Concepts of Solutions in the Thermodynamics of Compressible Fluids

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Abstract

The objective of this chapter is to highlight the recent development of the *mathematical theory* of complete fluids. The word *complete* means the governing system of equations is rich enough to incorporate the basic physical principles, in particular the first, second, and third laws of thermodynamics, in a correct and integral way into the mathematical model. In the whole text, the platform of classical *continuum mechanics* is adopted, where the fluid motion is described in terms of observable macroscopic quantities: the mass density, the (absolute) temperature, and the (bulk) velocity.

1 Introduction

This chapter reviews the recent development of the *mathematical theory* of complete fluid systems, where the word *complete* refers to the governing system of equations that is rich enough to incorporate the basic underlying physical principles: the first, second, and third laws of thermodynamics in the mathematical model. The standard platform of classical *continuum mechanics* is systematically used, where the fluid motion is described in terms of observable macroscopic quantities: the mass density, the (absolute) temperature, and the (bulk) velocity.

Continuum mechanics describes a fluid in motion in terms of numerical values of macroscopic quantities – fields – depending on the time t and the spatial position x. Throughout the whole text, the *Eulerian description* is used, where the coordinate frame is attached to the physical domain Ω occupied by the fluid. The fields are interrelated through a system of *field equations* – balance laws – reflecting the underlying physical principles of conservation of mass, momentum, energy, as well as other quantities as the case may be. The material properties of a specific fluid are characterized by *constitutive relations*. The interaction of the fluid with the outer world is specified through *boundary conditions*.

1.1 State Variables

In accordance with the general principles delineated above, the state of a fluid at any instant t is characterized by its *mass density* $\varrho = \varrho(t,x)$ and the *absolute temperature* $\vartheta = \vartheta(t,x)$. The motion is described by means of the macroscopic velocity field $\mathbf{u} = \mathbf{u}(t,x)$. Accordingly, the fluid moves along streamlines – the spatial curves $\mathbf{X} = \mathbf{X}(t)$ solving

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{X} = \mathbf{u}(t,\mathbf{X}).$$

Such a choice of state variables gives rise to a rather limited but still sufficiently rich class of fluids considered in this chapter. Obviously, more complex models are necessary and can be developed to handle more complicated real-world applications.

1.2 Conservation/Balance Laws

The conservation/balance laws in continuum mechanics are usually written in a general differential form

$$\partial_t d(t,x) + \mathrm{div}_x \mathbf{F}(t,x) = s(t,x).$$

A conservation/balance law reflects the underlying physical principle relating the changes of a volume density of a physical quantity d to its flux $\mathbf{F}$ and a possible source term s as the case may be. In the Eulerian coordinate system, the flux $\mathbf{F}$ consists of a convective (conservative) component $d\mathbf{u}$ and, at least for certain physical quantities, a diffusive (dissipative) part proportional to spatial derivatives of d.

1.2.1 Equation of Continuity, Mass Conservation

A mathematical formulation of the physical principle of mass conservation reads

$$\partial_t \varrho(t,x) + \mathrm{div}_x \left(\varrho(t,x)\mathbf{u}(t,x)\right) = 0. \tag{1}$$

The mass flux is purely convective and the source term is absent in (1).

1.2.2 Momentum Equation, Newton's Second Law

The time evolution of the momentum $\varrho\mathbf{u}$ is governed by the system of equations

$$\partial_t (\varrho(t,x)\mathbf{u}(t,x)) + \mathrm{div}_x \left(\varrho(t,x)\mathbf{u}(t,x) \otimes \mathbf{u}(t,x)\right) = \mathrm{div}_x \mathbb{T}(t,x) + \varrho(t,x)\mathbf{f}(t,x), \tag{2}$$

where $\mathbb{T}$ denotes the Cauchy stress specified below, and $\mathbf{f}$ is the volume density of the external forces acting on the fluid.

1.2.3 Energy Balance, First Law of Thermodynamics

The volume density of the total energy of the fluid

$$E = \frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e$$

consists of the kinetic energy component $\frac{1}{2}\varrho|\mathbf{u}|^2$ and the internal energy ϱe. In accordance with the specific choice of state variables, the (specific) internal energy $e = e(\varrho, \vartheta)$ is a function of the density ϱ and the temperature ϑ.

A mathematical formulation of the first law of thermodynamics reads:

$$\partial_t \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e \right] (t,x) + \mathrm{div}_x \left(\left[\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e \right] (t,x) \mathbf{u}(t,x) \right)$$
$$+ \mathrm{div}_x \mathbf{q}(t,x) - \mathrm{div}_x (\mathbb{T}(t,x) \cdot \mathbf{u}(t,x)) = \varrho(t,x) \mathbf{f}(t,x) \cdot \mathbf{u}(t,x)$$
$$+ \varrho(t,x) \mathcal{Q}(t,x), \tag{3}$$

where $\mathbf{q}$ denotes the diffusive part of the internal energy flux and $\mathcal{Q}$ the volume density of the external heat sources.

1.3 Constitutive Relations for Fluids

Fluids are characterized by Stokes' law:

$$\mathbb{T} = \mathbb{S} - p\mathbb{I}, \tag{4}$$

where $\mathbb{S}$ is the viscous stress tensor and p is a scalar quantity termed pressure. Similarly to the specific energy e, the pressure $p = p(\varrho, \vartheta)$ is a function of the state variables ϱ, ϑ.

1.3.1 Entropy, Second Law of Thermodynamics

The second law of thermodynamics postulates the existence of another thermodynamic function – entropy. Here the specific entropy $s = s(\varrho, \vartheta)$ is supposed to be interrelated with the internal energy e and the pressure p by means of Gibbs' equation:

$$\vartheta D s(\varrho, \vartheta) = D e(\varrho, \vartheta) + p(\varrho, \vartheta) D \left(\frac{1}{\varrho} \right), \tag{5}$$

where D stands for the differential with respect to ϱ and ϑ (see [28]).

Internal Energy Equation

In view of Stokes' relation (4), the total energy balance may be rewritten as the internal energy equation:

$$\partial_t (\varrho e(\varrho, \vartheta)) + \mathrm{div}_x (\varrho e(\varrho, \vartheta) \mathbf{u}) + \mathrm{div}_x \mathbf{q} = \mathbb{S} : \nabla_x \mathbf{u} - p(\varrho, \vartheta) \mathrm{div}_x \mathbf{u} + \varrho \mathcal{Q}, \tag{6}$$

or, equivalently, in the form of thermal energy balance

$$\varrho c_v(\varrho, \vartheta) \Big(\partial_t \vartheta + \mathbf{u} \cdot \nabla_x \vartheta \Big) + \mathrm{div}_x \mathbf{q} = \mathbb{S} : \nabla_x \mathbf{u} - \varrho \frac{\partial p(\varrho, \vartheta)}{\partial \vartheta} \mathrm{div}_x \mathbf{u} + \varrho \mathcal{Q}, \tag{7}$$

where we have introduced the specific heat at constant volume:

$$c_v(\varrho, \vartheta) = \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta}.$$

Note that the passage from (6) to (7) uses the equation of continuity (1).

In addition, the *thermodynamic stability hypothesis* imposes further restrictions on $p = (\varrho, \vartheta)$ and $e(\varrho, \vartheta)$, specifically,

$$\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0, \ c_v(\varrho, \vartheta) = \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0 \tag{8}$$

for all $\varrho > 0$, $\vartheta > 0$ (see [2]).

Entropy Production

The internal energy balance (6) divided by ϑ, together with the equation of continuity (1) and Gibbs' relation (5), gives rise to the entropy equation:

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x\,(\varrho s(\varrho, \vartheta)\mathbf{u}) + \mathrm{div}_x\left(\frac{\mathbf{q}}{\vartheta}\right) = \frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta}\right) + \frac{\varrho}{\vartheta}\mathcal{Q}. \tag{9}$$

Here, the quantity

$$\sigma = \frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta}\right) \geq 0 \tag{10}$$

is termed entropy production rate and, in accordance with the second law of thermodynamics, is always nonnegative. This may (and will) imply some structural restrictions to be satisfied by the constitutive equations for $\mathbb{S}$ and $\mathbf{q}$ discussed below.

2 Basic Equations of Fluid Dynamics

In order to close the system of fluid dynamic equations, certain constitutive equations relating the viscous stress $\mathbb{S}$ and the internal energy flux $\mathbf{q}$ to the basic phase variables are needed.

2.1 Euler System, Ideal Fluids

Ideal fluids are those for which $\mathbb{S} = 0$, $\mathbf{q} = 0$. The associated system of equations is usually called *Euler system* (see, e.g., [14]):

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{11}$$

$$\partial_t(\varrho\mathbf{u}) + \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) + \nabla_x p(\varrho,\vartheta) = 0, \tag{12}$$

$$\partial_t\left[\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\right] + \mathrm{div}_x\left(\varrho\left[\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\right]\mathbf{u}\right) + \mathrm{div}_x(p(\varrho,\vartheta)\mathbf{u}) = 0, \tag{13}$$

where, for the sake of simplicity, the effect of external sources is omitted in (12), (13).

2.2 Navier-Stokes-Fourier System, Viscous and Heat-Conducting Fluids

Ideal fluids introduced in the previous section may and should be seen as a hypothetical limit state of real fluids that are both viscous and heat conducting. In such a case, the viscous stress $\mathbb{S}$ and the internal energy flux $\mathbf{q}$ depend effectively on the velocity gradient $\nabla_x\mathbf{u}$ and the temperature gradient $\nabla_x\vartheta$, respectively.

2.2.1 Newtonian Fluids

For Newtonian or linearly viscous fluids, the viscous stress tensor is a linear function of the velocity gradient.

Newton's Law

The viscous stress tensor $\mathbb{S}$ for a Newtonian fluid is given by *Newton's rheological law* (see [11]):

$$\mathbb{S}(\nabla_x\mathbf{u}) = \mu\left(\nabla_x\mathbf{u} + \nabla_x^t\mathbf{u} - \frac{2}{3}\mathrm{div}_x\mathbf{u}\mathbb{I}\right) + \eta\,\mathrm{div}_x\mathbf{u}\mathbb{I}, \tag{14}$$

where the scalar quantities μ and η are termed the shear and bulk viscosity coefficient, respectively. In accordance with the second law of thermodynamics enforced through (10), μ and η are nonnegative and may depend on the state variables ϱ, ϑ as the case may be.

Fourier's Law

Similarly to (14), the internal energy flux $\mathbf{q}$ of a linearly viscous fluid is a linear function of $\nabla_x\vartheta$ determined by *Fourier's law*:

$$\mathbf{q} = -\kappa\nabla_x\vartheta, \tag{15}$$

with the heat conductivity coefficient $\kappa \geq 0$ that may depend on ϱ and ϑ.

2.2.2 Navier-Stokes-Fourier System

In accordance with the previous discussion, the time evolution of a Newtonian heat-conducting fluid is determined by the *Navier-Stokes-Fourier system* (see, e.g., [29, 47]):

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{16}$$

$$\begin{aligned} &\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) \\ &= \mathrm{div}_x \left(\mu \left[\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right] + \eta \mathrm{div}_x \mathbf{u} \mathbb{I} \right), \end{aligned} \tag{17}$$

$$\begin{aligned} &\varrho c_v(\varrho, \vartheta) \Big(\partial_t \vartheta + \mathbf{u} \cdot \nabla_x \vartheta \Big) - \mathrm{div}_x \left(\kappa \nabla_x \vartheta \right) \\ &= \left(\mu \left[\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right] + \eta \mathrm{div}_x \mathbf{u} \mathbb{I} \right) : \nabla_x \mathbf{u} - \varrho \frac{\partial p(\varrho, \vartheta)}{\partial \vartheta} \mathrm{div}_x \mathbf{u}, \end{aligned} \tag{18}$$

where, similarly to the Euler system (11)–(13), the effect of the external sources has been omitted. It is worth noting that equation (18) is *formally* equivalent to the total energy balance (3), the internal energy balance (6), and even to the entropy balance (9).

3 Boundary Conditions

Fluids are usually confined to a bounded spatial domain Ω; the unbounded domains considered in certain mathematical models should be seen as an idealization of large fluid domains in the real world. There is a large variety of boundary behavior of both Eulerian and Navier-Stokes fluid determined by its interaction with the outer world. For definiteness, only a very simple situation will be considered in this chapter, where the kinematic boundary $\partial\Omega$ is at rest and impermeable, meaning

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0 \tag{19}$$

where the symbol $\mathbf{n}$ denotes the outer normal vector to $\partial\Omega$.

3.1 Slip vs. Stick

While the impermeability condition (19) is sufficient for the description on an inviscid fluid governed by the Euler system (11)–(13), an extra piece of information is needed if the fluid is viscous.

3.1.1 No-Slip Boundary Conditions

A commonly accepted hypothesis asserts that a viscous fluid adheres completely to the boundary, meaning, in addition to (19), also the tangential component of the velocity vanishes on $\partial\Omega$. This can be written concisely in the form of *no-slip* boundary condition:

$$\mathbf{u}|_{\partial\Omega} = 0. \tag{20}$$

3.1.2 No-Stick, Complete Slip Boundary Conditions

Under certain circumstances, e.g., for nanofluids (see [38]), it was observed that the no-slip condition (20) is no longer a relevant description of the fluid behavior. Instead, one may postulate the *no-stick or complete slip* condition:

$$(\mathbb{S} \cdot \mathbf{n}) \times \mathbf{n}|_{\partial\Omega} = 0. \tag{21}$$

In other words, the tangential component of the normal (viscous) stress vanishes on $\partial\Omega$.

3.1.3 Navier's Slip

A compromise between (20) and (21) is *Navier's slip* condition:

$$[\mathbb{S} \cdot \mathbf{n}]_{\tan} + \beta \varrho \mathbf{u}|_{\partial\Omega} = 0, \tag{22}$$

where β plays a role of a friction coefficient (see [9]).

3.2 Boundary Behavior of the Temperature, Heat Flux

For heat-conducting fluids, the boundary behavior of the temperature must be specified. In energetically insulated systems, the heat flux vanishes in the normal direction to $\partial\Omega$:

$$-\mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = \kappa \nabla_x \vartheta \cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{23}$$

Alternatively, the distribution of the temperature on the boundary can be prescribed yielding Dirichlet-type boundary conditions:

$$\vartheta|_{\partial\Omega} = \vartheta_b. \tag{24}$$

Of course, there are many other possibilities of the boundary behavior of ϑ including a combination of (23), (24) imposed on disjoint parts of $\partial\Omega$.

4 Well Posedness, Classical Solutions

A system of *evolutionary* partial differential equations, supplemented with suitable boundary conditions, is *well posed* provided it admits a unique solution for any admissible initial state. The initial state for the Navier-Stokes-Fourier or complete Euler system is given by specifying the initial distribution of the density, velocity, and temperature:

$$\varrho(0, \cdot) = \varrho_0, \ \vartheta(0, \cdot) = \vartheta_0, \ \mathbf{u}(0, \cdot) = \mathbf{u}_0 \text{ in } \Omega. \tag{25}$$

Alternatively, the initial momentum $(\varrho \mathbf{u})_0$, the initial internal energy e_0, and/or the entropy s_0 may be given. In view of the physical background, the initial data should obey certain admissibility conditions, in particular, the density and (absolute) temperature should be strictly positive, the total initial energy finite, among others.

4.1 Classical Solvability

Given smooth and physically admissible initial data, the problems in fluid dynamics are supposed to admit unique classical (smooth) solutions. This is (known to be) true, however, only on a possibly short time interval $[0, T_{\max})$. Whether or not $T_{\max} = \infty$ is in general an open question. Solutions of the (inviscid) Euler system (11)–(13) may develop discontinuities (shock waves) in a finite time no matter how smooth and even small the initial data are (see [41, Chapter 15]). Regularity of solutions to the Navier-Stokes-Fourier system (16)–(18) in the long run is a famous open problem (see [20, 43] for a thorough discussion in the context of *incompressible fluids*).

4.2 Local-in-Time Existence

There are many results concerning local-in-time existence of smooth solutions for both the Euler and the Navier-Stokes-Fourier system, for different choices of spatial geometries, boundary conditions, classes of initial data, etc.

4.2.1 Euler System: Classical Solutions

To avoid technicalities connected with the boundary behavior of solutions, the existence result for the Euler system will be stated in the physically relevant domain R^3 (see [5, Chapter 13, Theorem 13.1.]).

Theorem 1. *Let $\overline{\varrho} > 0$, $\overline{\vartheta} > 0$ be given. Suppose that the pressure $p = p(\varrho, \vartheta)$, $e(\varrho, \vartheta)$ are twice continuously differentiable functions satisfying Gibbs' relation (5) and the thermodynamic stability condition (8) in an open set $\mathcal{U} \subset (0, \infty)^2$ containing $[\overline{\varrho}, \overline{\vartheta}]$. Let the initial data ϱ_0, ϑ_0, $\mathbf{u}_0$ be given such that*

$$[\varrho_0(x), \vartheta_0(x)] \text{ belong to a compact subset of } \mathcal{U} \text{ for all } x \in R^3,$$

$$\varrho_0 - \overline{\varrho},\ \vartheta_0 - \overline{\vartheta} \in W^{k,2}(R^3),\ \mathbf{u}_0 \in W^{k,2}(R^3; R^3) \text{ for some } k > \frac{5}{2}.$$

Then there exists a positive time $T > 0$ such that the Euler system (11)–(13) admits a solution ϱ, ϑ, $\mathbf{u}$ unique in the class

$$\varrho - \overline{\varrho}, \vartheta - \overline{\vartheta} \in C([0, T]; W^{k,2}(R^3)) \cap C^1([0, T]; W^{k-1,2}(R^3)),$$

$$\mathbf{u} \in C([0, T]; W^{k,2}(R^3; R^3)) \cap C^1([0, T]; W^{k-1,2}(R^3, R^3)).$$

Remark 1. The symbol $W^{k,2}(R^3)$ denotes the *Sobolev space* of functions having (generalized) derivatives up to order k square integrable in R^3.

4.2.2 Navier-Stokes-Fourier System: Classical Solutions

A short-time existence result for the Navier-Stokes-Fourier system (11)–(13), endowed, for definiteness, with the boundary conditions (20), (23) may be stated as follows (see [44, Theorem A and Remark 3.3]).

Theorem 2. *Let* $\Omega \subset R^3$ *be a bounded domain of class* $C^{2+\nu}$, $\nu > 0$. *Let the initial data* $\varrho_0, \vartheta_0 \in W^{3,2}(\Omega)$, $\mathbf{u}_0 \in W^{3,2}(\Omega; R^3)$ *be given such that* $[\varrho_0(x), \mathbf{u}_0(x)]$ *belong to a compact subset of an open set* $\mathcal{U} \subset (0,\infty)^2$, *and satisfying the* compatibility conditions

$$\mathbf{u}_0|_{\partial\Omega} = 0,\ \nabla_x \vartheta_0 \cdot \mathbf{n}|_{\partial\Omega} = 0, \nabla_x p(\varrho_0, \vartheta_0)\Big|_{\partial\Omega}$$

$$= \mathrm{div}_x \left(\mu(\varrho_0, \vartheta_0) \left[\nabla_x \mathbf{u}_0 + \nabla_x^t \mathbf{u}_0 - \frac{2}{3} \mathrm{div}_x \mathbf{u}_0 \mathbb{I} \right] + \eta(\varrho_0, \vartheta_0) \mathrm{div}_x \mathbf{u}_0 \mathbb{I} \right) \Big|_{\partial\Omega}.$$

Suppose that the pressure $p = p(\varrho, \vartheta)$, *the specific heat at constant volume* $c_v = c_v(\varrho, \vartheta)$, *as well as the transport coefficients* $\mu = \mu(\varrho, \vartheta)$, $\eta = \eta(\varrho, \vartheta)$, *and* $\kappa = \kappa(\varrho, \vartheta)$ *are three times continuously differentiable in* $\mathcal{U}$ *and satisfy*

$$\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0,\ c_v(\varrho, \vartheta) > 0,\ \mu(\varrho, \vartheta) > 0,\ \eta(\varrho, \vartheta) \geq 0,\ \kappa(\varrho, \vartheta) > 0$$

for all $[\varrho, \vartheta] \in \mathcal{U}$.

Then there exists $T > 0$ *such that the Navier-Stokes-Fourier system (11)–(13), supplemented with the boundary conditions (20), (23), admits a unique solution in the class:*

$$\varrho,\ \vartheta \in C([0,T]; W^{3,2}(\Omega)) \cap C^1([0,T]; W^{2,2}(\Omega)),$$

$$\mathbf{u} \in C([0,T]; W^{3,2}(\Omega; R^3)) \cap C^1([0,T]; W^{2,2}(\Omega; R^3)).$$

Remark 2. It can be shown that any solution belonging to the class specified in Theorem 2 possesses all the necessary derivatives and is therefore a classical solution in the *open set* $(0,T) \times \Omega$.

4.3 Classical Solvability: Conclusion

The systems of equations considered in mathematical fluid dynamics are nonlinear and as such susceptible to develop singularities, either in the form of steep gradients (shock waves) or concentrations (mass collapse). Such phenomena have been rigorously verified for the inviscid Euler system. What is more, a mathematical theory based on global-in-time classical solutions is beyond the reach of the available mathematical methods and up-to-date knowledge, even for the Navier-Stokes-Fourier system. On the other hand, these problems are being solved numerically with continuously improving capacity of modern computers. *Some* concept of solution is therefore needed to perform a rigorous analysis of convergence of the numerical methods. The weak solutions discussed in the next part offer such alternative.

5 Weak Solutions

The idea of weak solutions is based on the concept of *generalized derivatives* or distributions. Classical functions are replaced by their *integral averages* or, more precisely

$$f : Q \mapsto R \approx \int_Q f\varphi,\ \varphi \in C_c^\infty(Q)$$

where the symbol $C_c^\infty(Q)$ denotes the set of infinitely differentiable functions with compact support in Q. Differential operators D can be conveniently expressed by means of a formal by-parts integration:

$$Df \approx -\int_Q fD\varphi,\ \varphi \in C_c^\infty(Q).$$

Accordingly, any (locally) integrable function possesses derivatives of arbitrary order! The Sobolev spaces $W^{k,2}$ used in the previous part are based on distributional derivatives.

5.1 Euler System: Weak Solutions

A trio of functions $[\varrho, \vartheta, \mathbf{u}]$ is a *weak solution* of the Euler system (11)–(13) in the set $(0, T) \times \Omega$ if:

$$\int_0^T \int_\Omega (\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi)\ \mathrm{d}x\ \mathrm{d}t = 0 \tag{26}$$

for any $\varphi \in C_c^\infty((0, T) \times \Omega)$;

$$\int_0^T \int_\Omega (\varrho \mathbf{u} \cdot \partial_t \varphi + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \varphi + p(\varrho, \vartheta)\mathrm{div}_x \varphi)\ \mathrm{d}x\ \mathrm{d}t = 0 \tag{27}$$

for any $\varphi \in C_c^\infty((0, T) \times \Omega; R^3)$;

$$\int_0^T \int_\Omega \left(\left[\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right] \partial_t \varphi \right.$$

$$+ \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) + p(\varrho,\vartheta)\right]\mathbf{u}\cdot\nabla_x\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t = 0 \tag{28}$$

for any $\varphi \in C_c^\infty((0,T)\times\Omega)$.

Note that the integral identities (26-28) are well defined as soon as all the compositions of ϱ, ϑ, $\mathbf{u}$ with all nonlinearities are at least locally integrable.

5.1.1 Weak Continuity, Initial and/or Boundary Conditions

Functions that are merely (locally) integrable do not posses traces on lower-dimensional structures in Ω, in particular, it is not clear how to define the initial and/or boundary conditions in the class of weak solutions. Fortunately, the necessary piece of information is already encoded in the weak formulation. For example, if ϱ is a weak solution of (26), the choice of a special test function $\varphi(t,x) = \psi(t)\phi(x)$, $\psi \in C_c^\infty(0,T)$, $\phi \in C_c^\infty(\Omega)$ gives rise to

$$\int_0^T \psi'(t)\int_\Omega \varrho(t,\cdot)\phi\ \mathrm{d}x\ \mathrm{d}t = -\int_0^T \psi(t)\int_\Omega \varrho\mathbf{u}(t,\cdot)\cdot\nabla_x\phi\ \mathrm{d}x\ \mathrm{d}t,$$

from which it follows that the function

$$t \mapsto \int_\Omega \varrho(t,\cdot)\phi\ \mathrm{d}x \text{ admits an integrable generalized derivate in } (0,T)$$

and as such can be represented, upon modification on a set of times of zero measure, by an absolutely continuous function. Thus the initial conditions can be interpreted in the sense of integral averages:

$$\varrho(0,\cdot) = \varrho_0\ \approx \int_\Omega \varrho(t,\cdot)\phi\ \mathrm{d}x \to \int_\Omega \varrho_0\phi\ \mathrm{d}x \text{ as } t\to 0+ \text{ for any } \phi \in C_c^\infty(\Omega).$$

The anticipated weak continuity in time makes possible to incorporate the initial conditions into the weak formulation, replacing (26-27) by

$$\int_0^T\int_\Omega (\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi)\ \mathrm{d}x\ \mathrm{d}t = -\int_\Omega \varrho_0\varphi(0,\cdot)\ \mathrm{d}x \tag{29}$$

for any $\varphi \in C_c^\infty([0,T)\times\Omega)$;

$$\int_0^T\int_\Omega (\varrho\mathbf{u}\cdot\partial_t\varphi + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla_x\varphi + p(\varrho,\vartheta)\mathrm{div}_x\varphi)\ \mathrm{d}x\ \mathrm{d}t$$
$$= -\int_\Omega \varrho_0\mathbf{u}_0\cdot\varphi(0,\cdot)\ \mathrm{d}x \tag{30}$$

for any $\varphi \in C_c^\infty([0,T)\times\Omega; R^3)$;

$$\int_0^T \int_\Omega \left(\left[\frac{1}{2}\varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right] \partial_t \varphi + \left[\frac{1}{2}\varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) + p(\varrho, \vartheta) \right] \mathbf{u} \cdot \nabla_x \varphi \right) \, \mathrm{d}x \, \mathrm{d}t = - \int_\Omega \left[\frac{1}{2}\varrho_0 |\mathbf{u}_0|^2 + \varrho_0 e(\varrho_0, \vartheta_0) \right] \varphi(0, \cdot) \, \mathrm{d}x \tag{31}$$

for any $\varphi \in C_c^\infty([0, T) \times \Omega)$.

Boundary conditions, or at least the normal traces of the fluxes, can be interpreted in a similar way. This issue will be discussed in the context of the Navier-Stokes-Fourier system.

Remark 3. As a matter of fact, the weak formulation can be derived directly (without passing from classical to generalized derivatives) from the underlying physical principles written in their natural *integral form* (see [25, Chapter 1]).

5.2 Navier-Stokes-Fourier System: Weak Solutions

In order to introduce a weak formulation of the Navier-Stokes-Fourier system, rewrite first the energy equation (18) in the conservative form

$$\partial_t(\varrho e(\varrho, \vartheta)) + \mathrm{div}_x(\varrho e(\varrho, \vartheta)\mathbf{u}) - \mathrm{div}_x \, (\kappa \nabla_x \vartheta) = \left(\mu \left[\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right] + \eta \mathrm{div}_x \mathbf{u} \mathbb{I} \right) : \nabla_x \mathbf{u} - p(\varrho, \vartheta) \mathrm{div}_x \mathbf{u}.$$

Note that this is possible as long as p, e, and $c_v = \partial_\vartheta e$ are interrelated through (5).

Accordingly, the weak formulation of the Navier-Stokes-Fourier system (16)–(18) reads as follows:

$$\int_0^T \int_\Omega (\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi) \, \mathrm{d}x \, \mathrm{d}t = - \int_\Omega \varrho_0 \varphi(0, \cdot) \, \mathrm{d}x \tag{32}$$

for any $\varphi \in C_c^\infty([0, T) \times \Omega)$;

$$\int_0^T \int_\Omega (\varrho \mathbf{u} \cdot \partial_t \varphi + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \varphi + p(\varrho, \vartheta) \mathrm{div}_x \varphi) \, \mathrm{d}x \, \mathrm{d}t$$
$$\int_0^T \int_\Omega \mu \left[\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right] : \nabla_x \varphi \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_\Omega \eta \mathrm{div}_x \mathbf{u} \mathrm{div}_x \varphi \, \mathrm{d}x \, \mathrm{d}t = - \int_\Omega \varrho_0 \mathbf{u}_0 \cdot \varphi(0, \cdot) \, \mathrm{d}x \tag{33}$$

for any $\varphi \in C_c^\infty([0,T) \times \Omega; R^3)$;

$$\int_0^T \int_\Omega (\varrho e(\varrho,\vartheta)\partial_t \varphi + \varrho e(\varrho,\vartheta)\mathbf{u} \cdot \nabla_x \varphi) \ \mathrm{d}x \ \mathrm{d}t - \int_0^T \int_\Omega \kappa \nabla_x \vartheta \cdot \nabla_x \varphi \ \mathrm{d}x \ \mathrm{d}t$$
$$= -\int_0^T \int_\Omega \left(\mu \left[\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I} \right] + \eta \mathrm{div}_x \mathbf{u}\mathbb{I} \right) : \nabla_x \mathbf{u}\varphi \ \mathrm{d}x \ \mathrm{d}t$$
$$- \int_0^T \int_\Omega p(\varrho,\vartheta)\mathrm{div}_x \mathbf{u}\varphi \ \mathrm{d}x \ \mathrm{d}t - \int_\Omega \varrho_0 e(\varrho_0,\vartheta_0)\varphi(0,\cdot) \ \mathrm{d}x \tag{34}$$

for any $\varphi \in C_c^\infty([0,T) \times \Omega)$. Similarly to the previous part, the weak formulation already includes the satisfaction of the initial conditions.

5.2.1 Boundary Conditions

The reader will have noticed that, in contrast with the Euler system, the weak formulation of the Navier-Stokes-Fourier system includes *first* derivatives of the velocity $\mathbf{u}$ as well as the temperature ϑ. Anticipating that the first derivatives are integrable functions, the fields $\mathbf{u}$ and ϑ have well-defined *traces* on the boundary $\partial\Omega$ (see, e.g., [48, Chapter 3]).

Thus the *Dirichlet-type* boundary conditions (20), (24) may be incorporated in the definition of the function spaces the solution belongs to. In particular, the no-slip condition (20) corresponds to the Sobolev space $W_0^{1,p}(\Omega)$ of functions with integrable first-order derivatives in power p and vanishing on the boundary.

The boundary conditions of *Neumann type* like (22), (23) can be accommodated in the weak formulation by extending the class of admissible test functions. Thus, for instance, the no-flux condition (23) is enforced by postulating (34) for any $\varphi \in C_c^\infty([0,T) \times \overline{\Omega})$. The complete slip (19), (21) requires $\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0$ and (33) to be satisfied for any $\varphi \in C_c^\infty([0,T) \times \overline{\Omega}; R^3)$, $\varphi \cdot \mathbf{n}|_{\partial\Omega} = 0$, etc.

5.3 A Disturbing Example

The class of weak solutions to a given problem is apparently much larger than required by the classical theory. In other words, it might be easier to establish *existence* but definitely more delicate to show *uniqueness* among all possible weak solutions emanating from the same initial data. Indeed there exist weak solutions to the (incompressible) variant of the Euler system that can be obtained in a completely non-constructive way by the method of *convex integration* recently developed in the context of fluid mechanics in [15]. A further adaptation of this technique provides a rather illustrative but at the same time quite disturbing example of non-uniqueness in the context of fluid thermodynamics. To this end, consider the so-called Euler-Fourier system:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{35}$$

$$\partial_t(\varrho\mathbf{u}) + \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) + \nabla_x(\varrho\vartheta) = 0, \tag{36}$$

$$\frac{3}{2}\left[\partial_t(\varrho\vartheta) + \mathrm{div}_x(\varrho\vartheta\mathbf{u})\right] - \Delta\vartheta = -\varrho\vartheta\,\mathrm{div}_x\mathbf{u}. \tag{37}$$

The system (35-37) is a special case of the Navier-Stokes-Fourier system with $p = \varrho\vartheta$, $c_v = \frac{3}{2}$, $\mu = \eta = 0$, $\kappa = 1$. Although a correct physical justification of an inviscid heat-conducting fluid may be dubious, the system has been used as a suitable approximation in certain models (see [46]).

For the sake of simplicity, the problem will be endowed with the spatially periodic boundary conditions, meaning the underlying spatial domain

$$\Omega = \mathcal{T}^3 = \left([-1,1]\,|_{\{-1;1\}}\right)^3$$

is the "flat" torus. The following result holds true (see [10, Theorem 3.1]).

Theorem 3. *Let $T > 0$ be given. Let the initial data satisfy*

$$\varrho_0, \vartheta_0 \in C^3(\mathcal{T}^3),\ \mathbf{u}_0 \in C^3(\mathcal{T}^3; R^3),\ \varrho_0 > 0,\ \vartheta_0 > 0 \textit{ in } \mathcal{T}^3.$$

Then the initial-value problem for the Euler-Fourier system (35-37) admits infinitely many *weak solutions in $(0,T)\times\Omega$ belonging to the class*

$$\varrho \in C^2([0,T]\times\Omega),\ \partial_t\vartheta \in L^p(0,T;L^p(\Omega)),\ \nabla_x^2\vartheta \in L^p(0,T;L^p(\Omega;R^{3\times 3}))$$

for any $1 \le p < \infty$,

$$\mathbf{u} \in C_{\mathrm{weak}}([0,T];L^2(\Omega;R^3)) \cap L^\infty((0,T)\times\Omega;R^3),\ \mathrm{div}_x\mathbf{u} \in C^2([0,T]\times\Omega).$$

The conclusion of Theorem 3 reveals the main drawback of the mathematical theory based on the concept of weak solutions, namely, the restrictions imposed by the weak formulation upon the class of possible solutions are too weak to ensure uniqueness. Apparently, the weak formulation must be augmented by certain *admissibility conditions* dictated by physics to pick up the relevant solution. On the other hand, the extra conditions should not be too strong to prevent global-in-time existence. This as well as other related issues will be addressed in the remaining part of this chapter devoted to the mathematical theory of the *complete* Navier-Stokes-Fourier system.

6 Mathematical Theory of Compressible, Viscous, and Heat-Conducting Fluids

There is an alternative weak formulation of the Navier-Stokes-Fourier system based on the second law of thermodynamics. The theory accommodates, in particular, the *energetically closed systems*, mechanically and thermally insulated from the outer world. Accordingly, the *conservative* boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0, \ \nabla_x \vartheta \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{38}$$

are imposed, in particular the total energy E is a constant of motion:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right] \mathrm{d}x = 0. \tag{39}$$

The total energy balance (39), together with

$$\partial_t \varrho + \mathrm{div}_x (\varrho \mathbf{u}) = 0, \tag{40}$$

$$\partial_t (\varrho \mathbf{u}) + \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) \tag{41}$$

and the entropy *inequality*

$$\partial_t (\varrho s(\varrho, \vartheta)) + \mathrm{div}_x (\varrho s(\varrho, \vartheta) \mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) \geq \frac{1}{\vartheta} \left(\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right) \tag{42}$$

will be used as a basis of a new weak formulation of the Navier-Stokes-Fourier system. Similarly to the above, the viscous stress and the heat flux are taken in the form

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}, \ \mathbf{q} = -\kappa \nabla_x \vartheta. \tag{43}$$

6.1 Finite Energy Weak Solutions to the Navier-Stokes-Fourier System

A trio of functions ϱ, ϑ, $\mathbf{u}$ is termed a *finite energy weak solution* to the Navier-Stokes-Fourier system (39-43), supplemented with the boundary conditions (38) if:

- $$\varrho \in L^\infty(0,T;L^\gamma(\Omega)),\ \vartheta \in L^\infty(0,T;L^q(\Omega)) \cap L^2(0,T;W^{1,2}(\Omega))$$

  for certain $\gamma > 1, q > 1$,

  $$\varrho \geq 0,\ \vartheta > 0 \text{ a.a. in } (0,T) \times \Omega,$$
  $$\mathbf{u} \in L^2(0,T;W_0^{1,r}(\Omega;R^3)) \text{ for a certain } r > 1;$$

- $$\int_0^T \int_\Omega (\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi)\ \mathrm{d}x\ \mathrm{d}t = -\int_\Omega \varrho_0 \varphi(0,\cdot)\ \mathrm{d}x \tag{44}$$

  for any $\varphi \in C_c^\infty([0,T) \times \Omega)$;
- $$\int_0^T \int_\Omega (\varrho \mathbf{u} \cdot \partial_t \varphi + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \varphi + p(\varrho,\vartheta)\mathrm{div}_x \varphi)\ \mathrm{d}x\ \mathrm{d}t$$
  $$= \int_0^T \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \varphi\ \mathrm{d}x\ \mathrm{d}t - \int_\Omega \varrho_0 \mathbf{u}_0 \cdot \varphi(0,\cdot)\ \mathrm{d}x \tag{45}$$

  for any $\varphi \in C_c^\infty([0,T) \times \Omega; R^3)$;
- $$\int_0^T \int_\Omega \left(\varrho s(\varrho,\vartheta)\partial_t \varphi + \varrho s(\varrho,\vartheta)\mathbf{u} \cdot \nabla_x \varphi + \frac{\mathbf{q} \cdot \nabla_x \varphi}{\vartheta} \right)\ \mathrm{d}x\ \mathrm{d}t$$
  $$+ \int_0^T \int_\Omega \frac{1}{\vartheta} \left(\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right) \varphi\ \mathrm{d}x\ \mathrm{d}t \leq -\int_\Omega \varrho_0 s(\varrho_0,\vartheta_0)\varphi(0,\cdot)\ \mathrm{d}x \tag{46}$$

  for any $\varphi \in C_c^\infty([0,T) \times \overline{\Omega}), \varphi \geq 0$.
- $$\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) \right](\tau,\cdot)\ \mathrm{d}x = \int_\Omega \left[\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + \varrho_0 e(\varrho_0,\vartheta_0) \right]\ \mathrm{d}x \tag{47}$$

  for a.a. $\tau \in (0,T)$.

The weak solutions satisfying (44-47) enjoy the important *compatibility property*, namely, any weak solution that is smooth satisfies the classical formulation of the Navier-Stokes-Fourier system (16-18) (see [25, Chapter 2]).

6.2 Global-in-Time Existence of Finite Energy Weak Solutions

The weak formulation of the Navier-Stokes-Fourier system based on the integral identities (inequalities) (44)–(47) is mathematically tractable. Under certain technical but still physically grounded restrictions imposed on the constitutive relations, the problem admits global-in-time solution for any finite energy initial data.

6.2.1 Hypotheses: Constitutive Relations

The thermodynamic functions $p = p(\varrho, \vartheta)$, $e = e(\varrho, \vartheta)$, and $s = s(\varrho, \vartheta)$ are interrelated through Gibbs' equation (5) and comply with the hypothesis of thermodynamics stability (8). In addition, it is required that the internal energy $e = e(\varrho, \vartheta)$ and the pressure take the form

$$e(\varrho, \vartheta) = e_m(\varrho, \vartheta) + \frac{a}{\varrho}\vartheta^4, \ p(\varrho, \vartheta) = p_m(\varrho, \vartheta) + \frac{a}{4}\vartheta^4, \ a > 0, \tag{48}$$

where e_m, p_m represent molecular components augmented in (48) by radiation (see [25, Chapter 1]). Moreover, p_m and e_m satisfy the monoatomic gas equation of state:

$$p_m(\varrho, \vartheta) = \frac{2}{3}\varrho e_m(\varrho, \vartheta). \tag{49}$$

It is easy to see that relation (49) is compatible with Gibbs' equation (5) provided

$$p_m(\varrho, \vartheta) = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right); \text{ whence } e_m(\varrho, \vartheta) = \frac{3}{2}\vartheta \frac{\vartheta^{3/2}}{\varrho} P\left(\frac{\varrho}{\vartheta^{3/2}}\right). \tag{50}$$

In this setting, the hypothesis of thermodynamics stability (8) gives rise to

$$P(0) = 0, \ P'(Z) > 0, \ 0 < \frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z} < c \text{ for any } Z > 0, \tag{51}$$

where, in addition, the specific heat at constant volume is required to be uniformly bounded.

Finally, by virtue of (51), the function $Z \mapsto \frac{P(Z)}{Z}$ is nonincreasing; whence it is possible to suppose that

$$\lim_{Z \to \infty} \frac{P(Z)}{Z^{5/3}} = p_\infty > 0. \tag{52}$$

6.2.2 Hypotheses: Transport Coefficients

Transport coefficients $\mu = \mu(\vartheta)$, $\eta = \eta(\vartheta)$, and $\kappa = \kappa(\vartheta)$ appearing in (43) are effective functions of the absolute temperature, specifically:

$$\underline{\mu}(1+\vartheta^\alpha) \leq \mu(\vartheta) \leq \overline{\mu}(1+\vartheta^\alpha),\ |\mu'(\vartheta)| \leq c \text{ for all } \vartheta > 0,\ \frac{2}{5} < \alpha \leq 1,\ \underline{\mu} > 0, \tag{53}$$

$$0 \leq \eta(\vartheta) \leq \overline{\eta}(1+\vartheta^\alpha) \text{ for all } \vartheta > 0, \tag{54}$$

and

$$\underline{\kappa}(1+\vartheta^3) \leq \kappa(\vartheta) \leq \overline{\kappa}(1+\vartheta^3) \text{ for all } \vartheta > 0,\ \underline{\kappa} > 0. \tag{55}$$

6.2.3 Existence of Finite Energy Weak Solutions

The following existence result was proved in [25, Chapter 3, Theorem 3.1]:

Theorem 4. *Let $\Omega \subset R^3$ be a bounded domain of class $C^{2+\nu}$. Suppose that the pressure p and the internal energy e are interrelated through (48)–(50), where $P \in C[0,\infty) \cap C^3(0,\infty)$ satisfies the structural hypotheses (51), (52). Let the transport coefficients μ, η, κ be continuously differentiable functions of the temperature ϑ satisfying (53)–(55). Finally, let the initial data ϱ_0, ϑ_0, $\mathbf{u}_0$ be given such that*

$$\varrho_0,\ \vartheta_0 \in L^\infty(\Omega),\ \varrho_0 > 0,\ \vartheta_0 > 0 \text{ a.a. in } \Omega, \mathbf{u}_0 \in L^2(\Omega; R^3). \tag{56}$$

Then the Navier-Stokes-Fourier system (39)–(43), supplemented with the boundary conditions (38), possesses a finite energy weak solution $\varrho, \vartheta, \mathbf{u}$ in $(0,T) \times \Omega$ in the sense specified in (44)–(47). The weak solution belongs to the class:

$$\begin{gathered} \varrho \geq 0 \text{ a.a. in } (0,T) \times \Omega, \\ \varrho \in C([0,T]; L^1(\Omega)) \cap L^\infty(0,T; L^{5/3}(\Omega)) \cap L^\beta((0,T) \times \Omega) \end{gathered} \tag{57}$$

for a certain $\beta > \frac{5}{3}$;

$$\vartheta > 0 \text{ a.a. in } (0,T) \times \Omega,\ \vartheta \in L^\infty(0,T; L^4(\Omega)) \cap L^2(0,T; W^{1,2}(\Omega)), \tag{58}$$

$$\vartheta^3, \log(\vartheta) \in L^2(0,T; W^{1,2}(\Omega)); \tag{59}$$

$$\mathbf{u} \in L^2(0,T; W_0^{1,\Lambda}(\Omega; R^3)),\ \Lambda = \frac{8}{5-\alpha},\ \varrho\mathbf{u} \in C_{\text{weak}}(0,T; L^{5/4}(\Omega; R^3)). \tag{60}$$

In the remaining part of this text, various properties of the finite energy weak solutions, the existence of which is guaranteed by Theorem 4, will be discussed. At this point, it is worth noting that an alternative approach based on the internal energy formulation (32)–(34) was proposed in [21]. Although mathematically less sophisticated and physically limited by more restrictive constitutive relations than in Theorem 4, the approach [21] proved to be convenient when studying stability and convergence properties of certain numerical methods [23].

The weak formulation of the Navier-Stokes-Fourier system based on the complete energy balance has also been studied in the framework of weak solutions. In [31], the authors established global existence for radially symmetric data in R^3. They also identified one of the main stumbling blocks in the analysis of the Navier-Stokes-Fourier system, namely, the (hypothetical) appearance of *vacuum zones*, where the density vanishes and the classical understanding of the equations breaks down. More recently, a new a priori bound on the density gradient was discovered in [6, 7] leading to global-in-time existence in the truly $3D$setting conditioned, unfortunately, by a very specific relation imposed on the density-dependent viscosity coefficients and a rather unrealistic formula for the pressure that has to be infinite (negative) for $\varrho \to 0$.

The constraint represented by (44)–(47) may seem too weak to ensure, at least formally, the *well posedness* of the problem, meaning uniqueness and possibly stability of solutions with respect to the initial data. Note, however, that this issue remains largely open even for the seemingly simpler *incompressible* Navier-Stokes system despite a concerted effort of generations of excellent mathematicians (see [20]). In the text below, a less ambitious but still interesting question will be discussed, namely, *the weak-strong uniqueness* principle. This principle asserts that weak and strong solutions emanating from the *same* initial data coincide as long as the latter exists. To attack the problem, more thermodynamics is needed encoded in the so-called relative energy inequality; the resulting concept of *dissipative solution* is discussed in the next section.

7 Dissipative Solutions

Motivated by the work [13], a *relative energy functional* associated to the Navier-Stokes-Fourier system will be introduced. Here again, the second law of thermodynamics, enforced through Gibbs' equation (7) and the hypothesis of thermodynamics stability (8), will play a crucial role.

7.1 Ballistic Free Energy

In accordance with the terminology of [18], the so-called ballistic free energy functionally takes the form

$$H_\Theta(\varrho, \vartheta) = \varrho e(\varrho, \vartheta) - \Theta \varrho s(\varrho, \vartheta).$$

The thermodynamic stability relation (8) gives rise to the following two properties of the function H_Θ:

$$\varrho \mapsto H_\Theta(\varrho, \Theta) \text{ is strictly convex,} \tag{61}$$

and

$$\vartheta \mapsto H_\Theta(\varrho, \vartheta) \text{ attains its global minimum at } \vartheta = \tilde{\vartheta}. \tag{62}$$

As observed by [2], the above properties are intimately related to *stability* of the equilibrium solutions to the Navier-Stokes-Fourier system. It will become clear that (61), (62) contain the necessary piece of information used later in the proof of weak-strong uniqueness.

7.2 Relative Energy

The relative energy is defined as

$$\mathcal{E}\left(\varrho, \vartheta, \mathbf{u} \middle| r, \Theta, \mathbf{U}\right)$$
$$= \int_\Omega \left(\frac{1}{2}\varrho\,|\mathbf{u}-\mathbf{U}|^2 + H_\Theta(\varrho,\vartheta) - \frac{\partial H_\Theta(r,\Theta)}{\partial \varrho}(\varrho - r) - H_\Theta(r,\Theta)\right)\,\mathrm{d}x,$$

where $\varrho, \vartheta, \mathbf{u}$ is a weak solution to the Navier-Stokes-Fourier system and $r, \Theta, \mathbf{U}$ is an arbitrary trio of functions satisfying the relevant compatibility conditions. More precisely, imposing the no-slip conditions (38) requires the test functions to satisfy

$$r > 0,\ \Theta > 0 \text{ and } \mathbf{U}|_{\partial\Omega} = 0. \tag{63}$$

Given the coercivity properties of the ballistic free energy stated in (61), (62), it is easy to see that $\mathcal{E}\left(\varrho, \vartheta, \mathbf{u} \middle| r, \Theta, \mathbf{U}\right)$ plays a role of "distance" between $[\varrho, \vartheta, \mathbf{u}]$ and $[r, \Theta, \mathbf{U}]$, meaning $\mathcal{E}\left(\varrho, \vartheta, \mathbf{u} \middle| r, \Theta, \mathbf{U}\right) \geq 0$ vanishing only if $[\varrho, \vartheta, \mathbf{u}] = [r, \Theta, \mathbf{U}]$.

7.3 Relative Energy Inequality, Dissipative Solutions

The strength of the mathematical theory based on the weak solutions in the setting (44)–(47) consists in the fact that it is possible to derive a functional relation for $\mathcal{E}\left(\varrho, \vartheta, \mathbf{u} \middle| r, \Theta, \mathbf{U}\right)$ reminiscent of the Gronwall inequality. Specifically, the following result holds:

$$\left[\mathcal{E}\left(\varrho,\vartheta,\mathbf{u}\middle|r,\Theta,\mathbf{U}\right)\right]_{t=0}^{t=\tau}$$
$$+\int_0^\tau\int_\Omega \frac{\Theta}{\vartheta}\left(\mathbb{S}(\vartheta,\nabla_x\mathbf{u}):\nabla_x\mathbf{u}-\frac{\mathbf{q}(\vartheta,\nabla_x\vartheta)\cdot\nabla_x\vartheta}{\vartheta}\right)\,\mathrm{d}x\,\mathrm{d}t$$
$$\le\int_0^\tau\int_\Omega \varrho(\mathbf{u}-\mathbf{U})\cdot\nabla_x\mathbf{U}\cdot(\mathbf{U}-\mathbf{u})\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_0^\tau\int_\Omega \varrho\Big(s(\varrho,\vartheta)-s(r,\Theta)\Big)\Big(\mathbf{U}-\mathbf{u}\Big)\cdot\nabla_x\Theta\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_0^\tau\int_\Omega \varrho\Big(\partial_t\mathbf{U}+\mathbf{U}\cdot\nabla_x\mathbf{U}\Big)\cdot(\mathbf{U}-\mathbf{u})\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_0^\tau\int_\Omega \left(\mathbb{S}(\vartheta,\nabla_x\mathbf{u}):\nabla_x\mathbf{U}-p(\varrho,\vartheta)\mathrm{div}_x\mathbf{U}\right)\,\mathrm{d}x\,\mathrm{d}t$$
$$-\int_0^\tau\int_\Omega \left(\varrho\Big(s(\varrho,\vartheta)-s(r,\Theta)\Big)\partial_t\Theta+\varrho\Big(s(\varrho,\vartheta)-s(r,\Theta)\Big)\mathbf{U}\cdot\nabla_x\Theta\right)\,\mathrm{d}x\,\mathrm{d}t$$
$$-\int_0^\tau\int_\Omega \frac{\mathbf{q}(\vartheta,\nabla_x\vartheta)}{\vartheta}\cdot\nabla_x\Theta\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_0^\tau\int_\Omega \left(\left(1-\frac{\varrho}{r}\right)\partial_t p(r,\Theta)-\frac{\varrho}{r}\mathbf{u}\cdot\nabla_x p(r,\Theta)\right)\,\mathrm{d}x\,\mathrm{d}t \tag{64}$$

for any finite energy weak solution of the Navier-Stokes-Fourier system (38)–(43) and any trio of (smooth) test functions satisfying the compatibility conditions (63) (see [26, Section 3]). Motivated by [35], where a similar definition is proposed for the incompressible Euler system, a trio of functions $\varrho,\vartheta,\mathbf{u}$ is called a *dissipative solution* to the Navier-Stokes-Fourier system (38)–(43) if

- $\varrho,\vartheta,\mathbf{u}$ belong to the regularity class specified in Theorem 4;
- $\varrho,\vartheta,\mathbf{u}$ satisfy the relative energy inequality (64) for any trio $r,\Theta,\mathbf{U}$ of sufficiently smooth (for all integrals in (64) to be well defined) test functions satisfying the compatibility conditions (63).

As observed in [26, Section 3], any finite energy weak solution of the Navier-Stokes-Fourier system is a dissipative solution. The reverse implication is an interesting open problem. The concept as well as a relevant existence theory in the framework of dissipative solutions can be extended to a vast class of physical spaces, including unbounded domains in R^3 (see [34]).

7.4 Weak-Strong Uniqueness

The important feature of the dissipative solutions is that they comply with the weak-strong uniqueness principle (see [22, Theorem 6.2] and [26, Theorem 2.1]).

Theorem 5. *In addition to the hypotheses of Theorem 4, suppose that*

$$s(\varrho,\vartheta) = S\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{4a}{3}\frac{\vartheta^3}{\varrho}, \ \textit{with } S(Z) \to 0 \textit{ as } Z \to \infty. \tag{65}$$

Let $\varrho, \vartheta, \mathbf{u}$ be a dissipative (weak) solution to the Navier-Stokes-Fourier system in the set $(0,T) \times \Omega$. Suppose that the Navier-Stokes-Fourier system admits a strong solution $\tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}}$ in the time interval $(0,T)$, emanating from the same initial data and belonging to the class:

$$\partial_t \tilde{\varrho},\ \partial_t \tilde{\vartheta},\ \partial_t \tilde{\mathbf{u}},\ \partial_x^m \tilde{\varrho},\ \partial_x^m \tilde{\vartheta},\ \partial_x^m \tilde{\mathbf{u}} \in L^\infty((0,T)\times\Omega),\ m = 0,1,2.$$

Then

$$\varrho \equiv \tilde{\varrho},\ \vartheta \equiv \tilde{\vartheta},\ \mathbf{u} \equiv \tilde{\mathbf{u}} \textit{ in } (0,T)\times\Omega.$$

Remark 4. The extra hypothesis (65) reflects the third law of thermodynamics and can be possibly relaxed (cf. [3,4]).

As stated in Theorem 2, the Navier-Stokes-Fourier system admits a local-in-time regular solution as soon as the initial data are regular. In view of Theorem 5, any weak solution coincides with this strong solution as long as the latter exists. On the other hand, by virtue of Theorem 4, the weak solutions exist globally in time and as such provide a possible alternative of extending the local smooth solution beyond its existence interval. Whether or not strong solutions exist globally in time is an interesting open question, for small data results in this direction see [36,37].

7.4.1 Back to the Euler-Fourier System

At this moment, it seems interesting and useful to go back to Theorem 3, where an example of a system (Euler-Fourier) possessing infinitely many weak solutions was produced. One can introduce the relative energy and define the dissipative solutions for the Euler-Fourier system (35)–(37), exactly as for the Navier-Stokes-Fourier system. It can be shown, see [10, Theorem 4.1], that the dissipative solutions of the Euler-Fourier system enjoy the property of weak-strong uniqueness similarly to the solutions of the Navier-Stokes-Fourier system. Still these restrictions do not lead to a well-posed problem as the following result shows (see [10, Theorem 4.2]):

Theorem 6. *Under the hypotheses of Theorem 3, let $T > 0$ be given, together with the initial data:*

$$\varrho_0, \vartheta_0 \in C^3(\mathcal{T}^3),\ \varrho_0 > 0,\ \vartheta_0 > 0 \textit{ in } \mathcal{T}^3.$$

Then there exists an initial velocity $\mathbf{u}_0 \in L^\infty(\mathcal{T}^3, R^3)$ *such that the corresponding initial-value problem for the Euler-Fourier system (35-37) admits infinitely many* dissipative *weak solutions in* $(0,T) \times \Omega$ *belonging to the class:*

$$\varrho \in C^2([0,T] \times \Omega),\ \partial_t \vartheta \in L^p(0,T; L^p(\Omega)),\ \nabla_x^2 \vartheta \in L^p(0,T; L^p(\Omega; R^{3\times 3}))$$

for any $1 \le p < \infty$,

$$\mathbf{u} \in C_{\rm weak}([0,T]; L^2(\Omega; R^3)) \cap L^\infty((0,T) \times \Omega; R^3),\ {\rm div}_x \mathbf{u} \in C^2([0,T] \times \Omega).$$

It is worth noting that the conclusion of Theorem 6 does not contradict the principle of weak-strong uniqueness as $\mathbf{u}_0$ *is not* smooth. The problem of "maximal" smoothness of such data is closely related to the so-called Onsager's conjecture that has been intensively studied in the context of the incompressible Euler system, see [16, 17], and, more recently, [8]. It is worth noting that application of the relative energy inequality may provide *positive* well-posed results even in the context of the Euler system (see [24]).

8 Conclusion

A conditional regularity criterion is a condition which, if satisfied by a weak solution to a given system, implies that the latter is regular. Similarly, such a condition may be applied to guarantee that a local (strong) solution can be extended to a given time interval. The most celebrated conditional regularity criteria are due to [1,39,40], and [12] in the context of the incompressible Navier-Stokes and Euler systems. Recently, similar conditions were obtained also for compressible barotropic fluids and the full Navier-Stokes-Fourier system, the reader may consult [19,33], [42,45], and also the references cited therein.

In view of the results of [30, 32], certain discontinuities imposed through the initial data in the compressible Navier-Stokes system propagate in time. In other words, unlike its incompressible counterpart, the hyperbolic-parabolic compressible Navier-Stokes system does not enjoy the smoothing property typical for purely parabolic equations. Analogously, a solution of the full Navier-Stokes-Fourier system can be regular only if regularity is enforced by a proper choice of the initial data.

8.1 Conditional Regularity via the Relative Energy

The chapter is concluded by a short discussion of *conditional regularity* of weak solutions to problems in fluid dynamics. These are additional restriction that, if satisfied by a weak solution, imply its regularity (smoothness). A possible approach to conditional regularity of *weak solutions* is to show that:

- the problem admits local-in-time strong solution;
- the problem enjoys the weak-strong uniqueness property;
- show conditional regularity for the strong solution.

This procedure applied in the context of the finite energy weak solutions to the Navier-Stokes-Fourier system gives rise to the following result (see [27, Theorem 2.1]).

Theorem 7. *Under the hypotheses of Theorem 5, let* $\varrho, \vartheta, \mathbf{u}$ *be a finite energy weak solution of the Navier-Stokes-Fourier system on the time interval* $(0, T)$ *belonging to the regularity class specified in Theorem 4, emanating from (regular) initial data satisfying the hypotheses of Theorem 2. Suppose, in addition, that*

$$\operatorname{ess}\sup_{t\in(0,T)} \|\nabla_x \mathbf{u}(t,\cdot)\|_{L^\infty(\Omega;R^{3\times 3})} < \infty.$$

Then $\varrho, \vartheta, \mathbf{u}$ *is a classical solution of the Navier-Stokes-Fourier system in* $(0, T) \times \Omega$.

9 Cross-References

- ▶ Blow-Up Criteria of Strong Solutions and Conditional Regularity of Weak Solutions for the Compressible Navier-Stokes Equations
- ▶ Derivation of Equations for Continuum Mechanics and Thermodynamics of Fluids
- ▶ Weak Solutions for the Compressible Navier-Stokes Equations: Existence, Stability, and Longtime Behavior
- ▶ Weak Solutions for the Compressible Navier-Stokes Equations with Density Dependent Viscosities
- ▶ Weak Solutions to 2D and 3D Compressible Navier-Stokes Equations in Critical Cases

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References

1. J.T. Beale, T. Kato, A. Majda, Remarks on the breakdown of smooth solutions for the 3-D Euler equations. Commun. Math. Phys. **94**(1), 61–66 (1984)
2. S.E. Bechtel, F.J. Rooney, M.G. Forest, Connection between stability, convexity of internal energy, and the second law for compressible Newtonian fuids. J. Appl. Mech. **72**, 299–300 (2005)

3. F. Belgiorno, Notes on the third law of thermodynamics, i. J. Phys. A **36**, 8165–8193 (2003)
4. F. Belgiorno, Notes on the third law of thermodynamics, ii. J. Phys. A **36**, 8195–8221 (2003)
5. S. Benzoni-Gavage, D. Serre, *Multidimensional Hyperbolic Partial Differential Equations, First Order Systems and Applications*. Oxford Mathematical Monographs (The Clarendon Press Oxford University Press, Oxford, 2007)
6. D. Bresch, B. Desjardins, Stabilité de solutions faibles globales pour les équations de Navier-Stokes compressibles avec température. C.R. Acad. Sci. Paris **343**, 219–224 (2006)
7. D. Bresch, B. Desjardins, On the existence of global weak solutions to the Navier-Stokes equations for viscous compressible and heat conducting fluids. J. Math. Pures Appl. **87**, 57–90 (2007)
8. T. Buckmaster, Onsager's conjecture almost everywhere in time. Commun. Math. Phys. **333**(3), 1175–1198 (2015)
9. M. Bulíček, J. Málek, K.R. Rajagopal, Navier's slip and evolutionary Navier-Stokes-like systems with pressure and shear- rate dependent viscosity. Indiana Univ. Math. J. **56**, 51–86 (2007)
10. E. Chiodaroli, E. Feireisl, O. Kreml, On the weak solutions to the equations of a compressible heat conducting gas. Anna. Inst. Poincaré, Anal. Nonlinear **32**, 225–243 (2015)
11. A.J. Chorin, J.E. Marsden, *A Mathematical Introduction to Fluid Mechanics* (Springer, New York, 1979)
12. P. Constantin, C. Fefferman, Direction of vorticity and the problem of global regularity for the Navier-Stokes equations. Indiana Univ. Math. J. **42**(3), 775–789 (1993)
13. C.M. Dafermos, The second law of thermodynamics and stability. Arch. Ration. Mech. Anal. **70**, 167–179 (1979)
14. S.R. de Groot, P. Mazur, *Nonequilibrium Thermodynamics* (Dover Publications, Inc., New York, 1984) Reprint of the 1962 original.
15. C. De Lellis, L. Székelyhidi, Jr., On admissibility criteria for weak solutions of the Euler equations. Arch. Ration. Mech. Anal. **195**(1), 225–260 (2010)
16. C. De Lellis, L. Székelyhidi, Jr., Dissipative continuous Euler flows. Invent. Math. **193**(2), 377–407 (2013)
17. C. De Lellis, L. Székelyhidi, Jr., Dissipative Euler flows and Onsager's conjecture. J. Eur. Math. Soc. (JEMS) **16**(7), 1467–1505 (2014)
18. J.L. Ericksen, *Introduction to the Thermodynamics of Solids*, revised ed. Applied Mathematical Sciences, vol. 131 (Springer, New York, 1998)
19. J. Fan, S. Jiang, Y. Ou, A blow-up criterion for compressible viscous heat-conductive flows. Ann. Inst. H. Poincaré Anal. Non Linéaire **27**(1), 337–350 (2010)
20. C.L. Fefferman, Existence and smoothness of the Navier-Stokes equation, in *The Millennium Prize Problems* (Clay Mathematical Inst., Cambridge, 2006), pp. 57–67
21. E. Feireisl, *Dynamics of Viscous Compressible Fluids* (Oxford University Press, Oxford, 2004)
22. E. Feireisl, Relative entropies in thermodynamics of complete fluid systems. Discret. Cont. Dyn. Syst. Ser. A **32**, 3059–3080 (2012)
23. E. Feireisl, T. Karper, A. Novotný, On a convergent numerical scheme for the full Navier-Stokes-Fourier system. IMA J. Numer. Math. **36**(4), 2016, 1477–1535
24. E. Feireisl, O. Kreml, A. Vasseur, Stability of the isentropic Riemann solutions of the full multidimensional Euler system. SIAM J. Math. Anal. **47**, 2416–2425 (2015)
25. E. Feireisl, A. Novotný, *Singular Limits in Thermodynamics of Viscous Fluids* (Birkhäuser-Verlag, Basel, 2009)
26. E. Feireisl, A. Novotný, Weak-strong uniqueness property for the full Navier-Stokes-Fourier system. Arch. Ration. Mech. Anal. **204**, 683–706 (2012)
27. E. Feireisl, A. Novotný, Y. Sun, A regularity criterion for the weak solutions to the Navier-Stokes-Fourier system. Arch. Ration. Mech. Anal. **212**(1), 219–239 (2014)
28. G. Gallavotti, *Statistical Mechanics: A Short Treatise* (Springer, Heidelberg, 1999)
29. G. Gallavotti, *Foundations of Fluid Dynamics* (Springer, New York, 2002)
30. D. Hoff, Dynamics of singularity surfaces for compressible viscous flows in two space dimensions. Commun. Pure Appl. Math. **55**, 1365–1407 (2002)

31. D. Hoff, H.K. Jenssen, Symmetric nonbarotropic flows with large data and forces. Arch. Ration. Mech. Anal. **173**, 297–343 (2004)
32. D. Hoff, M.M. Santos, Lagrangean structure and propagation of singularities in multidimensional compressible flow. Arch. Ration. Mech. Anal. **188**(3), 509–543 (2008)
33. X. Huang, J. Li, Y. Wang, Serrin-type blowup criterion for full compressible Navier-Stokes system. Arch. Ration. Mech. Anal. **207**(1), 303–316 (2013)
34. D. Jesslé, B.J. Jin, A. Novotný, Navier-Stokes-Fourier system on unbounded domains: weak solutions, relative entropies, weak-strong uniqueness. SIAM J. Math. Anal. **45**(3), 1907–1951 (2013)
35. P.-L. Lions, *Mathematical Topics in Fluid Dynamics, Vol.1, Incompressible Models* (Oxford Science Publication, Oxford, 1996)
36. A. Matsumura, T. Nishida, The initial value problem for the equations of motion of viscous and heat-conductive gases. J. Math. Kyoto Univ. **20**, 67–104 (1980)
37. A. Matsumura, T. Nishida, The initial value problem for the equations of motion of compressible and heat conductive fluids. Commun. Math. Phys. **89**, 445–464 (1983)
38. N.V. Priezjev, S.M. Troian, Influence of periodic wall roughness on the slip behaviour at liquid/solid interfaces: molecular versus continuum predictions. J. Fluid Mech. **554**, 25–46 (2006)
39. G. Prodi, Un teorema di unicità per le equazioni di Navier-Stokes. Ann. Mat. Pura Appl. **48**, 173–182 (1959)
40. J. Serrin, On the interior regularity of weak solutions of the Navier-Stokes equations. Arch. Ration. Mech. Anal. **9**, 187–195 (1962)
41. J. Smoller, *Shock Waves and Reaction-Diffusion Equations* (Springer, New York, 1967)
42. Y. Sun, C. Wang, Z. Zhang, A Beale-Kato-Majda criterion for three dimensional compressible viscous heat-conductive flows. Arch. Ration. Mech. Anal. **201**(2), 727–742 (2011)
43. T. Tao, Localisation and compactness properties of the Navier-Stokes global regularity problem. Anal. PDE **6**(1), 25–107 (2013)
44. A. Valli, An existence theorem for compressible viscous fluids. Ann. Mat. Pura Appl. (4) **130**, 197–213 (1982)
45. H. Wen, C. Zhu, Blow-up criterions of strong solutions to 3D compressible Navier-Stokes equations with vacuum. Adv. Math. **248**, 534–572 (2013)
46. C.H. Wilcox, *Sound Propagation in Stratified Fluids*. Applied Mathematics Series, vol. 50 (Springer, Berlin, 1984)
47. R.Kh. Zeytounian, *Asymptotic Modelling of Fluid Flow Phenomena*. Fluid Mechanics and Its Applications, vol. 64 (Kluwer Academic Publishers, Dordrecht, 2002)
48. W.P. Ziemer, *Weakly Differentiable Functions* (Springer, New York, 1989)