with

$$\|\mathrm{div}\,(v\otimes w+w\otimes V)\|_{\dot H^{-1}_{3/2}(\mathbb{R}^3)}\le C(\|v\|_{3,\infty,\mathbb{R}^3}+\|V\|_{3,\infty,\mathbb{R}^3})\|w\|_{3,3/2,\mathbb{R}^3}$$

by (17). Therefore, the term Tw in (107) belongs to $\dot H^1_{3/2}(\mathbb{R}^3)\hookrightarrow L^{3,3/2}(\mathbb{R}^3)$. By virtue of (110), the rest of the proof of existence of a solution of class (111) is the same as above. The proof is complete. □

Remark 5. If, in addition, div $F\in L^1(\Omega)\cap L^t_{\mathrm{loc}}(\overline{\Omega})$ for some $t>1$, then the equality $\int_\Omega f(y)\,dy=\int_\Omega f_0(y)\,dy+\int_{\partial\Omega}F\nu\,d\sigma$ is justified, so that (82) is equal to $M(0,0,f)$ given by (4).

Remark 6. If F is absent (so that $f=f_0$) and if $\alpha\ge 4$, the exponent q is chosen arbitrarily in the interval $(1,3/2)$ (as close to 1 as one wishes) and the small constant γ depends on the choice of q.

Finally, let us consider the Navier-Stokes system around a rotating obstacle

$$-\Delta u+\nabla p+u\cdot\nabla u-(\omega\times x)\cdot\nabla u+\omega\times u=f,\qquad \mathrm{div}\,u=0\qquad \text{in }\Omega,\tag{112}$$

where $\omega=ae_3$ with $a\in\mathbb{R}\setminus\{0\}$. As mentioned in [74, section 3], a scaling argument with (7) works for the case $\omega=0$ to see that, if solutions are asymptotically homogeneous of degree (-1), then their leading terms are the Landau solutions. But (112) is no longer invariant under the transformation (7) unless $\omega=0$. One thus needs another heuristic observation, which is based on knowledge of the linearized system (51). The point is the asymptotic expansion (58), which yields the leading term (59) of the linearized flow. Let $u\in L^{3,\infty}(\Omega)$ be the solution to the Navier-Stokes system (112). In view of features of (59), it is reasonable to expect that the leading term, denoted by U, still keeps symmetry about the axis of rotation (i.e., $\mathbb{R}e_3$) as well as homogeneity of degree (-1) and that the quantity $e_3\cdot M$ controls the rate of decay; here, $M=M(0,\omega,f)$ is given by (4), or

$$M=\int_{\partial\Omega}[T(u,p)-u\otimes(u-\omega\times y)-(\omega\times y)\otimes u+F]\,\nu\,d\sigma_y+\int_\Omega f_0(y)\,dy\tag{113}$$

when the external force is of the form $f=f_0+\mathrm{div}\,F$ satisfying the same assumptions as in Theorem 4. One can also expect, as in (60), that the leading term U together with some scalar field P solves

$$-\Delta U+\nabla P+U\cdot\nabla U-(\omega\times x)\cdot\nabla U+\omega\times U=(e_3\cdot M)\,e_3\delta,\qquad \mathrm{div}\,U=0\tag{114}$$

in $\mathcal{D}'(\mathbb{R}^3)$; however, this is reduced to

$$-\Delta U+\nabla P+U\cdot\nabla U=(e_3\cdot M)\,e_3\delta,\qquad \mathrm{div}\,U=0\tag{115}$$

because U satisfies (61) under the symmetry about $\mathbb{R}e_3$. Hence, U is a self-similar solution to (8), and it should be a Landau solution $U_{(e_3\cdot M)e_3}$. This observation can be justified along the same way as in the proof of Theorem 4, in which (94) and $(96)_2$ should be replaced by

$$\int_{\mathbb{R}^3} H(y)\,dy = (e_3\cdot M)e_3, \qquad e_3\cdot\int_{\mathbb{R}^3}\{g+(1-\psi)f_0-H\}(y)\,dy = 0. \tag{116}$$

Then one can use Lemma 5 to obtain $(101)_1$ for w_1, which combined with the result of [20] implies $(102)_1/(109)_1$ for w_1. This was done by Farwig and Hishida [19] for (112) with no external force under the no-slip boundary condition $u|_{\partial\Omega} = \omega\times x$. But their result can be extended to the case where the external force satisfies the same conditions as in Theorem 4 without assuming any boundary condition on $\partial\Omega$ (see [47, section 6] for the flux carrier). Note that the corresponding Landau pressure $P_{(e_3\cdot M)e_3}$ is not solely the leading term of the associated pressure unlike Theorem 4 for $\omega = 0$. This is because $(116)_2$ is not sufficient to get faster decay of the pressure, so that $(101)_2$ for ϑ_1 is not available. In addition to $P_{(e_3\cdot M)e_3}$, however, one can use $(32)_2$ to take the leading term of ϑ_1 given by (100) so that one gets

$$\begin{aligned} &P_{(e_3\cdot M)e_3} + Q(x)\cdot\int_{\mathbb{R}^3}\{g+(1-\psi)f_0-H\}(y)\,dy \\ &= P_{(e_3\cdot M)e_3} + Q(x)\cdot\{M-(e_3\cdot M)e_3\}, \end{aligned}$$

which is the leading term of the pressure, where $Q(x)$ is the fundamental solution (29). This was found by Farwig, Galdi, and Kyed [16]. In [16] the authors deduced the asymptotic expansion of solutions of the Leray class satisfying the energy inequality, which eventually decay like $|x|^{-1}$ (see [30]), as long as they are small enough.

6 Conclusion

The asymptotic structure at infinity as well as existence of 3D exterior stationary Navier-Stokes flows being in $L^{3,\infty}$ (weak-L^3) is discussed when the obstacle is at rest. The class $L^{3,\infty}$ is critical from both of the following points of view (one is essential, while the other would be technical). On the one hand, it is optimal summability of generic flows in the sense that better summability than $L^{3,\infty}$ necessarily implies the vanishing total net force, $M(0,0,f) = 0$ (see (4)). On the other hand, it is difficult to conclude the stability of flows with worse summability than $L^{3,\infty}$ (even if they are small enough) as long as one adopts the mathematical analysis developed until now.

The Stokes fundamental solution, which is the leading profile of the Stokes flow, is no longer the leading profile of the Navier-Stokes flow being in $L^{3,\infty}$ on account of the balance between the linear part and nonlinearity. The correct leading term is the Landau solution whose label is the net force $M(0,0,f)$ of given Navier-Stokes flow provided it is small enough. The reason comes essentially from Šverák's observation on structure of the set that consists of all homogeneous Navier-Stokes flows of degree (-1). One finds a contrast with the case where the obstacle is translating, in which the leading profile is described in terms of the linear part, that is, the Oseen fundamental solution. When the obstacle is rotating with constant angular velocity ω, the leading term is still a Landau solution; however, its label is given by $(\frac{\omega}{|\omega|}\cdot M)\frac{\omega}{|\omega|}$ with $M = M(0,\omega,f)$, which follows from a decay structure of the associated fundamental solution. One is able to specify a condition on the external force (which is not necessarily of bounded support) such that the conclusions above hold true.

Several open questions about the related issues are in order. The results above (except the case where the obstacle is translating) require smallness of the Navier-Stokes flow in $L^{3,\infty}$ because a perturbation argument is adopted. Asymptotic structure of large solutions of this class is much more involved and remains open. When $\omega = 0$, as pointed out by Šverák [74, section 3], once one establishes the asymptotic expansion with homogeneous leading term of degree (-1) without any smallness, it turns out by a scaling argument that the leading term must be a Landau solution.

If the external force f has less decay property (i.e., α is close to 3 and s is close to $3/2$, or even $F \in L^{3/2}(\Omega)$, in Theorem 4), it is then hopeless to find out the second term after the leading one (a Landau solution) in the asymptotic expansion. For the simple case $f = 0$, however, one can ask what the second term is. It is probably homogeneous of degree (-2).

As compared with the 3D problem, there are many open problems concerning exterior stationary Navier-Stokes flows in 2D (see Galdi [28, Chapter XII] for the details). The most difficult case is that the obstacle is at rest (unless assuming any symmetry), where the linearization method can no longer work because of the Stokes paradox. No one knows the asymptotic structure of the Navier-Stokes flow even if it is small enough; however, a remarkable conjecture based on numerical verification has been recently proposed by Guillod and Wittwer [36]. When the obstacle is translating, the problem is less difficult on account of decay structure of the 2D Oseen fundamental solution, which is the leading profile of the Navier-Stokes flows without restriction on the magnitude as in 3D (see Smith [72] and Galdi [28, Theorem XII.8.1]). But the stability/instability of such flows is far from clear.

The case when the obstacle is rotating in 2D has been much less studied. As for the linearized problem, it was found by Hishida [45] that the oscillation caused by rotation of the obstacle leads to the resolution of the Stokes paradox and that the leading term of the flow at infinity involves the profile $x^\perp/|x|^2$ whose coefficient is (not the net force but) the torque, where $x^\perp = (-x_2, x_1)^\top$. Very recently, Higaki, Maekawa, and Nakahara [41] showed that asymptotic structure of small Navier-Stokes flow around a slowly rotating obstacle exhibits the same profile as above. Such a structure has still remained open unless imposing smallness on the angular velocity. Note that the pair

$$u(x) = \frac{cx^\perp}{|x|^2}, \qquad p(x) = \frac{-c^2}{2|x|^2} \qquad (c \in \mathbb{R}), \tag{117}$$

is a self-similar solution to the Navier-Stokes system (8) in $\mathbb{R}^2 \setminus \{0\}$ and that it also satisfies (112) with $f = 0$ in $\mathbb{R}^2 \setminus \{0\}$ since the last two terms in the left-hand side vanish, that is, $-ax^\perp \cdot \nabla u + au^\perp = 0$ (by following the standard notation in 2D). By Theorem 2 of Šverák [74, section 5], under the zero flux condition $\int_{\mathbb{S}^1} \nu \cdot u \, d\sigma = 0$, the homogeneous Navier-Stokes flow of degree (-1) in 2D must be either the circular flow (117) or a particular Jeffery-Hamel flow (whose component tangent to the circle $\mathbb{S}^1$ vanishes). When the fluid region is in particular the exterior of a rotating disk, one refers to interesting papers by Hillairet and Wittwer [42] and by Maekawa [62]: the former finds the Navier-Stokes flow which is close to the solution (117) with large $|c|$ and whose leading profile is also given by $x^\perp/|x|^2$, and the latter successfully proves the L^2 stability of the solution (117) provided $|c|$ is sufficiently small. One can expect that this latter result would hold for small Navier-Stokes flow constructed in [41].

7 Cross-References

▸ Self-Similar Solutions to the Nonstationary Navier-Stokes Equations
▸ Steady-State Navier-Stokes Flow Around a Moving Body
▸ Time-Periodic Solutions to the Navier-Stokes Equations

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