Soft Computing Algorithm for Arithmetic Multiplication of Fuzzy Sets Based on Universal Analytic Models

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Abstract. This paper deals with the investigations of increasing dependability of computing operations for fuzzy numbers with triangular and bell-shape membership functions (MFs). Special attention is paid to the synthesis of analytic models of the MFs for the results of fuzzy arithmetic operations. New analytical models of the result's MFs with the description of synthesis procedures for the multiplication operation with triangular fuzzy numbers in R^+ and R are presented in a universal style. The general analytic models for determination of α-cuts parameters (direct and inverse approaches) for result fuzzy sets are given. Modeling results confirm the efficiency of the proposed models and fuzzy arithmetic algorithms for fuzzy information processing.

Keywords: Fuzzy number · Arithmetic operations · Multiplication · Direct model · Inverse model

1 Introduction

While performing the tasks of organizational control there always happen such situations when the original conditions of decision making are not clearly defined and characterized by insufficient awareness of the person who makes decisions, particularly in conflict situations or under extreme conditions. For mathematical formalization of processes and systems of this class there appeared a need to create a new mathematical approach. This approach is a theory of fuzzy sets developed by professor Zadeh [19]. Since the theory of fuzzy sets appeared, the specialists have had a great interest in it in terms of practical applications of mathematical methods in all fields of science and technology. The scientists around the world are aware of fundamental theoretical developments in the theory of fuzzy sets and fuzzy logic [2, 3, 11, 15, 17, 18].

Fuzzy set theory has a special notion of membership function [12, 15, 19] that exists in the interval [0,1]. Each element x of the fuzzy set, for example set $\underset{\sim}{A}$, corresponds to a specific value of the membership function $\mu_{\underset{\sim}{A}}(x) \in [0, 1]$. Thus, fuzzy set

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$\underset{\sim}{A}$ that is specified on the basis of the universal set E, is called [19] the set of pairs $(x, \mu_{\underset{\sim}{A}}(x))$, where $x \in E$, $\mu_{\underset{\sim}{A}}(x) \in [0, 1]$.

Fuzzy sets and fuzzy logic are used for tasks of decision making and control in uncertainty, in particular for problems of routes and trajectory optimization [8]. The solution of the problems causes the necessity of fulfilling the operations of fuzzy arithmetic, in particular operations with fuzzy sets including addition, subtraction, multiplication and division.

Inverse models of resulting membership functions that are based on using α-cuts [4] do not always provide high performance of computing operations and often lead to complications in solving control problems in real time. Thus, the development of generalized analytic models, based on the direct approach that allow to formalize fuzzy arithmetic operations to improve their operating speed and accuracy is an important direction of research that is associated with increased dependability [5] of intelligent systems.

One of the most difficult fuzzy arithmetic operations in terms of its mathematical formalization is an operation of multiplication.

Computational algorithms for the operations of multiplication on the basis of using α-cuts of the relevant fuzzy sets [4, 5, 9] (inverse approach) have high computational complexity, as it is performed in turn for all α- levels ($\alpha_i \in [0,\ 1]$, $i = 0,\ 1,\ 2,\ \ldots,\ r$, $\alpha_0 = 0$, $\alpha_r = 1$) with the step of discreteness $\Delta\alpha$, which value, taking into consideration that $\alpha_{i+1} = \alpha_i + \Delta\alpha$, significantly affects the accuracy and operating speed of the performance of computational procedures [5]. Therefore, α- cuts of the fuzzy set $\underset{\sim}{A} \in R$ is ordinary (in terms of conditions $\mu_{\underset{\sim}{A}}(x) \geq \alpha$) subset that contains elements $x \in R$ whose degree of membership to a set $\underset{\sim}{A}$ is not less than value α, that is $A_\alpha = \{x | \mu_{\underset{\sim}{A}}(x) \geq \alpha\}$, $\alpha \in [0, 1]$.

Subsets A_α та B_α that determine the appropriate α-cuts of fuzzy sets $\underset{\sim}{A}$ and $\underset{\sim}{B}$ can be written as follows: $A_\alpha = [a_1(\alpha),\ a_2(\alpha)]$, $B_\alpha = [b_1(\alpha),\ b_2(\alpha)]$, where $\alpha \in [0,\ 1]$, $\underset{\sim}{A}, \underset{\sim}{B} \in R^+$, and arithmetic operation of multiplication can be written as [4, 5, 9, 12, 15, 19]

$$\begin{aligned} A_\alpha(\cdot)B_\alpha &= [a_1(\alpha),\ a_2(\alpha)](\cdot)[b_1(\alpha),\ b_2(\alpha)] \\ &= [a_1(\alpha)b_1(\alpha),\ a_2(\alpha)b_2(\alpha)]. \end{aligned} \tag{1}$$

In addition to calculations based on the mentioned above α-cuts [4, 5] for implementation of fuzzy arithmetic operations where computational algorithms are often used that are realized through the use of max-min or min-max convolutions [4, 14] that in some cases leads to increased complexity and reduced operating speed of performance or to the moment of obtaining the resulting MFs that do not meet the requirements of convexity and normality of fuzzy sets.

The aim of this work is a synthesis of analytical models of resulting MFs. Their use in fuzzy arithmetic will give the opportunity to significantly reduce the volume, complexity and accuracy, and to improve their operating speed. Then a more detailed analysis of the properties of arithmetic operation of multiplication of fuzzy triangular numbers will be given. These properties are the most common while using the theory of fuzzy sets for designing control systems, decision making support systems and intelligence expert systems. This fuzzy triangular number is called fuzzy number $\underset{\sim}{A}$ whose MF $\mu_{\underset{\sim}{A}}(x)$ is of triangular shape and mathematical presentation of triangular fuzzy number has the form

$$\underset{\sim}{A} = (a_1, a_0, a_2),$$

where $\mu_{\underset{\sim}{A}}(a_1) = 0$; $\mu_{\underset{\sim}{A}}(a_0) = 1$; $\mu_{\underset{\sim}{A}}(a_2) = 0$.

Generalized model A_α, synthesized on the basis of inverse approach, and direct model in a form of a triangular membership function $\mu_{\underset{\sim}{A}}(x)$ of triangular fuzzy number $\underset{\sim}{A}$ are determined by the appropriate relevant dependencies (2) and (3):

$$A_\alpha = [a_1(\alpha),\ a_2(\alpha)] = [a_1 + \alpha(a_0 - a_1),\ a_2 - \alpha(a_2 - a_0)\,], \tag{2}$$

$$\mu_{\underset{\sim}{A}}(x) = \begin{cases} 0, \forall(x \le a_1) \cup (x \ge a_2) \\ (x - a_1)/(a_0 - a_1), \forall(a_1 < x \le a_0)\,. \\ (a_2 - x)/(a_2 - a_0), \forall(a_0 < x < a_2) \end{cases} \tag{3}$$

2 Universal Analytical Models for Multiplication of Triangular Fuzzy Numbers in R^+

2.1 Synthesis of Inverse Model for Multiplication of Two Fuzzy Numbers

We shall illustrate the methods of forming inverse $C_\alpha = [c_1(\alpha), c_2(\alpha)]$ and direct $\mu_C(x)$ generalized analytical models of resulting MF for the operation of fuzzy triangular numbers multiplication $\underset{\sim}{C} = \underset{\sim}{A}(\cdot)\underset{\sim}{B}$.

Firstly we shall form the inverse generalized model $A_\alpha = [a_1(\alpha), a_2(\alpha)]$ for a given (Fig. 1a) triangular fuzzy number $\underset{\sim}{A} = (a_1, a_0, a_2)$ in the set of non-negative real numbers R^+ [4, 5, 6, 9, 10] for the case $(a_1 < a_0 < a_2)$.

Let us analyze the left branch of the triangular fuzzy number $\underset{\sim}{A}$ for α- cut on the basis (2).

It is possible to write

$$\alpha = (a_1(\alpha) - a_1)/(a_0 - a_1),$$

where one can define

$$a_1(\alpha) = a_1 + (a_0 - a_1)\alpha,$$

where $a_1 = a_1(0) \ge 0$, $a_0 - a_1 > 0$, as $a_0 > a_1$, since $\underset{\sim}{A} \in R_0^+$.

Let us introduce the designations: $K_1 = a_0 - a_1$; $K_2 = a_1$, where $K_1 > 0$ and $K_2 \geq 0$, regarding this it can be written as:

$$a_1(\alpha) = K_2 + K_1\alpha.$$

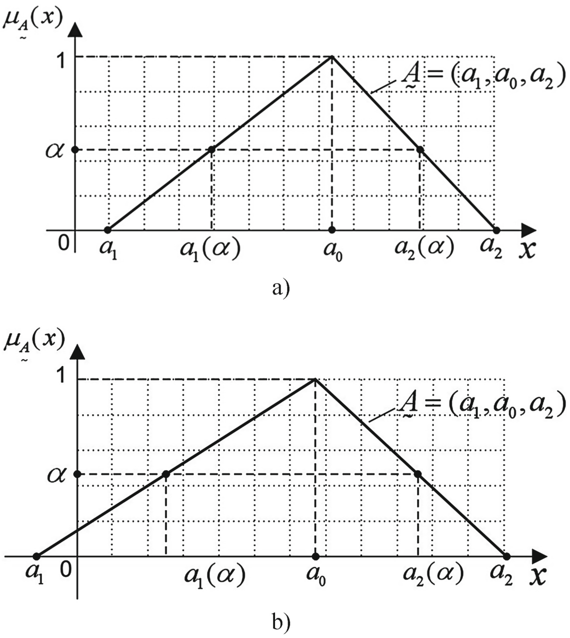

Fig. 1. Triangular Fuzzy Number $\underset{\sim}{A}$: (a) $\underset{\sim}{A} \in R^+$; (b) $\underset{\sim}{A} \in R$

Let us similarly analyze the right branch of a fuzzy number $\underset{\sim}{A}$:

$$\alpha = (a_2(\alpha) - a_2)/(a_0 - a_2),$$

$$a_2(\alpha) = a_2 + (a_0 - a_2)\alpha.$$

Having marked $K_3 = a_0 - a_2$, $K_4 = a_2$ ($K_3 < 0$, because $a_2 = a_2(0) > a_0$; $k_4 > 0$, as $\underset{\sim}{A} \in R_0^+$), we shall receive

$$a_2(\alpha) = K_4 + K_3\alpha.$$

However, as there is inequality $a_2 > |a_0 - a_2|$, then $K_4 > |K_3|$, respectively.

The α- cut for the left and right branches of the triangular fuzzy number $\underset{\sim}{A}$ can be represented as follows (taking into account coefficients K_i, $i = 1\ldots4 : K_1 > 0; K_2 > 0$; $K_3 < 0$; $K_4 > 0$; $K_4 > |K_3|$):

$$A_\alpha = [a_1(\alpha), a_2(\alpha)] = [K_2 + K_1\alpha,\ K_4 + K_3\alpha]. \tag{4}$$

Corresponding $\alpha-$ cut for triangular fuzzy number $\underset{\sim}{B} = (b_1, b_0, b_2)$ has the form

$$B_\alpha = [b_1(\alpha), b_2(\alpha)]$$

Having marked for the case $(b_1 < b_0 < b_2)$:

$$S_1 = b_0 - b_1, S_2 = b_1, S_3 = b_0 - b_2, S_4 = b_2,$$

we shall receive the modified $\alpha-$ cut B_α (taking into account the coefficients $S_i,\ i = 1\ldots4:\ S_1 > 0;\ S_2 \geq 0;\ S_3 < 0;\ S_4 > 0;\ S_4 > |S_3|$):

$$B_\alpha = [b_1(\alpha),\ b_2(\alpha)] = [S_2 + S_1\alpha,\ S_4 + S_3\alpha]. \tag{5}$$

Based on (4) and (5) we shall receive an inverse model for $\alpha-$ cut of a fuzzy set $\underset{\sim}{C} = \underset{\sim}{A}(\cdot)\underset{\sim}{B}$:

$$\begin{aligned} C_\alpha = A_\alpha(\cdot)B_\alpha &= [a_1(\alpha), a_2(\alpha)](\cdot)[b_1(\alpha), b_2(\alpha)] \\ &= [a_1(\alpha)b_1(\alpha), a_2(\alpha)b_2(\alpha)] \\ &= [(K_2 + K_1\alpha)(S_2 + S_1\alpha),\ (K_4 + K_3\alpha)(S_4 + S_3\alpha)] \\ &= \left[K_1S_1\alpha^2 + (K_1S_2 + K_2S_1)\alpha + K_2S_2,\ K_3S_3\alpha^2 + (K_3S_4 + K_4S_3)\alpha + K_4S_4\right] \\ &= [C_1(\alpha), C_2(\alpha)]. \end{aligned} \tag{6}$$

2.2 Synthesis of Direct Model for Multiplication of Two Fuzzy Numbers

Left branch of resulting MF. For the direct model $\mu_{\underset{\sim}{C}}(x)$ on the basis of proposed approach we shall consider in a more detailed way a constituent $C_1(\alpha)$ for $\alpha-$ cut (6) of the resulting fuzzy set $\underset{\sim}{C}$, formed by the operation of multiplying $\underset{\sim}{C} = \underset{\sim}{A}(\cdot)\underset{\sim}{B}$:

$$C_1(\alpha) = K_1S_1\alpha^2 + (K_1S_2 + K_2S_1)\alpha + K_2S_2.$$

The solution of the equation

$$K_1S_1\alpha^2 + (K_1S_2 + K_2S_1)\alpha + (K_2S_2 - C_1(\alpha)) = 0$$

will have the following roots

$$\alpha_{1,2} = \left(-(K_1S_2 + K_2S_1) \pm \sqrt{(K_1S_2 - K_2S_1)^2 + 4K_1S_1C_1(\alpha)}\right)/(2K_1S). \tag{7}$$

Let us analyze the roots α_1 and α_2 according to (7), which can be written as

$$\alpha_{1,2} = \left(-V_1 \pm \sqrt{Q_1}\right)/W_1 = -V_1/W_1 \pm \sqrt{Q_1}/W_1,$$

taking into consideration the following signs:

$$V_1 = K_1 S_2 + K_2 S_1,$$

$$Q_1 = (K_1 S_2 - K_2 S_1)^2 + 4K_1 S_1 C_1(\alpha),$$

$$W_1 = 2K_1 S_1.$$

To form a direct model of the resulting nonlinear membership function $\mu_{\underset{\sim}{c}}(x)$ it is necessary to check the performance of the condition $\alpha_{1,2} \in [0; 1]$.

Let us consider in details [8] the components V_1, Q_1, W_1 :

(a) $V_1 = K_1 S_2 + K_2 S_1$, taking into account correlations $K_1 > 0$, $K_2 > 0$, $S_1 > 0$, $S_2 > 0$ where there inequality $V_1 > 0$ takes place and respectively inequality $-V_1 < 0$;
(b) $Q_1 = (K_1 S_2 - K_2 S_1)^2 + 4K_1 S_1 C_1(\alpha) : C_1(\alpha) > 0$ as $\underset{\sim}{A}$, $\underset{\sim}{B} \in R_0^+$ because $K_1 > 0$, $S_1 > 0$, then always $Q_1 > 0$;
(c) $W_1 = 2K_1 S_1$, as $K_1 > 0, S_1 > 0$, then $W_1 > 0$.

As $-V_1/W_1 < 0$, and the roots $\alpha_{1,2}$ have requirements $\alpha_{1,2} \in [0; 1]$, then the root that satisfies the given condition under $V_1 > 0$, $Q_1 > 0$, $W_1 > 0$ will be

$$\alpha_1 = \left(-V_1 + \sqrt{Q_1}\right)/W_1,$$

that is

$$\alpha_1 = \left(-(K_1 S_2 + K_2 S_1) + \sqrt{(K_1 S_2 - K_2 S_1)^2 + 4K_1 S_1 C_1(\alpha)}\right)/(2K_1 S_1). \quad (8)$$

The root

$$\alpha_2 = \left(-V_1 - \sqrt{Q_1}\right)/W_1$$

does not satisfy the condition $\alpha_2 \in [0; 1]$, as there is always a condition $\alpha_2 < 0$.

The transition from inverse to direct approach [4, 5, 6, 8] shows that x is a parameter of the function $\alpha = f(C_1(\alpha))$, that is $\alpha = f(x)$, where $x = C_1(\alpha) \in [c_1, c_0]$, and then (8) can be represented as

$$\alpha_1 = \left(-(K_1 S_2 + K_2 S_1) + \sqrt{(K_1 S_2 - K_2 S_1)^2 + 4K_1 S_1 x}\right)/(2K_1 S_1). \quad (9)$$

Right branch of resulting MF. Let us consider in detail the second component of the inverse model (6)

$$C_2(\alpha) = K_3S_3\alpha^2 + (K_3S_4 + K_4S_3)\alpha + K_4S_4.$$

The solution of the corresponding equation

$$K_3S_3\alpha^2 + (K_3S_4 + K_4S_3)\alpha + (K_4S_4 - C_2(\alpha)) = 0$$

will be the roots

$$\alpha_{3,4} = \left(-(K_3S_4 + K_4S_3) \pm \sqrt{(K_3S_4 - K_4S_3)^2 + 4K_3S_3C_2(\alpha)}\right)/(2K_3S_3).$$

The analysis of roots α_3 and α_4 shows, that the root $\alpha_3 > 1$ does not satisfy the condition

$$\alpha_3 \in [0;1]$$

and therefore the only acceptable root will be the root α_4

The transition from inverse to direct approach [4, 5, 6, 8] allows to transform $\alpha = f(C_2(\alpha))$ into $\alpha = f(x)$, $x = C_2(\alpha) \in [c_0, c_2]$, and thus

$$\alpha_4 = \left(-(K_3S_4 + K_4S_3) - \sqrt{(K_3S_4 - K_4S_3)^2 + 4K_3S_3x}\right)/(2K_3S_3). \quad (10)$$

The direct model of resulting MF and modeling results. The analysis of the roots α_1 and α_4 allows to make a conclusion that nonlinear dependence $\alpha = f(x)$ under intervals $x \in [c_1, c_0], x \in [c_0, c_2]$ is one-valued function.

The investigations allow to form a direct analytical model for one-valued nonlinear resulting membership function $\mu_{\underset{\sim}{C}}(x)$ of fuzzy set $\underset{\sim}{C} = \underset{\sim}{A}(\cdot)\underset{\sim}{B}$ that is formed by multiplying triangular fuzzy numbers $\underset{\sim}{A}$ and $\underset{\sim}{B}$ in R^+:

$$\mu_{\underset{\sim}{C}}(x) = \begin{cases} 0, \\ \quad \text{for } (\forall x \le K_2S_2) \bigcup (\forall x \ge K_4S_4) \\ \frac{-(K_1S_2 + K_2S_1) + \sqrt{(K_1S_2 - K_2S_1)^2 + 4K_1S_1x}}{2K_1S_1}, \\ \quad \text{for } \forall x \in [K_2S_2, K_1S_1 + K_1S_2 + K_2S_1 + K_2S_2] \\ \frac{-(K_3S_4 + K_4S_3) - \sqrt{(K_3S_4 - K_4S_3)^2 + 4K_3S_3x}}{2K_3S_3}, \\ \quad \text{for } \forall x \in [K_3S_3 + K_3S_4 + K_4S_3 + K_4S_4, K_4S_4] \\ 1, \\ \quad \text{for } \forall x = c_0 = K_1S_1 + K_1S_2 + K_2S_1 + K_2S_2 \end{cases}, \quad (11)$$

where $c_0 = K_1S_1 + K_1S_2 + K_2S_1 + K_2S_2 = K_3S_3 + K_3S_4 + K_4S_3 + K_4S_4$.

The given direct approach allows to form nonlinear resulting membership function $\mu_{\underset{\sim}{C}}(x)$ on the basis of known coefficients $K_i,\ S_i(i = 1..4)$ of fuzzy numbers $\underset{\sim}{A}$ (4) and $\underset{\sim}{B}$ (5) in R^+.

By substituting previously set marks

$$K_1 = a_0 - a_1, K_2 = a_1, K_3 = a_0 - a_2, K_4 = a_2,$$

$$S_1 = b_0 - b_1, S_2 = b_1, S_3 = b_0 - b_2, S_4 = b_2,$$

we shall get (Table 1) a direct model $\mu_{\underset{\sim}{C}}(x)$ that is realized directly on the basis of parameters $(a_0,\ a_1,\ a_2)$, $(b_0,\ b_1,\ b_2)$ of triangular fuzzy numbers $\underset{\sim}{A}$ and $\underset{\sim}{B}$ in R^+.

The chart of the corresponding resulting membership function $\mu_{\underset{\sim}{C}}(x)$ under the realizing the operation of multiplication of triangular fuzzy numbers

$$\underset{\sim}{A} = (5,\ 7,\ 12)$$

and

$$\underset{\sim}{B} = (2,\ 9,\ 14)$$

using a developed direct model, presented in Table 1, shown in Fig. 2.

Table 1. Analytical model $\mu_{\underset{\sim}{C}}(x)$: direct approach.

Resulting membership function $\mu_{\underset{\sim}{C}}(x) =$
$\forall (x < a_1b_1 \cup x > a_2b_2),\quad = 0$
$\forall x \in [a_1b_1, a_0b_0),$ $= \frac{-[(a_0-a_1)b_1+a_1(b_0-b_1)]+\sqrt{[(a_0-a_1)b_1-a_1(b_0-b_1)]^2+4(a_0-a_1)(b_0-b_1)x}}{2(a_0-a_1)(b_0-b_1)}$
$\forall x = a_0b_0, = 1$
$\forall x \in (a_0b_0, a_2b_2], = \frac{-[(a_0-a_2)b_2+a_2(b_0-b_2)]-\sqrt{((a_0-a_2)b_2-a_2(b_0-b_2))^2+4(a_0-a_2)(b_0-b_2)x}}{2(a_0-a_2)(b_0-b_2)}$

3 Direct and Inverse Analytical Models of Fuzzy Sets with Bell-Shape Membership Functions

Let's consider the class of bell-shape membership functions with following direct model [14, 15], for example for fuzzy set $\underset{\sim}{A}$,

$$\mu_{\underset{\sim}{A}}(x) = 1/\left(1 + \left(\frac{x-b}{c}\right)^2\right) \tag{12}$$

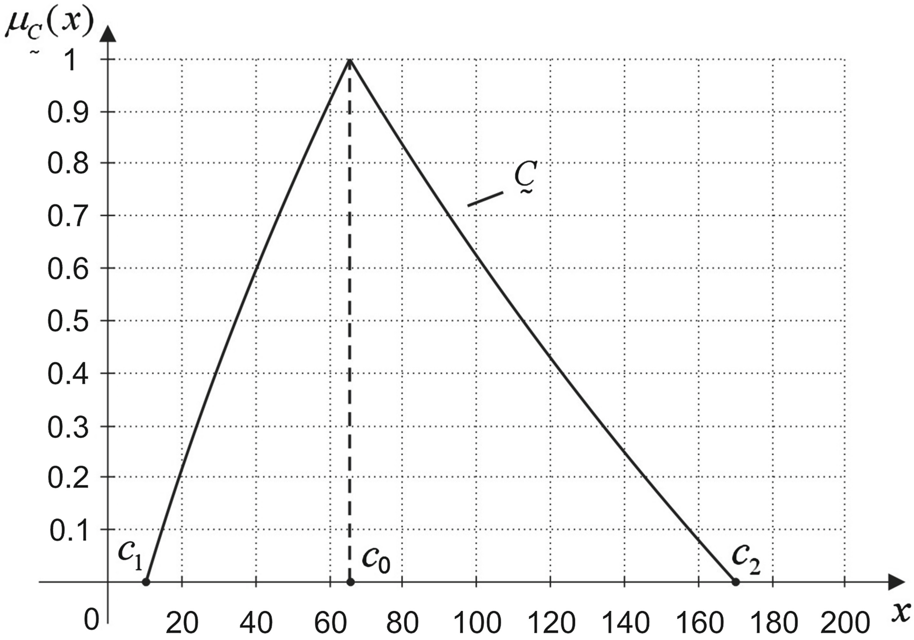

Fig. 2. Implementation of the direct model $\mu_{\underset{\sim}{C}}(x)$

and a synthesis procedure for analytic models of membership functions as results of arithmetic operations implementation. We will consider the cases when parameters of bell-shape MFs in model (12) are positive ($b > 0, c > 0$) and the approach is based on using α-cut models.

First of all, we form the inverse (horizontal) model of fuzzy set $\underset{\sim}{A}$

$$A_\alpha = [a_1(\alpha), a_2(\alpha)], \alpha \in [0,\ 1],$$

taking into account the equivalence relation between notations [4] in direct and inverse models and substituting the next parameters $\mu_{\underset{\sim}{A}}(x) = \alpha,\ \ x = a_{1,2}(\alpha)$ in direct model (12):

$$\alpha = 1/\left(1 + \left[\frac{a_{1,2}(\alpha) - b}{c}\right]^2\right), \tag{13}$$

where $a_{1,2}(\alpha) \in \{a_1(\alpha), a_2(\alpha)\}$.

At the next step it is necessary to find the roots of the Eq. (13) based on the following transformations (14)–(16)

$$\alpha = \frac{c^2}{c^2 + \left[a_{1,2}(\alpha) - b\right]^2}, \tag{14}$$

$$\alpha = \frac{c^2}{c^2 + a_{1,2}^2(\alpha) - 2ba_{1,2}(\alpha) + b^2}, \tag{15}$$

$$c^2 = c^2\alpha + a_{1,2}^2(\alpha)\alpha - 2ba_{1,2}(\alpha)\alpha + b^2\alpha. \tag{16}$$

As result we can obtain the square Eq. (17) corresponding to parameters $a_{1,2}(\alpha)$

$$\alpha a_{1,2}^2(\alpha) - 2\alpha b a_{1,2}(\alpha) + \left(b^2\alpha + c^2\alpha - c^2\right) = 0. \tag{17}$$

The roots of (17) can be calculated as (18)

$$a_{1,2}(\alpha) = \frac{2\alpha b \pm \sqrt{4\alpha^2 b^2 - 4\alpha(b^2\alpha + c^2\alpha - c^2)}}{2\alpha} \tag{18}$$

or as (19)

$$a_{1,2}(\alpha) = \frac{2\alpha b \pm \sqrt{4\alpha c^2 - 4\alpha^2 c^2}}{2\alpha} \tag{19}$$

The formula (19) can be transformed to (20)

$$a_{1,2}(\alpha) = b \pm c\sqrt{\frac{1}{\alpha} - 1}, \tag{20}$$

that means that roots of Eq. (17) can be finally calculated as:

$$a_1(\alpha) = b - c\sqrt{\frac{1}{\alpha} - 1} \tag{21}$$

and

$$a_2(\alpha) = b + c\sqrt{\frac{1}{\alpha} - 1}. \tag{22}$$

In this case the inverse (horizontal) model (17) of fuzzy set $\underset{\sim}{A}$ can be written in the following form:

$$A_\alpha = [a_1(\alpha), a_2(\alpha)] = \left[b - c\sqrt{\frac{1}{\alpha} - 1}, b + c\sqrt{\frac{1}{\alpha} - 1}\right]. \tag{23}$$

4 Universal Analytical Models for Multiplication of Bell-Shape Fuzzy Numbers in R^{+}

4.1 Synthesis of Inverse Analytical Model for Resulting Fuzzy Set

For analytic models synthesis of fuzzy sets which can result in fuzzy sets after fuzzy arithmetic operations implementation let's consider two fuzzy sets $\underset{\sim}{A}$ and $\underset{\sim}{B}$ with membership functions of type (12), which are presented by models of α-cuts [4]:

$$A_\alpha = [a_1(\alpha), a_2(\alpha)] \tag{24}$$

$$B_\alpha = [b_1(\alpha), b_2(\alpha)]. \tag{25}$$

Using the designations $b = p_1, c = s_1$ for model A_α in (24) and $b = p_2, c = s_2$ for model B_α in (25) and using the general horizontal model (23) we can determine:

$$A_\alpha = [a_1(\alpha), a_2(\alpha)] = \left[p_1 - s_1\sqrt{\frac{1}{\alpha} - 1}, p_1 + s_1\sqrt{\frac{1}{\alpha} - 1}\right], \tag{26}$$

$$B_\alpha = [b_1(\alpha), b_2(\alpha)] = \left[p_2 - s_2\sqrt{\frac{1}{\alpha} - 1}, p_2 + s_2\sqrt{\frac{1}{\alpha} - 1}\right]. \tag{27}$$

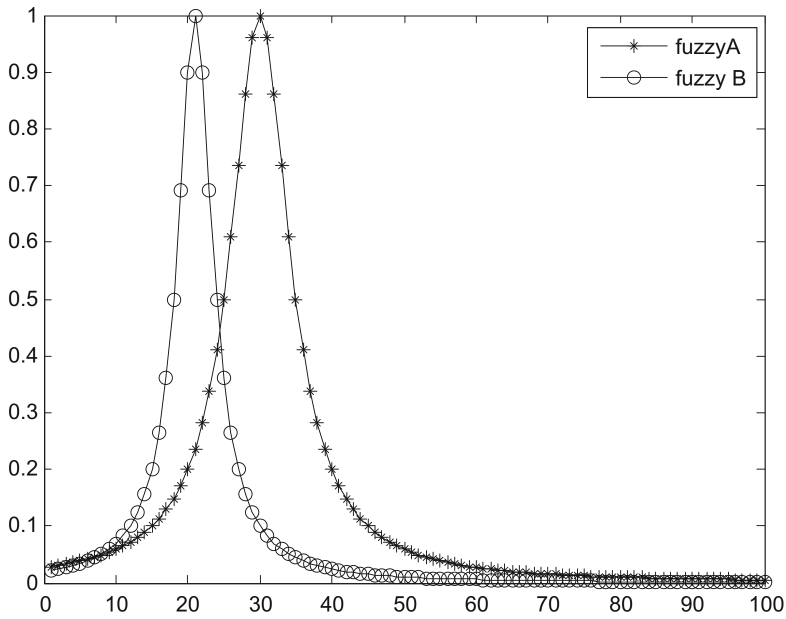

Fig. 3. Fuzzy sets $\underset{\sim}{A}, \underset{\sim}{B}$ with parameters of bell-shape membership functions: $p1 = 30, p2 = 21, s1 = 5, s2 = 3$

Let's consider the arithmetic operation of multiplication $\underset{\sim}{C} = \underset{\sim}{A}(\cdot)\underset{\sim}{B}$ for two fuzzy numbers $\underset{\sim}{A} \subset R^+$ and $\underset{\sim}{B} \subset R^+$ with bell-shape membership functions (12) according to the algorithm used in [2, 7, 11]:

$$C_\alpha = A_\alpha(\cdot)B_\alpha = [a_1(\alpha)(\cdot)b_1(\alpha), a_2(\alpha)(\cdot)b_2(\alpha)] \\ = [c_1(\alpha), c_2(\alpha)]. \tag{28}$$

Substituting the horizontal models (26) and (27) into (28) we can form the inverse model of resulting membership function for implementation of multiplication operation $\underset{\sim}{C} = \underset{\sim}{A}(\cdot)\underset{\sim}{B}$ for fuzzy sets $\underset{\sim}{A} \subset R^+$ and $\underset{\sim}{B} \subset R^+$:

$$\begin{aligned} C_\alpha = A_\alpha(\cdot)B_\alpha &= \begin{bmatrix} \left(p_1 - s_1\sqrt{\frac{1}{\alpha} - 1}\right)(\cdot)\left(p_2 - s_2\sqrt{\frac{1}{\alpha} - 1}\right), \\ \left(p_1 + s_1\sqrt{\frac{1}{\alpha} - 1}\right)(\cdot)\left(p_2 + s_2\sqrt{\frac{1}{\alpha} - 1}\right) \end{bmatrix} \\ &= \begin{bmatrix} p_1p_2 - \sqrt{\frac{1}{\alpha} - 1}(p_1s_2 + s_1p_2) + s_1s_2\left(\frac{1}{\alpha} - 1\right), \\ p_1p_2 + \sqrt{\frac{1}{\alpha} - 1}(p_1s_2 + s_1p_2) + s_1s_2\left(\frac{1}{\alpha} - 1\right) \end{bmatrix} \\ &= [c_1(\alpha), c_2(\alpha)]. \end{aligned} \tag{29}$$

The resulting fuzzy set, which was created on the basis of model (29) after multiplication $\underset{\sim}{C} = \underset{\sim}{A}(\cdot)\underset{\sim}{B}$ of two fuzzy sets (Fig. 3), is represented in Fig. 4.

4.2 Synthesis of Direct Model for Resulting MF

Let's introduce new designations:

$$k_1 = p_1p_2, k_2 = p_1s_2 + s_1p_2, k_3 = s_1s_2,$$

where $k_1 > 0$, $k_2 >$, $k_3 > 0$, as $b > 0$, $c > 0$ in (12) and correspondently, $p_i > 0$, $s_i > 0, (i = 1, 2)$ in (26) and (27).

In this case we can represent resulting membership function (29) for implementation of multiplication operation as following

$$\begin{aligned} C_\alpha &= \left[k_1 - k_2\sqrt{\frac{1}{\alpha} - 1} + k_3\left(\frac{1}{\alpha} - 1\right), k_1 + k_2\sqrt{\frac{1}{\alpha} - 1} + k_3\left(\frac{1}{\alpha} - 1\right)\right] \\ &= [c_1(\alpha), c_2(\alpha)]. \end{aligned} \tag{30}$$

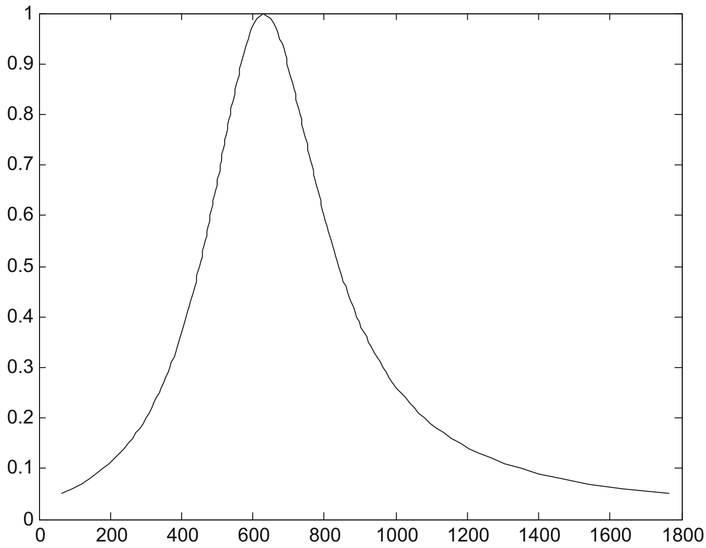

Fig. 4. Resulting fuzzy set for multiplication $\underset{\sim}{C} = \underset{\sim}{A}(\cdot)\underset{\sim}{B}$ based on the inverse model (29) with $\Delta\alpha = 0,01$

Investigation of the left branch and its properties for resulting MF. Let's consider in more details a left branch of Eq. (30)

$$c_1(\alpha) = k_1 - k_2\sqrt{\frac{1}{\alpha} - 1} + k_3\left(\frac{1}{\alpha} - 1\right), \tag{31}$$

which may be transformed to the form

$$k_1 + k_3\left(\frac{1}{\alpha} - 1\right) - c_1(\alpha) = k_2\sqrt{\frac{1}{\alpha} - 1}. \tag{32}$$

Square the left and right sides of the Eq. (32)

$$\left(k_1 + k_3\left(\frac{1}{\alpha} - 1\right) - c_1(\alpha)\right)^2 = \left(k_2\sqrt{\frac{1}{\alpha} - 1}\right)^2. \tag{33}$$

At the next step it is necessary to simplify the Eq. (33) based on the following transformations (34)–(36)

$$\begin{aligned} &k_1^2 + k_3^2\left(\frac{1}{\alpha} - 1\right)^2 + c_1^2(\alpha) + 2k_1k_3\left(\frac{1}{\alpha} - 1\right) - 2k_1c_1(\alpha) - 2k_3\left(\frac{1}{\alpha} - 1\right)c_1(\alpha) \\ &\quad = k_2^2\left(\frac{1}{\alpha} - 1\right) \end{aligned} \tag{34}$$

$$\begin{aligned} & k_1^2 + \frac{k_3^2}{\alpha^2} - 2\frac{k_3^2}{\alpha} + k_3^2 + c_1^2(\alpha) + \frac{2k_1k_3}{\alpha} - 2k_1k_3 \\ & \quad - 2k_1c_1(\alpha) - \frac{2k_3}{\alpha}c_1(\alpha) + 2k_3c_1(\alpha) - \frac{k_2^2}{\alpha} + k_2^2 = 0 \end{aligned}, \tag{35}$$

$$\begin{aligned} & \frac{k_3^2}{\alpha^2} + \left(2k_1k_3 - 2k_3^2 - 2k_3c_1(\alpha) - k_2^2\right)\frac{1}{\alpha} + k_1^2 + k_3^2 + c_1^2(\alpha) \\ & \quad - 2k_1k_3 - 2k_1c_1(\alpha) + 2k_3c_1(\alpha) + k_2^2 = 0 \end{aligned}. \tag{36}$$

Multiplying the left and right sides of the Eq. (36) for α^2

$$\begin{aligned} & (k_1^2 + k_3^2 + c_1^2(\alpha) - 2k_1k_3 - 2k_1c_1(\alpha) + 2k_3c_1(\alpha) + k_2^2)\alpha^2 \\ & \quad + \left(2k_1k_3 - 2k_3^2 - 2k_3c_1(\alpha) - k_2^2\right)\alpha + k_3^2 = 0 \end{aligned} \tag{37}$$

and find the roots $\alpha_{1,2}$ of Eq. (37)

$$\alpha_{1,2} = \frac{-\left(2k_1k_3 - 2k_3^2 - 2k_3c_1(\alpha) - k_2^2\right) \pm \sqrt{\begin{aligned} & \left(2k_1k_3 - 2k_3^2 - 2k_3c_1(\alpha) - k_2^2\right)^2 - 4\left(k_1^2 + k_3^2 + \right. \\ & \left. + c_1^2(\alpha) - 2k_1k_3 - 2k_1c_1(\alpha) + 2k_3c_1(\alpha) + k_2^2\right)k_3^2 \end{aligned}}}{2\left(k_1^2 + k_3^2 + c_1^2(\alpha) - 2k_1k_3 - 2k_1c_1(\alpha) + 2k_3c_1(\alpha) + k_2^2\right)}$$

or

$$\alpha_{1,2} = \left(-v_1 \pm \sqrt{q_1}\right)/w_1, \tag{38}$$

where indicated:

$$v_1 = 2k_1k_3 - 2k_3^2 - 2k_3c_1(\alpha) - k_2^2, \tag{39}$$

$$\begin{aligned} q_1 &= \left(2k_1k_3 - 2k_3^2 - 2k_3c_1(\alpha) - k_2^2\right)^2 \\ & \quad - 4\left(k_1^2 + k_3^2 + c_1^2(\alpha) - 2k_1k_3 - 2k_1c_1(\alpha) + 2k_3c_1(\alpha) + k_2^2\right)k_3^2 \\ &= \left(2k_1k_3 - 2k_3^2 - 2k_3c_1(\alpha) - k_2^2\right)^2 - 4\left((k_3 + c_1(\alpha) - k_1)^2 + k_2^2\right)k_3^2 \end{aligned}, \tag{40}$$

$$\begin{aligned} w_1 &= 2\left(k_1^2 + k_3^2 + c_1^2(\alpha) - 2k_1k_3 - 2k_1c_1(\alpha) + 2k_3c_1(\alpha) + k_2^2\right) \\ &= 2\left((k_3 + c_1(\alpha) - k_1)^2 + k_2^2\right) \end{aligned}. \tag{41}$$

To form a direct model of the resulting nonlinear membership function $\mu_{\underset{\sim}{c}}(x)$ it is necessary to check the performance of the condition $\alpha_{1,2} \in [0; 1]$.

Let us consider in details the components (39)–(41) in roots (38), in particular, v_1, q_1, w_1:

(a) for value v_1 the two cases should be considered:
 (1) $v_1 > 0$, if $2k_1k_3 > \left(2k_3^2 + 2k_3c_1(\alpha) + k_2^2\right)$ and
 (2) $v_1 < 0$, if $2k_1k_3 < \left(2k_3^2 + 2k_3c_1(\alpha) + k_2^2\right)$;

(b) the value q_1 theoretically may be positive or negative, but taking into account that it is necessary to calculate $\sqrt{q_1}$ and roots (38) are real numbers, then q_1 should be only positive number: $q_1 > 0$;

(c) analyzing (41) we may conclude that for any cases $w_1 > 0$.

According to (38) we can rewrite

$$\alpha_{1,2} = -v_1/w_1 \pm \sqrt{q_1}/w_1. \tag{42}$$

Let's analyze the situation when $v_1 > 0$ and in this case we have inequality

$$k_1 - k_3 > c_1(\alpha) + \frac{k_2^2}{2k_3}. \tag{43}$$

The Eq. (40) can be represented as

$$q_1 = z - y, \tag{44}$$

where components z and y may be transformed in the following way:

$$\begin{aligned} z &= \left(2k_1k_3 - 2k_3^2 - 2k_3c_1(\alpha) - k_2^2\right)^2 \\ &= 4k_1^2k_3^2 - 8k_1k_3^3 - 8k_1k_3^2c_1(\alpha) - 4k_1k_2^2k_3 + 4k_3^4 \\ &\quad + 8k_3^3c_1(\alpha) + 4k_2^2k_3^2 + 4k_3^2c_1^2(\alpha) + 4k_2^2k_3c_1(\alpha) + k_2^4 \end{aligned}; \tag{45}$$

$$\begin{aligned} y &= 4\left((k_3 + c_1(\alpha) - k_1)^2 + k_2^2\right)k_3^2 \\ &= 4k_1^2k_3^2 + 4k_3^4 + 4k_3^2c_1^2(\alpha) - 8k_1k_3^3 - 8k_1k_3^2c_1(\alpha) + 8k_3^3c_1(\alpha) + 4k_2^2k_3^2 \end{aligned}. \tag{46}$$

Substituting (45) and (46) into (44) we can obtain

$$q_1 = 4k_2^2k_3\left(c_1(\alpha) + \frac{k_2^2}{4k_3} - k_1\right). \tag{47}$$

Analyzing (43) and taking into account that $k_3 > 0$ it is possible to conclude

$$k_1 > c_1(\alpha) + \frac{k_2^2}{2k_3}$$

and respectively

$$c_1(\alpha) + \frac{k_2^2}{2k_3} - k_1 < 0. \tag{48}$$

The condition (48) means that in this case according to (47) we have

$$q_1 < 0. \tag{49}$$

Taking into account that q_1 should be always a positive number it is necessary to say the considered situation $(v_1 > 0)$ is uncorrected.

Let's analyze the second situation when $v_1 < 0$ for roots (42) and correlation between v_1 and w_1:

$$-v_1 = -2k_1k_3 + 2k_3^2 + 2k_3c_1(\alpha) + k_2^2, \tag{50}$$

$$\begin{aligned} w_1 &= 2\Big((k_3 + c_1(\alpha) - k_1)^2 + k_2^2\Big) \\ &= 2(c_1(\alpha) - k_1)^2 + 2k_3^2 - 4k_1k_3 + 4k_3c_1(\alpha) + 2k_2^2 \end{aligned}. \tag{51}$$

First of all, it is necessary to check the condition

$$w_1 > -v_1$$

or

$$w_1 + v_1 > 0. \tag{52}$$

Substituting (50) and (51) into (52) we will obtain

$$\begin{aligned} w_1 + v_1 &= 2(c_1(\alpha) - k_1)^2 + 2k_3^2 - 4k_1k_3 + 4k_3c_1(\alpha) + 2k_2^2 \\ &\quad + 2k_1k_3 - 2k_3^2 - 2k_3c_1(\alpha) - k_2^2 \\ &= 2(c_1(\alpha) - k_1)^2 - 2k_1k_3 + 2k_3c_1(\alpha) + k_2^2 > 0 \end{aligned}. \tag{53}$$

Taking into account that

$$k_1 > 0, k_3 > 0, k_2 > 0, c_1(\alpha) > 0$$

in the inequality (53), such components will be always positive

$$2(c_1(\alpha) - k_1)^2 > 0;\ 2k_3c_1(\alpha) > 0;\ k_2^2 > 0;\ 2k_1k_3 > 0$$

and only one component is negative

$$-2k_1k_3 < 0.$$

Let's compare one of the positive components in (53)

$$k_2^2 = (p_1s_2 + p_2s_1)^2 = (p_1s_2)^2 + 2p_1s_2p_2s_1 + (p_2s_1)^2$$

with negative component in (53)

$$-2k_1k_3 = -2(p_1p_2)(s_1s_2) = -2p_1p_2s_1s_2$$

in the following way

$$k_2^2 - 2k_1k_3 = (p_1s_2)^2 + 2p_1s_2p_2s_1 + (p_2s_1)^2 - 2p_1p_2s_1s_2 = (p_1s_2)^2 + (p_2s_1)^2 > 0. \quad (54)$$

The inequality (54) confirms that in (53)

$$2(c_1(\alpha) - k_1)^2 + 2k_3c_1(\alpha) + k_2^2 > 2k_1k_3$$

and inequalities and correlations

$$w_1 + v_1 > 0,\ w_1 > -v_1 \quad (55)$$

are correct.

In this case we can see that

$$-v_1/w_1 < 1 \quad (56)$$

and at the same time taking into account that $v_1 < 0$, $w_1 > 0$ we have

$$-v_1/w_1 > 0 \quad (57)$$

and finally

$$0 < (-v_1/w_1) < 1,\ -v_1/w_1 \in [0,\ 1].$$

Analyzing (39)–(41) we may represent q_1 and $\sqrt{q_1}$ in following forms

$$q_1 = v_1^2 - 2k_3^2w_1,$$

$$\sqrt{q_1} = \sqrt{v_1^2 - 2k_3^2w_1}.$$

Comparing $\sqrt{q_1}$ and $-v_1$ (for case, when $v_1 < 0$) we can see that $-v_1 > \sqrt{q_1}$ and, correspondently,

$$-v_1 - \sqrt{q_1} > 0. \quad (58)$$

Let's check the condition

$$\alpha_2 \in [0,\ 1] \quad (59)$$

for root $\alpha_2 = \left(-v_1 - \sqrt{q_1}\right)/w_1$ according to (38) and (42).

Based on (56)–(58) we can obtain the next inequalities:

$$w_1 > -v_1 - \sqrt{q_1},$$

$$0 < \left(-v_1 - \sqrt{q_1}\right)/w_1 < 1$$

and, correspondingly, $0 < \alpha_2 < 1$ or $\alpha_2 \in [0,\ 1]$.
Let's check the condition

$$\alpha_1 \in [0,\ 1] \tag{60}$$

for root $\alpha_1 = \left(-v_1 + \sqrt{q_1}\right)/w_1$ according to (38) and (42).
The condition (60) is correct for $-v_1 > 0$, $\sqrt{q_1} > 0$, $w_1 > 0$, if

$$w_1 \geq \left(-v_1 + \sqrt{q_1}\right), \tag{61}$$

$$(w_1 + v_1) \geq \sqrt{q_1}$$

or, correspondingly, for

$$(w_1 + v_1) \geq \sqrt{v_1^2 - 2k_3^2 w_1}. \tag{62}$$

Square the left and right sides of the inequality (62)

$$(w_1 + v_1)^2 \geq \left(v_1^2 - 2k_3^2 w_1\right)$$

and make the following sequence transformations

$$w_1^2 + 2w_1v_1 + v_1^2 \geq v_1^2 - 2k_3^2 w_1,$$

$$w_1^2 + 2w_1v_1 \geq -2k_3^2 w_1,$$

$$w_1 + 2v_1 + 2k_3^2 \geq 0. \tag{63}$$

Substituting (39) and (41) into (63) we can obtain

$$2((k_3 + C_1(\alpha) - k_1)^2 + k_2^2) + 2\left(2k_1k_3 - 2k_3^2 - 2k_3C_1(\alpha) - k_2^2\right) + 2k_3^2 \geq 0. \tag{64}$$

After dividing both sides of inequality (64) by 2, exponentiation and reduction of similar the expression (64) takes the following form

$$k_1^2 + c_1^2(\alpha) - 2k_1c_1(\alpha) \geq 0,$$

$$(c_1(\alpha) - k_1)^2 \geq 0. \tag{65}$$

As inequality (65) and corresponding inequality (61) are confirmed, then

$$\left(-v_1 + \sqrt{q_1}\right)/w_1 < 1,$$

and condition (60) is satisfied.
Taking into account that both conditions (59) and (60)

$$\alpha_1 \in [0,1],\ \alpha_2 \in [0,1]$$

are satisfied for roots $\alpha_1,\ \alpha_2$ of left branch's Eq. (31), the resulting value α_L^* can be find in the following way

$$\alpha_L^* = \max\{\alpha_1, \alpha_2\} : \tag{66}$$

$$\alpha_L^* = \frac{-\left(2k_1k_3 - 2k_3^2 - 2k_3c_1(\alpha) - k_2^2\right) + \sqrt{\begin{matrix}\left(2k_1k_3 - 2k_3^2 - 2k_3c_1(\alpha) - k_2^2\right)^2 - 4\left(k_1^2 + k_3^2 + \right.\\ \left. + c_1^2(\alpha) - 2k_1k_3 - 2k_1c_1(\alpha) + 2k_3c_1(\alpha) + k_2^2\right)k_3^2\end{matrix}}}{2\left(k_1^2 + k_3^2 + c_1(\alpha) - 2k_1k_3 - 2k_1c_1(\alpha) + 2k_3c_1(\alpha) + k_2^2\right)}.$$

Investigation of the right branch and its properties for resulting MF. Using proposed approach it is possible to make all similar transformation for right branch of Eq. (30)

$$c_2(\alpha) = k_1 + k_2\sqrt{\frac{1}{\alpha} - 1} + k_3\left(\frac{1}{\alpha} - 1\right), \tag{67}$$

which has two roots $\alpha_3,\ \alpha_4$.
As result we will obtain that both conditions

$$\alpha_3 \in [0,1],\ \alpha_4 \in [0,1]$$

are satisfied for roots $\alpha_3,\ \alpha_4$ of right branch of Eq. (30) and the resulting value α_R^* can be find in the following way

$$\alpha_R^* = \max\{\alpha_3, \alpha_4\} : \tag{68}$$

$$\alpha_R^* = \frac{-\left(2k_1k_3 - 2k_3^2 - 2k_3c_2(\alpha) - k_2^2\right) + \sqrt{\begin{matrix}(2k_1k_3 - 2k_3^2 - 2k_3c_2(\alpha) - k_2^2)^2 - 4(k_1^2 + k_3^2 + \\ + c_2^2(\alpha) - 2k_1k_3 - 2k_1c_2(\alpha) + 2k_3c_2(\alpha) + k_2^2)k_3^2\end{matrix}}}{2\left(k_1^2 + k_3^2 + c_2^2(\alpha) - 2k_1k_3 - 2k_1c_2(\alpha) + 2k_3c_2(\alpha) + k_2^2\right)}.$$

Generalized direct model of resulting MF for multiplication of two bell-shape fuzzy sets. The transition from inverse to direct approach allows to transform (66)

$\alpha_L^* = f(c_1(\alpha))$ into $\alpha_L^* = f(x)$ for interval $x \in [0, k_1]$ and (68) $\alpha_R^* = f(c_2(\alpha))$ into $\alpha_R^* = f(x)$ for interval $x \in [k_1, \infty]$.

The analysis of the roots α_L^* and α_R^* allows to make a conclusion that nonlinear dependence $\alpha = f(x)$ for $x \in R^+$ is one-valued function.

The investigations allow to form a direct analytical model (69)–(71) for one-valued nonlinear resulting membership function $\mu_{\underset{\sim}{C}}(x)$ of fuzzy set $\underset{\sim}{C} = \underset{\sim}{A}(\cdot)\underset{\sim}{B}$ that is formed by multiplying bell-shape fuzzy numbers $\tilde{\underset{\sim}{A}}$ та $\underset{\sim}{B}$ in R^+:

(a) $\forall x \in [0, p_1 p_2]$:

$$\mu_{\underset{\sim}{C}}(x) = \\ = \frac{-\left(2k_1k_3 - 2k_3^2 - 2k_3x - k_2^2\right) + \sqrt{\begin{array}{l}(2k_1k_3 - 2k_3^2 - 2k_3x - k_2^2)^2 - 4(k_1^2 + k_3^2 + \\ + x^2 - 2k_1k_3 - 2k_1x + 2k_3x + k_2^2)k_3^2\end{array}}}{2\left(k_1^2 + k_3^2 + x^2 - 2k_1k_3 - 2k_1x + 2k_3x + k_2^2\right)}; \quad (69)$$

(b) for $x = p_1 p_2$:

$$\mu_{\underset{\sim}{C}}(x) = 1; \quad (70)$$

(c) $\forall x \in [p_1 p_2, \infty]$:

$$\mu_{\underset{\sim}{C}}(x) = \\ = \frac{-\left(2k_1k_3 - 2k_3^2 - 2k_3x - k_2^2\right) + \sqrt{\begin{array}{l}(2k_1k_3 - 2k_3^2 - 2k_3x - k_2^2)^2 - 4(k_1^2 + k_3^2 + \\ + x^2 - 2k_1k_3 - 2k_1x + 2k_3x + k_2^2)k_3^2\end{array}}}{2\left(k_1^2 + k_3^2 + x^2 - 2k_1k_3 - 2k_1x + 2k_3x + k_2^2\right)}. \quad (71)$$

5 Inverse and Direct Analytic Models for Multiplication of Two Fuzzy Sets with Triangular Membership Functions in *R*

5.1 Synthesis of Inverse Model of Resulting MF

The task of synthesis of inverse and direct analytical models becomes complicated while performing the operation of multiplication of triangular fuzzy numbers $\underset{\sim}{A}$ (Fig. 1b) and $\underset{\sim}{B}$, that exist in the set of all real numbers R. We shall use the approach discussed above that is based on the analysis of the corresponding square roots for synthesis of inverse and direct models while implementing the operation of multiplication in R.

The algorithm of implementing the multiplication operation for triangular fuzzy numbers [4, 5, 9, 11]

$$\underset{\sim}{A} = (a_1, a_0, a_2)$$

and

$$\underset{\sim}{B} = (b_1, b_0, b_2)$$

while using α-cuts

$$A_\alpha = [a_1(\alpha),\ a_2(\alpha)] = [a_1 + (a_0 - a_1)\alpha,\ a_2 + (a_0 - a_2)\alpha]$$

and

$$B_\alpha = [b_1(\alpha),\ b_2(\alpha)] = [b_1 + (b_0 - b_1)\alpha,\ b_2 + (b_0 - b_2)\alpha]$$

is based on the next inverse model:

$$\begin{aligned} C_\alpha &= [c_1(\alpha),\ c_2(\alpha)] \\ &= \begin{bmatrix} \min\{a_1(\alpha)b_1(\alpha), a_2(\alpha)b_1(\alpha), a_1(\alpha)b_2(\alpha), a_2(\alpha)b_2(\alpha)\}, \\ \max\{a_1(\alpha)b_1(\alpha), a_2(\alpha)b_1(\alpha), a_1(\alpha)b_2(\alpha), a_2(\alpha)b_2(\alpha)\} \end{bmatrix} \\ &= \begin{bmatrix} \min\left\{ \begin{array}{c} [a_1 + (a_0 - a_1)\alpha][b_1 + (b_0 - b_1)\alpha], [a_2 + (a_0 - a_2)\alpha][b_1 + (b_0 - b_1)\alpha], \\ [a_1 + (a_0 - a_1)\alpha][b_2 + (b_0 - b_2)\alpha], [a_2 + (a_0 - a_2)\alpha][b_2 + (b_0 - b_2)\alpha] \end{array} \right\}, \\ \max\left\{ \begin{array}{c} [a_1 + (a_0 - a_1)\alpha][b_1 + (b_0 - b_1)\alpha], [a_2 + (a_0 - a_2)\alpha][b_1 + (b_0 - b_1)\alpha], \\ [a_1 + (a_0 - a_1)\alpha][b_2 + (b_0 - b_2)\alpha], [a_2 + (a_0 - a_2)\alpha][b_2 + (b_0 - b_2)\alpha] \end{array} \right\} \end{bmatrix} \end{aligned} \tag{72}$$

Where $c_1(0) = \min\{a_1b_1, a_2b_1, a_1b_2, a_2b_2\}$;

$c_2(0) = \max\{a_1b_1, a_2b_1, a_1b_2, a_2b_2\}$;

$c_1(1) = c_2(1) = a_0b_0$.

5.2 Synthesis of Direct Model of Resulting MF

Proposition 1. Using inverse model of resulting MF (72), we can form the direct model $\mu_C(x) = \mu_{A(\cdot)B}(x)$ in the following way:

$$\mu_C(x) = \begin{cases} 0, (\forall x < G_1) \bigcup (\forall x > G_2) \\ \alpha^* | \alpha^* \in \{\alpha_i,\}, i = 1 \ldots 8, (\forall x \in [G_1, a_0 b_0)) \bigcup (\forall x \in (a_0 b_0, G_2]) , \\ 1, \forall x = a_0 b_0 \end{cases} \quad (73)$$

Where $G_1 = \min\{a_1 b_1, a_2 b_1, a_1 b_2, a_2 b_2\};$

$$G_2 = \max\{a_1 b_1, a_2 b_1, a_1 b_2, a_2 b_2\}.$$

In the direct model (73) the roots of four square equations are used. These equations are formed while analyzing every of four components of the inverse model (72). In particular:

(a) for component

$$\begin{aligned} [a_1(\alpha) b_1(\alpha)] &= [a_1 + \alpha(a_0 - a_1)][b_1 + \alpha(b_0 - b_1)] \\ &= \alpha^2 (a_0 - a_1)(b_0 - b_1) + \alpha[(a_1(b_0 - b_1) + b_1(a_0 - b_1)] + a_1 b_1 \end{aligned}$$

we form the equation

$$\alpha^2 (a_0 - a_1)(b_0 - b_1) + \alpha[a_1(b_0 - b_1) + b_1(a_0 - a_1)] + (a_1 b_1 - x) = 0,$$

the roots of which are

$$\alpha_{1,2} = \frac{-[a_1(b_0 - b_1) + b_1(a_0 - a_1)]}{2(a_0 - a_1)(b_0 - b_1)} \pm \frac{\sqrt{[a_1(b_0 - b_1) + b_1(a_0 - a_1)]^2 - 4(a_0 - a_1)(b_0 - b_1)(a_1 . b_1 - x)}}{2(a_0 - a_1)(b_0 - b_1)}; \quad (74)$$

(b) for the component

$$[a_2(\alpha) b_1(\alpha)] = [a_2 + (a_0 - a_2)\alpha][b_1 + (b_0 - b_1)\alpha]$$

the roots of the formed equation

$$\alpha^2 (b_0 - b_2)(a_0 - a_1) + \alpha[a_1(b_0 - b_2) + b_2(a_0 - a_1)] + (a_1 b_2 - x) = 0$$

are

$$\alpha_{3,4} = \frac{-[b_2(a_0 - a_1) + a_1(b_0 - b_2)]}{2(b_0 - b_2)(a_0 - a_1)} \pm \frac{\sqrt{[b_2(a_0 - a_1) + a_1(b_0 - b_2)]^2 - 4(b_0 - b_2)(a_0 - a_1)(a_1 b_2 - x)}}{2(b_0 - b_2)(a_0 - a_1)} \quad (75)$$

(c) for the component

$$[a_1(\alpha)b_2(\alpha)] = [a_1 + (a_0 - a_1)\alpha][b_2 + (b_0 - b_2)\alpha]$$

the roots of the formed equation

$$\alpha^2(a_0 - a_2)(b_0 - b_1) + \alpha[a_2(b_0 - b_1) + b_1(a_0 - a_2)] + (a_2b_1 - x) = 0$$

are

$$\alpha_{5,6} = \frac{-[a_2(b_0 - b_1) + b_1(a_0 - a_2)]}{2(a_0 - a_2)(b_0 - b_1)} \pm \frac{\sqrt{[a_2(b_0 - b_1) + b_1(a_0 - a_2)]^2 - 4(a_0 - a_2)(b_0 - b_1)(a_2b_1 - x)}}{2(a_0 - a_2)(b_0 - b_1)}; \tag{76}$$

(d) for component

$$[a_2(\alpha)b_2(\alpha)] = [a_2 + (a_0 - a_2)\alpha][b_2 + (b_0 - b_2)\alpha]$$

the roots of the formed equation

$$\alpha^2(b_0 - b_2)(a_0 - a_1) + \alpha[a_1(b_0 - b_2) + b_2(a_0 - a_1) + (a_1b_2 - x) = 0$$

are

$$\alpha_{7,8} = \frac{-[a_2(b_0 - b_2) + b_2(a_0 - a_2)]}{2(a_0 - a_2)(b_0 - b_2)} \pm \frac{\sqrt{[a_2(b_0 - b_2) + b_2(a_0 - a_2)]^2 - 4(a_0 - a_2)(b_0 - b_2)(a_2b_2 - x)}}{2(a_0 - a_2)(b_0 - b_2)}. \tag{77}$$

The algorithm according to which the value of α^* is selected based on (74)–(77) has the following interpretation.

We define a subset of indices $I_1 \in I$ under the condition

$$I_1 = \{i | i \in I \cap \alpha_i \in [0, 1]\}$$

The subset of indices I_1 belongs to the set of indices

$$I = \{1, 2, 3, 4, 5, 6, 7, 8\}$$

of all roots (74)–(77) defined above: $\alpha_i,\ i = 1\ldots8$.

Proposition 2. The investigations made by the authors show that the presence of some roots that satisfy the condition $\alpha_i \in [0, 1]$ the parameter α^* is defined as follows

$$\alpha^* = \max_{i \in I_1}\{\alpha_i\}. \tag{78}$$

5.3 Modeling Results

Functional dependences of the roots $\alpha_i = f_i(x), i = 1\ldots8$ from the parameter x while implementing the operation of multiplication of triangular fuzzy numbers in $\boldsymbol{R}$:

$$\underset{\sim}{A} = (-3,\ 1,\ 8)$$

and

$$\underset{\sim}{B} = (-2,\ 2,\ 4)$$

are given in Fig. 5, and the chart of the resulting membership function for model (73) – in Fig. 6.

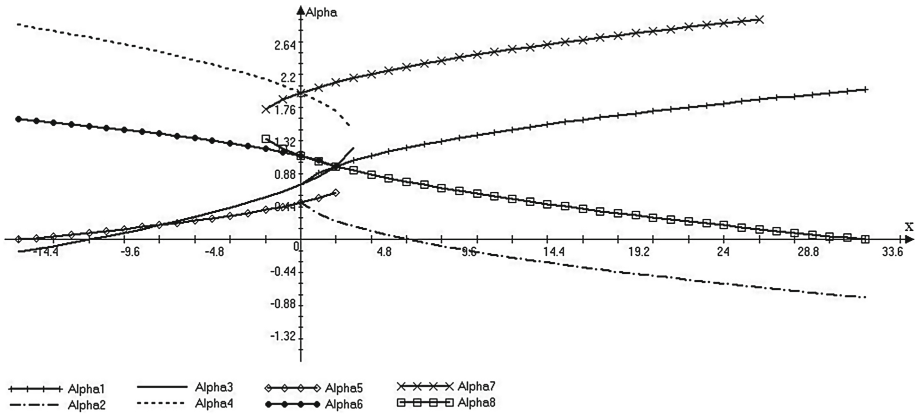

Fig. 5. Functional dependences of $\alpha_i = f_i(x), i = 1\ldots8$

Figure 6 illustrates that during the process of changing x from -16 to 32, the thunking α (Alpha) takes place according to the chain

$$\alpha_5 - \alpha_3 - \alpha_1 - \alpha_8.$$

(points 1, 2, 3 and 4 in Fig. 6).

6 Applied Aspects of Universal Analytic Models Implementation in Decision Support Processes

The multiplication of fuzzy sets is very important operation which is most complicated operation in fuzzy arithmetic and requires a lot of time for calculation processes. The implementation of developed direct analytic models for calculation of resulting membership functions $\mu_{\underset{\sim}{C}}(x)$ according to Table 1 and (70) allows using one step automation mode for operation $\underset{\sim}{C} = \underset{\sim}{A}(\cdot)\underset{\sim}{B}$. In some cases such direct analytic models $\mu_{\underset{\sim}{C}}(x) = \mu_{\underset{\sim}{A}(\cdot)\underset{\sim}{B}}(x)$ may have efficient introducing to evaluation, decision making and decision support processes.

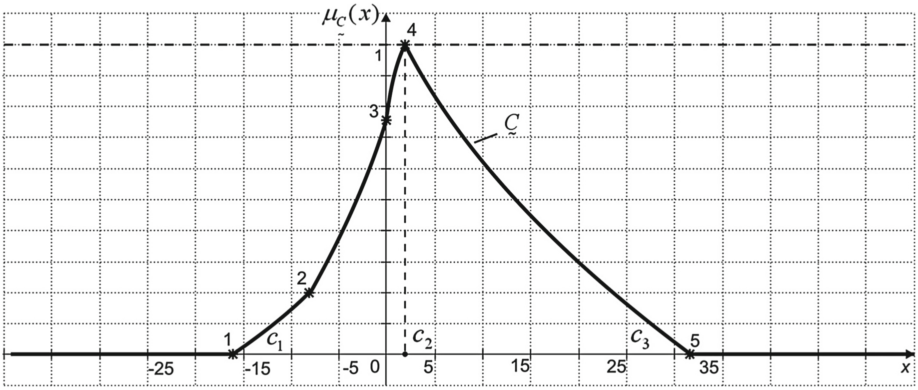

Fig. 6. Resulting fuzzy set $\underset{\sim}{C} = \underset{\sim}{A}(\cdot)\underset{\sim}{B} : \underset{\sim}{A} \subset R, \underset{\sim}{B} \subset R$

Let's consider several examples of real life problems solving [10] for decision support processes, where E is a set of alternatives

$$E = \{E_1, E_2, E_3, \ldots, E_i, \ldots, E_L\}. \tag{79}$$

6.1 Transportation Problem

The transportation problem is well-known operation research problem [7, 8] where it is necessary to find the best alternative solution $E^* \subset E$ for cargo transportation from several depots $N = \{N_1, N_2, \ldots, N_m\}$ to several customers $S = \{S_1, S_2, \ldots, S_n\}$.

The ship transportation problem (STP) can be formulated as a classical transportation problem which can be solved by linear programming methods with some modifications taking into account real conditions of marine environment.

At the general statement STP minimizes of the cost (goal/objective) function $Z(c, x)$ concerning to the transportation of various kind of cargoes (oil, coal, fuel etc.) from several (or one) supplying ports (deports) to various receiving ports (nodes):

$$\text{Min Z}(c, x) = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij,} \tag{80}$$

where: x_{ij} is a quantity of cargo which can be transported from deport i to port $j, i = 1, \ldots, m; j = 1, 2, \ldots, n,;$

c_{ij} is associated with a transported cost of cargo unit (for example, one ship's capacity) for cargo transportation from deport i to port $j, i = 1, \ldots, m; j = 1, 2, \ldots, n.$

It is very important to solve STP from the viewpoint of changeable character of marine conditions as problem in fuzzy environment.

Really, it is well-known that marine environment under consideration may create super-changeable conditions and the degree of fuzziness of such environment has very high level. So, in marine (or sea) environment each coefficient c_{ij} of goal function $Z(c, x)$ is, in general case, uncertain. It depends on the various external factors because each (*ij*)-separated service in STP includes transport penalty (cost) c_{ij}^{T}, unload penalty c_{ij}^{L}, penalties c_{ij}^{S} and c_{ij}^{E}, which represent, respectively, satisfaction of the safety's and ecology's requirements at the *j*-th port (*j*=1,2,…,*n*) during unloading operations and other components, for example, port's sanitary penalty or custom's penalty, which can be included to the one of the abovementioned cost components c_{ij}^{K} $(K = (T, L, S, E))$. In this case goal function (80) can be written as fuzzy function

$$Min\, \underset{\sim}{Z}(c, x) = \sum_{i=1}^{m} \sum_{j=1}^{n} \underset{\sim}{c}_{ij} x_{ij}, \tag{81}$$

where all coefficients $\underset{\sim}{c}_{ij}$ $(i = 1, 2, \ldots m; j = 1, 2, \ldots n)$ are fuzzy sets in the universal set of positive numbers R^{+}.

Let's consider in detail abovementioned disturbed factors for marine environment according to

$$\underset{\sim}{c}_{ij} = \underset{\sim}{c}_{ij}^{T} \oplus \underset{\sim}{c}_{ij}^{L} \oplus \underset{\sim}{c}_{ij}^{S} \oplus \underset{\sim}{c}_{ij}^{E}.$$

The distance between *i*-th deport and *j*-th port defines the value of transport penalty $\underset{\sim}{c}_{ij}^{T}$. It is most uncertain component, which depends on, first of all, weather conditions in served marine region:

(a) when weather is fine the captain of the supplied ship can choose the regular path for his ship from *i*-th deport to *j*-th port and this regular path, as usual (if no obstacles exist), has approximately straightforward character or, in the other words, it should be, desirably, a shortest way;
(b) when the weather has a tendency to change into storm conditions according to current meteorological prognosis the captain can choose another, from the viewpoint of ship's safety, path which as usual is more longer than regular path. Captain should form this new ship's path taking into account the distance between ship and bay (or various bays, which lie in the nearest distances from the ship's path). Sometimes the captain makes a decision to interrupt ship movement from the reason of ship's safety and to stay at the nearest bay for uncertain period which depends on improving of weather conditions.

In some cases the captain changes course of ship according to appearances of some temporary restrictive zones or dynamical obstacles etc. Finally, any type of time delay influences to the increasing of total distance of ship's path and fuel consumption and leads to increasing of penalty component $\underset{\sim}{c}_{ij}^{T}$. The power (force) and directions of wind and sea current have as usual changeable character that also influences to penalty component $\underset{\sim}{c}_{ij}^{T}$, in some cases, by increasing or decreasing of $\underset{\sim}{c}_{ij}^{T}$ according to value $\underset{\sim}{c}_{ij}^{T}$ with grade of membership function $\mu_{\underset{\sim}{c}_{ij}^{T}}\left(\hat{c}_{ij}^{T}\right) = 1$.

The uncertainness of unloading penalty $\underset{\sim}{c}_{ij}^{L}$ depends on quantity of receiving cargo at the j-th port and this parameter in STP is uncertain and "a priori" identified as approximately in terms "about $\hat{c}_{ij}^{L}$ " or "average $\hat{c}_{ij}^{L}$ " for real situations.

Penalty components $\underset{\sim}{c}_{ij}^{S}$ and $\underset{\sim}{c}_{ij}^{E}$ depend on the j-th port rules as well as the weather conditions also. It is often necessary to provide the certain requirements to ensure limits of ship's hill, trim and stress for hull e.g. [7, 8] to install a special type of floating protection around sea's unloaded area for restriction of fuel dissemination at the sear surface if any damages may happen in unloading operations. Sometimes (and it depends on cargo type) penalty $\underset{\sim}{c}_{ij}^{S}$ includes the expenses for fire-ship providing a safety of unloading operations during unloading time.

Analysing all coefficients $\underset{\sim}{c}_{ij}$, we can form matrix $\underset{\sim}{C}$ ($\underset{\sim}{c}_{ij} \neq \underset{\sim}{c}_{ij}, i = 1, 2, \ldots m$; $j = 1, 2, \ldots n$) as asymmetrical fuzzy matrix of penalties $\underset{\sim}{c}_{ij}$ for solving of STP

$$\underset{\sim}{C} = \left\| \underset{\sim}{c}_{ij} \right\|_{(m \times n)} = \begin{vmatrix} \underset{\sim}{c}_{11} & \underset{\sim}{c}_{12} & \ldots & \underset{\sim}{c}_{1n} \\ \underset{\sim}{c}_{21} & \underset{\sim}{c}_{22} & \ldots & \underset{\sim}{c}_{2n} \\ . & . & \ldots & . \\ \underset{\sim}{c}_{m1} & \underset{\sim}{c}_{m2} & \ldots & \underset{\sim}{c}_{mn} \end{vmatrix}. \tag{82}$$

It should be mentioned that in some real situations cargo demands $\sum_{j=1}^{n} \underset{\sim}{x}_{ij}$ of several ports (destinations) are uncertain values (preplanned demand is a such uncertain type as "about VALUE", "Approximately VALUE", "between VALUE_1 and VALUE_2") and respectively the value of total quantity of supplied cargo for each deport is also uncertain.

In such situation the goal function (81) can be transformed to such fuzzy function as

$$Min\ \underset{\sim}{Z}(c, x) = \sum_{i=1}^{m} \sum_{j=1}^{n} \underset{\sim}{c}_{ij} \underset{\sim}{x}_{ij}, \tag{83}$$

where both components $\underset{\sim}{c}_{ij}$, $\underset{\sim}{x}_{ij}$ are corresponding fuzzy sets.

Solving optimization problem (83) with corresponding restrictions for real transportation task it is necessary multi-timely to calculate the results

$$\underset{\sim}{c}_{ij}(\cdot)\underset{\sim}{x}_{ij}$$

of multiplication operation for different pairs of fuzzy sets $\underset{\sim}{c}_{ij}, \underset{\sim}{x}_{ij} (i = 1\ldots m; j = 1\ldots n)$.

In the cases of decision making process when the fuzzy sets $\underset{\sim}{c}_{ij}, \underset{\sim}{x}_{ij}$ are represented as triangular fuzzy numbers it is very efficient to use developed analytic models $\mu_{\underset{\sim}{C}}(x) = \mu_{\underset{\sim}{A}(\cdot)\underset{\sim}{B}}(x)$ in Sects. 2, 4 and 5 of this chapter.

6.2 Decision Making in Agriculture Sector

In some cases it is necessary to find best alternative from (79) based on the prognosis of total profit after agriculture season using multiplication operation

$$\underset{\sim}{C}_i = \underset{\sim}{A}_i(\cdot)\underset{\sim}{B}_i, \tag{84}$$

where $\underset{\sim}{A}_i$ is triangular fuzzy number which corresponds to future output value of i-th agriculture product, $(i = 1\ldots L)$; $\underset{\sim}{B}_i$ is triangular fuzzy number which corresponds to future price value for a unit of i-th agriculture product, $(i = 1\ldots L)$; $\underset{\sim}{C}_i$ is fuzzy number which corresponds to future profit value at the end of agriculture season, in particular, after realization of i-th agriculture product, $(i = 1\ldots L)$.

7 Conclusions

The usage of the developed analytical models (11), (69)–(71), (73), (78) has significant advantage for accuracy of calculations, time of modeling and program implementation of the formed models in comparison with step by step models of multiplication operation of triangular fuzzy numbers based on the algorithms of sorting and max-min convolutions [4, 12]. Suggested approach can be used for different types of MFs and fuzzy models, in particular, for triangular MFs, bell-shape MFs, trapezoidal MFs, etc. Modeling results for multiplication of different fuzzy numbers with triangular and bell-shape membership functions confirm the efficiency of proposed universal analytic models for different applications, in particular, for soft computing based on reconfigurable technology [13], risk analysis in the test diagnosis of the digital components for systems of critical applications [1], and solving real life decision support problems [2, 3, 6, 7, 16], partly presented in Sect. 6 of the chapter.

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