Truthful Approximations to Range Voting

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Abstract. We consider the fundamental mechanism design problem of *approximate social welfare maximization* under *general cardinal preferences* on a finite number of alternatives and *without money*. The well-known *range voting* scheme can be thought of as a non-truthful mechanism for exact social welfare maximization in this setting. With m being the number of alternatives, we exhibit a randomized truthful-in-expectation ordinal mechanism with approximation ratio $\Omega(m^{-3/4})$. On the other hand, we show that for sufficiently many agents, the approximation ratio of any truthful-in-expectation ordinal mechanism is $O(m^{-2/3})$. We supplement our results with an upper bound for any truthful-in-expectation mechanism. We get tighter bounds for the natural special case of $m = 3$, and in that case furthermore obtain separation results concerning the approximation ratios achievable by natural restricted classes of truthful-in-expectation mechanisms. In particular, we show that the best cardinal truthful mechanism strictly outperforms all ordinal ones.

1 Introduction

We consider the fundamental mechanism design problem of *approximate social welfare maximization* under *general cardinal preferences* and *without money*. In this setting, there is a finite set of agents (or *voters*) $N = \{1, \ldots, n\}$ and a finite set of alternatives (or *candidates*) $M = \{1, \ldots, m\}$. Each voter i has a private valuation function $u_i : M \to \mathbb{R}$ that can be arbitrary, except that it is injective, i.e., it induces a total order on candidates. Standardly, the function u_i is considered well-defined only up to positive affine transformations. That is, $x \to au_i(x)+b$, for $a > 0$ and any b, is considered to be a different representation of u_i. Given this, we fix a canonical representation of u_i. The two most widely used cannonical representations in literature are *unit-range* (i.e. $\forall j, u_i(j) \in [0, 1]$ and $\max_j u_i(j) = 1$, $\min_j u_i(j) = 0$) [1–4, 6, 8, 9] and *unit-sum* (i.e. $\forall j, u_i(j) \in [0, 1]$ and $\sum_j u_i(j) = 1$) [7, 8, 13]. In this paper we will assume that all u_i are

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canonically represented using the unit-sum representation, because it is arguably more suited for social welfare maximization. Intuitively, agents should not be "punished" for liking a lot of different outcomes. We shall let V_m denote the set of all unit-range canonically represented valuation functions.

We shall be interested in *direct revelation mechanisms without money* that elicit the *valuation profile* $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ from the voters and based on this elect a candidate $J(\mathbf{u}) \in M$. We shall allow mechanisms to be randomized and $J(\mathbf{u})$ is therefore in general a random map. In fact, we shall define a mechanism simply to be a random map $J : {V_m}^n \to M$. We prefer mechanisms that are *truthful-in-expectation*, by which we mean that the following condition is satisfied: For each voter i, and all $\mathbf{u} = (u_i, u_{-i}) \in {V_m}^n$ and $\tilde{u}_i \in V_m$, we have $E[u_i(J(u_i, u_{-i})] \geq E[u_i(J(\tilde{u}_i, u_{-i})]$. That is, if voters are assumed to be expected utility maximizers, the optimal behavior of each voter is always to reveal their true valuation function to the mechanism. As truthfulness-in-expectation is the only notion of truthfulness of interest to us in this paper, we shall use "truthful" as a synonym for "truthful-in-expectation" from now on. Furthermore, we are interested in mechanisms for which the expected *social welfare*, i.e., $E[\sum_{i=1}^n u_i(J(\mathbf{u}))]$, is as high as possible, and we shall in particular be interested in the *approximation ratio* of the mechanism, trying to achieve mechanisms with as high an approximation ratio as possible. Note that for $m = 2$, the problem is easy; a majority vote is a truthful mechanism that achieves optimal social welfare, i.e., it has approximation ratio 1, so we only consider the problem for $m \geq 3$.

A mechanism without money for general cardinal preferences can be naturally interpreted as a *cardinal voting scheme* in which each voter provides a *ballot* giving each candidate $j \in M$ a numerical score between 0 and 1. A winning candidate is then determined based on the set of ballots. With this interpretation, the well-known *range voting scheme* is simply the determinstic mechanism that elects the socially optimal candidate $\text{argmax}_{j \in M} \sum_{i=1}^n u_i(j)$, or, more precisely, elects this candidate *if* ballots are reflecting the true valuation functions u_i. In particular, range voting has by construction an approximation ratio of 1. However, range voting is not a truthful mechanism.

As our first main result, we exhibit a randomized truthful mechanism with an approximation ratio of $0.37m^{-3/4}$. The mechanism is *ordinal*: Its behavior depends only on the *rankings* of the candidates on the ballots, not on their numerical scores. We also show a negative result: For sufficiently many voters and any truthful ordinal mechanism, there is a valuation profile where the mechanism achieves at most an $O(m^{-2/3})$ fraction of the optimal social welfare in expectation. The negative result also holds for non-ordinal mechanisms that are *mixed-unilateral*, by which we mean mechanisms that elect a candidate based on the ballot of a single randomly chosen voter. Finally, we prove that no truthful mechanism can achieve an approximation ratio of 0.94.

We get tighter bounds for the natural case of $m = 3$ candidates and for this case, we also obtain separation results concerning the approximation ratios achievable by natural restricted classes of truthful mechanisms. In particular, the best mixed-unilateral mechanism strictly outperforms all ordinal ones, even the non-unilateral ordinal ones. The mixed-unilateral mechanism that establishes this is a convex combination of *quadratic-lottery*, a mechanism presented by Freixas [9] and Feige and Tennenholtz [6] and *random-favorite*, the mechanism that picks a voter uniformly at random and elects his favorite candidate.

1.1 Background, Related Research and Discussion

Characterizing strategy-proof social choice functions (a.k.a., truthful direct revelation mechanisms without money) under general preferences is a classical topic of mechanism design and social choice theory. The class of truthful deterministic mechanisms is limited to *dictatorships*, as proven by the celebrated Gibbard-Satterthwaite theorem [10, 20]. On the other hand, the class of randomized truthful mechanisms is much richer [2], as suggested by the following characterization for randomized ordinal mechanisms:

Theorem 1. *[11] The ordinal mechanisms without money that are truthful under general cardinal preferences*[1] *are exactly the convex combinations of truthful* unilateral *ordinal mechanisms and truthful* duple *mechanisms.*

Here, a *unilateral* mechanism is a randomized mechanism whose output depends on the ballot of a single distinguished voter i^* only. A *duple* mechanism is an ordinal mechanism for which there are two distinguished candidates so that all other candidates are elected with probability 0, for all valuation profiles.

One of the main conceptual contributions from computer science to mechanism design is the suggestion of a measure for comparing mechanisms and finding the best one, namely the notion of worst case approximation ratio [16, 18] relative to some objective function. Following this research program, and using Gibbard's characterization, Procaccia [17] gave in a paper conceptually very closely related to the present one but he only considered objective functions that can be defined ordinally (such as, e.g., Borda count), and did in particular not consider approximating the optimal social welfare, as we do in the present paper.

Social welfare maximization is indeed a very standard objective in mechanism design. In particular, it is very widespread in *quasi-linear* settings (where valuations are measured in monetary terms)(see [15, 19]). On the other hand, in the setting of social choice theory, the valuation functions are to be interpreted as von Neumann-Morgenstern utilities (i.e, they are meant to encode orderings on lotteries), and in particular are only well-defined up to affine transformations. In this setting, the social welfare has to be defined as above, as the result of adding up the valuations of all players, *after* these are normalized by scaling to, say, the interval [0,1]. A significant amount of work considers social welfare maximization in the von Neumann-Morgenstern setting. [5, 7, 8, 13].

[1] without ties, i.e., valuation functions must be injective, as we require throughout this paper, except in Theorem 6. If ties were allowed, the characterization would be much more complicated.

It is often argued that while an underlying cardinal utility structure exists, it is more reasonable to only ask individuals to only provide a *ranking* of the candidates, rather than exact numerical values. This makes ordinal mechanisms particularly appealing. The limitations of this class of mechanisms were considered recently by Boutilier *et al.* [5] in a very interesting paper closely related to the present one, but crucially, they did not require truthfulness of the mechanisms in their investigations. On the other hand, truthfulness is the pivotal property in our approach. Perhaps the most interesting question to ask is whether truthful cardinal mechanisms [4, 9, 22], can outperform truthful ordinal ones, in terms of social welfare. We answer this in the positive for the case of three alternatives. Exploring the limitations of truthful cardinal mechanisms is impaired by the lack of a characterization similar to the one of Theorem 1 for the general case. Obtaining such a characterization is a major open problem in social choice theory and several attempts have been made throughout the years [3, 4, 9, 12]. Our results imply that one could be able to sidestep the need for such a characterization and use direct manipulation arguments to obtain upper bounds and at the same time highlight what to look for and what to avoid, when trying to come up with "good" cardinal truthful mechanisms.

Our investigations are very much helped by the work of Feige and Tennenholtz [6] and Freixas [9]. In particular, our construction establishing the gap between the approximation ratios for cardinal and ordinal mechanisms for three candidates is based on the *quadratic lottery*, first presented in [9] and later in [6]. Most of the proofs are omitted due to lack of space but appear in the full version.

2 Preliminaries

We let V_m denote the set of canonically represented valuation functions on $M = \{1, 2, \ldots, m\}$. That is, V_m is the set of injective functions $u : M \to [0, 1]$ with the property that 0 as well as 1 are contained in the image of u.

We let $\mathrm{Mech}_{m,n}$ denote the set of truthful mechanisms for n voters and m candidates. That is, $\mathrm{Mech}_{m,n}$ is the set of random maps $J : {V_m}^n \to M$ with the property that for voter $i \in \{1, \ldots, n\}$, and all $\mathbf{u} = (u_i, u_{-i}) \in {V_m}^n$ and $\tilde{u}_i \in V_m$, we have $E[u_i(J(u_i, u_{-i})] \geq E[u_i(J(\tilde{u}_i, u_{-i})]$. Alternatively, instead of viewing a mechanism as a random map, we can view it as a map from ${V_m}^n$ to Δ_m, the set of probability density functions on $\{1, \ldots, m\}$. With this interpretation, note that $\mathrm{Mech}_{m,n}$ is a convex subset of the vector space of all maps from ${V_m}^n$ to $\mathbb{R}^m$. We shall be interested in certain special classes of mechanisms. In the following definitions, we throughout view a mechanism J as a map from ${V_m}^n$ to Δ_m.

An *ordinal* mechanism J is a mechanism with the property: $J(u_i, u_{-i}) = J(u'_i, u_{-i})$, for any voter i, any preference profile $\mathbf{u} = (u_i, u_{-i})$, and any valuation function u'_i with the property that for all pairs of candidates j, j', it is the case that $u_i(j) < u_i(j')$ if and only if $u'_i(j) < u'_i(j')$. Informally, the behavior of an ordinal mechanism only depends on the ranking of candidates on each ballot; not on the numerical valuations. We let $\mathrm{Mech}^{\mathbf{O}}_{m,n}$ denote those mechanisms in $\mathrm{Mech}_{m,n}$ that are ordinal.

Following Barbera [2], we define an *anonymous* mechanism J as one that does not depend on the names of voters. Formally, given any permutation π on N, and any $\mathbf{u} \in (V_m)^n$, we have $J(\mathbf{u}) = J(\pi \cdot \mathbf{u})$, where $\pi \cdot \mathbf{u}$ denotes the vector $(u_{\pi(i)})_{i=1}^n$. Similarly following Barbera [2], we define a *neutral* mechanism J as one that does not depend on the names of candidates. Formally, given any permutation σ on M, any $\mathbf{u} \in (V_m)^n$, and any candidate j, we have $J(\mathbf{u})_{\sigma(j)} = J(u_1 \circ \sigma, u_2 \circ \sigma, \ldots, u_n \circ \sigma)_j$.

Following [4, 11], a *unilateral* mechanism is a mechanism for which there exists a single voter i^* so that for all valuation profiles (u_{i^*}, u_{-i^*}) and any alternative valuation profile u'_{-i^*} for the voters except i^*, we have $J(u_{i^*}, u_{-i^*}) = J(u_{i^*}, u'_{-i^*})$. Note that i^* is *not* allowed to be chosen at random in the definition of a unilateral mechanism. In this paper, we shall say that a mechanism is *mixed-unilateral* if it is a convex combination of unilateral truthful mechanisms. Mixed-unilateral mechanisms are quite attractive seen through the "computer science lens": They are mechanisms of *low query complexity*; consulting only a single randomly chosen voter, and therefore deserve special attention in their own right. We let $\mathrm{Mech}^{\mathbf{U}}_{m,n}$ denote those mechanisms in $\mathrm{Mech}_{m,n}$ that are mixed-unilateral. Also, we let $\mathrm{Mech}^{\mathbf{OU}}_{m,n}$ denote those mechanisms in $\mathrm{Mech}_{m,n}$ that are ordinal as well as mixed-unilateral.

Following Gibbard [11], a *duple* mechanism J is an ordinal[2] mechanism for which there exist two candidates j_1^* and j_2^* so that for all valuation profiles, J elects all other candidates with probability 0.

We next give names to some specific important mechanisms. We let $U^q_{m,n} \in \mathrm{Mech}^{\mathbf{OU}}_{m,n}$ be the mechanism for m candidates and n voters that picks a voter uniformly at random, and elects uniformly at random a candidate among his q most preferred candidates. We let *random-favorite* be a nickname for $U^1_{m,n}$ and *random-candidate* be a nickname for $U^m_{m,n}$. We let $D^q_{m,n} \in \mathrm{Mech}^{\mathbf{O}}_{m,n}$, for $\lfloor n/2 \rfloor + 1 \le q \le n+1$, be the mechanism for m candidates and n voters that picks two candidates uniformly at random and eliminates all other candidates. It then checks for each voter which of the two candidates he prefers and gives that candidate a "vote". If a candidate gets at least q votes, she is elected. Otherwise, a coin is flipped to decide which of the two candidates is elected. We let *random-majority* be a nickname for $D^{\lfloor n/2 \rfloor+1}_{m,n}$. Note also that $D^{n+1}_{m,n}$ is just another name for *random-candidate*. Finally, we shall be interested in the following mechanism Q_n for three candidates shown to be in $\mathrm{Mech}^{\mathbf{U}}_{3,n}$ by Feige and Tennenholtz [6]: Select a voter uniformly at random, and let α be the valuation of his second most preferred candidate. Elect his most preferred candidate with probability $(4-\alpha^2)/6$, his second most preferred candidate with probability $(1+2\alpha)/6$ and his least preferred candidate with probability $(1-2\alpha+\alpha^2)/6$. We let *quadratic-lottery* be a nickname for Q_n. Note that *quadratic-lottery* is not ordinal. Feige and Tennenholtz [6] in fact presented several explicitly given non-ordinal one-voter truthful mechanisms, but *quadratic-lottery* is particularly amenable to an

2 Barbera *et al.* [4] gave a much more general definition of duple mechanism; their duple mechanisms are not restricted to be ordinal. In this paper, "duple" refers exclusively to Gibbard's original notion.

approximation ratio analysis due to the fact that the election probabilities are quadratic polynomials.

We let ratio(J) denote the approximation ratio of a mechanism $J \in \text{Mech}_{m,n}$, when the objective is social welfare. That is,

$$\text{ratio}(J) = \inf_{\mathbf{u} \in V_m{}^n} \frac{E[\sum_{i=1}^n u_i(J(\mathbf{u}))]}{\max_{j \in M} \sum_{i=1}^n u_i(j)}.$$

We let $r_{m,n}$ denote the best possible approximation ratio when there are n voters and m candidates. That is, $r_{m,n} = \sup_{J \in \text{Mech}_{m,n}} \text{ratio}(J)$. Similarly, we let $r^{\mathbf{C}}_{m,n} = \sup_{J \in \text{Mech}^{\mathbf{C}}_{m,n}} \text{ratio}(J)$, for $\mathbf{C}$ being either $\mathbf{O}$, $\mathbf{U}$ or $\mathbf{OU}$. We let r_m denote the asymptotically best possible approximation ratio when the number of voters approaches infinity. That is, $r_m = \liminf_{n \to \infty} r_{m,n}$, and we also extend this notation to the restricted classes of mechanisms with the obvious notation $r^{\mathbf{O}}_m, r^{\mathbf{U}}_m$ and $r^{\mathbf{OU}}_m$.

The importance of neutral and anonymous mechanisms is apparent from the following simple lemma:

Lemma 1. *For all $J \in \text{Mech}_{m,n}$, there is a $J' \in \text{Mech}_{m,n}$ such that J' is anonymous and neutral and so that* $\text{ratio}(J') \geq \text{ratio}(J)$. *Similarly, for all $J \in \text{Mech}^{\mathbf{C}}_{m,n}$, there is $J' \in \text{Mech}^{\mathbf{C}}_{m,n}$ so that J' is anonymous and neutral and so that* $\text{ratio}(J') \geq \text{ratio}(J)$, *for* $\mathbf{C}$ *being either* $\mathbf{O}$, $\mathbf{U}$ *or* $\mathbf{OU}$.

Lemma 1 makes the characterizations of the following theorem very useful.

Theorem 2. *The set of anonymous and neutral mechanisms in* $\text{Mech}^{\mathbf{OU}}_{m,n}$ *is equal to the set of convex combinations of mechanisms $U^q_{m,n}$, for $q \in \{1, \ldots, m\}$. Also, the set of anonymous and neutral mechanisms in* $\text{Mech}_{m,n}$ *that can be obtained as convex combinations of duple mechanisms is equal to the set of convex combinations of the mechanisms $D^q_{m,n}$, for $q \in \{\lfloor n/2 \rfloor + 1, \lfloor n/2 \rfloor + 2, \ldots, n, n+1\}$.*

The following corollary is immediate from Theorem 1 and Theorem 2.

Corollary 1. *The ordinal, anonymous and neutral mechanisms in* $\text{Mech}_{m,n}$ *are exactly the convex combinations of the mechanisms $U^q_{m,n}$, for $q \in \{1, \ldots, m\}$ and $D^q_{m,n}$, for $q \in \{\lfloor n/2 \rfloor + 1, \lfloor n/2 \rfloor + 2, \ldots, n\}$.*

We next present some lemmas that allow us to understand the asymptotic behavior of $r_{m,n}$ and $r^{\mathbf{C}}_{m,n}$ for fixed m and large n, for $\mathbf{C}$ being either $\mathbf{O}$, $\mathbf{U}$ or $\mathbf{OU}$.

Lemma 2. *For any positive integers n, m, k, we have $r_{m,kn} \leq r_{m,n}$ and $r^{\mathbf{C}}_{m,kn} \leq r^{\mathbf{C}}_{m,n}$, for* $\mathbf{C}$ *being either* $\mathbf{O}$, $\mathbf{U}$ *or* $\mathbf{OU}$.

Lemma 3. *For any $m, n \geq 2, \epsilon > 0$ and all $n' \geq (n-1)m/\epsilon$, we have $r_{m,n'} \leq r_{m,n} + \epsilon$ and $r^{\mathbf{C}}_{m,n'} \leq r^{\mathbf{C}}_{m,n} + \epsilon$, for* $\mathbf{C}$ *being either* $\mathbf{O}$, $\mathbf{U}$, *or* $\mathbf{OU}$.

In particular, Lemma 3 implies that $r_{m,n}$ converges to a limit as $n \to \infty$.

2.1 Quasi-Combinatorial Valuation Profiles

It will sometimes be useful to restrict the set of valuation functions to a certain finite domain $R_{m,k}$ for an integer parameter $k \geq m$. Specifically, we define:

$$R_{m,k} = \left\{ u \in V_m | u(M) \subseteq \{0, \frac{1}{k}, \frac{2}{k}, \ldots, \frac{k-1}{k}, 1\} \right\}$$

where $u(M)$ denotes the image of u. Given a valuation function $u \in R_{m,k}$, we define its *alternation number* $a(u)$ as

$$a(u) = \#\{j \in \{0, \ldots, k-1\} | [\frac{j}{k} \in u(M)] \oplus [\frac{j+1}{k} \in u(M)]\},$$

where $\oplus$ denotes exclusive-or. That is, the alternation number of u is the number of indices j for which exactly one of j/k and $(j+1)/k$ is in the image of u. Since $k \geq m$ and $\{0, 1\} \subseteq u(M)$, we have that the alternation number of u is at least 2. We shall be interested in the class of valuation functions $C_{m,k}$ with minimal alternation number. Specifically, we define:

$$C_{m,k} = \{u \in R_{m,k} | a(u) = 2\}$$

and shall refer to such valuation functions as *quasi-combinatorial valuation functions*. Informally, the quasi-combinatorial valuation functions with sufficiently small k, have all valuations as close to 0 or 1 as possible.

The following lemma will be very useful in later sections. It formalizes the intuition that for ordinal mechanisms, the worst approximation ratio is achieved in extreme profiles, when all valuations are either very close to 1 or very close to 0.

Lemma 4. *Let $J \in \mathrm{Mech}_{m,n}$ be ordinal and neutral. Then*

$$\mathrm{ratio}(J) = \liminf_{k \to \infty} \min_{\mathbf{u} \in (C_{m,k})^n} \frac{E[\sum_{i=1}^n u_i(J(\mathbf{u}))]}{\sum_{i=1}^n u_i(1)}.$$

The idea of the proof is that starting from a valuation profile, we can iductively "shift" blocks of valuations towards 0 or 1, without increasing the approximation ratio and this way transform the profile into a quasi-combinatorial profile. This is possible because since the mechanism is ordinal, this "shift" leaves the outcome unchanged. See the full version for the full proof.

3 Mechanisms and Negative Results for the Case of Many Candidates

We can now analyze the approximation ratio of the mechanism $J \in \mathrm{Mech}^{\mathbf{OU}}_{m,n}$ that with probability 3/4 elects a uniformly random candidate and with probability 1/4 uniformly at random picks a voter and elects a candidate uniformly at random from the set of his $\lfloor m^{1/2} \rfloor$ most preferred candidates.

Theorem 3. *Let* $n \geq 2, m \geq 3$. *Let* $J = \frac{3}{4}U_{m,n}^{m} + \frac{1}{4}U_{m,n}^{\lfloor m^{1/2} \rfloor}$. *Then,* $\text{ratio}(J) \geq 0.37m^{-3/4}$.

Proof. For a valuation profile $\mathbf{u} = (u_i)$, we define $g(\mathbf{u}) = \frac{E[\sum_{i=1}^{n} u_i(J(\mathbf{u}))]}{\sum_{i=1}^{n} u_i(1)}$. By Lemma 4, since J is ordinal, it is enough to bound from below $g(\mathbf{u})$ for all $\mathbf{u} \in (C_{m,k})^n$ with $k \geq 1000(nm)^2$. Let $\epsilon = 1/k$. Let $\delta = m\epsilon$. Note that all functions of $\mathbf{u}$ map each alternative either to a valuation smaller than δ or a valuation larger than $1 - \delta$.

Since each voter assigns valuation 1 to at least one candidate, and since J with probability 3/4 picks a candidate uniformly at random from the set of all candidates, we have $E[\sum_{i=1}^{n} u_i(J(\mathbf{u}))] \geq 3n/(4m)$. Suppose $\sum_{i=1}^{n} u_i(1) \leq 2m^{-1/4}n$. Then $g(\mathbf{u}) \geq \frac{3}{8}m^{-3/4}$, and we are done. So we shall assume from now on that

$$\sum_{i=1}^{n} u_i(1) > 2m^{-1/4}n. \tag{1}$$

Obviously, $\sum_{i=1}^{n} u_i(1) \leq n$. Since J with probability 3/4 picks a candidate uniformly at random from the set of all candidates, we have $E[\sum_{i=1}^{n} u_i(J(\mathbf{u}))] \geq \frac{3}{4m} \sum_{i,j} u_i(j)$. So if $\sum_{i,j} u_i(j) \geq \frac{1}{2}nm^{1/4}$, we have $g(\mathbf{u}) \geq \frac{3}{8}m^{-3/4}$, and we are done. So we shall assume from now on that

$$\sum_{i,j} u_i(j) < \frac{1}{2}nm^{1/4}. \tag{2}$$

Still looking at the fixed quasi-combinatorial $\mathbf{u}$, let a voter i be called *generous* if his $\lfloor m^{1/2} \rfloor + 1$ most preferred candidates are all assigned valuation greater than $1 - \delta$. Also, let a voter i be called *friendly* if he has candidate 1 among his $\lfloor m^{1/2} \rfloor$ most preferred candidates. Note that if a voter is neither generous nor friendly, he assigns to candidate 1 valuation at most δ. This means that the total contribution to $\sum_{i=1}^{n} u_i(1)$ from such voters is less than $n\delta < 0.001/m$. Therefore, by equation (1), the union of friendly and generous voters must be a set of size at least $1.99m^{-1/4}n$.

If we let g denote the number of generous voters, we have $\sum_{i,j} u_i(j) \geq gm^{1/2}(1-\delta) \geq 0.999gm^{1/2}$, so by equation (2), we have that $0.999gm^{1/2} < \frac{1}{2}nm^{1/4}$. In particular $g < 0.51m^{-1/4}n$. So since the union of friendly and generous voters must be a set of size at least a $1.99m^{-1/4}n$ voters, we conclude that there are at least $1.48m^{-1/4}n$ friendly voters, i.e. the friendly voters is at least a $1.48m^{-1/4}$ fraction of the set of all voters. But this ensures that $U_{m,n}^{\lfloor m^{1/2} \rfloor}$ elects candidate 1 with probability at least $1.48m^{-1/4}/m^{1/2} \geq 1.48m^{-3/4}$. Then, J elects candidate 1 with probability at least $0.37m^{-3/4}$ which means that $g(\mathbf{u}) \geq 0.37m^{-3/4}$, as desired. This completes the proof. □

We next state our negative result. We show that any convex combination of (not necessarily ordinal) unilateral and duple mechanisms performs poorly. The proof is omitted due to lack of space, but can be found in the full version of the paper.

Theorem 4. *Let* $m \geq 20$ *and let* $n = m - 1 + g$ *where* $g = \lfloor m^{2/3} \rfloor$. *For any mechanism* J *that is a convex combination of unilateral and duple mechanisms in* $\mathrm{Mech}_{m,n}$, *we have* $\mathrm{ratio}(J) \leq 5m^{-2/3}$.

Corollary 2. *For all* m, *and all sufficiently large* n *compared to* m, *any mechanism* J *in* $\mathrm{Mech}^{\mathbf{O}}_{m,n} \cup \mathrm{Mech}^{\mathbf{U}}_{m,n}$ *has approximation ratio* $O(m^{-2/3})$.

Proof. Combine Theorem 1, Lemma 3 and Theorem 4. □

As followup work to the present paper, in a working manuscript, Lee [14] states a lower bound of $\Omega(m^{-2/3})$ that closes the gap between our upper and lower bounds. The mechanism achieving this bound is a convex combination of random-favorite and the mixed unilateral mechanism that uniformly at random elects one of the $m^{1/3}$ most preferred candidates of a uniformly chosen voter. The main question that we would like to answer is how well one can do with (general) cardinal mechanisms. The next theorem provides a weak upper bound.

Theorem 5. *All mechanisms* $J \in \mathrm{Mech}_{m,n}$ *with* $m, n \geq 3$ *have* $\mathrm{ratio}(J) < 0.94$.

Recall that in the definition of valuation functions u_i, we required u_i to be injective, i.e. no ties are not allowed. This requirement is made primarily for convenience, and all results of this paper can also be proved for the setting with ties. In contrast, allowing ties seems crucial for the proof of the follwing theorem, yielding a much stronger upper bound on the approximation ratio of any truthful mechanism than Theorem 5:

Theorem 6. *Any voting mechanism in the settting with ties, for* m *alternatives and* n *agents with* $m \geq n^{\lfloor \sqrt{n} \rfloor + 2}$, *has approximation ratio* $O(\log\log m / \log m)$.

The proof uses a black-box reduction from the *one-sided matchings problem* for which Filos-Ratsikas et al. [8] proved a $O(1/\sqrt{n})$ upper bound. Ideally, we want all negative result to hold for the setting without ties and the positive ones to hold for the setting with ties. We leave a "no-ties" version of Theorem 6 for future work.

4 Mechanisms and Negative Results for the Case of Three Candidates

In this section, we consider the special case of three candidates $m = 3$. To improve readability, we shall denote the three candidates by A, B and C, rather than by 1,2 and 3. When the number of candidates m as well as the number of voters n are small constants, the exact values of $r^{\mathbf{O}}_{m,n}$ and $r^{\mathbf{OU}}_{m,n}$ can be determined. We describe a general method for how to exactly and mechanically compute $r^{\mathbf{O}}_{m,n}$ and $r^{\mathbf{OU}}_{m,n}$ and the associated optimal mechanisms for small values of m and n. The key is to apply *Yao's principle* [21] and view the construction of a randomized mechanism as devising a strategy for Player I in a two-player zero-sum game G played between Player I, the mechanism designer, who picks a mechanism J and

Player II, the adversary, who picks an input profile $\mathbf{u}$ for the mechanism, i.e., an element of $(V_m)^n$. The payoff to Player I is the approximation ratio of J on $\mathbf{u}$. Then, the value of G is exactly the approximation ratio of the best possible randomized mechanism. In order to apply the principle, the computation of the value of G has to be tractable. In our case, Theorem 2 allows us to reduce the strategy set of Player I to be finite while Lemma 4 allows us to reduce the strategy set of Player II to be finite. This makes the game into a matrix game, which can be solved to optimality using linear programming. The details follow.

For fixed m, n and $k > 2m$, recall that the set of quasi-combinatorial valuation functions $C_{m,k}$ is the set of valuation functions u for which there is a j so that $\Im(u) = \{0, \frac{1}{k}, \frac{2}{k}, \ldots, \frac{m-j-1}{k}\} \cup \{\frac{k-j+1}{k}, \frac{k-j+2}{k}, \ldots, \frac{k-1}{k}, 1\}$. Note that a quasi-combinatorial valuation function u is fully described by the value of k, together with a partition of M into two sets M_0 and M_1, with M_0 being those candidates close to 0 and M_1 being those sets close to 1 together with a ranking of the candidates (i.e., a total ordering $<$ on M), so that all elements of M_1 are greater than all elements of M_0 in this ordering. Let the *type* of a quasi-combinatorial valuation function be the partition and the total ordering $(M_0, M_1, <)$. Then, a quasi-combinatorial valuation function is given by its type and the value of k. For instance, if $m = 3$, one possible type is $(\{B\}, \{A, C\}, \{B < A < C\})$, and the quasi-combinatorial valuation function u corresponding to this type for $k = 1000$ is $u(A) = 0.999$, $u(B) = 0$, $u(C) = 1$. We see that for any fixed value of m, there is a finite set T_m of possible types. In particular, we have $|T_3| = 12$. Let $\eta : T_m \times \mathbb{N} \to C_{m,k}$ be the map that maps a type and an integer k into the corresponding quasi-combinatorial valuation function.

For fixed m, n, consider the following matrices G and H. The matrix G has a row for each of the mechanisms $U^q_{m,n}$ for $q = 1, \ldots, m$, while the matrix H has a row for each of the mechanisms $U^q_{m,n}$ for $q = 1, \ldots, m$ as well as for each of the mechanisms $D^q_{m,n}$, for $q = \lfloor n/2 \rfloor + 1, \lfloor n/2 \rfloor + 2, \ldots, n$. Both matrices have a column for each element of $(T_m)^n$. The entries of the matrices are as follows: Each entry is indexed by a mechanism $J \in \text{Mech}_{m,n}$ (the row index) and by a type profile $\mathbf{t} \in (T_m)^n$ (the column index). We let that entry be

$$c_{J,\mathbf{t}} = \lim_{k \to \infty} \frac{E[\sum_{i=1}^n u_i(J(\mathbf{u}^k))]}{\max_{j \in M} \sum_{i=1}^n u_i^k(j)},$$

where $u_i^k = \eta(t_i, k)$. Informally, we let the entry be the approximation ratio of the mechanism on the quasi-combinatorial profile of the type profile indicated in the column and with $1/k$ being "infinitisimally small". Note that for the mechanisms at hand, despite the fact that the entries are defined as a limit, it is straightforward to compute the entries symbolically, and they are rational numbers. We now have the following lemma.

Lemma 5. *The value of G, viewed as a matrix game with the row player being the maximizer, is equal to $r^{\mathbf{OU}}_{m,n}$. The value of H is equal to $r^{\mathbf{O}}_{m,n}$. Also, the optimal strategies for the row players in the two matrices, viewed as convex combinations of the mechanisms corresponding to the rows, achieve those ratios.*

When applying Lemma 5 for concrete values of m, n, one can take advantage of the fact that all mechanisms corresponding to rows are anonymous and neutral. This means that two different columns will have identical entries if they correspond to two type profiles that can be obtained from one another by permuting voters and/or candidates. This makes it possible to reduce the number of columns drastically. After such reduction, we have applied the theorem to $m = 3$ and $n = 2, 3, 4$ and 5, computing the corresponding optimal approximation ratios and optimal mechanisms. We leave the details for the full version and instead now turn our attention to the case of three candidates and arbitrarily many voters. In particular, we shall be interested in $r_3^{\mathbf{O}} = \liminf_{n\to\infty} r_{3,n}^{\mathbf{O}}$ and $r_3^{\mathbf{OU}} = \liminf_{n\to\infty} r_{3,n}^{\mathbf{OU}}$. By Lemma 3, we in fact have $r_3^{\mathbf{O}} = \lim_{n\to\infty} r_{3,n}^{\mathbf{O}}$ and $r_3^{\mathbf{OU}} = \lim_{n\to\infty} r_{3,n}^{\mathbf{OU}}$. We present a family of ordinal and mixed-unilateral mechanisms J_n with ratio$(J_n) > 0.610$. In particular, $r_3^{\mathbf{OU}} > 0.610$. The coefficents c_1 and c_2 were found by trial-and-error; we present more information about how in the full version.

Theorem 7. *Let* $c_1 = \frac{77066611}{157737759} \approx 0.489$ *and* $c_2 = \frac{80671148}{157737759} \approx 0.511$. *Let* $J_n = c_1 \cdot U_{m,n}^1 + c_2 \cdot U_{m,n}^2$. *For all* n, *we have* ratio$(J_n) > 0.610$.

Proof. By Lemma 4, we have that

$$\text{ratio}(J_n) = \liminf_{k\to\infty} \min_{\mathbf{u}\in(C_{3,k})^n} \frac{E[\sum_{i=1}^n u_i(J_n(\mathbf{u}))]}{\sum_{i=1}^n u_i(A)}.$$

Recall the definition of the set of *types* T_3 of quasi-combinatorial valuation functions on three candidates and the definintion of η preceding the proof of Lemma 5. From that discussion, we have $\liminf_{k\to\infty} \min_{\mathbf{u}\in(C_{m,k})^n} \frac{E[\sum_{i=1}^n u_i(J_n(\mathbf{u}))]}{\sum_{i=1}^n u_i(A)} = \min_{\mathbf{t}\in(T_3)^n} \liminf_{k\to\infty} \frac{E[\sum_{i=1}^n u_i(J_n(\mathbf{u}))]}{\sum_{i=1}^n u_i(A)}$, where $u_i = \eta(t_i, k)$. Recall that $|T_3| = 12$. Since J_n is anonymous, to determine the approximation ratio of J_n on $\mathbf{u} \in (C_{m,k})^n$, we observe that we only need to know the value of k and the *fraction* of voters of each of the possible 12 types. In particular, fixing a type profile $\mathbf{t} \in (C_{m,k})^n$, for each type $k \in T_3$, let x_k be the fraction of voters in $\mathbf{u}$ of type k. For convenience of notation, we identify T_3 with $\{1, 2, \ldots, 12\}$ using the scheme depicted in Table 1. Let $w_j = \lim_{k\to\infty} \sum_{i=1}^n u_i(i)$, where $u_i = \eta(t_i, k)$, and let $p_j = \lim_{k\to\infty} \Pr[E_j]$, where E_j is the event that candidate j is elected by J_n in an election with valuation profile $\mathbf{u}$ where $u_i = \eta(t_i, k)$. We then have $\liminf_{k\to\infty} \frac{E[\sum_{i=1}^n u_i(J_n(\mathbf{u}))]}{\sum_{i=1}^n u_i(A)} = (p_A \cdot w_A + p_B \cdot w_B + p_C \cdot w_C)/w_A$. Also, from Table 1 and the definition of J_n, we see:

$$\begin{aligned}
w_A &= n(x_1 + x_2 + x_3 + x_4 + x_5 + x_9) \\
w_B &= n(x_1 + x_5 + x_6 + x_7 + x_8 + x_{11}) \\
w_C &= n(x_4 + x_7 + x_9 + x_{10} + x_{11} + x_{12}) \\
p_A &= (c_1 + c_2/2)(x_1 + x_2 + x_3 + x_4) + (c_2/2)(x_5 + x_6 + x_9 + x_{10}) \\
p_B &= (c_1 + c_2/2)(x_5 + x_6 + x_7 + x_8) + (c_2/2)(x_1 + x_2 + x_{11} + x_{12}) \\
p_C &= (c_1 + c_2/2)(x_9 + x_{10} + x_{11} + x_{12}) + (c_2/2)(x_3 + x_4 + x_7 + x_8)
\end{aligned}$$

Table 1. Variables for types of quasi-combinatorial valuation functions with ϵ denoting $1/k$

Candidate/Variable	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}
A	1	1	1	1	$1-\epsilon$	ϵ	0	0	$1-\epsilon$	ϵ	0	0
B	$1-\epsilon$	ϵ	0	0	1	1	1	1	0	0	$1-\epsilon$	ϵ
C	0	0	ϵ	$1-\epsilon$	0	0	$1-\epsilon$	ϵ	1	1	1	1

Thus we can establish that $\text{ratio}(J_n) > 0.610$ for all n, by showing that the *quadratic program "Minimize* $(p_A \cdot w_A + p_B \cdot w_B + p_C \cdot w_C) - 0.610 w_A$ *subject to* $x_1 + x_2 + \cdots + x_{12} = 1, x_1, x_2, \ldots, x_{12} \geq 0$", where $w_A, w_B, w_C, p_A, p_B, p_C$ have been replaced with the above formulae using the variables x_i, has a strictly positive minimum (note that the parameter n appears as a multiplicative constant in the objective function and can be removed, so there is only one program, not one for each n). This was established rigorously by solving the program symbolically in Maple by a facet enumeration approach (the program being non-convex), which is easily feasible for quadratic programs of this relatively small size. □

We next present a family of ordinal mechanisms J'_n with $\text{ratio}(J'_n) > 0.616$. In particular, $r_3^{\mathbf{O}} > 0.616$. The proof idea is the same as in the proof of Theorem 7 althought the existence of duple mechanisms requires some additional care. The details are left for the full version.

Theorem 8. *Let* $c'_1 = 0.476, c'_2 = 0.467$ *and* $d = 0.057$ *and let* $J_n = c'_1 \cdot U^1_{3,n} + c'_2 U^2_{3,n} + d \cdot D^{\lfloor n/2 \rfloor + 1}_{m,n}$. *Then* $\text{ratio}(J_n) > 0.616$ *for all* n.

We next show that $r_3^{\mathbf{OU}} \leq 0.611$ and $r_4^{\mathbf{O}} \leq 0.641$. By Lemma 3, it is enough to show that $r_{3,n^*}^{\mathbf{OU}} \leq 0.611$ and $r_{3,n^*}^{\mathbf{O}} \leq 0.641$ for some fixed n^*. Therefore, the statements follow from the following theorem. We leave the proof for the full version.

Theorem 9. $r_{3,23000}^{\mathbf{OU}} \leq \frac{32093343}{52579253} < 0.611$ *and* $r_{3,23000}^{\mathbf{O}} \leq \frac{41}{64} < 0.641$.

We finally show that $r_3^{\mathbf{U}}$ is between 0.660 and 0.750. The upper bound follows from the following proposition and Lemma 3.

Proposition 1. $r_{3,2}^{\mathbf{U}} \leq 0.75$.

The lower bound follows from an analysis of the *quadratic-lottery* [6, 9]. The main reason that we focus on this particular cardinal mechanism is given by the following lemma. The proof is a simple modification of the proof of Lemma 4 and is omitted here.

Lemma 6. *Let* $J \in \text{Mech}_{3,n}$ *be a convex combination of* Q_n *and any ordinal and neutral mechanism. Then*

$$\text{ratio}(J) = \liminf_{k \to \infty} \min_{\mathbf{u} \in (C_{m,k})^n} \frac{E[\sum_{i=1}^n u_i(J(\mathbf{u}))]}{\sum_{i=1}^n u_i(1)}.$$

Theorem 10. *The limit of the approximation ratio of Q_n as n approaches infinity, is exactly the golden ratio, i.e.,* $(\sqrt{5}-1)/2 \approx 0.618$*. Also, let J_n be the mechanism for n voters that selects* random-favorite *with probability* 29/100 *and* quadratic-lottery *with probability* 71/100*. Then,* $\text{ratio}(J_n) > \frac{33}{50} = 0.660$.

Proof. (sketch) Lemma 6 allows us to proceed completely as in the proof of Theorem 7, by constructing and solving appropriate quadratic programs. As the proof is a straightforward adaptation, we leave out the details. □

Mechanism J_n of Theorem 10 achieves an approximation ratio strictly better than 0.641. In other words, the best truthful cardinal mechanism for three candidates strictly outperforms all ordinal ones.

5 Conclusion

By the statement of Lee [14], mixed-unilateral mechanisms are asymptotically no better than ordinal mechanisms. Can a cardinal mechanism which is not mixed-unilateral beat this approximation barrier? Getting upper bounds on the performance of general cardinal mechanisms is impaired by the lack of a characterization of cardinal mechanisms a la Gibbard's. Can we adapt the proof of Theorem 6 to work in the general setting without ties? For the case of $m = 3$, can we close the gaps for ordinal mechanisms and for mixed-unilateral mechanisms? How well can cardinal mechanisms do for $m = 3$? Theorem 5 holds for $m = 3$ as well, but perhaps we could prove a tighter upper bound for cardinal mechanisms in this case.

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