Nash Stability in Fractional Hedonic Games*

Vittorio Bilò[1], Angelo Fanelli[2], Michele Flammini[3,4], Gianpiero Monaco[3], and Luca Moscardelli[5]

[1] Department of Mathematics and Physics - University of Salento, Italy
vittorio.bilo@unisalento.it
[2] CNRS, (UMR-6211), France
angelo.fanelli@unicaen.fr
[3] DISIM - University of L'Aquila, Italy
{michele.flammini,gianpiero.monaco}@univaq.it
[4] Gran Sasso Science Institute, L'Aquila, Italy
[5] Department of Economic Studies - University of Chieti-Pescara, Italy
luca.moscardelli@unich.it

Abstract. Cluster formation games are games in which self-organized groups (or clusters) are created as a result of the strategic interactions of independent and selfish players. We consider fractional hedonic games, that is, cluster formation games in which the happiness of each player in a group is the average value she ascribes to its members. We adopt Nash stable outcomes, where no player can improve her utility by unilaterally changing her own group, as the target solution concept and study their existence, complexity and performance for games played on general and specific graph topologies.

1 Introduction

Hedonic games, introduced in [6], are games in which players have preferences over the set of all possible player partitions (called clusterings). In particular, the utility of each player only depends on the composition or structure of the cluster she belongs to. Cluster formation is of fundamental importance in a variety of social, economic, and political problems. Therefore, a big stream of research considered this topic from a strategic cooperative point of view [5,7,9]. Nevertheless, studying strategic solutions under a non-cooperative scenario becomes important when considering huge environments (like the Internet) lacking a social planner or where the cost of coordination is tremendously high. In this setting, a clustering is Nash stable if no player can improve her utility by unilaterally changing her own cluster. A non-cooperative research on hedonic games can be found in [8].

A notably class of hedonic games is that of *additively separable* ones [2,5], in which the utility of a player is given by the sum of the weights of the edges

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being incident to the other players belonging to the same cluster. Moreover, within this class of games, the *symmetric* case, where the weights are given by an undirected edge-weighted graph in which nodes represent players and edge weights measure the happiness of the players for belonging to the same cluster, has received significant attention [3,5].

In this paper, we consider the class of (symmetric) *fractional hedonic games* recently introduced in [1]. The main difference with respect to additive separable hedonic games is that, in the fractional model, the utility of each player in a cluster is divided by the number of players belonging to it. In such a way, fractional hedonic games model natural behavioral dynamics in social environments that are not captured by additive separable ones: one usually prefers having a couple of good friends in a cluster composed by few other people rather than being part of a crowded cluster populated by uninteresting guys. We analyze this class of games from a non-cooperative perspective, with the aim of understanding the existence, computability and performance of Nash stable clusterings.

We first show that in presence of negative edge weights, Nash stable clusterings are not guaranteed to exist, while, if edge weights are non-negative, the basic outcome in which all players belong to the same cluster (*basic Nash stable clustering*) is Nash stable. Then, we evaluate their performance by means of the widely used notions of price of anarchy and price of stability. We give an upper bound of $O(n)$ on the price of anarchy for weighted graphs and show that it is asymptotically tight even for unweighted paths. We also prove a lower bound of $\Omega(n)$ on the price of stability holding even for weighted stars. We observe that, being the basic Nash stable clustering the responsible for such a bad performance, one may ask whether Nash stable clusterings of better quality may exist and be efficiently computed. To this aim, we show that Nash stable clusterings may not be reached by independent selfish agents unless some kind of centralized control is enforced in the game (that is, uncoordinated best-response dynamics may not converge to stable outcomes), even for unweighted bipartite graphs. This last result, in particular, rises the question of the existence of efficient algorithms for the determination of good quality Nash stable clusterings. To this aim, however, we prove that computing the best quality Nash stable clustering, as well as an optimal (non necessarily stable) one, is an NP-hard problem. Given the above negative and impossibility results, we focus on fractional hedonic games played on particular graph topologies such as unweighted bipartite graphs and unweighted trees which already pose challenging questions and require non-trivial approaches. For bipartite graphs we show that the price of stability is strictly greater than 1 and provide a polynomial time algorithm computing a Nash stable clustering approximating the social optimum by a factor strictly smaller than 2 (thus proving that 2 is an upper bound to the price of stability in this setting). For trees, we prove that the price of stability is 1 and show how to constructively compute in polynomial time an optimal Nash stable clustering.

Due to space limitations, most of the proofs are omitted. All details can be found in the full version of the paper.

2 Definitions, Notation and Preliminaries

For an integer $n > 0$, denote with $[n]$ the set $\{1, \ldots, n\}$. Let $G = (N, E, w)$, with $w : E \to \mathbb{R}_{>0}$ (we consider positive weights because in Lemma 1 we prove that Nash stable clusterings may not exist with negative weights), be an edge weighted connected undirected graph. We denote with $n = |N|$ and with $w_{u,v}$ the weight of edge $(u, v) \in E$. Furthermore, given any set of edges $X \subseteq E$, let $W(X) = \sum_{(u,v)\in X} w_{u,v}$. We say that G is unweighted when $w_{u,v} = 1$ for each $(u, v) \in E$. Given a subset of nodes $S \subseteq N$, $G_S = (S, E_S)$ is the subgraph of G induced by the set S, i.e., $E_S = \{(u, v) \in E : u, v \in S\}$. $N_u(S)$ denotes the neighbors of u in S, i.e., $N_u(S) = \{v \in S : (u, v) \in E\}$, and $E_u(S)$ the edges in E_S being incident to u, i.e., $E_u(S) = \{(u, v) \in E : (u, v) \in E_S\}$.

The *fractional hedonic game* induced by G, denoted as $\mathcal{G}(G)$, is the non-cooperative strategic game in which each node $u \in N$ is associated with a selfish player (or agent) and each player chooses to join a certain *cluster* (assuming that candidate clusters are numbered from 1 to n). Hence, a state of the game, that we will call in the sequel a *clustering*, is a partition of the agents into n clusters $\mathbf{C} = \{C_1, C_2, \ldots, C_n\}$ such that $C_j \subseteq N$ for each $j \in [n]$, $\bigcup_{j\in[n]} C_j = N$ and $C_i \cap C_j = \emptyset$ for any $i, j \in [n]$ with $i \neq j$. Notice that every cluster does not need to be necessarily non-empty. If $u \in C_i$, we say that u is a member of C_i. We denote by $C(u)$ the cluster in $\mathbf{C}$ of which agent u is a member. In a clustering $\mathbf{C}$, the payoff (or utility) of agent u is defined as $p_u(\mathbf{C}) = \frac{W(E_u(C(u)))}{|C(u)|}$. Each agent chooses the cluster she belongs to with the aim of maximizing her payoff. We denote by $(\mathbf{C}, u, j)$, the new clustering obtained from $\mathbf{C}$ by moving agent u from $C(u)$ to C_j; formally, $(\mathbf{C}, u, j) = \mathbf{C} \setminus \{\mathbf{C}(u), C_j\} \cup \{\mathbf{C}(u) \setminus \{u\}, C_j \cup \{u\}\}$. An agent *deviates* if she changes the cluster she belongs to. Given a clustering $\mathbf{C}$, an *improving move* (or simply a *move*) for player u is a deviation to any cluster C_j that strictly increases her payoff, i.e., $p_u((\mathbf{C}, u, j)) > p_u(\mathbf{C})$. Moreover, player u performs a *best-response* in clustering $\mathbf{C}$ by choosing a cluster providing her the highest possible payoff (notice that a best-response is also a move when there exists a cluster C_j such that $p_u((\mathbf{C}, u, j)) > p_u(\mathbf{C})$). An agent is *stable* if she cannot perform a move; a clustering is *Nash stable* (or is a *Nash equilibrium*) if every agent is stable. An *improving dynamics* is a sequence of moves, while a *best-response dynamics* is a sequence of best-responses. A game has the *finite improvement path property* if it does not admit an improvement dynamics of infinite length. Clearly, a game possessing the finite improvement path property always admits a Nash stable clustering. We denote with $\mathsf{NSC}(\mathcal{G}(G))$ the set of Nash stable clusterings of $\mathcal{G}(G)$. The *social welfare* of a clustering $\mathbf{C}$ is the summation of the players' payoffs, i.e., $\mathsf{SW}(\mathbf{C}) = \sum_{u\in N} p_u(\mathbf{C})$. We overload the social welfare function by applying it also to single clusters to obtain their contribution to the social welfare, i.e., $\mathsf{SW}(C_i) = \sum_{u\in C_i} p_u(\mathbf{C})$ so that $\mathsf{SW}(\mathbf{C}) = \sum_{i\in[n]} \mathsf{SW}(C_i)$.

Given a game $\mathcal{G}(G)$, an *optimal* clustering $\mathbf{C}^*$ is one that maximizes the social welfare of $\mathcal{G}(G)$. We denote $\mathsf{SW}(\mathbf{C}^*)$ as $\mathsf{SW}^*(\mathcal{G}(G))$. A clustering $\mathbf{C}$ is *feasible* if G_{C_i} is connected, for every $i \in [n]$. Notice that an optimal configuration is always feasible. The *price of anarchy* of a fractional hedonic game $\mathcal{G}(G)$ is defined as the worst-case ratio between the social welfare of a Nash stable clustering and the social optimum: $\mathsf{PoA}(\mathcal{G}(G)) = \max_{\mathbf{C}\in\mathsf{NSC}(\mathcal{G}(G))} \frac{\mathsf{SW}^*(\mathcal{G}(G))}{\mathsf{SW}(\mathbf{C})}$. Analogously, the *price of stability* of $\mathcal{G}(G)$ is defined as the best-case ratio between the social welfare of a Nash stable clustering and the social optimum: $\mathsf{PoS}(\mathcal{G}(G)) = \min_{\mathbf{C}\in\mathsf{NSC}(\mathcal{G}(G))} \frac{\mathsf{SW}^*(\mathcal{G}(G))}{\mathsf{SW}(\mathbf{C})}$.

Next two lemmas characterize the existence of Nash stable clustering in fractional hedonic games.

Lemma 1. *There exists a graph G containing edges with negative weights such that $\mathcal{G}(G)$ admits no Nash stable clusterings.*

Lemma 2. *For any weighted graph G (with positive weights),* $\mathsf{NSC}(\mathcal{G}(G)) \neq \emptyset$.

3 General Graphs

This section is devoted to results concerning general graph topologies.

Theorem 1. *For any weighted graph G,* $\mathsf{PoA}(\mathcal{G}(G)) \leq n - 1$.

Theorem 2. *For any integer $n \geq 2$, there exists an unweighted path G_n such that* $\mathsf{PoA}(\mathcal{G}(G_n)) = \Omega(n)$.

Theorem 3. *For any integer $n \geq 2$, there exists a weighted star G_n such that* $\mathsf{PoS}(\mathcal{G}(G_n)) = \Omega(n)$.

Theorem 4. *There exists an unweighted bipartite graph G such that $\mathcal{G}(G)$ does not possess the finite improvement path property even under best-response dynamics.*

Theorem 5. *Given a fractional hedonic game, the problem of computing a Nash stable clustering of maximum social welfare is NP-hard, as well as the problem of computing an optimal (not necessarily stable) clustering.*

4 Bipartite Graphs

In this section, we focus on games played on unweighted bipartite graphs.

Theorem 6. *There exists an unweighted bipartite graph G such that* $\mathsf{PoS}(\mathcal{G}(G)) > 1$.

We now show how to constructively compute in polynomial time a Nash stable clustering for such graph topology with good social welfare. As an implication, we obtain an upper bound to the price of stability. Given an unweighted bipartite graph G, let VC be a minimum vertex cover of G, and let $\overline{VC} = N \setminus VC$. It is well known that $\overline{VC}$ is a maximum independent set of G. Moreover, due to the König's theorem, we know that, in a bipartite graph, the number of vertices in a minimum vertex cover equals the number of edges in a maximum matching and the minimum vertex cover can be computed in polynomial time. We will construct a Nash stable clustering that is composed by $|VC|$ non-empty clusters where, for each player $u \in VC$, we have a different cluster C_u that is a star graph having node u as its center. We obtain such a clustering by considering a particular dynamics of the game $\mathcal{G}(G)$. Candidate clusters are numbered from 1 to $|VC|$, that is, one cluster for each node in VC. Let C_u be the cluster associated to player $u \in VC$. We fix the strategy of each player $u \in VC$ to C_u and let only players in $\overline{VC}$ move in the dynamics. The strategy set of a player $u \in \overline{VC}$ is $\{C_v | (u, v) \in E\}$. We consider the dynamics starting from a clustering where each cluster contains at least 2 nodes whose existence is guaranteed by König's theorem, i.e., for each edge (u, v) of the maximum matching associated to VC, without loss of generality we assume $u \in VC$ and $v \in \overline{VC}$, therefore we have a cluster C_u and the starting strategy of $v \in \overline{VC}$ is the cluster C_u. In this section, we refer to such a dynamics as $\mathcal{D}$. The following property holds.

Property 1. At each step of the dynamics $\mathcal{D}$, for each $u \in VC$, we have $|C_u| \geq 2$.

Lemma 3. *The dynamics $\mathcal{D}$ converges after a number of moves which is polynomial in the number of players.*

Next lemma claims that, once the dynamics reaches a stable clustering (which is guaranteed by Lemma 3) henceforth called $\mathbf{C}^{\mathcal{D}}$, the players in VC are also stable and therefore $\mathbf{C}^{\mathcal{D}}$ is Nash stable for the game $\mathcal{G}(G)$.

Lemma 4. $\mathbf{C}^{\mathcal{D}}$ *is Nash stable for* $\mathcal{G}(G)$.

We conclude by proving the approximation guarantee yielded by $\mathbf{C}^{\mathcal{D}}$

Theorem 7. *The Nash stable clustering* $\mathbf{C}^{\mathcal{D}}$ *is such that* $\frac{\mathsf{SW}(\mathbf{C}^*)}{\mathsf{SW}(\mathbf{C}^{\mathcal{D}})} < 2$.

Proof. Let VC be the minimum vertex cover of G used to define the dynamics $\mathcal{D}$. By Property 1, we get that the contribution to the social welfare of any cluster $C^{\mathcal{D}}(u)$, where $u \in VC$, is at least 1, i.e., $\mathsf{SW}(C^{\mathcal{D}}(u)) \geq 1$ for any $u \in VC$. Let C_i^* be a non-empty cluster of an optimal clustering $\mathbf{C}^*$. We partition the nodes of C_i^* in two sets $X_i^{VC} = C_i^* \cap VC$ and $X_i^{\overline{VC}} = C_i^* \cap \overline{VC}$. We distinguish between two cases:

i) $X_i^{VC} = \emptyset$; it follows that $C_i^* \subseteq \overline{VC}$. Therefore, since $\overline{VC}$ is an independent set, it follows that $\mathsf{SW}(C_i^*) = 0$.

ii) $X_i^{VC} \neq \emptyset$; in this case the total number of edges in C_i^* is at most $|X_i^{VC}||X_i^{\overline{VC}}| + \frac{1}{2}|X_i^{VC}|^2$.

Hence, the contribution to the optimal social welfare of cluster C_i^* verifies

$$\mathsf{SW}(C_i^*) \leq 2\frac{|X_i^{VC}||X_i^{\overline{VC}}| + \frac{1}{2}|X_i^{VC}|^2}{|X_i^{VC}| + |X_i^{\overline{VC}}|} = 2|X_i^{VC}|\frac{|X_i^{\overline{VC}}| + \frac{1}{2}|X_i^{VC}|}{|X_i^{VC}| + |X_i^{\overline{VC}}|}. \quad (1)$$

On the other hand, in the Nash stable clustering $\mathbf{C}^{\mathcal{D}}$, for any $u \in X_i^{VC}$, there is a cluster $C^{\mathcal{D}}(u)$ whose contribution to the social welfare is at least one; thus, we get

$$\sum_{u \in X_i^{VC}} \mathsf{SW}(C^{\mathcal{D}}(u)) \geq |X_i^{VC}|. \tag{2}$$

Dividing (1) by (2) we obtain

$$\frac{\mathsf{SW}(C_i^*)}{\sum_{u \in X_i^{VC}} \mathsf{SW}(C^{\mathcal{D}}(u))} \leq 2 \frac{|X_i^{\overline{VC}}| + \frac{1}{2}|X_i^{VC}|}{|X_i^{\overline{VC}}| + |X_i^{VC}|} < 2.$$

By summing over all the non-empty clusters C_i^* of the optimal clustering $\mathbf{C}^*$ the theorem follows. □

5 Trees

In this section, we focus on games played on unweighted trees.

Theorem 8. *For any unweighted tree graph G, $\mathsf{PoS}(\mathcal{G}(G)) = 1$. Moreover, an optimal clustering for $\mathcal{G}(G)$ can be computed in polynomial time.*

6 Conclusions

There are several open problems that still need to be addressed. For instance, some of the provided upper and lower bounds are not tight, so there are some gaps that need to be closed. Among them, the major one is that requiring the determination of significant bounds to the price of stability for general unweighted graphs. Another interesting research direction would be considering directed graphs where the weight of a directed arc (u, v) denotes the value player u has for player v.

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