Matching Dynamics with Constraints*

Martin Hoefer[1] and Lisa Wagner[2]

[1] Max-Planck-Institut für Informatik and Saarland University, Germany
mhoefer@mpi-inf.mpg.de
[2] Dept. of Computer Science, RWTH Aachen University, Germany
lwagner@cs.rwth-aachen.de

Abstract. We study uncoordinated matching markets with additional local constraints that capture, e.g., restricted information, visibility, or externalities in markets. Each agent is a node in a fixed matching network and strives to be matched to another agent. Each agent has a complete preference list over all other agents it can be matched with. However, depending on the constraints and the current state of the game, not all possible partners are available for matching at all times.

For correlated preferences, we propose and study a general class of hedonic coalition formation games that we call coalition formation games with constraints. This class includes and extends many recently studied variants of stable matching, such as locally stable matching, socially stable matching, or friendship matching. Perhaps surprisingly, we show that all these variants are encompassed in a class of "consistent" instances that always allow a polynomial improvement sequence to a stable state. In addition, we show that for consistent instances there always exists a polynomial sequence to every reachable state. Our characterization is tight in the sense that we provide exponential lower bounds when each of the requirements for consistency is violated.

We also analyze matching with uncorrelated preferences, where we obtain a larger variety of results. While socially stable matching always allows a polynomial sequence to a stable state, for other classes different additional assumptions are sufficient to guarantee the same results. For the problem of reaching a *given* stable state, we show NP-hardness in almost all considered classes of matching games.

1 Introduction

Matching problems are at the basis of many important assignment and allocation tasks in computer science, operations research, and economics. A classic approach in all these areas is *stable matching*, as it captures distributed control and rationality of participants that arise in many assignment markets. In the standard two-sided variant, there is a set of men and a set of women. Each man (woman) has a preference list over all women (men) and strives to be matched to one woman (man). A (partial) matching M has a blocking pair (m, w) if both m and w prefer each other to their current partner in M (if any). A matching

* Supported by DFG Cluster of Excellence MMCI and grant Ho 3831/3-1.

T.-Y. Liu et al. (Eds.): WINE 2014, LNCS 8877, pp. 161–174, 2014.

M is stable if it has no blocking pair. A large variety of allocation problems in markets can be analyzed using variants and extensions of stable matching, e.g., the assignment of jobs to workers, organs to patients, students to dormitory rooms, buyers to sellers, etc. In addition, stable matching problems arise in the study of distributed resource allocation problems in networks.

In this paper, we study uncoordinated matching markets with dynamic matching constraints. An underlying assumption in the vast majority of works on stable matching is that matching possibilities are always available – deviations of agents are only restricted by their preferences. In contrast, many assignment markets in reality are subject to additional (dynamic) constraints in terms of information, visibility, or externalities that prohibit the formation of certain matches (in certain states). Agents might have restricted information about the population and learn about other agents only dynamically through a matching process. For example, in scientific publishing we would not expect any person to be able to write a joint paper with a possible collaborator instantaneously. Instead, agents first have to get to know about each other to engage in a cooperation. Alternatively, agents might have full information but exhibit externalities that restrict the possibility to form certain matches. For example, an agent might be more reluctant to accept a proposal from the current partner of a close friend knowing that this would leave the friend unmatched.

Recent work has started to formalize some of these intuitions in generalized matching models with dynamic restrictions. For example, the lack of information motivates *socially* [5] or *locally stable matching* [4], externalities between agents have been addressed in *friendship matching* [3]. On a formal level, these are matching models where the definition of blocking pair is restricted beyond the condition of mutual improvement and satisfies additional constraints depending on the current matching M (expressing visibility/externalities/...). Consequently, the resulting stable states are supersets of stable matchings. Our main interest in this paper are convergence properties of dynamics that evolve from iterative resolution of such restricted blocking pairs. Can a stable state be reached from every initial state? Can we reach it in a polynomial number of steps? Will randomized dynamics converge (with probability 1 and/or in expected polynomial time)? Is it possible to obtain a particular stable state from an initial state (quickly)? These questions are prominent also in the economics literature (for a small survey see below) and provide valuable insights under which conditions stable matchings will evolve (quickly) in uncoordinated markets. Also, they highlight interesting structural and algorithmic aspects of matching markets.

Perhaps surprisingly, there is a unified approach to study these questions in all the above mentioned scenarios (and additional ones) via a novel class of coalition formation games with constraints. In these games, the coalitions available for deviation in a state are specified by the interplay of generation and domination rules. We provide a tight characterization of the rules that allow to show polynomial-time convergence results. They encompass all above mentioned matching models and additional ones proposed in this paper. In addition, we provide lower bounds in each model.

Contribution and Outline. A formal definition of stable matching games, socially stable, locally stable, and friendship matching can be found in Section 1.1. In addition, we describe a novel scenario that we term *considerate matching*.

In Section 2 we concentrate on stable matching with correlated preferences, in which each matched pair generates a single number that represents the utility of the match to both agents. Blocking pair dynamics in stable matching with correlated preferences give rise to a lexicographical potential function [1, 2]. In Section 2.1 we present a general approach on *coalition formation games with constraints*. These games are hedonic coalition formation games, where deviating coalitions are characterized by sets of generation and domination rules. We concentrate on classes of rules that we term *consistent*. For correlated preferences all matching scenarios introduced in Section 1.1 can be formulated as coalition formation games with constraints and consistent rules. For games with consistent rules we show that from every initial coalition structure a stable state can be reached by a sequence of polynomially many iterative deviations. This shows that for every initial state there is always *some* stable state that can be reached efficiently. In other words, there are polynomial "paths to stability" for all consistent games. Consistency relies on three structural assumptions, and we show that if either one of them is relaxed, the result breaks down and exponentially many deviations become necessary. This also implies that in consistent games random dynamics converge with probability 1 in the limit. While it is easy to observe convergence in expected polynomial time for socially stable matching, such a result is impossible for all consistent games due to exponential lower bounds for locally stable matching. The question for considerate and friendship matching remains an interesting open problem.

In Section 2.2 we study the same question for a given initial state and a *given stable state*. We first show that if there is a sequence leading to a given stable state, then there is also another sequence to that state with polynomial length. Hence, there is a polynomial-size certificate to decide if a given (stable) state can be reached from an initial state or not. Consequently, this problem is in NP for consistent games. We also provide a generic reduction in Section 2.2 to show that it is NP-complete for all, socially stable, locally stable, considerate, and friendship matching, even with strict correlated preferences in the two-sided case. Our reduction also works for traditional two-sided stable matching with either correlated preferences and ties, or strict (non-correlated) preferences.

In Section 3 we study general preferences with incomplete lists and ties that are not necessarily correlated. We show that for socially and classes of considerate and friendship matching we can construct for every initial state a polynomial sequence of deviations to a stable state. Known results for locally stable matching show that such a result cannot hold for all consistent games.

Related Work. For a general introduction to stable matching and variants of the model we refer to textbooks in the area [26]. Over the last decade, there has been significant interest in dynamics, especially in economics, but usually there is no consideration of worst-case convergence times or computational complexity. While the literature is too broad to survey here, a few directly related works are as follows. If agents iteratively resolve blocking pairs in the two-sided stable marriage problem, dynamics can cycle [25]. On the other hand, there is always a "path to stability", i.e., a sequence of (polynomially many) resolutions converging to a stable matching [28]. If blocking pairs are chosen uniformly at random at each step, the worst-case convergence time is exponential. In the case of weighted or correlated matching, however, random dynamics converge in expected polynomial time [2,27]. More recently, several works studied convergence time of random dynamics using combinatorial properties of preferences [20], or the probabilities of reaching certain stable matchings via random dynamics [8].

In the roommates problem, where every pair of players is allowed to match, stable matchings can be absent, but deciding existence can be done in polynomial time [23]. If there exists a stable matching, there are also paths to stability [13]. Similar results hold for more general concepts like P-stable matchings that always exist [21]. Ergodic sets of the underlying Markov chain have been studied [22] and related to random dynamics [24]. Alternatively, several works have studied the computation of (variants of) stable matchings using iterative entry dynamics [7,9,10], or in scenarios with payments or profit sharing [3,6,18].

Locally stable matching was introduced by [4] in a two-sided job-market model, in which links exist only among one partition. More recently, we studied locally stable matching with correlated preferences in the roommates case [16], and with strict preferences in the two-sided case [19]. For correlated preferences, we can always reach a locally stable matching using polynomially many resolutions of local blocking pairs. The expected convergence time of random dynamics, however, is exponential. For strict non-correlated preferences, no converging sequence might exist, and existence becomes NP-hard to decide. Even if they exist, the shortest sequence might require an exponential number of steps. These convergence properties improve drastically if agents have random memory.

Friendship and other-regarding preferences in stable matching games have been addressed by [3] in a model with pairwise externalities. They study existence of friendship matchings and bound prices of anarchy and stability in correlated games as well as games with unequal sharing of matching benefits. In friendship matching, agents strive to maximize a weighted linear combination of all agent benefits. In addition, we here propose and study considerate matching based on a friendship graph, where no agent accepts a deviation that deteriorates a friend. Such ordinal externalities have been considered before in the context of resource selection games [17].

Our general model of coalition formation games with constraints is related to hedonic coalition formation games [11, 12, 14]. A prominent question in the literature is the existence and computational complexity of stable states (for details and references see, e.g., a recent survey [29]).

1.1 Preliminaries

A *matching game* consists of a graph $G = (V, E)$ where V is a set of vertices representing *agents* and $E \subseteq \{\{u, v\} \mid u, v \in V, u \neq v\}$ defines the *potential matching edges*. A *state* is a matching $M \subseteq E$ such that for each $v \in V$ we have $|\{e \mid e \in M, v \in e\}| \leq 1$. An edge $e = \{u, v\} \in M$ provides *utilities* $b_u(e), b_v(e) > 0$ for u and v, respectively. If for every $e \in E$ we have some $b_u(e) = b_v(e) = b(e) > 0$, we speak of *correlated preferences*. If no explicit values are given, we will assume that each agent has an order $\succeq$ over its possible matching partners, and for every agent the utility of matching edges is given according to this ranking. In this case we speak of *general preferences*. Note that in general, the ranking is allowed to be an incomplete list or to have ties. We define $B(M, u)$ to be $b_u(e)$ if $u \in e \in M$ and 0 otherwise. A *blocking pair* for matching M is a pair of agents $\{u, v\} \notin M$ such that each agent u and v is either unmatched or strictly prefers the other over its current partner (if any). A *stable matching* M is a matching without blocking pair.

Unless otherwise stated, we consider the *roommates case* without assumptions on the topology of G. In contrast, the *two-sided* or *bipartite* case is often referred to as the *stable marriage problem*. Here V is divided into two disjoint sets U and W such that $E \subseteq \{\{u, w\} | \ u \in U, w \in W\}$. Further we will consider matchings when each agent can match with up to k different agents at the same time.

In this paper, we consider broad classes of matching games, in which additional constraints restrict the set of available blocking pairs. Let us outline a number of examples that fall into this class and will be of special interest.

Socially Stable Matching. In addition to the graph G, there is a (social) *network of links* (V, L) which models static visibility. A state M has a *social blocking pair* $e = \{u, v\} \in E$ if e is blocking pair and $e \in L$. Thus, in a social blocking pair both agents can strictly increase their utility by generating e (and possibly dismissing some other edge thereby). A state M that has no social blocking pair is a *socially stable matching*. A *social improvement step* is the resolution of such a social blocking pair, that is, the blocking pair is added to M and all conflicting edges are removed.

Locally Stable Matching. In addition to G, there is a network (V, L) that models dynamic visibility by taking the current matching into account. To describe stability, we assume the pair $\{u, v\}$ is *accessible* in state M if u and v have hop-distance at most 2 in the graph $(V, L \cup M)$, that is, the shortest path between u and v in $(V, L \cup M)$ is of length at most 2 (where we define the shortest path to be of length ∞, if u and v are not in the same connected component). A state M has a *local blocking pair* $e = \{u, v\} \in E$ if e is blocking pair and u and v are accessible. Consequently, a *locally stable matching* is a matching without local blocking pair. A *local improvement step* is the resolution of such a local blocking pair, that is, the blocking pair is added to M and all conflicting edges are removed.

Considerate Matching. In this case, the (social) network (V, L) symbolizes friendships and consideration. We assume the pair $\{u, v'\}$ is *not accessible* in state M if there is agent v such that $\{u, v\} \in M$, and (a) $\{u, v\} \in L$ or (b) $\{v, v'\} \in L$. Otherwise, the pair is called accessible in M. Intuitively, this implies a form of consideration – formation of $\{u, v'\}$ would leave a friend v unmatched, so (a) u will not propose to v' or (b) v' will not accept u's proposal. A state

M has a *considerate blocking pair* $e = \{u, v\} \in E$ if e is blocking pair and it is accessible. A state M that has no considerate blocking pair is a *considerate stable matching*. A *considerate improvement step* is the resolution of such a considerate blocking pair.

Friendship Matching. In this scenario, there are numerical values $\alpha_{u,v} \geq 0$ for every unordered pair $u, v \in V$, $u \neq v$, representing how much u and v care for each other's well-being. Thus, instead of the utility gain through its direct matching partner, u now receives a *perceived utility* $B_p(M, u) = B(M, u) + \sum_{v \in V \setminus \{u\}} \alpha_{u,v} B(M, v)$. In contrast to all other examples listed above, this definition requires cardinal matching utilities and cannot be applied directly on ordinal preferences. A state M has a *perceived blocking pair* $e = \{u, v\} \in E$ if $B_p(M, u) < B_p((M \setminus \{e' \mid e \cap e' \neq \emptyset\}) \cup \{e\}, u)$ and $B_p(M, v) < B_p((M \setminus \{e' \mid e \cap e' \neq \emptyset\}) \cup \{e\}, v)$. A state M that has no perceived blocking pair is a *perceived* or *friendship stable matching*. A *perceived improvement step* is the resolution of such a perceived blocking pair.

2 Correlated Preferences

2.1 Coalition Formation Games with Constraints

In this section, we consider correlated matching where agent preferences are correlated via edge benefits $b(e)$. In fact, we will prove our results for a straightforward generalization of correlated matching – in correlated coalition formation games that involve coalitions of size larger than 2. In such a *coalition formation game* there is a set N of agents, and a set $\mathcal{C} \subseteq 2^N$ of hyper-edges, the *possible coalitions*. We denote $n = |N|$ and $m = |\mathcal{C}|$. A *state* is a *coalition structure* $\mathcal{S} \subseteq \mathcal{C}$ such that for each $v \in N$ we have $|\{C \mid C \in \mathcal{S}, v \in C\}| \leq 1$. That is, each agent is involved in at most one coalition. Each coalition C has a weight or benefit $w(C) > 0$, which is the profit given to each agent $v \in C$. For a coalition structure $\mathcal{S}$, a *blocking coalition* is a coalition $C \in \mathcal{C} \setminus \mathcal{S}$ with $w(C) > w(C_v)$ where $v \in C_v \in \mathcal{S}$ for every $v \in C$ which is part of a coalition in $\mathcal{S}$. Again, the resolution of such a blocking coalition is called an improvement step. A stable state or *stable coalition structure* $\mathcal{S}$ does not have any blocking coalitions. Correlated matching games are a special case of coalition formation games where $\mathcal{C}$ is restricted to pairs of agents and thereby defines the edge set E.

To embed the classes of matching games detailed above into a more general framework, we define *coalition formation games with constraints*. For each state $\mathcal{S}$ we consider two sets of rules – *generation rules* that determine candidate coalitions, and *domination rules* that forbid some of the candidate coalitions. The set of undominated candidate coalitions then forms the blocking coalitions for state $\mathcal{S}$. Using suitable generation and domination rules, this allows to describe socially, locally, considerate and friendship matching in this scenario.

More formally, there is a set $T \subseteq \{(\mathcal{T}, C) \mid \mathcal{T} \subset \mathcal{C}, C \in \mathcal{C}\}$ of *generation rules*. If in the current state $\mathcal{S}$ we have $\mathcal{T} \subseteq \mathcal{S}$ and $C \notin \mathcal{S}$, then C becomes a candidate coalition. For convenience, we exclude generation rules of the form $(\emptyset, C)$ from T

and capture these rules by a set $\mathcal{C}_g \subseteq \mathcal{C}$ of self-generating coalitions. A coalition $C \in \mathcal{C}_g$ is a candidate coalition for all states $\mathcal{S}$ with $C \notin \mathcal{S}$. In addition, there is a set $D \subseteq \{(\mathcal{T}, C) \mid \mathcal{T} \subset \mathcal{C}, C \in \mathcal{C}\}$ of *domination rules*. If $\mathcal{T} \subseteq \mathcal{S}$ for the current state $\mathcal{S}$, then C must not be inserted. To capture the underlying preferences of the agents, we assume that D always includes at least the set $D_w = \{(\{C_1\}, C_2) \mid w(C_1) \geq w(C_2), C_1 \cap C_2 \neq \emptyset, C_1 \neq C_2\}$ of all weight domination rules.

The undominated candidate coalitions represent the blocking coalitions for $\mathcal{S}$. In particular, the latter assumption on D implies that a blocking coalition must at least yield strictly better profit for every involved agent. Note that in an improvement step, one of these coalitions is inserted, and every coalition that is dominated in the resulting state is removed. By assumption on D, we remove at least every overlapping coalition with smaller weight. A coalition structure is stable if the set of blocking coalitions is empty.

Note that we could also define coalition formation games with constraints for general preferences. Then $D_w = \{(\{C_1\}, C_2) \mid C_1 \cap C_2 \neq \emptyset, C_1 \neq C_2, \exists v \in C_1 : w_v(C_1) \geq w_v(C_2)\}$. However, a crucial point in our proofs is that in a chain of succeeding deletions no coalition can appear twice. This is guaranteed for correlated preferences as coalitions can only be deleted by more worthy ones. For general preferences on the other hand there is no such argument.

In the following we define *consistency* for generation and for domination rules. This encompasses many classes of matching cases described above and is key for reaching stable states (quickly).

Definition 1. *The generation rules of a coalition formation game with constraints are called* consistent *if $T \subseteq \{(\{C_1\}, C_2) \mid C_1 \cap C_2 \neq \emptyset\}$, that is, all generation rules have only a single coalition in their precondition and the candidate coalition shares at least one agent.*

Definition 2. *The domination rules of a coalition formation game with constraints are called* consistent *if $D \subseteq \{(\mathcal{S}, C) \mid \mathcal{S} \subset \mathcal{C}, C \in \mathcal{C}, C \notin \mathcal{S}, \exists S \in \mathcal{S} : S \cap C \neq \emptyset\}$, that is, at least one of the coalitions in $\mathcal{S}$ overlaps with the dominated coalition. Note that weight domination rules are always consistent.*

Theorem 1. *In every correlated coalition formation game with constraints and consistent generation and domination rules, for every initial structure $\mathcal{S}$ there is a sequence of polynomially many improvement steps that results in a stable coalition structure. The sequence can be computed in polynomial time.*

Proof. At first we analyze the consequences of consistency in generation and domination rules. For generation rules we demand that there is only a single precondition coalition and that this coalition overlaps with the candidate coalition. Thus if we apply such a generation rule we essentially replace the precondition with the candidate. The agents in the intersection of the two coalitions would not approve such a resolution if they would not improve. Therefore, the only applicable generation rules are those where the precondition is of smaller value than the candidate.

Now for domination rules we allow an arbitrary number of coalitions in the precondition, but at least one of them has to overlap with the dominated coalition. In consequence a larger set of coalitions might dominate a non-existing coalition, but to remove a coalition they can only use the rules in D_w. That is due to the fact that when a coalition C already exists, the overlapping coalition of the precondition cannot exist at the same time. But this coalition can only be created if C does not dominate it. Especially C has to be less worthy than the precondition. Thus the overlapping precondition alone can dominate C via weight.

The proof is inspired by the idea of the edge movement graph [15]. Given a coalition formation game with consistent constraints and some initial coalition structure $\mathcal{S}_0$, we define an object movement hypergraph

$$G_{mov} = (V, V_g, T_{mov}, D_{mov}).$$

A coalition structure corresponds to a marking of the vertices in G_{mov}. The vertex set is $V = \{v_C \mid C \in \mathcal{C}\}$, and $V_g = \{v_C \mid C \in \mathcal{C}_g\}$ the set of vertices which can generate a marking by themselves. The directed exchange edges are $T_{mov} = \{(v_{C_1}, v_{C_2}) \mid (\{C_1\}, C_2) \in E, w(C_1) < w(C_2)\}$. The directed domination hyperedges are given by $D_{mov} = D_1 \cup D_w$, where $D_1 = \{(\{v_S \mid S \in \mathcal{S}\}, v_C) \mid (\mathcal{S}, C) \in D\}$. This covers the rule that a newly inserted coalition must represent a strict improvement for all involved agents. The initial structure is represented by a marking of the vertices $V_0 = \{v_C \mid C \in \mathcal{S}_0\}$.

We represent improvement steps by adding, deleting, and moving markings over exchange edges to undominated vertices of the object movement graph. Suppose we are given a state $\mathcal{S}$ and assume we have a marking at every v_C with $C \in \mathcal{S}$. We call a vertex v in G_{mov} currently *undominated* if for every hyperedge $(U, v) \in D_{mov}$ at least one vertex in U is currently unmarked. An improvement step that inserts coalition C is represented by marking v_C. For this v_C must be unmarked and undominated. We can create a new marking if $v_C \in V_g$. Otherwise, we must move a marking along an exchange edge to v_C. Note that this maps the generation rules correctly as we have seen, that we exchange the precondition for the candidate. To implement the resulting deletion of conflicting coalitions from the current state, we delete markings at all vertices which are now dominated through a rule in D_{mov}. That is, we delete markings at all vertices v with $(U, v) \in D$ and every vertex in U marked.

Observe that T_{mov} forms a DAG as the generation of the candidate coalition deletes its overlapping precondition coalition and thus the rule will only be applied if the candidate coalition yields strictly more profit for every agent in the coalition.

Lemma 1. *The transformation of markings in the object movement graph correctly mirrors the improvement dynamics in the coalition formation game with constraints.*

Proof. Let $\mathcal{S}$ be a state of the coalition formation game and let C be a blocking coalition for $\mathcal{S}$. Then C can be generated either by itself (that is, $C \in \mathcal{C}_g$) or through some generation rule with fulfilled precondition $C' \in \mathcal{S}$, and is not dominated by any subset of $\mathcal{S}$ via D. Hence, for the set of marked vertices $V_{\mathcal{S}} = \{v_C \mid C \in \mathcal{S}\}$ it holds that v_C can be generated either because $v_C \in V_g$ or because there is a marking on some $v_{C'}$ with $(v_{C'}, v_C) \in T_{mov}$, and is further not dominated via D. Hence, we can generate a marking on v_C. It is straightforward to verify that if v_C gets marked, then in the resulting deletion step only domination rules of the form $\{(\{v_S\}, v_T) \mid S, T \in \mathcal{C}, S \cap T \neq \emptyset$ and $w(S) \geq w(T)\}$ are relevant. Thus, deletion of markings is based only on overlap with the newly inserted coalition C. These are exactly the coalitions we lose when inserting C in $\mathcal{S}$.

Conversely, let $V_{\mathcal{S}}$ be a set of marked vertices of G_{mov} such that $\mathcal{S} = \{C \mid v_C \in V_{\mathcal{S}}\}$ does not violate any domination rule (i.e., for every $(\mathcal{U}, C) \in D$, we have $\mathcal{U} \not\subseteq \mathcal{S}$ or $C \notin \mathcal{S}$). Then $\mathcal{S}$ is a feasible coalition structure. Now if v_C is an unmarked vertex in G_{mov}, then $C \notin \mathcal{S}$. Furthermore, assume v_C is undominated and can be marked, because $v_C \in V_g$ or because some marking can be moved to v_C via an edge in T_{mov}. Thus for every $\{\mathcal{S}_C, C\} \in D$ v_C being undominated implies $\mathcal{S}_C \not\subset \mathcal{S}$. The property that v_C can be marked implies that C is self-generating or can be formed from $\mathcal{S}$ using a generation rule. Hence C is a blocking coalition in $\mathcal{S}$. The insertion C again causes the deletion of exactly the coalitions whose markings get deleted when v_C is marked. □

To show the existence of a short sequence of improvement steps we consider two phases.

Phase 1. In each round we check whether there is an exchange edge from a marked vertex to an undominated one. If this is the case, we move the marking along the exchange edge and start the next round. Otherwise for each unmarked, undominated $v \in V_g$ we compute the set of reachable positions. This can be done by doing a forward search along the exchange edges that lead to an unmarked undominated vertex. Note that the vertex has to remain undominated when there are the existing markings and a marking on the source of the exchange edge. If we find a reachable position that dominates an existing marking, we create a marking at the associated $v \in V_g$ and move it along the exchange edges to the dominating position. Then we start the next round. If we cannot find a reachable position which dominates an existing marking, we switch to Phase 2.

Phase 2. Again we compute all reachable positions from $v \in V_g$. We iteratively find a reachable vertex v_C with highest weight $w(C)$, generate a marking at the corresponding $v \in V_g$ and move it along the path of exchange edges to v_C. We repeat this phase until no reachable vertex remains.

To prove termination and bound the length, we consider each phase separately. In Phase 1 in each round we replace an existing marking by a marking of higher value either by using an exchange edge or by deleting it through domination by weight. Further the remaining markings either stay untouched or get deleted. Now the number of improvements that can be made per marking is limited by m and the number of markings is limited by n. Hence, there can be at most

mn rounds in Phase 1. Additionally, the number of steps we need per round is limited m again, as we move the marking along the DAG structure of exchange edges. Thus, phase 1 generates a total of $O(n \cdot m^2)$ steps.

If in Phase 1 we cannot come up with an improvement, there is no way to (re)move the existing markings, no matter which other steps are made in subsequent iterations. This relies on the fact that the presence of additional markings can only restrict the subgraph of reachable positions. For the same reason, iteratively generating the reachable marking of highest weight results in markings that cannot be deleted in subsequent steps. Thus, at the end of every iteration in Phase 3, the number of markings is increased by one, and all markings are un(re)movable. Consequently, in Phase 2 there are $O(m \cdot n)$ steps.

For computation of the sequence, the relevant tasks are constructing the graph G_{mov}, checking edges in T_{mov} for possible improvement of markings, or constructing subgraphs and checking connectivity of single vertices to V_g. Obviously, all these tasks can be executed in time polynomial in n, m, $|T|$ and $|D|$ using standard algorithmic techniques. □

Next, we want to analyze whether consistency of generation and domination rules is necessary for the existence of short sequences or can be further relaxed.

Proposition 1. *If the generation rules contain more than one coalition in the precondition-set, there are instances and initial states such that every sequence to a stable state requires an exponential number of improvement steps.*

The proof uses a coalition formation game with constraints and inconsistent generation rules obtained from locally stable matching, when agents are allowed to match with partners at a hop distance of at most $\ell = 3$ in $(V, L \cup M)$. For this setting in [16, Theorem 3] we have given an instance such that every sequence to a stable state requires an exponential number of improvement steps. Note that the example is minimal in the sense that now we have at most 2 coalitions in the precondition-set. The detailed proof can be found in the full version.

Proposition 2. *If the generation rules have non-overlapping precondition- and target-coalitions, there are instances and initial states such that every sequence to a stable state requires an exponential number of improvement steps.*

The construction used for the proof exploits the fact that if precondition- and target-coalition do not overlap the precondition can remain when the target-coalition is formed. Then the dynamics require additional steps to clean up the leftover precondition-coalitions which results in an exponential blow-up. The entire proof as well as a sketch of the resulting movement graph can be found in the full version.

Proposition 3. *If the domination rules include target-coalitions that do not overlap with any coalition in the precondition, there are instances and starting states such that every sequence cycles.*

Consistent generation and domination rules arise in a large variety of settings, not only in basic matching games but also in some interesting extensions.

Corollary 1. *Consistent generation and domination rules are present in*

- *locally stable matching if agents can create* $k = 1$ *matching edges and have lookahead* $\ell = 2$ *in* $G = (V, M \cup L)$.
- *socially stable matching, even if agents can create* $k \geq 1$ *matching edges.*
- *considerate matching, even if agents can create* $k \geq 1$ *matching edges.*
- *friendship matching, even if agents can create* $k \geq 1$ *matching edges.*

Due to space restrictions we cannot give a detailed description of the embedding into coalition formation games with constraints. In most cases the embedding is quite straightforward. Agents and edge set are kept as well as the benefits. The generation and domination rules often follow directly from the definitions. The exact embedding for every type of game can be found in the full version. Additionally an exemplar proof for correctness is stated.

Unlike for the other cases, for locally stable matching we cannot guarantee consistent generation rules if we increase the number of matching edges. The same holds for lookahead > 2. In both cases the accessibility of an edge might depend on more than one matching edge. There are exponential lower bounds in [16, 19] for those extensions which proves that it is impossible to find an embedding with consistent rules even with the help of auxiliary constructions.

2.2 Reaching a Given Matching

In this section we consider the problem of deciding reachability of a *given* stable matching from a given initial state. We first show that for correlated coalition formation games with constraints and consistent rules, this problem is in NP. If we can reach it and there exists a sequence, then there always exists a polynomial-size certificate due to the following result.

Theorem 2. *In a correlated coalition formation game with constraints and consistent generation and domination rules, for every coalition structure* $\mathcal{S}^*$ *that is reachable from an initial state* $\mathcal{S}_0$ *through a sequence of improvement steps, there is also a sequence of polynomially many improvement steps from* $\mathcal{S}_0$ *to* $\mathcal{S}^*$.

For the proof we analyze an arbitrary sequence of improvement steps from $\mathcal{S}_0$ to $\mathcal{S}^*$ and show that, if the sequence is too long, there are unnecessary steps, that is, coalitions are created and deleted without making a difference for the final outcome. By identifying and removing those superfluous steps we can reduce every sequence to one of polynomial length. The detailed proof can be found in the full version.

For locally stable matching, the problem of reaching a given locally stable matching from a given initial matching is known to be NP-complete [19]. Here we provide a generic reduction that shows NP-completeness for socially, locally, considerate, and friendship matching, even in the two-sided case. Surprisingly, it also applies to ordinary two-sided stable matching games that have either correlated preferences with ties, or non-correlated strict preferences. Observe that the problem is trivially solvable for ordinary stable matching and correlated preferences without ties, as in this case there is a unique stable matching that can always be reached using the greedy construction algorithm [2].

Theorem 3. *It is* NP*-complete to decide if for a given matching game, initial matching* M_0 *and stable matching* M^**, there is a sequence of improvement steps leading form* M_0 *to* M^**. This holds even for bipartite games with strict correlated preferences in the case of*

1. *socially stable matching and locally stable matching,*
2. *considerate matching, and*
3. *friendship matching for symmetric* α*-values in* $[0, 1]$*.*

In addition, it holds for ordinary bipartite stable matching in the case of

4. *correlated preferences with ties,*
5. *strict preferences.*

3 General Preferences

In this section we consider convergence to stable matchings in the two-sided case with general preferences that may be incomplete and have ties. For locally stable matching it is known that in this case there are instances and initial states such that no locally stable matching can be reached using local blocking pair resolutions. Moreover, deciding the existence of a converging sequence of resolutions is NP-hard [19].

We here study the problem for socially, considerate, and friendship matching. Our positive results are based on the following procedure from [2] that is known to construct a sequence of polynomial length for unconstrained stable matching. The only modification of the algorithm for the respective scenarios is to resolve "social", "considerate" or "perceived blocking pairs" in both phases.

Phase 1. Iteratively resolve only blocking pairs involving a matched vertex $w \in W$. Phase 1 ends when for all blocking pairs $\{u, w\}$ we have $w \in W$ unmatched.

Phase 2. Choose an unmatched $w \in W$ that is involved in a blocking pair. Resolve one of the blocking pairs $\{u, w\}$ that is most preferred by w. Repeat until there are no blocking pairs.

It is rather straightforward to see that the algorithm can be applied directly to build a sequence for socially stable matching.

Theorem 4. *In every bipartite instance of socially stable matching* $G = (V = U \dot{\cup} W, E)$ *with general preference lists and social network* L*, for every initial matching* M_0 *there is a sequence of polynomially many improvement steps that results in a socially stable matching. The sequence can be computed in polynomial time.*

For extended settings the algorithm still works for somewhat restricted social networks. For considerate matching we assume that the link set is only within $L \subseteq (U \times U) \cup (U \times W)$, i.e., no links within partition W.

Theorem 5. *In every bipartite instance of considerate matching $G = (V = U \dot{\cup} W, E)$ with general preference lists and social network L such that $\{w, w'\} \notin L$ for all $w, w' \in W$, for every initial matching M_0 there is a sequence of polynomially many improvement steps that results in a considerate matching. The sequence can be computed in polynomial time.*

We also apply the algorithm to friendship matching in case there can be arbitrary friendship relations $\alpha_{u,u'}, \alpha_{u',u} \geq 0$ for each pair $u, u' \in U$. Here we allow asymmetry with $\alpha_{u,u'} \neq \alpha_{u',u}$. Otherwise, for all $u \in U, w, w' \in W$ we assume that $\alpha_{u,w} = \alpha_{w,u} = \alpha_{w,w'} = 0$, i.e., friendship only exists within U.

Theorem 6. *In every bipartite instance of friendship matching $G = (V = U \dot{\cup} W, E)$ with benefits b and friendship values α such that $\alpha_{u,u'} > 0$ only for $u, u' \in U$, for every initial matching M_0 there is a sequence of polynomially many improvement steps that results in a friendship matching. The sequence can be computed in polynomial time.*

The algorithm works fine with links between partitions U and W for the considerate setting, but it fails for positive α between partitions in the friendship case. We defer a discussion to the full version of the paper.

Acknowledgment. Part of this research was done at the Institute for Mathematical Sciences of NUS Singapore, and at NTU Singapore. The authors thank Edith Elkind for suggesting to study considerate matching.

References

1. Abraham, D., Levavi, A., Manlove, D., O'Malley, G.: The stable roommates problem with globally ranked pairs. Internet Math. 5(4), 493–515 (2008)
2. Ackermann, H., Goldberg, P., Mirrokni, V., Röglin, H., Vöcking, B.: Uncoordinated two-sided matching markets. SIAM J. Comput. 40(1), 92–106 (2011)
3. Anshelevich, E., Bhardwaj, O., Hoefer, M.: Friendship and stable matching. In: Bodlaender, H.L., Italiano, G.F. (eds.) ESA 2013. LNCS, vol. 8125, pp. 49–60. Springer, Heidelberg (2013)
4. Arcaute, E., Vassilvitskii, S.: Social networks and stable matchings in the job market. In: Leonardi, S. (ed.) WINE 2009. LNCS, vol. 5929, pp. 220–231. Springer, Heidelberg (2009)
5. Askalidis, G., Immorlica, N., Kwanashie, A., Manlove, D.F., Pountourakis, E.: Socially stable matchings in the hospitals/Residents problem. In: Dehne, F., Solis-Oba, R., Sack, J.-R. (eds.) WADS 2013. LNCS, vol. 8037, pp. 85–96. Springer, Heidelberg (2013)
6. Biró, P., Bomhoff, M., Golovach, P.A., Kern, W., Paulusma, D.: Solutions for the stable roommates problem with payments. Theoret. Comput. Sci. 540, 53–61 (2014)

7. Biró, P., Cechlárová, K., Fleiner, T.: The dynamics of stable matchings and half-matchings for the stable marriage and roommates problems. Int. J. Game Theory 36(3-4), 333–352 (2008)
8. Biró, P., Norman, G.: Analysis of stochastic matching markets. Int. J. Game Theory 42(4), 1021–1040 (2013)
9. Blum, Y., Roth, A., Rothblum, U.: Vacancy chains and equilibration in senior-level labor markets. J. Econom. Theory 76, 362–411 (1997)
10. Blum, Y., Rothblum, U.: "Timing is everything" and martial bliss. J. Econom. Theory 103, 429–442 (2002)
11. Bogomolnaia, A., Jackson, M.: The stability of hedonic coalition structures. Games Econom. Behav. 38, 201–230 (2002)
12. Cechlárova, K.: Stable partition problem. In: Encyclopedia of Algorithms (2008)
13. Diamantoudi, E., Miyagawa, E., Xue, L.: Random paths to stability in the roommates problem. Games Econom. Behav. 48(1), 18–28 (2004)
14. Hajduková, J.: Coalition formation games: A survey. Intl. Game Theory Rev. 8(4), 613–641 (2006)
15. Hoefer, M.: Local matching dynamics in social networks. In: Aceto, L., Henzinger, M., Sgall, J. (eds.) ICALP 2011, Part II. LNCS, vol. 6756, pp. 113–124. Springer, Heidelberg (2011)
16. Hoefer, M.: Local matching dynamics in social networks. Inf. Comput. 222, 20–35 (2013)
17. Hoefer, M., Penn, M., Polukarov, M., Skopalik, A., Vöcking, B.: Considerate equilibrium. In: Proc. 22nd Intl. Joint Conf. Artif. Intell. (IJCAI), pp. 234–239 (2011)
18. Hoefer, M., Wagner, L.: Designing profit shares in matching and coalition formation games. In: Chen, Y., Immorlica, N. (eds.) WINE 2013. LNCS, vol. 8289, pp. 249–262. Springer, Heidelberg (2013)
19. Hoefer, M., Wagner, L.: Locally stable marriage with strict preferences. In: Fomin, F.V., Freivalds, R., Kwiatkowska, M., Peleg, D. (eds.) ICALP 2013, Part II. LNCS, vol. 7966, pp. 620–631. Springer, Heidelberg (2013)
20. Hoffman, M., Moeller, D., Paturi, R.: Jealousy graphs: Structure and complexity of decentralized stable matching. In: Chen, Y., Immorlica, N. (eds.) WINE 2013. LNCS, vol. 8289, pp. 263–276. Springer, Heidelberg (2013)
21. Inarra, E., Larrea, C., Moris, E.: Random paths to P-stability in the roommates problem. Int. J. Game Theory 36(3-4), 461–471 (2008)
22. Inarra, E., Larrea, C., Moris, E.: The stability of the roommate problem revisited. Core Discussion Paper 2010/7 (2010)
23. Irving, R.: An efficient algorithm for the "stable roommates" problem. J. Algorithms 6(4), 577–595 (1985)
24. Klaus, B., Klijn, F., Walzl, M.: Stochastic stability for roommate markets. J. Econom. Theory 145, 2218–2240 (2010)
25. Knuth, D.: Marriages stables et leurs relations avec d'autres problemes combinatoires. Les Presses de l'Université de Montréal (1976)
26. Manlove, D.: Algorithmics of Matching Under Preferences. World Scientific (2013)
27. Mathieu, F.: Acyclic preference-based systems. In: Shen, X., Yu, H., Buford, J., Akon, M. (eds.) Handbook of Peer-to-Peer Networking. Springer (2010)
28. Roth, A., Vate, J.V.: Random paths to stability in two-sided matching. Econometrica 58(6), 1475–1480 (1990)
29. Woeginger, G.J.: Core stability in hedonic coalition formation. In: van Emde Boas, P., Groen, F.C.A., Italiano, G.F., Nawrocki, J., Sack, H. (eds.) SOFSEM 2013. LNCS, vol. 7741, pp. 33–50. Springer, Heidelberg (2013)