

Statics of Historic Masonry Constructions: An Essay

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Abstract I feel honoured to present the findings published in my recent book *Statica delle Costruzioni Storiche in Muratura* to the Association Edoardo Benvenuto. I like to add that, during the phase of preparation for this present essay, the English edition of the book, *Statics of Historic Masonry Constructions*, has also been published by Springer. My research took shape gradually, during thirty years of research, professional experience and teaching. The book firstly gives fundamentals of statics of the masonry solid from its mathematical groundings and then applies them to the study of the static behaviour of arches, piers and vaults. Further, combining engineering and architecture and through an interdisciplinary approach, my research highlights the deep connections existing between statics and architecture and investigates the static behaviour of many historic monuments, as the Pantheon, the Colosseum, the domes of S. Maria del Fiore in Florence and of St. Peter in Rome, the Tower of Pisa, the Gothic cathedrals etc. In the end the book considers the behaviour of masonry buildings under seismic actions. Here I will discuss the adopted hypotheses and some key passages of the main issues involved.

Keywords Strength and deformability of masonry materials • Deformation and equilibrium of masonry solids • Static behaviour of arches and vaults

1 Special Features of Masonry Behaviour

Under a given loading path a masonry structure can reach a collapse condition solely due to loss of equilibrium, that is to say, in the absence of any material failure. Such a condition, due to the very low—near zero—material tensile strength, can thus arise even in masonry with infinite compression strength. Masonry structures can suffer, in fact, cracks or detachments that may in turn generate displacement fields, called *mechanisms*, which develop without any internal opposition from the material. So, as soon as the pushing loads begin to exceed the action of the resistant loads

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along one of these mechanisms, the structure fails. Moreover, if a small settlement occurs at one of the external constraints of a masonry structure, it freely follows the settlement, maintaining constant its stresses and constraint reactions. It is thus easy to understand how the presence of a negligible tensile strength can disrupt the behaviour of structures as compared to the common elastic ones. These are the essentials of the masonry behaviour, fully realized by ancient builders and which have shaped the course of architecture from the origins up to the nineteenth century.

2 Heyman Assumptions

The constitutive assumptions that control the masonry behaviour, discussed in depth in Como (2010, 2013), were originally formulated by Heyman (1966) and are as follows:

- (i) *masonry is incapable of withstanding tensions;*
- (ii) *stresses are so low that masonry has effectively an unlimited compressive strength;*
- (iii) *shear strains cannot occur*

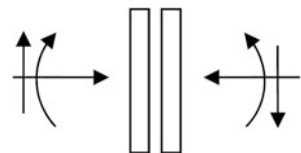
The other assumption: *elastic strains are negligible*, was not directly expressed by Heyman but constantly considered.

The foregoing assumptions turn out to be very clear if we refer to the elementary resistant cell of the masonry structure, represented by two idealized rigid masonry bricks compressed one against the other by the stress vector Σ , whose components are the more or less eccentric axial load N and the shear force T (Fig. 1). The two rigid bricks of the unit resistant cell cannot deform internally, but they can detach from each other. A *crack* can occur in the cell.

The first two of Heyman's assumptions involve stresses; the last one strains. The first and the second assumptions are the most important. The third assumption can be considered a consequence of the first two. We can make reference in fact to the Coulomb criterion (1776). In this framework the ratio between compression and tensile strengths σ_{rc} and σ_{rt} can be expressed in the following form:

$$\frac{\sigma_{rc}}{\sigma_{rt}} = \frac{1 + \sin \phi}{1 - \sin \phi}$$

Fig. 1 The ideal resistant masonry cell and the corresponding components of the stress vector Σ



where ϕ is the angle of the internal friction. By gradually reducing the ratio σ_{rt}/σ_{rc} , at the limit, we obtain

$$\frac{\sigma_{rc}}{\sigma_{rt}} \rightarrow \infty \Rightarrow \phi \rightarrow \frac{\pi}{2}$$

The internal friction strength, depending on $\text{tg}\phi$, becomes unbounded. The first two Heyman assumptions thus imply unbounded sliding strength (Como and Grimaldi 1985). This result will be considered further on.

Following the above assumptions, Statics of masonry constructions moves immediately towards the Limit Analysis. We remark that according to the above assumptions no local failures in the masonry structures are considered.

3 Extension of Heyman Assumptions to Masonry Continuum

A lack of knowledge reveals, on the other hand, as soon as the behaviour of the general masonry solid is inquired. A vast number of researches spread to fill this gap. In-depth studies into the behaviour of *elastic no-tension* bodies have been conducted by many authors, among whose works I recall Di Pasquale (1984), Del Piero (1989), Lucchesi et al. (2008), Romano and Romano (1985), Romano and Sacco (1984), Baratta (1999), Angelillo et al. (2010), Trovalusci and Masiani (2005) and Bacigalupo and Gambarotta (2010). All have addressed the general problem of the elastic equilibrium of no-tension bodies and numerous, noteworthy stress solutions have been provided (Lucchesi et al. 2008). Nevertheless, the much more complex goal of solutions expressed in terms of displacement and strain fields remains still today substantially unsolved. These difficulties stem from the fact that the no-tension elastic model cannot easily account for the presence of shear strains. In order to overcome these difficulties (Como 2010, 2013) assumes the rigid-in-compression no-tension material and aims to extend the Heyman model to the masonry continuum, on the wake of some previous results presented in Como (1992). This extension, which allows to go into the equilibrium of the masonry solid with a suitable mathematical formulation, wants also to pay homage to the outstanding description of the behaviour of masonry constructions given by Heyman in the far 1966. I will outline its main points of this extension in what follows.

A masonry solid can be considered an assemblage of rigid particles held together by the compressive stresses produced by loads. The small size of the stones compared to the dimensions of the body enables it to be considered a continuous body instead of a discrete system of many individual particles. When the compression stresses that held stones together cancel out in some regions of the masonry solid, it can get deformed. Cracks can thus occur in the masonry mass: they represent discontinuities or detachments of the displacement fields $\mathbf{u}(\mathbf{P})$, describing the deformation of the body. The research of compatibility conditions that the

functions $\mathbf{u}(P)$, called *mechanisms*, have to satisfy to describe the deformation of the solid, is then tackled in Como (2010, 2013).

The definition of the *impenetrability condition* is the starting point: it requires that the displacement function $\mathbf{u}(P)$ cannot produce any contraction between points connected by segments *entirely contained within the body*. Thus, if (P_1, P_2) is such a pair of points in Ω , the region occupied by the body, and (Q_1, Q_2) is the corresponding pair after the transformation $\mathbf{u}(P)$, we have

$$d(Q_1, Q_2) \geq d(P_1, P_2) \tag{1}$$

where $d(Q_1, Q_2)$ denotes the distance of the segment connecting the points Q_1, Q_2 (Como 1992) (Fig. 2). According to this condition *no internal sliding can occur*. Impenetrability condition (1) in a different form still represents both the assumptions (i) no tension, and (ii) the infinite compression strength.

In short, masonry material can only be *widened or opened*. Thus, the relative displacement between a pair of points located across the line of a crack will occur only along the direction *normal* to the crack. Let us consider the line f of the crack and its two edges f^- and f^+ (Fig. 3). We choose the point P^- on the edge f^- and P^+ on the other edge f^+ of the crack. These points are obtained by intersecting f^- and f^+ with the direction of the unit vector \mathbf{n}^- , located along the outward normal to f^- and passing through P^- . Cracks can thus *open* only along the direction of \mathbf{n}^- (or of \mathbf{n}^+). We can thus define, for instance, the *crack opening vector* or the *detachment*

Fig. 2 The impenetrability condition

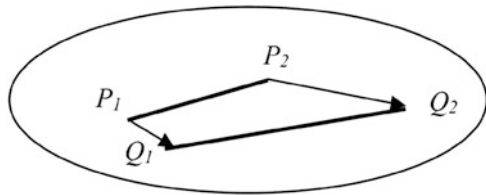
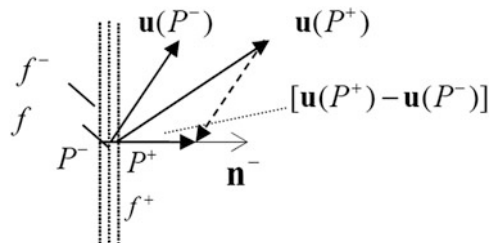


Fig. 3 The opening of a crack



vector as follows:

$$\Delta^{(n^-)}\mathbf{u}(P) = [u(P^+) - u(P^-)]\mathbf{n}^-,$$

with

$$[\mathbf{u}(P^+) - \mathbf{u}(P^-)] \cdot \mathbf{n}^- = u(P^+) - u(P^-) > 0$$

and where $u(P^+)$ and $u(P^-)$ are the scalar values of $\mathbf{u}(P^+)$ and $\mathbf{u}(P^-)$. This is the first local kinematical compatibility condition to be satisfied by the mechanism displacement $\mathbf{u}(P)$. Consequently, we can define the scalar *crack opening* by means of the *positive* quantity

$$\Delta^{(n^-)}u(P) = u(P^+) - u(P^-) > 0. \tag{2}$$

The stress vector is *null* along the crack. From this result other kinematical compatibility conditions follow. A displacement field $\mathbf{u}(P)$ satisfying all these kinematical conditions, defined in detail in Como (2010, 2013), represents a *mechanism* and M is the set of all the mechanisms. Likewise, other local compatibility conditions involving stresses and loads are also given.

4 The Principle of Virtual Work for Masonry Bodies

An important topic tackled in Como (2010, 2013) is the definition of the admissible equilibrium state for the masonry solid. Developing a general equilibrium analysis of masonry bodies is a very difficult task due to the discontinuities present in the corresponding displacement functions $\mathbf{u}(P)$. The idea of Vol’pert and Hujiadev’s (1985) for the study of discontinuous functions of including the set of all discontinuity points within the body’s boundary, turns out to be quite fruitful. Following this suggestion and in step with Como (1992), we can consider the set

$$\Gamma(\mathbf{u})$$

of all the points of discontinuities, that is, the set of all the cracks, *each with its two edges*, for any mechanism $\mathbf{u}(P)$ of the masonry body. This set becomes a *new part* of the boundary of the body, generated by the cracks associated to $\mathbf{u}(P)$. Consequently, we can define, the *free* cracks region $\Omega(\mathbf{u})$, associated to mechanism $\mathbf{u}(P)$

$$\Omega(\mathbf{u}) = \Omega / \Gamma(\mathbf{u})$$

Only in this region $\Omega(\mathbf{u})$ will the displacement fields $\mathbf{u}(P)$ be represented by regular functions, for instance, continuous with their first derivatives, so that strains

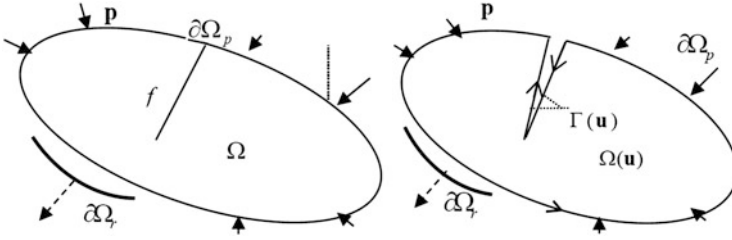


Fig. 4 The boundary of the masonry body and the new boundary of the cracked body corresponding to mechanism \mathbf{u}

$\boldsymbol{\varepsilon}(P)$ can be defined in $\Omega(\mathbf{u})$. The new boundary of the cracked body, corresponding to the mechanism displacement $\mathbf{u}(P)$, is thus represented as

$$\partial\Omega(\mathbf{u}) = \partial\Omega \cup \Gamma(\mathbf{u})$$

As per customary representations, the left-hand scheme in Fig. 4 shows the boundary of the masonry body crossed by the crack f ; the right-hand scheme instead shows the boundary $\partial\Omega(\mathbf{u})$ that includes the two edges of the crack f . We can cover the entire boundary $\partial\Omega(\mathbf{u})$ by circling the region $\Omega(\mathbf{u})$, for instance, in the counter clockwise direction, that is, with region $\Omega(\mathbf{u})$ always remaining on the left.

The equilibrium of the body is governed by the principle of *virtual work*. This principle will take a *particular form* for the compressionally rigid no-tension bodies, analysed in Como (2010, 2012, 2013) along the lines previously set forth in Como (1992).

Let us consider a masonry body under the action of the loads \mathbf{p} in an admissible equilibrium state. The body occupies the region Ω , whose boundary is denoted as $\partial\Omega$, which we assume to be sufficiently regular. The body is loaded by mass and surface loadings $\boldsymbol{\rho}(\Omega)$ and \mathbf{p} . The loaded part of the body surface $\partial\Omega$ is $\partial\Omega_p$.

Let $\delta\mathbf{u}(P) \in M$ be a mechanism field, representing a kinematically admissible virtual displacement of the body. Cracks will arise during the development of the virtual mechanism $\delta\mathbf{u}(P)$ and $\Gamma(\delta\mathbf{u})$ will be the region representing the cracks' boundaries. At any point P within the region $\Omega(\delta\mathbf{u})$, the stress field $\boldsymbol{\sigma}$ and the body forces $\boldsymbol{\rho}$ will satisfy the associated compatibility inequalities and the following internal equilibrium equation:

$$\sigma_{ij,j} + \rho_i = 0 \quad (3)$$

Now let dV be a generic volume element around P in $\Omega(\delta\mathbf{u})$. The virtual work done to displace this element is

$$(\sigma_{ij,j} + \rho_i) \delta u_i dV.$$

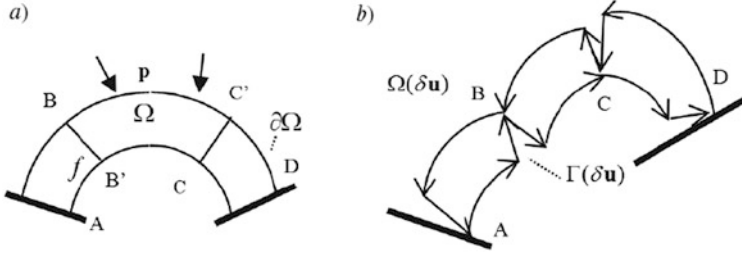


Fig. 5 The boundary of the arch and of the cracked arch with its new boundary associated to the virtual mechanism $\delta \mathbf{u}$

According to the equilibrium equation (3), this work vanishes. Integration of (9) over the volume $\Omega(\mathbf{u})$ thus yields

$$\int_{\Omega(\delta \mathbf{u})} (\sigma_{ij,j} + \rho_i) \delta u_i dV = 0. \quad (4)$$

Applying the Gauss-Green theorem (Fig. 5), together with some tensor calculations and the previous specifications, enables us to obtain

$$\int_{\Omega(\delta \mathbf{u})} \sigma_{ij,j} \delta \varepsilon_{ij} dV = \int_{\partial \Omega(\delta \mathbf{u})} t_i^{(\mathbf{n})} \delta u_i dS + \int_{\Omega(\delta \mathbf{u})} \rho_i \delta u_i dV, \quad (5)$$

where \mathbf{n} is the unit vector along the outward normal to the crack surface, Fig. 5a shows a masonry arch in an admissible equilibrium state under the action of loads \mathbf{p} and internal stress $\boldsymbol{\sigma}$. Figure 5b also shows the displacement field $\delta \mathbf{u}$ with hinges A, B, C and D, together with the corresponding internal cracks BB' and CC'. Figure 5a, b also show:

- the cracks' boundaries $\Gamma(\delta \mathbf{u})$;
- the region $\Omega(\delta \mathbf{u}) = \Omega / \Gamma(\delta \mathbf{u})$ lacking cracks;
- the overall boundary of the body, including the crack boundaries $\partial \Omega(\delta \mathbf{u}) = \partial \Omega \cup \Gamma(\delta \mathbf{u})$.

The entire boundary can also be specified by the union of the boundaries $\Gamma(\delta \mathbf{u})$, $\partial \Omega_r$ and $\partial \Omega_p$

The internal work can now be written in a more explicit form. In fact, according to (5), we have

$$\int_{\Omega(\delta \mathbf{u})} \sigma_{ij,j} \delta \varepsilon_{ij} dV = \int_{\Gamma(\delta \mathbf{u})} t_i^{(\mathbf{n})} \delta u_i dS + \int_{\partial \Omega_r} r_i^{(\mathbf{n})} \delta u_i dS + \int_{\partial \Omega_p} p_i^{(\mathbf{n})} \delta u_i dS + \int_{\Omega(\delta \mathbf{u})} \rho_i \delta u_i dV \quad (6)$$

To work out the first integral in the second member of (6), by moving around the whole contour of the body, the virtual work of the interactions $t_i^{(n)}$ can be evaluated along each of the two edges of the cracks (Fig. 5b). For the sake of simplicity, we can refer to a single crack alone and write

$$\Gamma(\delta \mathbf{u}) = \Gamma_1(\delta \mathbf{u}) \cup \Gamma_2(\delta \mathbf{u}),$$

where $\Gamma_1(\delta \mathbf{u})$ and $\Gamma_2(\delta \mathbf{u})$ are the two equal surfaces representing the two edges of the crack. Evaluating the first integral in the second member of (6) thus gives

$$\int_{\Gamma(\delta u)} t_i^{(n)} \delta u_i dS = \int_{\Gamma_1(\delta u)} t_i^{(n^-)} \delta u_i(P^-) dS + \int_{\Gamma_2(\delta u)} t_i^{(n^+)} \delta u_i(P^+) dS. \quad (7)$$

On the other hand, using expression (2) for the crack opening $\Delta^{(n^-)} u(P)$, we have

$$\delta u_i(P^-) = \delta u_i(P^+) - \Delta^{(n^-)} \delta u_i(P); \quad (8)$$

Substituting (8) into (7) gives

$$\begin{aligned} \int_{\Gamma(\delta u)} t_i^{(n)} \delta u_i dS &= \int_{\Gamma_1(\delta u)} t_i^{(n^-)} \delta u_i(P^+) dS - \int_{\Gamma_1(\delta u)} t_i^{(n^-)} \Delta^{(n^-)} \delta u_i(P) dS \\ &+ \int_{\Gamma_2(\delta u)} t_i^{(n^+)} \delta u_i(P^+) dS. \end{aligned}$$

Furthermore, by taking into account that

$$t_i^{(n^-)} = -t_i^{(n^+)},$$

we get

$$\begin{aligned} \int_{\Gamma(\delta u)} t_i^{(n)} \delta u_i dS &= - \int_{\Gamma_1(\delta u)} t_i^{(n^+)} \delta u_i(P^+) dS - \int_{\Gamma_2(\delta u)} t_i^{(n^-)} \Delta^{(n^-)} \delta u_i(P) dS \\ &+ \int_{\Gamma_2(\delta u)} t_i^{(n^+)} \delta u_i(P^+) dS. \end{aligned}$$

On the other hand,

$$\int_{\Gamma_1(\delta u)} t_i^{(n^+)} \delta u_i(P^+) dS = \int_{\Gamma_2(\delta u)} t_i^{(n^+)} \delta u_i(P^+) dS.$$

In fact, the integral is evaluated on the same surface because $\Gamma_1(\delta\mathbf{u})$ and $\Gamma_2(\delta\mathbf{u})$ are equal. Hence

$$\int_{\Gamma(\delta\mathbf{u})} t_i^{(n)} \delta u_i dS = - \int_{\Gamma_2(\delta\mathbf{u})} t_i^{(n^-)} \Delta^{(n^-)} \delta u_i$$

or

$$\int_{\Gamma(\delta\mathbf{u})} t_i^{(n)} \delta u_i dS = \int_{\Gamma_1(\delta\mathbf{u})} t_i^{(n^+)} \Delta^{(n^-)} \delta u_i dS$$

Summing up the work along all the crack surfaces, we get the virtual work equation

$$\begin{aligned} \int_{\Omega(\delta\mathbf{u})} \sigma_{ij,j} \delta \varepsilon_{ij} dV &= \sum_k \int_{\Gamma_k^1(\delta\mathbf{u})} t_i^{(n^+)} \Delta^{(n^-)} \delta u_i dS + \int_{\partial\Omega_r} r_i^{(n)} \delta u_i dS + \int_{\partial\Omega_p} p_i^{(n)} \delta u_i dS \\ &+ \int_{\Omega(\delta\mathbf{u})} \rho_i \delta u_i dV \end{aligned} \quad (9)$$

With the following definitions:

$$\{\mathbf{t}^{(n^+)}, \Delta^{(n^-)} \delta \mathbf{u}\} = \sum_k \int_{\Gamma_k^1(\delta\mathbf{u})} t_i^{(n^+)} \Delta^{(n^-)} \delta u_i dS; \langle \mathbf{r}, \delta \mathbf{u} \rangle = \int_{\partial\Omega_r} r_i^{(n)} \delta u_i dS;$$

$$\langle \mathbf{p}, \delta \mathbf{u} \rangle = \int_{\partial\Omega_p} p_i^{(n)} \delta u_i dS + \int_{\Omega(\delta\mathbf{u})} \rho_i \delta u_i dV; \langle \boldsymbol{\sigma}, \delta \boldsymbol{\varepsilon} \rangle = \int_{\Omega(\delta\mathbf{u})} \sigma_{ij,j} \delta \varepsilon_{ij} dV,$$

condition (9) becomes

$$\langle \boldsymbol{\sigma}, \delta \boldsymbol{\varepsilon} \rangle = \{\mathbf{t}^{(n^+)}, \Delta^{(n^-)} \delta \mathbf{u}\} + \langle \mathbf{r}, \delta \mathbf{u} \rangle + \langle \mathbf{p}, \delta \mathbf{u} \rangle \quad \forall \delta \mathbf{u} \in M \quad (10)$$

together with the associated compatibility conditions

$$\langle \boldsymbol{\sigma}, \delta \boldsymbol{\varepsilon} \rangle \leq 0 \quad \{\mathbf{t}^{(n^+)}, \Delta^{(n^-)} \delta \mathbf{u}\} \geq 0 \quad \langle \mathbf{r}, \delta \mathbf{u} \rangle \geq 0 \quad (11)$$

where the symbol in parentheses is the integral of the product of stress tensors, stress vectors and reactions with the corresponding virtual strains or detachments. Vice versa, working back from eq. (10), we arrive at equation (39). The two systems of forces and deformations, respectively statically and kinematically admissible, are together connected by the virtual work equation (10) (Fig. 6). Conditions (10) and (11) are *necessary* and *sufficient* for the existence of the admissible equilibrium between external and internal forces.

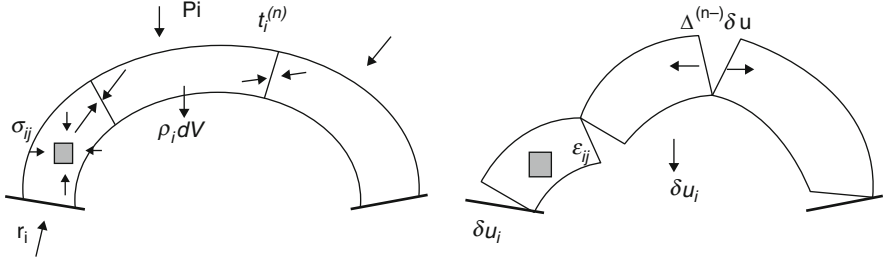


Fig. 6 The two systems of forces and deformations, respectively statically and kinematically compatible, connected together by the virtual work equation

Comparing the current formulation of the same principle for the linear elastic solids with this one for no-tension bodies, the difference is that here the work of the *stress vectors* on the *virtual detachments* $\Delta \delta \mathbf{u}$ must be added, as must also be associated the compatibility conditions (16).

Many relevant properties originate from (10) and (11): particularly the nonexistence of *self-equilibrated stresses*. In this case we speak of deformable masonry structures that can be considered statically determinate systems.

In Como (2010, 2013) I pay special attention to the passage from the general masonry body to the common masonry structures, generally made by assembling piers and arches. Como (2010, 2013) shows that all the foregoing conditions governing the admissible equilibrium of masonry bodies take simpler forms when referred to a one-dimensional structure. For one-dimensional systems distributions of stretching strains in the voussoirs lead to displacements negligible with respect to those produced by relative rotations at hinges. In defining the corresponding mechanisms, it is thus possible to consider only detachments $\Delta^{(n^-)} \mathbf{u}$ arising among voussoirs, where hinges can develop, and consequently neglect any strain $\boldsymbol{\varepsilon}$ that may spread into the voussoirs. We can also assume that the external constraints are fixed. Hence, in this simple case neither the work of the reactions \mathbf{r} , nor the work of stresses $\boldsymbol{\sigma}$ on the strains $\boldsymbol{\varepsilon}$, distributed internally in the voussoirs, will appear in the virtual work equation. With these restrictions, Eq. (10) takes the simpler form

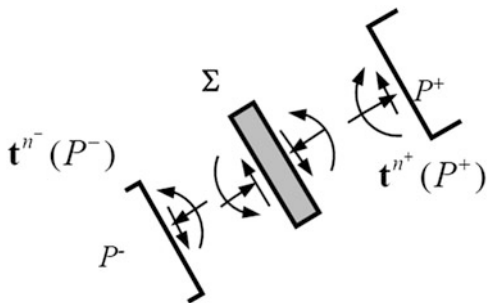
$$\left\{ \mathbf{t}^{(n^+)}, \Delta^{(n^-)} \delta \mathbf{u} \right\} + \langle \mathbf{p}, \delta \mathbf{u} \rangle = 0 \quad \forall \delta \mathbf{u} \in M \quad (10')$$

associated to the admissibility condition

$$\left\{ \mathbf{t}^{(n^+)}, \Delta^{(n^-)} \delta \mathbf{u} \right\} \geq 0. \quad (11')$$

With reference, for instance, to a masonry arch, the forces acting on the lateral sections of a small element of the arch are equal and opposite to the resultant of the stress vectors, $\mathbf{t}^{(n^+)}$ and $\mathbf{t}^{(n^-)}$, acting on the anterior sections (Fig. 7). Consequently, if the work of $\mathbf{t}^{(n^+)}$ on the detachment $\Delta^{(n^-)} \delta \mathbf{u}$ is non-negative, the work of the

Fig. 7 Actions and reactions inside the masonry arch



equal and opposite actions on the detachments themselves will be non-positive. The resultant of forces $\mathbf{t}^{(n+)}$ or $\mathbf{t}^{(n-)}$ acting on the transverse sections delimiting the detachment, where a hinge is formed, can be decomposed into the components axial force N , bending moment M and shear T of the resultant vector Σ .

At the same time, the detachments, $\Delta^{(n-)}\delta\mathbf{u}$, can, in turn, be expressed in terms of the virtual deformation vector, $\mathbf{E}(\delta\mathbf{u})$, whose components are the axial displacement $\delta\Delta$ and the relative rotation $\delta\phi$. In brief we can write

$$\{\mathbf{t}^{(n+)}, \Delta^{(n-)}\delta\mathbf{u}\} = -\langle \Sigma, \mathbf{E}(\delta\mathbf{u}) \rangle,$$

and the equation of virtual work (10') becomes

$$\langle \mathbf{p}, \delta\mathbf{u} \rangle = \langle \Sigma, \mathbf{E}(\delta\mathbf{u}) \rangle \quad \forall \delta\mathbf{u} \in M, \tag{10''}$$

and the admissibility conditions on the stresses are

$$\langle \Sigma, \mathbf{E}(\delta\mathbf{u}) \rangle \leq 0. \tag{11''}$$

The virtual work equation (10'') thus takes the typical simple form.

Although conditions (10) and (11) or (10'') and (11''), are necessary and sufficient to guarantee the existence of admissible equilibrium, they must be satisfied by both the loads and the internal stresses. However, these latter may be *a priori* unknown. It is on the other hand possible to prove that the variational inequality on loads \mathbf{p} alone

$$\langle \mathbf{p}, \delta\mathbf{u} \rangle \leq 0, \quad \forall \delta\mathbf{u} \in M \tag{12}$$

is necessary and sufficient to guarantee the existence of the admissible equilibrium state. It should be noted that the mechanisms $\delta\mathbf{u}$ represent the various deformation modes of the body. Inequality (12) thus simply means that the body is in an admissible equilibrium state under loads \mathbf{p} iff the work of these loads \mathbf{p} is not positive along any possible deformation of the body. Necessity follows immediately from (10) and (11). In the context of elastic no-tension models, proofs of the

sufficiency of condition (12) have been furnished in Romano and Romano (1985) and Romano and Sacco (1984). A simple proof, in the framework of the rigid no-tension model, was presented in Como (1992). The main lines of this latter proof, analysed in depth in Como (2010, 2013), are the following.

If the variational inequality (12) was only necessary, but insufficient, it could be also satisfied by unsustainable loads \mathbf{p} . Such a situation is however impossible. It is in fact shown that any load \mathbf{p} that is unsustainable by the body and that consequently sets the body in motion, does *positive* work on displacement \mathbf{v} along which the body begins to move. This contradiction with the assumption proves the statement.

5 Weight and Geometry: Essential Resources of Masonry Strength

Loads can be considered composed as

$$\mathbf{p} = \mathbf{g} + \lambda \mathbf{q} \quad (13)$$

where \mathbf{g} and \mathbf{q} are respectively the dead and live loads and λ a load multiplier. Properties of these loads differ considerably. Live loads \mathbf{q} , affected by the loading parameter λ , can exert a pushing action along some mechanism. As a rule, the weight, \mathbf{g} , on the contrary, represents the resistant load for a masonry structure. This statement is frequently stressed in Como (2010, 2013). Consequently, recalling condition (12), the structure will certainly be *safe* under the action of its own weight \mathbf{g} if the following condition is satisfied:

$$\langle \mathbf{g}, \mathbf{v} \rangle < 0, \quad \forall \mathbf{v} \in M \quad (14)$$

The weight will always *oppose* any deformation of a *safe* masonry structure. For a safe arch, for instance, the pressure line corresponding to the weight alone will always be contained within the arch: it can never touch the arch extrados or intrados, at any section. The contribution to strength of the weight \mathbf{g} comes by virtue of the structure's *geometry*. Masonry structures must be designed so that the mechanisms produce vertical displacements in which lifting is always dominant, thereby satisfying condition (14) for any mechanism. It is the *geometry* that ensures that the structure's weight counters the emergence of any mechanisms.

Weight and geometry represent the essential elements in the strength of masonry structures. More precisely, it is the *proportions* among a structure's various constituent parts and the structure itself that define its geometry, irrespective of the actual absolute dimensions. This aspect of the masonry behaviour is examined in particular depth in Como (2013).

6 Mechanism State

Como (2010, 2013) examines various admissible equilibrium states that occur in a masonry body. Among them the *mechanism state* is the more relevant. In this condition a structure at an admissible equilibrium under loads \mathbf{p} can be *freely deformed* along a given mechanism displacement $k\mathbf{u}_m$, defined by an arbitrary, but small, constant k . In such a state internal stresses and constraint reactions do not counter the emergence of the mechanism. Consequently also the external loads \mathbf{p} also offer no opposition to the development of the mechanism displacement \mathbf{u}_m and

$$\langle \mathbf{p}, \mathbf{u}_m \rangle = 0. \quad (15)$$

7 Collapse State

Como (2013) also shows that the *collapse state* is a particular mechanism state. Let us, in fact, consider a masonry structure under a loading path $\mathbf{p}(\lambda)$, where λ is the loading parameter. According to (13), the loads $\mathbf{p}(\lambda)$ will be made up of the resistant component \mathbf{g} , i.e., the dead loads, and of the pushing forces $\lambda\mathbf{q}$. At some stage of the loading process, when λ attains a critical value λ_c , the structure will *reach a mechanism state* defined by the mechanism \mathbf{u}_c . The work done by the forces $\mathbf{p} = \mathbf{g} + \lambda_c\mathbf{q}$ vanishes along the mechanism \mathbf{u}_c , which is to say

$$\langle \mathbf{g} + \lambda_c\mathbf{q}, \mathbf{u}_c \rangle = 0, \quad \lambda_c > 0, \quad \mathbf{u}_c \in M.$$

We admit that the live loads \mathbf{q} *push* along \mathbf{u}_c , so that

$$\langle \mathbf{q}, \mathbf{u}_c \rangle > 0. \quad (16)$$

Condition (16) evidences the presence of a pushing action by live loads \mathbf{q} along displacement \mathbf{u}_c , the *failure mechanism*. Thus, as soon as the loading parameter λ is further increased beyond λ_c , we have

$$\left(\frac{d}{d\lambda} \langle \mathbf{p}(\lambda), \mathbf{u}_c \rangle \right)_{\lambda_c} = \langle \mathbf{q}, \mathbf{u}_c \rangle > 0.$$

Accordingly, condition (12), necessary and sufficient for the *existence of an admissible equilibrium state*, is violated and the structure *fails*. At this collapse state an exchange occurs from conditions of existence to those of non-existence of the admissible equilibrium state. The *development* of the failure mechanism can be represented by a sequence $k\mathbf{u}_c$ of mechanisms of increasing amplitude. Collapse thus occurs under *constant loads*, because by gradually increasing the constant

$k > 0$, we consistently have

$$\langle \mathbf{g}, k\mathbf{u}_c \rangle + \lambda_c \langle \mathbf{q}, k\mathbf{u}_c \rangle = 0, \quad k\mathbf{u}_c \in M, \quad k > 0$$

for any amplitude of mechanism $k\mathbf{u}_c$. Constant loads also imply constant stresses. The failure mechanism thus develops under frozen loads and stresses. Como (2010, 2013) points out the *extraordinary situation* that occurs at the collapse. There is no energy dissipation. Nevertheless, the masonry structure is able to maintain its limit strength during the development of the failure mechanism, as occurs for a steel bar upon yielding. Despite the lack of dissipation, the behaviour at collapse of masonry structures is similar to that of ductile steel structures, as predicted by Limit Analysis.

8 The Theory of Proportionality in Architecture

Let us examine the two similar arches a and A in Fig. 8. Arch A , on the right, is k times larger than the arch a , on the left; in other words, arch A is a k times *magnified copy* of arch a . In the transverse direction, i.e., in the direction orthogonal to their plane, the structures have the same width s . Each segment in structure A is thus k times longer than the corresponding segment in structure a .

Let us now assume that structure a is *stable* under its own weight g , as defined according to (14). Thus we have

$$\langle g, v \rangle = \sum_1^N g_i v_i < 0$$

for any mechanism v . The work $\langle g, v \rangle$ is evaluated considering the work of the weight forces g_i of the various voussoirs composing the arch on the corresponding

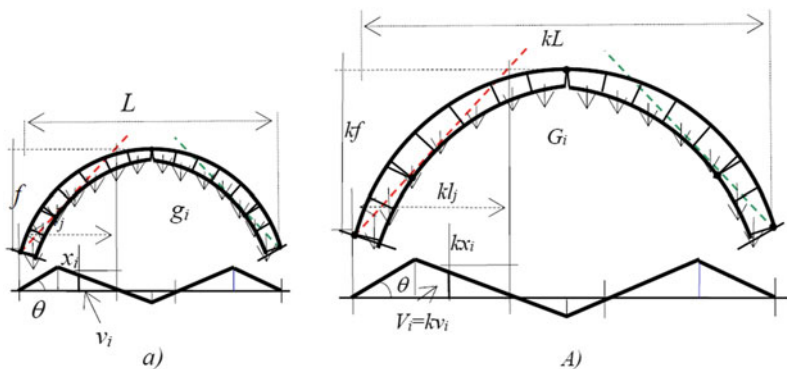


Fig. 8 Geometries of two similar arches and of the two corresponding mechanisms governed by the dimension ratio k

vertical displacements v_i of the mechanism. Consequently the k magnified structure A is thus also *stable* under its own weight. In fact, for any mechanism V corresponding to v , we will have

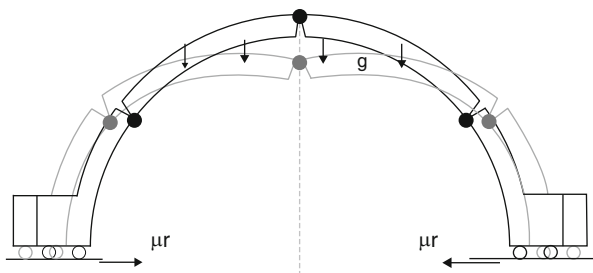
$$\langle G, V \rangle = \sum_1^N G_i V_i = k^3 \sum_1^N g_i v_i < 0.$$

Thus, to conclude, *if a structure under its own weight is stable, a k times magnified copy will also be stable*. The same outcome holds in a more general sense. These results, proved in Como (2013) as a direct consequence of the essential features of the masonry behaviour, were already known to architects of the past and formed the basis for their fundamental rules of construction. As set down in the theory of proportions by Andrea Palladio and Leon Battista Alberti, statics of masonry structures is governed solely by their geometry and, consequently, by their basic measurement, *the modulus*, irrespective of their absolute measurements. Knowledge of the most suitable proportions among the various components of a masonry structure, often jealously guarded by past masters, represented the essence of the art of construction. These results arrived to us through centuries of long experience, and are direct consequence of the unique, fundamental masonry behaviour. This theory of proportions was strongly opposed by Galileo Galilei in his *Discorsi e dimostrazioni matematiche intorno a due nuove scienze* (1638) by means some sharp arguments involving the local states of stresses in the material but not pertaining to the behaviour of masonry structures, behaviour only marginally influenced by local stresses. The argument of the theory of proportions has been also debated by other scholars (Heyman 1997; Huerta 2006; Di Pasquale 1996; Benvenuto 1981, 1991; Baratta 1999). We observe that compressions increase in the larger structure by increasing the ratio k , while the masonry strength remains constant. Hence the proportionality rule holds as long as compression stresses remain low. This is indeed all that it happens in masonry structures, as clearly shown by Heyman (1997) and Huerta (2006).

9 Settlement State: The Minimum Thrust Theorems

The case that a mechanism state can be attained at a settlement state is frequently met in Como (2010, 2013). This is another aspect of the masonry equilibrium. Let us consider a masonry structure that is at a *safe admissible equilibrium* state at the configuration C_i under the actions of loads \mathbf{g} . Inequality (14) thus holds. The structure becomes now deformed as a consequence of a *slight settlement* occurring at one of its external constraints. The structure deforms with the mechanism displacement \mathbf{v}_s , due to this settlement. By way of example, consider the arch in Fig. 9, which undergoes a slight increase in span due to settling.

Fig. 9 The settled arch



For sake of simplicity in what follows we will make reference to the case of one-dimensional structures and, therefore, to the simpler formulation (10'') of the virtual work equation. However, the internal stresses, represented by the vectors Σ having components N , M and T , and the strain vectors \mathbf{E} , with components Δ and ϕ , for sake of simplicity, will still be denoted by σ and ϵ .

Let C_s be the configuration taken on by the structure once the settlement has occurred. By assuming that C_s is *very near to* C_i , we can refer to the geometry of the initial configuration C_i when expressing the equilibrium equations. The settlement mechanism, \mathbf{v}_s is the displacement field that moves the structure from C_i to C_s . As the settlement occurs, the structure's internal equilibrium shifts from initial configuration C_i to the displaced one C_s . Changes in the internal stresses and constraint reactions will occur during the transition from C_i to C_s , so that the initial stress state σ_i is altered and becomes σ_s .

This internal stress state, σ_s , which accounts for settlement \mathbf{v}_s , is statically admissible and thus satisfies the inequality

$$\langle \sigma_s, \epsilon(\delta \mathbf{u}) \rangle \leq 0.$$

Likewise, the settled constraint which, *before* the settling, produced the reaction $\mu_i \mathbf{r}$, *after* the settling produces the new reaction

$$\mu_s \mathbf{r},$$

where \mathbf{r} is a given force having the direction of reaction of the settled constraint and μ is the corresponding multiplier. In brief, during the development of the settlement mechanism \mathbf{v}_s the structure will remain in a state of admissible equilibrium while the stresses vary from σ_i to σ_s , the corresponding pressure line shifts from π_i to π_s and the reaction of the settled constraint changes from $\mu_i \mathbf{r}$ to $\mu_s \mathbf{r}$. For instance, in the case of the masonry arch that has undergone a slight increase in span, its pressure line π_s will pass through the hinges corresponding to mechanism \mathbf{v}_s . Consequently, no work will be done by the internal stresses σ_s on the deformations corresponding to \mathbf{v}_s . The same occurs for any structure that is deformed by a mechanism and adapts itself to settling. Thus, at the settlement state the following *mechanism state* holds

$$\langle \sigma_s, \epsilon(\mathbf{v}_s) \rangle = 0. \quad (17)$$

We can *release* the structure by removing the settled constraint by applying the reaction $\mu_s \mathbf{r}$ to the eliminated constraint. The set of all the mechanisms of the released structure is denoted by \overline{M} . In the released structure at the settled state, the applied loads are represented by both the weights \mathbf{g} and the reactions $\mu_s \mathbf{r}$. Thus, considering the released structure at the admissible settled equilibrium state, from the virtual work equation (10'') we get

$$\langle \mathbf{g}, \delta \mathbf{u} \rangle + \mu_s \langle \mathbf{r}, \delta \mathbf{u} \rangle = \langle \boldsymbol{\sigma}_s, \boldsymbol{\varepsilon}(\delta \mathbf{u}) \rangle, \quad \forall \delta \mathbf{u} \in \overline{M}$$

which for $\delta \mathbf{u} = \mathbf{v}_s$, according to (17), yields

$$\langle \mathbf{g}, \mathbf{v}_s \rangle + \mu_s \langle \mathbf{r}, \mathbf{v}_s \rangle = 0.$$

Loads \mathbf{g} perform positive work along the mechanism displacements \mathbf{v}_s while the reaction $\mu_s \mathbf{r}$ of the released constraint opposes settling, so that

$$\langle \mathbf{g}, \mathbf{v}_s \rangle > 0$$

and

$$\mu_s \langle \mathbf{r}, \mathbf{v}_s \rangle < 0. \quad (18)$$

In spite of the settling that occurred, the work of the loads due to any mechanism $\delta \mathbf{u}$ is still the same as the work evaluated at the initial configuration C_i , assuming that displacements \mathbf{v}_s are *very small*, as above stated, and that the changes in geometry are consequently *negligible*. Thus, if at the initial state C_i , the admissibility condition $\langle \mathbf{g}, \delta \mathbf{u} \rangle < 0, \quad \forall \delta \mathbf{u} \in M$ is satisfied, the same condition will still be satisfied by the new configuration C_s . In this regard we meet Heyman's statement: "*if the foundations of a stone structure are liable to small movements, such movements will never, of themselves, promote the collapse of the structure*" (Heyman 1966, p. 255). Moreover, if settlement \mathbf{v}_s increases and becomes

$$k \mathbf{v}_s, \quad k > 1$$

the static arrangement of the structure will not change, and the internal stresses will remain fixed at $\boldsymbol{\sigma}_s$. In short, the structure freely follows any increase in the settlement, maintaining its configuration in admissible equilibrium. Settling develops with frozen internal stresses $\boldsymbol{\sigma}_s$ and constraint reactions, $\mu_s \mathbf{r}$. The actual degree of settling is difficult to quantify. Despite this uncertainty, the internal stress state of the structure is, to the contrary, well-defined. No equilibrium loss will occur during the settling. This is a peculiar aspect of masonry structures that can explain the great durability and longevity of so many historic buildings.

How do we evaluate this stress state and the corresponding reaction of the settled restraint? (Como 2013) shows that we have, in a reversed form, the static and the kinematical theorems of the minimum thrust. The static theorem of the minimum

thrust affirms that

$$\mu(\boldsymbol{\sigma}) \geq \mu_s \quad \boldsymbol{\sigma} \in S$$

i.e., the multiplier, μ_s of the settled thrust \mathbf{r} is thus *lower* than all the statically admissible multipliers μ . This general finding (Como 1996, 1998) relates to the particular property of the masonry arch that undergoes an increase in span due to settling at its springings: such an arch is at the state of minimum thrust because its pressure line corresponds to the minimum span and the maximum sag, as shown by Heyman (1966).

Como (2010, 2013) examines specifically the settlement equilibrium from a *kinematical* point of view. This last point is very interesting and produces *new approaches* to tackle the problem of the static analysis of settled structures. The actual settlement mechanism is unknown: for instance, for the case of the arch of Fig. 9, we cannot know the position of the internal hinge of the settlement mechanism. We only know that, during the development of the mechanism, loads \mathbf{g} will do positive work, while the work of the reaction of the settled constraint is, to the contrary, negative. Let us consider any settlement mechanism

$$\mathbf{v} \in \overline{M}$$

of the released structure. The loads \mathbf{g} will push along \mathbf{v} and consequently

$$\langle \mathbf{g}, \mathbf{v} \rangle > 0. \quad (19)$$

We define the *kinematic multiplier* λ of the reaction \mathbf{r} of the settled constraint as that multiplier able to ensure equilibrium of the structure along the assumed settlement mechanism \mathbf{v} , or, in other terms, such that the following condition holds

$$\langle \mathbf{g}, \mathbf{v} \rangle + \lambda \langle \mathbf{r}, \mathbf{v} \rangle = 0.$$

Reaction $\lambda \mathbf{r}(\mathbf{v})$ opposes the development of settling \mathbf{v} , given that, by taking (18) into account, we get

$$\lambda \langle \mathbf{r}, \mathbf{v} \rangle < 0. \quad (20)$$

The kinematical multiplier $\lambda(\mathbf{v})$ of reaction \mathbf{r} is thus defined as

$$\lambda(\mathbf{v}) = -\frac{\langle \mathbf{g}, \mathbf{v} \rangle}{\langle \mathbf{r}, \mathbf{v} \rangle}, \quad \mathbf{v} \in \overline{M}. \quad (21)$$

With these definitions it is easy to prove that (Como 1996, 1998)

$$\lambda(\mathbf{v}) \leq \mu_s \quad \forall \mathbf{v} \in \overline{M}. \quad (22)$$

For any settlement mechanism, $\mathbf{v} \in \overline{M}$, the corresponding kinematic multiplier, $\lambda(\mathbf{v} \in \overline{M})$, can never be *greater* than the actual settlement multiplier μ_s . Thus μ_s is *the maximum* of all kinematic multipliers, $\lambda(\mathbf{v} \in \overline{M})$, for varying \mathbf{v} in the set of all settlement mechanisms \overline{M} , or in other terms

$$\mu_s = \text{MAX} \left(- \frac{\langle \mathbf{g}, \mathbf{v} \rangle}{\langle \mathbf{r}, \mathbf{v} \rangle} \right) \quad \mathbf{v} \in \overline{M}. \tag{23}$$

This result makes it possible to analyse the actual equilibrium states of structures with a new easier approach than the static one, that make use of funicular polygons.

10 Actual Equilibrium States of Masonry Structures

Stress analysis of a masonry construction, rigidly constrained to a rigid environment, cannot be developed in the context of the rigid-in-compression no-tension model. The admissible equilibrium equations alone are not enough to evaluate the internal stresses, as such a problem is statically indeterminate. To this end, additional equations are required: the compatibility equations, as in the case of elastic structures. On the other hand, the assumption of constraints rigidly connected to a rigid external environment is physically meaningless. A certain degree of settlement, as a rule, occurs in the external constraints of the structure. The same deformation of the supporting structures drives displacements of the structure placed above.

The presence of elastic strains in the analysis of settled structures is really insignificant. As soon as the settlement starts, after a very negligible elastic stage, the masonry structure transforms immediately into a mechanism. The same occurs for the rigid in compression no tension structure. This aspect of the problem is particularly studied in Como (2010, 2013). Useful information can thus be obtained in the same context of the simple rigid no-tension model providing that the deformability of the structure constraints is taken into account. A *minimal thrust state* takes place. The degree of settling can be predicted only with difficulty. Thankfully, the compatibility equations expressing the occurrence of settling do not require defining the magnitude of the settlement, but only indication of the settled constraints. According to this approach, the problem of the determination of the actual stress state in masonry structures becomes statically determinate and Limit Analysis can once again be fruitfully applied, as thoroughly shown in Como (2010, 2013).

11 Minimum Thrust States in Masonry Domes: Application of the Kinematical Approach

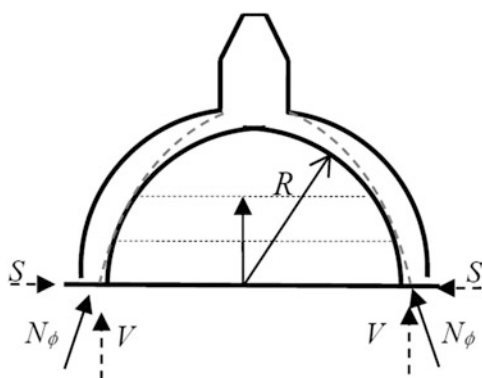
Useful applications of this last approach are described in Como (2010, 2013), as in the study of the actual equilibrium states of arches, domes, cross and cloister vaults, with particular attention to the more relevant cases in architecture.

With reference to domes, for instance, the occurrence of unavoidable deformations of their supporting structures yield the vanishing of hoop stresses and meridian cracks, starting from the dome springings, will occur. The cracked dome tends to open along a large band breaking up into slices and behaves as a set of independent pairs of semi-arches leaning on each other. Predictably, cracking brings about a profound change in the dome's statics.

A small cap at the top of each slice will be subjected to the thrusting action transmitted by the other slices, which will be transmitted all the way to the springing. Figure 10 shows an approximate sketch of the pressure curve of a cracked hemispherical dome. The dotted line shows the position of this curve, which inclines towards the horizontal at the springing. The horizontal component of the reaction of the supports represents the thrust S per unit length of the dome's base circumference. The thrust thus occurs in the passage of the stresses from the initial membrane state to the no tension state.

The emergence of thrust in the dome represents the most consequential outcome of meridian cracking in typical masonry round domes. Loaded by the dome's thrust, the sustaining structures, e.g., the drum or underlying piers, deform and splay. The slices, no longer restrained from deforming by rings, bend under the loads and can form mechanisms. The weight of a particularly heavy lantern, for example, could even cause the dome to fail on cracking. Thrust yields a more or less relevant further deformation of the dome supporting structures. The settled dome mobilizes a thrust that it is the *minimum* from among all the thrusts S transmitted by statically admissible pressure curves. The minimum thrust S_{Min} can be obtained via the *static*, as well as the *kinematic* approach. The *static* approach calls for tracing the statically admissible funicular curves of the loads. In the settled state the pressure curve passes

Fig. 10 Rising thrust due to meridian cracking



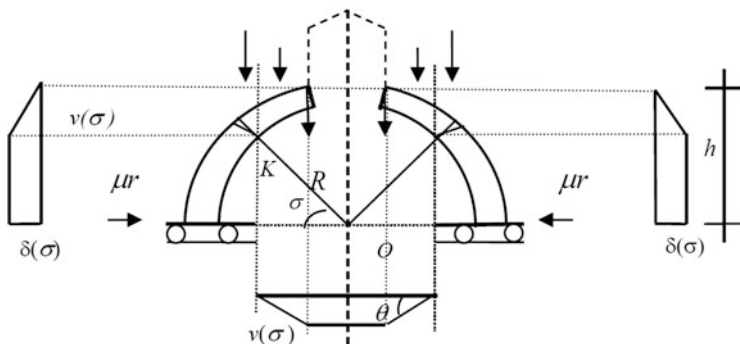


Fig. 11 Minimum thrust evaluation according to the kinematic approach

through the extrados at the key section of the slices and then runs within their interior, *skimming* over the intrados of the dome (Fig. 10). The *kinematic* approach is dual with respect to the static one and is ruled by (21) and (23) given earlier. In (23) $\langle g, v \rangle$ represents the work, undoubtedly positive, of the dead loads on the vertical displacements of mechanism v , and $\langle r, v \rangle$ the work, undoubtedly negative, performed by the thrust on the corresponding horizontal displacement. Figure 11 shows a generic dome mechanism produced by a base widening. In this mechanism the position of the internal hinge K is unknown.

The set of all these kinematically admissible mechanisms is described by varying the position of the hinge K between the springing and the key section of the slice. Identifying the maximum of function $\lambda(v)$ by varying the position of hinge K enables us to obtain the sought-for thrust. Many applications of this approach are described in Como (2010, 2013). It is, in fact, a relatively simple matter to apply the kinematic approach to evaluate the minimum thrust of masonry domes. The settlement mechanisms are obtained releasing the slices by positioning hinges to allow horizontal sliding of the dome at its springings. Hinges must thus be positioned (Fig. 11):

- at the *extrados*, on the section linking the slice with the central closing ring sustaining the lantern;
- at the *intrados*, at the haunches. The position of this hinge is unknown and is indicated by the angle σ (Fig. 11). Thus, the minimum thrust $\mu_{min}S$ is evaluated by seeking the maximum of the function

$$\mu_{min}S = Max \frac{\langle g, v(\sigma) \rangle}{\delta(\sigma)}$$

by varying angle σ along the intrados and where

$$\delta(\sigma) = (h - R \sin \sigma) \theta$$

is the horizontal displacement of the slice at springing. According to the kinematic theorem the search for the minimum thrust thus translates into searching for the maximum of the function

$$\mu S(\sigma) = \text{Max} \frac{(g, v(\sigma))}{(h - R \sin \sigma) \theta}$$

by varying the angle σ along the dome intrados. This approach has been applied to study the statics of the domes of S. Maria del Fiore in Florence and of St. Peter's in Rome.

12 Book Contents

Como (2010, 2013) is divided into nine chapters, each of which begins with historical notes and an introduction highlighting the main aspects of the topics covered. The strength and deformability of masonry materials are addressed in the first chapter. The second chapter deals with the deformation and equilibrium of masonry solids. The third and fourth chapters examine the static behaviour of the main basic masonry structures, such as arches and vaults. By way of example, static analysis are conducted of a number of renowned examples from the world's architecture heritage, such as ancient Mycenaean domes, the Pantheon in Rome, the large cross vaults of the Baths of Diocletian, and the domes of Santa Maria del Fiore in Florence and Saint Peter's in Rome. The fifth chapter turns to a detailed analysis of the statics of the Colosseum in Rome and examines the reasons for its actual state of damage. The sixth chapter describes and analyzes the statics of cantilevered stairways, a typical element whose structural behaviour is still somewhat unknown. Chapter seven then takes up the structural analysis of walls, piers and towers under vertical loads. The stability of such structures is heavily affected by the non-linear interactions between the destabilizing effects of the axial loads and masonry's no-tension response. The instability of towers, leaning towers in particular, is addressed in a specific section of the chapter. In this regard, a detailed stability analysis is conducted of the famous leaning Tower of Pisa, which has recently undergone a successful restoration work. The eighth chapter then analyzes the statics of Gothic cathedrals, with particular reference to analysis of their resistance to wind actions. The 1,294 collapse of the Beauvais cathedral is also examined in depth. The last chapter deals with the seismic behaviour of historic masonry buildings.

Como (2010, 2013) is addressed especially to researchers, engineers and architects operating in the field of masonry structures and of their consolidation and restoration, as well as to students of civil engineering and architecture.

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