To Save Or Not To Save: The Fisher Game

Ruta Mehta[1], Nithum Thain[2], László A. Végh[3], and Adrian Vetta[4]

[1] College of Computing, Georgia Institute of Technology
rmehta@cc.gatech.edu
[2] Department of Mathematics and Statistics, McGill University
nithum@gmail.com
[3] London School of Economics
L.Vegh@lse.ac.uk
[4] Department. of Mathematics and Statistics, and School of Computer Science, McGill University
vetta@math.mcgill.ca

Abstract. We examine the Fisher market model when buyers, as well as sellers, have an intrinsic value for money. We show that when the buyers have oligopsonistic power they are highly incentivized to act strategically with their monetary reports, as their potential gains are unbounded. This is in contrast to the bounded gains that have been shown when agents strategically report utilities [5]. Our main focus is upon the consequences for social welfare when the buyers act strategically. To this end, we define the *Price of Imperfect Competition (PoIC)* as the worst case ratio of the welfare at a Nash equilibrium in the induced game compared to the welfare at a Walrasian equilibrium. We prove that the PoIC is at least $\frac{1}{2}$ in markets with CES utilities with parameter $0 \leq \rho \leq 1$ – this includes the classes of Cobb-Douglas and linear utility functions. Furthermore, for linear utility functions, we prove that the PoIC increases as the level of competition in the market increases. Additionally, we prove that a Nash equilibrium exists in the case of Cobb-Douglas utilities. In contrast, we show that Nash equilibria need not exist for linear utilities. However, in that case, good welfare guarantees are still obtained for the best response dynamics of the game.

1 Introduction

General equilibrium is a fundamental concept in economics, tracing back to 1872 with the seminal work of Walras [20]. Traditionally, the focus has been upon *perfect competition*, where the number of buyers and sellers in the market are so huge that the contribution of any individual is infinitesimal. In particular, the participants are *price-takers.*

In practice, however, this assumption is unrealistic. This observation has motivated researchers to study markets where the players have an incentive to act strategically. A prominent example is the seminal work of Shapely and Shubik [17]. They defined *trading post games* for exchange markets and examined whether Nash equilibria there could implement competitive equilibrium prices

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and allocations. Another example, and a prime motivator of our research, is the Cournot-Walras market model introduced by Codognato and Gabszewicz [6] and Gabszewicz and Michel [10], which extends oligopolistic competition into the Arrow-Debreu setting. The importance of this model was demonstrated by Bonniseau and Florig [2] via a connection, in the limit, to traditional general equilibria models under the standard economic technique of agent *replication.* More recently, in the computer science community, Babaioff et al [3] extended Hurwicz's framework [12] to study the welfare of Walrasian markets acting through an auction mechanism.

Our interest is in analyzing the robustness of the pricing mechanism against strategic manipulation. Specifically, our primary goal is to quantify the loss in social welfare due price-making rather than price-taking behaviour. To do this, we define the *Price of Imperfect Competition (PoIC)* as the ratio of the social welfare at the worst Nash equilibrium to the social welfare at the perfectly-competitive Walrasian equilibrium.

Two remarks are pertinent here. First, we are interested in changes in the welfare produced by the market mechanism under the two settings of price-takers and price-makers. We are not interested in comparisons with the optimum social welfare, which requires the mechanism to possess the unrealistic power to perform total welfare redistribution. In particular, we are not concerned here with the *Price of Anarchy* or *Price of Stability.* Interestingly, though, the groundbreaking Price of Anarchy results of Johari and Tzitsiklis [15] on the proportional allocation mechanism for allocating one good (bandwidth) can be seen as the first Price of Imperfect Competition results. This is because in their setting the proportional allocation mechanism will produce optimal allocations in non-stategic settings; in contrast, for our markets, Walrasian equilibrium can be arbitrarily poor in comparison to optimal allocations.

Second, in some markets the Price of Imperfect Competition may actually be larger than one. Thus, strategic manipulations by the agents can lead to improvements in social welfare! Indeed, we will discuss examples where the social welfare increases by an arbitrarily large factor when the agents act strategically.

In this paper, we analyze the Price of Imperfect Competition in Fisher markets with strategic buyers, a special case of the Cournot-Walras model. This scenario models the case of an oligopsonistic market, where the price-making power lies with the buyers rather than the sellers (as in an oligopoly).[1] Adsul et al. [1] study Fisher markets where buyers can lie about their preferences. They gave a complete characterization of its symmetric Nash equilibria (SNE) and showed that market equilibrium prices can be implemented at one of the SNE. Later Chen et. al. [5] studied *incentive ratios* in such markets to show that a buyer can gain no more than twice by strategizing in markets with linear, Leontief and Cobb-Douglas utility functions. In upcoming work, Branzei et al [4] study the Price of Anarchy in the game of Adsul et al. and prove polynomial lower and

[1] The importance of oligopsonies was recently highlighted by the price-fixing behaviour of massive technology companies in San Francisco.

upper bounds for it. Furthermore, they show Nash equilibria exist for linear, Leontieff, and Cobb-Douglas utilities.

In the above games (and the Fisher model itself), only the sellers have an intrinsic utility for money. In contrast, we postulate that buyers (and not just sellers) have utility for money. Thus, buyers may also benefit by saving money for later use. This incentivizes buyers to withhold money from the market. This defines our *Fisher Market Game*, where agents strategize on the amount of money they wish to spend, and obtain utility one from each unit of saved money. Contrary to the bound of two on gains when strategizing on utility functions [5], we observe that strategizing on money may facilitate unbounded gains (see the full paper). These incentives can induce large variations between the allocations produced at a Market equilibrium and at a Nash equilibrium. Despite this, we prove the Price of Imperfect Competition is at least $\frac{1}{2}$ for Fisher markets when the buyers' utility functions belong to the utililty class of Constant Elasticity of Substitution (CES) with the weak gross substitutability property – this class includes linear and Cobb-Douglas functions.

1.1 Overview of Paper

In Section 2, we define the Fisher Game, give an overview of CES utility functions, and present our welfare metrics. In Section 3, we prove that Price of Imperfect Competition is at least $\frac{1}{2}$, for CES utilities which satisfy the weak gross substitutability property. In Section 4, we apply the economic technique of replication to demonstrate that, for linear utilities, the PoIC bound improves as the level of competition in the market increases. In Section 5, we turn our attention to the question of existence of Nash equilibria. We establish that Nash equilibria exist for the subclass of Cobb-Douglas utilities. However, they need not exist for all CES utilities. In particular, Nash equilibria need not exist for linear utilities. To address this possibility of non-existence, in Section 6, we examine the dynamics of the linear Fisher Game and provide logarithmic welfare guarantees.

2 Preliminaries

We now define the Fisher market model and the corresponding game where agents strategize on how much money to spend. We require the following notation. Vectors are shown in bold-face letters, and are considered as column vectors. To denote a row vector we use $\boldsymbol{x}^T$. The i^{th} coordinate of $\boldsymbol{x}$ is denoted by x_i, and $\boldsymbol{x}_{-i}$ denotes the vector $\boldsymbol{x}$ with the i^{th} coordinate removed.

2.1 The Fisher Market

A Fisher market $\mathcal{M}$, introduced by Irving Fisher in his 1891 PhD thesis, consists of a set $\mathcal{B}$ of buyers and and a set $\mathcal{G}$ of divisible goods (owned by sellers). Let $n = |\mathcal{B}|$ and $g = |\mathcal{G}|$. Buyer i brings m_i units of money to the market and wants to buy a bundle of goods that maximizes her utility. Here, a non-decreasing, concave function $U_i : \mathbb{R}_+^g \to \mathbb{R}_+$ measures the utility she obtains from a bundle of goods. Without loss of generality, the aggregate quantity of each good is one.

Given prices $\boldsymbol{p} = (p_1, \ldots, p_g)$, where p_j is price of good j, each buyer demands a utility maximizing (an optimal) bundle that she can afford. The prices $\boldsymbol{p}$ are said to be a *market equilibrium* (ME) if agents can be assigned an optimal bundle such that demand equals supply, *i.e.* the market clears. Formally, let x_{ij} be the amount of good j assigned to buyer i. So $\boldsymbol{x}_i = (x_{i1}, \ldots, x_{ig})$ is her bundle. Then,

1. `Supply = Demand`: $\forall j \in \mathcal{G},\ \sum_i x_{ij} = 1$ whenever $p_j > 0$.
2. `Utility Maximization`: $\boldsymbol{x}_i$ is a solution of $\max U_i(\boldsymbol{z})$ s.t $\sum_j p_j z_{ij} \le m_i$.

We denote by y_{ij} the amount of money player i invests in item j after prices are set. Thus $y_{ij} = p_j x_{ij}$. Equivalently y_{ij} can be thought of as player i's demand for item j in monetary terms.

Utility Functions

An important sub-class of Fisher markets occurs when we restrict utility functions to what are known as *Constant Elasticity of Substitution (CES)* utilities [18]. These functions have the form:

$$U_i(\boldsymbol{x}_i) = (\sum_j u_{ij} x_{ij}^{\rho})^{\frac{1}{\rho}}$$

for some fixed $\rho \le 1$ and some coefficients $u_{ij} \ge 0$. The elasticity of substitution for these markets are $\frac{1}{1-\rho}$. Hence, for $\rho = 1$, *i.e.* linear utilities, the goods are perfect substitutes; for $\rho \to -\infty$, the goods are perfect complements. As $\rho \to 0$, we obtain the well-known Cobb-Douglas utility function:

$$U_i(\boldsymbol{x}_i) = \prod_j x_{ij}^{u_{ij}}$$

where each $u_{ij} \ge 0$ and $\sum_j u_{ij} = 1$. In this paper, we will focus on the cases of $0 < \rho \le 1$ and the case $\rho \to 0$. These particular markets satisfy the property of *weak gross substitutability*, meaning that increasing the price of one good cannot decrease demand for other goods. It is also known that for these particular markets, one can determine the market prices and allocations by solving the Eisenberg-Gale convex program (see [8], [9], [14]):

$$\max\ \big(\sum_i m_i \log U_i(\boldsymbol{x}_i) : \sum_i x_{ij} \le 1, \forall j;\ x_{ij} \ge 0, \forall i, j.\big) \tag{1}$$

2.2 The Fisher Game

An implicit assumption within the Fisher market model is that money has an intrinsic value *to the sellers*, stemming from its potential use outside of the market or at a later date. Thus, money is not just a numéraire. We assume this intrinsic value applies to all market participants including the buyers. This assumption induces a strategic game in which the buyers may have an incentive to save some of their money.

This *Fisher Game* is a special case of the general Cournot-Walras game introduced by Codognato, Gabszewicz, and Michel ([6], [10]). Here the buyers can choose some strategic amount of money $s_i < m_i$ to bring to the market, which will affect their budget constraint. They gain utility both from the resulting market equilibria (with s_i substituted for m_i) and from the money they withhold from the market. Observe, in the Fisher market model, the sellers have no value for the goods in the market. Thus, in the corresponding game, they will place all their goods on sale as their only interest is in money. (Equivalently, we may assume the sellers are non-strategic.)

Thus, we are in an oligopsonistic situation where buyers have indirect price-making power. The set of strategies available to buyer i is $M_i = \{s \geq 0 \mid s \leq m_i\}$. When each buyer decides to spend $s_i \in M_i$, then $\boldsymbol{p}(\boldsymbol{s})$ and $\boldsymbol{x}(\boldsymbol{s})$ are the prices and allocations, respectively, produced by the Fisher market mechanism. These can be determined from the Eisenberg-Gale program (1) by substituting s_i for m_i. Thus, total payoff to buyer i is

$$T_i(\boldsymbol{s}) = U_i(\boldsymbol{x}_i(\boldsymbol{s})) + (m_i - s_i) \tag{2}$$

Our primary tool to analyze the Fisher Game is via the standard solution concept of a Nash equilibrium. A strategy profile $\boldsymbol{s}$ is said to be a *Nash equilibrium* if no player gains by deviating unilaterally. Formally, $\forall i \in \mathcal{B}$, $T_i(\boldsymbol{s}) \geq T_i(s', \boldsymbol{s}_{-i})$, $\forall s' \in M_i$. For the market game defined on market $\mathcal{M}$, let $NE(\mathcal{M})$ denote its set of NE strategy profiles.

The incentives in the Fisher Game can be high. In particular, in the full paper, we show that for any $L \geq 0$, there is a market with linear utility functions where an agent improve his payoff by a multiplicative factor of L by acting strategically.

The Price of Imperfect Competition

The social welfare of a strategy is the aggregate payoff of both buyers and sellers. At a state $\boldsymbol{s}$, with prices $\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{s})$ and allocations $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{s})$, the social welfare is:

$$\mathcal{W}(\boldsymbol{s}) \;=\; \sum_{i\in\mathcal{B}}(U_i(\mathbf{x_i}) + m_i - s_i) + \sum_{j\in\mathcal{G}} p_j \;=\; \sum_{i\in\mathcal{B}} U_i(\mathbf{x_i}) + \sum_{i\in\mathcal{B}} m_i \tag{3}$$

Note, here, that the cumulative payoff of sellers is $\sum_{j\in\mathcal{G}} p_j = \sum_{i\in\mathcal{B}} s_i$.

The focus of this paper is how strategic manipulations of the market mechanism affect the overall social welfare. Thus, we must compare the social welfare of the strategic Nash equilibrium to that of the unstrategic market equilibrium where all buyers simply put all of their money onto the market. This latter equilibrium is the *Walrasian equilibrium (WE)*. This comparison gives rise to a welfare ratio, which we term the *Price of Imperfect Competition (PoIC)*, the ratio of the minimum welfare amongst strategic Nash equilibria in the market game to the welfare of the unstrategic Walrasian equilibrium. Formally, for a given market $\mathcal{M}$,

$$\text{PoIC}(\mathcal{M}) = \min_{\boldsymbol{s} \in NE(\mathcal{M})} \frac{\mathcal{W}(\boldsymbol{s})}{\mathcal{W}(\boldsymbol{m})}$$

Thus the Price of Imperfect Competition is a measure of how robust, with respect to social welfare, the market mechanism is against oligopsonist behaviour. Observe that the Price of Imperfect Competition could be either greater or less than 1. Indeed, in the full paper, we show that a Nash Equilibrium may produce arbitrarily higher welfare than a Walrasian Equilibrium. Of course, one may expect that welfare falls when the mechanism is gamed and we do present an example in the full paper where the welfare at a Nash Equilibrium is slightly lower than at the Walrasian Equilibrium. This leads to the question of whether the welfare at a Nash can be much worse than at a market equilibrium. We will show that the answer is no; a Nash always produces at least a constant factor of the welfare of a market equilibrium.

3 Bounds on the Price of Imperfect Competition

In this section we establish bounds on the PoIC for the Fisher Game for CES utilities with $0 < \rho \leq 1$ and for Cobb-Douglas utilities. The example discussed above shows that there is no upper bound on PoIC for the Fisher Game. Thus, counterintuitively, even for linear utilities, it may be extremely beneficial to society if the players are strategic.

In the rest of this section, we demonstrate a lower bound of $\frac{1}{2}$ on the PoIC.

Consider a market with Cobb-Douglas or CES utility functions (where $0 < \rho \leq 1$). The key to proving the factor $\frac{1}{2}$ lower bound on the PoIC is the following lemma showing the monotonicity of prices.

Lemma 1. *Given two strategic allocations of money* $\mathbf{s}^* \leq \mathbf{s}$*, then the corresponding equilibrium prices satisfy* $\boldsymbol{p}^* \leq \boldsymbol{p}$*, where* $\boldsymbol{p}^* = \boldsymbol{p}(\boldsymbol{s}^*)$ *and* $\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{s})$*.*

Proof. We break the proof up into three classes of utility function.
`(i) Cobb-Douglas Utilities`
The case of Cobb-Douglas utility functions is simple. To see this, recall a result of Eaves [7]. He showed that, when buyer i spends s_i, the prices and allocations for the Fisher market are given by

$$p_j = \sum_i u_{ij} s_i \qquad x_{ij} = \frac{u_{ij} s_i}{\sum_k u_{kj} s_k} \tag{4}$$

It follows that if strategic allocations of money increase, then so must prices.

`(ii) CES Utilities with` $0 < \rho < 1$
Recall that market equilibria for CES Utilities can be calculated via the Eisenberg-Gale convex program (1). From the KKT conditions of this program, where p_j is the dual variable of the budget constraint, we observe that:

$$\begin{array}{ll} \forall j, & p_j > 0 \Rightarrow \sum_i x_{ij} = 1 \\ \forall (i,j), & \frac{s_i u_{ij}}{U_i(x)^\rho x_{ij}^{1-\rho}} \leq p_j \quad \text{and} \quad x_{ij} > 0 \Rightarrow \frac{s_i u_{ij}}{U_i(x)^\rho x_{ij}^{1-\rho}} = p_j \end{array} \tag{5}$$

Claim. If players have CES utilities with $0 < \rho < 1$ and $\boldsymbol{s} \geq 0$, then $x_{ij} > 0,\ \forall (i,j)$ with $u_{ij} > 0$.

Proof. Consider the derivative of U_i with respect to x_{ij} as $x_{ij} \to 0$:

$$\lim_{x_{ij}\to 0} \frac{\partial U_i(\boldsymbol{x}_i)}{\partial x_{ij}} = \lim_{x_{ij}\to 0} \frac{u_{ij} U_i(\boldsymbol{x}_i)^{1-\rho}}{x_{ij}^{1-\rho}} = +\infty \tag{6}$$

The claim follows since $p_j \leq \sum_i s_i$ and is, thus, finite. □

We may now proceed by contradiction. Suppose $\exists k$ s.t. $p_k < p_k^*$. Choose a good j such that $\frac{p_j}{p_j^*}$ is minimal and therefore less than 1, by assumption. Take any player i such that $u_{ij} > 0$. By the above claim, we have $x_{ij}, x_{ij}^* > 0$. Consequently, by the KKT conditions (5), we have:

$$\frac{u_{ij}}{p_j x_{ij}^{1-\rho}} = \frac{U_i(\boldsymbol{x}_i)^\rho}{s_i} \quad \text{and} \quad \frac{u_{ij}}{p_j^* x_{ij}^{*1-\rho}} = \frac{U_i(\boldsymbol{x}_i^*)^\rho}{s_i^*} \tag{7}$$

Taking a ratio gives:

$$\frac{p_j x_{ij}^{1-\rho}}{p_j^* x_{ij}^{*1-\rho}} = \frac{U_i(\boldsymbol{x}_i^*)^\rho s_i}{U_i(\boldsymbol{x}_i)^\rho s_i^*} \tag{8}$$

Indeed, this equation also holds for every good $t \in \mathcal{G}$ with $u_{it} > 0$. Next consider the following two cases:

Case 1: $x_{ij} \leq x_{ij}^*$ **for some player** i.
From (8) we must then have that $U_i(\boldsymbol{x}_i) > U_i(\boldsymbol{x}_i^*)$. However, by the minimality of $\frac{p_j}{p_j^*}$, and since (8) holds for every $t \in \mathcal{G}$ with $u_{it} > 0$, we obtain $x_{it} \leq x_{it}^*$ for all such t. This implies $U_i(\boldsymbol{x}_i) \leq U_i(\boldsymbol{x}_i^*)$, a contradiction.
Case 2: $x_{ij} > x_{ij}^*$ **for every player** i.
Since $p_j^* > p_j$, we must have $p_j^* > 0$. By (5) it follows that $\sum_i x_{ij}^* = 1$. But now we obtain the contradiction that demand must exceed supply as $\sum_i x_{ij} > \sum_i x_{ij}^* = 1$.

`(iii) Linear Utilities`

We begin with some notation. Let $S_i = \{j \in \mathcal{G} : x_{ij} > 0\}$ be the set of goods purchased by buyer i at strategy $\boldsymbol{s}$. Let $\beta_{ij} = \frac{u_{ij}}{p_j}$ be the *rate-of-return* of good j for buyer i at prices $\mathbf{p}$. Let $\beta_i = \max_{j\in\mathcal{G}} \beta_{ij}$ be the *bang-for-buck* buyer i can obtain at prices $\mathbf{p}$. It can be seen from the KKT conditions of the Eisenberg-Gale program (1) that at $\{\boldsymbol{p}, \boldsymbol{x}\}$, every good $j \in S_i$ will have a rate-of-return equal to the bang-for-buck (see, for example, [19]). Similarly, let S_i^*, β_i^* be correspondingly defined for strategy s^*.

Note that, assuming for each good j, $\exists i,\ u_{ij} > 0$, we have that $\boldsymbol{p}, \boldsymbol{p}^* > 0$. Thus, we can partition the goods into groups based on the *price ratios* $\frac{p_j^*}{p_j}$. Suppose there are k distinct price ratios over all the goods (thus $k \leq g$), then partition the goods into k groups, say $\mathcal{G}_1, \ldots, \mathcal{G}_k$ such that all the goods in a group have the same ratio. Let the ratio in group j be λ_j and let $\lambda_1 < \lambda_2 < \cdots < \lambda_k$. Thus $\mathcal{G}_1$ are the goods whose prices have fallen the most (risen the least) and $\mathcal{G}_k$ are the goods whose prices have fallen the least (risen the most).

Let $\mathcal{I}_k = \{i : \exists j \in \mathcal{G}_k, x_{ij} > 0\}$ and $\mathcal{I}_k^* = \{i : \exists j \in \mathcal{G}_k, x_{ij}^* > 0\}$. Thus $\mathcal{I}_k$ and $\mathcal{I}_k^*$ are the collections of buyers that purchase goods in $\mathcal{G}_k$ in each of the allocations. Take any buyer $i \in \mathcal{I}_k^*$; so there is some good $j \in S_i^* \cap \mathcal{G}_k$.

If $S_i \cap \bigcup_{\ell=1}^{k-1} \mathcal{G}_\ell \neq \emptyset$ then buyer i would not desire good j at prices p_j^*. To see this, take a good $j' \in S_i \cap \bigcup_{\ell=1}^{k-1} \mathcal{G}_\ell$. Then $\beta_{ij'} = \beta_i \geq \beta_{ij}$. Therefore

$$\begin{aligned} \beta_i^* \geq \frac{u_{ij'}}{p_{j'}^*} \quad &\geq \quad \frac{u_{ij'}}{\lambda_{k-1} \cdot p_{j'}} \quad > \quad \frac{u_{ij'}}{\lambda_k \cdot p_{j'}} \\ = \frac{1}{\lambda_k} \cdot \frac{u_{ij'}}{p_{j'}} \quad &\geq \quad \frac{1}{\lambda_k} \cdot \frac{u_{ij}}{p_j} \\ = \frac{u_{ij}}{p_j^*} \quad &= \quad \beta_i^* \end{aligned}$$

This contradiction tells us that $S_i \subseteq \mathcal{G}_k$ and $\mathcal{I}_k^* \subseteq \mathcal{I}_k$. It follows that $\cup_{i \in \mathcal{I}_k^*} S_i \subseteq \mathcal{G}_k$. Putting this together, we obtain that

$$\sum_{i \in \mathcal{I}_k^*} s_i \;\leq\; \sum_{i \in \mathcal{I}_k} s_i \;\leq\; \sum_{j \in \mathcal{G}_k} p_j \tag{9}$$

Now recall that all goods must be sold by the market mechanism (as $\boldsymbol{p}, \boldsymbol{p}^* > 0$). Thus the buyers $\mathcal{I}_k^*$ must be able to afford all of the goods in $\mathcal{G}_k$. Thus

$$\sum_{i \in \mathcal{I}_k^*} s_i^* \;\geq\; \sum_{j \in \mathcal{G}_k} p_j^* \;=\; \lambda_k \cdot \sum_{j \in \mathcal{G}_k} p_j \tag{10}$$

But $s_i^* \leq s_i$ for all i. Consequently, Inequalities (9) and (10) imply that $\lambda_k \leq 1$. Thus no price in $\boldsymbol{p}^*$ can be higher than in $\boldsymbol{p}$. □

First we use Lemma 1 to provide lower bounds on the individual payoffs.

Lemma 2. *Let s_i be a best response for agent i against the strategies $\boldsymbol{s}_{-i}$. Then $T_i(\boldsymbol{s}) \geq \max(\hat{U}_i, m_i)$, where $\hat{U}_i$ is her utility at the Walrasian equilibrium.*

Proof. Clearly $T_i(\boldsymbol{s}) \geq m_i$, otherwise player i could save all her money and achieve a payoff of m_i. For $T_i(\boldsymbol{s}) \geq \hat{U}_i$, let $\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{m})$ and $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{m})$ be the prices and allocation at Walrasian equilibrium. If buyer i decides to spend all his money when the others play $\boldsymbol{s}_{-i}$, the resulting equilibrium prices will be less than $\boldsymbol{p}$, by Lemma 1. Therefore, she can afford to buy bundle $\boldsymbol{x}_i$. Thus, her best response payoff must be at least $\hat{U}_i$.

It is now easy to show the lower bound on the Price of Imperfect Competition.

Theorem 1. *In the Fisher Game with Cobb-Douglas or CES utilities with $0 < \rho \leq 1$, we have PoIC $\geq \frac{1}{2}$. That is, $\mathcal{W}(\boldsymbol{s}^*) \geq \frac{1}{2}\mathcal{W}(\boldsymbol{m})$, for any Nash equilibrium $\boldsymbol{s}^*$.*

Proof. Let $\boldsymbol{p}^* = \boldsymbol{p}(\boldsymbol{s}^*)$ and $\boldsymbol{x}^* = \boldsymbol{x}(\boldsymbol{s}^*)$. Let $\boldsymbol{p}$ and $\boldsymbol{x}$ be the Walrasian equilibrium prices and allocations, respectively. At the Nash equilibrium $\boldsymbol{s}^*$ we have $T_i(\boldsymbol{s}^*) \geq \max(m_i, U_i(\boldsymbol{x}_i))$ for each player i, by Lemma 2. Thus, we obtain:

$$2\sum_i T_i(\boldsymbol{s}^*) \geq \sum_i U_i(\boldsymbol{x}_i) + \sum_i m_i \tag{11}$$

Therefore $\mathcal{W}(\boldsymbol{s}^*) \geq \frac{1}{2}\mathcal{W}(\boldsymbol{m})$, as desired. □

4 Social Welfare and the Degree of Competition

In this section, we examine how the welfare guarantee improves with the degree of competition in the market. To model the degree of competition, we apply a common technique in the economics literature, namely *replication* [17]. In a replica economy, we take each buyer type in the market and make N duplicates (the budgets of each duplicate is a factor N smaller than that of the original buyer). The *degree of competition* in the resultant market is N. We now consider the Fisher Game with linear utility functions and show how the lower bound on Price of Imperfect Competition improves with N.

Theorem 2. *Let $\boldsymbol{s}^*$ be a NE in a market with degree of competition N. Then*

$$\mathcal{W}(\boldsymbol{s}^*) \geq (1 - \frac{1}{N+1}) \cdot \mathcal{W}(\boldsymbol{m})$$

In order to prove Theorem 2, we need a better understanding of how prices adjust to changes in strategy under different degrees of competition. Towards this goal, we need the following two lemmas.

Lemma 3. *Given an arbitrary strategic money allocation $\boldsymbol{s}$. If player i increases (resp. decreases) her spending from s_i to $(1+\delta)s_i$ then the price of any good increases (resp. decreases) by at most a factor of $(1+\delta)$.*

Proof. We focus on the case of increase; the argument for the decrease case is analogous. Suppose all players increase their strategic allocation by a factor of $(1+\delta)$. Then the allocations to all players would remain the same by the market mechanism and all prices would be scaled up by a factor of $(1+\delta)$. Then suppose each player $k \neq i$ subsequently lowers its money allocation back down to the original amount s_k. By Lemma 1, no price can now increase. The result follows. □

Lemma 4. *Given an arbitrary strategic money allocation $\boldsymbol{s}$ in a market with degree of competition N. Let buyer i be the duplicate player of her type with the smallest money allocation s_i. If she increases her spending to $(1+N\cdot\delta)s_i$ then the price of any good increases by at most a factor $(1+\delta)$.*

Proof. We utilize the symmetry between the N identical players. Let players $i_1 = i, i_2, ..., i_N$ be the replicas identical to player i. If each of these players increased their spending by a factor of $(1+\delta)$ then, by Lemma 3, prices would go up by at most a factor $(1+\delta)$. From the market mechanism's perspective, this is equivalent to player i increasing her strategic allocation to $s_i + \delta \cdot \sum_k s_{i_k}$. But this is greater than $(1 + N \cdot \delta)s_i$. Thus, by Lemma 1, the new prices are larger by a factor of at most $(1+\delta)$. □

Now let $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{m})$ and $\boldsymbol{x}^* = \boldsymbol{x}(\boldsymbol{s}^*)$. Since we have rational inputs, $\mathbf{x}$ and $\mathbf{x}^*$ must be rational [14]. Therefore, by appropriately duplicating the goods and scaling the utility coefficients, we may assume that there is exactly one unit of each good and that both $\mathbf{x}$ and $\mathbf{x}^*$ are $\{0,1\}$-allocations. Recall from the proof of Lemma 1 our definition of S_i, S_i^* and β_i, β_i^*. Under this assumption, $S_i = \{j \in \mathcal{G} : x_{ij} = 1\}$ and similarly for S_i^*. We are now ready to prove the following welfare lemma.

Lemma 5. *For any Nash equilibrium* $\{\mathbf{s}^*, \mathbf{p}^*, \mathbf{x}^*\}$ *and any Walrasian equilibrium* $\{\mathbf{s} = \mathbf{m}, \mathbf{p}, \mathbf{x}\}$*, we have*

$$\sum_{i \in \mathcal{B}} \sum_{j \in S_i^*} u_{ij} \geq \left(1 - \frac{1}{N}\right) \cdot \sum_{i \in \mathcal{B}} \sum_{j \in S_i} u_{ij} \tag{12}$$

Proof. To prove the lemma we show that total utility produced by goods at NE, after scaling by a factor $\frac{N}{N-1}$, is at least as much as the utility they produce at the Walrasian equilibrium. We do this by partitioning goods into the sets S_i. We then notice that for each good, the player who receives it at NE must receive utility from it in excess of the price he paid for it. In many cases, this price is more than the utility of the player who receives it in Walrasian equilibrium and we are done. Otherwise we will set up a transfer system where players in NE who receive more utility for the good than the price paid for it transfer some of this excess utility to players who need it. This will ultimately allow us to reach the desired inequality.

For the rest of this proof wlog we will restrict our attention to Nash equilibria where each identical copy of a certain type of player has the same strategy. We are able to do this as the market could treat the sum of these copies as a single player and thus we are able to manipulate the allocations between these players without changing market prices or the total utility derived from market allocations. Thus if our argument holds for Nash equilibria where identical players have the same strategy, it will also hold for heterogeneous Nash equilibria. Now take any player i. There are two cases:

Case 1: $s_i^* = m_i$.

By Lemma 1, we know that

$$\sum_{j \in S_i^* \cap S_i} p_j^* \leq \sum_{j \in S_i^* \cap S_i} p_j \tag{13}$$

Therefore, by the assumption that $s_i^* = m_i$, we have

$$\sum_{j \in S_i \setminus S_i^*} p_j = m_i - \sum_{j \in S_i^* \cap S_i} p_j = s_i^* - \sum_{j \in S_i^* \cap S_i} p_j \leq s_i^* - \sum_{j \in S_i^* \cap S_i} p_j^* = \sum_{j \in S_i^* \setminus S_i} p_j^* \tag{14}$$

Thus buyer i spends more on $S_i^* \setminus S_i$ than she did on $S_i \setminus S_i^*$. But, by Lemma 1, she also receives a better bang-for-buck on $S_i^* \setminus S_i$ than on $S_i \setminus S_i^*$, as $\beta_i^* \geq \beta_i$ (Lemma 1). Let $\beta_i^* = 1 + \epsilon_i^*$. Thus, at the Nash equilibrium, her total utility on $S_i^* \setminus S_i$ is

$$\sum_{j \in S_i^* \setminus S_i} u_{ij} \;=\; \sum_{j \in S_i^* \setminus S_i} \beta_i^* \cdot p_j^* \;=\; (1 + \epsilon_i^*) \cdot \sum_{j \in S_i^* \setminus S_i} p_j^*$$

Of this utility, buyer i will allocate p_j^* units of utility to each item $j \in S_i^* \setminus S_i$. The remaining $\epsilon_i^* \cdot p_j^*$ units of utility derived from good j is reallocated to goods in $S_i \setminus S_i^*$.

Consider the goods in S_i. Clearly goods in $S_i \cap S_i^*$ contribute the same utility to both the Walrasian equilibrium and the Nash equilibrium. So take the items in $S_i \setminus S_i^*$. The buyers of these items at NE have obtained at least $\sum_{j \in S_i \setminus S_i^*} p_j^*$ units of utility from them (as $\beta_d^* \geq 1,\ \forall d$). In addition, buyer i has reallocated $\epsilon_i^* \cdot \sum_{j \in S_i^* \setminus S_i} p_j^*$ to goods in $S_i \setminus S_i^*$. So the total utility allocated to goods in $S_i \setminus S_i^*$ is

$$\sum_{j \in S_i \setminus S_i^*} p_j^* + \epsilon_i^* \cdot \sum_{j \in S_i^* \setminus S_i} p_j^* \geq \sum_{j \in S_i \setminus S_i^*} p_j^* + \epsilon_i^* \cdot \sum_{j \in S_i \setminus S_i^*} p_j^* \;=\; (1 + \epsilon_i^*) \cdot \sum_{j \in S_i \setminus S_i^*} p_j^*$$
$$= \beta_i^* \cdot \sum_{j \in S_i \setminus S_i^*} p_j^* \;\geq\; \sum_{j \in S_i \setminus S_i^*} u_{ij}$$

Here the first inequality follows by (14) and the final inequality follows as $\beta_i^* \geq \frac{u_{ij}}{p_j^*}$, for any good $j \notin S_i^*$. Thus the reallocated utility on S_i at NE is greater than the utility it provides in the Walrasian equilibrium (even without scaling by $\frac{N}{N-1}$).

Case 2: $s_i^* < m_i$.

Suppose buyer i increases her spending from s_i^* to $(1 + N \cdot \delta) \cdot s_i^*$. Then the prices of the goods she buys increase by at most a factor $(1 + \delta)$ by Lemma 4. Thus her utility changes by

$$(m_i - (1 + \delta \cdot N) \cdot s_i^*) + s_i^* \cdot \beta_i^* \cdot \frac{1 + N \cdot \delta}{1 + \delta} - (m_i - s_i^*) - s_i^* \cdot \beta_i^* \;\leq\; 0$$

where the inequality follows as s^* is a Nash equilibrium. This simplifies to

$$s_i^* \cdot \left(-\delta \cdot N + \beta_i^* \cdot \left(\frac{1 + N \cdot \delta}{1 + \delta} - 1\right) \right) \leq 0$$

Now suppose (i) $s_i^* = 0$. In this case we must have $u_{ij}/p_j^* \leq 1$ for every good j. To see this, we argue by contradiction. Suppose $u_{ij}/p_j^* = 1 + \epsilon$ for some good

j. Notice that if player i changes s_i^* to γ the price of good j can go up by at most γ as we know each price increases by Lemma 1 and the sum of all prices is at most γ higher (by the market conditions). Thus, if player i puts $\gamma < \epsilon$ money onto the market then good j will still have bang-for-buck greater than 1 and so player i will gain more utility than the loss of savings. Thus, s_i^* cannot be an equilibrium, a contradiction.

Thus $u_{ij} \leq p_j^* \leq u_{i^*j}$ where i^* is the player who receives good j at NE. Therefore this player obtains more utility from good j than player i did in the Walrasian equilibrium, even without scaling or a utility transfer.

On the other hand, suppose (ii) $s_i^* > 0$. This can only occur if we have both $\beta_i^* \geq 1$ and

$$\beta_i^* \cdot \frac{(N-1)\cdot\delta}{1+\delta} \leq \delta \cdot N \tag{15}$$

Therefore $1 \leq \beta_i^* \leq (1+\delta)\cdot(1+\frac{1}{N-1})$. Since this holds for all δ, as we take $\delta \to 0$ we must have $\beta_i^* \leq \frac{N}{N-1}$. Thus $\frac{u_{ij}}{p_j^*} \leq \frac{N}{N-1}$ for every good j. Thus if we multiply the utility of the player receiving good j in the Nash equilibrium by $\frac{N}{N-1}$ he will be getting more utility from it than player i did in the Walrasian equilibrium. □

Proof of Theorem 2. Given the other buyers strategies $\boldsymbol{s}^*_{-\mathbf{i}}$ suppose buyer i sets $s_i = m_i$. Then, by Lemma 1, prices cannot be higher for $(m_i, \boldsymbol{s}^*_{-\mathbf{i}})$ than at the Walrasian equilibrium $\boldsymbol{p}(\boldsymbol{m})$. Therefore, by selecting $s_i = m_i$, buyer i could afford to buy the entire bundle S_i at the resultant prices. Consequently, her best response strategy $\boldsymbol{s}_i^*$ must offer at least that much utility. This is true for each buyer, so we have

$$\sum_{i\in\mathcal{B}}\left((m_i - s_i^*) + \sum_{j\in\mathcal{G}} u_{ij}\cdot x_{ij}^*\right) \geq \sum_{i\in\mathcal{B}}\sum_{j\in\mathcal{G}} u_{ij}\cdot x_{ij} \tag{16}$$

Thus

$$\begin{aligned}\mathcal{W}(\boldsymbol{s}^*) = \sum_{i\in\mathcal{B}}\sum_{j\in\mathcal{G}} u_{ij}\cdot x_{ij}^* + \sum_{i\in\mathcal{B}} m_i \ &= \ \sum_{i\in\mathcal{B}}\left((m_i - s_i^*) + \sum_{j\in\mathcal{G}} u_{ij}\cdot x_{ij}^*\right) + \sum_{i\in\mathcal{B}} s_i^* \\ \geq \sum_{i\in\mathcal{B}}\sum_{j\in\mathcal{G}} u_{ij}\cdot x_{ij} + \sum_{i\in\mathcal{B}} s_i^* & \end{aligned} \tag{17}$$

On the other hand, Lemma 5 implies that

$$\mathcal{W}(\boldsymbol{s}^*) \ = \ \sum_{i\in\mathcal{B}}\sum_{j\in\mathcal{G}} u_{ij}\cdot x_{ij}^* + \sum_{i\in\mathcal{B}} m_i \ \geq \ \left(1 - \frac{1}{N}\right)\cdot\sum_{i\in\mathcal{B}}\sum_{j\in\mathcal{G}} u_{ij}\cdot x_{ij} + \sum_{i\in\mathcal{B}} m_i \tag{18}$$

Taking a convex combination of Inequalities (17) and (18) gives

$$\mathcal{W}(\boldsymbol{s}^*) \geq \left(\alpha\cdot(1-\frac{1}{N}) + (1-\alpha)\right)\cdot\sum_{i\in\mathcal{B}}\sum_{j\in\mathcal{G}} u_{ij}\cdot x_{ij} + \alpha\cdot\sum_{i\in\mathcal{B}} m_i + (1-\alpha)\cdot\sum_{i\in\mathcal{B}} s_i^*$$

$$\geq \left(\alpha \cdot (1 - \frac{1}{N}) + (1-\alpha)\right) \cdot \sum_{i \in \mathcal{B}} \sum_{j \in \mathcal{G}} u_{ij} \cdot x_{ij} + \alpha \cdot \sum_{i \in \mathcal{B}} m_i$$

$$= \left(1 - \frac{\alpha}{N}\right) \cdot \sum_{i \in \mathcal{B}} \sum_{j \in \mathcal{G}} u_{ij} \cdot x_{ij} + \alpha \cdot \sum_{i \in \mathcal{B}} m_i \quad (19)$$

Thus plugging $\alpha = \frac{N}{N+1}$ in (19) gives

$$\mathcal{W}(\boldsymbol{s}^*) \geq \left(1 - \frac{1}{N+1}\right) \cdot \left(\sum_{i \in \mathcal{B}} \sum_{j \in \mathcal{G}} u_{ij} \cdot x_{ij} + \sum_{i \in \mathcal{B}} m_i\right) = \left(1 - \frac{1}{N+1}\right) \cdot \mathcal{W}(\boldsymbol{m}) \quad (20)$$

This completes the proof. □

5 Existence of Nash Equilibria

We have demonstrated bounds for the Price of Imperfect Competition in the Fisher Game under both CES and Cobb-Douglas utilities. However, these welfare results only apply to strategies that are Nash equilibria. In the full paper, we prove that Nash equilibria exist for the Cobb-Douglas case, but need not exist for linear utilities. For games without Nash equilibria, we may still recover some welfare guarantees; we discuss this in Section 6, by examining the dynamics of the Fisher Game with linear utilities.

6 Social Welfare under Best Response Dynamics

Whilst Nash equilibria need not exist in the Fisher Game with linear utilities, we can still obtain a good welfare guarantee in the dynamic setting. Specifically, in the dynamic setting we assume that in every round (time period), each player simultaneously plays a best response to what they observed in the previous round. Dynamics are a natural way to view how a game is played and a well-studied question is whether or not the game dynamics converge to an equilibrium. Regardless of the answer, it is possible to quantify the average social welfare over time of the dynamic process. This method was introduced by Goemans et al in [11] and we show how it can be applied here to bound the *Dynamic Price of Imperfect Competition* - the worst case ratio of the average welfare of states in the dynamic process to the welfare of the Walrasian equilibrium.

For best responses to be well defined in the dynamic Fisher Game, we need the concept of a minimum monetary allocation s_i. Thus we discretize the game by allowing players to submit strategies which are rational numbers of precision up to Φ. This has the added benefit of making the game finite. In the full paper, we prove the following bound on the Dynamic Price of Imperfect Competition.

Theorem 3. *In the dynamic Fisher Game with linear utilities, the Dynamic Price of Imperfect Competition is lower bounded by $\Omega(1/\log(\frac{M}{\phi}))$ where $M = \max_i m_i$.*

References

1. Adsul, B., Babu, C.S., Garg, J., Mehta, R., Sohoni, M.: Nash equilibria in Fisher market. In: Kontogiannis, S., Koutsoupias, E., Spirakis, P.G. (eds.) SAGT 2010. LNCS, vol. 6386, pp. 30–41. Springer, Heidelberg (2010)
2. Bonnisseau, J., Florig, M.: Existence and optimality of oligopoly equilibria in linear exchange economies. Economic Theory 22(4), 727–741 (2002)
3. Babaioff, M., Lucier, B., Nisan, N., Paes Leme, R.: On the efficiency of the Walrasian mechanism. In: EC 2014, pp. 783–800 (2014)
4. Branzei, S., Chen, Y., Deng, X., Filos-Ratsikas, A., Frederiksen, S., Zhang, J.: The Fisher market game: Equilibrium and welfare. To appear in AAAI (2014)
5. Chen, N., Deng, X., Zhang, H., Zhang, J.: Incentive ratios of Fisher markets. In: Czumaj, A., Mehlhorn, K., Pitts, A., Wattenhofer, R. (eds.) ICALP 2012, Part II. LNCS, vol. 7392, pp. 464–475. Springer, Heidelberg (2012)
6. Codognato, G., Gabszewicz, J.: Equilibre de Cournot-Walras dans une économie d'échange. Revue Économique 42(6), 1013–1026 (1991)
7. Eaves, B.: Finite solution of pure trade markets with Cobb-Douglas utilities. In: Economic Equilibrium: Model Formulation and Solution, pp. 226–239 (1985)
8. Eisenberg, E., Gale, D.: Consensus of subjective probabilities: The pari-mutuel method. The Annals of Mathematical Statistics 30(1), 165–168 (1959)
9. Eisenberg, E.: Aggregation of utility functions. Management Sciences 7(4), 337–350 (1961)
10. Gabszewicz, J., Michel, P.: Oligopoly equilibria in exchange economies. In: Trade, Technology and Economics. Essays in Honour of Richard G. Lipsey, pp. 217–240 (1997)
11. Goemans, M., Mirrokni, V., Vetta, A.: Sink equilibria and convergence. In: Foundations of Computer Science (2005)
12. Hurwicz, L.: On informationally decentralized systems. In: Decision and Organization: A Volume in Honor of Jacob Marschak. Studies in Mathematical and Managerial Economics, vol. 12, pp. 297–336 (1972)
13. Jain, K.: A polynomial time algorithm for computing an Arrow-Debreu equilibrium for linear utilities. SIAM Journal on Computing 37(1), 291–300 (2007)
14. Jain, K., Vazirani, V.: Eisenberg–Gale markets: Algorithms and game-theoretic properties. Games and Economic Behavior 70(1), 84–106 (2010)
15. Johari, R., Tsitsiklis, J.: Efficiency loss in network resource allocation game. Mathematics of Operations Research 57(4), 823–839 (2004)
16. Rosen, J.B.: Existence and Uniqueness of Equilibrium Points for Concave N-person Games. Econometrica 33(3), 520–534 (1965)
17. Shapley, L., Shubik, M.: Trade using one commodity as a means of payment. The Journal of Political Economy, 937–968 (1977)
18. Solov, R.: A contribution to the theory of economic growth. Quarterly Journal of Economics 70, 65–94 (1956)
19. Vazirani, V.: Combinatorial Algorithms for Market Equilibria. In: Algorithmic Game Theory, pp. 103–133 (2007)
20. Walras, L.: Principe d'une théorie mathématique de l'échange (1874)