Value-Based Network Externalities and Optimal Auction Design

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Abstract. We study revenue maximization in settings where agents' valuations exhibit positive network externalities. In our model, items have unlimited supply, and agents are unit demand. In a departure from previous literature, we assume agents have *value based* externalities, meaning that their valuation depends not only on their own signal, but also on the signals of other agents in their neighborhood who win the item. We give a complete characterization of ex-post incentive compatible and individually rational auctions in this setting. Using this characterization, we show that the optimal auction is in fact deterministic, and can be computed in polynomial time when the agents' signals are independent. We further show a constant factor approximation when the signals of agents are correlated, and an optimal mechanism in this case for a constant number of bidders.

1 Introduction

There are many goods and services for which the utility of an individual consumer increases with the number of consumers using the same good or service. This phenomenon is called *positive externalities* in the economics literature. There have been extensive studies on various settings of positive externalities in both the economics and computer science communities. Most of the literature so far has focused on the *cardinality based* utility model given by Katz *et al.* [11], where the utility of an agent is of the form $r + v(y) - p$. Here r is the agent's intrinsic *type*, *i.e.*, her private information about the good. The quantity y is the number of agents using the good; v is an non-decreasing function that measures the externalities by the number of agents using the good; p is the price for the good. Such a model of externality is motivated by several factors:

– The physical effect of the number of buyers on the quality of the good. For example choosing a telephone network over other competing brands depends on the number of users each network has.

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– An indirect effect that gives rise to consumption externalities. For example the amount of software available in different operating systems is a function of the number of people using them.
– The availability and quality of post-purchase services depend on the size of the community using the good.

1.1 Value Based Externality Model

The cardinality based externality model discussed above assumes the externality leads to an additive increase in value depending on the number of users who obtain service. This implicitly assumes agents are identical in terms of how much they use the service. In several scenarios, the extent to which agents use the service is itself a function of their intrinsic value for the good. This motivates us to introduce the value based externality model. Agents are unit demand. Given the agents' intrinsic types $\{s_i\}$, suppose the set of agents winning the item is W. Then, the valuation for agent $i \in W$ is $v_i(s_i, W) = h_i(s_i) + \sum_{j \in W\ j \neq i} g_{ij}(s_j)$. On the other hand, for $i \notin W$, we have $v_i(s_i, W) = 0$. Since we consider positive externalities, we assume the functions h_i and g_{ij} are non-negative and non-decreasing.

As an example to illustrate the usefulness of this model, suppose agent i is deciding to adopt a social network. The agent's type is her signal s_i. This signal stands for how much she plans to use social networks. Furthermore, the agent receives externalities if her friend j also uses the same social network. The amount of externality received by i from j is determined by how much j plans to use the same network ($g_{ij}(s_j)$). Therefore, under the value based utility model, the agent's utility depends linearly on her friends' private information about how much they use the social network.

Note that the value based utility model naturally captures the *network externality* case, where the agents are located in a network $G(V, E)$, and receive externality only from neighbors in the network. Our valuation function is not only a generalization of cardinality based externality functions, but also of the *weighted sum values* model introduced in [15,12], generalized to the setting with network externalities.

1.2 Summary of Results

In this paper, we consider the Bayesian setting, where the agents' intrinsic types are assumed to be drawn from a known distribution. We assume there is unlimited supply of the item, and agents are unit demand. The goal of the auctioneer is to design an incentive compatible and individually rational mechanism that optimizes expected revenue, where the expectation is over the distribution of types, as well as the randomness introduced by the mechanism. Our solution concept will be ex-post, meaning that even when agents know the signals of the other agents, truthfulness and rationality hold in expectation over the randomness introduced by the mechanism. For formal definitions, see Section 2.

We present a characterization of ex-post (randomized) mechanisms in Section 3. Using this characterization, we show in Section 4 that when the intrinsic types of the agents are drawn from a regular product distribution, the optimal auction can be computed in polynomial time. Our characterization allows us to define a *virtual value* on each agent's private signal in a standard way. However, the winning set can have agents to have negative virtual values if they produce enough externalities. We show that despite this difficulty, we can modify the densest subgraph algorithm [5] in an interesting way to compute the minimal densest subgraph, and this allows us to design the optimal, polynomial time computable mechanism.

In Section 5 we show how to achieve near optimal ex post revenue when the distributions are correlated. Our mechanism takes the better of two deterministic mechanisms. One mechanism focuses on extracting revenue from the intrinsic value, the other one from the externalities. We show that the better of these two mechanisms produces at least $\frac{1}{4}$ of the optimal revenue.

We finally show an LP formulation whose size is polynomial in the size of the support of the joint distribution of the signal space. We round the solution to this LP to obtain an optimal mechanism when there a constant number of agents with correlated signals. More interestingly, this algorithm shows that the optimal ex-post incentive compatible and individually rational mechanism is always deterministic, *i.e.*, there is no gap between the revenues of the optimal deterministic and randomized mechanisms.

1.3 Related Work

The seminal work by Myerson [13] pioneered the study of optimal auctions in the *Bayesian* setting, where the bidders' values are assumed to be drawn from known distributions. In this setting, Myerson showed that any incentive compatible and rational mechanism satisfies monotonicity of the expected allocations of a bidder, and a relation between price and allocation. However with value-based externalities, a bidder's value also depends on her neighbor's private information. We therefore assume an agent's *type* (a.k.a. signal) is drawn from a known distribution, and her value is determined by her own signal, together with her neighbors' signals. The common approach when an agent's value depends not only on her own type but also on others' types is called the *interdependent values* setting. The key difference between the interdependent value model and the externalities model is that, in the interdependent value model, an agent can influence another agent even when she does not win the auction. However in the externalities model, an agent can influence another agent only if both of them win the items. Nevertheless our study is closely related to the literature of interdependent values, here we mention the most relevant works [15,12,6]. Roughgarden *et al.* [15] develop an analog to the Myerson's characterization for interdependent values under a matroid constraint. We show a similar characterization in the value based utility model. In terms of approximation algorithms, Li [12] showed a constant approximation for MHR distributions with interdependent values under a matroid constraint and Chawla et al. [6] generalized this result to arbitrary distributions building on a result due to Ronen [14]. We show a constant approximation to the externality model when the signals are correlated using the result in [6] as a subroutine.

Network externalities effects received much attention in recent years. These models generalize the classical model of Katz et al. [11], the value of player is her intrinsic value plus a function of the total number of winners. The work of [10,3,1,8,9] extends this model to the setting where agents are located in a network, and derive utility from the set of winners in their neighborhood. However, in all this work, the externality function only takes the identity of the winners into account, and does not take the types (signals) of the neighbors into account. In particular, the work of Haghpanah et al. [9] considered two interesting valuation functions which they call *concave externalities* and *step-function externalities.* In concave externalities, the value of an agent is a function of the set of the neighboring winners; in step-function externalities, the value of an agent is her own type if she and one of her neighbors win at the same time. Under their valuation model they studied the near optimal Bayesian incentive compatible mechanisms. We depart from this literature in considering the setting where the externality depends on the *signals* of the winning agents, and not just their identities. As mentioned before, with type-dependent externality, the valuation of an agent becomes multi-dimensional (depending on the types of other agents).

We finally note that in the ex-interim setting, the Bayesian optimal mechanism can be designed in polynomial time using the techniques in [4,2], even when the agents' signals are correlated. Similarly, for independent signals, it is not too hard to compute a constant approximation using techniques such as [10,9]; the hard part is to obtain an optimal auction.

2 Preliminaries

In this paper, we consider *unconstrained environments*, where the auctioneer can serve any subset of the agents simultaneously. As mentioned before, in our setting there are n unit-demand agents, and an unlimited supply of a homogeneous item.

Player types. An agent's type stands for all the private information about the good that is available to the agent. We denote bidder i's type by s_i. We call $s = (s_1, s_2, \ldots, s_n)$ the *signal profile* of all the bidders. And we denote (s'_i, s_{-i}) as the signal profile when we change bidder i's signal from s_i to s'_i and keep signals of all other bidders the same.

As in the optimal auction design literature, we assume that the signals s is drawn from a known distribution with probability density function (PDF) f. In Section 4 the distribution is a product distribution over the bidders, and is regular. In Section 5 we allow the distribution to be general and possibly correlated across bidders. We denote the marginal PDF and CDF of bidder i as $f_i(s_i|s_{-i})$ and $F_i(s_i|s_{-i})$ respectively, where we drop s_{-i} if the marginals are independent. For analytic convenience, we assume the type space is continuous unless otherwise stated; our results easily extend to discrete type spaces.

Player values. As mentioned in the introduction, given an unconstrained environment and a signal profile s, when the winning set is W the valuation for a winning agent $i \in W$ is $v_i(s_i, W) = h_i(s_i) + \sum_{j \in W \ j \neq i} g_{ij}(s_j)$. If i does not win, $i \notin W$, we have $v_i(s_i, W) = 0$. The functions h_i and g_{ij} are non-negative and non-decreasing.

Auction. An auction or a mechanism is specified by an allocation rule x and a payment rule p. We allow the mechanisms to be randomized, so that when the reported signal profile is s, we denote $x_i(s)$ the probability that agent i wins. Denote $p_i(s)$ as the payment of agent i. The utility of an agent is her value minus her payment. An auction is *deterministic* if $x_i(s)$ only takes value 0 or 1. Denote $y_{i,j}(s)$ the probability that both agent i and j win.

Solution Concepts. In this paper we focus on *ex post incentive compatible (IC)* and *ex post individual rational (IR)* auctions. There are three popular notions of equilibria. In this paper we focus on ex post IC and IR mechanisms. We denote W_1 the winning set when agent i tells the truth s_i, and W_2 is the winning set when she misreports s_i'. An auction is *Ex post IC* if for all agents i, reported signal profiles s,

$$x_i(s)v(s, W_1) - p_i(s) \geq x(s_i', s_{-i})v(s, W_2) - p_i(s_i', s_{-i})$$

Note that this solution concept is defined agnostic to the prior distribution, which may not even be common knowledge. Ex post individually rational (IR) means that the agents do not receive negative utility, so that the condition $x_i(s)v(s, W_1) - p_i(s) \geq 0$ always holds.

Optimal Auction Design. The total expected revenue of an auction is $E_s[\sum_i p_i(s)]$. The expectation is taken over all the possible signal profile s, according to the distribution with PDF f, as well as the randomness introduced by the mechanis. In this paper we focus on achieving optimal expected revenue when s is drawn from independent regular distributions and near optimal expected revenue when s is drawn from correlated distributions.

3 Characterization of Ex-Post Mechanisms

In this section we develop a characterization of ex post IC and IR mechanisms for the value based utility model. We note that this characterization holds for both independent and correlated distributions over the signal profile.

Theorem 1 (characterization). *A (possibly randomized) mechanism is ex Post IC IR if and only if it satisfies the following two conditions:*

1. *(**Monotonicity**) $x_i(s) \leq x_i(s')$ for all $s = (s_i, s_{-i})$ and $s' = (s_i', s_{-i})$ where $s_i < s_i'$;*

2. (**Payment Identity**) *Fixing the signals of the other agents, the expected payment of agent i is*

$$p_i(s) = h(s_i)x_i(s) - \int_0^{s_i} h'(z)(x_i(z, s_{-i}))dz + \sum_{j \neq i} g(s_j)y_{ij}(s)$$

Proof. We focus on agent i and drop the subscripts in $h_i(s_i)$ and $g_{ij}(s_j)$. Consider any ex-post IC and IR mechanism. The IC conditions imply:

$$h(s_i) \cdot x_i(s) + \sum_{j \neq i} g(s_j) \cdot y_{ij}(s) - p_i(s) \geq h(s_i) \cdot x_i(s') + \sum_{j \neq i} g(s_j) \cdot y_{ij}(s') - p_i(s')$$

$$h(s'_i) \cdot x_i(s') + \sum_{j \neq i} g(s_j) \cdot y_{ij}(s') - p_i(s') \geq h(s'_i) \cdot x_i(s) + \sum_{j \neq i} g(s_j) \cdot y_{ij}(s) - p_i(s)$$

From these two inequalities it is easy to derive that:

$$x_i(s)(h(s_i)) - h(s'_i)) \geq x_i(s')(h(s_i) - h(s'_i))$$

Therefore $x_i(s) \leq x_i(s')$ since $s_i \leq s'_i$, showing monotonicity.

The above two inequalities imply:

$$\begin{aligned} &\sum_{j \neq i}(y_{ij}(s') - y_{ij}(s))g(s_j) + (x_i(s') - x_i(s))h(s_i) \\ &\leq p_i(s') - p_i(s) \leq \sum_{j \neq i}(y_{ij}(s') - y_{ij}(s))g(s_j) + (x_i(s') - x_i(s))h(s'_i) \end{aligned}$$

Since we assumed the type space is continuous, we have:

$$\frac{dp_i(s)}{ds_i} = \frac{d \sum_{j \neq i} g(s_j)y_{ij}(s)}{ds_i} + h(s_i)\frac{dx_i(s)}{ds_i}$$

Taking the integral, we have:

$$\begin{aligned} p_i(s) &= p_i(0, s_{-i}) + \int_0^{s_i} g(s_j)\ d\sum_{j \neq i} y_{ij}(z, s_{-i}) + \int_0^{s_i} h(z)\ dx_i(z, s_{-i}) \\ &= h(s_i)x_i(s)) - \int_0^{s_i} h'(z)x_i(z, s_{-i}))dz + \sum_{j \neq i} g(s_j)y_{ij}(s) \end{aligned}$$

The second equality is true because we assume i pays 0 when she reports 0. This shows that the payment identity holds.

Suppose a mechanism satisfies monotonicity and the payment identity. Plugging in the payment identity into the ex post IC condition, we need to show:

$$x_i(s'_i, s_{-i})(h(s'_i) - h(s_i)) \geq \int_{s_i}^{s'_i} h'(z)x_i(z, s_{-i})dz$$

The monotonicity condition now implies the above inequality directly. This shows that any mechanism that satisfies monotonicity and the payment identity is ex-post IC and IR.

4 Optimal Mechanism for Independent, Regular Signals

Using the characterization developed so far, we design an optimal auction when the distributions of the agents' signals are independent and regular. We show that the payment identity implies we can perform optimization for each signal profile individually. For each signal profile, the resulting problem is a densest subgraph problem, which has an optimal solution computable in polynomial time [5]. However, in order to preserve monotonicity, we need a densest subgraph with a specific property, that we term the minimum densest subgraph. We show that even this solution is poly-time computable, which yields the desired mechanism.

Definition 1. *Let f_i and F_i denote the PDF and CDF of the distribution of s_i, which are assumed to be independent for different i. We define $\varphi_i(s_i) = h_i(s_i) - \frac{1-F_i(s_i)}{f_i(s_i)}h_i'(s_i)$. We say $\varphi_i(s_i)$ is the virtual value for agent i when she reports s_i. A distribution is said to be* regular *if when $s_i \geq s_i'$, then $\varphi_i(s_i) \geq \varphi_i(s_i')$.*

4.1 Revenue Expression

Lemma 1. *Fixing s_{-i}, the expected payment of agent i is $E_{s_i}[p_i(s)] = E_{s_i}[\varphi_i(s_i)x_i(s) + \sum_{i\neq j} y_{ij}(s)g_{ij}(s_j)]$.*

Proof. Since we fix s_{-i} we replace $x_i(s)$, $p_i(s)$ and $y_{ij}(s)$ by $x_i(s_i)$, $p_i(s_i)$ and $y_{ij}(s_i)$ in this proof. By Theorem 1,

$$\begin{aligned} E_{s_i}[p_i(s_i)] &= \int_{z=0}^{h} p_i(z)f(z)dz \\ &= \int_{z=0}^{h} (x_i(z)h_i(z) - \int_{b=0}^{z} h_i'(b)x(b)db + \sum_{i\neq j} y_{ij}(s_z)g_{ij}(s_j))f(z)dz \end{aligned}$$

By integration by parts and changing the order of integration of the second term:

$$\int_{z=0}^{h} x_i(z)h_i(z)f(z)dz - \int_{b=0}^{h} x_i(b)\int_{z=b}^{h} f(z)dzh_i'(b)db + \int_{z=0}^{h}\sum_{i\neq j} y_{ij}(s_z)g_{ij}(s_j)f(z)dz$$

Renaming the variables, we get:

$$\begin{aligned} E_{s_i}[p_i(s_i)] &= \int_{z=0}^{h} x_i(z)h_i(z)f(z)dz - \int_{z=0}^{h} x_i(z)[1-F(z)]h_i'(z)dz \\ &\quad + \int_{z=0}^{h}\sum_{i\neq j} y_{ij}(s_z)g_{ij}(s_j)f(z)dz \\ &= \int_{z=0}^{h} \varphi_i(s_i)x_i(z)f(z)dz + \int_{z=0}^{h}\sum_{i\neq j} y_{ij}(s_z)g_{ij}(s_j)f(z)dz \\ &= E_{s_i}[\varphi_i(s_i)x_i(s) + \sum_{i\neq j} y_{ij}(s)g_{ij}(s_j)] \end{aligned}$$

Corollary 1. *The expected revenue of an ex post IC and IR mechanism is* $E_s[Rev] = E_s[\sum_i^n \varphi_i(s_i)x_i(s) + \sum_{i\neq j}(g_{ij}(s_j) + g_{ji}(s_i))y_{ij}(s)]$

We call the sum in the expectation *virtual surplus.*

4.2 Linear Program

We call following linear program as the linear program for signal profile s.

$$\begin{aligned}
&\text{maximize} \sum_{i=1}^{n} \varphi_i(s_i)x_i + \sum_{i\neq j}(g_{ji}(s_i) + g_{ij}(s_j))y_{ij} && \\
&\text{subject to } x_i \geq y_{ij} && , i \neq j \\
&\qquad\qquad\quad x_j \geq y_{ij} && , i \neq j \\
&\qquad\qquad\quad 0 \leq x_i \leq 1 && , \forall i
\end{aligned}$$

Lemma 2. *The optimal value of the linear program for signal profile s is an upper bound on the optimal expected revenue from any ex post IC IR mechanism.*

Proof. Take any ex post IC IR mechanism and set the linear program variables according to the allocation rules of the mechanism. We can easily see the linear program constraints are satisfied. By Corollary 1 we have that objective of the linear program is the expected revenue of the mechanism. Therefore the optimal value of the linear program for signal profile s is an upper bound for the optimal expected revenue from any ex post IC IR mechanisms.

The linear program for signal profile s encodes a relaxation of the densest subgraph problem (see [5]), which we define below.

Densest Subgraph Problem. Let $G = (V, E)$ be an weighted undirected graph, and let $S = (V_S, E_S)$ be a subgraph of G. We define the density to be the sum of the weights induced by S. The densest subgraph problem asks to find the subgraph S which maximizes the density. Note that there are different definitions of density in the literature. In a LP relaxation of the densest subgraph problem, we have a variable x_i for vertex i, which is 1 if i belongs to the solution, and a variable y_{ij} for edge (i, j) if both i and j are in the solution. The constraints are exactly those in the LP written above. We can therefore view the LP for signal profile s as solving a densest subgraph problem on the agents.

Lemma 3. *[5] The linear program relaxation for densest subgraph problem has an optimal integral solution which can be found in polynomial time.*

Proof. Take an fractional optimal solution to the above linear program $\hat{x_i}, \hat{y_{ij}}$ with optimal value v. Choose a number $r \in [0, 1]$. We call $V_r = \{i : x_i \geq r\}$ and $E_r = \{(ij) : y_{ij} \geq r\}$. Round x_i for all $i \in V_r$ to 1, and set all other x_i to 0. It is easy to see that $y_{ij} = 1$ if and only if $(ij) \in E_r$, since $y_{ij} = \min\{x_i, x_j\}$.

Define function $f(r) = \sum_{i \in V_r} \varphi_i(s_i) + \sum_{(i,j) \in E_r} (g_{ji}(s_i) + g_{ij}(s_j))$. We have that $\int_{r=0}^{1} f(r)dr = \sum_{i=1}^{n} \varphi_i(s_i)\hat{x_i} + \sum_{i \neq j} (g_{ji}(s_i) + g_{ij}(s_j))\hat{y_{ij}} = v$. Therefore there must be at least one r such that $f(r) \geq v$. In other words the LP has an integral optimal solution. To find the right r one need only to look for all distinct V_r sets. Note that there are at most n such sets. For details please see [5].

Definition 2. *We call the winning set found by the integral optimal solution (with optimal value v) found in Lemma 3 as a* densest subgraph*. Among all densest subgraphs let V_r be the set that has the smallest cardinality. We say that $V_r(s)$ is the* minimum densest subgraph *with density v for signal profile s.*

Lemma 4. *The minimum densest subgraph can be computed in polynomial time.*

Proof. Let G denote the original problem instance. Denote d be any number that is smaller than the difference between the optimal solution to G, and the solution with next highest density. Add a term $-\epsilon \sum_{i=1}^{n} x_i$ to the objective function in the linear program, where $\epsilon = d/n^2$. Modify this instance to G' where we subtract ϵ from the weights on all the vertices. By our choice of ϵ, it is easy to see that the densest subgraphs for G and G' differs at most d/n. Therefore the densest subgraph for instance G' can only be selected from the integral solutions for G. The modified LP finds the optimal integral solution for G', hence it computes the minimum cardinality solution to instance G.

Lemma 5. *For any signal profile s and agent i, if agent i belong to the minimum densest subgraph $V_r(s)$, fixing s_{-i}, if agent i increases her signal to $s_i' > s_i$, and denote the new signal profile s'. Then in the new instance, $i \in V_r(s')$.*

Proof. First, the minimum densest subgraph $V_r(s)$ is unique for any signal profile s. Suppose not. Let there be $V_r \neq V_r'$, and denote the induced edge sets E_r and E_r'. Suppose that

$$\sum_{i \in V_r} \varphi_i(s_i) + \sum_{(i,j) \in E_r} (g_{ji}(s_i) + g_{ij}(s_j)) = \sum_{i \in V_r'} \varphi_i(s_i) + \sum_{(i,j) \in E_r'} (g_{ji}(s_i) + g_{ij}(s_j)) = v$$

It is easy to see that since $g_{ji}(s_i) \geq 0 \ \forall i, j$, we have

$$\sum_{i \in V_r \cup V_r'} \varphi_i(s_i) + \sum_{(i,j) \in E_r \cup E_r'} (g_{ji}(s_i) + g_{ij}(s_j)) > v$$

unless

$$\sum_{i \in V_r - V_r'} \varphi_i(s_i) + \sum_{(i,j) \in E_r - E_r'} (g_{ji}(s_i) + g_{ij}(s_j)) = 0$$

However it contradicts the fact that both V_r and V_r' are minimum densest subgraphs.

Next, denote $\varphi_j'(s_j)$ the new virtual value for agent j when i increased her signal. Note that $\varphi_j'(s_j) = \varphi_j(s_j) \ \forall j \neq i$ since the signals are independent, and $\varphi_i(s_i) \leq \varphi_i'(s_i)$ by regularity. Suppose $i \in V_r(s)$ but $i \notin V_r(s')$, we have that:

$$\begin{aligned}
&\sum_{j\in V_r(s')} \varphi'_j(s_j) + \sum_{(j,k)\in E_r(s')} (g_{kj}(s_j) + g_{jk}(s_k)) \\
\geq &\sum_{j\in V_r(s)} \varphi'_j(s_j) + \sum_{(j,k)\in E_r(s)} (g_{kj}(s_j) + g_{jk}(s_k)) \\
\geq &\sum_{j\in V_r(s)} \varphi_j(s_j) + \sum_{(j,k)\in E_r(s)} (g_{kj}(s_j) + g_{jk}(s_k)) \\
\geq &\sum_{j\in V_r(s')} \varphi_j(s_j) + \sum_{(j,k)\in E_r(s')} (g_{kj}(s_j) + g_{jk}(s_k)) \\
= &\sum_{j\in V_r(s')} \varphi'_j(s_j) + \sum_{(j,k)\in E_r(s')} (g_{kj}(s_j) + g_{jk}(s_k))
\end{aligned}$$

The last equality holds because by our assumption, $i \notin V_r(s')$ and the fact that $\varphi'_j(s_j) = \varphi_j(s_j)\ \forall j \neq i$. This implies $V_r(s)$ and $V_r(s')$ are two distinct minimum densest subgraphs for signal profile s. This contradicts the fact that the minimum densest subgraph is unique for any input s. This completes the proof.

Algorithm 1. Mechanism 1 for Independent Signals

1. Ask for the signal profile $s = (s_1, s_2, \ldots, s_n)$.
2. Compute the φ virtual values based on s.
3. Solve the linear program for signal profile s. Calculate the threshold signals(corollary 4.2).
4. Allocate according to the minimal densest subgraph $V_r(s)$, and make payments $p_i = v_i((s_i^*, s_{-i}), V_r(s))$ where s_i^* is the threshold signal and $V_r(s)$ is the minimum densest subgraph for signal profile s.

Our mechanism is illustrated in Mechanism 1. We now show several properties of this mechanism. First, as a corollary of Lemma 5, we have that if we maximize the objective in the above linear program by choosing the minimum densest subgraph for all signal profiles s, then if an agent i increases her signal, her allocation is non-decreasing. Therefore, the allocation rule in Mechanism 1 satisfies the monotonicity condition in Theorem 1.

Fixing the signal profile s_{-i}, to calculate the correct payment for agent i we have to compute the minimum signal s_i^* for agent i for her to be selected by the minimum densest subgraph. As a direct consequence of Lemma 5, we have the following corollary.

Claim. Fixing s_{-i} the threshold signal s_i^* for which agent i remains in the minimum densest subgraph can be computed in polynomial time.

Proof. By lemma 5 we can use binary search to find the minimum s_i^* so that agent i remains in the minimum densest subgraph in at most $O(\log h)$ linear programming computations where h is the maximum possible value for any signals.

Putting all this together, we have the following theorem. As a corollary, we also observe that the optimal mechanism is also deterministic.

Theorem 2. *Mechanism 1 is ex post IC and IR, polynomial time computable, and it achieves optimal expected revenue among all ex post mechanisms when the signal profile s is drawn from a regular product distribution.*

5 Mechanisms for Correlated Signals

In this section we present ex post mechanisms when the signals of the agents are correlated. For small type spaces, this mechanism is optimal, while it is an approximation algorithm for implicitly specified type spaces. We use the *conditional virtual values* as defined in [15].

Definition 3. *[15] For correlated signals, we define*

$$\varphi_i(s_i|s_{-i}) = h_i(s_i) - \frac{1 - F_i(s_i|s_{-i})}{f_i(s_i|s_{-i})} h_i'(s_i)$$

as the conditional virtual value for agent i when she reports s_i. As before, we call the sum of the conditional virtual values together with the externalities the virtual surplus*.*

Using Theorem 1, we have the following lemma, whose proof is identical to Lemma 1.

Lemma 6. *Fixing s_{-i} the expected payment of agent i is*

$$E_{(s_i|s_{-i})}[p_i(s|s_{-i})] = E_{(s_i|s_{-i})}[\varphi_i(s_i|s_{-i})x_i(s) + \sum_{j \neq i} y_{ij}(s) g_{ij}(s_j)]$$

Here, $x_i(s)$ is the probability that agent i gets the item when the signal profile is s; and $y_{ij}(s)$ is the corresponding probability that both *agents i and j get the item. Therefore, the expected revenue of an ex post IC and IR mechanism with correlated signals is*

$$E[Rev] = E[\sum_i^n \varphi_i(s_i|s_{-i})x_i(s) + \sum_{i \neq j} (g_{ij}(s_j) + g_{ji}(s_i)) y_{ij}(s)]$$

The approach in the previous section does not extend to correlated signals. Since when a winning agent increases her signal, the conditional virtual values for the other agents in the winning set also change. As a result, the agent can be rule out from the new optimal winning set. Nevertheless we can develop optimal and near optimal ex post IC IR mechanisms below.

Definition 4. *For signal profile s we define the auction $H(s)$ to be the auction in which each agent's value is simply $h_i(s_i)$. We denote $Rev(H(s))$ the optimal expected ex post revenue for auction $H(s)$. Let $G(s) = \sum_{i \neq j} (g_{ij}(s_j) + g_{ji}(s_i))$.*

5.1 Optimal Mechanism for Constant Number of Bidders

We now present an optimal mechanism that runs in polynomial time under the following assumptions. Bidder i's signal $s_i \in Q_i = \{0, 1, 2, \ldots, R_i\}$, and let $T = \times_{i=1}^{n} Q_i$ denote the joint type space, that has size $O(R^n)$, where $R = \max_i R_i$. We assume n is constant, so that $|T| = O(R^n)$ is poly-bounded. We assume $f(s) > 0$ for all $s \in T$. The definition of conditional virtual value (Def. 3) easily extends to such discrete spaces. Using the revenue formula, Lemma 6, and the monotonicity characterization Theorem 1, it is easy to check that the following integer program encodes the optimal mechanism.

$$\text{Maximize} \quad E_{s\in T}\left[\sum_i \varphi_i(s_i|s_{-i})x_i(s) + \sum_{(ij),i\neq j} y_{ij}(s)(g_{ij}(s_j) + g_{ji}(s_i))\right]$$

$$\begin{array}{rl} x_i(s) \geq y_{ij}(s) & \forall i, j, s \in T \\ x_j(s) \geq y_{ij}(s) & \forall i, j, s \in T \\ x_i(s_i, s_{-i}) \geq x_i(s_i', s_{-i}) & \forall i, s_{-i}, R \geq s_i \geq s_i' \geq 0 \\ x_i(s), y_{ij}(s) \in \{0, 1\} & \forall i, j, s \in T \end{array}$$

Here, the third set of constraints encodes monotonicity; the first and second set of constraints simply encodes that agents obtain externality from each other only if they both receive the item. We relax the final constraints so that the variables are real values in $[0, 1]$. Since the support of the signals, $|T|$, is polynomial size, the above LP has polynomial size and can be solved in polynomial time. Using the same technique as in the proof of Lemma 3, we choose an r uniformly at random in $[0, 1]$. For every variable, if its value is at least r, we set that variable to 1, else we set it to 0. It is easy to check that the resulting integer solution satisfies all the constraints, and its objective is the same as the LP in expectation. Hence, we have a valid mechanism that satisfies monotonicity, and whose expected revenue has the same as the LP solution. We omit the details, and note that with some technical work, this result also extends to arbitrary discrete type spaces with poly-bounded support.

As a consequence, since the above LP has an integral optimum solution, we have the following.

Theorem 3. *For the value based utility model, there is no gap between the expected revenue of the optimal deterministic and randomized mechanisms.*

5.2 Approximation Algorithm for the General Case

We now present an approximation algorithm when there are polynomially many bidders, and the type space of the signals is continuous. We use the VCG-L mechanism of Chawla *et al.* [6] as a subroutine.

Note that for items with unlimited supply, the $VCG - L$ auction simply offers each agent a posted price which is the conditional monopoly price for that agents' marginal distribution, given the signals of the other agents. For the sake of completeness, we state the result of Chawla *et al.* [6].

Algorithm 2. Mechanism 2 for Correlated Signals

1. Run the mechanism that yields higher expected revenue (over all signal profiles).
2. **Either** allocate to all agents and charge each agent i $\sum_{j \neq i} g_{ij}(s_j)$; **or**
3. Run the $VCG - L$ [6] auction for correlated private values on $H(s)$.

Theorem 4. *[6] The $VCG - L$ auction with conditional monopoly reserves obtains at least half of the optimal revenue under a matroid feasibility constraint when the agent have correlated private values.*

Lemma 7. *The optimal expected ex post revenue for correlated signal profile s is at most $Rev(H(s)) + G(s)$.*

Proof. For any signal profile s, let M be any ex post IC IR mechanism for valuations which externalities. Denote M's allocation rule and payment rule by (x, p). We can construct a mechanism for auction $H(s)$ by keeping the allocation rule of M exactly the same and let the winning set be W. Subtract the externalities from the payments of all winning agents in M, that is the new payment for any winning agent i in $H(s)$ is $p'_i = p_i - \sum_{j \neq i,\ j \in W} g_{i,j}(s_j)$. We call the new mechanism M'. By Theorem 1, $p'_i = h(s^*_i) \leq h(s_i)$. Therefore M' is ex post IR for $H(s)$. On the other hand it is also easy to see M' is ex post IC for $H(s)$ since M' satisfies the two conditions in Theorem 1 for the auction $H(s)$ with zero externality. We denote the expected payment from M' by $E_s[\sum_i p'_i(s)]$ and the expected payment from M by $E_s[\sum_i p_i(s)]$. Therefore by definition of $Rev(H(s))$,we have $E_s[\sum_i p_i(s)] - G(s) \leq E_s[\sum_i p_i(s)] - \sum_{(i,j), j \neq i, i,j \in W}(g_{i,j}(s_j) + g_{j,i}(s_i)) = E_s[\sum_i p'_i(s)] \leq Rev(H(s))$ since $G(s)$ is an upper bound on the amount of externality subtracted. Therefore we have $E_s[\sum_i p_i(s)] - G(s) \leq Rev(H(s))$.

Theorem 5. *Mechanism 2 is ex post IC IR, and a 4 approximation to the expectation of the optimal ex post revenue.*

Proof. We first observe that mechanism 2 is ex post IC and IR. It takes the better of two deterministic ex post IC IR mechanisms. By Lemma 7, the optimal expected revenue is upper bounded $Rev(H(s)) + G(s)$. If step 2 is used, the revenue is $REV_1 = \sum_{i \neq j}(g_{i,j}(s_j) + g_{j,i}(s_i)) = G(s)$. If step 3 is used, by Theorem 4 we have the revenue $REV_2 \geq Rev(H(s))$. Overall we have the expected revenue $REV = \max(REV_1, REV_2) \geq 1/4 E_s[(G(s) + Rev(H(s)))]$.

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