Complexity and Kernels for Bipartition into Degree-bounded Induced Graphs

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Abstract. In this paper, we study the parameterized complexity of the problems of partitioning the vertex set of a graph into two parts V_A and V_B such that V_A induces a graph with degree at most a (resp., an a-regular graph) and V_B induces a graph with degree at most b (resp., a b-regular graph). These two problems are called Upper-Degree-Bounded Bipartition and Regular Bipartition respectively. First, we prove that the two problems are NP-complete with any nonnegative integers a and b except $a = b = 0$. Second, we show that the two problems with parameter k being the size of V_A of a bipartition (V_A, V_B) are fixed-parameter tractable for fixed integer a or b by deriving some problem kernels for them.

1 Introduction

In graph algorithms and graph theory, there is a series of important problems that require us to partition the vertex set of a graph into several parts such that each part induces a subgraph satisfying some degree constraints. For example, the k-coloring problem is to partition the graph into k parts each of which induces an independent set (a 0-regular graph). Most of these kinds of problems are NP-hard, even if the problems are to partition a given graph into only two parts, which is called a *bipartition*.

For bipartition with a degree constraint on each part, we can find many references related to this topic. Here is a definition of the problem:

Degree-Constrained Bipartition
Instance: A graph $G = (V, E)$ and four integers a, a', b and b'.
Question: Is there a partition (V_A, V_B) of V such that

$$a' \leq \deg_{V_A}(v) \leq a \quad \forall v \in V_A \quad \text{and} \quad b' \leq \deg_{V_B}(v) \leq b \quad \forall v \in V_B,$$

where $\deg_X(v)$ denotes the degree of a vertex v in the induced subgraph $G[X]$?

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There are three special cases of Degree-Constrained Bipartition. If there are no constraints on the upper bounds (resp., lower bounds) of the degree in Degree-Constrained Bipartition, i.e., $a = b = \infty$ (resp., $a' = b' = 0$), we call the problem Lower-Degree-Bounded Bipartition (resp., Upper-Degree-Bounded Bipartition). We call Degree-Constrained Bipartition with a special case of $a = a'$ and $b = b'$ Regular Bipartition.

Lower-Degree-Bounded Bipartition has been extensively studied in the literature. The problem with 4-regular graphs is NP-complete for $a' = b' = 3$ [7] and linear-time solvable for $a' = b' = 2$ [4]. More polynomial-time solvable cases with restrictions on the structure of given graphs and constraints on a' and b' can be found in [2,3,7,12,16].

For Regular Bipartition, when $a = b = 0$, the problem becomes a polynomial-solvable problem of checking whether a given graph is bipartite or not; when $a = 0$ and $b = 1$, the problem becomes Dominating Induced Matching, a well studied NP-hard problem also known as Efficient Edge Domination [11,14]. However, not many results are known about Upper-Degree-Bounded Bipartition and Regular Bipartition with other values of a and b.

In this paper, we first show that Upper-Degree-Bounded Bipartition and Regular Bipartition are NP-complete with any nonnegative integers a and b except $a = b = 0$. The major contributions of this paper are vertex kernels for these two problems, which also implies that for constants a and b they are fixed-parameter tractable (FPT) with parameter $k = |V_A|$. We also discuss the fixed-parameter intractability of our problems with parameter only $k = |V_A|$ where b is not fixed.

We also note some related problems, in which the degree constraint on one part of the bipartition changes to a constraint on the size of the part. Bounded-Degree Deletion asks us to delete at most k vertices from a graph to make the remaining graph having maximum vertex degree at most a. Maximum Regular Induced Subgraph asks us to delete at most k vertices from a graph to make the remaining graph an a-regular graph. These two problems can be regarded as such a kind of bipartition problems and have been well studied in parameterized complexity. They are FPT with parameters k and a and W[1]-hard with only parameter k [10,15,16]. Let tw denote the treewidth of an input graph. Betzler et. al. also proved that Bounded-Degree Deletion is FPT with parameters k and tw and W[2]-hard with only parameter tw [6]. The parameterized complexity of some other related problems, such as Minimum Regular Induced Subgraph are studied in [1].

The remaining parts of the paper are organized as follows: Section 2 introduces a notation system. Section 3 proves the NP-hardness of our problems. Section 4 gives the problem kernels, and Section 5 shows the fixed-parameter intractability. Finally, some concluding remarks are given in the last section. Proofs of some lemmas are omitted due to space limitation.

2 Preliminaries

In this paper, a graph stands for a simple undirected graph. We may simply use v to denote the set $\{v\}$ of a single vertex v. Let $G = (V, E)$ be a graph, and $X \subseteq V$ be a subset of vertices. The subgraph induced by X is denoted by $G[X]$, and $G[V \setminus X]$ is also written as $G \setminus X$. Let $E(X)$ denote the set of edges between X and $V \setminus X$. Let $N(X)$ denote the *neighbors* of X, i.e., the vertices $y \in V \setminus X$ adjacent to a vertex $x \in X$, and denote $N(X) \cup X$ by $N[X]$. The *degree* $\deg(v)$ of a vertex v is defined to be $|N(v)|$. A vertex in X is called an *X-vertex*, and a neighbor $u \in X$ of a vertex v is called an *X-neighbor* of v. The number of X-neighbors of v is denoted by $\deg_X(v)$; i.e., $\deg_X(v) = |N(v) \cap X|$. The vertex set and edge set of a graph H are denoted by $V(H)$ and $E(H)$, respectively. When X is equal to $V(H)$ of some subgraph H of G, we may denote $V(H)$-vertices by H-vertices, $V(H)$-neighbors by H-neighbors, and $\deg_{V(H)}(v)$ by $\deg_H(v)$ for simplicity. For a subset $E' \subseteq E$, let $G - E'$ denote the subgraph obtained from G by deleting edges in E'. For an integer $p \geq 1$, a star with $p + 1$ vertices is called a *p-star*. The unique vertex of degree > 1 in a p-star with $p > 1$ is called the *center* of the star, and any vertex in a 1-star is a *center* of the star.

For a graph G and two nonnegative integers a and b, a partition of $V(G)$ into V_A and V_B is called *(a, b)-bounded* if $\deg_{V_A}(v) \leq a$ for all vertices in $v \in V_A$ and $\deg_{V_B}(v) \leq b$ for all vertices in $v \in V_B$. An (a, b)-bounded partition (V_A, V_B) is called *(a, b)-regular* if $\deg_{V_A}(v) = a$ for all vertices in $v \in V_A$ and $\deg_{V_B}(v) = b$ for all vertices in $v \in V_B$. An instance $I = (G, a, b)$ of Upper-Degree-Bounded Bipartition (resp., Regular Bipartition) consists of a graph G and two nonnegative integers a and b, and asks us to test whether an instance (G, a, b) admits an (a, b)-bounded partition (resp., (a, b)-regular partition) or not.

3 NP-Hardness

Theorem 1. Upper-Degree-Bounded Bipartition *is NP-complete for any nonnegative integers a and b except $a = b = 0$.*

Before proving Theorem 1, we first provide some properties on complete graphs in Upper-Degree-Bounded Bipartition. Without loss of generality we assume that $a \leq b$ and $b \geq 1$ in this section.

An $(a+1, b+1, a+1)$-*complete graph* W is defined to be the graph consisting of two complete graphs of size $a + b + 2$ that share exactly $b + 1$ vertices, where $|V(W)| = 2(a + b + 2) - (b + 1) = 2a + b + 3$ holds and the set of $b + 1$ vertices shared by the two complete graphs is denoted by $S(W)$.

Lemma 1. *Let (G, a, b) admit an (a, b)-bounded partition (V_A, V_B).*

(i) *If G contains a clique K of size $a + b + 2$, then $|V(K) \cap V_A| = a + 1$ and $|V(K) \cap V_B| = b + 1$; and*
(ii) *Assume that G contains an $(a + 1, b + 1, a + 1)$-complete graph W. Then $\{V(W) \cap V_A, V(W) \cap V_B\} = \{S(W), V(W) \setminus S(W)\}$ (or $V(W) \cap V_A = V(W) \setminus S(W)$ and $V(W) \cap V_B = S(W)$ when $a \neq b$), $N(V_A \cap V(W)) \setminus V(W) \subseteq V_B$ and $N(V_B \cap V(W)) \setminus V(W) \subseteq V_B$.*

We here construct a special graph that consists of an $(a+1, b+1, a+1)$-complete graph, several complete graphs with size $a+b+2$ and some edges joining them. Given two positive integers n and m, we first construct an $(a+1, b+1, a+1)$-complete graph W and $(n+m)$ complete graphs $X_1, X_2, \ldots, X_n$ and $C_1, C_2, \ldots, C_m$ with size $a+b+2$. Next we choose a vertex $v_A \in V(W) \setminus S(W)$ and a vertex $v_B \in S(W)$ arbitrarily, and add edges between $\{v_A, v_B\}$ and $\{X_1, \ldots, X_i, \ldots, X_n\} \cup \{C_1, \ldots, C_j, \ldots, C_m\}$ as follows:

1. For each X_i, join v_B to arbitrary a vertices $u_1, \ldots, u_a \in V(X_i)$ via new edges, and join v_A to arbitrary b vertices $u'_1, \ldots, u'_b \in V(X_i) \setminus \{u_1, \ldots, u_a\}$ via new edges;
2. For each C_j, join v_B to arbitrary a vertices $u_1, \ldots, u_a \in V(C_j)$ via new edges, and join v_A to arbitrary $(b-1)$ vertices $u'_1, \ldots, u'_{b-1} \in V(C_j) \setminus \{u_1, \ldots, u_a\}$ via new edges, where $b-1 \geq 0$ since $b \geq 1$ is assumed; and
3. Let $G_{n,m}$ denote the resulting graph.

Vertices in X_i $(i = 1, 2, \ldots, n)$ or C_j $(j = 1, 2, \ldots, m)$ not adjacent to v_A or v_B are called *free*. Each X_i contains exactly two free vertices, denoted by v_i and v'_i, and each C_j contains exactly three free vertices, denoted by v_j^1, v_j^2 and v_j^3.

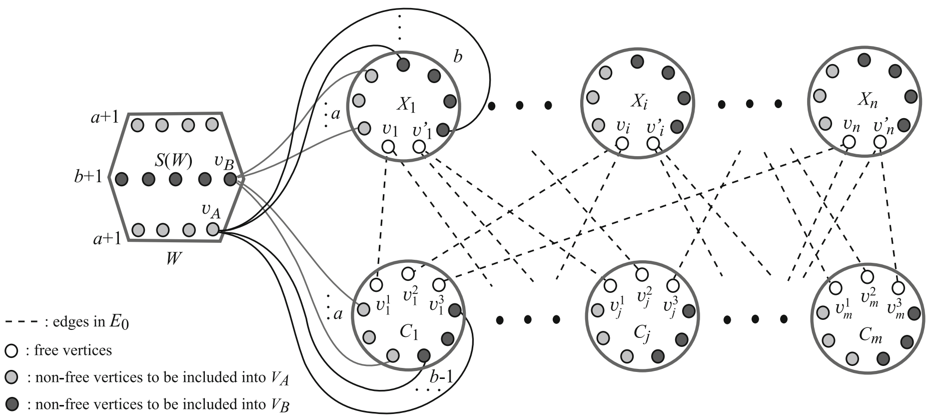

Fig. 1. Constructing graph $G_{n,m} + E_0$

Let E_0 be an arbitrary set of new edges between free vertices in $\cup_{1 \leq i \leq n} X_i$ and free vertices in $\cup_{1 \leq j \leq m} C_j$ in $G_{n,m}$. Let $G_{n,m} + E_0$ be the graph obtained from $G_{n,m}$ by adding the edges in E_0. See Figure 1. We have

Lemma 2. *Let (V_A, V_B) be a partition of $V(G_{n,m} + E_0)$, where if $a = b$ then we assume without loss of generality that $v_A \in V_A$. Then (V_A, V_B) is an (a, b)-bounded partition of $G_{n,m} + E_0$ if and only if the following hold:*

(i) *Every subgraph* $H \in \{W, X_1, \ldots, X_n, C_1, \ldots, C_m\}$ *satisfies that* $\deg_{V(H)\cap V_A}(v) = a$ *for all vertices* $v \in V(H) \cap V_A$ *and* $\deg_{V(H)\cap V_B}(v) = b$ *for all vertices* $v \in V(H) \cap V_B$*;*
(ii) $S(W) \subseteq V_B$*,* $V(W) \setminus S(W) \subseteq V_A$*,* $N(v_B) \setminus V(W) \subseteq V_A$*, and* $N(v_A) \setminus V(W) \subseteq V_B$*;*
(iii) *For each* X_i*, exactly one of the two free vertices in* X_i *is contained in* V_A *and the other is in* V_B*; and*
(iv) *For each* C_j*, exactly one of the three free vertices in* C_j *is contained in* V_A *and the other two are in* V_B*; and* (v) *For each* $uv \in E_0$*,* $|\{u, v\} \cap V_A| = |\{u, v\} \cap V_B| = 1$*.*

Now we are ready to prove Theorem 1. Clearly Upper-Degree-Bounded Bipartition is in NP. In what follows, we construct a polynomial reduction from the NP-complete problem One-In-Three 3SAT [12].

One-In-Three 3SAT
Instance: A set $\mathcal{C}$ of m clauses $c_1, c_2, \ldots, c_m$ on a set $\mathcal{X}$ of n variables $x_1, x_2, \ldots, x_n$ such that each clause c_j consists of exactly three literals ℓ_j^1, ℓ_j^2 and ℓ_j^3.
Question: Is there a truth assignment $\mathcal{X} \to \{\texttt{true}, \texttt{false}\}^n$ such that each clause c_j has exactly one true literal?

Given an instance $F = (\mathcal{C}, \mathcal{X})$ of One-In-Three 3SAT and nonnegative integers $a \le b$ (≥ 1), we will construct an instance $I_F = (G_F, a, b)$ of Upper-Degree-Bounded Bipartition such that I_F has an (a, b)-bounded partition if and only if F is feasible. Such an instance I_F is constructed on the graph $G_{n,m}$ by setting $G_F = G_{n,m} + E_0$, where a set E_0 of edges between $\{X_1, \ldots, X_i, \ldots, X_n\}$ and $\{C_1, \ldots, C_j, \ldots, C_m\}$ according to the relationship between $\mathcal{X}$ and $\mathcal{C}$ in F as follows:

For each clause $c_j = (\ell_j^1, \ell_j^2, \ell_j^3) \in \mathcal{C}$ and the k-th literal ℓ_j^k, $k = 1, 2, 3$, if ℓ_j^k is a positive (resp., negative) literal of a variable x_i, then join free vertex $v_j^k \in V(C_j)$ to free vertex $v_i \in V(X_i)$ (resp., $v_i' \in V(X_i)$) via a new edge.

Let $G_F = G_{n,m} + E_0$ be the resulting graph. We remark that X_i serves as a gadget for variable $x_i \in \mathcal{X}$ and C_j serves as a gadget for clause $c_j \in \mathcal{C}$.

This completes the construction of instance $I_F = (G_F, a, b)$. We interpret conditions (iii) and (iv) on free vertices in Lemma 2 as follows:

$$v_i \in V_B \text{ (resp., } v_i \in V_A) \Leftrightarrow \texttt{true} \text{ (resp., } \texttt{false}) \text{ is assigned to } x_i, \text{ and}$$

$$v_j^k \in V_A \text{ (resp., } v_j^k \in V_B) \Leftrightarrow \ell_j^k = \texttt{true} \text{ (resp., } \ell_j^k = \texttt{false}).$$

Hence we see by Lemma 2 that $I_F = (G_F = G_{n,m} + E_0, a, b)$ admits an (a, b)-bounded partition if and only if F is feasible. This completes a proof of Theorem 1.

By Lemma 2, F is feasible if and only if $I_F = (G_F = G_{n,m} + E_0, a, b)$ admits an (a, b)-regular partition. Hence the problem of testing whether an instance (G, a, b) admits an (a, b)-regular partition is also NP-complete for any nonnegative integers a and b except $a = b = 0$.

Corollary 1. Regular Bipartition *is NP-complete for any nonnegative integers a and b except $a = b = 0$.*

4 Kernelization

This section studies the parameterized complexity and kernels of our problems. For this, we introduce the following constrained versions of the problems.

Constrained Upper-Degree-Bounded Bipartition
Instance: A graph G, two subsets $A, B \subseteq V(G)$, and nonnegative integers a, b and k.
Question: Is there an (a, b)-bounded partition (V_A, V_B) of $V(G)$ such that $A \subseteq V_A$, $B \subseteq V_B$, and $|V_A| \leq k$?

In the same way, we can define Constrained Regular Bipartition by replacing "(a, b)-bounded partition" with "(a, b)-regular partition" in the above definition. Note that we do not assume $a \leq b$ in this section. We call a partition (V_A, V_B) satisfying the condition in the definitions of Constrained Upper-Degree-Bounded Bipartition and Constrained Regular Bipartition a *solution* to the problem instance. An instance (G, A, B, a, b, k) is called *feasible* if it admits a solution. A vertex in $V(G) \setminus (A \cup B)$ is called *undecided*, and we always denote $V(G) \setminus (A \cup B)$ by U. Clearly each of the two problems can be solved in $2^{|U|}|V|^{O(1)}$ time. We say that an instance (G, A, B, a, b, k) is *reduced* to an instance (G, A', B', a, b, k) such that (G, A, B, a, b, k) is feasible if and only if so is (G, A', B', a, b, k). Note that when it turns out that (G, A, B, a, b, k) is infeasible we can say that it is reduced to an infeasible instance (G, A', B', a, b, k) such as one with $A' \cap B' \neq \emptyset$.

In this paper, we say that a problem admits a kernel of size $O(f(k))$ if any instance of the problem can be reduced in polynomial time in n into an instance (G, A, B, a, b, k) with $|V(G)| = O(f(k))$ for a function $f(k)$ of k. The main results in this section are the following.

Theorem 2. Constrained Upper-Degree-Bounded Bipartition *admits a kernel of size $O((b+1)^2(b+k)k)$, and is fixed-parameter tractable with parameter k for a constant b.*

Theorem 3. Constrained Regular Bipartition *admits a kernel of size $O((b+1)(b+k)k^2)$ for $a \leq b$ or of size $O((b+1)(b+k)k^2 + (ak)^{(a-b+1)k})$ for $a > b$, and is fixed-parameter tractable with parameter k for constants a and b.*

4.1 Kernels for Constrained Upper-Degree-Bounded Bipartition

In this subsection, an instance always means the one of Constrained Upper-Degree-Bounded Bipartition. We have only five simple reduction rules to get a kernel to this problem.

Rule 1. *Conclude that an instance is infeasible if one of the following holds:* $A \cap B \neq \emptyset$; $|A| > k$; $\deg_A(v) > a$ *for some vertex* $v \in A$; *and* $\deg_B(u) > b$ *for some vertex* $u \in B$.

Rule 2. *Move to* B *any* U*-vertex* v *with* $\deg_A(v) > a$, *and move to* A *any* U*-vertex* u *with* $\deg_B(u) > b$.

If we include to B a U-vertex v with $\deg(v) > b+k$, then the instance cannot have a solution, because at least $k+1$ neighbors of v need to be included to A, implying that $|V_A|$ cannot be bounded by k.

Rule 3. *Move to* A *any* U*-vertex* v *with* $\deg(v) > b + k$.

Lemma 3. *Let* v *be a* $U \cup B$*-vertex in an instance* $I = (G, A, B, a, b, k)$ *such that* $\deg(u) \leq b$ *for all vertices* $u \in N[v]$. *Let* $I' = (G - \{v\}, A, B', a, b, k)$ *be the instance obtained from* I *by deleting the vertex* v, *where* $B' = B$ *if* $v \in U$ *and* $B' = B - \{v\}$ *if* $v \in B$. *The instance* I *is feasible if and only if so is* I'.

Proof. It is clear that if I has a solution then I' also has a solution, because deleting a vertex never increases the degree of any of the remaining vertices. Assume that I' admits a solution (V_A, V_B). We show that $(V_A, V_B \cup \{v\})$ is a solution to I. Note that adding v to V_B may increase the degree of a vertex only in $N[v]$. However, by the choice of the vertex v, for any vertex $u \in N[v]$ it holds $b \geq \deg(u) \geq \deg_{V_B \cup \{v\}}(u)$. Hence $(V_A, V_B \cup \{v\})$ is a solution to I. □

Rule 4. *Remove from the graph of an instance any* $U \cup B$*-vertex* v *such that* $\deg(u) \leq b$ *for all vertices* $u \in N[v]$.

Lemma 4. *An instance* $I = (G, A, B, a, b, k)$ *is infeasible if* G *contains more than* k *vertex-disjoint* $(b + 1)$*-stars.*

Proof. For a solution (V_A, V_B) to I, if there is a $(b + 1)$-star disjoint with V_A, then a center v of the star would satisfy $\deg_{V_B}(v) \geq b+1$. Hence V_A must contain at least one from each of more than k vertex-disjoint $(b+1)$-stars. This, however, contradicts $|V_A| \leq k$. □

Rule 5. *Compute a maximal set* $\mathcal{S}$ *of vertex-disjoint* $(b + 1)$*-stars in* G *of an instance* $I = (G, A, B, a, b, k)$ *(not only in* $G[U]$*). Conclude that the instance is infeasible if* $|\mathcal{S}| > k$.

Now we analyze the size $|V(G)|$ of an instance $I = (G, A, B, a, b, k)$ where none of the above five rules can be applied anymore. Assume that when Rule 4 is applied to a maximal set of vertex-disjoint $(b + 1)$-stars $\mathcal{S}$ in G, it holds $|\mathcal{S}| \leq k$ now. Let S_0 be the set of all vertices in $\mathcal{S}$, $S_1 = N(S_0)$ and $S_2 = N(S_1 \cup S_0) = N(S_1) \setminus S_0$. We first show that $V(G) = A \cup S_0 \cup S_1 \cup S_2$. By the maximality of $\mathcal{S}$, we know that there is no vertex of degree $\geq b + 1$ in the graph after deleting S_0. Then all vertices u with $\deg(u) \geq b + 1$ are in $S_0 \cup S_1$, and $|S_2| \leq b|S_1|$ holds. Since Rule 4 is no longer applicable, each $U \cup B$-vertex v with $\deg(v) \leq b$ is adjacent to a vertex u with $\deg(u) \geq b + 1$ that is in

$S_0 \cup S_1$. Then all $U \cup B$-vertices u with $\deg(u) \leq b$ are in $S_1 \cup S_2$. Hence $V(G) = A \cup S_0 \cup S_1 \cup S_2$. We have that $|A| \leq k$, $|S_0| \leq (b+2)|\mathcal{S}| \leq (b+2)k$, $|S_1| \leq (b+k)|S_0| \leq (b+k)(b+2)k$ by Rule 3 and $|S_2| \leq b|S_1| \leq b(b+k)(b+2)k$. Therefore $|V(G)| \leq |A| + |S_0| + |S_1| + |S_2| = O((b+1)^2(b+k)k)$. This proves Theorem 2.

4.2 Kernels for Constrained Regular Bipartition

In this subsection, an instance always stands for the one in Constrained Regular Bipartition. When we introduce a reduction rule, we assume that all previous reduction rules cannot be applied anymore.

We see that an instance $I = (G, A, B, a, b)$ is infeasible if one of the following conditions holds:

(i) $A \cap B \neq \emptyset$ or $|A| > k$;
(ii) There is a vertex $v \in V(G)$ with $\deg(v) < \min\{a, b\}$;
(iii) There is a vertex $v \in A$ with $\deg_{V(G)\setminus B}(v) < a$ or $\deg_A(v) > a$; and
(iv) There is a vertex $v \in B$ with $\deg_{V(G)\setminus A}(v) < b$ or $\deg_B(v) > b$.

Rule 6. *Conclude that an instance is infeasible if one of the above four conditions holds.*

Rule 7. *Move to B any U-vertex v with* $\deg_{V(G)\setminus B}(v) < a$ *or* $\deg_A(v) > a$ *or adjacent to a B-vertex u with* $\deg_B(u) + \deg_U(u) = b$. *Move to A any U-vertex v with* $\deg_{V(G)\setminus A}(v) < b$ *or* $\deg_B(v) > b$ *or adjacent to an A-vertex u with* $\deg_A(u) + \deg_U(u) = a$.

Rule 8. *Remove from the graph of an instance any edges between A and B. Delete the set $V(H)$ of vertices in any b-regular component H of G such that $V(H) \subseteq U \cup B$.*

Rule 9. *Move to A any U-vertex v with* $\deg(v) > b + k$.

We say that a vertex v is *tightly-connected* from a U-vertex u if there is a path P from u to v such that each vertex $w \in V(P) \setminus \{u\}$ is a U-vertex with $\deg_{V(G)\setminus A}(w) = b$. For each U-vertex u, let $T(u)$ denote the set U-vertices tightly-connected from u, which has the following property: when we include a U-vertex u to A, all the vertices $T(u)$ need to be included to A, because the degree of each vertex $v \in T(u) \setminus \{u\}$ in $G[U \cup B]$ will be less than b. Hence if we include a U-vertex u with $|T(u)| > k$, then $|A|$ will increase by $|T(u)| > k$ and the resulting instance cannot have a solution.

Rule 10. *Move to B any U-vertex u with $|T(u)| > k$.*

Rule 11. *Conclude that an instance is infeasible if $|B \cap N(U)| > bk$ or $|E(B)| > b(b+1)k$.*

In what follows, we assume that $b(b+1)k > |E(B)| \geq |N(B)|$. By Rule 10, it holds that $|T(u)| \leq k$ for each vertex $u \in N(B)$. Let $T^* = N(B) \cup (\cup_{u \in N(B)} T(u))$. Then $|T^*| \leq |N(B)|(k+1) \leq b(b+1)k(k+1)$. We have

Lemma 5. *When none of Rule 6-Rule 11 is applicable, it holds that* $|T^*| = O(b^2k^2)$.

We compute a maximal set $\mathcal{S}$ of vertex-disjoint $(b+1)$-stars in the induced graph $G[U]$. We see that an instance $I = (G, A, B, a, b)$ is infeasible if $G[U]$ contains more than k vertex-disjoint $(b+1)$-stars. This is because $|V_A| \leq k$ means that at least one $(b+1)$-star must become disjoint with V_A and a center v of the star would satisfy $\deg_{V_B}(v) \geq b+1$.

Rule 12. *Conclude that an instance is infeasible if* $|\mathcal{S}| > k$.

Let S_0 be the set of all vertices in the $(b+1)$-stars in $\mathcal{S}$. For each integer $i > 0$, we denote by S_i the set $U \cap N(S_{i-1}) \setminus (T^* \cup (\cup_{j=0}^{i-1} S_j))$. Let $S^* = \cup_{i \geq 0} S_i$.

Lemma 6. *When none of Rule 6-Rule 12 is applicable, every U-vertex u with* $\deg_U(u) \geq b+1$ *is in* $S_0 \cup S_1$*. For each vertex* $v \in U \setminus (T^* \cup S_0 \cup S_1)$*, it holds* $\deg_{V(G)\setminus A}(v) = \deg_U(v) = b$.

Lemma 7. *When none of Rule 6-Rule 12 is applicable, it holds that* $|S^*| = O((b+1)(b+k)k^2)$.

Lemma 8. *When none of Rule 6-Rule 12 is applicable, any* $U \setminus (T^* \cup S^*)$*-vertex is in a component H of $G[U]$ such that* $V(H) \subseteq U \setminus (T^* \cup S^*)$ *and* $V(H) \cap N(A) \neq \emptyset$.

We call a component H of $G[U]$ *residual* if $V(H) \subseteq U \setminus (T^* \cup S^*)$ and $V(H) \cap N(A) \neq \emptyset$. For a vertex u in a residual component H, it holds that $\deg_{V(G)\setminus A}(u) = \deg_U(u) = b$ for $u \in V(H) \cap N(A)$, and $\deg(u) = \deg_U(u) = b$ for $u \in V(H) \setminus N(A)$ by Lemma 6.

Lemma 9. *Let H be a residual component in $G[U]$ of an instance. Then any (a, b)-regular partition (V_A, V_B) satisfies either* $V(H) \subseteq V_A$ *or* $V(H) \subseteq V_B$.

Hence if a residual component H contains a vertex $u \in V(H) \cap N(A)$ with $\deg(u) \neq a$ or is adjacent to an A-vertex v with $\deg_H(v) > a$, then $V(H)$ cannot be contained in a set V_A of any (a, b)-regular partition (V_A, V_B).

Rule 13. *Move to B all vertices in a residual component H that satisfies one of the following:*

(i) *There is a vertex* $u \in V(H) \cap N(A)$ *with* $\deg(u) \neq a$*; and*
(ii) *There is an A-vertex v with* $\deg_H(v) > a$.

By Lemma 8, we know that each U-vertex is either in $T^* \cup S^*$ or a residual component. Note that for any vertex $u \in V(H) \cap N(A)$ in a residual component H, it holds $\deg(u) = \deg_U(u) + \deg_A(u) \geq b+1$, which indicates that $\deg(u) \geq b+1 > a$ if $a \leq b$. Hence when $a \leq b$, after Rule 13 is applied, there is no residual component. We get the following lemma by Lemma 5 and Lemma 7.

Lemma 10. *If $a \le b$, then the number $|U|$ of undecided vertices in the instance after applying all above rules is $O((b+1)(b+k)k^2)$.*

Lemma 11. *Assume that there is a residual component H in $G[U]$. Then $a > b$, $V(H) \subseteq N(A)$, $|V(H)| \le k$, and every vertex in $u \in V(H)$ satisfies $\deg_U(u) = b$ and $\deg_A(u) = a - b$.*

Next we consider the case that $a > b$. Let all the vertices in A be indexed by $w_1, w_2, \ldots, w_{|A|}$, and define the *code* $c(H)$ of a residual component H in $G[U]$ to be a vector

$$(\deg_H(w_1), \deg_H(w_2), \ldots, \deg_H(w_{|A|})),$$

where $0 \le \deg_H(w_i) \le a$ for each i. We say that two residual components H and H' are *equivalent* if they have the same code $c(H) = c(H')$, where we see that $|V(H)| = |V(H')|$ since each vertex u in a residual component has the same degrees in A and U by Lemma 11. Hence the feasibility of the instance is independent of the current graph structure among equivalent components. Moreover, if there are more than a equivalent components, then one of them is not contained in V_A of some (a,b)-regular partition when the instance is feasible.

Rule 14. *If there are more than a equivalent residual components for some code, choose arbitrarily one of them and include the vertices of the component to B.*

Lemma 12. *The number of vertices in all residual components is $O((ak)^{(a-b+1)k})$.*

By Lemma 5, Lemma 7, and Lemma 12, we have the following.

Lemma 13. *If $a > b$, the number $|U|$ of undecided vertices in any instance after applying all above rules is $O((b+1)(b+k)k^2 + (ak)^{(a-b+1)k})$.*

We finally derive an upper bound on the size of B in an instance I. Let $B_1 = B \cap N(U)$ and $B_2 = B \setminus B_1$, where $\deg_B(u) < b$ for each vertex $u \in B_1$ by Rule 6, and $\deg_B(u) = b$ for each vertex $u \in B_2$. Note that if $b \le 1$ then $B_2 = \emptyset$ by Rule 8, and that if $|E(B_1, B_2)|$ is odd then b is also odd since $b|B_2| - |E(B_1, B_2)| = 2|E(G[B_2])|$. Observe that the feasibility of I will not change even if we replace the subgraph $G[B_2]$ with a smaller graph G' of degree-b B-vertices as long as each vertex $u \in B_1$ has the same degree $\deg_{V(G')}(u) = \deg_{B_2}(u)$ as before. The next lemma ensures that there is such a graph G' with $O(|B_1| + b^2)$ vertices.

Lemma 14. *Let $b \ge 2$ be an integer, $V_1 = \{u_1, u_2, \ldots, u_n\}$ be a set of n vertices, and $\delta = (d_1, d_2, \ldots, d_n)$ be a sequence of nonnegative integers at most $b-1$ such that b is odd if $d = \sum_{1 \le i \le n} d_i$ is odd. Then there is a graph $G' = (V_2, E_2)$ with $|V_2| \le n + b^2 + b + 1$ and a set $E(V_1, V_2)$ of d edges between V_1 and V_2 such that after adding $E(V_1, V_2)$ between V_1 and V_2, it holds that $\deg_{V_2}(u_i) = d_i$ for each $u_i \in V_1$ and $\deg_{V_1 \cup V_2}(v_i) = b$ for each $v_i \in V_2$. Such a pair of graph G' and edge set $E(V_1, V_2)$ can be constructed in polynomial time in n.*

Rule 15. *When $b \geq 2$, remove the subgraph $G[B_2]$, and add a graph $G' = (V_2, E_2)$ with edge set $E(V_1 = B_1, V_2)$ according to Lemma 14, where $n = |B_1|$, $V_1 = B_1 = \{u_1, u_2, \ldots, u_n\}$ and $\delta = (\deg_{B_2}(u_1), \deg_{B_2}(u_2), \ldots, \deg_{B_2}(u_n))$.*

Lemma 15. *After applying all above rules, the number of vertices in A is at most k and the number of vertices in B is $O(bk + b^2)$.*

Proof. After Rule 6, the number of vertices in A is at most k. After Rule 15, all new vertices added in Rule 15 will form the new vertex set B_2. Then $|B| = |B_1| + |B_2| = |B_1| + |V_2| \leq 2|B_1| + b^2 + b + 1 = 2bk + b^2 + b + 1$. □

Lemma 10, Lemma 13, and Lemma 15 establish Theorem 3.

5 Fixed-Parameter Intractability

This section discusses the fixed-parameter intractability of our problems.

Theorem 4. Upper-Degree-Bounded Bipartition *is* W[2]*-hard with parameter $k = |V_A|$.*

For Upper-Degree-Bounded Bipartition, we give a reduction from Dominating Set, a well-known W[2]-hard problem. Dominating Set asks us to test whether a graph G admits a vertex subset $D \subseteq V(G)$ of size k such that each vertex in $V(G) \setminus D$ is adjacent to at least one vertex in D. Given an instance $I = (G, k)$ of Dominating Set with a graph G of maximum degree $d \geq 2$, we augment G to $G' = (V(G) \cup V_1, E(G) \cup E_1)$ so that each vertex $v \in V(G)$ will be of degree d by adding $d - \deg(v)$ new vertices adjacent to only v, where V_1 and E_1 are the sets of new added degree-1 vertices and edges, respectively. Let $I' = (G', a = d, b = d-1, k)$ be an instance of Upper-Degree-Bounded Bipartition. We prove that I is a yes-instance if and only if I' is feasible. If G has a dominating set D of size at most k, then $(V_A = D, V(G') \setminus D)$ is a solution to I', because each degree-d vertex in G' is adjacent to at least one vertex in D, and $\deg_{V(G')\setminus D}(u) \leq \max\{1, d-1\}$ holds for each vertex $u \in V(G') \setminus D$. When I' is feasible, we claim that I' always admits a solution (V_A, V_B) such that $V_A \subseteq V(G)$. The reason is that any vertex $v \in V_A \setminus V(G)$ must be a degree-1 vertex in G' whose unique neighbor u is in $V(G)$, and thereby we can replace v with u in V_A to get another solution to I'. For a solution $(V_A \subseteq V(G), V_B)$ to I', we see that V_A is a dominating set in the original graph G.

For Regular Bipartition, we will show that a special case of this problem is equivalent to Perfect Code in d-regular graphs. Perfect Code asks us to test whether G admits a set $S \subseteq V(G)$ of at most k vertices such that for each vertex $v \in V(G)$ there is precisely one vertex in $N[v] \cap S$. It is W[1]-hard when k is taken as the parameter [9]. It is easy to see that an instance (G, k) of Perfect Code in a d-regular graph G is yes if and only if the instance $(G, 0, d-1, k)$ of Regular Bipartition is feasible. It is quite possible that Perfect Code with parameter k remains W[1]-hard even if input graphs are restricted to regular graphs.

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