Network Cournot Competition*,**

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Abstract. Cournot competition, introduced in 1838 by Antoine Augustin Cournot, is a fundamental economic model that represents firms competing in a single market of a homogeneous good. Each firm tries to maximize its utility—naturally a function of the production cost as well as market price of the product—by deciding on the amount of production. This problem has been studied comprehensively in Economics and Game Theory; however, in today's dynamic and diverse economy, many firms often compete in more than one market simultaneously, i.e., each market might be shared among a subset of these firms. In this situation, a bipartite graph models the access restriction where firms are on one side, markets are on the other side, and edges demonstrate whether a firm has access to a market or not. We call this game *Network Cournot Competition* (NCC). Computation of equilibrium, taking into account a network of markets and firms and the different forms of cost and price functions, makes challenging and interesting new problems.

In this paper, we propose algorithms for finding pure Nash equilibria of NCC games in different situations. First, we carefully design a potential function for NCC, when the price function for each market is a linear function of it total production. This result lets us leverage optimization techniques for a single function rather than multiple utility functions of many firms. However, for nonlinear price functions, this approach is not feasible—there is indeed no single potential function that captures the utilities of all firms for the case of nonlinear price functions. We model the problem as a nonlinear complementarity problem in this case, and design a polynomial-time algorithm that finds an equilibrium of the game for strongly convex cost functions and strongly monotone revenue functions. We also explore the class of price functions that ensures strong monotonicity of the revenue function, and show it consists of a broad class of functions. Moreover, we discuss the uniqueness of equilibria in both these cases: our algorithms find the unique equilibria of the games. Last but not least, when the cost of production in one market is independent from the cost of production in other markets for all firms, the problem can be separated into several independent classical *Cournot Oligopoly* problems in which the firms compete over a single market. We give the first combinatorial algorithm for this widely studied problem. Interestingly, our algorithm is much simpler and faster than previous optimization-based approaches.

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1 Introduction

In the crude oil market the equilibrium price is set by the interplay of supply and demand. Since there are several ways for transporting crude oil from an oil-producing country to an oil-importing country, the market for crude oil seems to be an oligopoly with almost a single worldwide price[1]. In particular, the major portion of the market share belongs to the members of the Organization of Petroleum Exporting Countries (OPEC). The worldwide price for crude oil is mainly influenced by OPEC, and the few fluctuations in regional prices are negligible.

The power of an oil-producing country, in market for crude oil, mainly depends on its resources and its cost of production rather than its position in the network. [2]

The market for natural gas behaves differently from that for crude oil and matches our study well. Unlike the crude oil market with a world-wide price, the natural gas market is segmented and regional [40, 35]. Nowadays, pipelines are the most efficient way for transporting natural gas from one region to another. This fragments the market into different regional markets with their own prices. Therefore, the market for natural gas can be modeled by a network where the power of each country highly depends on its position in the network. For example, an importing country with access to only one exporting country suffers a monopolistic price, while an importing country having access to multiple suppliers enjoys a lower price as a result of the price competition. As an evidence, EU Commission Staff Working Document (2006) reports different prices for natural gas in different markets, varying from almost 0 to €300 per thousand cubic meters [15].

In this paper we study selling a utility with a distribution network—e.g., natural gas, water and electricity—in several markets when the clearing price of each market is determined by its supply and demand. The distribution network fragments the market into different regional markets with their own prices. Therefore, the relations between suppliers and submarkets form a complex network [11, 39, 10, 18, 15]. For example, a market with access to only one supplier suffers a monopolistic price, while a market having access to multiple suppliers enjoys a lower price as a result of the price competition.

Antoine Augustin Cournot introduced the first model for studying the duopoly competition in 1838. He proposed a model where two individuals own different springs of water, and sell it independently. Each individual decides on the amount of water to supply, and then the aggregate water supply determines the market price through an inverse demand function. Cournot characterizes the unique equilibrium outcome of the market when both suppliers have the same marginal costs of production, and the inverse demand function is linear. He argued that in the unique equilibrium outcome, the market price is above the marginal cost.

Joseph Bertrand 1883 criticized the Cournot model, where the strategy of each player is the quantity to supply, and in turn suggested to consider prices, rather than quantities, as strategies. In the Bertrand model each firm chooses a price for a homogeneous good, and the firm announcing the lowest price gets all the market share. Since the firm with the lowest price receives all the demand, each firm has incentive to price below the

[1] An oligopoly is a market that is shared between several sellers.

[2] However, political relations may also affect the power of a country.

current market price unless the market price matches its cost. Therefore, the market price will be equal to the marginal cost in an equilibrium outcome of the Bertrand model, assuming all marginal costs are the same and there are at least two competitors in the market.

The Cournot and Bertrand models are two basic tools for investigating the competitive market price, and have attracted much interest for modeling real markets; see, e.g., [11, 39, 10, 18]. In particular, the predictive power of each strongly depends on the nature of the market, and varies from application to application. For example, the Bertrand model explains the situation where firms literally set prices, e.g., the cellphone market, the laptop market, and the TV market. On the other hand, Cournot's approach would be suitable for modeling markets like those of crude oil, natural gas, and electricity, where firms decide about quantities rather than prices.

There are several attempts to find equilibrium outcomes of the Cournot or Bertrand competitions in the oligopolistic setting, where a small number of firms compete in only one market; see, e.g., [27, 36, 34, 21, 20, 41]. Nevertheless, it is not entirely clear what equilibrium outcomes of these games are when firms compete over more than one market. In this paper, we investigate the problem of finding equilibrium outcomes of the Cournot competition in a network setting where there are several markets for a homogeneous good and each market is accessible to a subset of firms.

The reader is referred to the full version of the paper to see a warm-up basic example for the Cournot competition in the network setting. In general due to interest of space, all missing proofs and examples are in the longer version of this paper on arXiv (`http://arxiv.org/abs/1405.1794`).

1.1 Related Work

Despite several papers that investigate the Cournot competition in an oligopolistic setting (see, e.g., [36, 21, 20, 41]), little is known about the Cournot competition in a network. Independently and in parallel to our work, Bimpikis et al. [6] (EC'14) study the Cournot competition in a network setting, and considers a network of firms and markets where each firm chooses a quantity to supply in each accessible market. The core of their work lies in building connections between the equilibrium outcome of the game and paths in the underlying network, and changes in profits and welfares upon coalition of two firms. While Bimpikis et al. [6] (EC'14) only consider the competition for linear inverse demand functions and quadratic cost functions (of total production), in this study, we consider the same model when the cost functions and the demand functions may have quite general forms. We show the game with linear inverse demand functions is a potential game and therefore has a unique equilibrium outcome. Furthermore, we present two polynomial-time algorithms for finding an equilibrium outcome for a wide range of cost functions and demand functions.

While we investigate the Cournot competition in networks, there is a paper which considers the Bertrand competition in network setting [3], albeit in a much more restricted case of only two firms competing in each market. While we investigate the Cournot competition in networks, there is a recent line of research exploring bargaining processes in networks; see, e.g., Bateni et al. [4], Kanoria et al. [24], Farczadi et al. [16], Chakraborty et al. [9]. Agents may cooperate to generate surplus to be divided based on an agreement. A bargaining process determines how this surplus is divided between the participants.

The final price of each market in the Cournot competition is one that clears the market. Finding a market clearance equilibrium is a well-established problem, and several papers propose polynomial-time algorithms for computing such equilibria. Examples include Arrow-Debreu market and its special case Fisher market (see related work on these markets [14, 13, 23, 12, 19]). The first polynomial-time algorithm for finding an Arrow-Debreu market equilibrium is proposed by Jain [23] for a special case with linear utilities. The Fisher market, a special case of the Arrow-Debreu market, attracted a lot of attention as well. Eisenberg and Gale [14] present the first polynomial-time algorithm by transferring the problem to a concave cost maximization problem. Devanur et al. [13] design the first combinatorial algorithm which runs in polynomial time and finds the market clearance equilibrium when the utility functions are linear. This result is later improved by Orlin [33].

For the sake of completeness, we refer to recent works in the computer science literature [22, 17], which investigate the Cournot competition in an oligopolistic setting. Immorlica et al. [22] study a coalition formation game in a Cournot oligopoly. In this setting, firms form coalitions, and the utility of each coalition, which is equally divided between its members, is determined by the equilibrium of a Cournot competition between coalitions. They prove the price of anarchy, which is the ratio between the social welfare of the worse stable partition and the social optimum, is $\Theta(n^{2/5})$ where n is the number of firms. Fiat et al. [17] consider a Cournot competition where agents may decide to be non-myopic. In particular, they define two principal strategies to maximize revenue and profit (revenue minus cost) respectively. Note that in the classic Cournot competition all agents want to maximize their profit. However, in their study each agent first chooses its principal strategy and then acts accordingly. The authors prove this game has a pure Nash equilibrium and the best response dynamics will converge to an equilibrium. They also show the equilibrium price in this game is lower than the equilibrium price in the standard Cournot competition.

1.2 Results and Techniques

We consider the problem of Cournot competition on a network of markets and firms (NCC) for different classes of cost and inverse demand functions. Adding these two dimensions to the classical Cournot competition which only involves a single market and basic cost and inverse demand functions yields an engaging but complicated problem that requires advanced techniques to analyze. For simplicity of notation we model the competition by a bipartite graph rather than a hypergraph: vertices on one side denote the firms, and vertices on the other side denote the markets. An edge between a firm and a market shows that the firm has access to the market. The complexity of finding the equilibrium, in addition to the number of markets and firms, depends on the classes that inverse demand and production cost functions belong to.

Cost functions	Inverse demand functions	Running time	Technique
Convex	Linear	$O(E^3)$	Convex optimization, Potential game formulation
Convex	Strongly monotone marginal revenue function[3]	poly(E)	Reduction to a nonlinear complementarity problem
Convex, separable	Concave	$O(n \log^2 Q_{\max})$	Supermodular optimization, nested binary search

We summarize our results in the above table, where E denotes the number of edges of the bipartite graph, n denotes the number of firms, and $Q_{\max}$ denotes the maximum possible total quantity in the oligopoly network at any equilibrium. In our results we assume the inverse demand functions are nonincreasing functions of total production in the market. This is the basic assumption in the classical Cournot Competition model: As the price in the market increases, it is reasonable to believe that the buyers drop out of the market and demand for the product decreases. The classical Cournot Competition model as well as many previous works on Cournot Competition model assumes linearity of the inverse demand function [6, 22]. In fact there is little work on generalizing the inverse demand function in this model. The second and third rows of the table show we have developed efficient algorithms for more general inverse demand functions satisfying concavity rather than linearity. The assumption of monotonicity of the inverse demand function is a standard assumption in Economics [2, 1, 30]. We assume cost functions to be convex which is the case in many works related to both Cournot Competition and Bertrand Network [28, 42]. In a previous work [6], the author considers NCC, however, assumes that inverse demand functions are linear and all the cost functions are quadratic function of the total production by the firm in all markets which is quite restrictive. Most of the results in other related works in Cournot Competition and Bertrand Network require linearity of the cost functions [3, 22]. Next comes a brief overview of our results.

Linear Inverse Demand Functions. In case inverse demand functions are linear and production costs are convex, we present a fast algorithm to obtain the equilibrium. This approach works by showing that NCC belongs to a class of games called *potential games*. In such games, the collective strategy of the independent players is to maximize a single potential function. The potential function is carefully designed so that changes made by one player reflects in the same way in the potential function as in their own utility function. Based on network structure, we design a potential function for the game, and establish the desired property. Moreover, in the case of convex cost functions, we prove concavity of the designed potential function (Theorem 6) concluding convex optimization methods can be employed to find the optimum and hence, the equilibrium of the original Cournot competition. We also discuss uniqueness of equilibria if the cost functions are strictly concave. We prove the following theorems in Section 3.

Theorem 1. *NCC with linear inverse demand functions forms a potential game.*

[3] Marginal revenue function is the vector function mapping production quantities on edges to marginal revenue along them.

Theorem 2. *Our designed potential function for NCC with linear inverse demand functions is concave provided that the cost functions are convex. Furthermore, the potential function is strictly concave if the cost functions are strictly convex, and hence the equilibrium for the game is unique. In addition, a polynomial-time algorithm finds the optimum of the potential function which describes the market clearance prices.*

The General Case. Since the above approach does not work for nonlinear inverse demand functions, we design another interesting but more involved algorithm to capture more general forms of inverse demand functions. We show that an equilibrium of the game can be computed in polynomial time if the production cost functions are convex and the revenue function is monotone. Moreover, we show under strict monotonicity of the revenue function, the solution is unique, and therefore our results in this section is structural; i.e., we find the one and only equilibrium[4]. For convergence guarantee we also need Lipschitz condition on derivatives of inverse demand and cost functions. We start the section by modeling our problem as a complementarity problem. Then we prove how holding the aforementioned conditions for cost and revenue functions yields satisfying *Scaled Lipschitz Condition* (SLC) and semidefiniteness for matrices of derivatives of the profit function. SLC is a standard condition widely used in convergence analysis for scalar and vector optimization [43]. Finally, we present our algorithm, and show how meeting these new conditions by inverse demand and cost functions helps us to guarantee polynomial running time of our algorithm. We also give examples of classes of inverse demand functions satisfying the above conditions. These include many families of inverse demand functions including quadratic functions, cubic functions and entropy functions. The following theorem is the main result of Section 4 which summarizes the performance of our algorithm.

Theorem 3. *A solution to NCC can be found in polynomial number of iterations under the following conditions:*

1. *The cost functions are (strongly) convex.*
2. *The marginal revenue function is (strongly*[5]*) monotone.*
3. *The first derivative of cost functions and inverse demand functions and the second derivative of inverse demand functions are Lipschitz continuous.*

Furthermore, the solution is unique assuming only the first condition. Therefore, our algorithm finds the unique equilibrium of NCC.

Cournot Oligopoly. Another reasonable model for considering cost functions of the firms is the case where the production cost in a market depends only on the quantity produced by the firm in that specific market (and not on quantities produced by this firm in other markets). In other words, the firms have completely independent sections

[4] It is worth mentioning that Bimpikis et al. [6] prove the uniqueness of the equilibrium in a concurrent work.

[5] For at least one of the first two conditions, strong version of condition should be satisfied, i.e., either cost functions should be strongly convex or the marginal revenue function should be strongly monotone.

for producing different goods in various markets, and there is no correlation between production cost in separate markets. In this case the competitions are separable; i.e., equilibrium for NCC can be found by finding the quantities at equilibrium for each market individually. This motivates considering Cournot game where the firms compete over a single market. We present a new algorithm for computing equilibrium quantities produced by firms in a Cournot oligopoly, i.e., when the firms compete over a single market. Cournot Oligopoly is a well-known model in Economics, and computation of its Cournot Equilibrium has been subject to a lot of attention. It has been considered in many works including [37, 25, 32, 29, 8] to name a few. The earlier attempts for calculating equilibrium for a general class of inverse demand and cost functions are mainly based on solving a Linear Complementarity Problem or a Variational Inequality. These settings can be then turned into convex optimization problems of size $O(n)$ where n is the number of firms. This means the runtime of the earlier works cannot be better than $O(n^3)$ which is the runtime of the most efficient algorithm known for convex optimization. We give a novel combinatorial algorithm for this important problem when the quantities produced are integral. Our algorithm runs in time $O(n \log^2(Q_{\max}))$ where $Q_{\max}$ is an upper bound on total quantity produced at equilibrium. The following is the main result of Section 5.

Theorem 4. *A polynomial-time algorithm successfully computes the quantities produced by each firm at an equilibrium of the Cournot oligopoly if the inverse demand function is nonincreasing, and the cost functions are convex. In addition, the algorithm runs in time $O(n \log^2(Q_{\max}))$ where $Q_{\max}$ is the maximum possible total quantity in the oligopoly network at any equilibrium.*

2 Notations

Suppose we have a set of n firms denoted by $\mathcal{F}$ and a set of m markets denoted by $\mathcal{M}$. A single good is produced in each market. Each firm may or may not be able to supply a particular market. A bipartite graph is used to demonstrate these relations. In this graph, the markets are denoted by the numbers $1, 2, \ldots, m$ on one side, and the firms are denoted by the numbers $1, 2, \ldots, n$ on the other side. For simplicity, throughout the paper we use the notation $i \in \mathcal{M}$ meaning the market i, and $j \in \mathcal{F}$ meaning firm j. For firm $j \in \mathcal{F}$ and market $i \in \mathcal{M}$ there exists an edge between the corresponding vertices in the bipartite graph if and only if firm j is able to produce the good in market i. This edge will be denoted (i, j). The set of edges of the graph is denoted by $\mathcal{E}$, and the number of edges in the graph is shown by E. For each market $i \in \mathcal{M}$, the set of vertices $N_{\mathcal{M}}(i)$ is the set of firms that this market is connected to in the graph. Similarly, $N_{\mathcal{F}}(j)$ denotes the set of neighbors of firms j among markets. The edges in $\mathcal{E}$ are sorted and numbered $1, \ldots, E$, first based on the number of their corresponding market and then based on the number of their corresponding firm. More formally, edge $(i, j) \in \mathcal{E}$ is ranked above edge $(l, k) \in \mathcal{E}$ if $i < l$ or $i = l$ and $j < k$. The quantity of the good that firm j produces in market i is denoted by q_{ij}. The vector $\mathbf{q}$ is an $E \times 1$ vector that contains all the quantities produced over the edges of the graph in the same order that the edges are numbered.

The demand for good i denoted by D_i is $\sum_{j \in N_{\mathcal{M}}(i)} q_{ij}$. The price of good i, denoted by the function $P_i(D_i)$, is only a decreasing function of total demand for this good and not the individual quantities produced by each firm in this market. For a firm j, the vector $\boldsymbol{s_j}$ denotes the strategy of firm j, which is the vector of all quantities produced by this firm in the markets $N_{\mathcal{F}}(j)$. Firm $j \in \mathcal{F}$ has a cost function related to its strategy denoted by $c_j(\boldsymbol{s_j})$. The profit that firm j makes is equal to the total money that it obtains by selling its production minus its cost of production. More formally, the profit of firm j is $\pi_j = \sum_{i \in N_{\mathcal{F}}(j)} P_i(D_i) q_{ij} - c_j(\boldsymbol{s_j})$.

3 Cournot Competition and Potential Games

In this section, we design an efficient algorithm for the case where the price functions are linear. More specifically, we design an innovative *potential function* that captures the changes of all the utility functions simultaneously, and therefore, show how finding the quantities at the equilibrium would be equivalent to finding the set of quantities that maximizes this function. We use the notion of *potential games* as introduced in Monderer and Shapley [31]. In that paper, the authors introduce *potential games* as the set of games for which there exists a *potential function* P^* such that the pure strategy equilibrium set of the game coincides with the pure strategy equilibrium set of a game where every party's utility function is P^*.

Next we design a potential function for NCC if the price functions are linear. Interestingly, this holds for any cost function meaning NCC with arbitrary cost functions is a potential game as long as the price functions are linear. Furthermore, we show when the cost functions are convex, the potential function is concave, and hence any convex optimization method can find the equilibrium of such a Network Cournot Competition. In case cost functions are strictly convex, the potential function is strictly concave. We show the equilibrium that we find is the one and only equilibrium of the game. The pure strategy equilibrium set of any potential game coincides with the pure strategy equilibrium set of a game with the potential function P^* as all parties' utility function.

Theorem 5. *NCC with linear price functions (of quantities) is a potential game.*

We can efficiently compute the equilibrium of the game if the potential function P^* is easy to optimize. Below we show that this function is concave.

Theorem 6. *The potential function P^* from the previous theorem is concave provided that the cost functions of the firms are convex. Moreover, if the cost functions are strictly convex then the potential function is strictly concave.*

The following well-known theorem discusses the uniqueness of the solution to a convex optimization problem.

Theorem 7. *Let $f : \mathcal{K} \to \mathbb{R}^n$ be a strictly concave and continuous function for some finite closed convex space $\mathcal{K} \in \mathbb{R}^n$. Then the convex optimization problem* $\max f(x) :$ *$x \in \mathcal{K}$ has a unique solution.*

By Theorem 6, if the cost functions are strictly convex then the potential function is strictly concave and hence, by Theorem 7 the equilibrium of the game is unique.

Let $ConvexP(\mathcal{E}, (\alpha_1, \ldots, \alpha_m), (\beta_1, \ldots, \beta_m), (c_1, \ldots, c_n))$ be the following convex optimization program:

$$\begin{aligned} \min \quad & -\sum_{i \in \mathcal{M}} \left[\alpha_i \sum_{j \in N_{\mathcal{M}}(i)} q_{ij} - \beta_i \sum_{j \in N_{\mathcal{M}}(i)} q_{ij}^2 - \beta_i \sum_{\substack{k \leq j \\ k,j \in N_{\mathcal{M}}(i)}} q_{ij} q_{ik} - \sum_{j \in N_{\mathcal{M}}(i)} \frac{c_j(\boldsymbol{s_j})}{|N_{\mathcal{F}}(j)|} \right] \qquad (1) \\ \text{subject to} \quad & q_{ij} \geq 0 \ \ \forall (i,j) \in \mathcal{E}. \end{aligned}$$

Note that in this optimization program we are trying to maximize P^* for a bipartite graph with set of edges $\mathcal{E}$, linear price functions characterized by the pair (α_i, β_i) for each market i, and cost functions c_j for each firm j. This algorithm has a time complexity equal to the time complexity of a convex optimization algorithm with E variables. The best such algorithm has a running time $O(E^3)$ [7].

4 Finding Equilibrium for Cournot Game with General Cost and Inverse Demand Functions

In this section, we focus on a much more general class of price and cost functions. Our approach is based on reducing NCC to a polynomial time solvable class of Nonlinear Complementarity Problem (NLCP). First in 4.1, we introduce our marginal profit function as the vector of partial derivatives of all firms with respect to the quantities they produce. Then in 4.2, we show how this marginal profit function helps in reducing NCC to a general NLCP. We also discuss uniqueness of equilibrium in this situation. Unfortunately, in its most general form, NLCP is computationally intractable. For a large class of functions, though, these problems are polynomial time solvable. In 4.3, we rigorously define the conditions under which NLCP is polynomial time solvable. We then present our algorithm with a theorem, showing it converges in polynomial number of steps. To show the conditions required for quick convergence are not restrictive, we refer the reader to the full version of this paper on arXiv, where we explore a wide range of important price functions that satisfy them.

Assumptions. Throughout the rest of this section we assume that the price functions are decreasing and concave and the cost functions are strongly convex (*to be defined later*). We also assume that for each firm there is a finite quantity at which extra production ceases to be profitable even if that is the only firm operating in the market. Thus, all production hence supply quantities are finite. In addition, we assume Lipschitz continuity and finiteness of the first and the second derivatives of price and cost functions. We note that these Lipschitz continuity assumptions are very common for convergence analysis in convex optimization [7] and finiteness assumptions are implied by Lipschitz continuity. In addition, they are not very restrictive as we do not expect unbounded price/cost fluctuation with supply change. For sake of brevity, we use the terms inverse demand function and price function interchangeably.

4.1 Marginal Profit Function

For the rest of this section, we assume that P_i and c_i are twice differentiable functions of quantities. For a firm j and a market i such that $(i,j) \in \mathcal{E}$, we define $f_{ij} = -\frac{\partial \pi_j}{\partial q_{ij}} = -P_i(D_i) - \frac{\partial P_i(D_i)}{\partial q_{ij}} q_{ij} + \frac{\partial c_j}{\partial q_{ij}}$. Recall that the price function of a market is only a function of the total production in that market and not the individual quantities produced by individual firms. Thus $\frac{\partial P_i(D_i)}{\partial q_{ij}} = \frac{\partial P_i(D_i)}{\partial q_{ik}}$ $\forall j,k \in N_{\mathcal{M}}(i)$. Therefore, we replace these terms by $P_i'(D_i)$ to obtain $f_{ij} = -P_i(D_i) - P_i'(D_i)q_{ij} + \frac{\partial c_j}{\partial q_{ij}}$.

Let vector F be the vector of all f_{ij}'s corresponding to the edges of the graph in the same format that we defined the vector q. Note that F is a function of q. Moreover, we separate the part representing marginal revenue from the part representing marginal cost in function F. More formally, we split F into two functions R and S such that $F = R + S$, and the element corresponding to the edge $(i,j) \in \mathcal{E}$ in the *marginal revenue function* $R(q)$ is $r_{ij} = -\frac{\partial \pi_j}{\partial q_{ij}} = -P_i(D_i) - P'(D_i)q_{ij}$, whereas for the *marginal cost function* $S(q)$ is $s_{ij} = \frac{\partial c_j}{\partial q_{ij}}$.

4.2 Non-linear Complementarity Problem

We now formally define NLCP, and prove our problem is an NLCP.

Definition 1. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a continuously differentiable function on $\mathbb{R}^n_+$. The complementarity problem seeks a vector $x \in \mathbb{R}^n$ that satisfies $x \geq 0$, $F(x) \geq 0$, and $x^T F(x) = 0$.*

Theorem 8. *The problem of finding the vector q at equilibrium in the Cournot game is a complementarity problem.*

Definition 2. *$F : \mathcal{K} \to \mathbb{R}^n$ is said to be strictly monotone at x^* if $\langle [F(x) - F(x^*)]^T, x - x^* \rangle \geq 0, \forall x \in \mathcal{K}$. Then, F is said to be strictly monotone if it is strictly monotone at any $x^* \in \mathcal{K}$. Equivalently, F is strictly monotone if its Jacobian matrix is positive definite.*

The following theorem is a well known theorem for Complementarity Problems.

Theorem 9. *[26] Let $F : \mathcal{K} \to \mathbb{R}^n$ be a continuous and strictly monotone function with a point $x \in \mathcal{K}$ such that $F(x) \geq 0$ (i.e., there exists a potential solution to the CP). Then the Complementarity Problem introduced in Definition 1 characterized by function F has a unique solution.*

Hence, the Complementarity Problem characterized by function F has a unique solution under the assumption that the revenue function is strongly monotone (special case of strictly monotone). In the next subsection, we aim to find this unique equilibrium of the NCC problem.

4.3 Designing a Polynomial-Time Algorithm

In this subsection, we present an algorithm to find the equilibrium of NCC, and establish its polynomial time convergence by Theorem 10. This theorem requires the marginal profit function to satisfy Scaled Lipschitz Condition (SLC) and monotonicity. We first introduce SLC, and show how the marginal profit function satisfies SLC and monotonicity by Lemmas 1 to 5. We present in in Lemma 5 the conditions that the cost and price functions should have in order for the marginal profit function to satisfy SLC and monotonicity. Finally, in Theorem 10, we show convergence of our algorithm in polynomial time.

Before introducing the next theorem, we explain what the Jacobians ∇R, ∇S, and ∇F are for the Cournot game. First note that these are $E \times E$ matrices. Let $(i, j) \in \mathcal{E}$ and $(l, k) \in \mathcal{E}$ be two edges of the graph. Let e_1 denote the index of edge (i, j), and e_2 denote the index of edge (l, k) in the vector as we discussed in the first section. Then the element in row e_1 and column e_2 of matrix ∇R, denoted $\nabla R_{e_1 e_2}$, is equal to $\frac{\partial r_{ij}}{\partial q_{lk}}$. We name the corresponding elements in ∇F and ∇S similarly. We have $\nabla F = \nabla R + \nabla S$ as $F = R + S$.

Definition 3 (Scaled Lipschitz Condition (SLC)). *A function $G : D \mapsto \mathbb{R}^n$, $D \subseteq \mathbb{R}^n$ is said to satisfy* Scaled Lipschitz Condition (SLC) *if there exists a scalar $\lambda > 0$ such that $\forall\ h \in \mathbb{R}^n, \forall\ x \in D$, such that $\|X^{-1}h\| \leq 1$, we have $\|X[G(x + h) - G(x) - \nabla G(x)h]\|_\infty \leq \lambda |h^T \nabla G(x)h|$, where X is a diagonal matrix with diagonal entries equal to elements of the vector x in the same order, i.e., $X_{ii} = x_i$ for all $i \in \mathcal{M}$.*

Satisfying SLC and monotonicity are essential for marginal profit function in Theorem 10. In Lemma 5 we discuss the assumptions for cost and revenue function under which these conditions hold for our marginal profit function. We use Lemmas 1 to 5 to show F satisfies SLC. More specifically, we demonstrate in Lemma 1, if we can derive an upperbound for LHS of SLC for R and S, then we can derive an upperbound for LHS of SLC for $F = R + S$ too. Then in Lemma 2 and Lemma 3 we show LHS of S and R in SLC definition can be upperbounded. Afterwards, we show monotonicity of S in Lemma 4. In Lemma 5 we aim to prove F satisfies SLC under some assumptions for cost and revenue functions. We use the fact that LHS of SLC for F can be upperbounded using Lemma 3 and Lemma 2 combined with Lemma 1. Then we use the fact that RHS of SLC can be upperbounded using strong monotonicity of R and Lemma 4. Using these two facts, we conclude F satisfies SLC in Lemma 5.

Lemma 1. *Let F, R, S be three $\mathbb{R}^n \to \mathbb{R}^n$ functions such that $F(q) = R(q) + S(q)$, $\forall q \in \mathbb{R}^n$. Let R and S satisfy the following inequalities for some $C > 0$ and $\forall\ h$ such that $\|X^{-1}h\| \leq 1$:*

$$\|X[R(q + h) - R(q) - \nabla R(q)h]\|_\infty \leq C\|h\|^2,$$
$$\|X[S(q + h) - S(q) - \nabla S(q)h]\|_\infty \leq C\|h\|^2,$$

where X is the diagonal matrix with $X_{ii} = q_i$. Then we have:

$$\|X[F(q + h) - F(q) - \nabla F(q)h]\|_\infty \leq 2C\|h\|^2.$$

The following lemmas give upper bounds for LHS of the SLC for S and R respectively.

Lemma 2. *Assume X is the diagonal matrix with $X_{ii} = q_i$. $\forall$ h such that $\|X^{-1}h\| \leq 1$, there exists a constant $C > 0$ satisfying: $\|X[S(q+h) - S(q) - \nabla S(q)h]\|_\infty \leq C\|h\|^2$.*

Lemma 3. *Assume X is the diagonal matrix with $X_{ii} = q_i$. $\forall$ h such that $\|X^{-1}h\| \leq 1$, $\exists C > 0$ such that$\|X[R(q+h) - R(q) - \nabla R(q)h]\|_\infty \leq C\|h\|^2$.*

If R is assumed to be strongly monotone, we immediately have a lower bound on RHS of the SLC for R. The following lemma gives a lower bound on RHS of the SLC for S.

Lemma 4. *If cost functions are (strongly) convex, S is (strongly) monotone.*

The following lemma combines the results of Lemma 2 and Lemma 3 using Lemma 1 to derive an upper bound for LHS of the SLC for F. We bound RHS of the SLC from below by using strong monotonicity of R and Lemma 4.

Lemma 5. *F satisfies SLC and is monotone if: (1) Cost functions are convex. (2) Marginal revenue function is monotone. (3) Cost functions are strongly convex or marginal revenue function is strongly monotone.*

We wrap up with the description of the algorithm. The algorithm first constructs the vector F of length E. It then finds the initial feasible solution $(F(x_0), x_0)$ for the complementarity problem. (This solution should satisfy $x_0 \geq 0$ and $F(x_0) \geq 0$.) If finally run Algorithm 3.1 from [43] to find the solution $(F(x^*), x^*)$ to the CP characterized by F, which gives the vector q of quantities produced by firms at equilibrium. Lemma 5 guarantees that our problem satisfies the two conditions mentioned in Zhao and Han 1999. Therefore, we can prove the following theorem.

Theorem 10. *The algorithm converges to an equilibrium of Network Cournot Competition in time $O\big(E^2 \log(\mu_0/\epsilon)\big)$ under the following assumptions:*

1. *The cost functions are strongly convex.*
2. *The marginal revenue function is strongly monotone.*
3. *The first derivative of cost functions and price functions and the second derivative of price functions are Lipschitz continuous.*

This algorithm outputs an approximate solution $(F(q^*), q^*)$ satisfying $(q^*)^T F(q^*)/n \leq \epsilon$ where $\mu_0 = (q_0)^T F(q_0)/n$, and $(F(q_0), q_0)$ is the initial feasible point[6].

For a discussion of price functions that satisfy the convergence conditions for our algorithm, we refer the reader to the full version of the paper on arXiv.

5 Algorithm for Cournot Oligopoly

In this section we present a new algorithm for computing the equilibrium in a Cournot oligopoly, i.e., when the firms compete over a single market. Computation of Cournot

[6] Initial feasible solution can be trivially found. E.g., it can be the same production quantity along each edge, large enough to ensure losses for all firms. Such quantity can easily be found by binary search between [0, Q].

Equilibrium is an important problem in its own right. A considerable body of literature has been dedicated to this problem [37, 25, 29, 8]. All earlier work computing Cournot equilibrium for a general class of price and cost functions rely on solving a Linear Complementarity Problem or a Variational Inequality which in turn are set up as convex optimization problems of size $O(n)$ where n is the number of firms in oligopoly. Thus, the runtime guarantee of the earlier works is $O(n^3)$ at best. We give a novel combinatorial algorithm for this important problem when the quantities produced are integral. Our algorithm runs in time $n\log^2(Q_{max})$ where Q_{max} is an upper bound on total quantity produced at equilibrium. There is always an upper bound for Q_{max} since if $Q = \sum_{i\in\mathcal{F}} q_i$ is large enough the price function would become negative and no firm has any incentive to produce a higher quantity. We note that, for two reasons, the restriction to integral quantities is practically no restriction at all. Firstly, in real-world all commodities and products are traded in integral (or rational) units. Secondly, this algorithm can easily be adapted to compute approximate Cournot-Nash equilibrium for the continuous case and since the quantities at equilibrium may be irrational numbers, this is the best we can hope for.

With only a single market present, we simplify the notation. Let $[n] = \{1, \ldots, n\}$ be the set of firms competing over the single market. Let $\mathbf{q} = (q_1, q_2, \ldots, q_n)$ be the set of all quantities they produce, one quantity for each firm. Let $Q = \sum_{i\in[n]} q_i$. In this case, there is only a single inverse demand function $P : \mathbb{Z} \mapsto \mathbb{R}_{\geq 0}$, which maps total supply, Q, to market price. We assume that, P is a decreasing function of Q. For each firm $i \in [n]$, the function $c_i : \mathbb{Z} \mapsto \mathbb{R}_{\geq 0}$ denotes the cost to firm i for producing quantity q_i of the good. We assume convex cost functions. The profit of firm $i \in [n]$ as a function of q_i and Q, denoted $\pi_i(q_i, Q)$, is $P(Q)q_i - c_i(q_i)$. Also let $f_i(q_i, Q) = \pi_i(q_i + 1, Q + 1) - \pi_i(q_i, Q)$ be the marginal profit for firm i of producing one extra unit. Although the quantities are nonnegative integers, for simplicity we assume the functions c_i, P, π_i and f_i are zero whenever any of their inputs are negative. Also, we refer to the forward difference $P(Q + 1) - P(Q)$ by $P'(Q)$.

Polynomial Time Algorithm. We leverage the supermodularity of price functions and Topkis' Monotonicity Theorem [38] to design a nested binary search algorithm to find the Cournot equilibrium. Intuitively, the algorithm works as follows. At each point we guess Q' to be the total quantity of good produced by all the firms. Then we check how good this guess is by computing for each firm the set of quantities that it can produce at equilibrium if we assume the total quantity is the fixed integer Q'. We prove that for given Q', the set of possible quantities for each firm at equilibrium is a consecutive set of integers. Let $I_i = \{q_i^l, q_i^l + 1, \ldots, q_i^u - 1, q_i^u\}$ be the range of all possible quantities for firm $i \in [n]$ assuming Q' is the total quantity produced in the market. We can conclude Q' was too low a guess if $\sum_{i\in[n]} q_i^l > Q'$. This implies our search should continue among total quantities above Q'. Similarly, if $\sum_{i\in[n]} q_i^u < Q'$, we can conclude our guess was too high, and the search should continues among total quantities below Q'. If neither case happens, then for each firm $i \in [n]$, there exists a $q_i' \in I_i$ such that $Q' = \sum_{i\in[n]} q_i'$ and firm i has no incentive to change this quantity if the total quantity is Q' and we have found an equilibrium. The pseudocode for the algorithm and its correctness is proved in the full version of this paper (`http://arxiv.org/abs/1405.1794`).

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