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Preface

We are pleased to present the book *Masonry Structures*: *Between Mechanics and Architecture*, sponsored by the Associazione Edoardo Benvenuto per la ricerca sulla Scienza e l'Arte del Costruire nel loro sviluppo storico in collaboration with the Dipartimento di Scienze per l'Architettura of the University of Genoa.

The idea of a book on masonry structures arises from the privileged context in which the Associazione Edoardo Benvenuto has carried out its activities in recent years. In fact the Associazione has been able to count on the participation of scholars of international prestige to its research and editorial initiatives, under the honorary presidency of Jacques Heyman. The book belongs to the series *Between Mechanics and Architecture*, born in 1995 from the collaboration of several internationally renowned scholars, including Edoardo Benvenuto. The first book in the series was *Entre Mécanique et Architecture/Between Mechanics and Architecture*, edited by Patricia Radelet-de Grave and Edoardo Benvenuto (Birkhäuser 1995).

As is well known, the topic of masonry structures is very complex and subject to multiple interpretations. In addition to historical studies, the mechanical behaviour of masonry arches and structures has been studied according to different lines of research (structural analysis, limit analysis, elastic analysis, plasticity, mathematical approaches, etc.), sometimes difficult to reconcile, sometimes intertwined with each other and complementary. Although we are aware that it is not possible to include in a single book the diversity of the studies on masonry structures, we have tried to represent the main approaches in order to make it easier for the reader to compare and evaluate their significance and interest.

In addition to selecting the papers published here, the editors have also played the role of reviewers of the manuscripts in conformance with the standards of peer review. In one case, in which one of the co-editors was also the co-author of a contribution, recourse was made to an external referee of international experience.

The introductory chapter, "Between Mechanics and Architecture: The Quest for the Rules of the Art" by Salvatore D'Agostino, addresses a fascinating topic: the quest for the "rules of the art", that is, the methods and procedures defined by complex experiences and verified by a practice which may be centuries old.

Continuing in the context of the search for such rules, in "Designing by "Expérience": Lecreulx Model Tests for the Design of the Abutments of the Bridge of Fouchard", Santiago Huerta investigates the role of experimentation to assess the stability of masonry arches. In the 1770s, the French engineer Jean-Rodolphe Perronet introduced a new type of masonry bridge, with very slender piers and extremely surbased segmental arches. Huerta examines the tests made by François Michel Lecreulx in 1774 during the construction of the bridge of Fouchard. The results demonstrated the enormous danger of a catastrophic failure by sliding. Huerta points out that Fouchard's experiments must have been influential in the great increase of the size of buttresses from the original designs of the 1770s in all the bridges built (most of them completed after 1780). In the Appendix to his chapter, Huerta provides the transcription of the original Memoir by Lecreulx, never before published.

The complexity of the mechanics of masonry structures emerges clearly in the chapter by Mario Como, "Statics of Historic Masonry Constructions: An Essay", author of *Statics of Historic Masonry Constructions* (Springer 2013). Como discusses the adopted hypotheses and the key passages of the main issues involved: the special features of the masonry behaviour, Heyman assumptions and their extension to the masonry continuum, the definition of the admissible equilibrium for the masonry solid by employing the principle of virtual work for masonry bodies.

From a historical point of view, the first approach to the study of mechanical behaviour is limit analysis, rooted in the contributions of Philippe de La Hire and Charles-Augustin de Coulomb. According to this line of reasoning, the masonry structures, in particular the arches, are conceived as a system of rigid blocks, focusing on the collapse mechanism and the determination of the ultimate load. In the twentieth century, this type of approach was taken up by various scholars from the point of view of the modern theory of plasticity. On the other hand, elastic analysis starts from the work of Claude-Louis Navier and from subsequent studies by Francesco Crotti, Carlo Alberto Castigliano, Ferdinand Gros de Perrodil and Antonio Signorini that little by little have contributed to define masonry structures as statically indeterminate elastic structures. This approach aims to describe the evolution of the stress and strain fields with increasing applied loads. If the solution of Castigliano is the outcome of nineteenth-century research on the statics of masonry vaults conceived as systems with linear elastic behaviour, in the twentieth century the issue about an adequate modelling of masonry material arises. This topic has led—even recently—to a renewed interest in the study of no-tension materials and in nonlinear elastic analysis of masonry arches.

The present volume contains some contributions focused on the mechanics of arches and masonry constructions, providing an overview of the recent state of the art on the matter.

In "Equilibrium Analysis", Jacques Heyman underlines the fact that only rarely do deformations of a masonry structure need to be computed; deformations arise, almost without exception, from displacements imposed by movements of the environment (sinking of foundations, spread of abutments), and such deformations, notably cracking, do not depend on the elastic properties of the masonry.

Anna Sinopoli, in "A Semi-analytical Approach for Masonry Arch Dynamics", proposes an analytical approach, firstly applied to the plane dynamics of a rectangular block simply supported on a moving base and then extended to the case of the arch, where each element is characterized at most by a double extended contact. This approach constitutes a first step for performing dynamic analysis through either an event-driven or a time-stepping numerical procedure.

The chapters that follow examine the mechanical behaviour of masonry structures found in historical buildings. In " On the Statics of the Dome of the Basilica of S. Maria Assunta in Carignano, Genoa ", Andrea Bacigalupo, Antonio Brencich and Luigi Gambarotta study in depth the sixteenth-century dome designed by Galeazzo Alessi, in which meridian cracking, rather common in masonry domes, requires the assessment of the dome's safety. In order to set a general procedure for the assessment, limit analysis approaches are discussed and compared. On the basis of classic limit analysis, local (dome only) and global (dome-drum system) collapse mechanisms are examined considering the different behaviour of several structural elements (lantern, shells of the dome, drum, colonnade). Comparisons between the results obtained are carried out in order to discuss a general approach to the assessment of dome–drum systems based on both numerical tools and standard limit analyses approaches; they provide a first glance in the assessment of the dome of the Basilica.

In "The Panthéon's Stability Already Questioned by Pierre Patte in 1770" Patricia Radelet-de Grave analyses some aspects of great historical interest related to the construction of the Panthéon in Paris. Conceived and initiated by Jacques Germain Soufflot, the construction of the Pantheon was continued after his death by Jean Baptiste Rondelet. This impressive structure was the object of various publications. As early as 1770, Pierre Patte pointed out stability problems in his *Mémoire*. Rondelet, a spokesman for Soufflot, does not answer to Patte, but writes a few notes on his copy of Patte's *Mémoire*.

In the chapter that follows, "Transcription of Patte's 1770 *Mémoire* on the Panthéon's Stability Together with Rondelet's *Marginalia*", Radelet-de Grave provides her transcription of the historical text of 1770, along with the *Marginalia* written by Rondelet on Patte's *Mémoire*.

Other authors of chapter in this volume use the approach of elastic analysis to study different types of masonry arches and structures.

In "Notes on Limit and Nonlinear Elastic Analyses of Masonry Arches", Danila Aita, Riccardo Barsotti and Stefano Bennati suggest a parallel study of masonry arches via both non-linear elastic analysis, taking up the groundbreaking work of Signorini, and the so-called "method of stability areas", originally proposed by Alfred Durand-Claye in 1867. Rather than offering two alternative paths, the two approaches may be considered complementary points of view on the same problem: the stability area method represents a particularly simple means for determining collapse load under conditions of limited material compressive strength, whereas the non-linear elastic analysis provides a helpful and, in some aspects, essential check of the former's mechanical significance by following the evolution of the displacement field and the extension of the non-linear regions.

In his chapter "Some Aspects on the Statics of Masonry Arches", Elio Sacco writes the equilibrium equations for the arch subjected to a distribution of pointwise forces acting on nodes lying on the line of thrust. He then determines the line of thrust for a prescribed arch geometry and loading distribution by solving a non-linear constrained minimization problem and formulates the problem of the elastic arch making use of the force method. Furthermore, the effects of the horizontal settlement of the impost of the arch are investigated.

Massimiliano Lucchesi, Miroslav Šilhavý and Nicola Zani, in "A Direct Approach to Membrane Reinforced Bodies", deal with membrane reinforced bodies. The membrane is treated as a two-dimensional surface with concentrated material properties. Its response is linearized and depends linearly on the surface strain tensor. The response of the matrix is treated separately in three cases: as a non-linear material, as a linear material and finally as a no-tension material. An example presenting an admissible stress solution is given for a rectangular panel with membrane occupying the main diagonal plane.

The chapter by Piero Villaggio, "The Thrust of an Elastic Soil of Variable Density against a Rigid Wall", is one of the last works written by the Professor, who passed away in January 2014, and it is a great honour for us to publish it. Villaggio examines the thrust of an elastic soil of variable density against a rigid wall, with reference to soil mechanics and complex variable method in elasticity. The theory of the equilibrium of a wall retaining earth masses was formulated by Coulomb in 1773. This topic is of great interest today, since Coulomb's theory is still applied by engineers in order to design walls. However, while Coulomb assumes that the material is earthy, like sand of soft clay, in actual fact soil often behaves elastically, and thus the stress state inside the mass and the associated pressures on the retaining walls are different. Thus, the question arises of how to analyse the elastic stress state in a heavy medium in contact with a rigid plane, and how to determine the stress distribution at the interface. The chapter by Villaggio certainly provides an important perspective on this issue, which remains an open question to date.

We have chosen to conclude the book with a few pages written by Stefano Bennati to honour the memory of Piero Villaggio. Bennati, who worked with professor Villaggio for many years, offers us the opportunity to remember his selfless love for knowledge, his unconditional dedication to work and his rectitude and integrity. To Piero Villaggio, we are grateful for giving us a valuable paradigm of a scholar who is coherent, passionate and humble.

The present volume is intended to offer a useful tool and interesting insights for further research, since it contains important contributions to an overall picture of the state of the art on masonry structures. The reader is offered the possibility to compare different theoretical lines of inquiry (construction history, structural analysis, limit analysis, elastic analysis, plasticity, mathematical approaches, etc.) and is thus invited to go towards new horizons of research.

In closing, we wish to thank co-editor Kim Williams for her careful revision of all chapters following peer review.

Genova, Italy

Danila Aita
Orietta Pedemonte

Contents

Between Mechanics and Architecture: The Quest for the Rules of the Art

Salvatore D'Agostino

Abstract The ancient conception of construction, from the fourth millennium B.C. through the entire eighteenth century A.D., was based on the transmission of the 'rules of the art' of building. In the nineteenth century it was based on the development of mechanics applied to construction. It was revolutionised in the twentieth century by the creation of construction science and industrial material. Ancient architecture is now re-read in terms of mechanics, with the serious risk of betraying the ancient concepts. Instead, these should be examined with the aim of discerning the rules that governed the original construction.

Keywords Mechanics • Architecture • Masonry structures • Rules of the art

1 The Ancients' Concept of Construction

Living and building were the primordial requisites of *Homo sapiens* for a stable occupation of the territory and the construction of the earliest communities. He observed nature in its infinite configurations and continuous evolution, picking up ideas and hints about his own activities whether in hunting, agriculture, dwelling. The need to live together, grow produce and defend themselves prompted men to gather together in communities, which in turn tended to occupy the most strategic territories. In this long evolutionary process man drew on his powers of reasoning to conceive abstract forms suggested by natural shapes, and, in a lengthy rational process, man also drew on nature to tackle and solve his own needs.

In order to build, ancient man needed materials which he could only obtain from nature. Hence our use of those materials which, on account of their existence over millennia, we now define as traditional: earth, wood, the infinite variety of stone, followed by the first complex elaborations: mud and fired bricks, binding agents and metals. This is how the ancients' concept of construction evolved in its infinite formal varieties: volume conceived in space and defined by geometric forms which,

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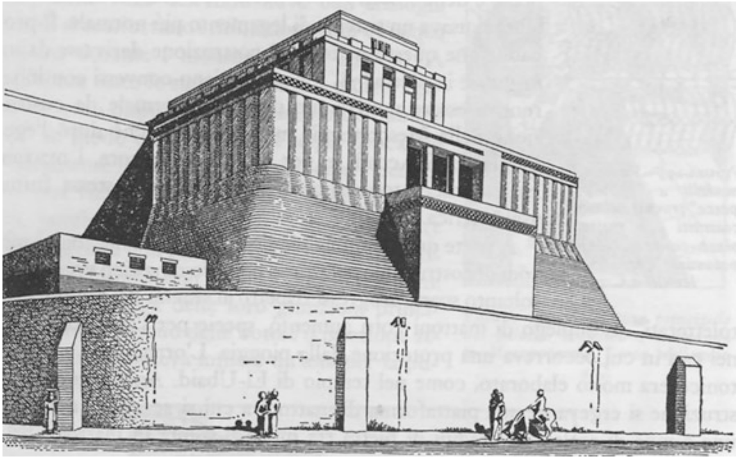

Fig. 1 Reconstruction of a temple at Eridu by Seton Lloyd, after a photograph by D. E. Woodall (Singer 1966)

through their dimensions, determined among other things by the quality of the materials, fulfil a range of functions in an indivisible unity, according to a spatial global conception in which load-bearing is just one of the functions that inform the project.

This conception spread with incredible tenacity from the fourth millennium B.C. through to the end of the eighteenth century (Fig. 1).

We can recall that this process also took the same course in civilizations whose practices developed in isolation, such as the pre-Columbian civilizations and those in the Far East. This lengthy process, which went hand in hand with man’s historical development, could not have come about except through repeated experimentations and their constant rationalization: in this way the “rules of the art” developed in all sectors of man’s activity, through failures, modifications, successes and evolutions, over the millennia (Cairoli Giuliani et al. 2007).

This process has left its mark on the evolution of human civilization; in particular, over five millennia it produced the built fabric and monumental constructions which form the material evidence of the evolution of the various civilizations. In fact, underlying the realization of both simple artefacts and of sophisticated monuments are the rules of the art (D’Agostino 2003).

2 The Rules of the Art

The rules of the art are methods and procedures defined by complex experiences and verified by practices which may be centuries old. They were formulated in response to material requirements and have informed everyday life since the dawn of time.

When they have specialised in the production of material products they have taken on a connotation of craftsmanship which gives rise to a professionalism that usually gained recognition as a specific trade.

A trade was acquired as standard practice by means of successive phases, and can aspire to ever higher levels of technical competence. When a craftsman both attains peaks of technical excellence and possesses a profound culture, he shows himself to be an artist capable of transmitting his own world view. This happened, and happens, in music, painting, sculpture, and so on. In architecture the process is the same, but in certain respects more complex, in that the realization of a work of architecture requires a range of processes which often take place over a lengthy period of time, involving a number of experts and kinds of expertise.

The rules of the art are still widespread and disseminated in today's world as "instructions for use". They take tangible form in a series of mechanical actions which cause a car, iPod or computer to function, without the user having to grasp the complex technical operations that enable him to exercise this control.

In the ancient world, on the contrary, the rules of the art developed through the slow, day by day acquisition of good practices passed down from one generation to the next, occasionally being improved by the genius of outstanding figures. Thus a trade was acquired not by means of an instructions manual, but through the everyday, laborious participation in the workshop or building site. In the artistic field this process actually survives in painting, sculpture and the so-called "minor arts", from ceramics to working with gold and silver, etc.

The slow acquisition of the rules of the art, together with outstanding personal abilities, created, as we have said, the "master craftsman" as well as, sometimes, the artist who realized an entire new work of art. In architecture the process was similar but not identical on account of the vast scale, complexity and often the lengthy time scale required for the completion of the work. In building the rules of the art sometimes manifested themselves in a simple, readily assimilable manner, and other times in a much more complex way, which may have involved strict secrecy. This gave rise, up until the mid-nineteenth century, to a widespread culture of building which enabled the peasant to make a house of his own, while complex, sophisticated rules, often revised in the course of operations by outstanding architects, informed the realization of large scale monumental complexes (Fig. 2).

This millennia-long process developed above all in the practice of construction, while with the advent of the Galilean revolution, both geometric forms and the resistance of the materials became objects of scientific interest, paving the way for the development of the disciplines of rational mechanics and building science (D'Agostino 2008).

Fig. 2 Palazzo Ducale, Urbino. Photo courtesy of Gastone Segala, 2008

3 The Tradition of Manuals

In antiquity we know of no treatises that set out the rules of the art and construction methods, with the partial exception of Vitruvius, who makes passing reference to them. In the Renaissance there were great architects like Alberti and Palladio who, rather than systematic rules, bequeathed certain pieces of evidence and annotations on the art of building. Only Leonardo, typically, gave us some prodigious intuitions, such as the one concerning the behaviour of arches in which, with remarkable prescience, he demonstrated the fundamental presence of thrust.

In practice, for over five millennia (up to 1500 A.D.), through a constant succession of new construction methods and materials, the rules of the art of building remained quite deliberately confined to an oral and material tradition, which, as we have said, in the most significant cases were kept secret. It was only with the advent of the Galilean scientific method that, above all in the triangle formed by Italy, France and Britain, a scientific reflection began to develop focusing on the fundamental construction elements—columns, arches, vaults—while not as yet paying any attention to the way they were assembled into a built organism.

Galileo Galilei (1564–1641), who chose to enquire into the world of construction in his *Discourses* (1638), was responsible for the first reflections on the behaviour of columns and curved beams (Fig. 3). Thereafter, in the new scientific spirit of the age, numerous treatises were written which sought, on primarily geometric grounds, to define the static behaviour of the construction elements.

In the meantime, the development of architecture continued its prodigious course and, in view of the new cultural stimulus for a rational and systematic analysis of human activities, works began to circulate, alongside the treatises we have mentioned, which sought to describe the complex art of construction by pursuing knowledge, both experimental and rational, of building materials and construction elements. A first series of manuals, appearing from the mid-seventeenth to the

Fig. 3 Galileo's depiction of a beam (1638, p. 114)

end of the eighteenth century, featured the study of construction elements through geometric constructions which in practice remained far removed from actual building.

The writings of Philippe de La Hire (1640–1718), in particular *Sur la construction des voûtes dans les édifices* of 1731, can be considered an important contribution to the rise of a theory embracing geometry and mechanics, but which has "no real practical counterpart and leads to various paradoxes" (Benvenuto 2006, p. 326, my trans.). In view of his geometric outlook La Hire can be considered the precursor of graphic statics, but over a century was to pass before this became, in the hands of engineers, a powerful method of calculation.

The manual by Bernard Forest de Bélidor (1693–1761), *Science des Ingénieurs dans la conduite des travaux de fortification et d'architecture civile*, published in Paris in 1719, proved to have more of an impact. In it, he developed de La Hire's theory of arches analytically, calculating the imposts and elaborating a first, incorrect, model of ground thrust. In addition he wrote about the construction of walls in fortifications, describing how the walls were erected.

The 1738 *Traité de la coupe des pierres* by J.B. de La Rue is full of interest for its analysis of the manufacture of the stone blocks that went into various construction elements, from vaulting to jack arches and flights of steps. The way in which stone was cut was highly important for the finished building but was even more crucial for

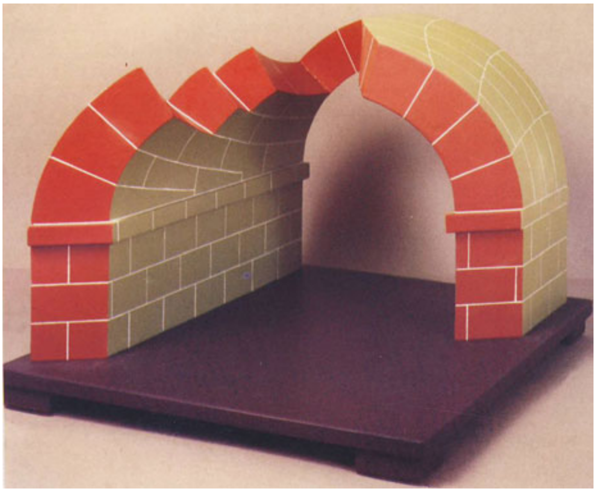

Fig. 4 Model of a vault. Photo: Centro Interdipartimentale di Ingegneria per i Beni Culturali, Cabinet of Structural Models, reproduced by permission

its resistance, since a construction made from well hewn-stone could be considered practically as isodomic, with a minimum of joints—perhaps none at all—requiring fixing (Fig. 4).

Even though the manual by Vincenzo Lamberti (1740?–1790), *Statica degli edifici*, published in Naples in 1781, described itself as an eminently theoretical work, the author was aware of the mystification of the art of building and sought to make the mathematical principles and general formulae available to builders (Cirillo 2007; Lippiello 2008). Lamberti anticipated the methods of modern experimental science, carrying out trials with tufa, piperno, mortar and pozzolan. He was also probably the first author to deal with the origin of lesions and map the development of cracks (Fig. 5).

The weighty tome by Jules Dupuit (1804–1866), *Traité de l'équilibre des voûtes et de la construction des ponts en maçonnerie* dates from 1870. Published after its author's premature demise, the manual starts from the mechanical properties of masonry, showing the influence of form and height on stability. Describing the practical evolution of a vault, it sets out a theory on the way the stress curve varies and introduces, for the limit state, the concept of pivot point, which would in time lead to the plastic pivot, the key to limit state calculation. In addition, it elaborates the conditions of stability for a set of vaults, discussing the question of thickness in order to ensure stability. It analyses the problem of thermic variations and defines the thickness of the shoulders, providing formulae for the keystones of vaulting. Lastly it describes a series of major stone bridges, including the Pont de l'Alma, the Pont d'Austerlitz and the Pont Napoleone III over the Seine in Paris (Fig. 6).

These then are examples of the extensive production of manuals concerning the interpretation of the art of building based on geometrics and mechanics, from

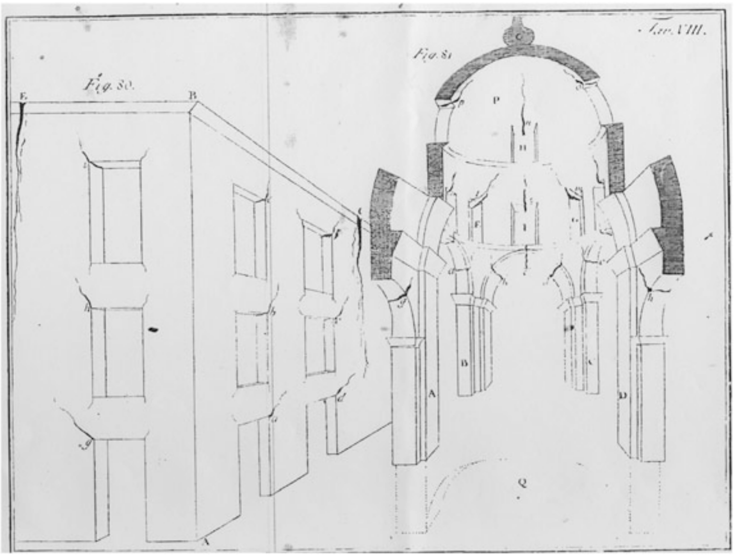

Fig. 5 Map of development of cracks (Lamberti 1781, Tav. 8)

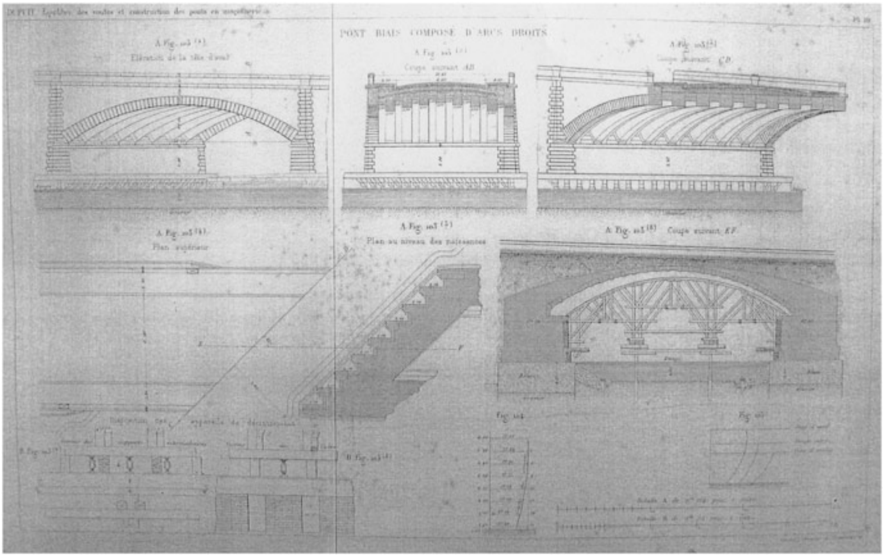

Fig. 6 Stone bridge as depicted by Dupuit (1870, Pl. 10)

the mid-seventeenth to the end of the nineteenth century. They illustrate the way in which the new scientific method was used to rationalise that art of building which for millennia had raised impressive constructions, although it was essential to take into account the behaviour of the construction in its entirety. While this

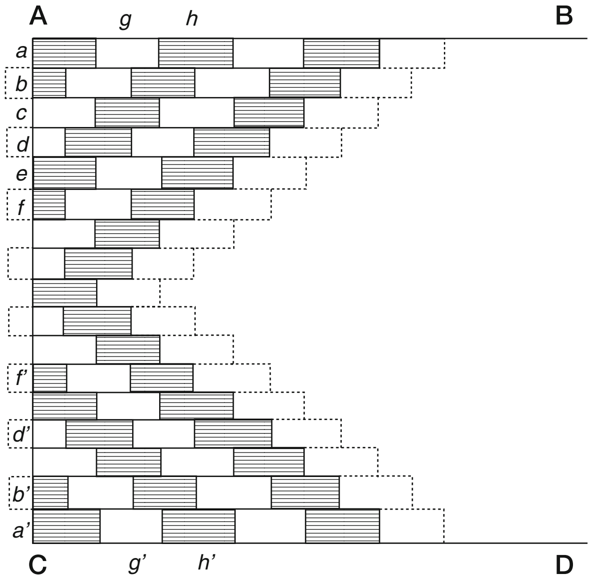

Fig. 7 Constructive scheme of a brick vault without rib, from Claudel and Laroque, *Pratique de l'art de construire*, 1870

rationalisation proved unable to measure up to the emerging "brave new world", some important studies appeared, in particular in the nineteenth century, illustrating the art of building and conserving its traditions. The *Traité théorique et practique de l'art de bâtir* (1817) by Jean Rondelet (1734–1829) was fundamental in this respect, describing the construction features and illustrating the rules of the art for gauging the size of load-bearing elements together with the modalities for their realization.

In Italy between 1864 and 1884 Giovanni Curioni (1831–1887) saw into print no fewer than eleven editions of his manual *L'Arte del construire*, undoubtedly the most popular textbook used for the formation of Italian engineers in the nineteenth century. It deals not only with civil constructions but also roads, aqueducts and railways, illustrating the most common theoretical and technical construction methods in use. In particular, it describes the design of a model building and verifies its stability.

In 1850 J. Claudel and L. Laroque brought out in Paris an elaborate volume called *Pratique de l'art de construire, maçonnerie, terrasse et platrerie* which enjoyed at least three later editions (1859, 1863 and 1870). It describes traditional materials, evaluating resistance and specifying the tools used in their employment (Fig. 7). In addition it speaks of some construction elements, in particular vaults, describing the collapse mechanisms, and outlines the Méry method for the graphic verification of arches.

I cannot end this brief summary without mentioning the general manual on civil constructions, *Allgemeine Bau-Constructions-Lehre*, by Gustav Adolf Breymann (1807–1859) (Breymann 1849), which was fundamental for the formation of engineers in the second half of the nineteenth century and still stands as a work of reference for the concept of construction and technical knowledge pertaining to

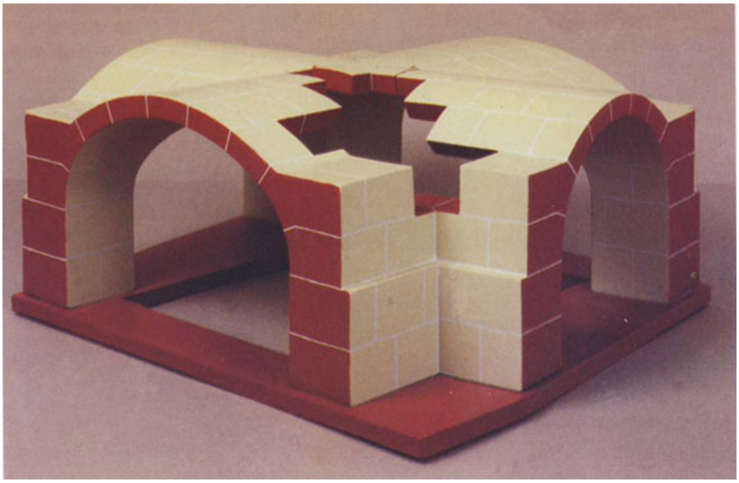

Fig. 8 Model of a vault. Photo: Centro Interdipartimentale di Ingegneria per i Beni Culturali, Cabinet of Structural Models, reproduced by permission

the historical built heritage. It sets out numerous rules of the art, many provided with graphic constructions and analytical calculations, and is even more important for its accurate description of construction techniques.

The dissemination of this culture did not stop at the fundamental production of comprehensive manuals but, in keeping with a venerable tradition, sought to communicate the sort of building being dealt with by means of models illustrating in detail the most complex construction issues; they still stand today as evidence of the refined artisanal sensibility of ancient culture (Fig. 8).

Regrettably, modern structural engineering has not been able to learn from these important manuals, which with the onset of the twentieth century were largely forgotten. This has played a dramatic part in the total ignorance of the ancients' concept of construction, having very serious consequences for the conservation and maintenance of the historical and archaeological built heritage. This has been possible because, in parallel with the process we have illustrated, another line of research into the resistance of materials and the analytical interpretation of structural behaviour was developing.

Isaac Newton (1642–1727) put forward a hypothesis concerning the elastic behaviour of materials as the macroscopic result of molecular actions, while in an address to the Académie des Sciences in 1773 Charles Augustin de Coulomb (1736–1806) set out a preliminary theory of the beam based on equilibrium equations, expounding a theory of the wedge to determine ground thrust.

In 1798 Pierre-Simon Girard (1765–1836) published his *Traité analytique de la résistance des solides et des solides d'égal résistance*, combining a theoretical and experimental approach. He investigated the stresses of traction, compression and bending and the relationship between stress and tension; he carried out extensive experiments on wooden beams, analysing movements and deformations. His manual was esteemed by Edoardo Benvenuto, who pointed out how Girard "perceived with singular clarity the 'epistemological' difference between general equilibrium equations and the constitutive equations linking tension and deformation" (2006: 284, my trans.).

A fundamental step forward was marked by the *Leçons sur l'application de la méchanique* published by Claude-Louis Navier (1785–1836) in 1826. This uses the results obtained by Jakob Bernoulli (1654–1705) and Leonhard Euler (1717–1783) in studies of elastic and flexible bodies to develop technical design procedures. This gave birth to building science in terms of both topics and methodology, applied to problems of the straight beam, reticular beams, arches and so on. In practice, with the memoir he presented to the Académie des Sciences in 1821, Navier laid the foundations for the theory of elasticity, whose fundamental law had been foreseen over a century earlier by Robert Hooke (1635–1702), and subjected to analytical enquiry by Robert Young (1773–1829), who introduced the elastic module foreseen by Euler.

Meanwhile, Carl Culmann (1821–1881) founded graphic statics, while in 1853 Arthur Morin (1795–1880) brought out *Resistance des Matériaux* in four parts, dealing with traction, compression, bending and torsion, with numerous experimental demonstrations. The time was ripe for the painstaking work of Jean-Claude Barré de Saint Venant (1797–1886) elaborating the modern theory of elasticity and the solution of the problem of beams in his analytical conception of a one-dimensional solid.

The years 1882 to 1884 saw the appearance in Turin of the *Manuale practico degli Ingegneri* by Alberto Castigliano (1847–1884). The conceptual approach relied on the new building science through the study of simple and composed stresses, given homogeneous and isotropic solids, and illustrating Barré de Saint Venant's formulae based on the mathematical theory of the elasticity of solids. Using this manual, engineers were able to calculate the structural elements involved in a one-dimensional solid.

All this was far removed, both epistemologically and practically, from the ancients' concept of construction, but times (and technology) had changed, quite apart from the theoretical process of the formation of building science. The nineteenth century saw the first constructions in ironwork: in 1811 the dome of the Halle au Blé in Paris by François-Joseph Bélanger (1744–1818), in 1818 the Royal Opera Arcade in London by John Nash (1852–1835), in 1825 an interior passageway roofed with ironwork in Palazzo San Giacomo, Naples, by Stefano Gasse (1788–1840), and in 1832 the exploit of the bridge over the Garigliano built by Luigi Giura (1795–1864).

There was a natural affinity between Barré de Saint Venant's one-dimensional solids and the structural elements of constructions in iron, and hence building science spread throughout construction practice, first in iron-based architecture and then in the new technology of reinforced concrete which, also during the nineteenth century, had undergone a lengthy phase of experimentation quite separately from the dictates of the new science.

4 The Modern Concept of Construction

As has always been the case for the art of building, technical development has been led by a succession of intuitions. In fact, with the new building science still in its infancy, in 1850–1851 the Crystal Palace in London was erected in the space of just 9 months with metal parts and other mass-produced components assembled on site, with hydraulic rams used for testing the girders (Fig. 9). Then in 1894–1895 a large steel building, the Marquette Building, was constructed in Chicago, while during the first decade of the twentieth century the Manhattan Bridge was built in New York with a central span of 447 m and two side spans measuring 210 m.

In 1850 the firm of Demarce and Conquety began large-scale production of cement, making it an economic material that was malleable and resistant to compression; shortly afterwards François Coignet (1814–1888) provided his factory with a concrete roof. In 1877 in the United States Thaddeus Hyatt (1816–1901) illustrated the characteristics of resistance of the solidified cement when combined with iron reinforcing, along with the protection that the concrete provided to the iron in case of fire. Ten years later Matthias Koenen (1849–1924) set out the first scientific theory of reinforced concrete, and in 1892 François Hennebique (1842–1921) patented reinforced concrete girders. These new construction elements eventually replaced the traditional construction in masonry, rapidly paving the way

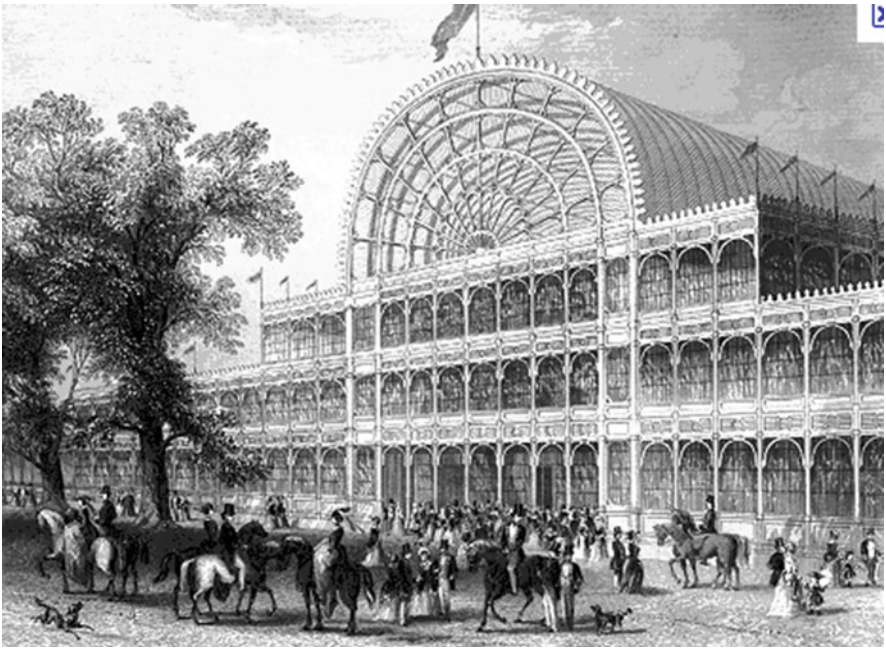

Fig. 9 Crystal palace

Fig. 10 Villa Savoye. Photo: Alessio Antonetti, 2006

for the emergence of "structure": an autonomous organism, calculated according to a technical theory and cogent norms, comprising above all standard elements which, once assembled, produce a composition that occupies space and responds exclusively to the load-bearing function.

The new material and technology spread very rapidly, and there were dramatic collapses which drew attention to the need for regulations. In fact, a new concept of construction had come into existence which, abandoning the unitary conception of ancient building work, was divided up into load-bearing elements (structure) and accessory elements (finishing). This marked the beginning of a process of industrialization in building which has become more prevalent ever since. This development has been very rapid and, in some respects, highly disconcerting.

The new conception evolved rapidly, giving rise to the rational architecture epitomised by Le Corbusier's Villa Savoye, denoting a new mode of occupancy (Fig. 10). Building in reinforced concrete became more and more common, while constructions using metal continued to spread in industrial complexes and infrastructures.

The first decades of the twentieth century were characterised, especially in day to day construction projects, by a significant pioneering spirit seen in the spread of constructions in a combination of masonry with some construction elements, in particular staircases, in reinforced concrete. The theoretical findings concerning the new building science did not spread so rapidly, even though as early as 1897 the first course featuring reinforced concrete was given in the École Nationale des Ponts et Chaussées, followed in 1900 by a course taught by Camillo Guidi in Turin, and in the same year the first Italian manual about reinforced concrete, *Costruzioni* *in calcestruzzo ed in cemento armato*, written by engineer Giuseppe Vacchelli and published by Hoepli.

In engineering schools the new building science took root together with the techniques of iron and reinforced concrete, while any reference to constructions in masonry rapidly fell by the wayside. In the first half of the twentieth century the methods used for calculations were essentially linked to graphic statics, while for obvious practical reasons there was very little analysis. This is clearly seen in Italian regulations issued in 1932, which require the calculation of pillars subject to normal stress and the beams of the frameworks calculated as being inserted into the supports and semi-inserted into clamps at mid-span.

The time was ripe for the emergence of structural engineering, in the years following the World War II. The new construction techniques radically excluded masonry from their terms of reference and in the space of a few decades the grand building tradition of the ancients was completely eliminated from the cultural and technical formation of engineers and architects. This was also the inevitable consequence of, first, the building boom associated with post-war reconstruction throughout Europe, and then the advent of the digital era, which as early as the 1960s had dispensed with graphic statics. In its most significant achievements the new specialization was capable of unprecedented architectural feats which have come increasingly to characterise the world as we know it.

5 A Theoretical Return to the Historic Built Fabric

In the second half of the twentieth century the discipline of structural engineering emerged, accompanied by the complete obsolescence of the ancient concept of construction. Building science has been "democratized", to use Pier Luigi Nervi's felicitous expression, entering the curriculum for both engineers and architects. At the same time the world of engineering has elaborated sets of regulations which follow hard on the heels of one another, providing coercive guidance for structural design. In its prevalently geometric definition, the structural aspects of this design are calculated using specialised software that entails the automatic observance of the regulations and proceeds directly to the formulation of the construction details. In this way the design process is fundamentally standardized and organized by the suppliers of software.

In parallel, from the mid-twentieth century structural engineers have had to cope with the conservation of the historical built heritage, and in particular its monumental and archaeological aspects. Unfortunately, having no knowledge of ancient architecture, they have intervened in confused and inappropriate ways, using some "consolidation" techniques which can lay no claim to being scientific and are in complete contrast to the construction concept of the ancients. Perforations, cement bindings, injections of cement, cladding and insertions using reinforced concrete have been introduced wholesale into the historical architecture throughout

Europe, doing irreparable damage to the material history of ancient building (Carbonara 1981–1984).

Following heated cultural debates it has been possible to limit the widespread cementification and introduce a more informed attitude concerning material history. In practice, for structural engineers the entire historical built heritage should be configured as an archaeological construct, since it belongs to a different, ancient civilization of construction. We have already seen how the research carried out over the course of recent centuries in the major manuals which characterised the evolution of the mechanics of masonry has had very little influence on the realization of historical architecture. The goal has always been to formulate a theory focusing on the construction elements that can be extrapolated from ancient construction, above all arches and vaults. This is in fact clearly set out in the fine volume by Antonio Becchi and Federico Foce, *Degli Archi e delle Volte* (2002). As the authors say: "our purpose is … to bring into focus the real innovative contributions thanks to which the theory of vaults … has acquired that character of rigour and generality required in a solid discipline of mechanics". In spite of this, the gap that exists between the tradition of treatises and the complex genius of ancient architecture is highlighted by Antonio Becchi:

> In 1676 work was finished on the construction of the Hotel de Ville in Arles and its unique vaulted roof designed by Mansart and built by Peytret. The achievement still stands, as proof of the soundness of both the construction and its conception. However, we lack the instruments to account for its durability: we no longer possess these instruments because they formed part of the stock of experience of those *maîtres maçons* who, from one generation to another, reinvented their own expertise by means of daring intuitions which were eccentric with respect to the best performance set out in the manuals of the time (2002, pp. 25–26, my trans.) (Fig. 11).

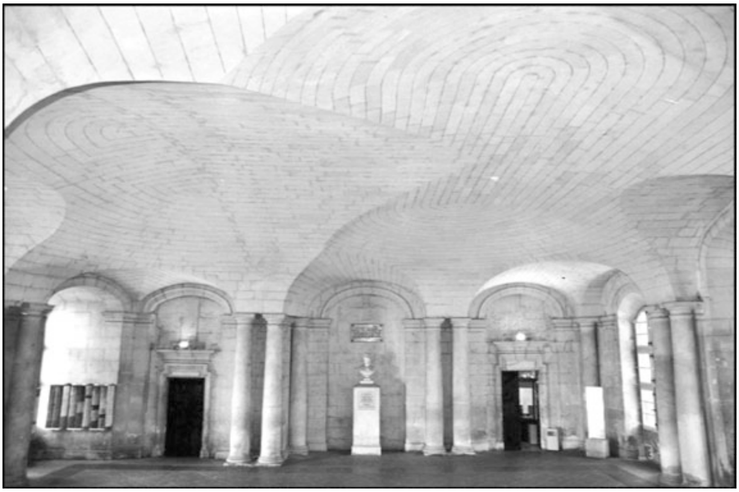

Fig. 11 Vault by Mansart ad Arles. Photo: Giuseppe Fallacara, reproduced by permission

More recently research has focused on the reinterpretation of the historical built fabric in terms of mechanics using the powerful resources of modern structural analysis. This has involved two orientations: theoretical research and numerical elaboration.

The theoretical research has been grounded in Jacques Heyman's classic *The Stone Skeleton* (1982), which right from the title sets out to reconsider the art of building in the light of the modern science and technique of constructions. The masonry fabric of walls, seen as a complex spatial whole, can thus be schematized as the wall block and panel. This produced a long theoretical development which regrettably has not had a considerable influence on the drawing up of the most recent regulations affecting the historical built fabric. This approach, which over recent decades has permeated the research of structural engineers, has recently produced an exhaustive exposition in the hefty volume by Mario Como (2013), on the statics of historical constructions in masonry. Case by case the author suggests a possible static behaviour for the various construction elements found in the edifices, identifying solutions which are statically admissible to justify its static efficiency. He goes on to examine the global behaviour of various monuments, demonstrating the reliability of their static conception. A clear example is the Colosseum, for which he presents a seismic verification showing that "the average acceleration of collapse is equivalent to circa 0.12 g, i.e. to 1.2 m/s^2, approximately 10 times greater than the acceleration of 0.136 m/s^2 which can have affected the masses of the Colosseum" (Como 2013, p. 372).

It is evident that the interpretative approach derives from the author's vision of static behaviour. Another emblematic example are the flights of steps referred to as "Roman-style" (Fig. 12).

For a long time, structural engineers considered, and in many respects still do today, such flights of steps to be insecure, inflicting a range of so-called standardizations on them. Even the Italian ministerial decree of 24 January 1986 regarding structures in seismic zones prescribed: "Non-loadbearing flights of steps in masonry (so-called "Roman-style") should as a rule be replaced by stairs in reinforced concrete or steel", persisting in an ill-informed static approach of the kind prescribed by legislation 219 of 14 May 1981, article 10, which actually prescribed the elimination of connecting arches between buildings, and wooden structures, suggesting that arches and vaults should be made so as not to exert strains, regardless of the millennia-old ancient concept of construction.

Recently, flights of steps have been carefully studied by both Alessandro Baratta (2007) and Mario Como (2013), but the static schemes they suggest differ widely (see Figs. 13, 14), and are certainly very distant from the rules of the art used by the ancient builders.

By virtue of his exemplary intellectual honesty Como has to say:

> Still today, in the technical literature, there is no sign of the presence of a unitary, consolidated approach to the analysis of the static behaviour of structures in masonry that can, in some way, be compared to what exists for constructions in reinforced concrete or steel (2013, vii).

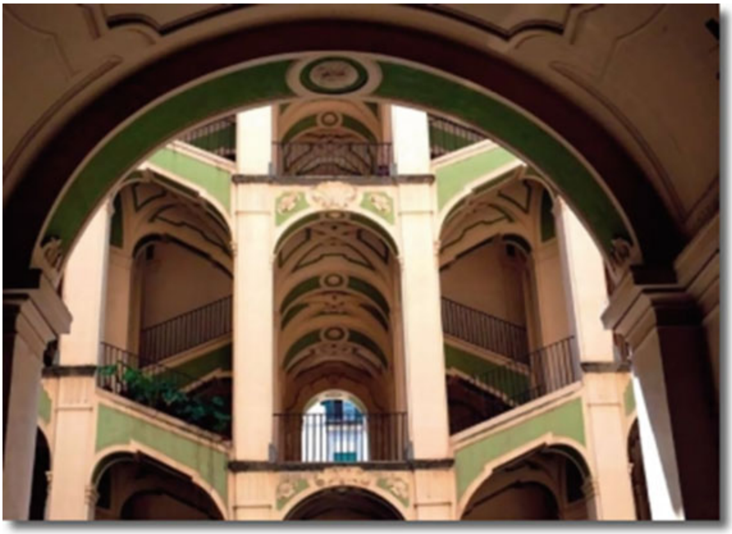

Fig. 12 Flight of steps referred to as "Roman-Style" in the Palazzo dello Spagnolo, Naples. Photo: courtesy of the Comune di Napoli

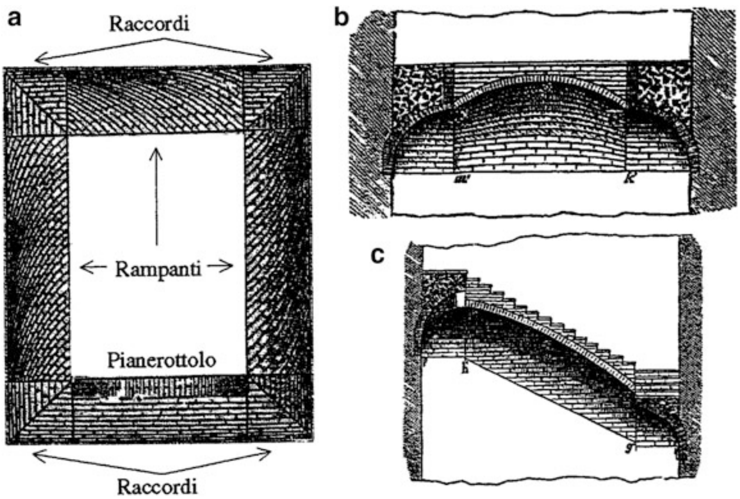

Fig. 13 Flight of steps referred to as "Roman-Style". Image: (Baratta 2007), reproduced by permission

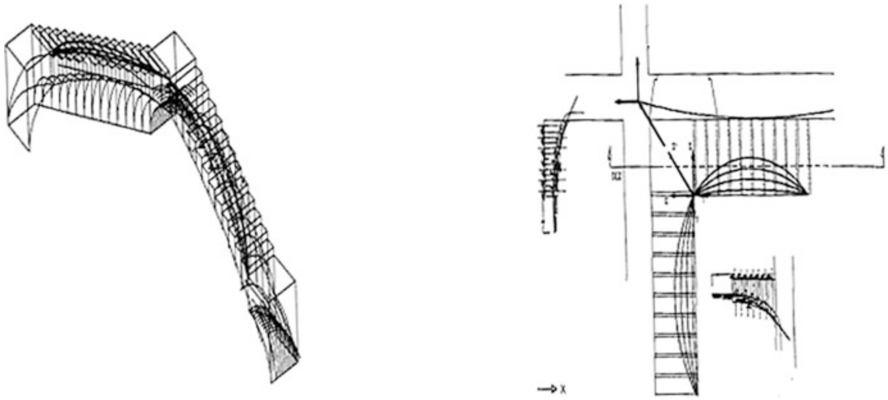

Fig. 14 Flight of steps referred to as “Roman-style”. Image: (Como 2013), reproduced by permission

This is due to the fact that works in masonry have always been approached in the framework of a historical and experimental conception which is profoundly different from the mechanics-based conception. The mechanics-based conception:

– designs and realises modern structures according to pre-established analytical models;
– analyses and studies constructions in masonry, envisaging “a posteriori” a compatible mechanical model.

A more common means for analysing constructions in masonry, widely used in recent decades, is the analysis of the finite elements of the construction in its entirety. Such analysis is carried out in the first place in a linear elastic phase, and since each material, even masonry, initially shows a behaviour that can be assimilated to elastic behaviour. This method made it possible, in most cases, to obtain an accurate vision of the tension state, hence also suggesting the static behaviour of the construction. Unfortunately, however, recently a mandatory set of seismic norms has been brought in for most of the Italian territory. These make the seismic verification of monuments in general and archaeological remains even more arid and systematic by imposing a numerical evaluation of the intended upgrading. All this has eliminated the on-going reflection which characterised structural design in the second half of the twentieth century, while permitting any technician to elaborate standardized calculations, with absolutely no regard to the original construction concept of the monument.

6 Conserving the Historic Built Fabric by Making Sense of the Past: Retrieving the Rules of the Art

As said, for more than half a century senseless interventions have cementified a large part of the monumental and archaeological heritage throughout Europe, in total ignorance of the ancients' construction concept.

In Italy, starting in the early 1980s, some voices were raised in alarm and protest, but ignorance and a 'get rich quick' attitude prevailed over the dissent. In addition a capricious fate brought to an untimely end Edoardo Benvenuto, Antonino Giuffrè, Salvatore Di Pasquale and Alfredo Corsanego, all authoritative scholars who championed with well-argued reasoning the cause of a conscious conservation of the historical built fabric. Yet all that was necessary for structural engineering was to recognise the problem and introduce courses in the statics of ancient buildings, building on the teaching of Breymann, Curioni, and so on, without allowing the widespread culture of construction based on experience of craftsmanship dating back centuries to be dispersed. It would have been sufficient for the Faculties of Architecture to undertake a systematic survey of the monuments, taking into account construction dimensions and quality of the materials so as to grasp the secrets of their deployment. All this would have set in motion a "virtuous circle", making it possible to retrieve the rules of the art and at the same time create engineers and architects well-versed in the ancients' concept of construction.

Some modest progress has been made, both in exorcising the all-pervading cementification and in the retrieval of the rules of the art that informed the realization of Roman constructions (Conforto and D'Agostino 1995, 2001), or residential architecture in the nineteenth century, even if this is all very limited with respect to an accurate knowledge of ancient architecture, in particular the religious edifices. Fortunately the industry of building materials has proven to be receptive to these instances, producing mortars which are much more compatible than cement-based fillers, as well as fibres in composite material that can be used to bind a construction together without excessive violence, even if the efficacy of these products over time has yet to be seen.

Furthermore, the new generations tend to be more aware of the need to safeguard the territory and landscape, as well as to conserve the built heritage. And there are interesting stimuli for design experts in the cultural sphere (D'Agostino et al. 2009). All this can and must serve to support the conservation authorities, who should not only dispense with undue deference towards contemporary technical expertise but unhesitatingly impose respect for the construction concept of the monumental heritage, its material conception and, at the end of the day, its integrity.

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Designing by "Expérience": Lecreulx Model Tests for the Design of the Abutments of the Bridge of Fouchard

Santiago Huerta

Abstract In the 1770s the French engineer J.-R. Perronet introduced a new type of masonry bridge, with very slender piers and extremely surbased segmental arches. Circa 20 bridges of these kind were designed by Perronet and his disciples. The ratio height of arch to span was between 1/9 and 1/18. The flat arches would have exerted a great inclined thrust. The piers were equilibrated, but the abutments at both ends of the bridge had to support the thrust. The danger of failure by some kind of fracture of the buttress on the upper part was evident; in particular then danger of sliding failure. The thrust could be calculated following the La Hire theory, which, though incorrect, was safe, i.e., led to greater, more inclined thrust. However, to know the effect of the thrust in the massive abutments it was necessary to make tests on models. These were carried out in 1774 by F.-M. Lecreulx during the construction of the bridge of Fouchard. The results demonstrated the enormous danger of a catastrophic failure by sliding. He then proposed some dispositions to avoid this danger, prolonging the voussoirs of the arch inside the mass of the arch. Fouchard's test must have been influential in the great increase of the size of buttresses from the original designs of the 1770s in all the bridges built (most completed after 1780). Eventually, this type of bridge was abandoned, but their short history (about 30 years) demonstrated the audacity, courage and faith in reason and experiment of Perronet and his disciples.

Keywords History of the theory of structures • Bridge design • Masonry arches • François Michel Lecreulx • Jean-Rodolphe Perronet • Surbased arches

In the second half of the eighteenth century the French engineer Jean-Rodolphe Perronet (1708–1794) initiated a revolution in bridge design. He proposed two radical changes: to reduce drastically the thickness of the piers, to facilitate the course of the water, and to use arches with a greater surbasement to ease the access to the bridge road, without the need of long ramps (Perronet 1777). The arches evolved from surbased ovals (*anse de panier*) with *cornes de vache* (the best example is the

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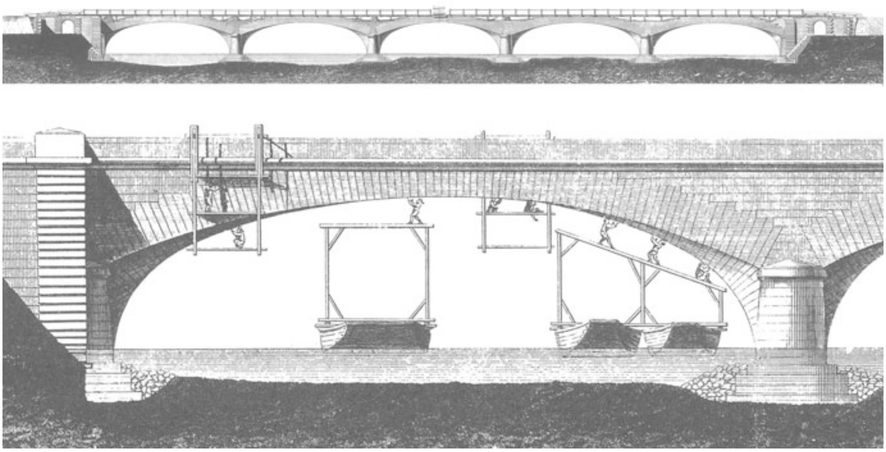


Fig. 1 Bridge of Neuilly, 1764–1774 (Perronet 1788: Plate IX)

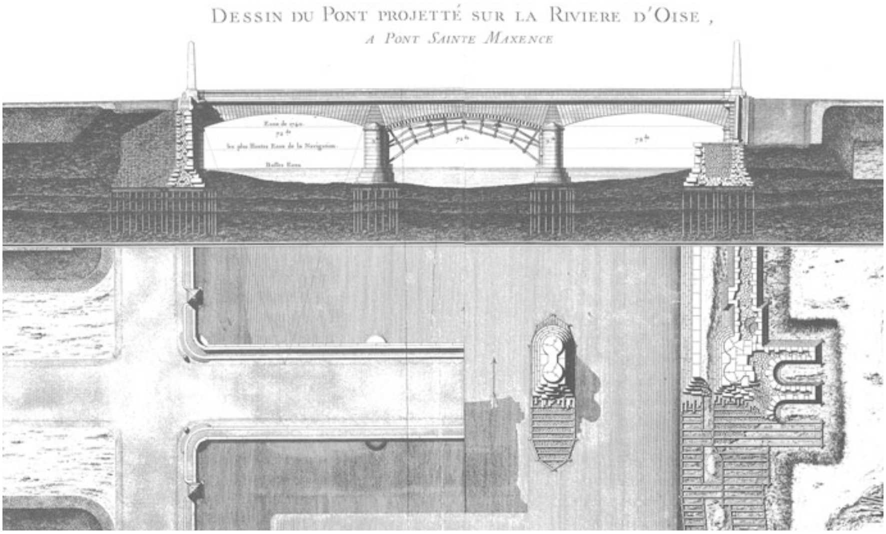


Fig. 2 Bridge of Saint-Maxence by Perronet (Perronet 1788: Plate XXX)

bridge of Neuilly by Perronet, 1768–1774, Fig. 1), to very flat segmental arches with the springings over the level of the maximum floods on slender piers (Fig. 2).

It was in the decade of the 1770s when Perronet set himself to design these flat segmental arch bridges. Never before had arch bridges been built with such extremely surbased proportions, with ratios height/span around 1/10 (to be more precise, between 1/9 and 1/18) supported on very slender piers (sometimes less than 1/10 of the span). He was followed enthusiastically by some of his disciples.[1]

The first design of this type was made by Perronet for the bridge of Nemours in 1771, with arches of 16.25 m span and a height of 1/15.6 of the span. It was an audacious design, with the manifest intention of breaking drastically with the past. However the bridge was constructed after Perronet’s death (1795–1804) by Louis-Charles Boistard (1763–1823). The first segmental arch bridge built was that of Pesmes, completed in 1772; it was designed by Philippe Bertrand and consisted of three arches of 13.70 m span and a height of 1.19 m, i.e., nearly 1/12 of the span. The same year, Jean-Baptiste de Voglie (1723?–1777) presented a design for the bridge of Fouchard (about which we will speak in detail below). It too consisted of three equal arches, with a span of 26 m and a height of 1/10 of the span. Shortly after, in 1774, Perronet began the construction of the bridge of Saint-Maxence over the Oise; the bridge was finished in 1784, with three arches of a 23.4 m span and a height of 1/12 (Fig. 2). Other bridges designed by Perronet in the early 1770s remained unbuilt: the bridge for Melun (one arch of 48.7 m surbased 1/10); the bridge of Pontoise over the Oise (3 arches of 29.24 m surbased 1/13); and the bridge of Moret over the Loing (3 arches of 25.34 m surbased 1/14). The highest *surbaissement* was attained in the bridge of Saint-Diez over the Meurthe (3 arches of 10 m surbased to 1/18, Fig. 3), designed by François Michel Lecreulx (1729–1812) in 1785 and built 1804–1821. In the nineteenth century this type of extremely flat arch disappeared almost as suddenly as it had appeared 30 years earlier.

These greatly surbased arches posed new problems, both for the practice of construction and the theory of arches. The thin piers implied the centering and construction of the whole bridge and the simultaneous decentering. The construction, usually over *cintres retroussés* (another invention by Perronet; the centering consists on a series of parallel arches or trusses which abut on the piers, leaving the river free of supports, as in the central span of Fig. 3) which showed some flexibility, made it obligatory to equilibrate the centering during the construction, placing provisional weights adequately; the great thrust produced a considerable lowering of the keystone that had to be considered. Further, some constructive precautions also had to be taken to avoid the chipping of the stones on the joints of rupture, etc. In the *Oeuvres* of Perronet (1788) there is a complete description of all these problems, which were carefully registered by the engineers during the construction and decentering of the bridges. Another invaluable source is the *Traité des ponts* (Gauthey 1809–1816) by Émiland-Marie Gauthey (1732–1806), edited and published posthumously by Claude-Louis Navier (1785–1836), with large annotations and additions; later editions in 1832 (revised and enlarged) and 1843 (reprint). Finally, Louis Bruyère (1758–1831) compiled some reports of engineers on the construction of stone bridges with the intention of completing the information of the previous works (Bruyère 1823).

[1]A detailed list of Perronet bridges with dates and dimensions, and a discussion of this type of design in Dartein (1906).

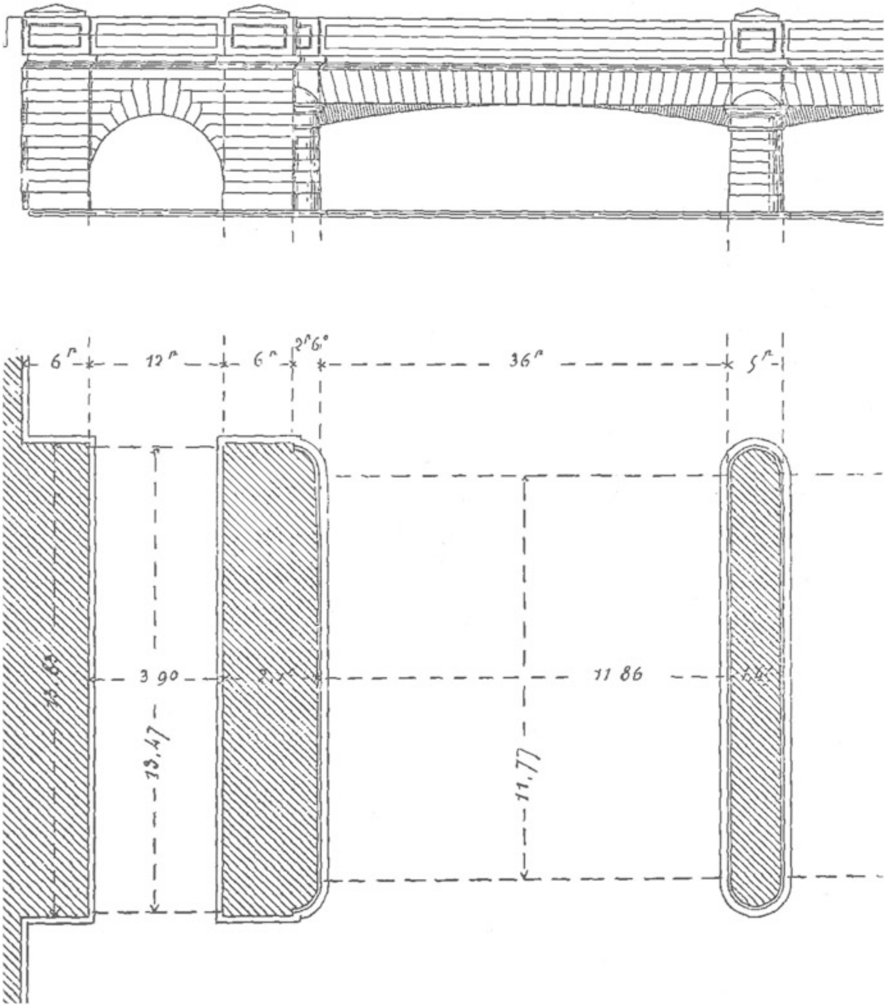


Fig. 3 Bridge of Saint-Diez over the Meurthe by Lecreulx (Dartein 1906)

As for the theory, these flat arches posed a new problem: never before had arches with such a degree of surbasement been designed. In the second half of the eighteenth century the French engineers still used La Hire's theory formulated in 1712. La Hire considered that when an arch collapses a joint of rupture forms at some point between the crown and the springings and that the thrust was tangent to the intrados. With these two conditions (plus the symmetry) it is possible to calculate the thrust. Contrary to common opinion, La Hire did not fix the position of the joint of rupture (Heyman 1998) (Fig. 4a).

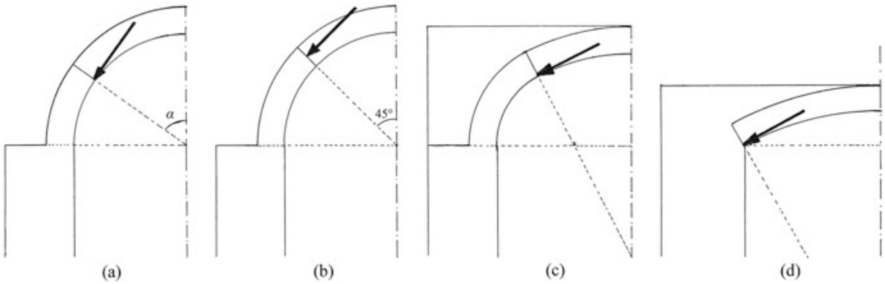


Fig. 4 The thrust of arches in the eighteenth century after La Hire’s theory: (**a**) La Hire (1712); (**b**) Bélidor (1729); (**c**) Perronet (ca. 1750); (**d**) Perronet/Lecreulx (ca. 1770). Image: author

Later, Bernard Forest de Bélidor (1698–1761) modified the method supposing the joint of rupture to be always at the mid-point between the intrados and the crown and the thrust acting tangent to the middle-line of the arch (Bélidor 1729) (Fig. 4b). Bélidor’s interpretation was that the upper part of the arch between the two symmetrical joints of rupture acted as a wedge sliding without friction over the joints, and therefore thrusts against the buttresses with forces normal to the joints. It was evident that the friction between stones is very high, but the theory gave dimensions similar to that of existing constructions and the empirical rules. As a consequence, the so-called “wedge theory” of La Hire-Bélidor was in use for more than 100 years (Huerta 2004).

In the 1750s Perronet (Perronet and Chezy 1810) studied the problem for surbased arches *en anse de panier* (ovals made of segment of circles) and concluded that the joint of rupture was at the point of change of curvature for an arch surbased to 1/3 and with the form proposed by Henri Pitot (1726) (Fig. 4c). For other forms of ovals he explicitly said it would be necessary to make trials which would involve long calculations.

For surbased segmental arches, the joint of rupture would be obviously at the springings (Fig. 4d). Perronet didn’t state this, but it was implicit in the observed cracking and movements of the arches after the decentering (Perronet 1773). As we shall see, Lecreulx stated this explicitly, in passim, as a matter of common knowledge among engineers.

Thus, to calculate the thrust of the new flat arches presented no problem (and as we will see is “safe”, i.e., the calculated thrust is greater than the “actual thrust”). The thickness assigned to the arches was substantial, between 1/15 and 1/20 of the span, and could reach the level of the road; in fact these proportions precluded the collapse of the arch, as long as the buttresses remained unmovable.[2] The problem concentrated on the abutments: What would the effect of such a great inclined thrust be on the massive abutments?

[2]There is no possible pattern of hinges leading to a collapse mechanism; see (Heyman 1982, pp. 40-42).

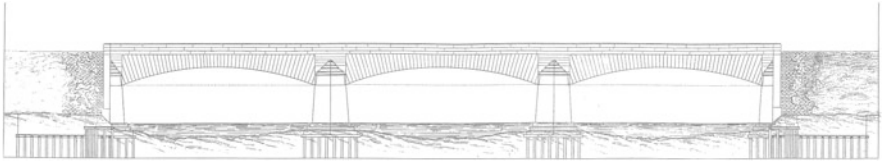


Fig. 5 Elevation of the bridge of Fouchard (Bruyère 1823)

1 Lecreulx's "*expériences*"

In this situation, it was logical to turn to the *expérience*. This word refers in French both to the observation of existing constructions and the actual performance of experiments. There were no examples to make a direct comparison and the only empirical rule available referred to straight arches (called flat arches or *plate-bandes*): according to Gautier (1717, p. 14) the buttress must be more than half the span (Huerta 2012, p. 405). Of course, the cracking and movements of previous surbased bridges during construction and after decentering had been carefully registered (Perronet 1773, 1788). Engineers were familiar with cracks, and the visible cracks at the springings confirmed the concentration of the thrust at this point, as predicted by the theory. But observations on the failure of buttresses were very difficult, if not impossible, as this failure would lead to an immediate and catastrophic collapse. The only way was, then, to make tests on models to study the different patterns of collapse and extract some clues for the design.

These kind of tests were carefully made by Lecreulx during the construction of the bridge of Fouchard in the year 1774 (Fig. 5). He registered and discussed the results in a detailed Memoir entitled, *Mémoire sur la nature de la poussée des voûtes, formées d'un seul arc de cercle, contre les culées, et expériences sur les éffêts qui en résultent* (1774). In addition, Bruyère (1823, pp. 10–12) published part of the Journal written by Lecreulx during the year 1774, corresponding to the construction of the vaults and abutments.

Lecreulx's Memoir was never published. Today the manuscript is preserved in the library of the École nationale des ponts et chaussées; we found the reference in the old catalogue of manuscripts published in 1886. The drawings are missing, but fortunately they were published by Bruyère (1823, Pl. 8) (Fig. 6), and the tests briefly described in a note.[3] The drawings have been grouped by *expériences* in Fig. 7. The Memoir has been transcribed and is reproduced in full in the Appendix.

[3]Navier quoted Lecreulx's tests in a footnote of the second edition of Gauthey's *Traité des ponts* (1832, vol. I, p. 244), confusing the name of the bridge ("Frouart" instead of "Fouchard"). The same quotation, with the same error, appears in the "Translations from Gauthey" in Weale (1843, p. 83). Eventually, Cresy (1865, pp. 1498-9) transcribed the note and reproduced Bruyère's drawings. Most recently the memoir was cited briefly by Marrey (1990, p. 58).

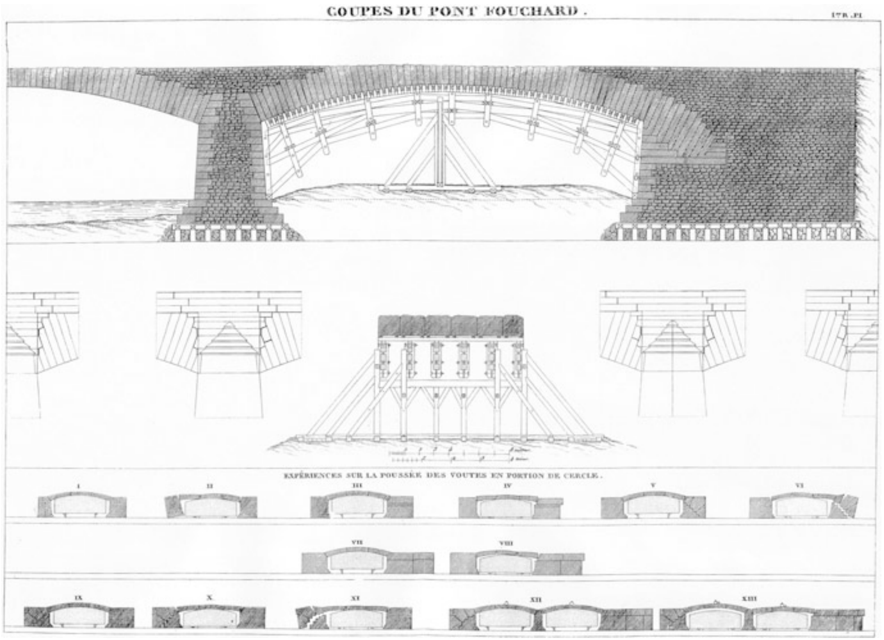


Fig. 6 Fouchard bridge (Bruyère 1823, Plate 8). At the bottom are the drawings of Lecreulx tests, reproduced below in Fig. 7, grouped by *expériences*

In the discussion that follows, the numbers in parentheses correspond to the pages of the volume in which the manuscript is bound.

The Memoir begins by discussing the thickness of the central piers. Lecreulx remarks that the piers support no horizontal thrust and should resist only the vertical load of the two demi-arches; the stone is of a good quality, *pierre de Champigny* with a specific weight of 25 kN/m^3. He states that they could support arches with spans more than three times greater, though he does not explain how he arrives at this figure.[4] He remarks that special attention should be given to the foundations (p. 81).

The rest of the Memoir is dedicated to discussing the results of the tests on the scale models. Lecreulx believes that the theory would make it possible to calculate the thrust. However, "as it often relied on assumptions that are not always consistent

[4]Perronet made *expériences* on the strength of different kinds of masonry, both by comparison with existing buildings and by using a machine of his invention. Following the idea of Gautier (1716, p. 102), he calculated the height of the equivalent column which will support at its base the same load of the pile and the two semi-arches, and, then compared this height with those obtained in existing buildings. The first systematic strength tests on stones ever published were made by Gauthey in 1774 (Huerta 2004, pp. 362-4).

with the nature" (p. 82) (he is probably referring to the absence of friction), he considers it necessary to make the tests to check and refine the theory.

Lecreulx planned the model "exactly conform with the proportions of the design" (p. 82), without the ornaments, so that the elevation represents a section of the bridge. Before going further, it will be helpful to explain the units of measurements used in the Memoire. For lengths and widths, Lecreulx cites the dimensions using *pieds*, *ponces*, *lignes*, and *toises*. 1 *pied* (or *pied-du-roi*) is equal to 32.48 cm in today's units, and was subdivided into 12 *pouces*; a *pouce* was further subdivided into 12 *lignes*. A *toise* is equal to 6 *pieds*. The scale of the models is 14 *lignes* to one *toise*, i.e., approximately 1:62. Therefore, the span of each of arches, cited as 80 *pieds* (p. 81), was equal to about 42 cm in the models. The breadth of the bridge is given as 19 *pieds* 26 *pouces*, and thus the breadth of the model was about 11 cm. For weight, instead, he cites the units of *livres* (equal to 489.51 g in today's units), subdivided into 16 *onces*.

The models were made of tufa stone, easier to carve and more homogeneous than that of Champigny which was employed for the bridge: "We have chosen this stone to make the model because it is more homogeneous and easy to carve, particularly in small pieces" (p. 82).

Lecreulx recognises that the tufa stone is lighter, but "as the purpose is to establish proportions, it would be possible to use any kind of stone" (p. 82). This statement is crucial: Lecreulx is conscious of the geometric character of the design of masonry structures. It is stability, which depends on form and not scale, which governs the design of masonry structures (Heyman 1995).

Next he explains the way the arches thrust against the abutments. Following the theory of La Hire/Perronet, he states that the action depends directly on the degree of surbasement of the arch: "It is easy to see that the vaults under consideration act against the abutments in a different way depending on whether the portion of the arc of the circle contains a greater or lesser number of degrees" (p. 82) (see Fig. 4d, above). Indeed, the direction of the thrust is indicated in the first drawing of the model, Fig. 7(1, I). He then gives the main geometric data: the radius (108 *pieds* or 35 m) and angle of aperture 44° 42′, and the corresponding lengths of the chord or span (82 *pieds*, 1 *pouce*, 6 *lignes*, or 26.68 m) and the height (8 *pieds*, 1 *pouce*, 4 *lignes*, or 2.62 m).

The vaults were mounted on a centering that could be raised or lowered at will. They are divided into three segments or voussoirs; the weight of the whole vault is 4 *livres* 6 *onces* (2.142 kg). (Due to this division, a slight yielding of the buttresses produces, instead of the usual crack at the crown, two symmetrical cracks on the joints of the central keystone. This reduces slightly the thrust with reference to an actual vault made of more voussoirs.)

Lecreulx is systematic in his exposition. First, he describes the model, giving the structure and dimensions of the abutments; afterwards, he discusses the *éfféts*, that is, the results of the thrust of the arch on the abutments.

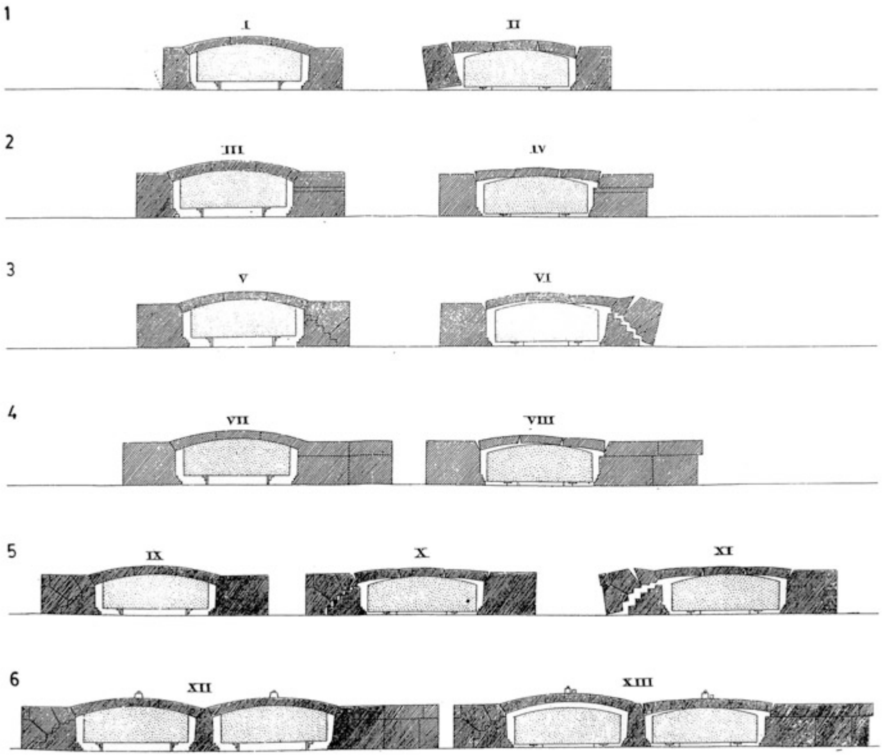


Fig. 7 Lecreulx tests on the strength (stability) of buttresses (re-elaborated from Fig. 6)

1.1 First "expérience"

The vault rests on two monolithic buttresses which have, on the left 18 *pieds* 6 *pouces*, and on the right 24 *pieds*, at the level of the springings [Fig. 7(1)]. The buttresses are monolithic.

After the decentering the vault remains standing; the buttress of 18 *pieds* 6 *pouces*, resists the thrust. But if 4 *onces* (120 g) are added (1/20 the weight of the vault), then the abutment fails. In Fig. 8a, the proportions of collapse are shown: a buttress of 17 *pieds* 2 *pouces* (7 % less than the buttress of the model), will be just in equilibrium supporting the vault. Therefore, Lecreulx had to put a small load to provoke the collapse. However, it seems clear that Lecreulx tried to obtain the limit proportion, probably making several trials.

It is interesting to note that the thickness of the right buttress, 24 *pieds*, corresponds to that obtained by applying the theory of La Hire (Fig. 8b). The thrust of the vault, tangent to the intrados at the springings, produces an overturning moment obtained by multiplying its magnitude by the lever arm respect the border of the base; this must be balanced by the moment of stability provided by the buttress weight. Of course, the actual collapse thickness corresponds to the minimum thrust

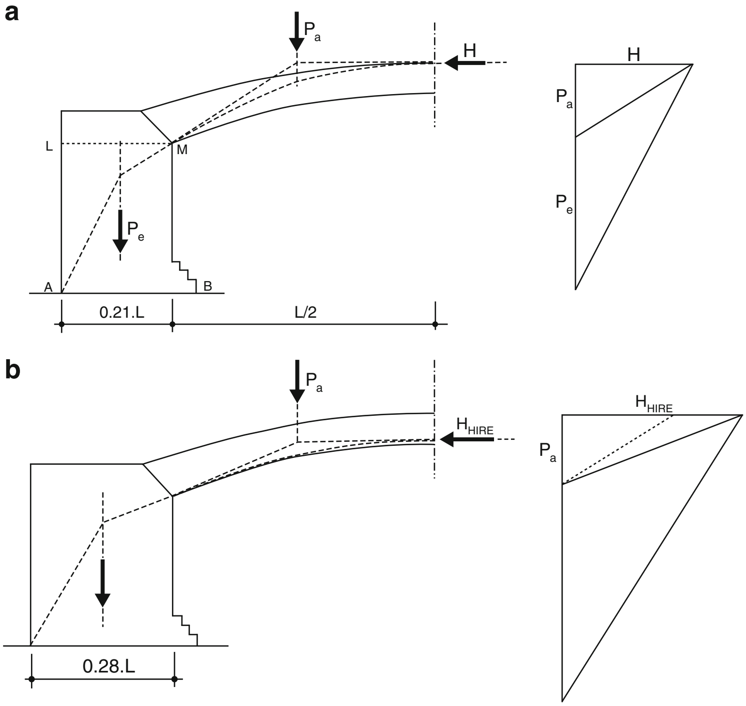


Fig. 8 (**a**) Limit buttress; (**b**) limit buttress for the thrust of La Hire. Image: author

of the vault, with the thrust line rising to the extrados at the crown. As can be seen by comparing Fig. 8a, b, the position of the thrust line varies significantly and the La Hire thrust is 65 % greater than the actual collapse thrust. La Hire's buttress has a thickness 33 % greater than that of collapse.

Lecreulx then performs another tests on a slightly thicker monolithic buttress. A buttress of 21 *pieds* 6 *pouces*, if sliding is precluded, supported a load on the crown of 19 *onces* (580 g); the La Hire buttress of 24 *pieds*, supported a load of 1 *livre*, 8 *onces* (734 g), until the failure occurred due to sliding on the base.

Looking again at Fig. 8a, b, it is easy to see that increasing the width of the buttress increases the collapse load by overturning, but as the angle of the thrust at the base of the buttress grows it may happen that the buttress fails by sliding instead of overturning, the thrust being outside the friction cone. In Fig. 8b, the angle of thrust with the horizontal is nearly 30° (tan 30 = 0.6), and the friction coefficient between stone and the wood is around 0.4 (angle of friction 22°).

Immediately, Lecreulx warns that these experiments have been made with monolithic abutments, which "is contrary to the real state [of the construction], which must be regarded, on the contrary, as formed by horizontal courses [of masonry]" (p. 84). He acknowledges that these courses will be laid with good mortar, but as the setting of the mortar could take years, it is not advisable to consider the cohesion of the masonry after the decentering, since, he maintains, "the moment just after the decentering is precisely when the vault exerts the greatest thrust" (p. 85).

He remarks that once the mortar joints have set, the buttress will form a monolith and the thrust will diminish with time until it will eventually disappear: "... after centuries, when the mortar will have acquired the hardness of stone, a barrel vault can be considered as a single piece, and its thrust reduced to nothing" (p. 85). However, he insists that he wishes to investigate the thrust of the vaults at the worst moment. Of course, the idea that the vault and the buttress together (or either of them singly) would form a monolith after the setting of the mortars is a fantasy, or perhaps the expression of a desire; even Roman vaults thrust and crack.[5]

1.2 Second "expérience"

The vault rests on the left side on the La Hire buttress of 24 *pieds*, and on the other, on a buttress of 36 *pieds*, consisting of three blocks with three horizontal joints, as shown in Figure 7(2). After the decentering the buttress of 36 *pieds* fails due to the sliding of the upper block. Lecreulx remarks that the number of joints under the joint of collapse is irrelevant: when the upper joint slides, the remaining joints make no movement.

In Fig. 9 a static analysis has been made; the inclined thrust forms with the joint an angle of 33°, similar to the friction angle between stones. In a small model the

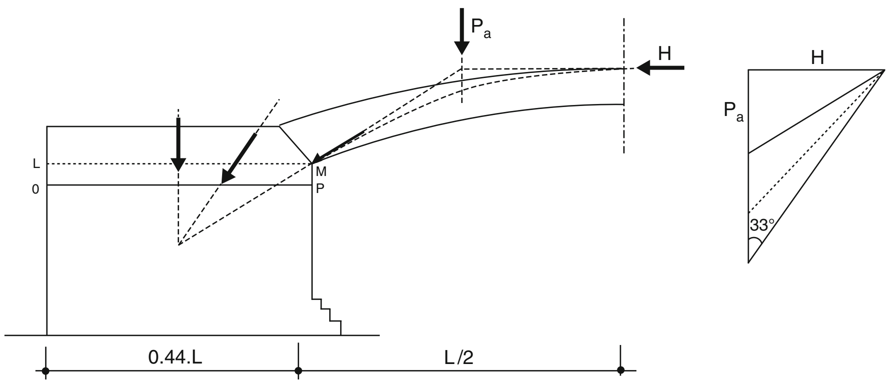


Fig. 9 Failure due to sliding of the upper part of the buttress. Image: author

[5]The idea of, and the desire for, monolithism (a masonry structure without thrusts) appears in the second half of the eighteenth century. It was not until the invention of reinforced concrete that this aspiration could be fulfilled. The same idea appears in the context of the theory of tile vaults, and can be traced to the present day (Huerta 2003).

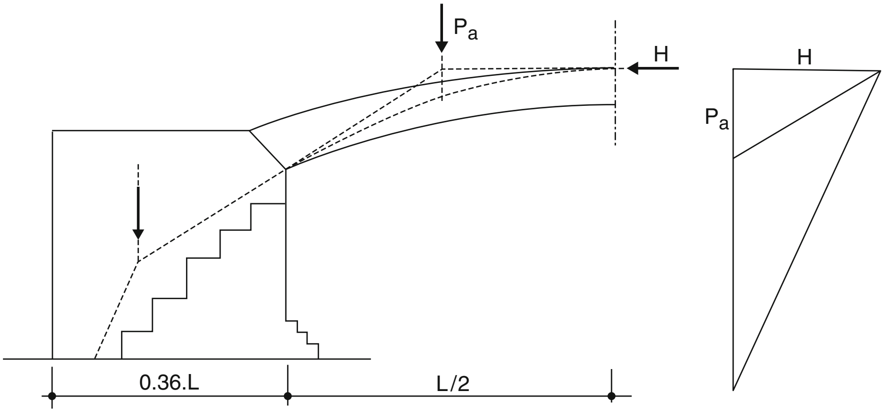


Fig. 10 Test on an already fractured buttress. A point load is needed to produce the collapse. Image: author

stones may have been polished so that the friction angle is less than 30°; this would explain the collapse.

It should be noted that Lecreulx places the joint of fracture OP (Fig. 9) about two *pieds* below the joint LM at the springings of the vault. He does not mention this. It may be that he thought that the height of the masonry courses would produce the fracture below. It also makes the tests easier.

1.3 Third "expérience"

In the third test the vault is supported on the left side on a monolithic buttress of 32 *pieds* and the other side on a buttress of 32 *pieds* divided into four pieces, of which he says, three are *en coupe*, that is, they have radial joints. After the decentering, the vault stood, and a weight of 10 *onces* (305 g), 1/8 of the weight of vault, must be added on the crown to provoke the collapse, as shown in Fig. 7(3). It should be noted that, due to the division of the buttress, the inclination of the thrust with the base has been reduced, and sliding does not occur (Fig. 10).

It is obvious that the form of collapse is determined by the joints dividing the buttress. Why this pattern of joints was chosen? Surely Lecreulx must be imitating or reproducing some observations which he doesn't cite. The buttress, trying to rotate around the exterior limit, breaks and leaves a wedge of masonry on the ground. This mode of fracture must have been known to any experienced engineer who have observed de collapse of buttresses or demolished some old bridge.

The first published observation about this fracture mode was made by Gauthey: *La chûte d'un pont ne pourrait guère arriver sans qu'il ne se fit quelques disjonctions dans ses culées*, "The collapse of a bridge can scarcely happen without there having been some disjunctions in its abutments" (Gauthey 1809, vol. I, p.

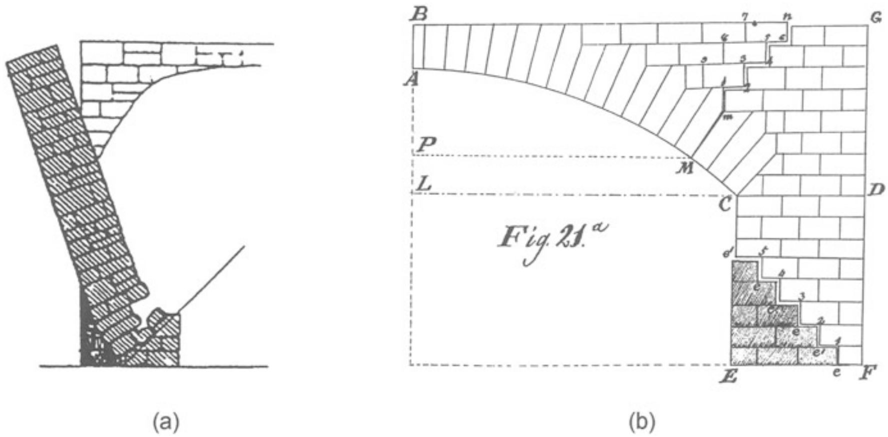


Fig. 11 Fracture of a buttress or retaining wall during the collapse: (**a**) Mayniel (1808); (**b**) Monasterio (ca. 1800). In this case, the form of the fracture is determined by the size and form of the stones. If they have a ratio of 1:2, as in the drawing, the crack will form an angle of 45°. Image: author

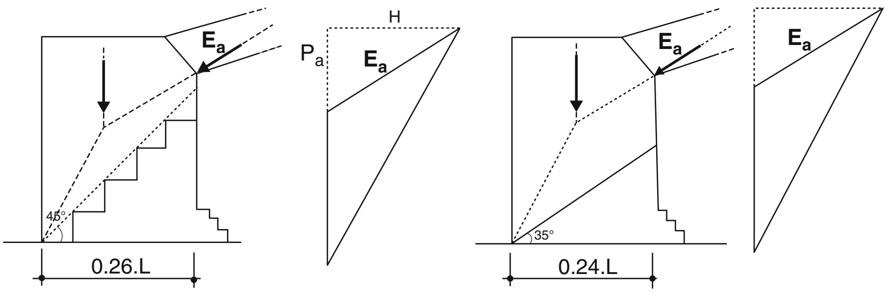


Fig. 12 Limit thickness of masonry buttress considering the failure by overturning: (**a**) assuming the plane of fracture at 45°; (**b**) exact solution. Image: author

24). This fact should have been well known by the French engineers of the late eighteenth century. The first to study this phenomenon was the Spanish engineer Joaquín Monasterio around 1800 (Huerta and Foce 2003; Huerta 2010; Albuerne and Huerta 2010) (Fig. 11b). At about the same time the French military engineer K. Mayniel (1808) discovered the fracture in his tests on the thrust of soils against retaining walls (Fig. 11a).

If the inclination of the surface of fracture is known, it is straightforward to calculate the thrust and the collapse thickness of buttress. For an inclination of 45°, considered both by Lecreulx and Monasterio, the collapse thickness is 21 *pieds*, representing a reduction of 38 % of the thickness of buttress tested (Fig. 12a). The shape and inclination of the fracture can be calculated for a homogeneous masonry buttress which resists only compressive forces: it can be shown that the fracture surface is a plane. Knowing this it is easy to calculate its slope and the buttress thickness of collapse (Ochsendorf et al. 2004). In this case, the fracture is formed at 35° and the collapse thickness is just over 19 *pieds* (Fig. 12b).

1.4 Fourth "expérience"

This test is analogous to the second. This time, Lecreulx seems interested in finding out what the thickness of the buttress needs to be to preclude the sliding failure at the top. He observes that for a buttress of 65 *pieds* formed by four pieces [Fig. 7(4)], the sliding failure is about to happen. He adds more weight, increasing gradually from 4 to 9 *onces*, and the collapse occurs. This leads to an angle of friction between the pieces of about 24°. The buttress had an unusual proportion, since its thickness is 80 % of the span (see Table 1 below). Lecreulx makes no comment on this, but later he comes back to the solution *en coupe*, with radial joints.

In the Memoir some reference to letters in the corresponding figure is made. Although these letters are missing in the drawings of Bruyère, the meaning is quite clear. At the beginning of the decentering (probably the centre is slightly shaken), he affirms that "Just when the decentering begins the vault thrusts at the joint *ef*" (p. 87). Apparently he does not fix the point of action. Then, he resolved this thrust into two forces: "the force could be resolved into the two forces *OD* and *BD*, where the vertical force *BD* [sic OD] is annulled by the upper joint, and the force BD … tends to provoke the sliding of the upper part" (p. 87). He notes that at this first instant, "the keystone descends slightly, the joint *f* opens, and the vault exerts its thrust through *E*" (p. 87)). He then makes a crucial statement: due to the opening of the crack at the joint *ef*, the thrust is reduced and the movement stops: "… and since it has less advantage in continuing to make the superior parts slide, [the movement] stops" (p. 87). Lecreulx is describing the effect of a slight yielding of the abutments in the vault thrust, which acquires its minimum value. (There is detailed description of the decentering, cracking and movements of the vaults of the bridge of Fouchard in Lecreulx's Journal of 1774 (Bruyère 1823, p. 11)). The theory of the arch was not sufficiently developed to interpret this, but nonetheless his deep understanding of vault behaviour is remarkable.[6]

[6]The concept of minimum thrust and its relation with the cracking of arches was first exposed by Méry (1840). It is however clear in the detailed registers of bridge vault movements made by Perronet and his disciples, that the French engineers of the second half of the eighteenth century understood the relationship between cracks and movements on the vaults.

1.5 Fifth "expérience"

The buttresses are 36 *pieds* and the left one is divided *en coupe* in five pieces [Fig. 7(5)]. After the decentering the vault stands. Weight is added to the crown; when the load reaches 14 *onces* (1/5 of vault's weight), the collapse occurred, this time by sliding of the buttress as a monolith [Fig. 7(5.X)]. Lecreulx attributed this sliding failure to the low friction between the stone and the wooden base. He notes that the tests should be made on a stone surface to be reliable and advises against the use of polished wooden platforms; eventually he made this test "on an old flat plank; but the surface was only moderately smooth" (p. 88). Then, he fixed the external lower point to prevent the sliding, but not the overturning [Fig. 7(5.XI)]. In this situation , the load on the crown reached a value of three *livres* 6 *onces*, more than 3/4 of the weight of the vault.

1.6 Sixth "expérience"

In the sixth and final test, Lecreulx used two vaults, with a central pier of the same proportion of the actual design [Fig. 7(6)]. The left buttress is 36 *pieds* thick divided *en coupe*, as in the previous test; on the right side, the buttress is 72 *pieds* thick and consists of six pieces. The joint of failure is the same as in tests two and four. After the decentering both vaults remain standing. Then, both vaults were loaded at the same time at the crown: under a load of 6 *onces* the vaults stand, but under 7 *onces* failure occurs by sliding on the right side.

This result must have been quite disturbing. Although the buttress is considerably thicker than that of the test 4, collapse occurred for practically the same load. Moreover, as Lecreulx remarked, the collapse of the right vault implied the collapse of the left vault, because the central pier is unable to resist the vault thrust. The conclusion is clear: an abutment of 36 *pieds* with pieces *en coupe* resists much more (in a ratio of 1:9), than a 72 buttress *pieds* with horizontal joints: "It can be seen also that, in this hypothesis, an abutment of 36 *pieds* divided in radial parts [*en coupe*], resists more than an abutment of 72 *pieds* made of horizontal courses, even though we have considered the most unfavourable situation, giving the first one the possibility of sliding along the platform" (p. 90). Obviously, he says, the friction over the *platte-forme* above the foundation will be much greater than over the wooden table, but, he insists again that, if sliding is precluded, the division *en coupe* allows for a much greater load than the usual division by horizontal courses.

Lecreulx ends his description of his *expériences* by remarking that all of them have been made and repeated in the presence of other people, of the chief engineer de Voglie, young engineers and many *entrepreneurs des Ponts et Chaussées*. Eventually, he stresses the usefulness of going forward with the *expériences*.

2 Conclusions

Lecreulx concludes that the tests show clearly that the segmental surbased vaults exert an enormous thrust against the abutments: “the vaults made of a portion of a circle, have a very considerable horizontal action against the abutments” (p. 90). The disposition *en coupe*, with radial joints, allows the direction of the thrust to change, making it more vertical and mobilizing much more weight, “by diverting this action, and pointing it towards the platform, it opposes a more considerable mass to the action”. In this way the strength of the abutment increases considerably and the failure by sliding is precluded, “and it increases the resistance of the abutments; and there would also be a gain in preventing the foundation from slipping” (p. 90).

He then proposes a completely original design for the abutment. He takes the dimension of 36 *pieds* of *expériences* 4–6, adjoining to this mass two small counter-forts of 6 *pieds*, following the advice given to him by Perronet, reaching a thickness of 42 *pieds*.[7] (It is doubtful that this counter-forts would make any noticeable improvement, but it would probably have been unthinkable not to follow the suggestion of the greatest French engineer.)

The main feature is that the vault should be prolonged inside the mass of masonry of the abutment 12 *pieds*, reaching almost the center of gravity of the abutment (see Fig. 6, above).[8] In this way, he feels confident in maintaining that the abutments will resist with all its weight the thrust of the vaults: “I feel sufficiently authorized to establish that these abutments will resist with all their mass, and will be more than sufficient to support the thrust of the vaults of this bridge”.

In fact, the only weight which helps to prevent the sliding, is that which is above the springings of the internal arch, which is only 60% of the total weight of the buttress. In this situation, a friction coefficient of 0.40, equivalent to a friction angle of 22°, will be sufficient to avoid the sliding failure. The value is low enough so that the masonry, even just after the decentering, with the setting of some part of the lime

[7]The suggestion appears in a letter of Perronet (1774) to Lecreulx. Here he praises the *expériences* and describes the dimensions and gives a detailed description of the abutments with counterforts he had designed for the bridge of St. Maxence (which were considerably increased during construction, see below). He recommends the dimension of 42 *pieds* for the abutments of Fouchard’s bridge. He then discusses at length the kind of construction to be used to avoid the failure by sliding: making irregular courses, incorporating large stones (*libages*), etc. The recommendations are similar to that suggested later by Gauthey (see note 10 below), who probably read Perronet’s letter in the Library de l’ of the École.

[8]The same description appears in the Journal, where he attributes to this disposition the small yielding of the abutments after the decentering: *Il ne doute même pas que cette différence de tassement n’eût été bien plus grande, sans la précaution qu’on a prise de contrebutter le derriére de l’assise des culées qui reçoit les naissances, par de forts libages posés en coupe, et comme formant le prolongement de la voûte dans le massif de ces culées*... (He had no doubt that this differential settlement would have been much greater without the care they took of counterforting the rear of the seat of the abutments receiving springings, by strong blocks of stone laid radially, and as though forming an extension of the vault in the solid mass of the abutments...) (Bruyère 1823, p. 11).

mortar not yet completed (in the thick masses of abutments this may take months or even years), the abutment would be safe enough.

Actually, the friction between two surfaces of dry stone or masonry with mortar not yet dried, should be well above the friction between two surfaces of the same stone. (However, as far as I know, tests on structures of real size have never been made.) If this were true , it would explain the absence of catastrophic collapse in the construction of this type of highly surbased bridges.[9]

In any case, the dangers of such constructions were evident. Boistard (1822, p. 15) explains how Gauthey tried to dissuade him from continuing the original project by Perronet. Gauthey said that he had made tests on wooden models and that the bridge would collapse just after the decentering. On the other hand, Bruyère (1823, p. 16 note C) maintains that Pierre-Antoine Demoustier (1755–1803) was overcautious in giving the abutments of the bridge of St. Maxence a thickness of 60 *pieds* (19.5 m) (Fig. 13), instead of the 40.5 *pieds* (13.2 m) of the original design (see Fig. 3).

In fact, very few highly surbased arch bridges (say, with a surbasement under 1/8 of the span) were built. Navier, in the third edition of Gauthey's *Traité des ponts*, lists ten (Gauthey 1843, p. 197). In Table 1 we give the dimensions and main geometrical ratios of the best known. The bridge of Fouchard has the most slender buttresses, but it is still of unusual dimensions. The fear of sliding collapse can be easily seen.[10]

[9]It appears that the first to assess the effect of the friction and the cohesion of the masonry on the safety of masonry buttresses was Boistard (1822, pp. 132-134). He made shear tests to calculate the cohesion between stone prisms jointed with mortar and, also, friction tests. (The tests were made ca. 1800 (Boistard 1804), and the results were reported later by Gauthey (1809, pp. 339-344.) He then applied these results to assess the safety of the abutments of the bridge of Nemours. The analysis is correct in approach but plagued with numerical errors. Gauthey repeated, correctly, the calculations for a "standard" segmental vault with a span of 20 m, with a height of 3 m. He obtained extremely slender proportions for the abutments: 1/6.7, compared with the traditional rules which assigned 1/3-1/4 of the span. He was apparently unaware that he was obtaining the collapse dimension of the buttress. However, he concluded "*. . . on sera çonvaincu de la nécessité de les augmenter encore, et de se rapprocher ainsi des règles pratiques*" (. . . one will be convinced of the need to increase further, and thus get closer to the rules of practice) (Gauthey 1809, pp. 327-8). The first to handle the problem of buttress safety correctly, within the frame of Coulomb theory of vaults, was Audoy in 1820 (Audoy 1820; Huerta 2010).

[10]The necessity of linking firmly the different courses of masonry was expressed several times. Boistard, after his analysis remarked: *Ces calculs . . . justifient en même temps les précautions que nous nous proposons de prendre dans la construction des dernières assises qui doivent être faites en libages pleins, posés en liaison, tant avec les assises inférieures, qu'avec celles des murs d'épaulement* (These calculations . . . justify at the same time the precautions we propose to take in building the last beds to be made in solid stone, set in connection with both the lower courses, and those of shoulder walls) (Boistard 1810, p. 212). Gauthey insists also in the necessity of connecting firmly the courses of masonry placing vertically big stones and, even, to construct the core of the buttress of irregular rubble masonry: *Il serait sur-tout fort avantageux de distribuer dans l'intérieur de la maçonnerie des libages placés debout, qui relieraient les assises les unes avec les autres, et qui contribueraient puissamment à ce qu'elle ne formât presque qu'un seul corps. On tendrait au même but en évitant de construire l'intérieur des massifs par assises, et en le remplissant, au*

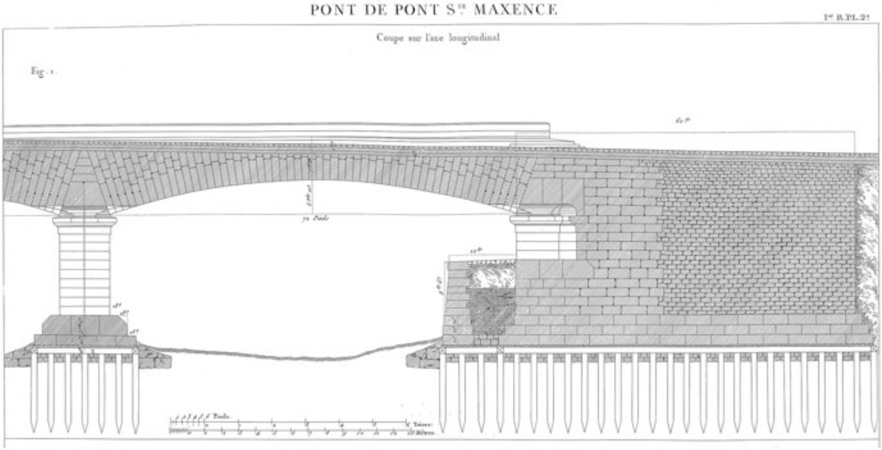
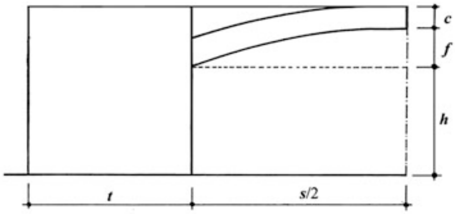


Fig. 13 Constructive bridge section Sainte Maxence (Bruyère 1823)

Table 1 Arch bridges with high surbasement



Bridge dimensions / proportions	***s*** (m)	***c*** (m) / $\alpha_1 = c/s$	***f*** (m) / $\alpha_2 = f/s$	$f_t = c + f$ (m) / $1/(\alpha_1+\alpha_2) = f_t/s$	***h*** (m) / $\alpha_3 = h/s$	**t** (m) / $\beta = t/s$
Nemours	16.2	0.98	1.12	2.10	4.22	10,2
(Perronet, Boistard; 1771, 1795-1804)		1/16.7	1/14.6	1/7.8	1/3.84	0.63
St. Maxence	23.4	1.46	1.95	3.41	5.84	19.5
(Perronet; 1774-1784)		1/16	1/12	1/6.9	1/4	0.83
Fouchard	26.0	1.3	2.63	3.93	5.20	14.50
(De Voglie, Lecreulx; 1772; 1774-1782)		1/20	1/10	1/6.6	1/5	0.56
Louis XVI-Concorde	28.6	1.41	3.00	4.41	5.84	19.5
(Perronet; 1786-1791)		1/20.3	1/9.5	1/6.5	1/4.9	0.68

s span; *c* thickness of keystone; *f* height of the vault; *h* height to the springings of the arch; *t* thickness of the abutments
Dimensions taken mainly from (Gauthey 1843, pp. 199, 251).

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contraire, en maçonnerie de blocage (It is above all very advantageous to distribute inside the masonry blocks of stone placed upright, which would connect the foundation with each other, and

The Memoir by Lecreulx describes “l’esprit” of the French engineers of the second half of eighteenth century, who, under the supervision of Perronet, full of courage and intelligence, revolutionised bridge design. It is true that the type of bridge at which they eventually arrived, with thin piers and extremely surbased vaults, was eventually abandoned, but this does not diminish the merit of one of the most audacious enterprises in the history of engineering.

Acknowledgments I would like to thank Antonio de las Casas for obtaining a reproduction of the original Memoir through the library of CEHOPU, when he was president of the Institution. Also to Rosa Senent who made a preliminary transcription of the document. Finally, I would like to express my thanks to Danila Aita for her encouragement and patience with the present author.

This article is based on a communication presented to the 4th Spanish Congress on Construction History (Huerta 2005). Since then I have continued to do research in the history of masonry buttress design. The text has been rewritten with a greater awareness of the importance of Lecreulx’s Memoir. Some errors have been corrected and new references added.

Appendix: *Mémoire sur la nature de la poussée des voutes, formées d’un seul arc de cercle, contre les culées, contenant des Experiences sur les éffëts qui en resultent* by François Michel Lecreulx

What follows is a diplomatic transcription of the original Memoir of 1744, Ms. 233, Tome 21 (pp. 81–90), Bibliothéque de l’École des Ponts et Chaussées. The numbers in square brackets refer to the page numbers of the manuscript in the volume in which it is bound. It should be noted that the numbering of the figures and some letters in them, when cited in the text, referred to the original illustrations, are missing in the figures reproduced by Bruyère (Fig. 6).

Mémoire sur la nature de la poussée des voutes, formées d’un seul arc de cercle, contre les culées, contenant des Experiences sur les éffëts qui en resultent

[p. 81] Il est évident que dans les voûtes dont il s’agit, chacune des piles etant également préssée de part et d’autre, et restant en equilibre entre ces deux pressions n’éprouvent aucune poussée mais leurs fonctions se reduisent à supporter chacune le poids de deux demi-voutes; c’est pour quoi, lorsque la pierre est de bonne qualité,

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that would contribute greatly so that it forms almost a single body. We tend to the same end, without building within the mass in courses, and filling, to the contrary, in stone masonry) (Gauthey 1809, pp. 330-1).

leur épaisseur est beaucoup plus que suffisante pour porter le poids des voutes. Dans le pont que l'on construit où les arches ont chacune 80 pieds, et les piles 12 pieds au dessus des retraites, reduites à 9 pieds 4 pouces à la naissance, on est convaincu qu'avec la qualité de la pierre de Champigny, dont on se sert, qui pese environt 180 L le pied cube, les piles sont en état de supporter des arches plus de trois fois plus grandes surtout avec l'attention de ne pas faire les joints des lits excessivement petits, et de les laisser ouverts, pour ne point laisser pincer les arrêtes; de sorte que les précautions doivent se borner à bien assurer les fondations, pour que l'assiette soit inébranlable et à l'abri des affouillemens; c'est pour quoi, après s'être bien assuré de la nature et qualité du terrein, et s'être établi sur un pilottage bien battu au refus, l'on a observé au dessus des eaux trois larges retraites, d'un pied chacune.

La poussée des voûtes agissant principalement sur les culées, on a cru de la derniere importance de réflechir sur les éffets [p. 82] qui en peuvent resulter; et quoique le théorie les fasse aisement prévoir, comme elle est souvent appuyée sur des hypotheses qui ne sont pas toujours conformes à la nature, l'on a crû qu'il etoit important de s'assurer des résultats par de nouvelles experiences que l'on a tenté et que d'autres pourront perfectionner.

Pour y parvenir on a fait un modele du Pont Fouchart sur une échelle de 14 lignes par toise, exactement conforme aux dimensions du project: on a retranché les ornemens des têtes du pont a fin de rendre les éfféts plus uniformes; de sorte que l'élévation resemble à une coupe du dit pont, prise sur la longueur, à la quelle on a donné, suivant l'échelle, une épaisseur égale de 19 pieds 26 pouces: Ce modele est fait en pierre tendre de Tuffeau autrement dit Bourré: on a choisi cette pierre pour le modele de préference à celle de Champigny, tant parce qu'elle est plus facile à tailler, que parce qu'elle est plus homogene, surtout en petit volume; et comme il n'est question que d'établir des rapports, on pourra également les appliquer ensuite à toute nature de pierre.

Nota. Il est facile de reconnoitre que les voûtes dont est question agissent differemment sur les culées, suivant que la portion d'arc de cercle contient un plus grand ou un moindre nombre de dégrès. Dans le cas dont il s'agit ici, les voûtes sont décrites à l'intrados par un rayon de 108 pieds, comprenant un arc de 44 degrés 42 minutes; ce qui donne à la corde mesurée à la naissance 82 pieds 1 pouce 6 lignes, en égard au talud des pie-droits des piles et culées; par ce moyen la montée de l'arc se trouve de 8 pieds, 1 pouce, 4 lignes; et le developpement du dit arc 84 pieds, 3 pouces, 9 lignes, 10 points, suivant le calcul. Pour faire les experiences cy après, les voutes sont posées sur des cintres que l'on leve ou que l'on baisse à volonté, et de façon à causer aux voûtes le moins de mouvement qu'il est possible, cy [p. 83] décintrant. Les figures sont faites sur une échelle qui est le tiers de celle du modele.

Prémiere Experience

On suppose une voûte des dimensions susdites, soutenue, d'une part par une culée de 24 pieds d'epaisseur au dessus des retraites; et de l'autre par une autre culée de 18 pieds, 6 pouces d'epaisseur au dessus des mêmes retraites, posée sur des cintres, qui sont disposés pour s'abaisser ou se relever à volonté; toutes les parties de la voûte dans le modele pesoient ensemble 4 livres 6 onces. On suppose aussi les culées faites d'une seule piece et homogenes.

Effets

Lors qu'on abaisse les cintres la voute se soutient: la culée de 18 pieds, 6 pouces d'épaisseur au dessus des retraites suffit pour l'équilibre mais 3 ou 4 onces de charge sur son milieu; c'est à dire environ la 20e partie de son poids, la font écrouler; et elle prend, en écroulant la figure que l'on voit. On a éprouvé qu'une culée d'un pied moins épaisse ne pouvoit soutenir l'équilibre.

On voit, en observant l'éffet, que l'éffort de la voûte se partageant sur les deux culées, la portion de voûte qui agit contre la culée *AEB*, la pousse dans la direction *EB*, que le mouvement se faisant a lors au tour du point *A*, l'effort de la voûte a pour bras le levier *AD* determiné par la distance de la direction du centre de gravité de la culée, au point d'appuy *A*. La quelle hypothése a été employée, par divers auteurs, pour base de la théorie, et des calculs qu'ils ont donnés; mais il faut, pour ce cas que la culée puisse être supposée d'une seule piece sans des unions et l'on verra par la suite que les éffêts sont fort differents quand les culées sont composées d'une grande quantité de parties.

Si l'on avoit donné 21 pieds, 6 pouces d'épaisseur au dessus [p. 84] des retraites à la culée la plus foible, toujours supposée d'une seule pièce; et si on l'empêchoit de glisser sur la platte-forme, a lors la voûte pourroit porter sur son milieu 19 onces, avant de renverser; mais elle a de la peine à resister à la poussée, sans glisser sur la platte-forme; de sorte que l'éffort de la voûte agit dans ce cas, plus fortement pour faire glisser cette culée, que pour la renverser. Si la même culée avoit 24 pieds d'épaisseur, la voûte porteroit, avant d'écrouler 1 Livre 8 onces, et alors la culée ne renverseroit pas; mais glisseroit sur la platte-forme.

Nous avons supposé dans l'experience précedente que les culées etoient faites d'un seul morceau, ce qui est contraire à l'état des choses: on doit les regarder au contraire, comme composées d'assises horizontales. Il est bien vray que les assises etant scellées avec mortier, elles ont entr'elles une adhésion qui augmente par laps de tems; mais les mortiers sont longs à secher dans l'interieur des grosses mâçonneries, et quoique la consistance du bon mortier augmente jusqu'à devenir plus dure que la pierre; il faut des siécles pour qu'ils acquierent cette dureté. Tout le monde sait aujourd'huy que c'est le tems qui a donné de la réputation au mortier des anciens Romains qui n'employient pas d'autres élemens que nous; C'est pour quoi, si l'on a tant de facilité à démolir les grosses mâçonneries faites avec le meilleur mortier, lors qu'elles n'ont que deux ou trois ans; s'il est vray, comme on l'assure, que l'on trouve encore alors dans le milieu des grosses masses, des mortiers humides; peut être paroitra-t'il prudent dans le calcul, d'avoir peu d'égard à l'adhésion des mortiers dans la resistance des culées contre la poussée des voûtes: vû, d'ailleurs, qu'il reste une autre consideration qui peut compenser ce que l'on néglige sur l'adhérence des mortiers: c'est que nous avons supposé dans nos Experiences, [p. 85] que les culées etoient homogenes avec les voûtes; cependant, on fait le corps entier des voûtes en pierre de taille dure tandis qu'il n'y a que les parements des culées qui soyent pareillement en pierre-de-taille; le reste devant être fait en fort libage et moëlon qui laisse beaucoup plus de vuide pour le mortier; il est évident que ces derniers especes de mâçonneries pesent moins que celle en pierre de taille; et dans ce cas, la resistance de la culée se trouveroit au dessous de l'éxperience, sans l'adhésion des mortiers que l'on a négligé.

On ajoutera encore que le moment qui suit le décintrement, en étant toujours celui de la plus grande poussée des voûtes, son éffet doit diminuer de jour en jour, à mesure que les mortiers prennent de la consistance; de sorte, qu'après des siècles, lorsque le mortier aura acquis la dureté de la pierre, une voûte en plein cintre pourroit être considerée comme formée d'un seul morceau, et sa poussée reduite à rien.

Mais, comme il est question ici, de considerer l'éffét de la poussée des arcs de cercle, dans le moment le plus désavantageux on va suivre les éffets des experiences cy après.

Deuxiême Experience

La même voûte que dessus, posée sur les cintres est appuyée d'une part par une culée de 24 pieds d'épaisseur au dessus des retraites, faite d'un seul morceau; de l'autre part, soutenue par une culée de 36 pieds d'épaisseur de même au dessus des retraites, faite de trois morceaux; savoir, une pièce inferieure, coupée horizontalement, à 12 pieds 6 pouces au dessus des dites rétraittes; ensuite une assise horizontale de [p. 86] deux pieds; et en fin un morceau superieur de dix pieds de hauteur, terminé de même horizontalement, et qui peut équivaloir à 8 à 9 assises contre les quelles la voûte s'appuye.

Efféts

Aussitôt le décintrement, la culée de 24 pieds reste fixe, et la partie superieure de la culée de 36 pieds glisse horizontalement, et la voûte écroule en prenant la forme decrite par la figure.

Nota. Soit que la partie *OB* soit d'une seule piéce, ou soit composée de plusieurs assises horizontales l'éffét est le même: et lorsque la partie superieure glisse les assises inferieures ne font aucune mouvement.

Troisiême Experience

On suppose la même voûte, posée sur les cintres, appuyée d’une part, par une culée de 32 pieds au dessus des retraites d’un seul morceau, servant de point fixe: Et de l’autre part, par une culée de 32 pieds, composée de 4 piéces, dont 3 sont en coupe.

Éfféts

Après le décintrement, la culée de plusieurs morceaux resiste ainsi que celle d’une piece; et la voûte se soutient: elle porte 9 onces, sans tomber; c’est à dire, le 8^{e}. de son propre poids; et elle n’écroule qu’avec 10 onces.

La figure 3 fais voir l’éffet, lors de l’écroulement.

Quatriême Experience

La même voûte etant sur les cintres, on suppose d’une part une culée de 36 pieds au dessus des retraites, d’un seul morceau, [p. 87]servant de point fixe: et de l’autre une culée de 65 pieds d’épaisseur mesurés de même au dessus des retraites; mais formée de 4 morceaux, suivant la figure où l’on voit que les deux parties superieures ont alors ensemble 62 pieds 6 pouces de longueur, reduite.

Prémier Effet

Au moment où l’on commence le décintrement, l’éffort de la voûte pressant le joint *ef* dans la direction de *AO*, la pression peut se decomposer dans les deux forces *OD* et *BD*; ou la force *BD* verticale est detruite par le plan superieur: et la force *BD*, horizontale, située dans la direction du centre de gravité des portions superieures de la culée, tend à les faire glisser: En fin se fait, au prémier moment, un petit mouvement: la cléf baisse un peu, le joint s’oeuvre en *f*; et la voûte ne presse plus qu’en *E*; et comme elle a alors moins d’avantage pour continuer à faire glisser les parties superieures, elle s’arrête.

Deuxième Effet

On achêve la décintrement, et la voûte se soutient; mais aussitôt qu’on la charge de 4 à 9 onces, elle écroule conformement à la figure.

Cinquiême Experience

La même voûte posée sur les cintres et appuyée d'une part contre un culée de 36 pieds d'épaisseur au dessus de retraites, d'un seul morceau, servant de point fixe: Et de l'autre par une culée aussi, de 36 pieds; mais formée de 5 morceaux

Effét Prémier

Après le décintrement, la voûte se soutient: on la charge [p. 88] de plusieurs petits poids successivement, et elle porte 14 onces; c'est à dire le cinquiême de son poids: Ensuite la partie de derriere glisse; et la voûte s'écroule, suivant l'éffet décrit dans la figure.

Nota. Il faut éviter de faire ces Experiences sur des planches nouvellement corroyées et polies au Rabeau. Il faudroit même, a fin que les circonstances du Frottement fussent semblables à l'Experience cy dessus, où l'on a fait glisser une portion de culée, que dans le cas present la culée glissât sur de la pierre; néanmoins on s'est contente de faire cette cinquième Experience, sur une vieille table de niveau; mais dont la surface etoit médiocrement unie.

Deuxième Éffét

On a mis derriere la culée faite de plusieurs piéces, un point fixe capable de l'empêcher de glisser sur la platte forme sans nuire à son renversement: Et après avoir décintré, on a chargé la voûte de plusieurs poids; et elle a porté, avant d'écrouler 3 livres 6 onces: c'est à dire, une charge de plus de trois quarts de son poids.

Sixiême Experience

On suppose deux voutes semblables à celle cy dessus, separées par une pile, des dimensions sus dites de projet. Les deux voûtes posées sur les cintres, sont appuyées l'une contre un culée de 72 pieds d'épaisseur, en six morceaux, posés horizontalement: Et l'autre, contre une culée de 36 pieds composée, comme dans la précedente experience de cinq morceaux; mais en liberté de glisser sur la Platte-forme.

Prémier Éffet

On baisse ensemble les cintres des deux voûtes: et elles se soutiennent: ensuite on charge ensemble ces deux voûtes, avec des petits poids égaux: on met d’abord 4 onces sur le milieu de chaque voûte; puis on ajoute deux onces sur chacune; Et les [p. 89] voûtes sont en équilibre et prêtes à écrouler.

Deuxième Effet

On ajoute une once sur chaque voûte; ce qui fait en total 14 onces sur les deux voûtes: alors la partie superieure de la culée de 72 pieds glisse; et une des voûtes s’écroule: Et la voûte qui repond à la culée en coupe reste en place.

Nota 1. Le poids de la portion glissante etoit d’environ 6 livres: et la moitié de la voûte pesoit 2 livres, 3 onces.

Nota 2. Il est évident que, si les cintres, après l’écroulement n’avoient pas appuyé les morceaux de la voûte écroulée qui butoient encore contre la pile elle n’auroit pû, seule, en soutenir la poussée.

On voit aussi que, dans l’hypothese dont il s’agit une culée de 36 pieds dont les parties sont en coupe; resiste plus qu’une culée de 72 pieds dont les assises sont horizontales, quoiqu’on ait pris le cas la plus désavantageux qui est celui ou la culée de 36 pieds a la liberté de glisser sur la platte-forme; néanmoins, on ne peut disconvenir qu’un modele n’ait plus de facilité à glisser sur une table, qu’un culée sur la platte-forme, toutes considerations faites et l’on a vû cy dessus qu’en l’empechant de glisser, la culée en coupe portoit la voûte avec une charge de 54 onces sur son milieu.

On croit devoir prévenir que toutes les experiences cy dessus ont été faites et repetées souvent devant plusieurs personnes, et notamment devant M. De Voglie, Messr. Le Grand et M.Benoit, éleves; ainsi que plusieurs entrepreneurs des Ponts et Chaussées; et que les resultáts ont été les mêmes, ou avec des differences très petites.

On avoit médité, pour completter les observations, dont les cas cy dessus, paroissent susceptibles, d’entreprendre dans un ordre déterminé des experiences que l’on auroit suivi pendant quelques années dans un ordre détermine et qui auroient pû par la suite être continuées [p. 90] par quelques autres qui y auroient pris le même interêt.

Pour resumer, on voit par les resultat des Experiences cy dessus, que les voûtes faites d’une portion de cercle, ont une action horizontale très considerable contre les culées; et qu’en dêtournant cette action, et en la dirigeant vers la platte forme, on oppose une masse plus considerable a l’action: et l’on augmente, la resistance des culées; et qu’il y auroit aussi à gagner en empechant les assises de glisser.

La culée du Pont-Fouchard, à Tours, a trente six pieds d’épaisseur au dessus des retraites, et avec les contreforts ajoutés par l’avis de M. Perronet, de 42 pieds, et les assises de la voûte doivent être prolongées en coupe dans l'interieur des culées, sur douze pieds de developpement, suivant le dévis; ce qui les fait arriver à peu près au centre de gravité des dites culées; moyenant quoy, l'on se croit suffisamment authorisé à établir, que ces culées resisteront avec toute leurs masse, et seront plus que suffisantes pour soutenir la poussée des voûtes de ce Pont.

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Statics of Historic Masonry Constructions: An Essay

Mario Como

Abstract I feel honoured to present the findings published in my recent book *Statica delle Costruzioni Storiche in Muratura* to the Association Edoardo Benvenuto. I like to add that, during the phase of preparation for this present essay, the English edition of the book, *Statics of Historic Masonry Constructions*, has also been published by Springer. My research took shape gradually, during thirty years of research, professional experience and teaching. The book firstly gives fundamentals of statics of the masonry solid from its mathematical groundings and then applies them to the study of the static behaviour of arches, piers and vaults. Further, combining engineering and architecture and through an interdisciplinary approach, my research highlights the deep connections existing between statics and architecture and investigates the static behaviour of many historic monuments, as the Pantheon, the Colosseum, the domes of S. Maria del Fiore in Florence and of St. Peter in Rome, the Tower of Pisa, the Gothic cathedrals etc. In the end the book considers the behaviour of masonry buildings under seismic actions. Here I will discuss the adopted hypotheses and some key passages of the main issues involved.

Keywords Strength and deformability of masonry materials • Deformation and equilibrium of masonry solids • Static behaviour of arches and vaults

1 Special Features of Masonry Behaviour

Under a given loading path a masonry structure can reach a collapse condition solely due to loss of equilibrium, that is to say, in the absence of any material failure. Such a condition, due to the very low—near zero—material tensile strength, can thus arise even in masonry with infinite compression strength. Masonry structures can suffer, in fact, cracks or detachments that may in turn generate displacement fields, called *mechanisms*, which develop without any internal opposition from the material. So, as soon as the pushing loads begin to exceed the action of the resistant loads

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along one of these mechanisms, the structure fails. Moreover, if a small settlement occurs at one of the external constraints of a masonry structure, it freely follows the settlement, maintaining constant its stresses and constraint reactions. It is thus easy to understand how the presence of a negligible tensile strength can disrupt the behaviour of structures as compared to the common elastic ones. These are the essentials of the masonry behaviour, fully realized by ancient builders and which have shaped the course of architecture from the origins up to the nineteenth century.

2 Heyman Assumptions

The constitutive assumptions that control the masonry behaviour, discussed in depth in Como (2010, 2013), were originally formulated by Heyman (1966) and are as follows:

(i) *masonry is incapable of withstanding tensions*;
(ii) *stresses are so low that masonry has effectively an unlimited compressive strength*;
(iii) *shear strains cannot occur*

The other assumption: *elastic strains are negligible*, was not directly expressed by Heyman but constantly considered.

The foregoing assumptions turn out to be very clear if we refer to the elementary resistant cell of the masonry structure, represented by two idealized rigid masonry bricks compressed one against the other by the stress vector $\boldsymbol{\Sigma}$, whose components are the more or less eccentric axial load N and the shear force T (Fig. 1). The two rigid bricks of the unit resistant cell cannot deform internally, but they can detach from each other. A *crack* can occur in the cell.

The first two of Heyman's assumptions involve stresses; the last one strains. The first and the second assumptions are the most important. The third assumption can be considered a consequence of the first two. We can make reference in fact to the Coulomb criterion (1776). In this framework the ratio between compression and tensile strengths σ_{rc} and σ_{rt} can be expressed in the following form:

$$\frac{\sigma_{rc}}{\sigma_{rt}} = \frac{1 + \sin\phi}{1 - \sin\phi}$$

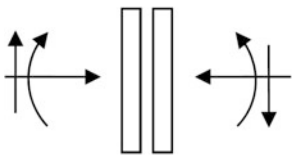


Fig. 1 The ideal resistant masonry cell and the corresponding components of the stress vector $\boldsymbol{\Sigma}$

where ϕ is the angle of the internal friction. By gradually reducing the ratio σ_{rt}/σ_{rc}, at the limit, we obtain

$$\frac{\sigma_{rc}}{\sigma_{rt}} \to \infty \Rightarrow \phi \to \frac{\pi}{2}$$

The internal friction strength, depending on tgϕ, becomes unbounded. The first two Heyman assumptions thus imply unbounded sliding strength (Como and Grimaldi 1985). This result will be considered further on.

Following the above assumptions, Statics of masonry constructions moves immediately towards the Limit Analysis. We remark that according to the above assumptions no local failures in the masonry structures are considered.

3 Extension of Heyman Assumptions to Masonry Continuum

A lack of knowledge reveals, on the other hand, as soon as the behaviour of the general masonry solid is inquired. A vast number of researches spread to fill this gap. In-depth studies into the behaviour of *elastic no-tension* bodies have been conducted by many authors, among whose works I recall Di Pasquale (1984), Del Piero (1989), Lucchesi et al. (2008), Romano and Romano (1985), Romano and Sacco (1984), Baratta (1999), Angelillo et al. (2010), Trovalusci and Masiani (2005) and Bacigalupo and Gambarotta (2010). All have addressed the general problem of the elastic equilibrium of no-tension bodies and numerous, noteworthy stress solutions have been provided (Lucchesi et al. 2008). Nevertheless, the much more complex goal of solutions expressed in terms of displacement and strain fields remains still today substantially unsolved. These difficulties stem from the fact that the no-tension elastic model cannot easily account for the presence of shear strains. In order to overcome these difficulties (Como 2010, 2013) assumes the rigid-in-compression no-tension material and aims to extend the Heyman model to the masonry continuum, on the wake of some previous results presented in Como (1992). This extension, which allows to go into the equilibrium of the masonry solid with a suitable mathematical formulation, wants also to pay homage to the outstanding description of the behaviour of masonry constructions given by Heyman in the far 1966. I will outline its main points of this extension in what follows.

A masonry solid can be considered an assemblage of rigid particles held together by the compressive stresses produced by loads. The small size of the stones compared to the dimensions of the body enables it to be considered a continuous body instead of a discrete system of many individual particles. When the compression stresses that held stones together cancel out in some regions of the masonry solid, it can get deformed. Cracks can thus occur in the masonry mass: they represent discontinuities or detachments of the displacement fields **u**(P), describing the deformation of the body. The research of compatibility conditions that the functions **u**(P), called *mechanisms*, have to satisfy to describe the deformation of the solid, is then tackled in Como (2010, 2013).

The definition of the *impenetrability condition* is the starting point: it requires that the displacement function **u**(P) cannot produce any contraction between points connected by segments *entirely contained within the body*. Thus, if (P_1, P_2) is such a pair of points in Ω, the region occupied by the body, and (Q_1, Q_2) is the corresponding pair after the transformation **u**(P), we have

$$d\,(Q_1, Q_2) \geq d\,(P_1, P_2) \tag{1}$$

where $d(Q_1, Q_2)$ denotes the distance of the segment connecting the points Q_1, Q_2 (Como 1992) (Fig. 2). According to this condition *no internal sliding can occur*. Impenetrability condition (1) in a different form still represents both the assumptions (i) no tension, and (ii) the infinite compression strength.

In short, masonry material can only be *widened or opened*. Thus, the relative displacement between a pair of points located across the line of a crack will occur only along the direction *normal* to the crack. Let us consider the line f of the crack and its two edges f^- and f^+ (Fig. 3). We choose the point P^- on the edge f^- and P^+ on the other edge f^+ of the crack. These points are obtained by intersecting f^- and f^+ with the direction of the unit vector $\mathbf{n}^-$, located along the outward normal to f^- and passing through P^-. Cracks can thus *open* only along the direction of $\mathbf{n}^-$ (or of $\mathbf{n}^+$). We can thus define, for instance, the *crack opening vector* or the *detachment*

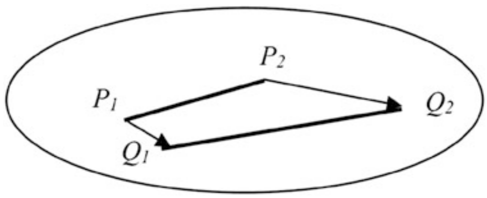


Fig. 2 The impenetrability condition

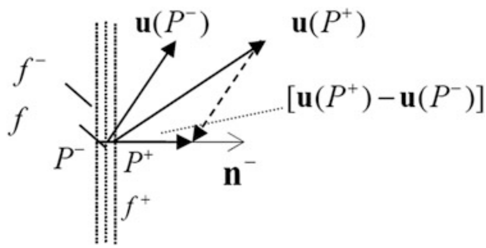


Fig. 3 The opening of a crack

vector as follows:

$$\Delta^{(n^-)}\mathbf{u}(P) = \left[u\left(P^+\right) - u\left(P^-\right)\right]\mathbf{n}^-,$$

with

$$\left[\mathbf{u}\left(P^+\right) - \mathbf{u}\left(P^-\right)\right] \cdot \mathbf{n}^- = u\left(P^+\right) - u\left(P^-\right) > 0$$

and where $u\left(P^+\right)$ and $u\left(P^-\right)$ are the scalar values of $\mathbf{u}\left(P^+\right)$ and $\mathbf{u}\left(P^-\right)$. This is the first local kinematical compatibility condition to be satisfied by the mechanism displacement $\mathbf{u}(P)$. Consequently, we can define the scalar *crack opening* by means of the *positive* quantity

$$\Delta^{(n^-)}u(P) = u\left(P^+\right) - u\left(P^-\right) > 0. \tag{2}$$

The stress vector is *null* along the crack. From this result other kinematical compatibility conditions follow. A displacement field $\mathbf{u}$(P) satisfying all these kinematical conditions, defined in detail in Como (2010, 2013), represents a *mechanism* and M is the set of all the mechanisms. Likewise, other local compatibility conditions involving stresses and loads are also given.

4 The Principle of Virtual Work for Masonry Bodies

An important topic tackled in Como (2010, 2013) is the definition of the admissible equilibrium state for the masonry solid. Developing a general equilibrium analysis of masonry bodies is a very difficult task due to the discontinuities present in the corresponding displacement functions $\mathbf{u}(P)$. The idea of Vol'pert and Hujiadev's (1985) for the study of discontinuous functions of including the set of all discontinuity points within the body's boundary, turns out to be quite fruitful. Following this suggestion and in step with Como (1992), we can consider the set

$$\Gamma\left(\mathbf{u}\right)$$

of all the points of discontinuities, that is, the set of all the cracks, *each with its two edges*, for any mechanism $\mathbf{u}(P)$ of the masonry body. This set becomes a *new part* of the boundary of the body, generated by the cracks associated to $\mathbf{u}(P)$. Consequently, we can define, the *free* cracks region $\Omega(\mathbf{u})$, associated to mechanism $\mathbf{u}(P)$

$$\Omega\left(\mathbf{u}\right) = \Omega / \Gamma\left(\mathbf{u}\right)$$

Only in this region $\Omega(\mathbf{u})$ will the displacement fields $\mathbf{u}(P)$ be represented by regular functions, for instance, continuous with their first derivatives, so that strains

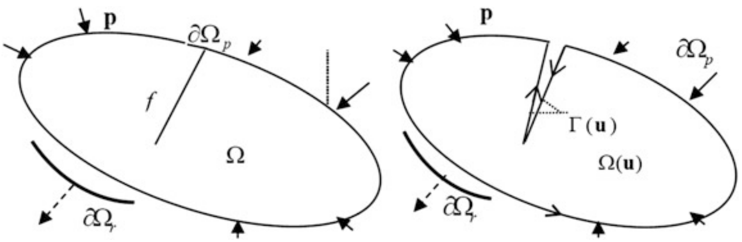


Fig. 4 The boundary of the masonry body and the new boundary of the cracked body corresponding to mechanism **u**

$\boldsymbol{\varepsilon}(P)$ can be defined in $\Omega(\mathbf{u})$. The new boundary of the cracked body, corresponding to the mechanism displacement $\mathbf{u}(P)$, is thus represented as

$$\partial\Omega\,(\mathbf{u}) = \partial\Omega \cup \Gamma\,(\mathbf{u})$$

As per customary representations, the left-hand scheme in Fig. 4 shows the boundary of the masonry body crossed by the crack f; the right-hand scheme instead shows the boundary $\partial\Omega\,(\mathbf{u})$ that includes the two edges of the crack f. We can cover the entire boundary $\partial\Omega\,(\mathbf{u})$ by circling the region $\Omega(\mathbf{u})$, for instance, in the counter clockwise direction, that is, with region $\Omega(\mathbf{u})$ always remaining on the left.

The equilibrium of the body is governed by the principle of *virtual work*. This principle will take a *particular form* for the compressionally rigid no-tension bodies, analysed in Como (2010, 2012, 2013) along the lines previously set forth in Como (1992).

Let us consider a masonry body under the action of the loads **p** in an admissible equilibrium state. The body occupies the region Ω, whose boundary is denoted as $\partial\Omega$, which we assume to be sufficiently regular. The body is loaded by mass and surface loadings $\boldsymbol{\rho}(\Omega)$ and **p**. The loaded part of the body surface $\partial\Omega$ is $\partial\Omega_p$.

Let $\delta\mathbf{u}(P) \in M$ be a mechanism field, representing a kinematically admissible virtual displacement of the body. Cracks will arise during the development of the virtual mechanism $\delta\mathbf{u}(P)$ and $\Gamma(\delta\mathbf{u})$ will be the region representing the cracks' boundaries. At any point P within the region $\Omega(\delta\mathbf{u})$, the stress field $\boldsymbol{\sigma}$ and the body forces $\boldsymbol{\rho}$ will satisfy the associated compatibility inequalities and the following internal equilibrium equation:

$$\sigma_{ij,j} + \rho_i = 0 \tag{3}$$

Now let dV be a generic volume element around P in $\Omega(\delta\mathbf{u})$. The virtual work done to displace this element is

$$\left(\sigma_{ij,j} + \rho_i\right)\delta u_i dV.$$

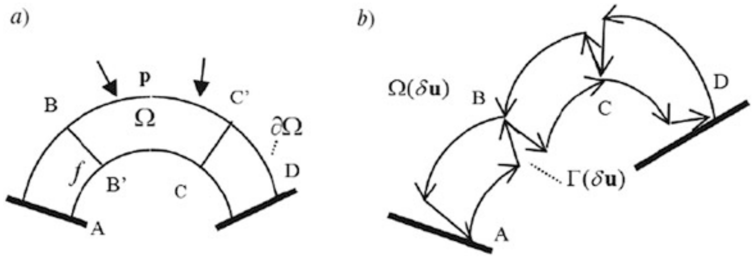


Fig. 5 The boundary of the arch and of the cracked arch with its new boundary associated to the virtual mechanism δ**u**

According to the equilibrium equation (3), this work vanishes. Integration of (9) over the volume Ω(**u**) thus yields

$$\int_{\Omega(\delta u)} \left(\sigma_{ij,j} + \rho_i\right) \delta u_i dV = 0. \tag{4}$$

Applying the Gauss-Green theorem (Fig. 5), together with some tensor calculations and the previous specifications, enables us to obtain

$$\int_{\Omega(\delta u)} \sigma_{ij,j} \delta \varepsilon_{ij} dV = \int_{\partial\Omega(\delta u)} t_i^{(\mathbf{n})} \delta u_i dS + \int_{\Omega(\delta u)} \rho_i \delta u_i dV, \tag{5}$$

where **n** is the unit vector along the outward normal to the crack surface, Fig. 5a shows a masonry arch in an admissible equilibrium state under the action of loads **p** and internal stress **σ**. Figure 5b also shows the displacement field δ**u** with hinges A, B, C and D, together with the corresponding internal cracks BB′ and CC'. Figure 5a, b also show:

- the cracks' boundaries Γ(δ**u**);
- the region Ω (δ**u**) = Ω/ Γ (δ**u**) lacking cracks;
- the overall boundary of the body, including the crack boundaries ∂Ω (δ**u**) = ∂Ω ∪ Γ (δ**u**).

The entire boundary can also be specified by the union of the boundaries Γ(δ**u**), $\partial\Omega_r$ and $\partial\Omega_p$

The internal work can now be written in a more explicit form. In fact, according to (5), we have

$$\int_{\Omega(\delta u)} \sigma_{ij,j} \delta \varepsilon_{ij} dV = \int_{\Gamma(\delta u)} t_i^{(\mathbf{n})} \delta u_i dS + \int_{\partial\Omega_r} r_i^{(\mathbf{n})} \delta u_i dS + \int_{\partial\Omega_p} p_i^{(\mathbf{n})} \delta u_i dS + \int_{\Omega(\delta u)} \rho_i \delta u_i dV \tag{6}$$

To work out the first integral in the second member of (6), by moving around the whole contour of the body, the virtual work of the interactions $t_i^{(\mathbf{n})}$ can be evaluated along each of the two edges of the cracks (Fig. 5b). For the sake of simplicity, we can refer to a single crack alone and write

$$\Gamma\left(\delta\mathbf{u}\right) = \Gamma_1\left(\delta\mathbf{u}\right) \cup \Gamma_2\left(\delta\mathbf{u}\right),$$

where $\Gamma_1(\delta\mathbf{u})$ and $\Gamma_2(\delta\mathbf{u})$ are the two equal surfaces representing the two edges of the crack. Evaluating the first integral in the second member of (6) thus gives

$$\int\limits_{\Gamma(\delta u)} t_i^{(\mathbf{n})}\delta u_i dS = \int\limits_{\Gamma_1(\delta u)} t_i^{(\mathbf{n}^-)}\delta u_i\left(P^-\right)dS + \int\limits_{\Gamma_2(\delta u)} t_i^{(\mathbf{n}^+)}\delta u_i\left(P^+\right)dS. \tag{7}$$

On the other hand, using expression (2) for the crack opening $\Delta^{(n^-)}u(P)$, we have

$$\delta u_i\left(P^-\right) = \delta u_i\left(P^+\right) - \Delta^{(n^-)}\delta u_i(P); \tag{8}$$

Substituting (8) into (7) gives

$$\int\limits_{\Gamma(\delta u)} t_i^{(\mathbf{n})}\delta u_i dS = \int\limits_{\Gamma_1(\delta u)} t_i^{(\mathbf{n}^-)}\delta u_i\left(P^+\right)dS - \int\limits_{\Gamma_1(\delta u)} t_i^{(\mathbf{n}^-)}\Delta^{(n^-)}\delta u_i(P)dS$$
$$+ \int\limits_{\Gamma_2(\delta u)} t_i^{(\mathbf{n}^+)}\delta u_i\left(P^+\right)dS.$$

Furthermore, by taking into account that

$$t_i^{(\mathbf{n}^-)} = -t_i^{(\mathbf{n}^+)},$$

we get

$$\int\limits_{\Gamma(\delta u)} t_i^{(\mathbf{n})}\delta u_i dS = -\int\limits_{\Gamma_1(\delta u)} t_i^{(\mathbf{n}^+)}\delta u_i\left(P^+\right)dS - \int\limits_{\Gamma_2(\delta u)} t_i^{(\mathbf{n}^-)}\Delta^{(n^-)}\delta u_i(P)dS$$
$$+ \int\limits_{\Gamma_2(\delta u)} t_i^{(\mathbf{n}^+)}\delta u_i\left(P^+\right)dS.$$

On the other hand,

$$\int\limits_{\Gamma_1(\delta u)} t_i^{(\mathbf{n}^+)}\delta u_i\left(P^+\right)dS = \int\limits_{\Gamma_2(\delta u)} t_i^{(\mathbf{n}^+)}\delta u_i\left(P^+\right)dS.$$

In fact, the integral is evaluated on the same surface because $\Gamma_1(\delta\mathbf{u})$ and $\Gamma_2(\delta\mathbf{u})$ are equal. Hence

$$\int_{\Gamma(\delta u)} t_i^{(\mathbf{n})}\delta u_i dS = -\int_{\Gamma_2(\delta u)} t_i^{(\mathbf{n}^-)}\Delta^{(n^-)}\delta u_i$$

or

$$\int_{\Gamma(\delta u)} t_i^{(\mathbf{n})}\delta u_i dS = \int_{\Gamma_1(\delta u)} t_i^{(\mathbf{n}^+)}\Delta^{(n^-)}\delta u_i dS$$

Summing up the work along all the crack surfaces, we get the virtual work equation

$$\int_{\Omega(\delta u)} \sigma_{ij,j}\delta\varepsilon_{ij}dV = \sum_k \int_{\Gamma_k^1(\delta u)} t_i^{(\mathbf{n}^+)}\Delta^{(n^-)}\delta u_i dS + \int_{\partial\Omega_r} r_i^{(\mathbf{n})}\delta u_i dS + \int_{\partial\Omega_p} p_i^{(\mathbf{n})}\delta u_i dS + \int_{\Omega(\delta u)} \rho_i \delta u_i dV \qquad (9)$$

With the following definitions:

$$\left\{\mathbf{t}^{(\mathbf{n}^+)}, \Delta^{(\mathbf{n}^-)}\delta\mathbf{u}\right\} = \sum_k \int_{\Gamma_k^1(\delta u)} t_i^{(\mathbf{n}^+)}\Delta^{(n^-)}\delta u_i dS;\ \langle\mathbf{r}, \delta\mathbf{u}\rangle = \int_{\partial\Omega_r} r_i^{(\mathbf{n})}\delta u_i dS;$$

$$\langle\mathbf{p}, \delta\mathbf{u}\rangle = \int_{\partial\Omega_p} p_i^{(\mathbf{n})}\delta u_i dS + \int_{\Omega(\delta u)} \rho_i\delta u_i dV;\ \langle\boldsymbol{\sigma}, \delta\boldsymbol{\varepsilon}\rangle = \int_{\Omega(\delta u)} \sigma_{ij,j}\delta\varepsilon_{ij}dV,$$

condition (9) becomes

$$\langle\boldsymbol{\sigma}, \delta\boldsymbol{\varepsilon}\rangle = \left\{\mathbf{t}^{(\mathbf{n}^+)}, \Delta^{(\mathbf{n}^-)}\delta\mathbf{u}\right\} + \langle\mathbf{r}, \delta\mathbf{u}\rangle + \langle\mathbf{p}, \delta\mathbf{u}\rangle \qquad \forall\delta\mathbf{u} \in M \qquad (10)$$

together with the associated compatibility conditions

$$\langle\boldsymbol{\sigma}, \delta\boldsymbol{\varepsilon}\rangle \le 0\ \left\{\mathbf{t}^{(\mathbf{n}^+)}, \Delta^{(\mathbf{n}^-)}\delta\mathbf{u}\right\} \ge 0\ \langle\mathbf{r}, \delta\mathbf{u}\rangle \ge 0 \qquad (11)$$

where the symbol in parentheses is the integral of the product of stress tensors, stress vectors and reactions with the corresponding virtual strains or detachments. Vice versa, working back from eq. (10), we arrive at equation (39). The two systems of forces and deformations, respectively statically and kinematically admissible, are together connected by the virtual work equation (10) (Fig. 6). Conditions (10) and (11) are *necessary* and *sufficient* for the existence of the admissible equilibrium between external and internal forces.

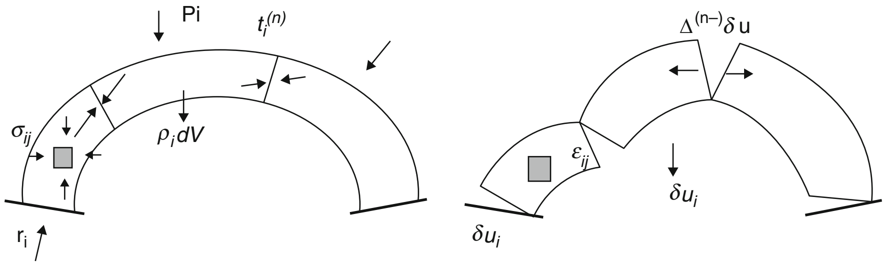


Fig. 6 The two systems of forces and deformations, respectively statically and kinematically compatible. connected together by the virtual work equation

Comparing the current formulation of the same principle for the linear elastic solids with this one for no-tension bodies, the difference is that here the work of the *stress vectors* on the *virtual detachments* $\Delta\delta\mathbf{u}$ must be added, as must also be associated the compatibility conditions (16).

Many relevant properties originate from (10) and (11): particularly the nonexistence of *self-equilibrated stresses*. In this case we speak of deformable masonry structures that can be considered statically determinate systems.

In Como (2010, 2013) I pay special attention to the passage from the general masonry body to the common masonry structures, generally made by assembling piers and arches. Como (2010, 2013) shows that all the foregoing conditions governing the admissible equilibrium of masonry bodies take simpler forms when referred to a one-dimensional structure. For one-dimensional systems distributions of stretching strains in the voussoirs lead to displacements negligible with respect to those produced by relative rotations at hinges. In defining the corresponding mechanisms, it is thus possible to consider only detachments $\Delta^{(\mathbf{n}^-)}\mathbf{u}$ arising among voussoirs, where hinges can develop, and consequently neglect any strain $\boldsymbol{\varepsilon}$ that may spread into the voussoirs. We can also assume that the external constraints are fixed. Hence, in this simple case neither the work of the reactions $\mathbf{r}$, nor the work of stresses $\boldsymbol{\sigma}$ on the strains $\boldsymbol{\varepsilon}$, distributed internally in the voussoirs, will appear in the virtual work equation. With these restrictions, Eq. (10) takes the simpler form

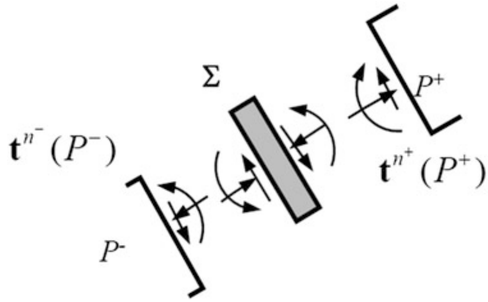
$$\left\{\mathbf{t}^{(\mathbf{n}^+)}, \Delta^{(\mathbf{n}^-)}\delta\mathbf{u}\right\} + \langle\mathbf{p}, \delta\mathbf{u}\rangle = 0 \qquad \forall\delta\mathbf{u} \in M \tag{10'}$$

associated to the admissibility condition

$$\left\{\mathbf{t}^{(\mathbf{n}^+)}, \Delta^{(\mathbf{n}^-)}\delta\mathbf{u}\right\} \geq 0. \tag{11'}$$

With reference, for instance, to a masonry arch, the forces acting on the lateral sections of a small element of the arch are equal and opposite to the resultant of the stress vectors, $\mathbf{t}^{(\mathbf{n}^+)}$ and $\mathbf{t}^{(\mathbf{n}^-)}$, acting on the anterior sections (Fig. 7). Consequently, if the work of $\mathbf{t}^{(\mathbf{n}^+)}$ on the detachment $\Delta^{(\mathbf{n}^-)}\delta\mathbf{u}$ is non-negative, the work of the

Fig. 7 Actions and reactions inside the masonry arch



equal and opposite actions on the detachments themselves will be non-positive. The resultant of forces $\mathbf{t}^{(\mathbf{n}^+)}$ or $\mathbf{t}^{(\mathbf{n}^-)}$ acting on the transverse sections delimiting the detachment, where a hinge is formed, can be decomposed into the components axial force *N*, bending moment *M* and shear *T* of the resultant vector $\boldsymbol{\Sigma}$.

At the same time, the detachments, $\Delta^{(\mathbf{n}^-)}\delta\mathbf{u}$, can, in turn, be expressed in terms of the virtual deformation vector, $\mathbf{E}(\delta\boldsymbol{u})$, whose components are the axial displacement $\delta\Delta$ and the relative rotation $\delta\phi$. In brief we can write

$$\left\{\mathbf{t}^{(\mathbf{n}^+)}, \Delta^{(\mathbf{n}^-)}\delta\mathbf{u}\right\} = -\left\langle \boldsymbol{\Sigma}, \mathbf{E}\left(\delta\mathbf{u}\right)\right\rangle,$$

and the equation of virtual work (10′) becomes

$$\langle \mathbf{p}, \delta\mathbf{u}\rangle \quad = \quad \langle \boldsymbol{\Sigma}, \mathbf{E}\left(\delta\mathbf{u}\right)\rangle \ \forall \delta\mathbf{u} \in M, \tag{10''}$$

and the admissibility conditions on the stresses are

$$\langle \boldsymbol{\Sigma}, \mathbf{E}\left(\delta\mathbf{u}\right)\rangle \le 0. \tag{11''}$$

The virtual work equation (10″) thus takes the typical simple form.

Although conditions (10) and (11) or (10″) and (11″), are necessary and sufficient to guarantee the existence of admissible equilibrium, they must be satisfied by both the loads and the internal stresses. However, these latter may be *a priori* unknown. It is on the other hand possible to prove that the variational inequality on loads $\mathbf{p}$ alone

$$\langle \mathbf{p}, \delta\mathbf{u}\rangle \le 0, \ \forall \delta\mathbf{u} \in M \tag{12}$$

is necessary and sufficient to guarantee the existence of the admissible equilibrium state. It should be noted that the mechanisms $\delta\mathbf{u}$ represent the various deformation modes of the body. Inequality (12) thus simply means that the body is in an admissible equilibrium state under loads $\mathbf{p}$ *iff* the work of these loads $\mathbf{p}$ *is not positive* along any possible deformation of the body. Necessity follows immediately from (10) and (11). In the context of elastic no-tension models, proofs of the sufficiency of condition (12) have been furnished in Romano and Romano (1985) and Romano and Sacco (1984). A simple proof, in the framework of the rigid no-tension model, was presented in Como (1992). The main lines of this latter proof, analysed in depth in Como (2010, 2013), are the following.

If the variational inequality (12) was only necessary, but insufficient, it could be also satisfied by unsustainable loads **p**. Such a situation is however impossible. It is in fact shown that any load **p** that is unsustainable by the body and that consequently sets the body in motion, does *positive* work on displacement **v** along which the body begins to move. This contradiction with the assumption proves the statement.

5 Weight and Geometry: Essential Resources of Masonry Strength

Loads can be considered composed as

$$\mathbf{p} = \mathbf{g} + \lambda \mathbf{q} \tag{13}$$

where **g** and **q** are respectively the dead and live loads and λ a load multiplier. Properties of these loads differ considerably. Live loads **q**, affected by the loading parameter λ, can exert a pushing action along some mechanism. As a rule, the weight, **g**, on the contrary, represents the resistant load for a masonry structure. This statement is frequently stressed in Como (2010, 2013). Consequently, recalling condition (12), the structure will certainly be *safe* under the action of its own weight **g** *if* the following condition is satisfied:

$$\langle \mathbf{g}, \mathbf{v} \rangle < 0, \quad \forall \mathbf{v} \in M \tag{14}$$

The weight will always *oppose* any deformation of a *safe* masonry structure. For a safe arch, for instance, the pressure line corresponding to the weight alone will always be contained within the arch: it can never touch the arch extrados or intrados, at any section. The contribution to strength of the weight **g** comes by virtue of the structure's *geometry*. Masonry structures must be designed so that the mechanisms produce vertical displacements in which lifting is always dominant, thereby satisfying condition (14) for any mechanism. It is the *geometry* that ensures that the structure's weight counters the emergence of any mechanisms.

Weight and geometry represent the essential elements in the strength of masonry structures. More precisely, it is *the proportions* among a structure's various constituent parts and the structure itself that define its geometry, irrespective of the actual absolute dimensions. This aspect of the masonry behaviour is examined in particular depth in Como (2013).

6 Mechanism State

Como (2010, 2013) examines various admissible equilibrium states that occur in a masonry body. Among them the *mechanism state* is the more relevant. In this condition a structure at an admissible equilibrium under loads **p** can be *freely deformed* along a given mechanism displacement $k\mathbf{u}_m$, defined by an arbitrary, but small, constant k. In such a state internal stresses and constraint reactions do not counter the emergence of the mechanism. Consequently also the external loads **p** also offer no opposition to the development of the mechanism displacement $\mathbf{u}_m$ and

$$\langle \mathbf{p}, \mathbf{u}_m \rangle = 0. \tag{15}$$

7 Collapse State

Como (2013) also shows that the *collapse state* is a particular mechanism state. Let us, in fact, consider a masonry structure under a loading path $\mathbf{p}(\lambda)$, where λ is the loading parameter. According to (13), the loads $\mathbf{p}(\lambda)$ will be made up of the resistant component **g**, i.e., the dead loads, and of the pushing forces $\lambda\mathbf{q}$. At some stage of the loading process, when λ attains a critical value λ_c, the structure will *reach a mechanism state* defined by the mechanism $\mathbf{u}_c$. The work done by the forces $\mathbf{p} = \mathbf{g} + \lambda_c\mathbf{q}$ vanishes along the mechanism $\mathbf{u}_c$, which is to say

$$\langle \mathbf{g} + \lambda_c\mathbf{q}, \mathbf{u}_c \rangle \quad = \quad 0, \quad \lambda_c > 0, \quad \mathbf{u}_c \in M.$$

We admit that the live loads **q** *push* along $\mathbf{u}_c$, so that

$$\langle \mathbf{q}, \mathbf{u}_c \rangle > 0. \tag{16}$$

Condition (16) evidences the presence of a pushing action by live loads **q** along displacement $\mathbf{u}_c$, the *failure mechanism*. Thus, as soon as the loading parameter λ is further increased beyond λ_c, we have

$$\left(\frac{d}{d\lambda} \langle \mathbf{p}(\lambda), \mathbf{u}_c \rangle \right)_{\lambda_c} = \langle \mathbf{q},\ \mathbf{u}_c \rangle > 0.$$

Accordingly, condition (12), necessary and sufficient for the *existence of an admissible equilibrium state*, is violated and the structure *fails*. At this collapse state an exchange occurs from conditions of existence to those of non-existence of the admissible equilibrium state. The *development* of the failure mechanism can be represented by a sequence $k\mathbf{u}_c$ of mechanisms of increasing amplitude. Collapse thus occurs under *constant loads*, because by gradually increasing the constant

$k > 0$, we consistently have

$$\langle \mathbf{g},\ k\mathbf{u}_c\rangle + \lambda_c \langle \mathbf{q},\ k\mathbf{u}_c\rangle \ = 0, \quad k\mathbf{u}_c \in M,\ \ k > 0$$

for any amplitude of mechanism $k\mathbf{u}_c$. Constant loads also imply constant stresses. The failure mechanism thus develops under frozen loads and stresses. Como (2010, 2013) points out the *extraordinary situation* that occurs at the collapse. There is no energy dissipation. Nevertheless, the masonry structure is able to maintain its limit strength during the development of the failure mechanism, as occurs for a steel bar upon yielding. Despite the lack of dissipation, the behaviour at collapse of masonry structures is similar to that of ductile steel structures, as predicted by Limit Analysis.

8 The Theory of Proportionality in Architecture

Let us examine the two similar arches *a* and *A* in Fig. 8. Arch *A*, on the right, is *k* times larger than the arch *a*, on the left; in other words, arch *A* is a *k* times *magnified copy* of arch *a*. In the transverse direction, i.e., in the direction orthogonal to their plane, the structures have the same width *s*. Each segment in structure *A* is thus *k* times longer than the corresponding segment in structure *a*.

Let us now assume that structure *a* is *stable* under its own weight *g*, as defined according to (14). Thus we have

$$\langle g, v\rangle = \sum_1^N g_i v_i < 0$$

for any mechanism v. The work $\langle g, v\rangle$ is evaluated considering the work of the weight forces g_i of the various voussoirs composing the arch on the corresponding

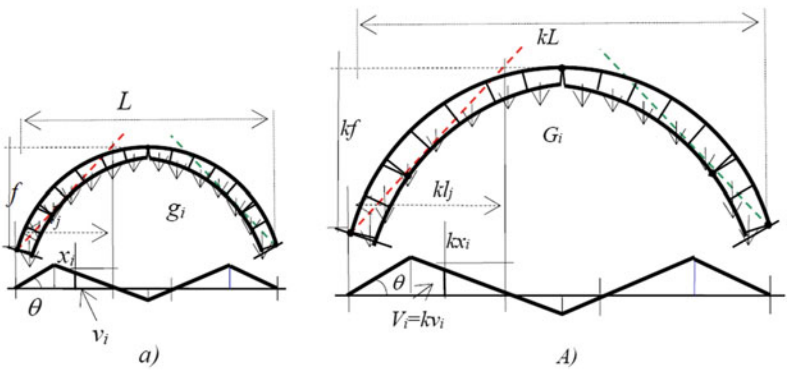


Fig. 8 Geometries of two similar arches and of the two corresponding mechanisms governed by the dimension ratio *k*

vertical displacements v_i of the mechanism. Consequently the k magnified structure A is thus also *stable* under its own weight. In fact, for any mechanism V *corresponding* to v, we will have

$$\langle G, V \rangle = \sum_{1}^{N} G_i V_i = k^3 \sum_{1}^{N} g_i v_i < 0.$$

Thus, to conclude, *if a structure under its own weight is stable*, *a k times magnified copy will also be stable*. The *same* outcome holds in a more general sense. These results, proved in Como (2013) as a direct consequence of the essential features of the masonry behaviour, were already known to architects of the past and formed the basis for their fundamental rules of construction. As set down in the theory of proportions by Andrea Palladio and Leon Battista Alberti, statics of masonry structures is governed solely by their geometry and, consequently, by their basic measurement, *the modulus*, irrespective of their absolute measurements. Knowledge of the most suitable proportions among the various components of a masonry structure, often jealously guarded by past masters, represented the essence of the art of construction. These results arrived to us through centuries of long experience, and are direct consequence of the unique, fundamental masonry behaviour. This theory of proportions was strongly opposed by Galileo Galilei in his *Discorsi e dimostrazioni matematiche intorno a due nuove scienze* (1638) by means some sharp arguments involving the local states of stresses in the material but not pertaining to the behaviour of masonry structures, behaviour only marginally influenced by local stresses. The argument of the theory of proportions has been also debated by other scholars (Heyman 1997; Huerta 2006; Di Pasquale 1996; Benvenuto 1981, 1991; Baratta 1999). We observe that compressions increase in the larger structure by increasing the ratio k, while the masonry strength remains constant. Hence the proportionality rule holds as long as compression stresses remain low. This is indeed all that it happens in masonry structures, as clearly shown by Heyman (1997) and Huerta (2006).

9 Settlement State: The Minimum Thrust Theorems

The case that a mechanism state can be attained at a settlement state is frequently met in Como (2010, 2013). This is another aspect of the masonry equilibrium. Let us consider a masonry structure that is at a *safe admissible equilibrium* state at the configuration C_i under the actions of loads $\boldsymbol{g}$. Inequality (14) thus holds. The structure becomes now deformed as a consequence of a *slight settlement* occurring at one of its external constraints. The structure deforms with the mechanism displacement $\mathbf{v}_s$, due to this settlement. By way of example, consider the arch in Fig. 9, which undergoes a slight increase in span due to settling.

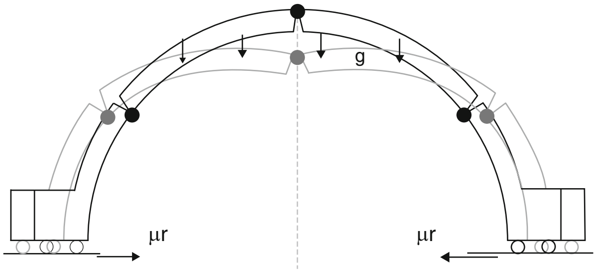


Fig. 9 The settled arch

For sake of simplicity in what follows we will make reference to the case of one-dimensional structures and, therefore, to the simpler formulation (10″) of the virtual work equation. However, the internal stresses, represented by the vectors $\mathbf{\Sigma}$ having components N, M and T, and the strain vectors $\mathbf{E}$, with components Δ and ϕ, for sake of simplicity, will still be denoted by $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$.

Let C_s be the configuration taken on by the structure once the settlement has occurred. By assuming that C_s is *very near to* C_i, we can refer to the geometry of the initial configuration C_i when expressing the equilibrium equations. The settlement mechanism, $\mathbf{v}_s$ is the displacement field that moves the structure from C_i to C_s. As the settlement occurs, the structure's internal equilibrium shifts from initial configuration C_i to the displaced one C_s. Changes in the internal stresses and constraint reactions will occur during the transition from C_i to C_s, so that the initial stress state $\boldsymbol{\sigma}_i$ is altered and becomes $\boldsymbol{\sigma}_s$.

This internal stress state, $\boldsymbol{\sigma}_s$, which accounts for settlement $\mathbf{v}_s$, is statically admissible and thus satisfies the inequality

$$\langle \boldsymbol{\sigma}_s, \boldsymbol{\varepsilon}\,(\delta \mathbf{u})\rangle \ \leq 0\,.$$

Likewise, the settled constraint which, *before* the settling, produced the reaction $\mu_i\mathbf{r}$, *after* the settling produces the new reaction

$$\mu_s\mathbf{r},$$

where $\mathbf{r}$ is a given force having the direction of reaction of the settled constraint and μ is the corresponding multiplier. In brief, during the development of the settlement mechanism $\mathbf{v}_s$ the structure will remain in a state of admissible equilibrium while the stresses vary from $\boldsymbol{\sigma}_i$ to $\boldsymbol{\sigma}_s$, the corresponding pressure line shifts from π_i to π_s and the reaction of the settled constraint changes from $\mu_i\mathbf{r}$ to $\mu_s\mathbf{r}$. For instance, in the case of the masonry arch that has undergone a slight increase in span, its pressure line π_s will pass through the hinges corresponding to mechanism $\mathbf{v}_s$. Consequently, no work will be done by the internal stresses $\boldsymbol{\sigma}_s$ on the deformations corresponding to $\mathbf{v}_s$. The same occurs for any structure that is deformed by a mechanism and adapts itself to settling. Thus, at the settlement state the following *mechanism state* holds

$$\langle \boldsymbol{\sigma}_s, \boldsymbol{\varepsilon}\,(\mathbf{v}_s)\rangle = 0. \tag{17}$$

We can *release* the structure by removing the settled constraint by applying the reaction $\mu_s \mathbf{r}$ to the eliminated constraint. The set of all the mechanisms of the released structure is denoted by $\overline{M}$. In the released structure at the settled state, the applied loads are represented by both the weights $\mathbf{g}$ and the reactions $\mu_s \mathbf{r}$. Thus, considering the released structure at the admissible settled equilibrium state, from the virtual work equation (10″) we get

$$\langle \mathbf{g}, \delta\mathbf{u} \rangle + \mu_s \langle \mathbf{r}, \delta\mathbf{u} \rangle = \langle \boldsymbol{\sigma}_s, \boldsymbol{\varepsilon}(\delta\mathbf{u}) \rangle, \quad \forall \delta\mathbf{u} \in \overline{M}$$

which for $\delta\mathbf{u} = \mathbf{v}_s$, according to (17), yields

$$\langle \mathbf{g}, \mathbf{v}_s \rangle + \mu_s \langle \mathbf{r}, \mathbf{v}_s \rangle = 0.$$

Loads $\mathbf{g}$ perform positive work along the mechanism displacements $\mathbf{v}_s$ while the reaction $\mu_s \mathbf{r}$ of the released constraint opposes settling, so that

$$\langle \mathbf{g}, \mathbf{v}_s \rangle > 0$$

and

$$\mu_s \langle \mathbf{r}, \mathbf{v}_s \rangle < 0. \tag{18}$$

In spite of the settling that occurred, the work of the loads due to any mechanism $\delta\mathbf{u}$ is still the same as the work evaluated at the initial configuration C_i, assuming that displacements $\mathbf{v}_s$, are *very small*, as above stated, and that the changes in geometry are consequently *negligible*. Thus, if at the initial state C_i, the admissibility condition $\langle \mathbf{g}, \delta\mathbf{u} \rangle < 0, \quad \forall \delta\mathbf{u} \in M$ is satisfied, the same condition will still be satisfied by the new configuration C_s. In this regard we meet Heyman's statement: "*if the foundations of a stone structure are liable to small movements, such movements will never, of themselves, promote the collapse of the structure*" (Heyman 1966, p. 255). Moreover, if settlement $\mathbf{v}_s$ increases and becomes

$$k\mathbf{v}_s, \quad k > 1$$

the static arrangement of the structure will not change, and the internal stresses will remain fixed at $\boldsymbol{\sigma}_s$. In short, the structure freely follows any increase in the settlement, maintaining its configuration in admissible equilibrium. Settling develops with frozen internal stresses $\boldsymbol{\sigma}_s$ and constraint reactions, $\mu_s \mathbf{r}$. The actual degree of settling is difficult to quantify. Despite this uncertainty, the internal stress state of the structure is, to the contrary, well-defined. No equilibrium loss will occur during the settling. This is a peculiar aspect of masonry structures that can explain the great durability and longevity of so many historic buildings.

How do we evaluate this stress state and the corresponding reaction of the settled restraint? (Como 2013) shows that we have, in a reversed form, the static and the kinematical theorems of the minimum thrust. The static theorem of the minimum thrust affirms that

$$\mu(\boldsymbol{\sigma}) \geq \mu_s \quad \boldsymbol{\sigma} \in S$$

i.e., the multiplier, μ_s of the settled thrust **r** is thus *lower* than all the statically admissible multipliers μ . This general finding (Como 1996, 1998) relates to the particular property of the masonry arch that undergoes an increase in span due to settling at its springings: such an arch is at the state of minimum thrust because its pressure line corresponds to the minimum span and the maximum sag, as shown by Heyman (1966).

Como (2010, 2013) examines specifically the settlement equilibrium from a *kinematical* point of view. This last point is very interesting and produces *new approaches* to tackle the problem of the static analysis of settled structures. The actual settlement mechanism is unknown: for instance, for the case of the arch of Fig. 9, we cannot know the position of the internal hinge of the settlement mechanism. We only know that, during the development of the mechanism, loads **g** will do positive work, while the work of the reaction of the settled constraint is, to the contrary, negative. Let us consider any settlement mechanism

$$\mathbf{v} \in \overline{M}$$

of the released structure. The loads **g** will push along **v** and consequently

$$\langle \mathbf{g}, \mathbf{v} \rangle > 0. \tag{19}$$

We define the *kinematic multiplier* λ of the reaction **r** of the settled constraint as that multiplier able to ensure equilibrium of the structure along the assumed settlement mechanism **v**, or, in other terms, such that the following condition holds

$$\langle \mathbf{g}, \mathbf{v} \rangle + \lambda \langle \mathbf{r}, \mathbf{v} \rangle = 0.$$

Reaction $\lambda \mathbf{r}(\mathbf{v})$ opposes the development of settling **v**, given that, by taking (18) into account, we get

$$\lambda \langle \mathbf{r}, \mathbf{v} \rangle < 0. \tag{20}$$

The kinematical multiplier $\lambda(\mathbf{v})$ of reaction **r** is thus defined as

$$\lambda(\mathbf{v}) = -\frac{\langle \mathbf{g}, \mathbf{v} \rangle}{\langle \mathbf{r}, \mathbf{v} \rangle}, \quad \mathbf{v} \in \overline{M}. \tag{21}$$

With these definitions it is easy to prove that (Como 1996, 1998)

$$\lambda(\mathbf{v}) \leq \mu_s \quad \forall \mathbf{v} \in \overline{M}. \tag{22}$$

For any settlement mechanism, $\mathbf{v} \in \overline{M}$, the corresponding kinematic multiplier, $\lambda\left(\mathbf{v} \in \overline{M}\right)$, can never be *greater* than the actual settlement multiplier μ_s. Thus μ_s is *the maximum* of all kinematic multipliers, $\lambda\left(\mathbf{v} \in \overline{M}\right)$, for varying $\mathbf{v}$ in the set of all settlement mechanisms $\overline{M}$, or in other terms

$$\mu_s = MAX\left(-\frac{\langle \mathbf{g}, \mathbf{v}\rangle}{\langle \mathbf{r}, \mathbf{v}\rangle}\right) \quad \mathbf{v} \in \overline{M}. \tag{23}$$

This result makes it possible to analyse the actual equilibrium states of structures with a new easier approach than the static one, that make use of funicular polygons.

10 Actual Equilibrium States of Masonry Structures

Stress analysis of a masonry costruction, rigidly constrained to a rigid environment, cannot be developed in the context of the rigid-in-compression no-tension model. The admissible equilibrium equations alone are not enough to evaluate the internal stresses, as such a problem is statically indeterminate. To this end, additional equations are required: the compatibility equations, as in the case of elastic structures. On the other hand, the assumption of constraints rigidly connected to a rigid external environment is physically meaningless. A certain degree of settlement, as a rule, occurs in the external constraints of the structure. The same deformation of the supporting structures drives displacements of the structure placed above.

The presence of elastic strains in the analysis of settled structures is really insignificant. As soon as the settlement starts, after a very negligible elastic stage, the masonry structure transforms immediately into a mechanism. The same occurs for the rigid in compression no tension structure. This aspect of the problem is particularly studied in Como (2010, 2013). Useful information can thus be obtained in the same context of the simple rigid no-tension model providing that the deformability of the structure constraints is taken into account. A *minimal thrust state* takes place. The degree of settling can be predicted only with difficulty. Thankfully, the compatibility equations expressing the occurrence of settling do not require defining the magnitude of the settlement, but only indication of the settled constraints. According to this approach, the problem of the determination of the actual stress state in masonry structures becomes statically determinate and Limit Analysis can once again be fruitfully applied, as thoroughly shown in Como (2010, 2013).

11 Minimum Thrust States in Masonry Domes: Application of the Kinematical Approach

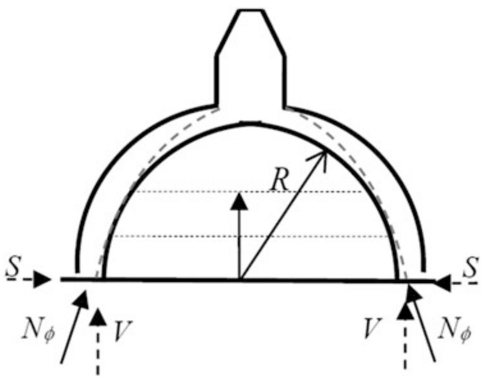
Useful applications of this last approach are described in Como (2010, 2013), as in the study of the actual equilibrium states of arches, domes, cross and cloister vaults, with particular attention to the more relevant cases in architecture.

With reference to domes, for instance, the occurrence of unavoidable deformations of their supporting structures yield the vanishing of hoop stresses and meridian cracks, starting from the dome springings, will occur. The cracked dome tends to open along a large band breaking up into slices and behaves as a set of independent pairs of semi-arches leaning on each other. Predictably, cracking brings about a profound change in the dome's statics.

A small cap at the top of each slice will be subjected to the thrusting action transmitted by the other slices, which will be transmitted all the way to the springing. Figure 10 shows an approximate sketch of the pressure curve of a cracked hemispherical dome. The dotted line shows the position of this curve, which inclines towards the horizontal at the springing. The horizontal component of the reaction of the supports represents the thrust S per unit length of the dome's base circumference. The thrust thus occurs in the passage of the stresses from the initial membrane state to the no tension state.

The emergence of thrust in the dome represents the most consequential outcome of meridian cracking in typical masonry round domes. Loaded by the dome's thrust, the sustaining structures, e.g., the drum or underlying piers, deform and splay. The slices, no longer restrained from deforming by rings, bend under the loads and can form mechanisms. The weight of a particularly heavy lantern, for example, could even cause the dome to fail on cracking. Thrust yields a more or less relevant further deformation of the dome supporting structures. The settled dome mobilizes a thrust that it is the *minimum* from among all the thrusts S transmitted by statically admissible pressure curves. The minimum thrust S_{Min} can be obtained via the *static*, as well as the *kinematic* approach. The *static* approach calls for tracing the statically admissible funicular curves of the loads. In the settled state the pressure curve passes

Fig. 10 Rising thrust due to meridian cracking



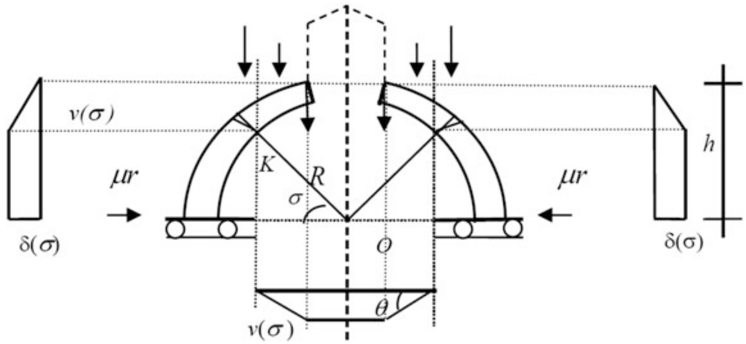


Fig. 11 Minimum thrust evaluation according to the kinematic approach

through the extrados at the key section of the slices and then runs within their interior, *skimming* over the intrados of the dome (Fig. 10). The *kinematic* approach is dual with respect to the static one and is ruled by (21) and (23) given earlier. In (23) $\langle g, v\rangle$ represents the work, undoubtedly positive, of the dead loads on the vertical displacements of mechanism v, and $\langle r, v\rangle$ the work, undoubtedly negative, performed by the thrust on the corresponding horizontal displacement. Figure 11 shows a generic dome mechanism produced by a base widening. In this mechanism the position of the internal hinge K is unknown.

The set of all these kinematically admissible mechanisms is described by varying the position of the hinge K between the springing and the key section of the slice. Identifying the maximum of function $\lambda(v)$ by varying the position of hinge K enables us to obtain the sought-for thrust. Many applications of this approach are described in Como (2010, 2013). It is, in fact, a relatively simple matter to apply the kinematic approach to evaluate the minimum thrust of masonry domes. The settlement mechanisms are obtained releasing the slices by positioning hinges to allow horizontal sliding of the dome at its springings. Hinges must thus be positioned (Fig. 11):

- at the *extrados*, on the section linking the slice with the central closing ring sustaining the lantern;
- at the *intrados*, at the haunches. The position of this hinge is unknown and is indicated by the angle σ (Fig. 11). Thus, the minimum thrust $\mu_{min}S$ is evaluated by seeking the maximum of the function

$$\mu_{\min} S = Max \frac{\langle g, v(\sigma)\rangle}{\delta(\sigma)}$$

by varying angle σ along the intrados and where

$$\delta(\sigma) = (h - R \sin \sigma)\,\theta$$

is the horizontal displacement of the slice at springing. According to the kinematic theorem the search for the minimum thrust thus translates into searching for the maximum of the function

$$\mu S(\sigma) = Max \frac{\langle g, v(\sigma) \rangle}{(h - R \sin \sigma)\, \theta}$$

by varying the angle σ along the dome intrados. This approach has been applied to study the statics of the domes of S. Maria del Fiore in Florence and of St. Peter's in Rome.

12 Book Contents

Como (2010, 2013) is divided into nine chapters, each of which begins with historical notes and an introduction highlighting the main aspects of the topics covered. The strength and deformability of masonry materials are addressed in the first chapter. The second chapter deals with the deformation and equilibrium of masonry solids. The third and fourth chapters examine the static behaviour of the main basic masonry structures, such as arches and vaults. By way of example, static analysis are conducted of a number of renowned examples from the world's architecture heritage, such as ancient Mycenaean domes, the Pantheon in Rome, the large cross vaults of the Baths of Diocletian, and the domes of Santa Maria del Fiore in Florence and Saint Peter's in Rome. The fifth chapter turns to a detailed analysis of the statics of the Colosseum in Rome and examines the reasons for its actual state of damage. The sixth chapter describes and analyzes the statics of cantilevered stairways, a typical element whose structural behaviour is still somewhat unknown. Chapter seven then takes up the structural analysis of walls, piers and towers under vertical loads. The stability of such structures is heavily affected by the non-linear interactions between the destabilizing effects of the axial loads and masonry's no-tension response. The instability of towers, leaning towers in particular, is addressed in a specific section of the chapter. In this regard, a detailed stability analysis is conducted of the famous leaning Tower of Pisa, which has recently undergone a successful restoration work. The eighth chapter then analyzes the statics of Gothic cathedrals, with particular reference to analysis of their resistance to wind actions. The 1,294 collapse of the Beauvais cathedral is also examined in depth. The last chapter deals with the seismic behaviour of historic masonry buildings.

Como (2010, 2013) is addressed especially to researchers, engineers and architects operating in the field of masonry structures and of their consolidation and restoration, as well as to students of civil engineering and architecture.

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Equilibrium Analysis

Jacques Heyman

Abstract There are occasional opportunities for the design of masonry—of a new vault, for example, or of a highway bridge. However, the structural analysis of masonry is concerned in the main with the determination of the state of an existing structure. Analysis is of great theoretical interest, but it is also of practical importance. In repair work it may be necessary to replace a major structural element, and it is clear that estimates must be made of the structural forces. Indeed, the main objective of a structural analysis is the determination of such forces. It is only rarely that deformations of a masonry structure need to be computed; deformations arise, almost without exception, from displacements imposed by movements of the environment (sinking of foundations, spread of abutments), and such deformations, notably cracking, do not depend on the elastic properties of the masonry. An elastic analysis will, in fact, shed no light on the deformation of a masonry structure. Equally, the magnitudes and distribution of the internal structural forces are determined by the (in general, unknown) movements imposed by the environment, and again an elastic analysis will be of no help in estimating these forces. It is fortunate that an "equilibrium" analysis, making no reference to elastic properties of the masonry, can nevertheless be made to give reliable values for the key structural quantities.

Keywords Equilibrium analysis • Masonry structures • Vaults • Deformation

It is in the nature of a masonry structure that it should be cracked. Individual components (brick, stone) may have good tensile strength, but weak (or no) mortar in the joints between these components will not allow tensile forces to be transmitted. By contrast, the compressive stresses in a large span masonry bridge, or in a cathedral, are very low compared with the crushing stress of the masonry. These observations have enabled the establishment of a structural theory on the assumption that masonry is a "unilateral" material, having zero tensile strength and infinite compressive strength. In applying this theory to the analysis of a real

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structure, it is possible to make adjustments to allow for the actual as opposed to assumed material properties, but in general the simple theory gives meaningful and useful results. Further, since stresses are low, so are internal deformations, and the material of the structural components is virtually rigid (Heyman 1966).

The cracking of such an idealized structure (and the cracking of a real structure) results from movements imposed by the external environment. A single-span masonry bridge will thrust against the river banks; the abutments will inevitably give way, and the arch must accommodate an increased span (Heyman 1982), or the footing beneath one of the piers supporting a crossing tower will settle, and cracking will be observed in the neighbouring bays of a cathedral. Such cracking is not due to the potential development of internal tensile stresses in the structure, but results solely from a disturbance to its original geometry. In the case of the simple voussoir arch bridge, an increase in span results in the well known pattern of three "hinges" between voussoirs, and this pattern is independent of the magnitude of the small movements of the abutments.

Such anomalous movements imposed on a structure by the environment are unpredictable: they are unknown and by their nature unknowable. It is however a fact that even tiny geometrical imperfections can have a grossly disproportionate effect on the "state" of a structure, that is, on the values of the internal stress resultants induced by given loading.

1 Elastic Analysis

This sensitivity to small imperfections is at once evident from the results of conventional elastic analysis. Such an analysis is not possible for a structure made of rigid material; in addition to the master equations of statics (internal stress resultants must be in equilibrium with the external loads), a statement must be made of the elastic properties of the material. Finally, boundary conditions must be satisfied; a masonry arch must fit exactly between its abutments. It is precisely the positions of those abutments that are unknown, and the engineer using elastic theory is forced to assume that they are perfectly fixed. It is when the engineer examines the consequences of a 2 or 3 mm displacement of an arch abutment that the theory shows the sensitivity of the calculations to such a movement.

2 The Purpose of Structural Analysis

A masonry structure (and in fact any structure) responds almost invisibly but violently to small movements of the environment; these movements are not known, so that, inevitably, as said, the "actual" state of the structure is also unknowable. Moreover, the state is impermanent; the passage of a heavy load, an earth tremor, a hurricane, may all profoundly change the internal stress resultants in the structure.

What, then, is the purpose of a structural analysis? Any calculations will result in a solution which could not be observed in a real structure, and the values of internal stress resultants do not correlate with the observed defects imposed by movements of the environment. An answer to the question may be found by examination of the historical record of attempts by scientists to analyse the masonry structure.

3 Brief Historical Notes

As is well known, structural design of masonry, from the earliest times to the end of the seventeenth century, was incorporated in empirical rules of proportion (Heyman 1998). That these rules were effective is evidenced by the continued survival of ancient and medieval buildings (there were of course failures). In one sense there is no purpose in demonstrating by structural analysis that a particular building is stable, since its very presence confirms its stability. However, such modern analyses do at least confirm that rules of proportion, proper shapes, geometry, are precisely the criteria needed for safe designs.

The science of mechanics was added to that of geometry late in the seventeenth century, for example in the work of La Hire (1695, 1712), and of Couplet (1731, 1732). It is in these studies that the proper end of structural analysis of masonry becomes evident. The abutments of a masonry arch must be designed, and one of the major objectives of eighteenth-century work was the determination of the value of the thrust exerted by an arch on its abutments. La Hire used the positions of the hinges in a cracked arch to unlock the statics of the problem, and he was able to demonstrate that a value of the thrust could be found. His calculations were of course correct, but there is no consideration of the fact that a different hinge pattern would give a different value of the thrust.

It is noteworthy that Coulomb (1776) made no attempt to calculate the “actual” state of the arch that he studied (Heyman 1997). He introduced another mathematical tool, that of “analysis”, where the word is used in its technical sense to denote the use of algebra, and above all, calculus. He was able to show that the value of a structural quantity (say, the thrust of an arch) could be contained between calculable limits; indeed, the title of his paper includes the words “maximum and minimum”. Implicitly, although he did not express these ideas, he was demonstrating that a safe state could be found for the equilibrium of an arch, even though that “actual” state could be determined only within limits.

Such calculations are of great importance to those engaged in the repair of masonry structures. As a single example, a flying buttress may have to be dismantled and rebuilt, and temporary props may be designed with confidence to continue to support the high vault of a great church.

4 Equilibrium Design

The concept of safety lies at the heart of the modern theories of plastic (or limit) analysis. The labels "plastic" and "limit" are misleading, implying as they do that a structure is on the point of collapse; they result from the twentieth-century development of structural analysis, in which indeed ultimate states were examined. However, it is the concern of the engineer that a structure should be stable in its working state, in comfortable equilibrium with whatever loading it is required to carry. The "plastic" engineer examines only the equations of statics, ensuring that the internal stress resultants in masonry are everywhere compressive; no use is made of the "boundary conditions" (e.g., the positions of the abutments of an arch). The master theorem of this twentieth-century analysis is that if such a solution can be found, then the structure is safe (moreover, it is possible to calculate a "factor of safety" for the design).

The power of this theorem lies in the fact that if the engineer can find any satisfactory state of equilibrium, then this gives absolute assurance that the (unknowable) actual state of the structure is also satisfactory. This whole process should now perhaps be called "equilibrium analysis".

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A Semi-analytical Approach for Masonry Arch Dynamics

Anna Sinopoli

Abstract A semi-analytical approach is proposed for modelling the plane dynamics of a masonry arch, treated as a system of rigid elements with friction and unilateral contacts at each joint. By generalising the method proposed in previous research, the analytical approach is firstly applied to the plane dynamics of a rectangular block simply supported on a moving base. In this case, where the contact although sometimes extended is unique, dynamics is formulated as a frictional contact problem, and conditions for onset of motion according to various mechanisms are fully analytically identified; moreover, criteria for evaluating contact reactions during either smooth or non-smooth dynamics are outlined. The method is then extended to the case of the arch, where each element is characterized at most by a double extended contact; criteria for the onset of motion and evaluation for each element of contact reactions during the dynamic evolution are then identified. The approach proposed constitutes a first step for performing dynamic analysis through either an event-driven or a time-stepping numerical procedure.

Keywords Signorini-Coulomb law • Non-smooth contact dynamics • Masonry arch

1 Introduction

The development of the plastic theory in the 1950s and Heyman's basic idea of transferring its philosophy from the steel to stone skeleton (Heyman 1969) made it possible to state the limit analysis of masonry arches for standard behaviour as a linear complementarity problem (Gilbert and Melbourne 1994); in this case, bounding theorems allow for determining collapse from either below or from above without distinction. In the presence of finite friction, on the contrary, the normality rule does not hold and non-associated flow rule invalidates bounding theorems; modified criteria must then be defined, on the basis of which computational strategies can be

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adopted (Livesley 1978; Sinopoli et al. 1997; Baggio and Trovalusci 2000; Orduna and Lourenço 2005). The problem is generally formulated by using non-linear programming, and searching for the optimal solution is difficult not only because of numerical calculations; the absence of stability criteria makes it possible to find an optimal solution which is not a global minimum.

Recently, a simple method has been proposed (Sinopoli et al. 2007), in the framework of contact mechanics, for the collapse analysis with finite friction of a semicircular arch under its own weight. In this respect, it is worthwhile observing that, if the collapse condition is considered as that separating equilibrium from starting motion, the best framework for identifying it is in dynamics; the collapse condition thus transforms into the starting mechanism of the motion. Moreover, dynamics formulated as a contact problem, through the decomposition in sub-problems for which appropriate flow rules can be defined, eliminates the typical indeterminacy of any static approach.

While numerous papers have been produced on limit analysis, it is surprising that masonry arch dynamics is a subject to which little attention has been paid in the literature, despite the fact that the preservation of the monumental patrimony against seismic risk has promoted research during the last decades on stone or masonry structures, modelled as assemblage of blocks with frictional contacts. Starting with Housner's model (Housner 1963), the main aspects of dynamics became a new field of research: classical questions of Coulomb friction (Lötstedt 1982); structural behaviour (Spanos and Koh 1984; Sinopoli 1987); stability of the responses and transition to chaos (Hogan 1990; Ageno and Sinopoli 2005, 2010); impact and dynamic modelling (Sinopoli 1987, 1997; Moreau 1988, 1999, 2005; Augusti and Sinopoli 1992); and, finally, numerical codes (Glocker and Pfeiffer 1992; Jean 1999).

Oppenheim (1992) was the first to propose a semi-analytical model to investigate the dynamics of an arch, treated as a four-link mechanism—that is, a single degree of freedom system—subjected to a base impulse acceleration; the dynamics equation was obtained by a classical Lagrangian formulation and numerically integrated. The same model was adopted in later papers (Clemente 1998; De Lorenzis et al. 2007). De Lorenzis et al. (2007) also made a comparison with results obtained using the commercial program UDEC (Cundall and Strack 1979). More recently, a stone arched structure has been investigated (Rafiee et al. 2008) with a discrete elements method using the platform LMGC90, based on the non-smooth contact dynamics method (Jean 1999).

It is obvious that difficulties inherent in analytical modelling encourage the use of numerical methods. Nevertheless, although mechanical modelling of un-reinforced masonry remains a challenging problem, the focusing of numerical methods has mainly been concentrated in discussing computational algorithms.

In this present paper I extend my previous research on the dynamics of a rigid body simply supported on a moving boundary (Sinopoli 1997). The new approach for the dynamics, formulated as a contact problem governed by Signorini's and Coulomb's laws (Sinopoli 2010), is oriented to the dynamical modelling of the masonry arch treated as a system of n rigid voussoirs, with frictional and unilateral contacts at each joint. First, the modelling is given for the single block, to guide the understanding on how contact laws can be checked, by identifying at the contact point both the value of the generalised reaction and persistency or variation of the activated mechanism; we refer to the relationships between initial conditions, active forces and contact reactions. The formulation is then extended to the arch; new questions related to its typology—indeterminacy due to double contacts and elements number—are then discussed and solved. The formulation allows the implementation of a numerical platform, based on either implicit time-stepping or explicit event-driven numerical method for integrating the equations of motion; the platform, outside the scope of the present paper due to its complexity, is a target of future research.

2 Single Block Dynamics

Consider a rectangular rigid block free-standing with Coulomb friction on rigid ground Γ, which moves by a translational horizontal motion $\ddot{x}_O = k_s(t)g$, where g is the gravity acceleration. Refer the dynamics to the system (O, x, y) fixed on Γ, with which unit vectors $(\boldsymbol{t}, \boldsymbol{n})$ are associated; $\boldsymbol{n}$ is outwards oriented (For the definition of the symbols used in what follows, see the Appendix "List of Symbols").

2.1 Kinematics

Assume the position of the mass centre G and rotation angle to be Lagrangian coordinates; thus, the motion of the body transforms into the path of its representative point in the configuration space, where the kinetic energy metrics is assumed in order to preserve the Euclidean structure (Moreau 1988; Sinopoli 1997). Through the linear mapping induced by Lagrangian coordinates, the velocity of any point P of the body becomes:

$$\dot{\boldsymbol{r}}_{\mathrm{P}} = \mathbf{N}_{\mathrm{P}}^{\mathrm{T}}\,\dot{\boldsymbol{u}} \tag{1}$$

where $\mathbf{N}_{\mathrm{P}}^{\mathrm{T}}$ is the $[2 \times 3]$ gradient operator of the mapping, $\dot{\boldsymbol{u}}$ the generalised velocity and apex T indicates transposition. According to (1), tangential and normal components of $\dot{r}_P$ are:

$$\begin{aligned} \dot{r}_{\mathrm{P,t}} &= \boldsymbol{N}_{\mathrm{P,t}}{}^{T}\dot{\boldsymbol{u}} \\ \dot{r}_{\mathrm{P,n}} &= \boldsymbol{N}_{\mathrm{P,n}}{}^{T}\dot{\boldsymbol{u}} \end{aligned} \tag{2}$$

Note that $\boldsymbol{N}_{\mathrm{P,t}}$ and $\boldsymbol{N}_{\mathrm{P,n}}$ are generalised directions starting at P and associated with $\boldsymbol{t}$ and $\boldsymbol{n}$, respectively (Fig. 1).

If the body, initially at rest, is in contact with the ground along the side AB, the boundary impenetrability allows only positive or null values of the normal

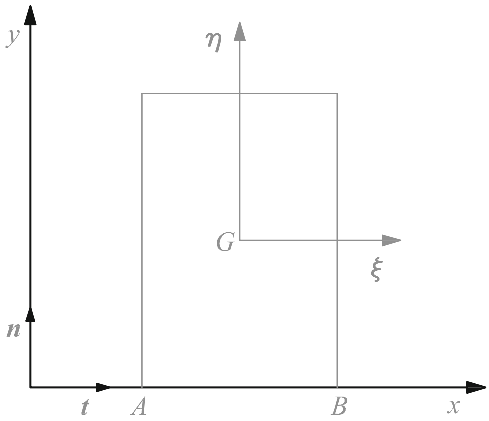

Fig. 1 The rigid block

virtual displacement of contact points. Since virtual displacements and velocities have the same structure, the impenetrability transforms into a unilateral constraint to be satisfied by admissible velocities $\bar{\dot{\boldsymbol{u}}}$ of the body:

$$\overline{\dot{r}_{\mathrm{P,n}}} = \boldsymbol{N}^{\mathrm{T}}_{\mathrm{P,n}} \bar{\dot{\boldsymbol{u}}} \geq 0 \quad \forall P \in AB \tag{3}$$

The set (3) contains the velocity at instant t, if it exists; on the contrary, it must be interpreted as the right-sided velocities set if velocity does not exist, as is the case of an impact occurring when either one or multiple points enter into contact coming from a no-contact situation.

2.2 Normal and Tangential Contact Laws

Normal contact (3) is guaranteed by reaction $R_{\mathrm{n}} \geq 0$, acting at the unknown contact centre Q. According to mapping, forces are transformed through the invariance of the work, so that normal and tangential generalised reactions become:

$$\begin{aligned} \boldsymbol{\Psi}_{\mathrm{Q,t}} &= R_{\mathrm{t}}\, \boldsymbol{N}_{\mathrm{Q,t}} \\ \boldsymbol{\Psi}_{\mathrm{Q,n}} &= R_{\mathrm{n}}\, \boldsymbol{N}_{\mathrm{Q,n}} \end{aligned} \tag{4}$$

At any time t, contact is lost if $\dot{r}_{\mathrm{Q,n}} > 0$, so that $R_{\mathrm{n}} = 0$; while contact is maintained if $\dot{r}_{\mathrm{Q,n}} = 0$, with $R_{\mathrm{n}} \geq 0$. The law of normal contact is thus the well-known Signorini's law (Fig. 2):

$$\begin{aligned} &\dot{r}_{\mathrm{Q,n}} \geq 0 \\ &R_{\mathrm{n}} \geq 0 \\ &\boldsymbol{\Psi}^{\mathrm{T}}_{\mathrm{Q,n}} \dot{\boldsymbol{u}} = 0 \end{aligned} \tag{5}$$

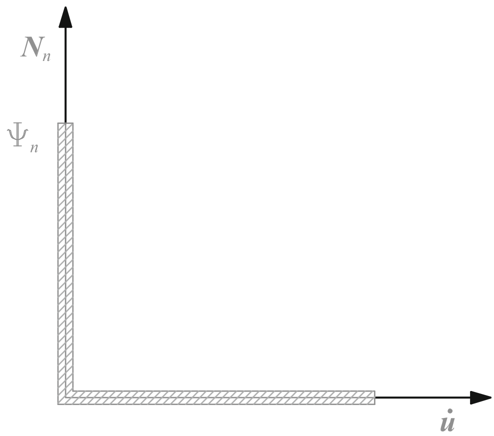

Fig. 2 Signorini's law

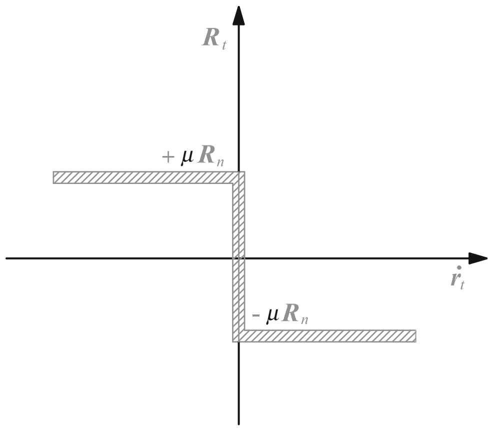

Fig. 3 Coulomb's law

Note that vector $\boldsymbol{N}_{Q,n}$ is orthogonal to $\dot{\boldsymbol{u}}$, while values of R_n determines a positive flux along $\boldsymbol{N}_{Q,n}$ giving $\boldsymbol{\Psi}_{Q,n}$.

The tangential contact is governed by Coulomb's law (Fig. 3):

$$\begin{aligned} & R_t = -\boldsymbol{\mu}\, sgn(\dot{r}_{Q,t}) R_n && \text{for}: \dot{r}_{Q,t} \neq 0 \\ & |R_t| \leq \boldsymbol{\mu}\, R_n && \text{for}: \dot{r}_{Q,t} = 0 \\ & \boldsymbol{\Psi}_{Q,t}{}^{T} \dot{\boldsymbol{u}} \leq \boldsymbol{0} \end{aligned} \tag{6}$$

corresponding to maximal dissipation if $\boldsymbol{\Psi}^{*}_{Q,t}$ is a generic solution belonging to Coulomb's cone:

$$\left(\boldsymbol{\Psi}^{*}_{Q,t} - \boldsymbol{\Psi}_{Q,t}\right)^{T} \dot{\boldsymbol{u}} \geq \boldsymbol{0} \tag{7}$$

Normal and tangential contact laws also hold during an impact, where reactions are associated with Dirac's distributions. By assuming inelastic impact, that is when the impact ends as soon as the normal velocity of Q becomes zero, contact laws in terms of right-sided velocities (apex $+$) are, respectively:

$$\begin{aligned} &\dot{r}^{+}_{\mathrm{Q,n}} = 0 \\ &I_{\mathrm{n}} \geq 0 \\ &\Xi^{\mathrm{T}}_{\mathrm{Q,n}} \dot{\boldsymbol{u}}^{+} = 0 \end{aligned} \tag{8}$$

and:

$$\begin{aligned} &I_{\mathrm{t}} = -\boldsymbol{\mu}\, sgn\left(\dot{r}^{+}_{\mathrm{Q,t}}\right) I_{\mathrm{n}} && \text{for}: \ \dot{r}^{+}_{\mathrm{Q,t}} \neq 0 \\ &|I_{\mathrm{t}}| \leq \boldsymbol{\mu}\, I_{\mathrm{n}} && \text{for}: \ \dot{r}^{+}_{\mathrm{Q,t}} = 0 \\ &\Xi^{\mathrm{T}}_{\mathrm{Q,t}} \dot{\boldsymbol{u}}^{+} \leq 0 && \end{aligned} \tag{9}$$

The unilateral frictional contact results thus into two sub-problems which are connected to each other since tangential reaction depends on the normal one, and reciprocally. Dynamics with friction and unilateral contact can therefore be tackled by solving in turns the two sub-problems until convergence.

2.3 Dynamics Equation

Derive the equation of dynamics from d'Alembert's principle; in terms of generalised forces—active $\boldsymbol{S}$ and reactive $\boldsymbol{\Psi}_{\mathrm{Q}}$—and virtual velocity $\overline{\dot{\boldsymbol{u}}}$ it is:

$$\ddot{\boldsymbol{u}}^{\mathrm{T}} \overline{\dot{\boldsymbol{u}}} = \left(\boldsymbol{S} + \boldsymbol{\Psi}_{\mathrm{Q,t}} + \boldsymbol{\Psi}_{\mathrm{Q,n}}\right)^{\mathrm{T}} \overline{\dot{\boldsymbol{u}}} \tag{10}$$

Since the contact laws do not reduce the degrees of freedom, the equation of motion can have a time-variant structure; thus, the transition from one mechanism to another consistent with contact laws induces a strongly non-linear character of dynamics, which can also exhibit phases of impulsive motion. In this last case, the dynamics equation (10) becomes:

$$\Delta \dot{\boldsymbol{u}}^{\mathrm{T}} \overline{\dot{\boldsymbol{u}}}^{+} = \left(\Xi_{\mathrm{Q,t}} + \Xi_{\mathrm{Q,n}}\right)^{\mathrm{T}} \overline{\dot{\boldsymbol{u}}}^{+} \tag{11}$$

A unilateral frictional contact introduces two kinds of difficulties: firstly, contact reactions are unknown; secondly, when the system is at rest or after an impact, the starting mechanism is also unknown. The starting mechanism is extremely important for practical purposes, as is the protection of art objects or technical instruments, which can be damaged during earthquakes. Knowing how the starting mechanism depends on the excitation and system features could provide useful indications for passive protection devices.

2.4 Starting Motion and Dynamic Balance

With reference to Eq. (10), observe that reactive forces are requested to modulate dynamics in accordance with contact laws; so, their virtual power cannot be positive. The necessary condition for starting motion is therefore that in correspondence to an admissible mechanism $\bar{\dot{\boldsymbol{u}}}$:

$$S^{\mathrm{T}}\bar{\dot{\boldsymbol{u}}} \geq 0 \tag{12}$$

For conservative active forces and friction large enough, inequality (12) satisfied as equality corresponds to a stability criterion; thus, if satisfied for the lowest value of $\boldsymbol{S}$:

$$\min_{\boldsymbol{S}} \left(\boldsymbol{S}^{\mathrm{T}}\bar{\dot{\boldsymbol{u}}} \right) = 0 \tag{13}$$

it gives the closest unstable equilibrium configuration. For finite friction, on the contrary, relationship (13) must be interpreted as a necessary but not sufficient condition; among admissible mechanisms $\bar{\dot{\boldsymbol{u}}}$ it selects the candidates with no sliding at the contact point. The activation of the actual mechanism—the sufficient condition—requires that contact laws be satisfied. For example, in the case of the block of Fig. 1 subjected to its own weight and excited by a horizontal ground motion $\ddot{x}_{\mathrm{O}} = k_{\mathrm{s}}(0)g$ with $k_{\mathrm{s}}(0) > 0$, in accordance with inequality (3), admissible mechanisms maintaining normal contact without sliding are: the null mechanism $\dot{\boldsymbol{u}} = \boldsymbol{0}$, corresponding to resting and contact at any point of AB, and the rocking mechanism $\dot{\boldsymbol{u}}_{\mathrm{A}} \neq \boldsymbol{0}$ with contact at point A. This means that identification of the starting mechanism coincides with that of unknown contact centre Q; therefore, consider mechanism $\dot{\boldsymbol{u}}_{\mathrm{Q}}$ with the centre of instantaneous rotation at Q, and express the tangential position of Q as a linear combination of A and B positions:

$$r_{\mathrm{Q,t}} = (1 - e)\, r_{\mathrm{A,t}} + e\, r_{\mathrm{B,t}} \qquad 0 \leq e \leq 1 \tag{14}$$

Mechanism $\dot{\boldsymbol{u}}_{\mathrm{Q}}$ in Eq. (13) allows for identifying the value of e and contact centre Q, that is, the mechanism as a function of $k_{\mathrm{s}}(0)$. If b and h are the base and height lengths, respectively, and friction is large enough, for $k_{\mathrm{s}}(0) < b/h$ the contact centre Q is internal to AB, with $\dot{\boldsymbol{u}}_{\mathrm{Q}} = \boldsymbol{0}$, so that the block remains at rest; while the contact centre is at point A, corresponding to both resting $\dot{\boldsymbol{u}} = \boldsymbol{0}$ and rocking mechanism $\dot{\boldsymbol{u}}_{\mathrm{A}}$ for $k_{\mathrm{s}}(0) = b/h$, so that the equilibrium configuration becomes unstable and rocking can start for $k_{\mathrm{s}}(0) \geq b/h$.

If the sticking assumption is removed, admissible sliding-rocking mechanisms $\bar{\dot{\boldsymbol{u}}} \neq \boldsymbol{0}$ exist with normal contact maintained at A. Thus, Eq. (13) gives, for any force centre internal to AB, that is for $k_{\mathrm{s}}(0) < b/h$, sliding-rocking mechanisms corresponding to indifferent equilibrium condition, that is to resting; the unique sliding-rocking mechanism corresponding to unstable equilibrium configuration is

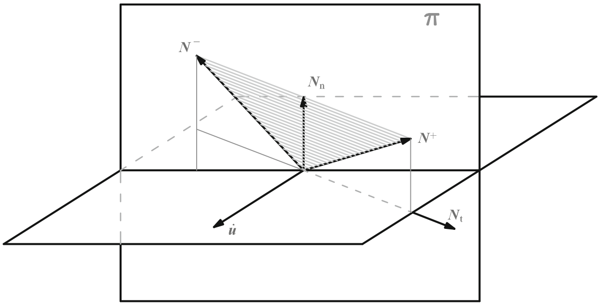

Fig. 4 Vectors $\boldsymbol{N}_{\text{A,t}}$ and $\boldsymbol{N}_{\text{A,n}}$ and Coulomb cone, in plane π

$\dot{\boldsymbol{u}}_{\text{A}}$, for $k_s(0) = b/h$, so that necessary conditions to activate sliding-rocking and rocking coincide with contact at point A and $k_s(0) \geq b/h$.

To identify the sufficient condition, perform the dynamic balance of equation (10) for rocking with contact at point A:

$$\ddot{\boldsymbol{u}}_{\text{A}} = \boldsymbol{S} + \boldsymbol{\Psi}_{\text{A,t}} + \boldsymbol{\Psi}_{\text{A,n}} \tag{15}$$

The analysis here proposed, although aimed at identifying the starting motion, holds also at any time of dynamics according to a given mechanism, if the tangential velocity of the contact point is zero. Assume $k_s(0) \geq b/h$, and observe that, if both tangent and normal contact have to be maintained, from relationships:

$$\begin{aligned} \dot{r}_{\text{A,t}} &= \boldsymbol{N}_{\text{A,t}}{}^{\text{T}}\dot{\boldsymbol{u}}_{\text{A}} = 0 \\ \dot{r}_{\text{A,n}} &= \boldsymbol{N}_{\text{A,n}}{}^{\text{T}}\dot{\boldsymbol{u}}_{\text{A}} = 0 \end{aligned} \tag{16}$$

it follows that both vectors $\boldsymbol{N}_{\text{A,t}}$ and $\boldsymbol{N}_{\text{A,n}}$ lie on the plane π orthogonal to the mechanism $\dot{\boldsymbol{u}}_{\text{A}}$ (Fig. 4).

Moreover, both acceleration components of point A consists of two terms:

$$\begin{aligned} \ddot{r}_{\text{A,t}} &= \boldsymbol{N}_{\text{A,t}}{}^{\text{T}}\ddot{\boldsymbol{u}}_{\text{A}} + \dot{\boldsymbol{N}}_{\text{A,t}}{}^{\text{T}}\dot{\boldsymbol{u}}_{\text{A}} = 0 \\ \ddot{r}_{\text{A,n}} &= \boldsymbol{N}_{\text{A,n}}{}^{\text{T}}\ddot{\boldsymbol{u}}_{\text{A}} + \dot{\boldsymbol{N}}_{\text{A,n}}{}^{\text{T}}\dot{\boldsymbol{u}}_{\text{A}} = 0 \end{aligned} \tag{17}$$

the first parallel to the corresponding component of $\ddot{r}_A$ and the second—the centripetal acceleration—depending on initial conditions and oriented from A to G. For the reciprocity between points A and G considered as rotating each with respect to the other, it follows that acceleration $\ddot{\boldsymbol{u}}_{\text{A}}$ is composed of two terms, the first parallel to $\dot{\boldsymbol{u}}_{\text{A}}$ and the second lying on plane π with components:

$$\begin{aligned} \ddot{\boldsymbol{u}}_{\text{A,t}} &= -\dot{\boldsymbol{N}}_{\text{A,t}}{}^{\text{T}}\dot{\boldsymbol{u}}_{\text{A}} \\ \ddot{\boldsymbol{u}}_{\text{A,n}} &= -\dot{\boldsymbol{N}}_{\text{A,n}}{}^{\text{T}}\dot{\boldsymbol{u}}_{\text{A}} \end{aligned} \tag{18}$$

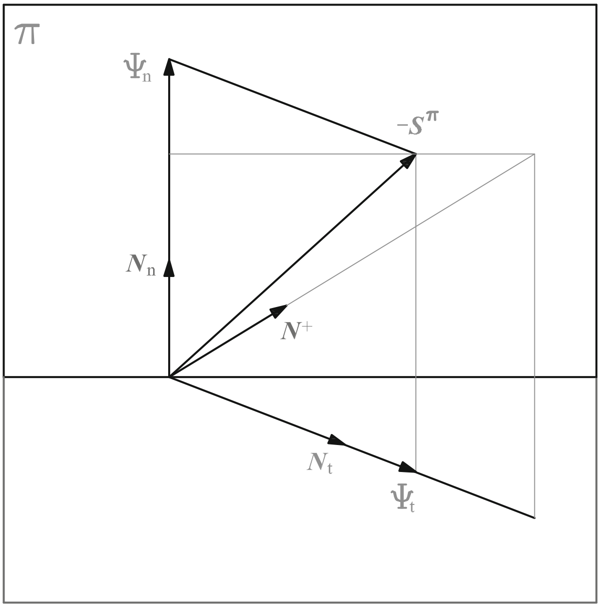

Fig. 5 Dynamic balance in the plane π

both strictly negative and oppositely directed with respect to $\ddot{N}_{A,t}$ and $\ddot{N}_{A,n}$, respectively.

By projecting Eq. (15) on plane π, that is, ignoring active force and acceleration components parallel to $\dot{\boldsymbol{u}}_{\mathrm{A}}$, which are allowed for maintaining contact, the dynamic balance becomes:

$$\ddot{\boldsymbol{u}}_{\mathrm{A,t}} + \ddot{\boldsymbol{u}}_{\mathrm{A,n}} - \boldsymbol{S}^{\pi} = \boldsymbol{\Psi}_{\mathrm{A,t}} + \boldsymbol{\Psi}_{\mathrm{A,n}} \tag{19}$$

where $\ddot{\boldsymbol{u}}_{\mathrm{A,t}}$ and $\ddot{\boldsymbol{u}}_{\mathrm{A,n}}$—the centripetal acceleration components—are known quantities and $\boldsymbol{\Psi}_{\mathrm{A,t}} = R_{\mathrm{t}}\boldsymbol{N}_{\mathrm{A,t}}$ and $\boldsymbol{\Psi}_{\mathrm{A,n}} = R_{\mathrm{n}}\boldsymbol{N}_{\mathrm{A,n}}$ are the unknowns to be determined according to Signorini's and Coulomb's laws. As shown in Figs. 4 and 5 (without subscript indicating contact point), vectors $\boldsymbol{N}_{\mathrm{A,t}}$ and $\boldsymbol{N}_{\mathrm{A,n}}$ identified in plane π Coulomb's cone through its boundaries:

$$\begin{aligned} \boldsymbol{N}_{\mathrm{A}}{}^{+} &= \boldsymbol{N}_{\mathrm{A,n}} + \mu\boldsymbol{N}_{\mathrm{A,t}} \\ \boldsymbol{N}_{\mathrm{A}}{}^{-} &= \boldsymbol{N}_{\mathrm{A,n}} - \mu\boldsymbol{N}_{\mathrm{A,t}} \end{aligned} \tag{20}$$

which are symmetric with respect to $\boldsymbol{N}_{\mathrm{A,n}}$ along the conjugate direction $\boldsymbol{N}_{\mathrm{A,t}}$.

Since centripetal terms are known, the dynamic balance is of a merely static nature. In particular, if the system is initially at rest, with null centripetal accelerations, and $-\boldsymbol{S}^{\pi}$ is internal to Coulomb's cone, generalised reactions are obtained by decomposing $-\boldsymbol{S}^{\pi}$ along $\boldsymbol{N}_{\mathrm{A,t}}$ and $\boldsymbol{N}_{\mathrm{A,n}}$ (Fig. 5).

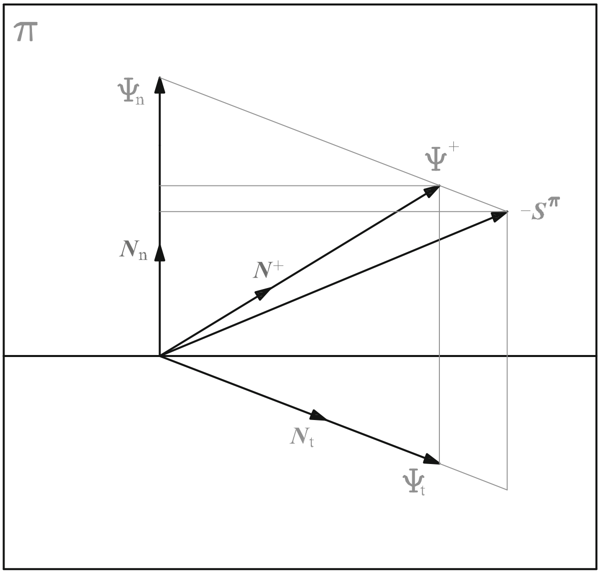

Fig. 6 Unbalance between active force and generalised reaction, for sliding-rocking

Note that since $\boldsymbol{N}_{\mathrm{A,t}}$ and $\boldsymbol{N}_{\mathrm{A,n}}$ are not orthogonal, the projection of $\boldsymbol{\Psi}_{\mathrm{A,t}}$ along $\boldsymbol{N}_{\mathrm{A,n}}$ gives the variation of $\boldsymbol{\Psi}_{\mathrm{A,n}}$ due to friction; this is the interplay between tangent and normal reactions of a frictional contact.

If the contact tangential velocity is zero, the sufficient condition for activating rocking is that: $\boldsymbol{\Psi}_{\mathrm{A,t}} \leq \mu R_{\mathrm{n}} \boldsymbol{N}_{\mathrm{A,t}}$, otherwise tangential contact cannot be maintained and tangential reaction is along the boundary of Coulomb's cone; both tangential and normal reactions are varied with respect to rocking, and the unbalanced component of $\boldsymbol{S}_{\pi}$ determines an acceleration opposite to $\boldsymbol{N}_{\mathrm{A,t}}$ (Fig. 6).

Similarly, for $k_s(0) < b/h$ the possibility of resting or the activation of sliding can be checked. The regions where each mechanism of the block, initially at rest, is activated are reported in Sinopoli (1997).

In conclusion, having satisfied the necessary condition to activate rocking and sliding-rocking, friction alone decides the activated mechanism. Thus, it is nonsense to say that sliding-rocking requires different values of $k_{\mathrm{s}}(0)$ with respect to rocking, since the energy dissipated by friction is balanced by the acceleration variation along $\boldsymbol{N}_{\mathrm{A,n}}$.

2.5 Dynamic Evolution for Smooth and Non-smooth Dynamics

During the motion, until both tangential and normal contact are maintained, the dynamic balance is in plane π, where Eq. (19) holds; in this case, vector $\dot{\boldsymbol{u}}_{\mathrm{A}}$ lies on

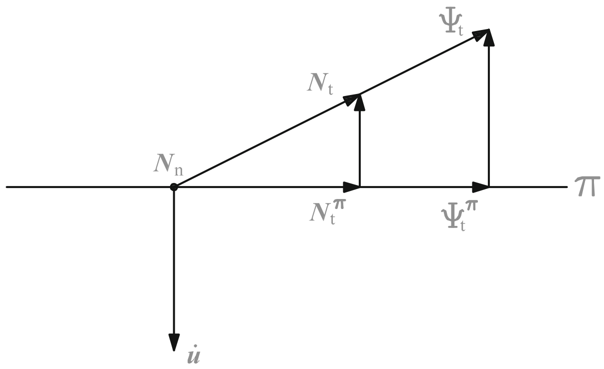

Fig. 7 Decomposition of tangential reaction, for sliding-rocking

the plane orthogonal to $\boldsymbol{N}_{A,n}$, while $\boldsymbol{N}_{A,t}$ and $\boldsymbol{N}_{A,n}$ belong to the plane π orthogonal to $\dot{\boldsymbol{u}}_A$ (Fig. 4). For lost tangential contact, that is, for $\ddot{r}_{A,t}$ —projection of $\ddot{\boldsymbol{u}}_A$ along $\boldsymbol{N}_{A,t}$—different from zero, $\boldsymbol{N}_{A,n}$ is still orthogonal to $\dot{\boldsymbol{u}}_A$, but vectors $\boldsymbol{N}_{A,t}$ and $\boldsymbol{N}_{A,n}$ lie on a plane π^* which is not orthogonal to $\dot{\boldsymbol{u}}_A$. Nevertheless, since the contact balance is always on the positive hemi-space associated with $\boldsymbol{N}_{A,n}$, the friction reaction $\boldsymbol{\Psi}_{A,t} = R_t \boldsymbol{N}_{A,t}$ can be evaluated by decomposing it into two components (Fig. 7): the first parallel to $\dot{\boldsymbol{u}}_A$, which determines a reduction of the velocity, and the second lying on plane π orthogonal to $\dot{\boldsymbol{u}}_A$; both components depend on R_t through the corresponding decomposition of $\boldsymbol{N}_{A,t}$. Thus, the problem can be solved on plane π by determining first R_t and then the resultant $\boldsymbol{\Psi}_{A,t}$ along $\boldsymbol{N}_{A,t}$.

In the case of an impact, the main problem is the identification of the post-impact mechanism. In accordance with contact laws (8) and (9), the dynamic balance (11) is:

$$\dot{\boldsymbol{u}}^{+^{T}} \overline{\dot{\boldsymbol{u}}^{+}} = \left(\dot{\boldsymbol{u}}^{-} + \Xi_{Q,t} + \Xi_{Q,n}\right)^{T} \overline{\dot{\boldsymbol{u}}^{+}} \tag{21}$$

By comparing Eqs. (21) and (10), it follows that during an impact the role of the acceleration of smooth dynamics is played by the post-impact velocity, and that of the active force by pre-impact velocity. The procedure will therefore be the same followed for the starting mechanism of smooth dynamics, by treating pre-impact velocity $\dot{\boldsymbol{u}}^{-}$ as "active force", which makes it possible to determine both the velocity of post-impact motion and generalised impulses.

3 Arch Dynamics as an Assemblage of Discrete Elements

Consider an arch made of n rigid voussoirs, free standing with Coulomb friction on rigid ground Γ, which moves with a translational horizontal motion. Refer the dynamics of each voussoir to the system (O, x, y) fixed on Γ, with which unit vectors $(\boldsymbol{t}_0, \boldsymbol{n}_0)$ are associated; $\boldsymbol{n}_0$ is outwards oriented (Fig. 8).

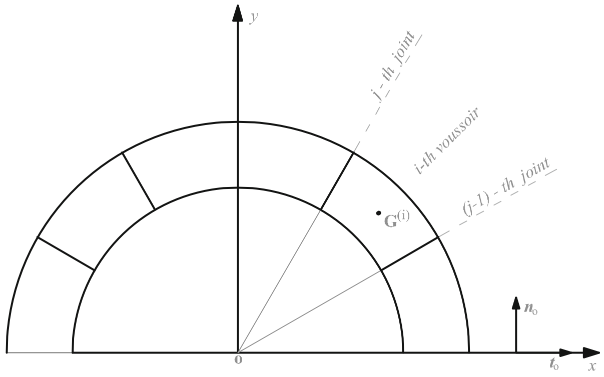

Fig. 8 Scheme of the arch as an assemblage of rigid voussoirs

Denote by i the counter of voussoirs, and number them from 1 to n; similarly, denote by j the counter of joints, and number them from 0 to n. When the arch is at rest, j-*th* joint is unique. During the motion, on the contrary, j-*th* joint can open and be split into two, the $j^{(i)}$-*th* and $j^{(i+1)}$-*th* joints, bounding i-*th* and $(i + 1)$-*th* voussoirs, respectively.

3.1 Kinematics and Contact Laws

Assume for generic i-*th* voussoir the position of the mass centre $G^{(i)}$ and rotation angle as Lagrangian coordinates, evaluated with respect to the system (O, x, y) fixed on Γ, and assume the kinetic energy metrics. If the arch is initially at rest, each voussoir is in full contact with two adjacent ones. In particular, at j-*th* joint, contact is maintained at point $Q^{(i+1)}$, belonging to the lower boundary of $(i + 1)$-*th* voussoir, and at point $P^{(i)}$ belonging to the upper boundary of i-*th* voussoir; denote points $Q^{(i+1)}$ and $P^{(i)}$ as candidate and antagonist contact points at j-*th* joint. Thus, the couples of points maintaining contact for the i-*th* voussoir are $(Q^{(i+1)}, P^{(i)})$ at the j-*th* joint, and $(Q^{(i)}, P^{(i-1)})$ at $(j-1)$-*th* joint (Fig. 9). Through the mapping, velocities of points $Q^{(i+1)}$ and $P^{(i)}$ become:

$$\begin{aligned} \dot{\boldsymbol{r}}_{\mathrm{Q}}^{(i+1)} &= \mathbf{N}_{\mathrm{Q}}^{(i+1)}\, \dot{\boldsymbol{u}}^{(i+1)} \\ \dot{\boldsymbol{r}}_{\mathrm{P}}^{(i)} &= \mathbf{N}_{\mathrm{P}}^{(i)}\, \dot{\boldsymbol{u}}^{(i)} \end{aligned} \tag{22}$$

with analogous expressions for points $(Q^{(i)}, P^{(i-1)})$ at $(j-1)$-*th* joint.

The impenetrability condition in this case bounds the normal velocity of the candidate point with respect to that of antagonist one, both evaluated on the local reference system fixed on the antagonist, and reciprocally; the unilateral constraint at each joint thus splits into two conditions. As an example, the impenetrability

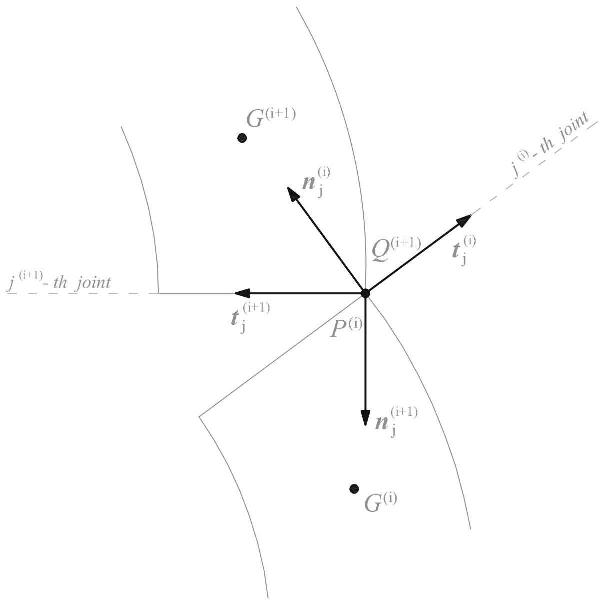

Fig. 9 Scheme of the contact, between *i-th* and (*i*+1)-*th* voussoirs

condition at *j-th* joint requires admissible velocities, evaluated in the local unit vectors system ($\boldsymbol{t}_{\mathrm{j}}^{(\mathrm{i})}$, $\boldsymbol{n}_{\mathrm{j}}^{(\mathrm{i})}$), fixed on $P^{(\mathrm{i})}$:

$$\begin{aligned} & \dot{r}^{(\mathrm{i+1})}_{\mathrm{Q,n_j^{(i)}}} - \dot{r}^{(\mathrm{i})}_{\mathrm{P,n_j^{(i)}}} = \boldsymbol{n}_{\mathrm{j}}^{(\mathrm{i})\mathrm{T}} \left(\dot{\boldsymbol{r}}_{\mathrm{Q}}^{(\mathrm{i+1})} - \dot{\boldsymbol{r}}_{\mathrm{P}}^{(\mathrm{i})}\right) = \boldsymbol{n}_{\mathrm{j}}^{(\mathrm{i})\mathrm{T}} \left(\mathbf{N}_{\mathrm{Q}}^{(\mathrm{i+1})^{\mathrm{T}}} \dot{\boldsymbol{u}}^{(\mathrm{i+1})} - \mathbf{N}_{\mathrm{P}}^{(\mathrm{i})^{\mathrm{T}}} \dot{\boldsymbol{u}}^{(\mathrm{i})}\right) = \\ & = \boldsymbol{H}^{(\mathrm{i+1})^{\mathrm{T}}}_{\mathrm{Q,n_j^{(\mathit{i})}}} \dot{\boldsymbol{u}}^{(\mathrm{i+1})} - \boldsymbol{H}^{(\mathrm{i})^{\mathrm{T}}}_{\mathrm{P,n_j^{(\mathit{i})}}} \dot{\boldsymbol{u}}^{(\mathrm{i})} \geq 0 \end{aligned} \tag{23a}$$

and, reciprocally, in the local system $\left(\boldsymbol{t}_{\mathrm{j}}^{(\mathrm{i+1})},\ \boldsymbol{n}_{\mathrm{j}}^{(\mathrm{i+1})}\right)$:

$$\begin{aligned} & \dot{r}^{(\mathrm{i})}_{\mathrm{P,n_j^{(i+1)}}} - \dot{r}^{(\mathrm{i+1})}_{\mathrm{Q,n_j^{(i+1)}}} = \boldsymbol{n}_{\mathrm{j}}^{(\mathrm{i+1})\mathrm{T}} \left(\dot{\boldsymbol{r}}_{\mathrm{P}}^{(\mathrm{i})} - \dot{\boldsymbol{r}}_{\mathrm{Q}}^{(\mathrm{i+1})}\right) = \boldsymbol{n}_{\mathrm{j}}^{(\mathrm{i+1})\mathrm{T}} \left(\mathbf{N}_{\mathrm{P}}^{(\mathrm{i})^{\mathrm{T}}} \dot{\boldsymbol{u}}^{(\mathrm{i})} - \mathbf{N}_{\mathrm{Q,n}}^{(\mathrm{i+1})^{\mathrm{T}}} \dot{\boldsymbol{u}}^{(\mathrm{i+1})}\right) = \\ & = \boldsymbol{H}^{(\mathrm{i})^{\mathrm{T}}}_{\mathrm{P,n_j^{(i+1)}}} \dot{\boldsymbol{u}}^{(\mathrm{i})} - \boldsymbol{H}^{(\mathrm{i+1})^{\mathrm{T}}}_{\mathrm{Q,n_j^{(i+1)}}} \dot{\boldsymbol{u}}^{(\mathrm{i+1})} \geq 0 \end{aligned} \tag{23b}$$

where vectors $\boldsymbol{H}$ are generalised directions associated at contact points with local unit vectors $\boldsymbol{n}_{\mathrm{j}}^{(\mathrm{i})}$ and $\boldsymbol{n}_{\mathrm{j}}^{(\mathrm{i+1})}$. Inequalities (23a) and (23b) are in general different; they coincide only if the joint is closed. The consequence is that the contact laws imply different generalised directions $\boldsymbol{H}$ and reactions $\boldsymbol{\Phi}$. Relationship (23a) gives

Signorini's contact law:

$$R^{(i,i+1)}_{n_j^{(i)}}\left(\dot{r}^{(i+1)}_{Q,n_j^{(i)}}-\dot{r}^{(i)}_{P,n_j^{(i)}}\right)=\left(\boldsymbol{\Phi}^{(i+1)^T}_{Q,n_j^{(i)}}\dot{\boldsymbol{u}}^{(i+1)}-\boldsymbol{\Phi}^{(i)}_{P,n_j^{(i)}}{}^{T}\dot{\boldsymbol{u}}^{(i)}\right)=0 \tag{24}$$

while Coulomb's law governing friction is:

$$R^{(i,i+1)}_{t_j^{(i)}}\left(\dot{r}^{(i+1)}_{Q,t_j^{(i)}}-\dot{r}^{(i)}_{P,t}\right)=\left(\boldsymbol{\Phi}^{(i+1)^T}_{Q,t_j^{(i)}}\dot{\boldsymbol{u}}^{(i+1)}-\boldsymbol{\Phi}^{(i)}_{P,t_j^{(i)}}{}^{T}\dot{\boldsymbol{u}}^{(i)}\right)\leq 0 \tag{25}$$

Analogous, although different, expressions are obtained from (23b).

3.2 Dynamics Equations and Outlines of the Method Proposed

With reference to system (O, x, y) fixed on Γ consider the equation of motion for the arch, obtained from d'Alembert's principle:

$$\sum_{i=1}^{n}\left(-\ddot{\boldsymbol{u}}^{(i)}+\boldsymbol{S}^{(i)}+\boldsymbol{\Psi}^{(i+1,i)}_{P}+\boldsymbol{\Psi}^{(i-1,i)}_{Q}\right)^{T}\overline{\dot{\boldsymbol{u}}^{(i)}}=0 \tag{26}$$

where $\ddot{\boldsymbol{u}}^{(i)}$ is the acceleration, $\boldsymbol{S}^{(i)}$ the active force, $\overline{\dot{\boldsymbol{u}}^{(i)}}$ the admissible velocity, and $\boldsymbol{\Psi}^{(i+1,i)}_{P}$ and $\boldsymbol{\Psi}^{(i-1,i)}_{Q}$ the reactions transmitted to *i-th* voussoir by the contiguous ones.

Since Signorini's and Coulomb's laws do not reduce the degrees of freedom, it seems that the dynamics of *i-th* voussoir:

$$\left(-\ddot{\boldsymbol{u}}^{(i)}+\boldsymbol{S}^{(i)}+\boldsymbol{\Psi}^{(i+1,i)}_{P}+\boldsymbol{\Psi}^{(i-1,i)}_{Q}\right)^{T}\overline{\dot{\boldsymbol{u}}^{(i)}}=0 \quad i=1,\ldots,n \tag{27}$$

cannot be solved, as for the single block, unless the whole arch dynamics is, since reactions transmitted by the two contiguous voussoirs depend on their dynamics. A classical procedure (Lötstedt 1982; Glocker and Pfeiffer 1992) would be that of adopting an augmented Lagrangian multipliers method, by using reactions as multipliers and Eqs. (23a)–(25) as constraints. In this case, the fulfilment of Signorini's and Coulomb's laws is equivalent to the Kuhn-Tucker condition corresponding to the optimal solution of a quadratic programming problem; an iterative procedure is thus necessary and the mathematical problem requires inversion of matrices of great dimension, equal to the number of degrees of freedom plus constraints. In the case of the arch, however, matrices have elements different from zero only at the sides of the principal diagonal, since only contiguous voussoirs interact with each other; furthermore, activated mechanisms involve a number of mega-voussoirs lower than that of all the voussoirs.

Moreover, since the dynamics evolves starting from given initial positions and velocities according to a given mechanism, the instantaneous rotation centres are given. This means that the starting of relative either rocking or sliding-rocking, as of relative sticking or sliding, corresponds—as for the single block—to the same contact centres; whether the mechanism changes in the following motion depends only on the values and directions of reactions, which determine possible new instantaneous rotation centres. This is the reason why limit analysis with finite friction seems undetermined, unless contact laws are verified.

Both persistency and variation of the activated mechanism thus depend on either the fulfilment of Signorini's and Coulomb's laws or reaching the limit of their admissible sets; both require the determination of contact reactions—the main unknowns of the problem.

It will be demonstrated that the indeterminacy of arch dynamics can be eliminated by appropriate considerations on the relationship between the centre of instantaneous rotation of a central mega-voussoir and allowed reactions transmitted to it. A hierarchical iterative procedure is thus required, concerning at each step a number of joints at most equal to that of activated mega-voussoirs. A first level of iteration concerns only the boundary joints of mega-voussoirs; while a second level—at which intermediate joints are checked—can restart the first level of iteration. Observe that any numerical algorithm of mathematical programming requires an iterative procedure, involving at each step a number of unknowns equal to the number of all the degrees of freedom plus constraints.

Once the indeterminacy at boundary joints of mega-voussoirs has been eliminated—albeit tentatively –, contact reactions at intermediate joints can be determined by propagating the dynamic balance in the respect of contact laws. Note that it is the reaching of Coulomb's cone limit that requires the use of the iterative procedure, while that of Signorini's corresponds to loss of contact and then to probable collapse of the arch.

The starting of sliding-rocking of a mega-voussoir with respect to the contiguous one can be detected; since the new mechanism starts with null relative velocity at contact points, sliding-rocking effects are obtained in the motion that follows as a consequence of the dynamics and reaction along Coulomb's cone. Only for open joints, since Signorini's and Coulomb's laws are both split into two, which contact point slides with respect to the joint of contiguous mega-voussoir can be foreseen. In this case, the reaction is along Coulomb's cone at one joint, and internal at the other one; otherwise, if both reactions are along Coulomb's cone, as for closed joints, which point slides is given only by dynamics.

With reference to intermediate joints, the starting of either relative sliding or sliding-rocking for reaching Coulomb's cone limit depends on the contact points position, internal to the joint or at its boundary, respectively; the last case occurs when one contact centre at the boundary of a mega-voussoir is split into two, the second lying at the nearest joint and at the same side (extrados or intrados). The imposed reactions both along Coulomb's cone at the considered intermediate joint—since it is initially closed—vary with respect to those of relative sticking and determine a variation also at successive joints until the boundary of the mega-voussoir is reached. As a consequence, the reaction acting on the opposite boundary also varies, and both the first and second levels of iteration restart until convergence.

In contrast, when the contact centre bounding a mega-voussoir is split into two—the second lying on same side at the extrados or intrados—and Coulomb's law is respected the iterative procedure is not required.

The greatest number of steps of iteration are therefore required only when Coulomb's limit is reached for variation of both size and number of elements of a mega-voussoir, while any numerical algorithm of mathematical programming is characterised by a number of steps exponentially proportional to that of unknowns.

Finally, note that numerical platforms of the literature use the discrete elements method with great difficulties related to the elimination of high frequencies, while the present formulation is based on the rigidity assumption.

These among others are the reasons why an analytical approach is proposed by propagating dynamics balance from a mega-voussoir to its contiguous and all intermediate joints.

3.3 *Starting Motion and Dynamic Balance: First Level Iteration*

By referring to the results obtained for the single block, remember that both rocking and sliding-rocking mechanisms require contact centre at joint boundaries, while resting and sliding correspond to contact centre internal to the joint. Moreover, remember that it is friction, which decides the loss of tangential contact, so that we can assume tentatively that it is maintained unless contact laws impose the contrary.

Admissible velocities for the generic voussoir require that admissible mechanisms for the whole arch be identified. Moreover, in any rotational arch mechanism, if *i-th* voussoir rotates around intrados at $(j-1)$-*th* joint, its maximal interaction with $(i+1)$-*th* voussoir is at the extrados of *j-th* joint. Therefore, rotational mechanisms are characterized by alternate hinges at the extrados and intrados respectively, even if each of such hinges can be split into two for a dynamical thrust line going out from a single voussoir; while, if the arch is at rest and external excitation is increasing starting from zero, the most probable mechanism is characterised by the lowest number of alternate hinges at the extrados and intrados.

Consider the dynamics excited by increasing horizontal ground motion $\ddot{x}_O = k_s(t)g$ starting from zero, and determine the rotational starting mechanism; by disregarding reactions since contacts are governed by normality rules, the necessary condition for starting motion becomes:

$$\min_{k_s} \sum_{i=1}^{n} \boldsymbol{S}^{(i)^T} \overline{\dot{\boldsymbol{u}}^{(i)}} = 0 \tag{28}$$

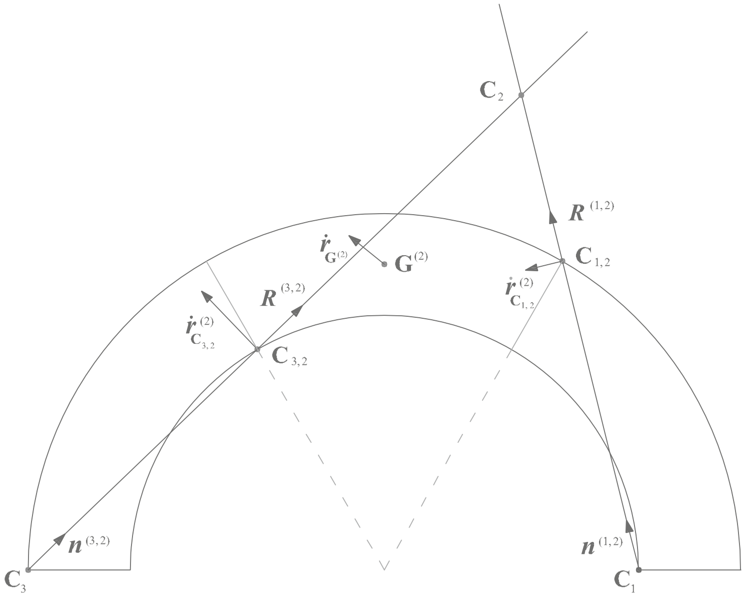

Fig. 10 Scheme of the starting rotational mechanism

By increasing the ground acceleration, null mechanisms $\dot{\boldsymbol{u}}^{(i)}$ are firstly admissible; they correspond to thrust lines with at most three alternate hinges at the extrados and intrados. In this case, the possibility of a mechanism with pure relative sliding at some intermediate joint, even improbable, could be examined; for arches subjected only to their own weight this is the mixed mechanism identified in (Sinopoli *et al.* 2007).

Once four alternate hinges are formed, the well-known mechanism for asymmetric loads usually determined by an upper approach appears. The arch can thus be considered as made of only three mega-voussoirs; label them by index k, with $k = 1,2,3$, and contact centres positions (Fig. 10) by C_1, $C_{1,2}$, $C_{2,3}$, and C_3. Contact centres define the instantaneous rotation centre C_2 of second mega-voussoir, lying at the intersection of straight lines crossing points C_1 and $C_{1,2}$, and C_3 and $C_{2,3}$, respectively; C_2 does not vary in the following motion only if tangential contacts at all voussoirs contact centres are maintained.

The identification of contact centres at intermediate joints inside each mega-voussoir and checking of Signorini's and Coulomb's conditions at each joint require that reactions acting on each mega-voussoir be determined. To this aim, since the main role is played by second mega-voussoir, consider its dynamics equation:

$$\ddot{\boldsymbol{u}}^{(2)} - \boldsymbol{S}^{(2)} = +\boldsymbol{\Psi}^{(3,2)}_{C_{2,3}} + \boldsymbol{\Psi}^{(1,2)}_{C_{1,2}} \tag{29}$$

If Coulomb's law is verified at $C_{1,2}$ and $C_{2,3}$, but also at C_1 and C_3, reactions $\boldsymbol{R}^{(1,2)}$ and $\boldsymbol{R}^{(3,2)}$ evaluated in the system ($\boldsymbol{t}_o$, $\boldsymbol{n}_o$) are parallel to unit vector $\boldsymbol{n}^{(1,2)}$ and $\boldsymbol{n}^{(3,2)}$, in line with points $C_{1,2}$ and C_2, and $C_{2,3}$ and C_2, respectively; thus: $\boldsymbol{R}^{(1,2)} = -\boldsymbol{R}^{(2,1)} = R^{(1,2)}\boldsymbol{n}^{(1,2)}$ and, similarly: $\boldsymbol{R}^{(3,2)} = -\boldsymbol{R}^{(2,3)} = R^{(3,2)}\boldsymbol{n}^{(3,2)}$. In fact, since the dynamics of second mega-voussoir is modulated by reactions $\boldsymbol{R}^{(1,2)}$ and $\boldsymbol{R}^{(3,2)}$ crossing contact centres $C_{1,2}$ and $C_{2,3}$ and second mega-voussoir rotates around C_2, the resultant of $\boldsymbol{R}^{(1,2)}$ and $\boldsymbol{R}^{(3,2)}$ necessarily crosses the instantaneous rotation centre C_2.

Thus, since inequalities (23a) and (23b) give: $\dot{\boldsymbol{r}}^{(1)}_{\mathrm{C}_{1,2}} = \dot{\boldsymbol{r}}^{(2)}_{\mathrm{C}_{1,2}}$, reactions $\boldsymbol{R}^{(1,2)}$ and $\boldsymbol{R}^{(2,1)}$ are orthogonal to $\dot{\boldsymbol{r}}^{(2)}_{\mathrm{C}_{1,2}}$ and $\dot{\boldsymbol{r}}^{(1)}_{\mathrm{C}_{1,2}}$, respectively; and, similarly, $\boldsymbol{R}^{(3,2)}$ and $\boldsymbol{R}^{(2,3)}$ are orthogonal to $\dot{\boldsymbol{r}}^{(2)}_{\mathrm{C}_{2,3}}$ and $\dot{\boldsymbol{r}}^{(3)}_{\mathrm{C}_{2,3}}$. A normality rule therefore governs generalised reactions and velocities at relative contact centres:

$$\begin{aligned} \boldsymbol{\Psi}^{(1,2)^{\mathrm{T}}}_{\mathrm{C}_{1,2}} \dot{\boldsymbol{u}}^{(2)} &= \boldsymbol{\Psi}^{(2,1)^{\mathrm{T}}}_{\mathrm{C}_{1,2}} \dot{\boldsymbol{u}}^{(1)} = 0 \\ \boldsymbol{\Psi}^{(3,2)^{\mathrm{T}}}_{\mathrm{C}_{2,3}} \dot{\boldsymbol{u}}^{(2)} &= \boldsymbol{\Psi}^{(2,3)^{\mathrm{T}}}_{\mathrm{C}_{2,3}} \dot{\boldsymbol{u}}^{(3)} = 0 \end{aligned} \tag{30}$$

Moreover, since reactions $\boldsymbol{\Psi}^{(1,2)}_{\mathrm{C}_{1,2}}$ and $\boldsymbol{\Psi}^{(3,2)}_{\mathrm{C}_{2,3}}$ can be rewritten as:

$$\begin{aligned} \boldsymbol{\Psi}^{(1,2)}_{\mathrm{C}_{1,2}} &= R^{(1,2)}\mathbf{N}^{(2)}_{\mathrm{C}_{1,2}}\boldsymbol{n}^{(1,2)} = R^{(1,2)}\boldsymbol{H}^{(2)}_{\mathrm{C}_{1,2}} \\ \boldsymbol{\Psi}^{(3,2)}_{\mathrm{C}_{2,3}} &= R^{(3,2)}\mathbf{N}^{(2)}_{C_{2,3}}\boldsymbol{n}^{(3,2)} = R^{(3,2)}\boldsymbol{H}^{(2)}_{\mathrm{C}_{2,3}} \end{aligned} \tag{31}$$

scalar values of reactions $R^{(1,2)}$ and $R^{(3,2)}$ become flux along the generalised directions $\boldsymbol{H}^{(2)}_{\mathrm{C}_{1,2}}$ and $\boldsymbol{H}^{(2)}_{\mathrm{C}_{2,3}}$ associated with $\boldsymbol{n}^{(1,2)}$ and $\boldsymbol{n}^{(3,2)}$, and similarly for $R^{(2,1)}$ and $R^{(2,3)}$. Equation (29) assumes therefore a form very useful for dynamic balance:

$$\ddot{\boldsymbol{u}}^{(2)} - \boldsymbol{S}^{(2)} = R^{(3,2)}\boldsymbol{H}^{(2)}_{\mathrm{C}_{2,3}} + R^{(1,2)}\boldsymbol{H}^{(2)}_{\mathrm{C}_{1,2}} \tag{32}$$

In fact, from (30) to (31) it follows that both $\boldsymbol{H}^{(2)}_{\mathrm{C}_{1,2}}$ and $\boldsymbol{H}^{(2)}_{\mathrm{C}_{2,3}}$, as $\boldsymbol{\Psi}^{(1,2)}_{\mathrm{C}_{1,2}}$ and $\boldsymbol{\Psi}^{(3,2)}_{\mathrm{C}_{2,3}}$, belong to the plane $\pi^{(2)}$ orthogonal to $\dot{\boldsymbol{u}}^{(2)}$. By projecting (32) on plane $\pi^{(2)}$:

$$\ddot{\boldsymbol{u}}^{(2)}_{\pi}\text{-}\boldsymbol{S}^{(2)}_{\pi} = R^{(3,2)}\boldsymbol{H}^{(2)}_{\mathrm{C}_{2,3}} + R^{(1,2)}\boldsymbol{H}^{(2)}_{\mathrm{C}_{1,2}} \tag{33}$$

since the centripetal acceleration $\ddot{\boldsymbol{u}}^{(2)}_{\pi} = -\dot{\boldsymbol{N}}^{(2)^{\mathrm{T}}}_{\mathrm{C}_2} \dot{\boldsymbol{u}}^{(2)}$ is a known quantity, the dynamic balance assumes a static character; values $R^{(1,2)}$ and $R^{(3,2)}$ of transmitted reactions, like their generalised expressions $\boldsymbol{\Psi}^{(1,2)}_{\mathrm{C}_{1,2}}$ and $\boldsymbol{\Psi}^{(3,2)}_{\mathrm{C}_{2,3}}$, can be evaluated.

Contact laws must now be verified at each of the two joints bounding second voussoir, by decomposing $\boldsymbol{\Psi}^{(1,2)}_{\mathrm{C}_{1,2}}$ and $\boldsymbol{\Psi}^{(3,2)}_{\mathrm{C}_{2,3}}$ along the corresponding generalised normal and tangential directions; having satisfied contact laws, which can also imply starting of sliding-rocking, reactions $\boldsymbol{\Psi}^{(1,2)}_{\mathrm{C}_{1,2}}$ and $\boldsymbol{\Psi}^{(3,2)}_{\mathrm{C}_{2,3}}$ are tentatively determined.

By a similar procedure through the dynamic balance of first and third mega-voussoirs, the opposites of $R^{(1,2)}$ and $R^{(3,2)}$ allow us to determine $\boldsymbol{\Psi}_{C_1}^{(0,1)}$ and $\boldsymbol{\Psi}_{C_3}^{(4,3)}$ and, if Coulomb's limit of admissible solutions is reached at either or both joints, both $\boldsymbol{\Psi}_{C_{1,2}}^{(1,2)}$ and $\boldsymbol{\Psi}_{C_{2,3}}^{(3,2)}$ vary and the iteration continues until convergence.

Having tentatively determined $\boldsymbol{\Psi}_{C_1}^{(0,1)}$ and $\boldsymbol{\Psi}_{C_3}^{(4,3)}$, and $\boldsymbol{\Psi}_{C_{1,2}}^{(1,2)}$ and $\boldsymbol{\Psi}_{C_{2,3}}^{(3,2)}$ at the first level, the evaluation of contact reactions and checking of contact laws can be extended at intermediate joints of second mega-voussoir, as to those of first and third mega-voussoirs.

3.4 Second Level of Iteration at Intermediate Contact Centres

The evaluation of reactions acting at intermediate joints is performed starting from either boundary of second mega-voussoir, and checking all the joints up to the other boundary. At the first joint nearest either boundary, $\boldsymbol{\Psi}_{C_{1,2}}^{(1,2)}$ or $\boldsymbol{\Psi}_{C_{2,3}}^{(3,2)}$ is treated as a known force, which determines an augmented $\boldsymbol{S}^{(i)*}$ and gives the reaction at the successive joint. In this case, however, candidate and antagonist contact centres are unknown; they are identified by checking the possibility of relative motion, as for a starting mechanism. As an example, with reference to *j-th* joint by starting from the $(j-1)$-*th* one, express the tangential position of the unknown centre $P^{(i)}$ as a linear combination of intrados $A^{(i)}$ and extrados $B^{(i)}$ positions:

$$r_{P,t}^{(i)} = (1-e)\, r_{A,t}^{(i)} + e\, r_{B,t}^{(i)} \qquad 0 \le e \le 1 \tag{34}$$

and determine the contact centre of the virtual mechanism of relative rotation $\overline{\dot{\boldsymbol{u}}_P^{(i,i+1)}}$ satisfying:

$$\overline{\dot{\boldsymbol{u}}_P^{(i,i+1)}}^{T} \boldsymbol{S}^{(i)*} = 0 \tag{35}$$

The value of e so obtained identifies the contact centre $P^{(i)}$ and the reaction $\boldsymbol{\Psi}_P^{(i,i+1)}$ transmitted at *j-th* joint by *i-th* to $(i+1)$-*th* voussoir. If point $P^{(i)}$ is internal to the joint and Coulomb's law is satisfied, the mechanism does not change in the motion which follows, and checking can continue at successive intermediate joints; alternatively, pure sliding can start and the reaction along Coulomb's cone determines the variation of contact reactions from that joint to successive ones, until the boundary of mega-voussoir is reached. As a consequence, the reaction at the opposite boundary varies, and both first and second levels of iteration restart until convergence. On the contrary, if point $P^{(i)}$ lies either at extrados or intrados of the joint nearest that bounding the mega-voussoir, at same side of its contact centre, the mechanism changes in the following motion with no need of iterative procedure unless sliding-rocking is activated.

Similarly, reactions transmitted at all intermediate joints of first and third mega-voussoirs can be evaluated and checked up to the springing, and if necessary the iterative procedure is restarted until convergence. Having determined contact reactions through the method proposed, the equation of motion for each mega-voussoir can finally be integrated.

3.5 Dynamic Evolution and Mechanism Variation

The analysis here proposed, although aimed at identifying the starting motion, also holds during dynamics evolution according to a given mechanism.

The dynamics evolves starting from given initial positions and velocities; thus, if the activated mechanism is known, adherence to Signorini's and Coulomb's laws can be checked at each contact centre and, if verified, allows the maintainance of the mechanism in the motion which follows.

On the contrary, if the reaction required at any contact centre reaches Coulomb's limit of admissible solutions, the mechanism changes in the following motion depending on the position of the contact centre.

If it is either at the extrados or intrados of contiguous mega-voussoirs, with open contact joints, the projection of the determined contact reactions along the two local unit vectors systems identifies which contact point slides with respect to the contiguous joint, only if the reaction is along Coulomb's cone at either joint, and internal at the other one; otherwise, if both reactions are along Coulomb's cone, as for a closed joint, which contact point slides is given by dynamics. If the contact centre belongs to an intermediate joint of a mega-voussoir, relative either sliding—if allowed—or sliding-rocking can start with consequent variation of the number of mega-voussoirs involved in the mechanism.

Finally, it is worthwhile to note that if at any time and at any joint Signorini's set reaches a zero gap, the local contact is lost with consequent probable collapse of the arch.

4 Conclusions

The paper extends previous research of the author (Sinopoli 1997) on the dynamic modelling of a rigid body simply supported on a moving boundary. Its main purpose is the dynamic modelling of the masonry arch, treated as a system of n voussoirs with frictional unilateral contacts at each joint and dynamics governed by Signorini's and Coulomb's laws. The difficulties of the dynamics formulation are:

(a) the choice of the reference system and Lagrangian coordinates to write the equations of motion for each voussoir and each mega-voussoir, obtained by assembling one or many voussoirs in the activated mechanism;

(b) the evaluation of the principal unknowns of the dynamics, namely, the contact reactions acting at the two boundary joints of each mega-voussoir—and then of each voussoir—on which accelerations and velocities of the following motion depend.

An analytical procedure is proposed to write the equations of motion and to identify, for each voussoir in accordance with Signorini's and Coulomb's laws, the contact reactions as a function of active forces and instantaneous velocities of contact points. First, the modelling is given for the single block, to guide the understanding on how Signorini's and Coulomb's laws can be checked at contact point by identifying both the value of the generalised reaction and the persistency or variation of the activated mechanism. The formalism is then extended to the arch; new theoretical questions related to the arch typology are discussed and solved. The method is based on the key idea of transforming the dynamical modelling into an equivalent problem of "static" balance, by using for each element projecting techniques in the configuration space.

The indeterminacy of the arch is eliminated through an iterative procedure by propagating the dynamic balance from a selected mega-voussoir to contiguous ones, and then to all the voussoirs. The iterative procedure is activated only if Coulomb's cone limit is reached at any joint; it concerns a number of joints at most equal to that of the activated mega-voussoirs. The iterative procedure is thus not limiting and not time consuming with respect to other methods; it does not alter the solution, since the tentative solution at the first step of iteration is that for relative sticking or rocking, and corresponds at any step to instantaneous values of active forces, positions and velocities.

Having determined the contact reactions at each joint, the equations of motion can be integrated by either an explicit or implicit numerical procedure; in particular, an implicit procedure does not require evaluation of the acceleration, since the velocity of the following motion is obtained by that of the previous one plus the integral of active forces and reactions.

In conclusion, the paper presents a consistent analytical formulation, indicating also the computational procedure to implement a numerical platform for the dynamic analysis of the masonry arch.

Appendix: List of Symbols

C_k, $C_{k,k+1}$	Absolute and relative instantaneous rotation centres in the arch mechanism
g	Gravity acceleration
G	Mass centre of the block
$G^{(i)}$	Mass centre of *i-th* voussoir
$\boldsymbol{H}^{(\mathrm{i})}_{\mathrm{P},\mathrm{n}_\mathrm{j}^{(i)}}$	Generalised direction associated with unit vector $\boldsymbol{n}_\mathrm{j}^{(\mathrm{i})}$ at point $P^{(\mathrm{i})}$ of *i-th* mega-voussoir

$\boldsymbol{H}_{C_{k,k+1}}^{(k+1)}$	Generalised direction associated with $\boldsymbol{n}^{(k,k+1)}$ at point $C_{k,k+1}$ of $(k+1)$-*th* mega-voussoir
i, j	Counters of arch voussoirs and joints, respectively
I_n, I_t	Normal and tangential impulsive reactions
k	Counter of mega-voussoirs in the arch mechanism
$k_s(t)$	Acceleration of the ground motion in g units
$\boldsymbol{n}^{(k,k+1)}$	Unit vector lined with contact and rotation centres of k-*th* and $(k+1)$-*th* mega-voussoirs
$\mathbf{N}_P$	Gradient operator of the position of point P
$\boldsymbol{N}_{P,n}, \boldsymbol{N}_{P,t}$	Normal and tangential vectors of the gradient operator $\mathbf{N}_P$
$\dot{\boldsymbol{N}}_{P,n}, \dot{\boldsymbol{N}}_{P,t}$	Normal and tangential vectors of derivative of $\mathbf{N}_P$
$\boldsymbol{N}_A{}^-, \boldsymbol{N}_A{}^+$	Negative and positive generalised Coulomb's boundaries for contact at point A
$\mathbf{N}_P^{(i)}$	Gradient operator of the position of point $P^{(i)}$ belonging to i-*th* voussoir of the arch
$\boldsymbol{N}_{P,n}^{(i)}, \boldsymbol{N}_{P,t}^{(i)}$	Normal and tangential vectors of gradient operator $\mathbf{N}_P^{(i)}$ of i-*th* voussoir
(O, x, y)	Reference system fixed on boundary Γ
P	Generic point of the block
Q	Centre of contact for the block
$P_j^{(i)}, Q_j^{(i+1)}$	Antagonist and candidate contact points at j-*th* joint of the arch
$r_{P,t}$	Tangential position of point P
$\dot{\boldsymbol{r}}_P$	Velocity of point P
$\dot{r}_{P,n}, \dot{r}_{P,t}$	Normal and tangential velocities of point P
$\ddot{r}_{P,n}, \ddot{r}_{P,t}$	Normal and tangential accelerations of point P
$\dot{r}_{Q,n}^{+}, \dot{r}_{Q,t}^{+}$	Post-impact normal and tangential velocities of point Q
$\dot{\boldsymbol{r}}_P^{(i)}, \dot{\boldsymbol{r}}_Q^{(i)}$	Velocity of points $P^{(i)}$ and $Q^{(i)}$ belonging to i-*th* voussoir of the arch
$\dot{r}_{P,n_j^{(i)}}^{(i)}, \dot{r}_{Q,n_j^{(i)}}^{(i+1)}$	Normal velocity of antagonist $P^{(i)}$ and candidate $Q^{(i+1)}$ points in the system $(\boldsymbol{t}_j^{(i)}, \boldsymbol{n}_j^{(i)})$
R_n, R_t	Normal and tangential reactions at contact point
$\boldsymbol{R}^{(i,i+1)}$	Reaction transmitted by i-*th* to $(i+1)$-*th* voussoir in $(\boldsymbol{t}_o, \boldsymbol{n}_o)$
$R_n^{(i,i+1)}, R_t^{(i,i+1)}$	Normal and tangential reactions transmitted by i-*th* to $(i+1)$-*th* voussoir in $(\boldsymbol{t}_o, \boldsymbol{n}_o)$
$R_{n_j^{(i)}}^{(i,i+1)}, R_{t_j^{(i)}}^{(i,i+1)}$	Normal and tangential reactions transmitted by i-*th* to $(i+1)$-*th* voussoir in $(\boldsymbol{t}_j^{(i)}, \boldsymbol{n}_j^{(i)})$
$\boldsymbol{S}$	Generalised force active on the block
$\boldsymbol{S}^{\pi}$	Generalised active force in plane π
$\boldsymbol{S}^{(i)}$	Generalised force active on the i-*th* voussoir
t	Time instant
$(\boldsymbol{t}, \boldsymbol{n}), (\boldsymbol{t}_o, \boldsymbol{n}_o)$	Unit vectors associated with system (O, x, y) for the block and arch, respectively

$(\boldsymbol{t}_j^{(i)}, \boldsymbol{n}_j^{(i)})$	Local unit vectors system associated with *i-th* voussoir at *j-th* joint
$\overline{\dot{\boldsymbol{u}}}$	Generalised admissible velocity of the block
$\dot{\boldsymbol{u}}, \ddot{\boldsymbol{u}}$	Generalised velocity and acceleration of the block
$\overline{\dot{\boldsymbol{u}}^{+}}$	Generalised admissible post-impact velocity
$\dot{\boldsymbol{u}}^{-}, \dot{\boldsymbol{u}}^{+}$	Generalised pre-impact and post-impact velocities
$\dot{\boldsymbol{u}}_{\mathrm{A}}$	Mechanism with contact at point A
$\ddot{\boldsymbol{u}}_{\mathrm{A}}$	Generalised acceleration with contact at point A
$\ddot{\boldsymbol{u}}_{\mathrm{A,n}}, \ddot{\boldsymbol{u}}_{\mathrm{A,t}}$	Normal and tangential generalised accelerations in plane π for contact at point A
$\overline{\dot{\boldsymbol{u}}^{(i)}}$	Generalised admissible velocity of *i-th* voussoir
$\dot{\boldsymbol{u}}^{(i)}, \ddot{\boldsymbol{u}}^{(i)}$	Generalised velocity and acceleration of *i-th* voussoir
$\ddot{x}_{\mathrm{O}}$	Acceleration of ground motion
$\Delta\dot{\boldsymbol{u}}$	Generalised velocity variation
Γ	Boundary of the rigid ground
μ	Friction coefficient
π	Plane to which $\boldsymbol{N}_{\mathrm{A,n}}$ and $\boldsymbol{N}_{\mathrm{A,t}}$ belong for $\dot{r}_{\mathrm{A,t}}$ equal to zero
$\pi\,*$	Plane to which $\boldsymbol{N}_{\mathrm{A,n}}$ and $\boldsymbol{N}_{\mathrm{A,t}}$ belong for $\dot{r}_{\mathrm{A,t}}$ different from zero
$\pi^{(2)}$	Plane orthogonal to mechanism $\dot{u}^{(2)}$ of the second mega-voussoir
$\boldsymbol{\Psi}_{\mathrm{C}_{1,2}}^{(1,2)}$	Generalised reaction transmitted by first to second mega-voussoir at $C_{1,2}$
$\boldsymbol{\Phi}_{\mathrm{P},n_j^{(i)}}^{(i)}, \boldsymbol{\Phi}_{\mathrm{P},t_j^{(i)}}^{(i)}$	Local generalised normal and tangential reactions at point $P^{(i)}$ of *i-th* voussoir
$\boldsymbol{\Psi}_{\mathrm{P}}^{(i,i+1)}$	Generalised reaction transmitted at point $P^{(i)}$ by *i-th* to $(i+1)$-*th* voussoir
$\boldsymbol{\Psi}_{\mathrm{Q,n}}, \boldsymbol{\Psi}_{\mathrm{Q,t}}$	Generalised normal and tangential reactions at point Q
$\boldsymbol{\Psi}_{\mathrm{Q,t}}^{*}$	Generic generalised reaction belonging to Coulomb's cone
$\boldsymbol{\Xi}_{\mathrm{Q,n}}, \boldsymbol{\Xi}_{\mathrm{Q,t}}$	Generalised normal and tangential impulses

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On the Statics of the Dome of the Basilica of S. Maria Assunta in Carignano, Genoa

Andrea Bacigalupo, Antonio Brencich, and Luigi Gambarotta

Abstract The paper deals with the dome of the Basilica of S. Maria Assunta in Carignano in Genoa, designed by Galeazzo Alessi and built in the sixteenth century, for which meridian cracking, rather common in masonry domes, requires the assessment of the dome. In order to set a general procedure for the assessment this structures, limit analysis approaches are here discussed and compared. On the basis of classic limit analysis, local (dome only) and global (dome-drum system) collapse mechanisms are considered considering the different behaviour of several structural elements (lantern, shells of the dome, drum, colonnade). A static (safe theorem) and a kinematic approach are applied to the structure by means of equilibrium limit conditions and kinematically admissible collapse mechanisms. Comparisons between the obtained results are carried out so as to: (i) discuss a general approach to the assessment of dome-drum systems based on both numerical tools and standard limit analyses approaches; (ii) provide a first glance in the assessment of the dome.

Keywords Masonry dome • Assessment • Limit analysis • Collapse mechanism • Safe theorem • Galeazzo Alessi • Kinematic procedures

1 Introduction

The large masonry domes, the dome of the Pantheon in Rome being the most famous example, are amazing structures coming from the past (Cowan 1977a, b, 1981; Di Pasquale 1996; Huerta 2001, 2008). It is often believed that ancient structures originate from experience, that is, through a trial and error evolution. This is not true, as can be recognized from the early (and rather detailed) studies on the safety of St. Peter's dome in Rome (Le Seur et al. 1743; Poleni 1748), where the basic concepts of equilibrium such as the inverted chain originated by Robert Hooke (Heyman 1988) and limit analysis are clearly recognized (Como 1997, 2010, 2013).

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The case of S. Peter's dome puts forward a typical feature of masonry domes: a rather diffused cracking along meridian lines, much more pronounced close to the drum, and sometimes also in the drum, disappearing as the crack approaches either the upper part of the dome, for solid domes, or the lantern in the other cases (Di Pasquale 1979; Heyman 1988; Como 2010, 2013). A serious concern regarding dome safety is the direct consequence of the crack pattern, which can sometimes be discerned from the nave of the church, as in the case of St. Peter's in Rome.

Difficulties in structural analysis of masonry domes come from the geometric complexity (the lack of axial symmetry), and from the material response (its intrinsic non-linear and non-isotropic response). As discussed in Huerta (2008), the structural analysis may rely either on non-linear and incremental finite element method (FEM) procedures or on limit analysis, for which the solution does not depend on the large number of mechanical properties required by FEM codes and difficult to identify.

The equilibrium approach (Huerta 2001), based on the "safe theorem" (Heyman 1966), was extended to masonry domes in (Heyman 1966, 1988) assuming that masonry: (i) is a no-tension material; (ii) has an infinite compressive strength; (iii) no shear collapse mechanism is activated. Starting from domes where the hoop stresses have been assumed to vanish, recent works take into account that the crown of the dome is transversally compressed (Heyman 1988; Oppenheim et al. 1989; Zessin et al. 2010). In this framework, optimum design of no-tension domes (Farshad 1977; Pesciullesi et al. 1997) and other equilibrium methods relying on the funicular analysis (O'Dwyer 1999; Andreu et al. 2007; Baratta and Corbi 2010; Fraternali 2010; Goshima et al. 2011) have been proposed. Usually, these approaches assume the dome to be axisymmetric and do not take into account the drum as a structure but only as an external support to the dome. Non axisymmetric domes have been considered in (Lucchesi et al. 2007) on the basis of a no-tension constitutive elastic model for masonry (Del Piero 1989; Como 1992). All these approaches neglect the actual texture of masonry, that is, the orthotropic directions of the material, so that it is always necessary to verify that the results are compatible with the actual brickwork texture (which is not easy to identify). Such a limit has been recently cleared up in (Milani et al. 2008), which takes into account the effect of masonry texture by means of proper homogenization techniques of the repetitive cell of solid clay brickwork.

In this present chapter we present the results of both a kinematic and a static approach of limit analysis of the dome of the Basilica of S. Maria Assunta in Carignano in Genoa, in order to verify the applicability of such procedures to complex geometries and to obtain information about the mechanical response of the dome-drum system and of some of its parts. The no-tension model assuming no shear collapse is assumed, along with the hypothesis of vanishing hoop stresses (see for instance, Oppenheim et al. 1989), due to the large meridian cracking. The analysis aims at identifying the effect of the geometric shape on the equilibrium of the dome and its dependence on the material strength under the dead weight only since it is by far the largest load acting on a dome.

Due to the geometric complexity of the drum, only axisymmetric collapse mechanism could be tested for the kinematic theorem, while the safe theorem has been applied taking into account as well a reduced compressive strength for masonry, assuming a 1/8 scale model of the dome.

2 The Basilica, The Dome, The Drum

The Basilica of S. Maria of Carignano (Fig. 1) was built between 1552 and 1602 in Genoa after a design by Galeazzo Alessi (Ghia 1999, 2010). The dome consists of two concentric ogival shells connected in their upper parts by the ring at the base of the lantern, supported by the drum, and connected by flying buttresses, steel chains and two helicoidal stairs, which represents a typical Renaissance dome very similar to the small temple by Bramante in S. Peter in Montorio, Rome. The interest in the monument is due to the widespread cracking in the dome and in the drum that was recorded approximately a century ago (De Gasperi et al. 1907) and has increased since then, raising serious concern about the building's safety. It is not known whether the cracks appeared shortly before 1907, or formed some years after the completion of the dome. Archival research has demonstrated that no structural analysis was performed on the basilica in either ancient times or in recent years, with the exception of Baldacci and De Maestri (1975).

Fig. 1 Basilica of S. Maria of Carignano

2.1 History of the Church

The history of the basilica (Ghia 1999, 2010) dates back to 16 October 1481, when Bendinelli I Sauli (d. 1482), head of one of the outstanding families of the city of Genoa, in his will leaves to the family 2410 Liras for the building of an important church; the will stipulated that only the interest on this sum only could be used to this aim, and not until 60 years after his death. The design of the church was entrusted to Galeazzo Alessi on 7 September 1549. The friendly collaboration of the Sauli family with Alessi—unusual for the times—lasted till 1570, when Alessi retired, and gave the architect almost total freedom in the design of the Basilica. The basilica was located on the Carignano Hill, where the Sauli family already had some properties. In those times the site lay outside the city, but nowadays it is in the city centre. The history of the church can be divided into several periods:

1549–1551: design, purchase of the areas, preparation of the building site
1552–1570: building of the most part of the church, except for the two bell towers and the dome
1570–1602: building of the bell towers and of the dome
1602–1900: completion of minor works and ordinary refurbishment
1900–present: concern for the crack pattern in the dome, WWII damage and related repairs.

Here is a brief list of the most relevant events:

10 March 1552: excavations for the foundations start. According to the account books, the excavations reached the depth of 8–10 m. in the north side and 14 m. almost uniform on the south side. Foundations completed by December 1558.
1561: the roofs are initiated, supported by a series of parallel walls supported by the vaults of the Basilica.
March 1563: the outer pillars are completed (Fig. 2).
1564–1565: financial problems almost stop the works.
1566–1567: large part of the roofs, the drum and the dome are still unbuilt. Alessi gives details of the drum, most of which were lost.
1568–1570: a dramatic plague in the city substantially stops the works. However, by the end of 1570 the drum is completed. In 1570Alessi retires.
30 December 1572: Galeazzo Alessi dies. By that time part of the roofs, the dome and the four bell towers were still unbuilt.
12 January 1574: building of the first bell tower is decided. Comparing the original drawing by Alessi (Fig. 3), to the actual bell tower (Fig. 1), we can deduce that financial problems forced the family to simplify the original decorations designed by Alessi. The original four bell towers were reduced to the two of the main façade are simplified.

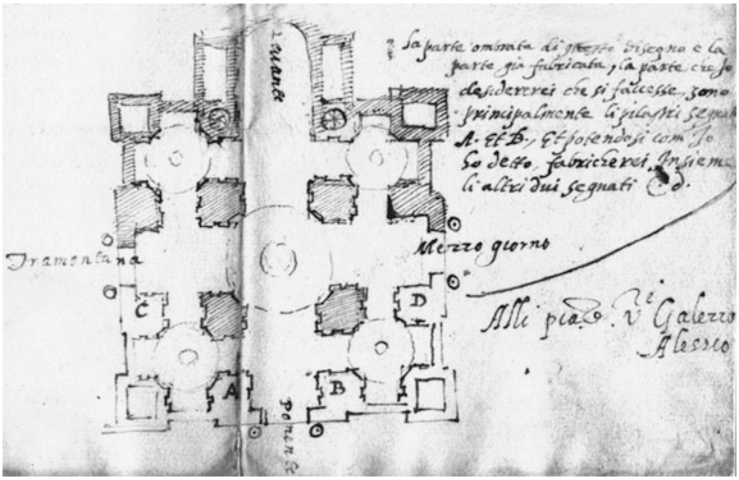

Fig. 2 Original drawing by Alessi showing the large pillars to be completed in 1562–1563 (Ghia 2010)

1574–1590: decorations and the second bell tower absorb all the financial capability of the Sauli family.

15 September 1594: the city council forces the Sauli family to complete the church with the last element, the dome, which is thus started 25 years after the last contribution by Alessi. This shows that any settlement of the foundations and creep response of the structures supporting the dome should not be considered as causes of the crack pattern.

Late 1595–spring 1596: the dome is started

1602: the dome is complete

When compared to similar structures by Alessi—above all Santa Maria degli Angeli in Assisi—the dome (solid clay brickwork) and the drum (stone masonry) clearly show his extraordinary design capacity: the circular drum is supported by four pendentives and arches on four pillars of irregular hexagonal shape (Fig. 4).

The dome consists of two concentric shells: the inner one, with a diameter of about 14 m and thickness of 1 m, is the main support of the lantern (diameter 5 m; height 9 m); the outer shell, 17.6 m in diameter, is much lighter, 25 to 30 cm thick and apparently carries only the covering. The two shells are connected by five elements: (i) the drum at the base through a stiff ring with a narrow internal inspection tunnel; (ii) the ring sustaining the lantern at the top; (iii) a dozen small, irregularly spaced flying buttresses; (iv) the stairs reaching the lantern from the drum and running in between the two shells along a helical route; (v) some steel chains (nowadays broken) of unclear origin.

Fig. 3 Galeazzo Alessi (attributed), original drawing, original bell tower (Ghia 2010)

The drum consists of two cylinders: the inner one, 1 m thick, is a cylinder with eight symmetrically distributed windows; the outer one, 20 m in diameter, is actually a colonnade, separated from the inner cylinder by an ambulatory, consisting of eight pairs of pillars connected one to the other by deep round arches; the two concentric elements are locally connected by radial masonry walls (Figs. 4 and 5). Further information about the history of the basilica and its geometry can be found in (Brencich et al. 2014).

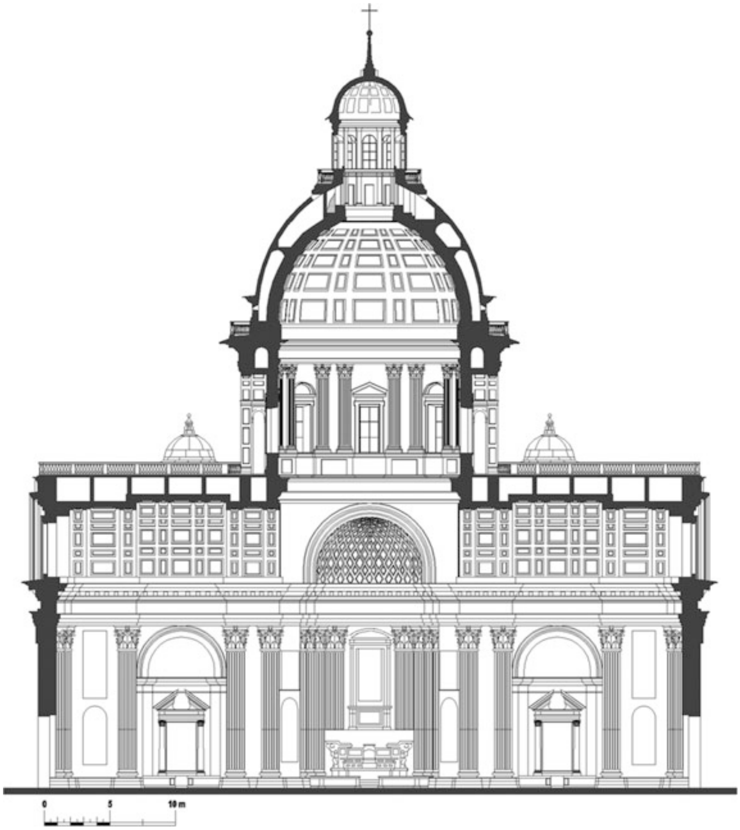

Fig. 4 North-south (approx.) section of the basilica

2.2 The Crack Pattern in the Dome

Several meridian cracks affect the inner dome, some of which extended up to the base of the lantern (upper ring), as in similar cases (Di Pasquale 1979; Heyman 1988; Como 2010, 2013), while many cracks originate in the drum (Figs. 6 and 7). This latter issue is quite unique to this dome and is not commonly reported for other similar cases. The crack pattern is not recent, since monitoring of the cracking started in approximately 1907 according to a detailed technical report on the crack pattern and on the safety of the structure (De Gasperi et al. 1907), where a detailed geometric description of the dome is provided, including the crack pattern, the irregular distribution of flying buttresses and the steel chains connecting the outer to the inner dome. Actually we know that the crack pattern in 1907 consisted in eight major cracks, one per window (no information is available on the crack opening), and the steel chains connecting the inner dome to the outer dome were all broken. Assuming that the cracks in the dome were dangerous for the dome safety, the

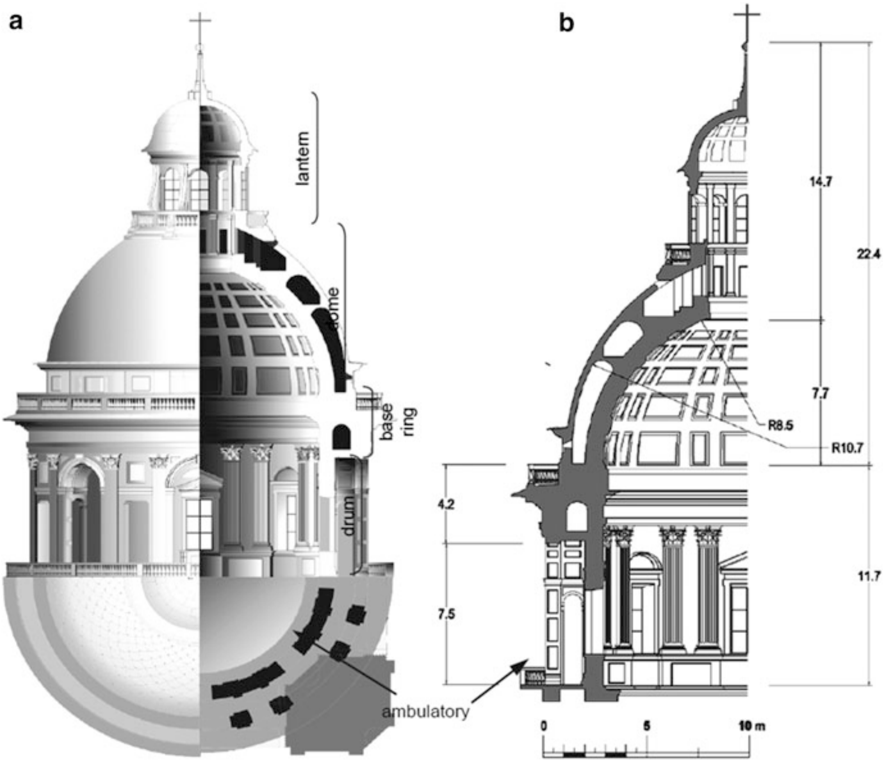

Fig. 5 (**a**) View, plan and (**b**) sections of the drum and dome

engineers suggested substituting the corroded chains, placing one confining chain at the base of inner dome on its outer surface and a second confining chain half the way to the top of the dome and connected to the first one by steel connections, and plastering all the cracks.

According to a handwritten note on the document, only the hooping tie at the base of the inner dome was done through a steel plate 250 × 10 mm hosted in the inspection tunnel at top of the drum. Sometime between 1907 and the present, it appears that some plastering of the cracks was performed.

The last analysis of the crack pattern of the dome dates back to 1975 (Baldacci and De Maestri 1975), when a detailed survey was performed and some structural analyses, including pioneering FEM analyses, were performed.

The crack pattern (Figs. 6 and 7) consists of: meridian cracks through the thickness of the inner dome from the base of the lantern to the drum, and cracks limited to the surface of the shell widespread across the dome except for the upper portion of the shell, which remains undamaged. There are essentially four penetrating large cracks (close to windows 1, 3, 5 and 7) which seem to be somehow correlated to the brick arches connecting the crowns. The continuity of some cracks

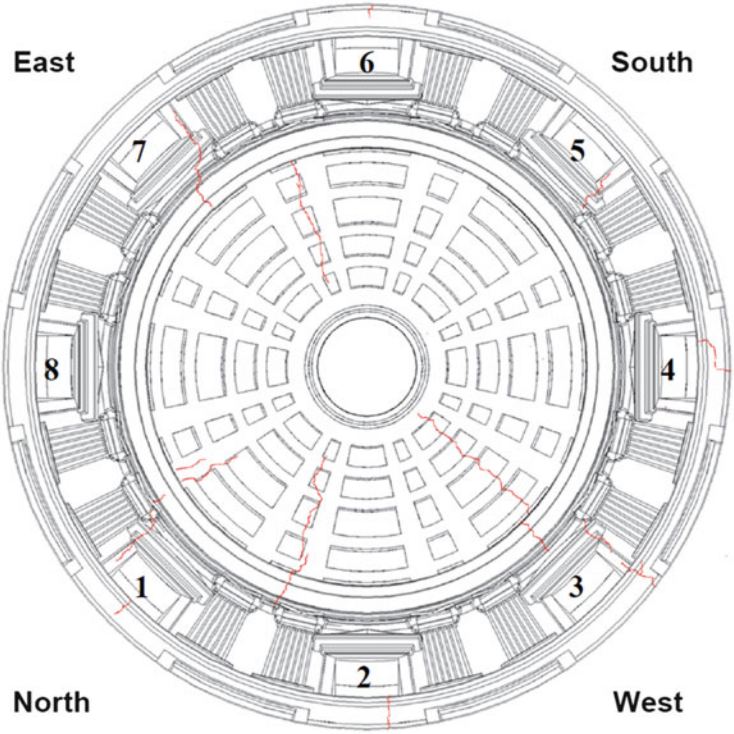

Fig. 6 Major cracks on the intrados of the (*inner*) dome. View from the floor of the basilica

from the drum to the base of the lantern suggests that these elements are not independent each other.

According to the available documents, it seems that cracks in the inner shell developed in some unidentified period between 1907 and 1975, while they seem stable from 1975 until now, which suggests that the increase of the crack pattern from 1907 is not be independent from the damage inflicted during World War II.

The crucial issue now—as for all the cracked domes—is whether the crack pattern affects the structural safety of the structure, which requires the causes of cracking to be identified or, at least, to be rationally conjectured. Material degradation, due to environmental actions, is another issue that the assessment of the dome needs to take into account. In what follows, some hints are discussed on the basis of classic limit analysis approaches, aiming both at discussing the dome safety and at identifying its mechanical response and the role of its elements.

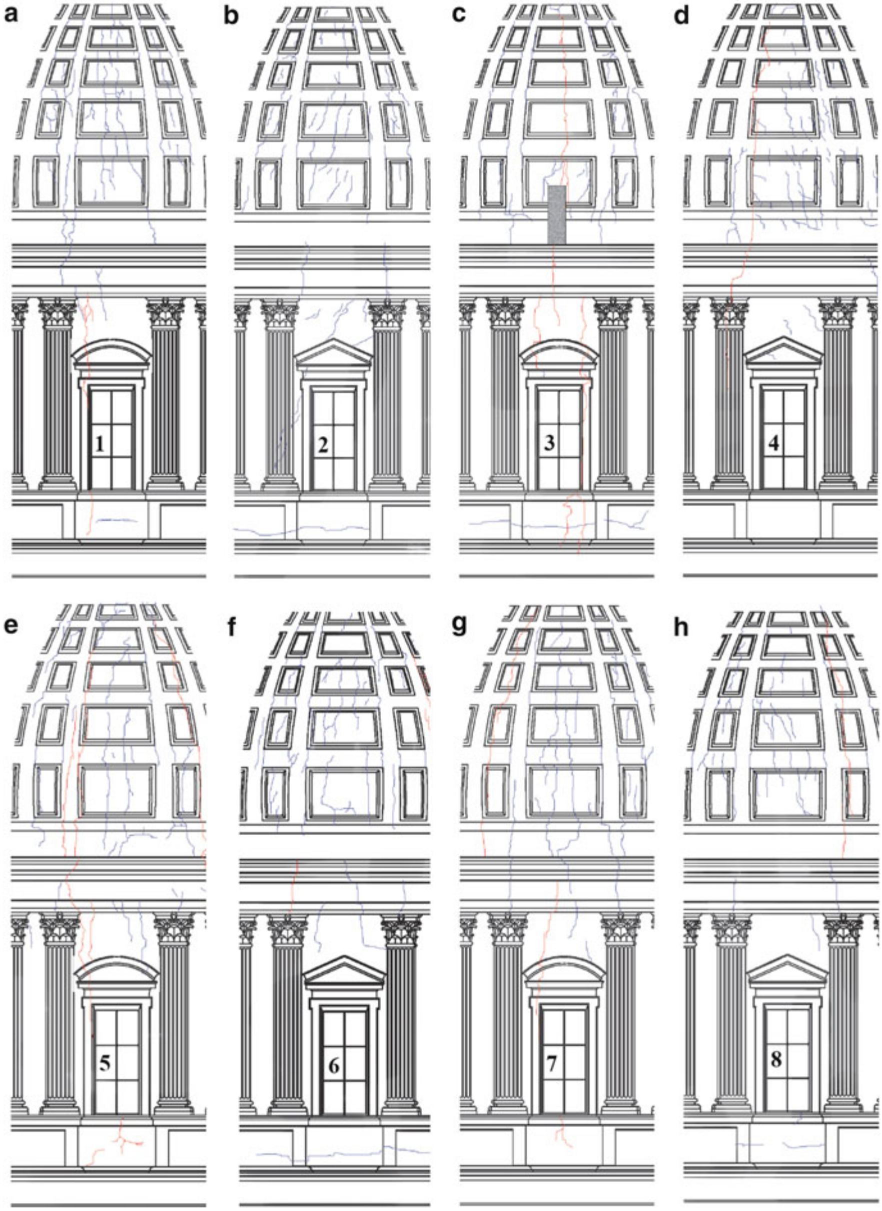

Fig. 7 Major (*red*) and minor (*blue*) cracks on the intrados of the drum and of the *inner dome*. Windows are numbered 1–8 as in Fig. 6

3 Simplified Limit Analysis of the Dome

The equilibrium of the dome-drum system, and the collapse mechanisms, are studied by limit analysis approaches that, despite their limits, may help and guide more detailed (but also computationally demanding) numerical methods. The applicability of limit analysis to masonry structures has already been discussed in (Kooharian 1952; Heyman 1966; Como 1992; Del Piero 1989) and is commonly accepted as a tool for the structural analysis of masonry structures. In what follows, only gravity loads are considered since snow and wind pressure are negligible in comparison to the self-weight of the dome. Further, seismic and thermal loading are not considered, since they require specific procedures that are outside the scope of this paper

Masonry is assumed to be a no-tension material: no shear collapse is activated (Heyman 1966) and the compressive strength is assumed either unbounded or is given a finite value. This latter constitutive model is given only to the columns of the outer colonnade of the drum, where material degradation is more likely, thus affecting the residual life of the dome. Under these assumptions, limit analysis may provide information on the existence of equilibrium conditions for the cracked dome-drum system, not the safety margin of the structure.

This approach appears to be quite simplified if compared to the detailed results provided by FEM models, but it must be noted that FEM models require a very detailed knowledge of the structure (internal geometry, mechanical parameters for all the materials, internal texture, building sequences; see Bacigalupo and Gambarotta 2012) which are not always available and, in some cases, impossible to obtain.

One of the simplifying assumptions requires the inner and outer shells to be independent from the base up to the lantern, ignoring the interaction between the shells due to the connecting flying buttresses and helical stairs and neglecting the shear transfer that may still take place across open cracks.

3.1 Collapse Mechanisms

For the kinematic approach to be applied different local and global collapse mechanisms need to be considered in order to investigate the dome-drum interaction, according to Como (1997) with reference to the case of St. Peter's in Rome. Collapse mechanisms are identified assuming meridian cracking, that is, the kinematic counterpart of the static hypothesis of vanishing hoop stresses $\sigma_h = 0$. Furthermore, we assume the dome-drum system to be axisymmetric, the arches above the ambulatory being included in the base ring of the dome by proper averaging of the cross section. Such an approximation simplifies the geometric model but makes it impossible to identify the path of the internal forces in the outer colonnade once the thrust line is identified in the averaged section. The axial-symmetric and the meridian cracking approximations allow a segment (an "orange slice") of the dome to be considered as the reference unit. Compatibility in the compressed parts require the hoop strain rate $\dot{\varepsilon}_h = \dot{v}_r/R$ to be positive, $\dot{v}_r$ being the radial velocity at the end of radius R. As a consequence only positive radial displacements (i.e., directed outside the dome) are admissible in a collapse mechanism, $\dot{v}_r \geq 0$. The upper dot for displacement and strain rates is omitted from now on for simplicity.

Under these assumptions, a collapse mechanism for the reference slice of the dome is obtained introducing enough hinges in the slice to reduce it to a mechanism. The concept of no-tension hinge in a dome slice is the extension of the concept of plastic hinge for arches (Heyman 1988): hinges are located in those sections in which relative rotations along a line (instead of a point, as for arches) are located. Plastic deformation may take place in the compressed part of the hinge due to the eccentricity of the axial thrust, even though the traditional approach assumed that the collapse mechanism is activated far from material compressive strength, thus plastic deformation is not expected or is expected to play a minor role.

If we call $\mathbf{b}$ the actions on the dome (i.e., gravity forces per unit volume, 17 kN/m^3), a kinematically admissible displacement rate field $\mathbf{v}$ allows the definition of the external power P_{ext} to be given as:

$$P_{ext} = \int_B \mathbf{b} \cdot \mathbf{v}\, dV = \int_{B^-} \mathbf{b} \cdot \mathbf{v}^-\, dV + \int_{B^+} \mathbf{b} \cdot \mathbf{v}^+\, dV = P^+_{ext} + P^-_{ext} \geq 0, \quad (1)$$

where B^- and B^+ are the parts of the structure B where the displacement rate field $\mathbf{v}$ acts equal ($\mathbf{b} \cdot \mathbf{v}^- \geq 0$, $\mathbf{v}^-$ downwards velocity, positive power) and opposite ($\mathbf{b} \cdot \mathbf{v}^+ \leq 0$, $\mathbf{v}^+$ upwards velocity, negative power) to the direction of the body forces $\mathbf{b}$, respectively. Defining P_{int} the power of internal forces due to the plastic deformation rates in the steel chain and in the compressed parts of the hinges, the kinematically admissible displacement rate field $\mathbf{v}$ turns out to be the collapse mechanism if $P_{ext} = P_{int}$. Conversely, the collapse is not attained for all the kinematically admissible displacement rates for which:

$$\frac{P_{int} - P^-_{ext}}{P^+_{ext}} = \frac{P_{res}}{P_{act}} = \eta > 1, \quad (2)$$

where the positive contribution of the external power P^+_{ext} is the *active power* P_{act} and the difference $P_{int} - P^-_{ext}$, which is an algebraic operation that sums up the stabilizing contributions of the resisting and internal forces, is called the *resisting power* P_{res}. The ratio η can be considered a safety parameter that need to exceed unity for structural safety to be guaranteed (Brencich et al. 2001; Gusella et al. 2012). Nevertheless, η is not a safety margin since Eq. (2) is simply a balance between stabilizing and non-stabilizing powers that does not refer to the ultimate load or to the actual collapse mechanism. In fact, we can expect that the larger the ratio η, the higher the structural safety, since large values for η mean that the specific displacement rate field considered is far from the actual one, but no direct correlation between η and a safety margin can be established since the actual collapse mechanism might not be (and in general is not) the mechanism considered in the kinematic approach, and might depend greatly on the material compressive strength.

If the material is assumed to be no-tension (no plastic deformation is included in the model), as is usual in applications of kinematic limit analysis, the power P_{int} of the internal forces vanishes and Eq. (2) results in a balance between the positive and negative powers of the external forces (dead loads in this case). In practical applications, Eq. (2) can be verified on a number of selected collapse mechanisms only, so that structural safety can only be conjectured from this kind of analysis.

Figures 8 and 9 show two kinematically admissible collapse mechanisms corresponding, respectively, to a local mechanism involving the two domes only and a global mechanism involving both the domes and the drum. For these mechanisms to be kinematically admissible two conditions need to be respected: (i) equal vertical displacement rates of the upper parts of the two shells due to compatibility conditions at the base of the lantern; (ii) local mechanisms involving only the outer shell are not possible since the two shells of the dome cannot approach one to another due to the flying buttresses and the helicoidal stairs in-between the two shells and due to compatibility conditions in the hoop direction. Among all the mechanisms related to the dome only, the one of Fig. 8 makes η 2; this suggests that, in the framework of kinematic limit analysis and under its assumptions, the dome should be safe. Since kinematic limit analysis is an upper bound approach, and due to the strong assumptions that are needed for limit analysis to be applicable to masonry structures, this value is not exactly comfortable.

The global mechanisms of Fig. 9a, involving both the shells and the drum, assume the vertical displacement rates of the lantern H and of the drum (ring D), named δ_1 and δ_2 respectively, as ruling quantities. The relative rotation centres between the domes-drum and drum-base are assumed exactly in the corners of the lower pillars (Fig. 9b), which is possible only under the assumption of infinite compressive strength of masonry. Further, in the kinematic model the power dissipated by the hooping circumferential tie (inside the tunnel at the base of the dome, resulting from the 1907 investigations) has been taken into account. For the steel chain is has been assumed $f_y = 100$ MPa, contributing to the internal dissipated power as $P_{\text{int}} = f_y \dot{\varepsilon}_h A_c$, with $A_c = 250\ \text{mm}^2$ and $\dot{\varepsilon}_h$ the hoop strain rate in the tie.

The assumption of rigid body motion for blocks E (drum) and F (external colonnade) maximizes the vertical uplift of the drum D and of most of the other parts of the dome slice. Thus the resisting power of the dead loads and the ratio η are maximized, the latter reaching a value not lower than 7. It seems that we can conclude that a structural failure due to dead loads has to be excluded but the assumption of infinite compressive strength for masonry is quite severe and unrealistic. For these reasons, another mechanism derived from that of Fig. 9 is considered, differing in that the hinge at the base of the drum and of the colonnade are located not on a corner but at 1/4 of the element width (Fig. 9c); the global mechanism remains that of Fig. 9a. Applying Eq. (2), a minimum value is obtained

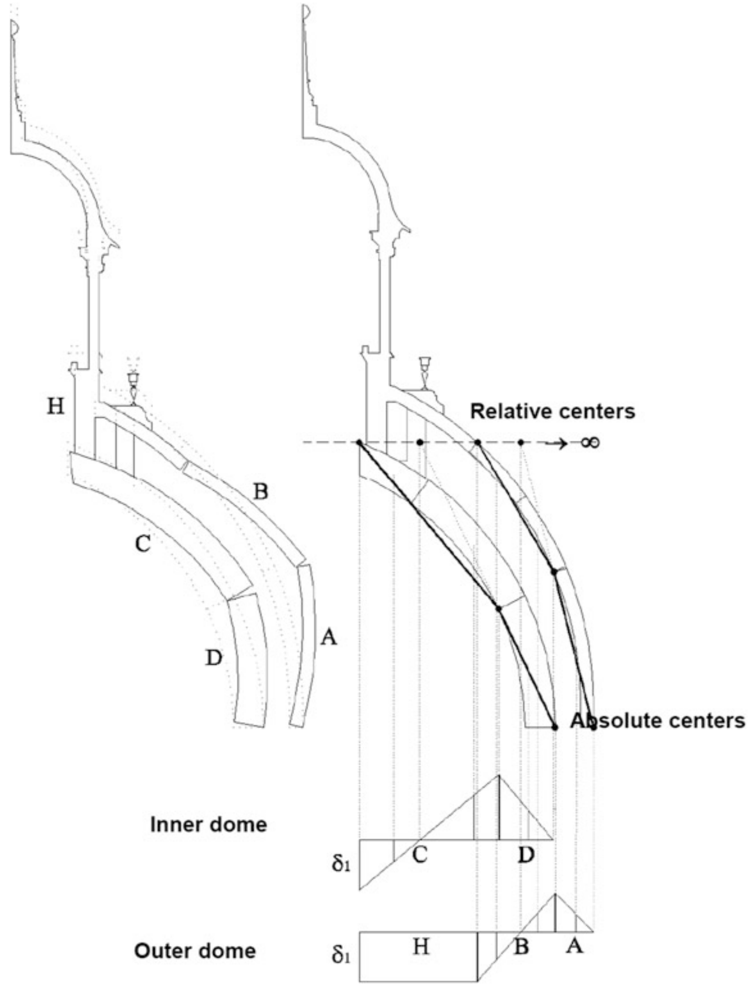

Fig. 8 Kinematically admissible mechanisms for the dome with hinges in both the shells. Diagrams represent the vertical component of the velocity field

for η 2.15. Even though the structure is still safe, such a great change η shows that the safety of the dome-drum system strongly depends on two factors: (i) the dead weight of the drum, which provides a stabilising contribution; (ii) the compressive strength of masonry of the external columns, which accounts for the position of the hinges, which appears to be of critical importance since the pillars are liable to environmental degradation due to exposure to rain and frost.

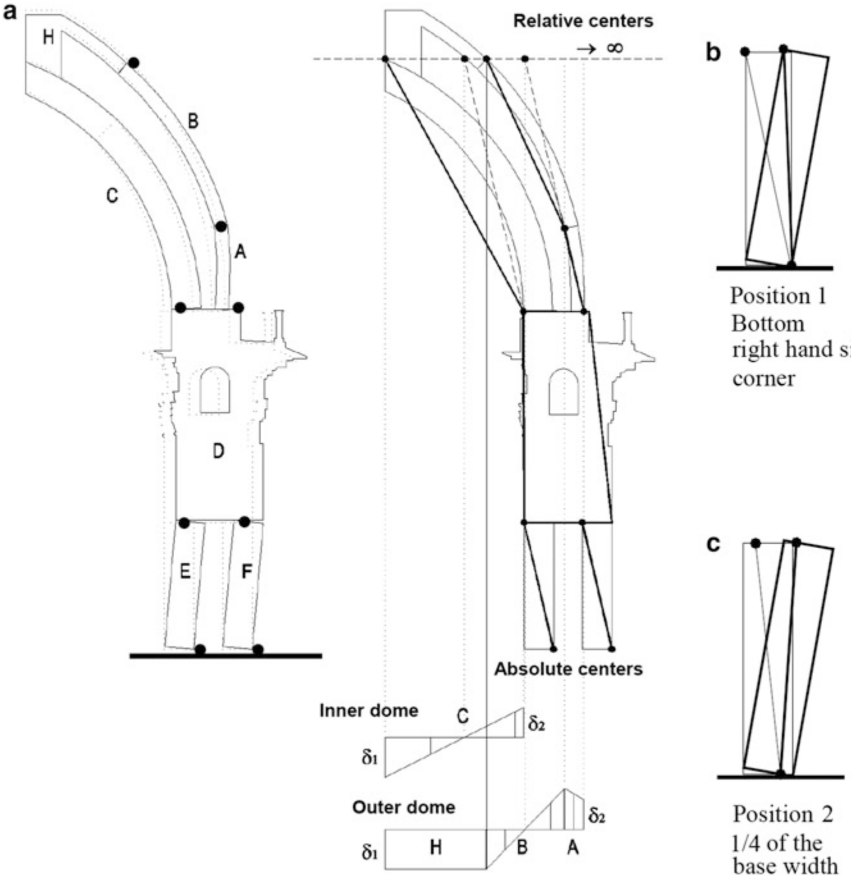

Fig. 9 (**a**) Kinematically admissible global mechanism for the dome-drum system; (**b**) and (**c**) different positions considered for the hinges at the base of the pillars

3.2 Admissible Equilibrium States

The safe theorem guarantees the structure to be safe provided that a statically admissible stress field can be found (Heyman 1988), that is, if an equilibrium configuration, coherent with the material constitutive load, can be found. In this case as well, only an estimate (but conservative, in this case) of the safety margin can be performed. This approach, which is admissible provided that frictional sliding is not activated (D'Ayala and Tomasoni 2011), is again applied assuming the constitutive model of a no-tension material with unbounded compressive strength for masonry. In the case of the dome and drum of the Basilica of S. Maria Assunta in Carignano the set of equilibrated forces is complex due to the geometric complexity of the structure and to the large number of restraints of the system.

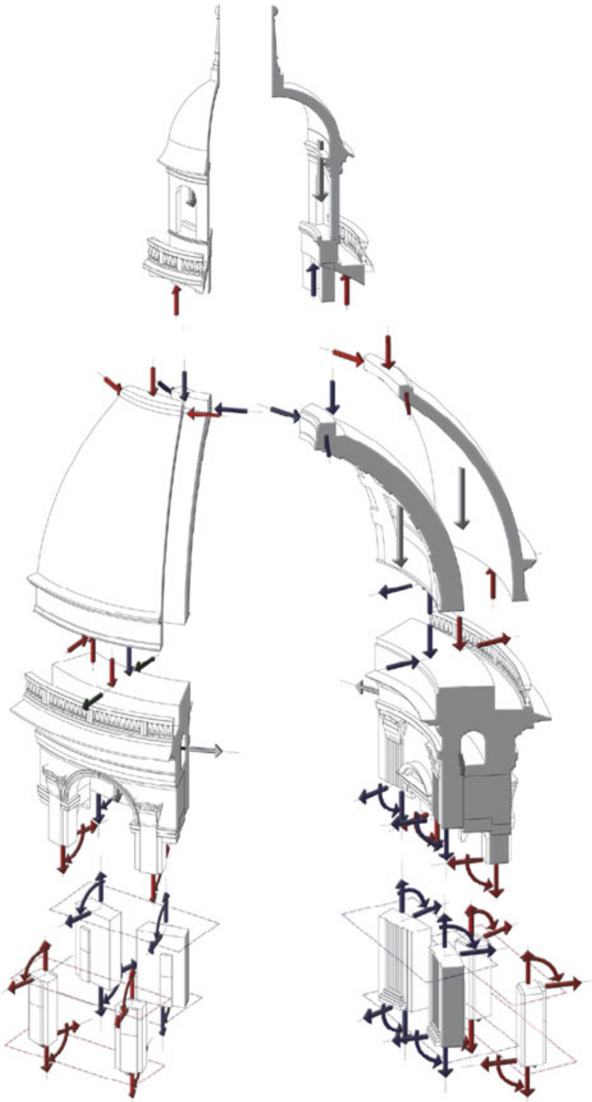

Fig. 10 1/8 slice of the dome, blocks into which it has been divided and internal forces

The whole set of external and internal forces that need to be considered for the equilibrium of the dome-drum system is represented in Fig. 10. The regularity of the openings and of the colonnade allow consideration of 1/8 of the dome ($\pi/4$ angle). The statically admissible distribution of internal and external forces equilibrated with the dead loads, assume vanishing hoop stresses, either tensile and compressive. This latter assumption would be quite strong for solid domes but is reasonable for domes with an oculus, such as the one analysed in this paper, where the weight of the lantern makes the transversally compressed part of the dome reduced almost to the ring at its base (Oppenheim et al. 1989).

According to the material density (17 kN/m^3) the weight of the main parts of the dome are: (i) ring at the base of the dome: $W_A = 2595$ kN; (ii) lantern: $W_L = 150$ kN; (iii) outer shell: ${W_S}^e = 553$ kN; (iv) inner shell: ${W_S}^i = 755$ kN.

The statically admissible distribution of internal and external forces, coherent with the external restraints, is built from the upper part of the dome imposing the equilibrium of the lantern on the top of the two shells (Fig. 11).

We assume that the stress state is axysimmetric; as an exception to the vanishing hoop stress assumption, we need to assume compressive force, C_i and C_e, on the top of the shells to sustain the lantern. At this point, forces W_L, W_S^e, W_S^i are known while forces $W_L^e = (1-\alpha)\, W_L$ and $W_L^i = \alpha W_L$ represent the ratio of the dead load of the

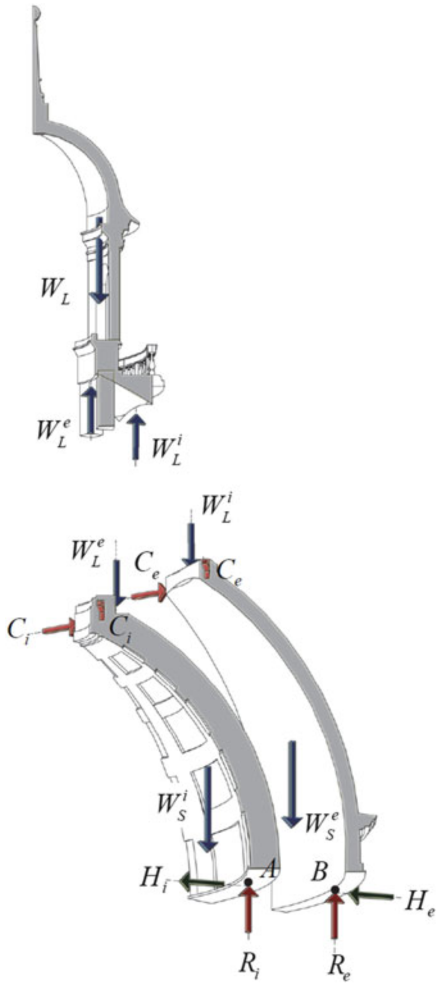

Fig. 11 Lantern and the two ogival shells of the dome

lantern sustained by the external and inner shell, respectively. The parameter α∈ [0, 1] is unknown, as are the vertical forces R_i and R_e, their application points and the horizontal forces H_i and H_e, representing the horizontal reaction of the drum on the dome.

If we impose the global equilibrium of the inner and outer shell (six independent equations and nine unknowns (α, C_i, C_e, R_i, R_e, A, B, H_i, H_e), the problem is three times over-constrained; for which α, H_i and H_e can be assumed as independent quantities. It is easy to observe that $H_a = 2C_a \sin(\pi/8) = \sqrt{1-\sqrt{2}}C_a$, $a = i, e$. The assumption of vanishing tensile strength for the material makes the thrust surface, which is a surface of revolution, lie within the meridian section of the shell. The thrust line in the shells can be found by means of a trial-and-error procedure:

i. the points at the top of the shells where the weight of the lantern is applied are assumed (Fig. 12);
ii. initial values for α, H_i and H_e are assumed and the subsequent thrust line is built;
iii. values for α, H_i and H_e are corrected in order to set the thrust line completely inside the meridian section of the shells or, at least, tangent to the internal or

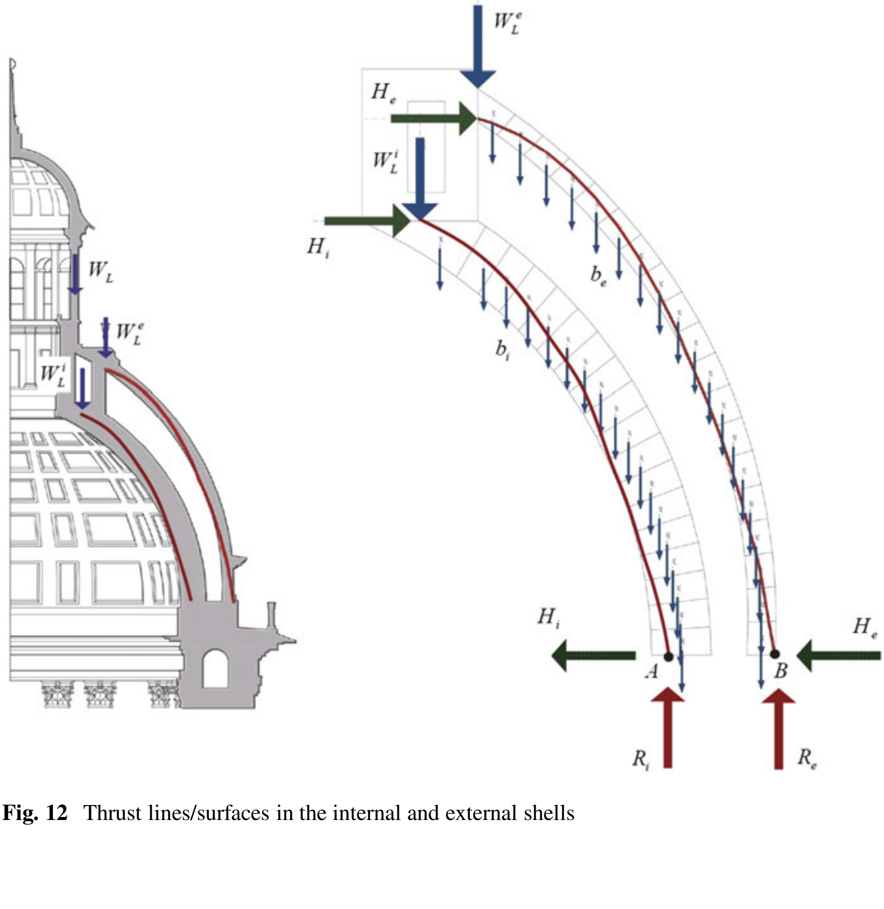

Fig. 12 Thrust lines/surfaces in the internal and external shells

external surface of the shells (Fig. 10), which can be expressed by a linear system of inequalities:

iv.

$$\mathbf{S}_d\, \mathbf{s}_d + \mathbf{f}_d \leq \underset{\sim}{0} \tag{3}$$

where $\mathbf{s}_d = \{\alpha, H_i, H_e\}$, $\mathbf{S}_d$ and $\mathbf{f}_d$ are the static matrix and the force vector of the weight of the lantern and of the dome, respectively.

The number of rows of $\mathbf{S}_d$ and $\mathbf{f}_d$ equals the number of sections considered. In Fig. 12 the thrust lines represent the case $\alpha = 0.8$, H_i =115.3 kN/rad, H_e = 61.6 kN/rad. The thrust lines are in good agreement with the collapse mechanism of Fig. 6 since the plastic hinges are located precisely where the axial thrust is tangent to the surfaces of the two shells. For the external shell, the admissible thrust line requires the horizontal force H_e to lie inside a very narrow range: $H_e \in$ [61.7 kN/rad; 61.9 kN/rad], while the admissible range for H_i is much larger: $H_i \in$ [115.3 kN/rad; 160.4 kN/rad].

The drum is first considered referring to its upper part, a ring sustaining the dome (Fig. 13) and block D of Fig. 9, inside which there is the tunnel where the steel tie has been found. The lower part of the drum and the columns of the outer colonnade are considered separately, as in Fig. 10. The equilibrium of the ring of Fig. 13 is given by the vertical R_i, R_e and horizontal forces H_i and H_e, transferred by the dome to the drum, the weight W_A of the ring, the tensile force T in the steel tie in the inner

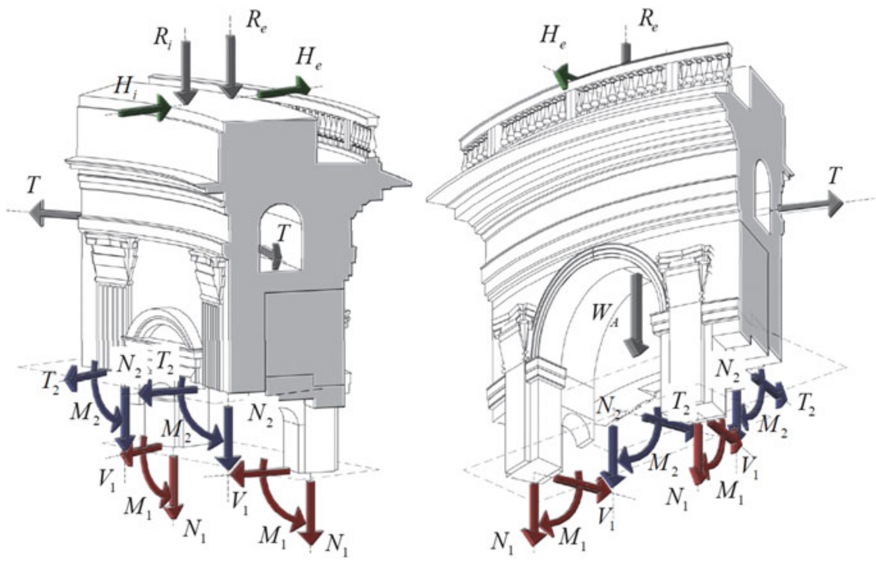

Fig. 13 Ring at the base of the dome and related forces

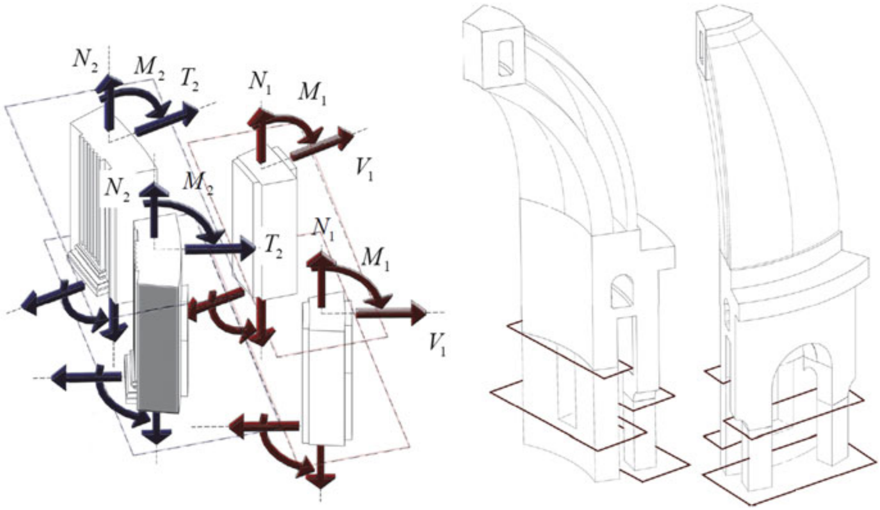

Fig. 14 (**a**, *left*) internal forces of the outer columns and of the lower part of the drum; (**b**, *right*) reference section for the lower part of the drum

tunnel and the internal forces N_1, V_1, M_1, and N_2, V_2, M_2 (Fig. 14), due to the outer pillars and to the lower part of the drum.

Equilibrium of the block of Fig. 13 provides three equations for seven unknowns (T, N_1, V_1, M_1, N_2, V_2, M_2); the tensile force in the steel tie T and the internal forces $\mathbf{s}_{c1} = \{N_1, V_1, M_1\}$ related to the outer columns (Fig. 14), are assumed as independent quantities. The internal forces at the base of the columns are linearly dependent on the internal forces on the upper section of the columns (Fig. 14a), that is, on the vector $\mathbf{s}_{c1}$. Therefore, the internal forces (N_i, M_i), $i = 1, 4$, at the top and at the base of the columns and of the drum, represented in the vector $\boldsymbol{\sigma} = \{N_1, M_1, N_2, M_2, N_3, M_3, N_4, M_4\}^T$ depend linearly on the vectors $\mathbf{s}_d$ and $\mathbf{s}_{c1}$ of the unknown forces and on the tensile force T in the steel tie:

$$\boldsymbol{\sigma} = \mathbf{A}_d \mathbf{s}_d + \mathbf{a}T + \mathbf{A}_c \mathbf{s}_{cl} + \mathbf{f}_c, \tag{4}$$

where $\mathbf{A}_d$ and $\mathbf{A}_c$ are static matrices, $\mathbf{a}$ is the static vector of the unknown forces $\mathbf{s}_d$, $\mathbf{s}_{c1}$ and the unknown tensile force in the steel tie T, vector $\mathbf{f}_c$ collecting the dead loads.

The plastic condition for the steel tie is directly expressed by inequality :

$$T \leq T_y, \tag{5}$$

with $T_y = A_c f_y, f_y = 100$ MPa and $A_c = 250\ \text{mm}^2$, internal forces have to be checked with regards to the vanishing tensile strength of the material and compressive plastic limit condition set for masonry. This latter condition is checked in the sections shown in Fig. 14b. For unbounded compressive strength, the compressive limit condition is:

$$f_1\,(N, M) = |M| - \frac{h}{2}N \leq 0, \tag{6}$$

where h is the thickness of the column or of the internal wall of the drum, both under eccentric loading. In case a finite compressive strength is given to the material, the plastic limit domain, in non-dimensional form, is described as:

$$f_2\left(\overline{N}, \overline{M}\right) = \overline{M} + 2\overline{N}\left(1 + \overline{N}\right) \leq 0, \tag{7}$$

where $\overline{\mathrm{N}} = \mathrm{N}/\mathrm{N_p}$, $\overline{\mathrm{M}} = \mathrm{M}/\mathrm{M_p}$, $\mathrm{N}_p = bh\ \mathrm{f_M}$, $\mathrm{m}_p = 1/4\ bh^2\ \mathrm{f_M}$, (Brencich et al. 2008). If the condition for plastic admissibility are combined with Eq. (1), related to the thrust surface, we obtain a series of inequalities on the independent static variables $\mathbf{s}_d$ and $\mathbf{s}_{c1}$ and on the tensile force T in the steel tie:

$$\mathbf{B}_d\ \mathbf{s}_d + \mathbf{b}T + \mathbf{B}_c\ \mathbf{s}_{cl} + \mathbf{g}_c \leq \mathbf{0}, \tag{8}$$

The feasible domain in the space of the unknown quantities {α, H_i, H_e, T, N_1, V_1, M_1} defined by Eq. (8) is a polytope and its existence as a non-void domain guarantees that equilibrium is possible and, therefore, the structure is safe in the sense provided by the safe theorem. Vice versa, if a polytope satisfying conditions (8) cannot be found, we cannot say that the structure is not safe (i.e., that equilibrium cannot be reached), and that other, more detailed models, looking for other plastically admissible states, need to be formulated.

We now assume $\alpha = 0.8$ and the minimum admissible horizontal thrust, $H_i = 115.3$ kN/rad, $H_e = 61.7$, which set the thrust surface contained inside the thickness of the outer and inner shell simultaneously, and estimate the effect of tensile force T on the polytope of the feasible domains in the reduced space of forces $\{N_1, V_1, M_1\}$ describing all the equilibrium states (N_1, V_1, M_1). Figures 15 and 16 show a brief description of the polytopes admitting equilibrium states; they are related to the dome-drum system for compressive strength of masonry f_M assumed of 10 and 2 MPa, respectively.

Each polytope is referred to a different value for T in the plane (N_1, V_1) for different values of M_1. The polytopes of Fig. 16 are almost the same as those for unbounded compressive strength (therefore omitted). If we compare the polytopes corresponding to a reasonable value of traction T, representing the effect of the hooping tie (left column of Fig. 15) to the polytopes in absence of the hooping tie (right column in Fig. 15), it is clear that the hooping tie greatly enlarges the admissible polytopes, that is, the number of possible settings providing equilibrium. Similar comments can be deduced from Fig. 16 where the reduced compressive strength for masonry greatly reduces the extension of the polytopes. A compressive strength lower and very close to 2 MPa would reduce the smaller polytope (last row, right column of Fig. 16), related to the dome without the hooping chain, to a

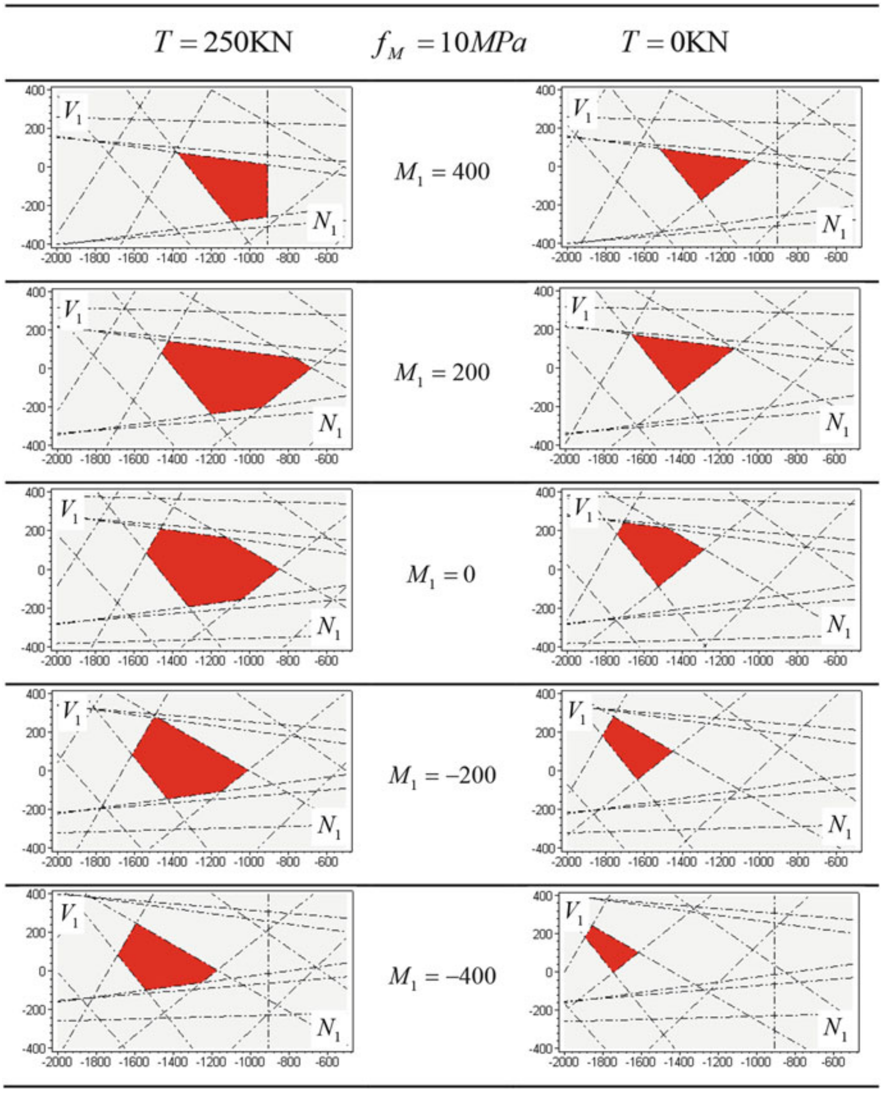

Fig. 15 Feasible domains in the $\{N_1, V_1, M_1\}$ space for different values of the tensile force T in the steel tie and for masonry compressive strength f_M of 10 MPa

vanishing area. According to the safe theorem, this does not mean that equilibrium is not possible, but that more detailed models are needed if safety has to be guaranteed.

Nevertheless, such an approach provides some useful information on the effects of material degradation due to environmental actions: if material strength decays, the possibilities of admissible polytopes is reduced, which is a crucial issue since the most intense degradation takes place in those parts of the drum and of the pillars where the stress state is more severe.

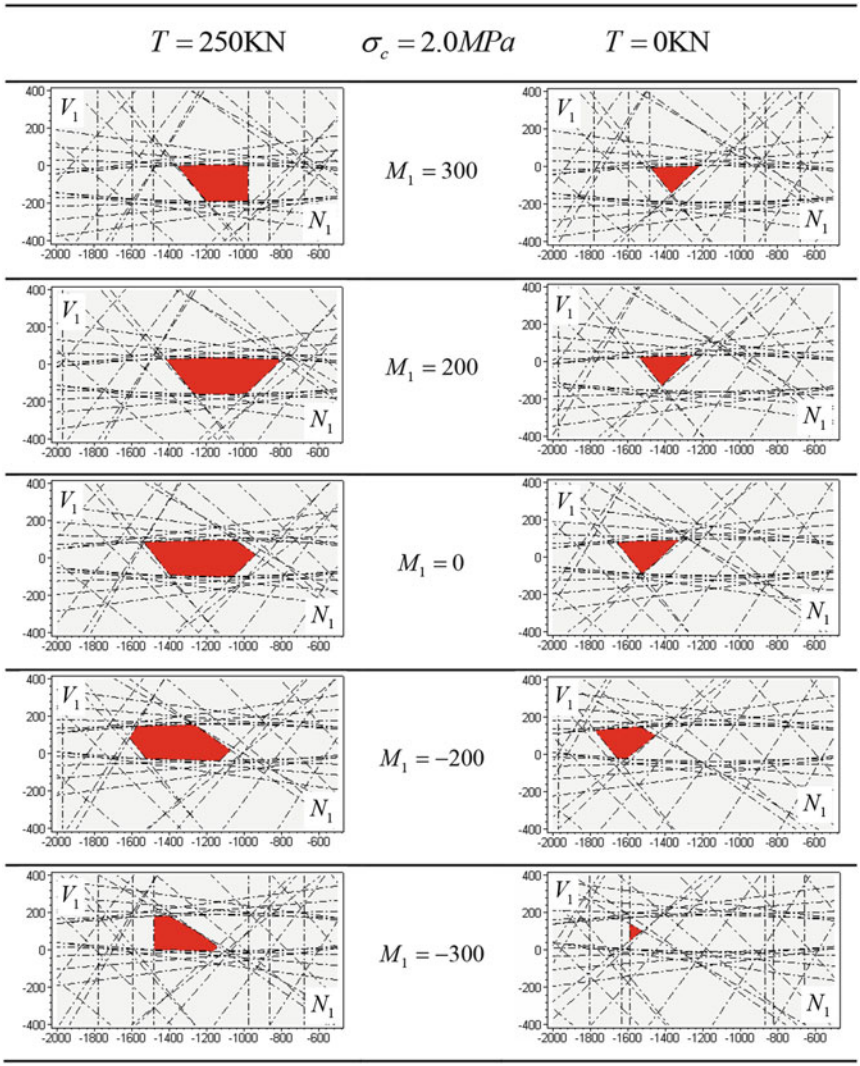

Fig. 16 Feasible domains in the $\{N_1, V_1, M_1\}$ space for different values of the tensile force T in the steel tie and for masonry compressive strength of 2 MPa

4 Conclusions

The assessment of masonry domes is a challenging and substantially unsolved issue for several reasons: (i) brickwork is anisotropic, quasi-brittle, and difficult to model; (ii) transversally compressed masonry, due to the high values of the friction coefficient, exhibits a non-vanishing tensile strength which largely depends on the amount of transversal compression; (iii) the actual brickwork texture in the shells is substantially unknown; (iv) equivalent mechanical properties of masonry are not easily defined. For all these reasons the assessment of a dome structure needs to assume several different structural models and assessment procedures, so that the final response can be based on the comparison between the different outcomes.

Assuming the dome to be axisymmetric and masonry to be no-tension with vanishing hoop stresses, an "orange slice" (Heyman 1967; Oppenheim et al. 1989) of the dome can be considered if dead loads are the main concern. The static approach showed that a thrust line contained within the shells of the dome-drum system can be found, providing an average compressive stress at the base of the drum of 0.5 MPa, thus concluding for the safety of the structure. The kinematic procedures confirm this conclusion showing that the ratio between stabilizing and active powers is never less than 2. Further, limit analysis procedures shed light on the stabilizing effect of the dead weight of the drum, which is shown to be a crucial element for the stability of the structural system, and of the hooping chain, thus indicating a strategy for retrofitting this type of structures without invasive works that would seriously threaten the cultural value of the monument.

The safety assessment of the dome-drum system is only partially fulfilled since this present analysis gained a deeper insight only into the effects of axisymmetric loads (i.e., dead loads), leaving completely open the issue of non-symmetric loads, such as wind and thermal loads.

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The Panthéon's Stability Already Questioned by Pierre Patte in 1770

Patricia Radelet-de Grave

Voyez le Panthéon, Voyez les Tuileries, Le Louvre et l'Odéon, Notre-Dame jolie. De tous ces monuments, la France est orgueilleuse, Elle en doit l'agrément, Aux maçons de la Creuse. (See the Pantheon, see the Tuileries, the Louvre and the Odeon, lovely Notre Dame. Of all these monuments, France is proud, She must give her approval, to the masons of the Creuse).

La chanson des maçons de la Creuse

Abstract Conceived and begun by architect Jacques-Germain Soufflot in 1755, the construction of the Church of Ste. Geneviéve (later the Panthéon) in Paris was continued after his death by Jean Baptiste Rondelet. This impressive structure was the object of various publications. As early as 1770, Pierre Patte pointed out problems of its stability in his *Mémoire sur la construction de la coupole projettée pour couronner la nouvelle église de Sainte Geneviève à Paris.* This was the beginning of a polemic regarding the structure's stability that involved some of the greatest scholars and architects of the day, and which was fundamentally a quarrel between tradition and new ideas.

Keywords Panthéon (Paris) • Pierre Patte • History of mechanics • Dome construction • Parallelogram method of forces • Jean Baptiste Rondelet • Jacques-Germaine Soufflot

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1 Preamble

Writing about the life, work and aesthetics of Jacques-Germain Soufflot (1713–1780), Jean Mondain-Monval had this to say:

> We might say that the church of Ste. Geneviève was the result of a lifetime of work, the fruit of thirty years of meditation: all the notes and measurements taken by Soufflot on many churches, both in France and Italy, all memories and reports presented or read by him at the Academies of Paris and Lyon were in his eyes only the justification for the great novelty he wanted to bring to Architecture.[1]

What novelty did Soufflot have in mind? In a letter written to le Comte de la Billarderie d'Angiviller, *Directeur et ordonnateur general des bâtiments du roi,* Jean Baptiste Rondelet (1734–1829) gives the answer:

> The main purpose of Mr. Soufflot in building his church was to unite, in one of the most beautiful forms, the lightness of Gothic buildings with the purity and magnificence of Greek architecture.[2]

Although it may appear rather surprising to talk of lightness when speaking about the Panthéon, I'll nevertheless try to explain the sense in which this quotation must be understand. But first, here is another quotation to consider. Writing some 75 years after Mondain-Monval, Robin Middleton, Jean Rondelet's biographer, seems to underline a completely different aspect of Soufflot's work:

> The church of Ste. Geneviève, now known as the Panthéon, provoked more interest, stirred more polemic, than any other building in the second half of the eighteenth century. The interest, moreover, was wide in range. The formal qualities of its architecture were as demanding of attention as its structural daring. It radically changed the contemporary understanding of architecture (Middleton 1993, p. 224).

Is it possible to find a link between those two affirmations? Leaving aside the formal qualities, as well as historical and sociological discussions, I'll try to show in what manner the polemic around the Panthéon's structural daring radically changed the contemporary understanding of stability, paving a new way to lightness. I won't go into the crucial discussions about the resistance of materials also raised by that polemic, but will instead restrict myself to the fundamental laws of statics used to study stability. I also won't enter into historical details, as this was accurately

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[1] *On peut dire que l'église Sainte-Geneviève a été le résultat de toute une vie de travail, le fruit de trente ans de méditation : toutes les notes et mesures prises par Soufflot sur de nombreuses églises, tant en France qu'en Italie, les mémoires et rapports présentés ou lus par lui aux Académies de Lyon et de Paris n'étaient à ses yeux que la justification de la grande nouveauté qu'il voulait apporter en architecture* (Mondain-Monval 1918, p. 423).

[2] *Le principal objet de M. Soufflot en bâtissant son église a été de réunir, sous une des plus belles formes, la légèreté de la construction des édifices gothiques avec la pureté et la magnificence de l'architecture grecque.* The letter is reproduced in (Middleton and Baudouin-Matuszek 2007, pp. 302–306). The sentence is frequently quoted (Mondain-Monval 1918, p. 423) attributes it to Brebion, another pupil of Soufflot.

documented by Robin Middleton and Marie-Nöelle Baudouin-Matuszek in their biography of Jean Rondelet (2007), Soufflot's spokesman.

This present chapter will be divided into the following parts:

1. Presentation of the polemic and of its protagonists;
2. A brief summary of the history of the mechanical principles of statics independent of science of construction;
3. The eight anonymous articles in the *Histoire de l'Académie royale des sciences* as evidence of the dissemination of the parallelogram law of forces;
4. A less brief summary of the history of stability in the context of architecture;
5. The polemic as quarrel between tradition and new ideas.
6. Conclusion

2 Presentation of the Polemic and of Its Protagonists

In 1755, Jacques-Germain Soufflot was put in charge of elaborating plans for a church dedicated to Ste. Genviève to be erected on the Ste. Geneviève hill, near the Jardins du Luxembourg. The same year, Soufflot gave a first version of the plans, which showed isolated columns supporting the vaults and four piers underpinning the dome; iron armatures inserted into the masonry were to reinforce the structure.

The following year, when work started to hollow out the foundations, the remains of a pottery factory were found, including 75 wells that have been dug to extract the clay to make the pottery. Soufflot bricked them up to insure the foundations, obtaining a series of 75 piles.[3] Rondelet explained this to the Comte d'Angerviller in 1780:

> The first care of Mr. Soufflot was to ensure the ground; in doing research on the subject, he found that the space his Church was to occupy, was riddled with an infinity of filled wells, which had been dug earlier to extract the earth for pots ... all of these, some of which were to 80 feet deep, were excavated again and filled with solid masonry made of *moilons* [blocks] and *libages* [large stones] to the height of the lowest foundations, which are about 20 feet from the pavement of the square.[4]

[3]Patte gives the number as 150 piles in his *Mémoires* (1769, p. 178).

[4]*Le premier soin de Monsieur Soufflot a été de s'assurer du sol ; en faisant des recherches à ce sujet, il trouva que l'espace que devait occuper son Eglise, etoit criblé d'une infinité de puits comblés, qui avoient autrefois été creusés pour tirer de la terre à pots ... tous ces puits dont quelques-uns avoient jusqu'à 80 pieds de profondeur furent fouillés de nouveau et remplis de maçonnerie solide en moilons et libages, jusqu'à la hauteur des plus basses fondations, qui sont environ à 20 pieds du pavé de la place* (Rondelet 1780; see also Middleton and Baudouin-Matuszek 2007, p. 302).

In doing so Soufflot respects the first point of his list of criteria for a building's stability:

The strength of a building like Ste. Geneviève depends on four principals causes, which are

1. The firmness of the soil;
2. The good construction of the foundations that establishes it;
3. The appropriate proportion of its walls and its points of support relative to the load that must be borne;
4. The equilibrium of vaults and the intimate union of all these parts.[5]

Rondelet continues with his description of the way the foundations were built:

> The foundation of all walls and solid parts were built of *libages* [large stones] and below all free-standing columns cut stone piers six feet square were erected, with rusticated facing, which formed links in all directions, and to bind and maintain these together, they built between them block walls three feet thick. Although these walls were erected on the foundation block, they were built on two beds of stones cut into voussoirs that together formed a reversed double arch. These arches were built to make pressure more uniform and advantageous, acting on a larger area and to transfer part of the load of a pier, which would carry a bigger weight, to those surrounding it.[6]

Even the architect Pierre Patte (1723–1814), *architecte de S.A.S. Mgr. le Prince Palatin Duc régnant de Deux-Ponts*, as he described himself, and a fierce opponent of Soufflot's Ste. Geneviève, as we will see below, praised these foundations in his book *Mémoires sur les objets les plus importants de l'Architecture*, giving a careful description of the foundations and of the piles and an illustration (Patte 1769, pp. 176–187 and Pl. IV) (Fig. 1).

Slowed down by the Seven Years' War, progress was delayed until 1764, the year in which King Louis XV laid the foundation stone, before the foundations were finished and work could begin with the surrounding walls as well as with the

[5]*La solidité d'un Edifice tel que Ste Geneviève depend de quatre causes principals qui sont*

1° La fermeté du sol ;

2° de la bonne construction des fondemens qu'on etablit dessus ;

3° de la juste proportion de ses murs et de ses points d'apuis relativement à la charge qu'ils doivent porter ;

4° de l'équilibre des voûtes et de l'union intime de toutes ces parties (Rondelet, quoted in Middleton and Baudouin-Matuszek 2007, p. 302).

[6]*Les fondations de tous les murs et massifs furent construites en libages et audessous de toutes les colonnes isolées on eleva des piliers de six pieds au quarré, en pierre de taille, a paremens rustiqués qui formoient liaisons sur tous sens, et pour lier et entretenir ces piliers les uns avec les autres, on construisit entre eux des murs en moilons de trois pieds d'epaisseur; quoique ces murs fussent posés sur le massif general ils furent erigés sur deux assises de pierres taillées en voussoirs formant ensemble un double arc renversé, on construisit ces arcs pour rendre la pression plus uniforme et plus avantageuse en agissant sur une plus grande superficie et pour transmettre une partie de la charge d'un pilier qui porteroit un plus grand poid sur ceux qui seroit autour* (Rondelet quoted in Middleton and Baudouin-Matuszek 2007, p. 302).

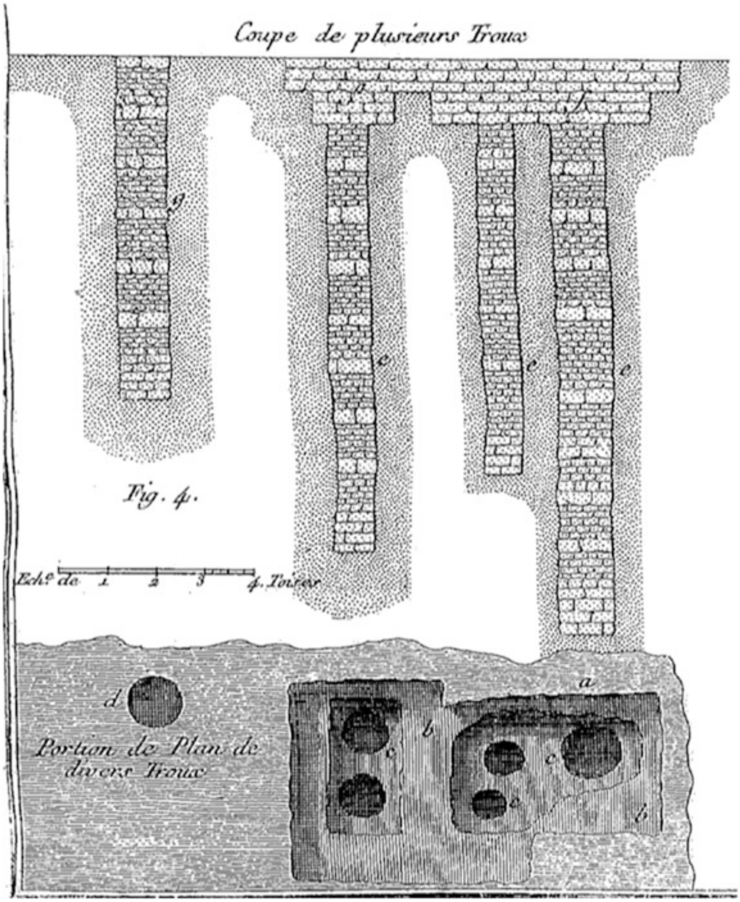

Fig. 1 The piles of Soufflot's foundations for Ste. Geneviève, from figure 5 in Pl. IV in Patte's *Mémoires sur les objets les plus importants de l'Architecture* (1769)

columns that had to be inserted in them (Mondain-Monval 1918, p. 448):

> On such solid foundations Mr. Soufflot could execute a light construction that gave all the walls and support points only the dimensions necessary relative to their heights with respect to the weight that they ought to bear. That is why he lightened his walls by piercing them with arches, but he connected the construction in such an ingenious way, especially the perpendiculars of architraves and vaults, that in these places it forms a kind of circular net that supports all parts of the Church. The arrangement of the vaults is so well combined that the forces of each are directed so as to eliminate the forces of the others.[7]

[7] *Sur des fondemens aussi solide M. Soufflot a pu executer une construction legere en ne donnant a tous les murs et points d'appuis que les dimensions necessaires relativement a leurs elevations et aux poids qu'il devoient porter; c'est pourquoi il a elegit ses murs par des percés des arcades, mais il a relié sa construction d'une maniere si ingenieuse, surtout au droit des architraves et des voutes,*

In 1769, the walls were erected and the plan of the building became easier to discern. The next year, Patte published a rather long article criticizing Soufflot's work. This work, entitled *Mémoire sur la construction de la coupole projettée pour couronner la nouvelle église de Sainte Geneviève* (Patte 1770a) is said to be the origin of one of the most important disputes concerning architecture. As Middleton and Baudouin-Matuszek put it:

> … the dispute was to erupt again and again, throughout the remaining years of the century, for a very serious issue was involved: whether established procedures should remain the basis of structural design, or whether new theory of structures might be formulated, based on calculation and experimental study of strength of materials (Middleton and Baudouin-Matuszek 2007, p. 43).

They are right when underlining two aspects of the dispute. The first is theoretical and began during the building of Ste. Geneviève; the second is experimental and began later, in 1797, when it was decided to transform the church into a *Panthéon*. When new cracks appeared in the piles, rekindling the old quarrel about their ability to sustain the dome, new articles appeared, such as that of architect Charles François Viel de Saint Maux (1745–1819), *Moyens pour la restauration des pilliers du dome du panthéon françois* (1797).

In what follows I shall concentrate on the theoretical aspects of the first part of the dispute, beginning right after the publication of Patte's *Mémoire,* analysing the reactions to it by Rondelet, Émiland-Marie Gauthey (1732–1806) and Jean-Rodolphe Perronet (1708–1794). The quarrel is the eternal one opposing *les anciens et les modernes,* tradition and new ideas. Those new ideas have to do with the mechanical principles of stability or of statics. I shall attempt to show why did such principles remain hidden in the Gothic cathedrals so admired by Soufflot, despite the fact that the designers surely knew them at least intuitively.

3 A Brief Summary of the History of the Mechanical Principles of Statics, Independently of Science of Construction

There are two fundamental laws of statics, namely, the equilibrium of the lever the parallelogram law of forces. The first one corresponds to balance to rotations and the other one to balance to translations. The first one is translated mathematically by the sum of momenta being zero: $\sum \overrightarrow{M} = 0$ and the other one by the sum of forces being zero as well: $\sum \overrightarrow{F} = 0$.

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qu'a ces endroits elle forme une espece d'erayure qui entretient toutes les parties de son Eglise. La Disposition des voutes est même si bien combinée que l'effort des unes est dirigé ensorte qu'il sert a detruire l'effort des autres (Rondelet quoted in Middleton and Baudouin-Matuszek 2007, p. 302).

The first one has been known since Antiquity and was enunciated by Aristotle (4th c. B.C.) and proven by Archimedes (3rd c. B.C.). The second one has a more tortuous story. A parallelogram law has been known for the displacements and for the velocities since Antiquity. It was not found for forces until the end of the Renaissance with Simon Stevin (1548–1620). Nevertheless, it is impossible to believe that the Gothic cathedrals could have been built without any notion of that law. Strangely, just as the law of the lever was discovered in Antiquity in the context of construction, the law of composition of forces was discovered experimentally by Stevin, Gilles Personne de Roberval (1602–1675) and Christiaan Huygens (1629–1695), all of whom were studying the way stones or weights could be kept in equilibrium using ropes.

In one of his main works, *Wisconstighe Gedachtenissen* (1608), Simon Stevin gives following explanation (Fig. 2):

> But to come to set forth the properties of weights hanging on cords, let AB be a prism, whose centre be C and which be hanging in the two fixed points D, E, with two lines CD, CE coming from the centre of gravity C; these lines CD and CE are centre line of gravity of the prism by the fifth definition. Therefore, if HI were drawn between DC and CF, parallel to CE, ... So as CI is to CH, so is the weight of the whole prism to the weight acting on D. And in the same way, the weight acting on E is also found, provided there be drawn from I to CE, the line IK, parallel to DC; we can then say: as the vertical lifting line CI is to the oblique lifting line CK, so is the weight of the whole prism, to the weight acting on E.[8]

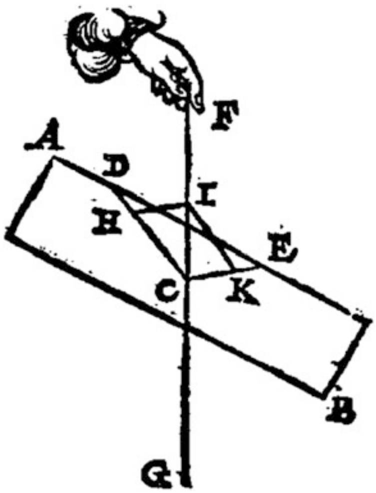

Fig. 2 Stevin's parallelogram law

[8] *Or pour venir à la déclaration de pesanteur suspendüe par cordage. Soit AB une colonne de laquelle C soit le centre, suspendüe à deux lignes CD, CE (venant dudit centre C) ès poinct fermes D, E, ... parquoy menant HI entre DC, CF, parallele à CE, ... Donc comme CI à CH, ainsi le poids de la colonne entière, au poids qui advient en D ; et de même trouvera-on le poids qui advient en E, en menant de I jusques à CE, la ligne IK, parallele à DC ; & disant, comme l'élévation droite CI, à l'élévation oblique CH, ainsi le poids de la colonne, au poids qui advient sur E* (Stevin, 1608,

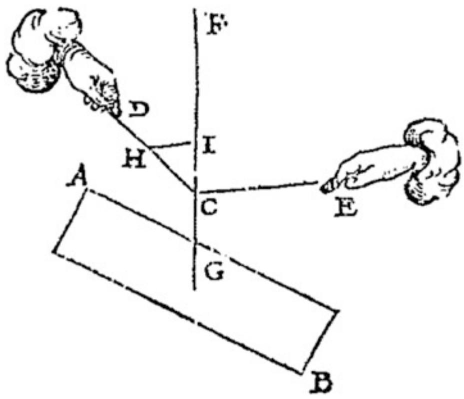

Fig. 3 Stevin's explanation of the parallelogram law

Two years after the publication in 1634 of a French translation of that work of Stevin, Roberval gives, without any figure a new expression, a slightly more generalised statement of Stevin's theorem which perfectly fits another figure of Stevin (Fig. 3).

> If from any point taken in the direction line of the weight, one leads a line parallel to one of the cords to the other rope, the sides of the triangle formed will be homologous to the weight and to the two powers.[9]

Finally, in a manuscript of Huygens (1667), one finds the decomposition of a weight into more than two components, namely, seven (Fig. 4).

It wasn't until 1687, the year in which Isaac Newton (1642–1727) published the *Principia* (1687), that Pierre Varignon (1654–1722) gave the parallelogram law of forces a fundamental status in statics. In his *Projet d'une nouvelle mécanique* (1687), and later, in his *Nouvelle mécanique* (1725), Varignon proposed replacing the law of the lever, which had been the fundamental law of statics since Antiquity, with the parallelogram law of forces.

Finally, a year after the posthumous publication of the *Nouvelle mechanique* by Varignon, in which he reiterates and develops his proposal of 1687, Daniel Bernoulli (1700–1782) gave the first demonstration of the parallelogram law in the case of forces (1726).

This brief summary shows how long that story was and how difficult it was to measure the generality and the importance of that fundamental law. It took almost 2000 years to observe the geometrical way in which forces act and to realise that

Byvough der Weeghconst, p. 182; English translation in (Dijskerhuis 1955, pp. 534–535); French translation in (Girard 1634, p. 505).

[9] *Si, de quelque point pris en la ligne de direction du poids, on mène la ligne parallèle à l'une des cordes jusqu'à l'autre corde, les côtés du triangle ainsi formé seront homologues au poids et aux deux puissances* (Roberval 1636, p. 28).

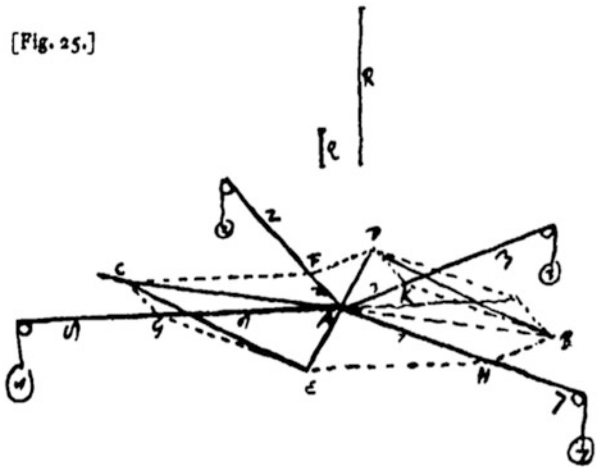

Fig. 4 Huygens exercise with the parallelogram law

forces and velocities obeyed the same geometrical law. It then took eighty more years to give this law its fundamental character, and another forty years to give a proof of this law.

4 The Eight Anonymous Articles in the *Histoire de l'Académie royale des sciences* as Evidence of the Dissemination of the Parallelogram Law of Forces

It is only around 1702 that more people began to be interested in the way forces behave. Between 1702 and 1730, Bernard Le Bouyer (or Le Bovier) de Fontenelle (1657–1757) published anonymously in the *Histoire de l'Académie royale des sciences* six articles concerning constructions: five concerning domes, one concerning the parallelogram law and two concerning friction, in which he speaks also of the parallelogram law.[10] Each of these articles corresponds to a more elaborate article in the *Mémoires*. The authors of these were Antoine Parent (1666–1716), Philippe de La Hire (1640–1718), Pierre Couplet (1670?–1744), Pierre Varignon and Henri Pitot (1695–1771). We may conclude that the link between the parallelogram law and the stability of domes was at the centre of the preoccupations of many members of the Académie des sciences de Paris.

[10]These articles appear in the bibliography as follows: Anon. [Fontenelle] (1702, 1704a, 1704b, 1714a, 1714b, 1726, 1729, 1730).

As early as 1702, Fontenelle, in his introduction to an article by Varignon concerning the resistance of solids, draws attention to the fact that forces act along a certain direction and that they compose in a particular mode:

> Whenever several forces, united or linked together, or finally changing each other in any way, act at the same time either to impart motion to a body or to impart to him different or opposite ones, none of these forces exerts its action through the same line, or what is the same, in the same direction it would have had, if it had acted alone, but of all specific & simple directions a composite one is formed, which alone moves the body.[11]

In 1704, Fontenelle entitled his second article "Sur la figure de l'extrados d'une voûte circulaire dont tous les voussoirs sont en equilibre entr'eux" (Anon. [Fontenelle] 1704a). It shows his particular interest in domes, because the article doesn't correspond to any article included in the *Mémoires*. It in fact corresponds to an article read to the Académie by Antoine Parent on 7 May 1704, entitled "Des charges qu'il faut donner aux Voûtes, afin qu'elles tendent à s'affermir le plus qu'il est possible". The text was finally published by Parent himself in his *Essais et Recherches de mathématiques et de physique* (1704a).

Fontenelle explains Parent's main results.

> Mr. Parent has sought what would be the external curvature or Extrados of a vault whose Intrados would be circular and all the voussoirs in equilibrium because of their weight, according to the rule of Mr. de la Hire, because it is clear that all these voussoirs unequal but in a certain proportion, would form outside some regular curve. He has found it by points, but in a very simple way, so that by his method one could easily build a vault of which one would be sure that all voussoirs are in balance.[12]

He also explains that, identifying the voussoirs to wedges considered as double inclined plane, Parent applies to the voussoir what he calls the *Règle de la Hire*, which I shall comment on in greater detail in the next section. In that way he finds

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[11] *Toutes les fois que plusieurs Puissances unies, ou liées ensemble, ou enfin se modifiant les unes les autres de quelque manière que ce soit, agissent en même-temps ou pour imprimer un mouvement à un Corps, ou pour lui en imprimer de différens ou d'opposés, aucune de ces Puissances n'exerce son action par la même ligne, ou ce qui est la même chose, selon la même direction qu'elle eût eue, si elle eût agi seule ; mais de toutes les directions particuliéres & simples, il s'en forme une composée, qui est la seule selon laquelle le Corps est mû* (Anon. [Fontenelle] 1702, pp. 108–109).

[12] *M. Parent a cherché quelle seroit la courbure extérieures ou l'Extrados d'une voute dont l'Intrados seroit circulaire, & tous les voussoirs en équilibre par leur pesanteur, selon la règle de M. de la Hire, car il est clair que tous ces voussoirs inégaux dans une certaine proportions feroient en dehors une certaine courbure régulière. Il ne l'a trouvée que par points, mais d'une manière fort simple, de sorte que par sa méthode on pourroit assés facilement construire une voute, dont on seroit sûr que tous les voussoirs seroient en équilibre* (Anon. [Fontenelle] 1704a, p. 95).

his rule giving the ratio of the thrust to the weight of the vault:

> A significant result of Mr. Parent's research is that he discovered at the same time the measurement of the thrust of the vault, or the report of this thrust to the weight of the entire vault. We only knew that this force was very large, and we opposed large masses of stone to it, or abutments, rather too strong than too weak, but we did not know exactly what was the reason for it. We can know it now; the Arts are always affected by the progress of Geometry.[13]

These final words on the influence of geometry conclude his article.

Another article by Fontenelle in the same volume (Anon. [Fontenelle] 1704b) introduces an article on friction, again by Parent, on friction (Parent 1704b), in which Parent reintroduces the terms *ligne de direction* to indicate the direction of a force. This expression was introduced by Niccolò Tartaglia (1499?–1557) in 1554; Roberval also used it, in the article cited in the previous section (Roberval 1636).

Despite the fact that Fontenelle introduces Parent's article in the *Mémoires* of 1704, the text was not published in the Paris *Mémoires*. Parent finally published himself in a volume entitled *Essais et Recherches de mathématique et de physique* (1713).[14] (Some later authors quote this text as published in the *Mémoires de Paris*, thus proving that they have not read it.) This paper is very important,[15] not only because of the useful results it gives about the thrust, but also because in it Parent applies the general parallelogram law to the calculation of the thrust—despite the fact that it doesn't appear in the figure—combined with graphical properties of the radius of curvature. He obtains an equilibrium when, for each voussoir, the resultant of the weight of the voussoir and of the horizontal reaction of the wall opposed to the thrust, is in the direction of the radius of curvature (Fig. 5).

Ten years later, this same Fontenelle introduced an article by Varignon on the parallelogram law in general. His title "Sur l'action de plusieurs puissances qui tirent à la fois un même corps ou point" (Anon. [Fontenelle] 1714a) is much more precise than that of Varignon, "Solution d'un problème de statique, avec la manière d'en résoudre une infinité d'autres de la même espèce" (Varignon 1714). But Varignon's title already shows the fundamental character of that law. This article doesn't require any further comment, a glance at the plate illustrating it, shows immediately the nature of the work (Fig. 6).

[13]*Un fruit considérable de la recherche de M. Parent, c'est qu'il a découvert en même temps la mesure de la poussée de la voute, ou quel rapport a cette poussée au poids de la voute entière. On sçavoit seulement que cet effort étoit trés-grand, & on y opposoit de grosses masses de pierres, ou de culées, plûtôt trop fortes, que trop foibles, mais on ne sçavoit point précisément où il s'en falloit tenir. On pourra le sçavoir présentement, les Arts se sentent toûjours du progrés de la Geometrie* (Anon. [Fontenelle] 1704a, p. 96).

[14]The volume is rather difficult to find and Google digitized it without unfolding the plates, rendering the text impossible to read.

[15]I hope to have the opportunity of publishing Parent's text together with a more extensive comment, but to go into it here would lead us too far away from the story of the Panthéon quarrel.

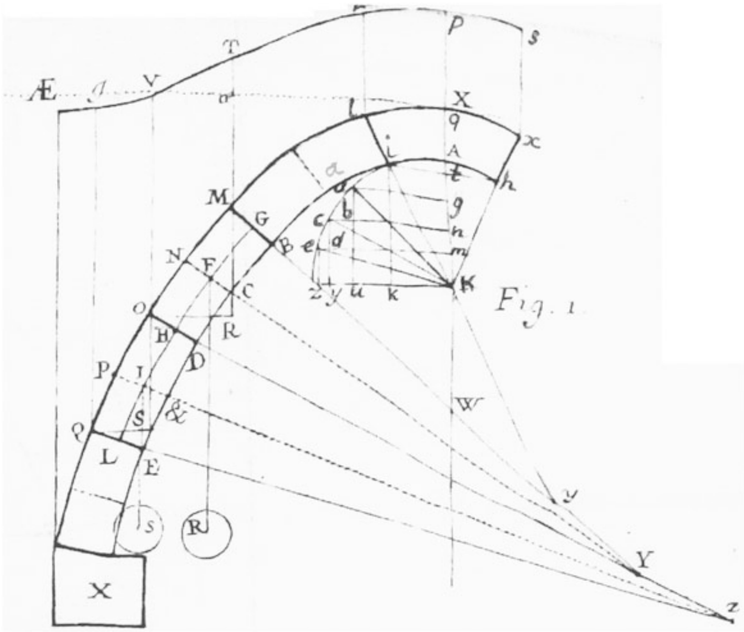

Fig. 5 Parent's study of the thrust

The next article by Fontenelle on constructions was published much later, and concerns *la force des ceintres* (Anon. [Fontenelle] 1726). It introduces a work of Henri Pitot concerning the force to be given to centring used in the construction of large arched bridges (1726):

> As to the position of pieces of which the greatest part are necessarily inclined, changing & weakening their absolute resistance according to the angles of inclination which are different, Mr. Pitot computes them by the theory of compound movements, or what is the same, by the diagonals of the late Mr. Varignon.[16]

We could summarise the results by saying that Pitot uses the parallelogram law for vectors.

Three and four years later, Fontenelle contributed two articles with the same title, namely, "Sur les voûtes" (Anon. [Fontenelle] 1729, 1730). He introduces separately the two parts of a long article by Pierre Couplet (1729), which I shall comment on in the next section. The author comments on a result he obtains about the pressure of a dome on its piers with the following remark: "Conclusion, which we had already

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[16] *Quant à la position des pieces dont la plûpart sont nécessairement inclinées, ce qui modifie & affoiblit leurs résistances absolues selon que les angles d'inclinaison sont différens, M. Pitot en fait le calcul par la théorie des mouvemens composés, ou, ce qui est la même chose, par les diagonales de feu M. Varignon* (Anon. [Fontenelle] 1726, p. 67).

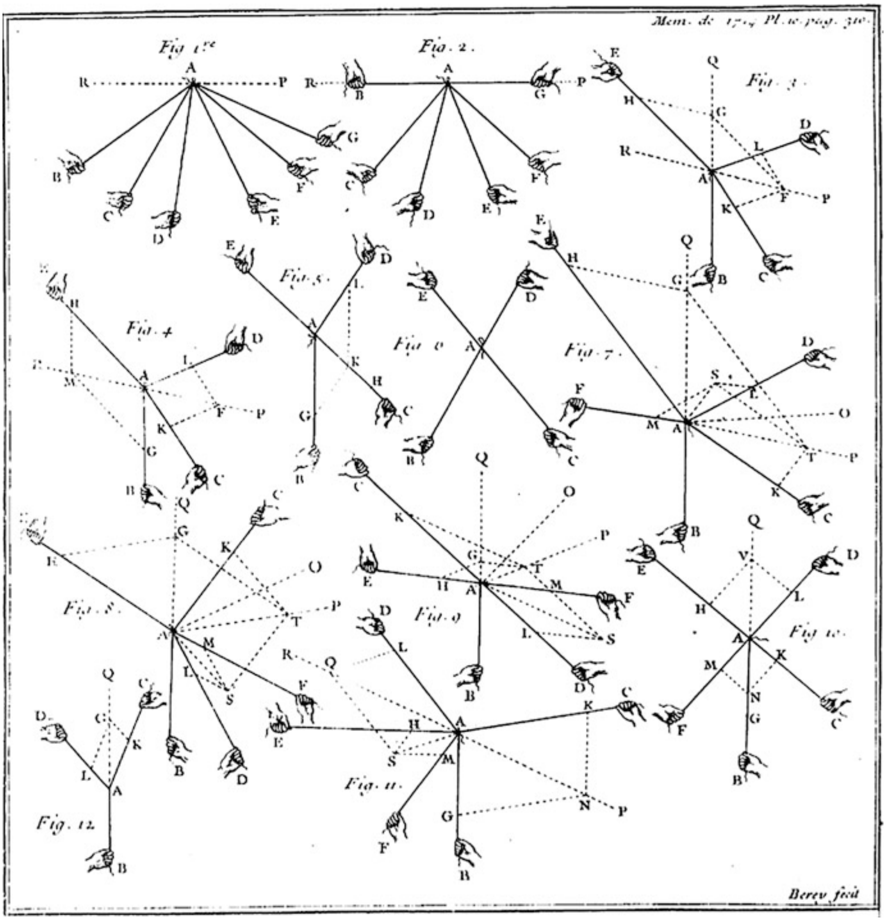

Fig. 6 Varignon's illustrations of the parallelogram law

reached in 1704 using some other way".[17] This comment reveals that the anonymous author of all the articles we just listed is one and the same person: Fontenelle. In 1704 he used La Hire's law, and now he replaces it with the parallelogram law.

This lengthy enumeration shows how the vectorial behaviour of forces and the parallelogram law slowly became common knowledge for the members of the Academie des sciences de Paris. In the next section I will show how the same thing happened in the world of architecture.

[17] *Conclusion où nous étions déjà arrivés en 1704 par une autre voie* (Anon. [Fontenelle] 1729, p. 77).

5 A Less Brief Summary of the History of Stability in the Context of Architecture

How is the situation in publications dedicated to architects? Is it really different? Do these authors know about the theoretical works of their colleagues? Why is the development slower there?

A first name naturally comes to mind: Villard de Honnecourt. Although a glance at his *Carnet* (13th cen.) seems to confirm that there is nothing about forces in it, there is that marvellous drawing of the flying buttresses of the cathedral in Reims (Fig. 7).

How could it have been possible to think of such a solution to ensure the stability of a construction without any intuition of the way forces act? What is the meaning of the sloped bar near the bottom that also seems to retain the wall? Those flying

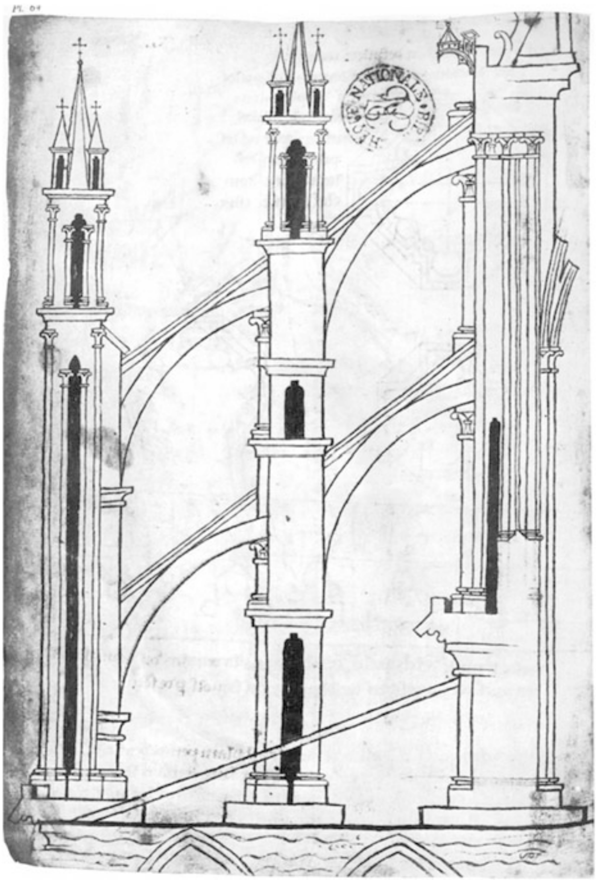

Fig. 7 Villard de Honnecourt's flying buttresses

buttresses will lead to the expression *rejeter le poids* (literally, to reject the weights) which will be often used in the texts of our quarrel. For example, in his *Mémoire* of 1770 Patte writes, "All the weight and the thrust ... are rejected in the direction of principal points of support".[18]

In the article "Arc de décharge" in the *Dictionnaire raisonné de l'Architecture* by Eugène Viollet-le-Duc (1814–1879), we read that this type of construction is used to "carry the weight of constructions above onto support points whose stability is ensured".[19] Viollet-le-Duc observes that those constructions of arches embedded into the masonry were already used during the Roman period:

> In Roman buildings.... we often encounter relieving brick arches and stonework embedded in the middle of a wall to carry the weights onto some points of the foundations and basements established more firmly than the rest of the building.[20]

This is exactly what Soufflot did in his foundation under the places where he intended to build the piers supporting the dome.

Viollet-le-Duc gives a most credible description of the intuitive knowledge that architects of the Roman and Gothic periods had of the behaviour of forces. But such a study would lead us too far afield, let us restrict to theoricians of architecture.

Let us first mention Philibert de l'Orme (1514–1570), whose *Architecture* (1567) is dedicated to stonecutting and stereometry and has more to do with projective geometry than with mechanics. But de l'Orme seems to be the first to aim at applying Euclid's geometry to architecture:

> Which [time], with God's help, I'll also use to review Euclid and accommodate his theory to the practice of our architecture, accompanying him with Vitruvius, and reducing it to a certain method, which I notice in his books to be strongly indigestible and confused. ... Some may say that it is without cause and for nothing that I employ myself to review Euclid ... seeing that there are so many learned men whose profession it is to read and divinely interpret Euclid. What I aim to do ... is to conjoin the practice of architecture to the theory of Euclid.[21]

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[18] *Tout le poids & la poussée ... étant rejettés vers des points d'appui principaux* (Patte 1770a, p. 13).

[19] *...reporter le poids des constructions supérieures sur des points d'appuis dont la stabilité est assurée* (Viollet-le-Duc 1873, tome 1, p. 83).

[20] *Dans les constructions romaines ...., on rencontre souvent des arcs de décharges en brique et en moellons noyés en plein mur, afin de reporter les pesanteurs sur des points des fondations et soubassements établis plus solidement que le reste de la bâtisse* (Viollet-le-Duc 1873, tome 1, p. 83).

[21] *Lequel [temps], avec l'aide de Dieu, j'employray aussi à revoir Euclide & accomoder sa theorique avec la pratique de nostre architecture luy accompagnant Vitruve, & le reduisant à une certaine methode, laquelle j'apercois en ses livres estre fort indigeste & confuse. ... Quelques-uns pourront dire que sans cause et pour néant ie m'emploiray à revoir Euclide ...veu qu'il y a tant d'hommes doctes qui font profession de lire & interpréter divinement bien ledit Euclide. ... ce que ie prétend, ... est de conioindre la pratique d'architecture, avec la theorique du dit Euclide* (De l'Orme 1567, p. 62).

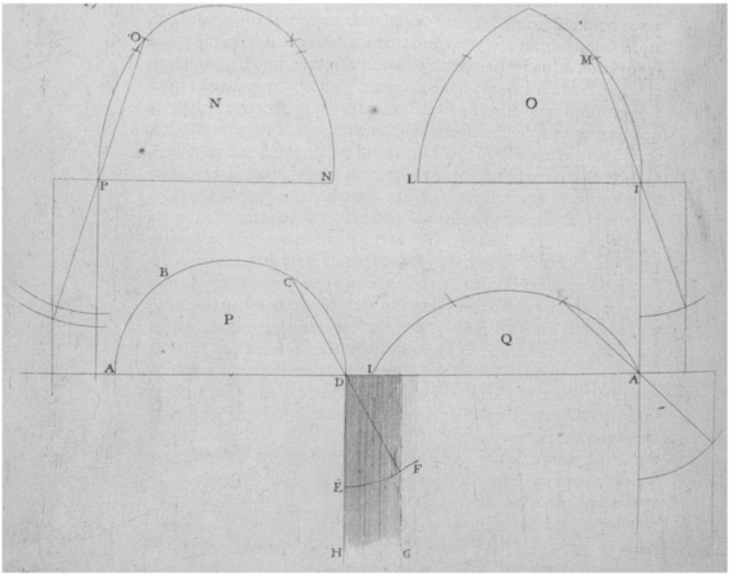

Fig. 8 Derand's rule

A beautiful example of what de l'Orme is aiming at is given by the very well known rule published by François Derand (1588?–1644) in his *Architecture des voûtes ou l'art des traits et coupe des voûtes* (1643). The word 'rule' that is used universally is very well chosen because it is not a law but rather a way of finding the width to be given to the piers in order to be sure they will resist the vault's thrust. Derand doesn't give any justification, he just assures us that it works and that geometry renders it easy to apply on a building site (Fig. 8).

Just over 30 years later, Nicolas-François Blondel (1618–1686) repeated the same drawings in his well known *Cours d'architecture* and added a more algebraic formulation, namely:

> From this [Derand's] rule it follows, according with what we have taught previously, that the width of piers of semicircular arches must be the quarter of their diameters, and that for surmounted arches one needs less thick piers, as on the contrary one needs thicker ones for surbased arches.[22]

In his bilingual (Italian–Latin, a custom at the time) book entitled, *Il tempio Vaticano e la sua origine ... Templum vaticanum et ipsius origo,* published in 1694, Carlo Fontana (1638–1714) gives some "occult rules ... for building a dome and

[22]*Par cette règle [de Derand] il s'ensuit, conformement à ce que nous avons enseigné cy-devant, que la largeur des piles aux Arcs à demi-cercle doit estre du quart de leurs diametres, & qu'aux Arcs surmontez il faut moins d'épaisseur de piédroits, comme au contraire il en faut plus à ceux qui sont surbaissez* (Blondel 1675, p. 419).

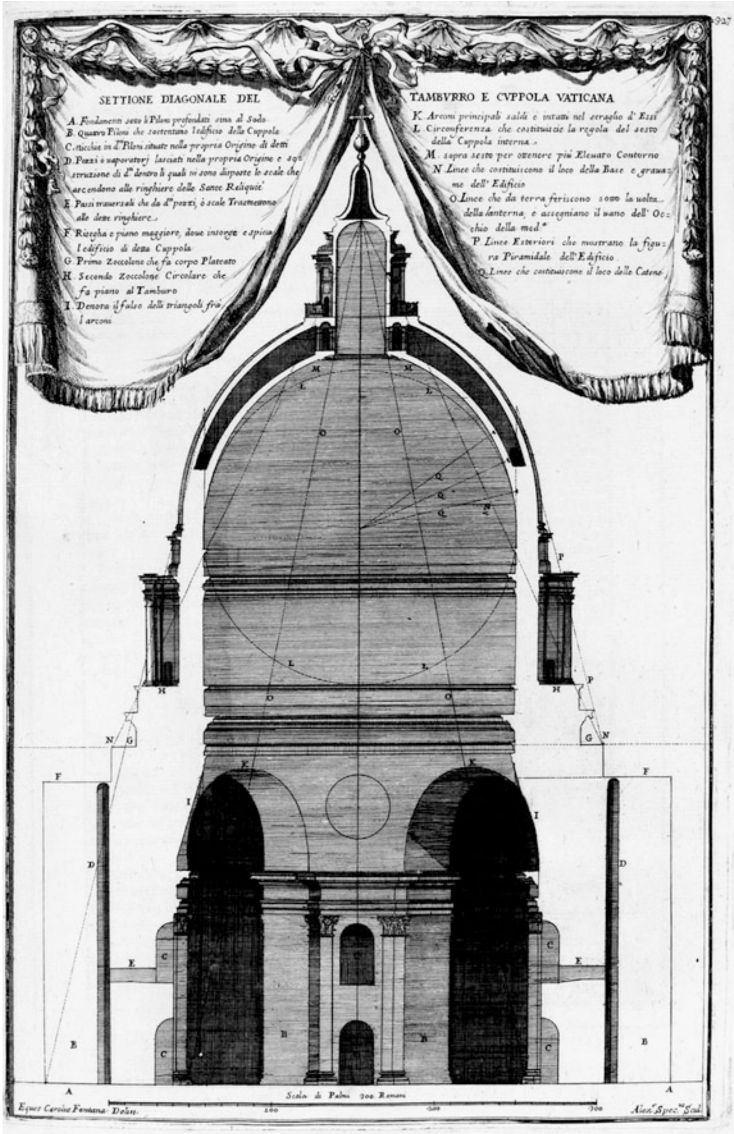

Fig. 9 Fontana's triangles

its supports".[23] Those rules are purely geometric: they are given by a circle and some vertical or oblique lines, can be seen in Fig. 9. One of those lines P reaches the base of the dome at the limit of the construction. It makes with that base and the

[23] *Capitolo XV. Regole occulte, che si mostrano nella seguente Tavola, per il Composto della Cupola e i suoi sostegni* (Fontana 1694, tome V, ch. XV, p. 325).

vertical a triangle showing “the pyramidal stability of the construction”: the line P, which cuts the internal perpendicular of the drum, proves the pyramidal support of the construction.[24]

Experience taught that the pyramid was the most stable construction, so it was believed that any stable edifice had to be inscribable in a pyramid.

In 1695, one man—Philippe de La Hire—tried to change the situation, introducing mechanical principles into architecture (Radelet-de Grave 2013; Becchi 2013). His work marks the origin of a revolution in architecture and, at the same time, the origin of the quarrel involving Patte, Soufflot and Rondelet. La Hire’s *Traité de mécanique* (1695), enjoyed no success among theoretical mechanists, but in contrast, and despite the fact that it only contains three sections about domes, every educated architect knew it.

La Hire’s master is Archimedes, as the first words of the Preface to his book show: “Of all the works of mechanics that remain of the Ancients, there are only those of Archimedes in which the principles of this science are treated thoroughly”.[25] In the spirit of his master he considers “the lever, which can be regarded as the fundamental proposition of all mechanics, since all other parts are easily reduced to it”.[26] His aim is thus to built all of mechanics on the law of the equilibrium of the lever. He writes:

> I have tried in this book to show all the proposals in the manner of ancient Geometers, without making use of any other axiom or basic proposition except those that all of those who have written about Mechanics have assumed first, & to make it even more obvious I proved it in my first proposal by means of another that is more universal and of which there is no doubt in physics, which is that in the force of the powers all things being equal on one side & on the other, forces are equal.[27]

Thus he describes the way he constructs mechanics, but that is not the end of the story: “The final proposals of this book contain what we found most curious in Physics with respect to Mechanics”.[28] His object is not restricted to mechanics, because mechanics is part of the foundations of physics. He then enumerates the most important problems of the time—percussion, oscillations—after which he

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[24] *La linea P, che va a intersecare con la perpendicolare interiore del Tamburo, dimostra il piramidale sostenimento dell’edifizio* (Fontana 1694, tome V, ch. XV , p. 335).

[25] *Entre les Ouvrages de Mécanique qui nous restent des Anciens, il n’y a que ceux d’Archiméde, où les principes des cette Science soient traités à font* (La Hire 1695, Preface, n.p. (p. 1)).

[26] *... le levier qu’on peut regarder comme la proposition fondamentale de toute la Mécanique, puisque les autres parties s’y peuvent réduire facilement* (La Hire 1695, Preface, n.p. (p. 4)).

[27] *J’ay tâché dans cet ouvrage de démontrer toutes les propositions à la manière des anciens geométres, sans me servir d’autre Axiome ou proposition fondamentale que de celle que tous ceux qui ont écrit de Mécanique ont supposée d’abord; & pour la rendre encore plus évidente je la démontre dans ma première proposition par une autre qui est plus universelle & dont on ne fait aucun doute dans la physique, qui est que dans l’effort des puissances toutes choses étant égales d’un côté & d’autre, les efforts sont égaux* (La Hire 1695, Preface, n.p. (pp. 5–6)).

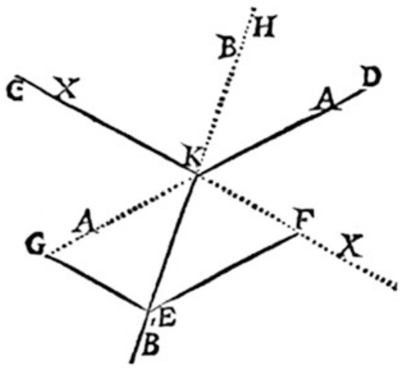
[28] *Les dernières propositions de cet ouvrage contiennent ce qu’on a trouvé de plus curieux dans la Physique par rapport à la Mécanique* (La Hire 1695, Preface, n.p. (p. 8)).

explains "what should be the figure of equally ponderous and flexible cords or lines ... and I give in the same place the solution of one of the most significant problems in the construction of the buildings".[29] The "significant problems" that he refers to are indeed essential to architecture: the problem of the catenary and its inverse, the most stable dome.

In his general definitions, he underlines the directional or vectorial character of forces: "II. The direction of a power is the straight line following which it exerts its force when it is applied to the machine".[30] This is a most important remark because it says that forces in general have a direction but also that weights may act in a direction different from the vertical depending on the way it acts on the machine. This is exactly the point that Patte doesn't fully understand. We will see that for him and for most of La Hire's predecessors, a weight acts vertically, or, as Aristotle said, weight tends towards the centre of the world. This way of thinking explains their predilection for the pyramid, which is the surest way to get all the weights acting vertically on the base of the construction, thus preventing it from falling down.

La Hire's message to the architects is clear and most important: the laws of mechanics and in particular the most fundamental one, namely the law of the lever, are able to explain the stability of buildings and, as he will show, the stability of domes. But there is another reason for La Hire's success in the world of architecture. From the law of the lever, La Hire derives a particular form of the parallelogram law of forces (see Radelet-de Grave 2003). His proposition XXI says, "We must find three powers AXB that pulling a point K in three directions given by CK, DK, EK, are in equilibrium with each other" (Fig. 10).[31]

Fig. 10 La Hire's parallelogram law



[29] *J'explique en suite quelle doit être la figure des cordes ou lignes également pesantes et flexibles ... & je donne dans le même endroit la solution d'un des plus considérables problèmes de la construction des bâtimens* (La Hire 1695, Preface, n.p. (p. 9)).

[30] *II. La direction d'une puissance est la ligne droite suivant laquelle elle fait son effort étant appliquée à la machine* (La Hire 1695, p. 8).

[31] *Il faut trouver trois puissances AXB, qui tirant un point K par trois directions données CK, DK, EK, soient en équilibre entre elles* (La Hire 1695, p. 70).

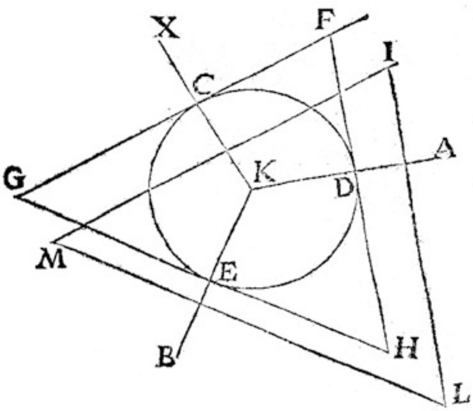


Fig. 11 La Hire's particular parallelogram law for vaults

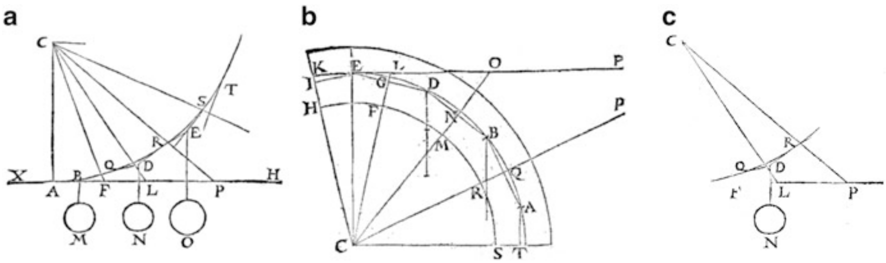


Fig. 12 La Hire relating the vault to the catenary

La Hire then gives another form to the theorem equating the proportion of the forces to that of the sides of a triangle having its sides perpendicular to those forces (Fig. 11).

This is the theorem that he uses to solve the problem of the vault (Fig. 12).

This form of the law is very well suited to the equilibrium of the dome, because the sides of the triangle on which he can read the proportion of the forces are parallel to the upper and lateral sides of the voussoir. However, there is more: he deduces that the weight of the different voussoirs have to be proportional respectively to KL, LO, OP taken on a horizontal line tangent to the top E of the vault.

Thus La Hire owes his success to three facts:

1. He made architects conscious of the fact that weights don't always act vertically but that they can also push laterally in a way expressed by the parallelogram law or by his triangle law.
2. The triangle law is particularly well suited to the study of the equilibrium of domes.
3. The triangle law gives a direct geometrical representation of the weights the voussoirs must have in order to put the dome in equilibrium.

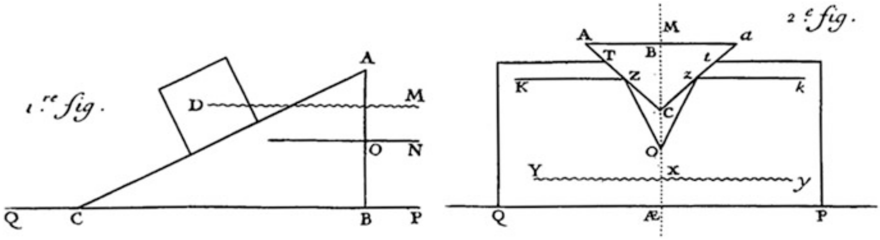


Fig. 13 Parent's comparison of the voussoir with the wedge

However, as is often the case, the majority did not immediately adopt those new ideas, as we will observe in the particular case of the construction of the dome of Ste. Geneviève.

Let us have a look at the repercussions of La Hire's book for the works published before Patte's Memoir of 1770.

The study of the apparent weight of an object on an inclined plane also contributed much to the understanding of the parallelogram law. It is in fact the study of a particular case of the parallelogram law. In that case the components of the apparent weight are perpendicular to each other. One of the components is eliminated by the reaction of the plane. It is thus perpendicular to the plane. The other component is parallel to the plane. Thus it is not surprising to see that one of the first ideas for studying the stability will be to consider, as Parent did, the keystone of a dome to be a wedge (Fig. 13). That is the reason for the importance of the second of Parent's articles of 1704, whose title translates as "Finding the force with which a wedge must be pushed to separate a body either directly, or on a fixed point or on two" (Parent 1704c). In this work Parent generalizes the law of the inclined plane to cases where the direction of the force that maintains the weight on the plane is not parallel to the plane. Roberval had also done this in 1636.

In a manuscript dated 5 November 1705, Jacob Bernoulli (1654–1705) studied the equilibrium of the dome and has found using the law of the lever that the best curve in order to build an infinitely thin dome is the catenary (Jacob Bernoulli 1704), but this text would exert an influence only after its publication in his *Opera omnia* in 1744. Moreover, although La Hire gave the parallelogram law of forces in his *Traité de mécanique* (1695), in his next article on the stability of domes, he will return to his fundamental law, the law of the lever. The title of his article, also published in the *Mémoires*, is "Sur la construction des voûtes dans les édifices" (La Hire 1712). The influence of this text will be very important in the domain of architecture. The reason of its success is the same as for the *Traité*. La Hire doesn't use abstract fundamental laws like the parallelogram law, but uses instead practical rules that are easy to explain to workmen on the site, who all knew what a lever is. In his *Traité* he had given a way to solve the problem of the form of a stable dome, while in this article he gives a way of finding the width of a pier in order for the dome to be stable, or of finding the width of the pier in order to resist the pressure of the dome:

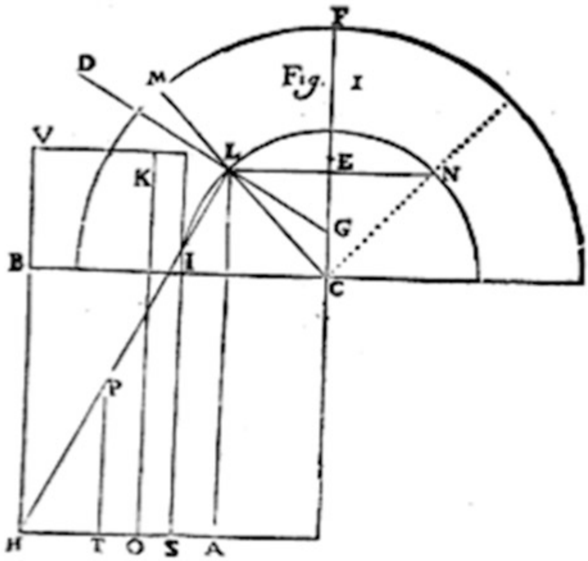


Fig. 14 La Hire's law for determining the thickness of piers sustaining an arch

"This is one of the most difficult Problems in Architecture, to find the force that the piers of arches must have to support the thrust" (Fig. 14).[32]

The method that was in use to solve that problem at the time was still Derand's rule.

La Hire goes on, "One calls the thrust of arches, the force created by all the stones that form them and which are cut in the form of wedge, called voussoirs, to pull apart the legs or piers that support the arch.[33] To solve the problem, La Hire decomposes the dome into three parts: a wedge, such as a keystone or as a bloc of voussoirs that stick together, as Parent had done, and two piers that also form unbreakable pieces (see Fig. 14). He then considers a lever with its fulcrum in H and its arms HL and HA. In L acts the half of the weight of the keystone LMF and in A acts the weight of the pier and that of ILM. He considers the width of the dome has to be uniform in order to consider surfaces instead of volumes. The equation he finds for the equilibrium of that lever is

$$\frac{1}{2}bfy^2 + \frac{1}{2}fyv^2 + fhv^2 = s^2eg - s^2fy - s^2fa$$

with LMF $= ss$; ILM $= vv$; LE $= f$; CE $= e$; IS $= b$; SA $= a$; TD $= h$ and HS $= y$.

[32] *C'est un Probleme des plus difficiles qu'il y ait dans l'Architecture, que de connoître la force que doivent avoir les pieds-droits des Voûtes pour en soutenir la poussée* (La Hire 1712, p. 69).

[33] *On appelle la poussée des Voûtes, l'effort que font toutes les pierres qui les forment & qui sont taillées en coin, qu'on appelle voussoirs, pour écarter les jambes ou pieds-droits qui soutiennent ces Voûte* (La Hire 1712, p. 69).

In this equation one needs to find y = HS, the thickness of the pier. He writes, "Although the equation I have just found is easy to build after it is reduced, it never allows itself to be composed because of the numbers of the terms it contains, and that is why it might even be shortened in practice . . . ".[34] La Hire shows us the importance of the fact that architects must collaborate with workmen on the building site and that this influences their choice of the laws they use. From the same point of view, the book by Hubert Gautier (1660–1737), *Dissertation sur l'épaisseur des culées des ponts, sur la largeur des piles, sur la portée des voussoirs* (1717) is even more interesting, because he criticizes La Hire:

> I ingenuously confess that I am not clever enough to understand [La Hire's article]. I have not even been able to follow his calculation as I find it too composed, and I look at everything he told us, as something which those less educated, and especially workmen, cannot understand. For in order to understand what he relates, it is necessary to know algebra perfectly, without the help of which I think no Stone-cutter, Fitter or Architect, for whom these kinds of books should be made & made easy, can ever enjoy it because usually these people do not apply themselves to this Science, as irrelevant to their profession, and because they have lots of other work to do. I am advised that when Mr. de la Hire may wish to resolve these difficulties, making them easy for anyone involved in building, he can do better than anyone else, since he is brighter, and this is what is to be hoped.[35]

He then considers Fontenelle's article (1704a) in which Fontenelle explains how the force, (namely the weight of a voussoir) is transmitted along the dome until it arrives at the pier using only the law giving the apparent weight on an inclined plane and explaining that the pressure becomes infinite on the pier. This way of explaining the transmission of the force appears to be much more concrete and adapted to workmen on the site, but it doesn't take the direction of its force into account. The apparent weight being always considered to be pressing down vertically, it has to be added to the weight of the previous voussoirs. That is the reason why it becomes infinite.

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[34] *Quoique l'Equation que je viens de trouver, soit facile à construire après qu'on l'aura réduite, elle ne laisse pas d'être composée à cause de la quantité des termes qui y sont ; c'est pourquoi on pourroit encore l'abréger dans la pratique* (La Hire 1712, p. 73).

[35] *J'avouë ingenuëment que je ne suis pas assez habile pour le* [l'article de La Hire] *comprendre. Je n'ay pas pû même suivre son Operation tant je la trouve composée ; & je regarde tout ce qu'il nous a dit, comme une chose dont les demi Sçavans, & surtout les Ouvriers, ne sçauroient comprendre. Car pour concevoir ce qu'il rapporte, il faut sçavoir absolument l'Algèbre, dont il emprunte les secours, je ne crois pas qu'un tailleur de pierres, Appareilleur, ni Architecte, pour qui ces sortes d'Ouvrages doivent être faits & rendus aisez, en puissent jamais profiter, parce que pour l'ordinaire ces Personnes ne s'appliquent pas à cette Science, comme inutile à leur Profession, & comme infiniment occupez ailleurs à leurs Ouvrages. Je suis prévenu que lorsque Mr de la Hire voudra bien resoudre ces Difficultez, pour les rendre aisées à tous ceux qui se mêlent de bâtir, il pourra le faire mieux qu'un autre, comme ayant plus de lumieres ; & c'est ce qui est bien à souhaiter* (Gautier 1717, p. 6).

Gautier also explains that before La Hire's *Traité*, architects had to use Derand's rule to find the width of the pier and criticizes the fact that this law is not proved, and that it only works because the measure it gives is far too large. He also recalls the important paper Parent had read at the Academy in 1704 (Parent 1704a).

In complete opposition to what we just have just seen, Pierre Couplet starts his article "De la poussée des voûtes" (1729) with a lemma on the law of composition of forces:

> Lemma. If the force x is decomposed into two forces y & z, these three forces will be together like the sides of a triangle formed by the perpendicular conducted on the directions of the three forces.[36]

This is the form that La Hire gave to the law. Surprisingly, Couplet starts his demonstration with the general parallelogram law.

> Since the force is decomposed in the two forces y and z, these two forces z & y are expressed by the sides AB, AD of the parallelogram ABCD where x is the diagonal, and these three forces x, y, z, will be to each other as those same lines AC, AB, AD.[37]

From this he easily derives La Hire's rule. Remarkable is the fact that Couplet uses both forms of the law of composition of forces. In his first problem, he uses the usual parallelogram law (Fig. 15), while in the theorem I he uses La Hire's form of the law (Fig. 16).

Like Jacob Bernoulli, Couplet realised that the vault most stable under its own weight is a reversed catenary. For that reason, he gives an illustration of the catenary together with that of the vault (Fig. 17).

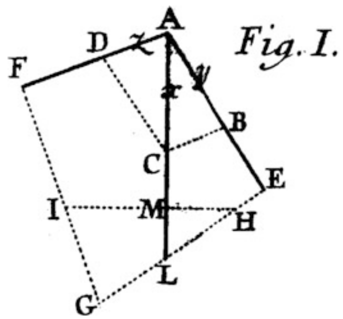

Fig. 15 Couplet's parallelogram law

[36] *Lemme. Si la force x se décompose en deux forces y & z, ces trois forces seront entre elles comme les côtés d'un Triangle formé par les perpendiculaires menées sur les directions des trois forces* (Couplet 1729, p. 80).

[37] *Puisque la force x se décompose dans les deux forces y & z, ces deux forces y & z seront exprimées par les côtés AB, AD, du parallelogramme ABCD, dont x est la diagonale, ainsi ces trois forces x, y, z, seront entre elles comme ces mêmes lignes AC, AB, AD* (Couplet 1729, p. 81).

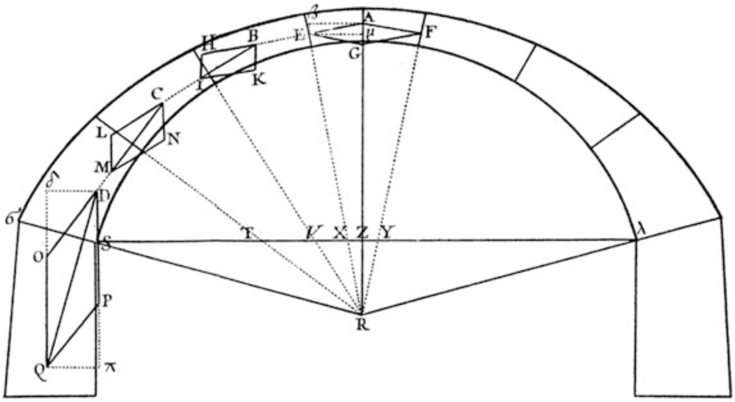

Fig. 16 Couplet's parallelogram law in the vault

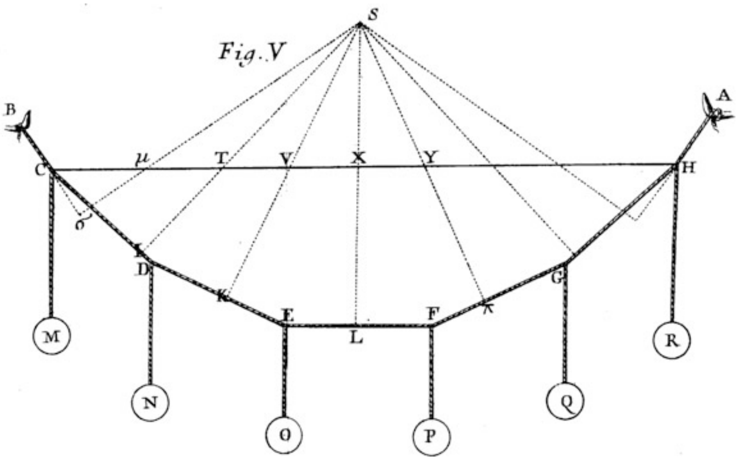

Fig. 17 Couplet's catenary

He states his first problem as follows:

> Problem I. Determine the ratio there must be between the weights of the Voussoirs forming any arch, and the thrust of Voussoirs so that they balance each other without the aid of mortar between their parts.[38]

In his solution, he shows how the forces are transmitted through the dome, with help of the parallelogram law. He seems to be the first to do that, and it is worth reading this in its entirety (refer to Fig. 15):

[38]*Problème I. Déterminer le rapport qu'il doit y avoir dans les pesanteurs des Voussoirs qui forment une Voûte quelconque, & quelle est la poussée des Voussoirs, afin qu'ils fassent équilibre entre eux sans le secours de l'engrément de leurs parties* (Couplet 1729, p. 81).

Let A, B, C, D, be the Voussoirs from the key to the pier and abutment, [the problem is] to determine the ratio there should be between the weight of those Voussoirs, that is to say, what their section should be, expressing their weight by their section. Imagine the verticals AG, BK, CN, DP, driven by the centres of gravity of Voussoirs somewhere where are the centres of gravity.

First, the key A, by its weight, exerts against its two neighbouring Voussoirs forces that will be perpendicular to the joints at which it touches them.

Thus if one draws from the centre of gravity of the Key A, taken on the vertical AG, the lines AE, AF perpendicular to its joints, and that around any position AG of the vertical, taken as diagonal, one completes the parallelogram AEGF.

For when expressing the gravity of the key A by the vertical diagonal AG, the sides AE, AF, of the parallelogram will express the forces that Key A exerts perpendicularly to the adjacent Voussoirs.

Now let the direction AE of the force that key A exerts on its neighbour Voussoir B be extended until H. And from point B where that line meets the vertical BK, BH = AE being taken, which is the force that Voussoir B has received from the Key A and let the parallelogram HIKB be completed. Then this Voussoir B will press its neighbour C with a force BI composed of gravity BK, BH and of the force he received from the Key A.

But this weight BK, and this force BI of the Voussoir B against the Voussoir C will be easily determined, if you consider that the composed force BI must be perpendicular to the joint, and that the force BH this Voussoir received from the key A is given = AE, with its direction AEBH.

So if we make BI perpendicular to the joint, and if through the given point H, one draws the vertical HI, and that from point I, where it meets BI one draws IK parallel to BH, there will be a parallelogram BHIK, whose vertical side BK express the gravity of the Voussoir B, side BH, the force the Voussoir received from the Key A and the diagonal BI, which is perpendicular to the joint, will express the composed force that the Voussoir B exerts against the Voussoir C.[39]

[39] *Soient les Voussoirs A, B, C, D, depuis la Clef jusqu'au pied-droit & pilier buttant, il s'agit de déterminer quel rapport il doit y avoir entre la pesanteur de ces Voussoirs, c'est-à-dire, quelle rapport il doit y avoir entre la pesanteur de ces Voussoirs, c'est-à-dire, quelle doit être la surface de leur coupe, en exprimant leur pesanteur par leur coupe. Imaginons les verticale[s] AG, BK, CN, DP, tirées par les centres de gravité des Voussoirs, quelque part où se trouvent ces centres de gravité.*

Premièrement la Clef A par sa pesanteur fera contre ses deux Voussoirs voisins des efforts perpendiculaires aux joints par lesquels elle les touche.

Ainsi du centre de gravité A de la Clef, pris sur la verticale AG, si l'on tire les lignes AE, AF, perpendiculaires sur ses joints, & qu'autour d'une position quelconque AG de la verticale, prise pour diagonale, l'on achève le parallélogramme AEGF.

Pour lors en exprimant la pesanteur de Clef A par la diagonale verticale AG, les côtés AE, AF, du parallelogramme exprimeront les efforts que cette Clef A fait perpendiculairement sur les Voussoirs voisins.

Maintenant soit prolongée la direction AE de l'effort que la Clef A fait sur son Voussoir voisin B jusqu'en H. Et du point B, où cette ligne rencontre la verticale BK, soit pris BH = AE, qui est l'effort que le Voussoir B a reçu de la Clef A, & soit achevé le parallelogramme HIKB, pour lors ce Voussoir B Pressera son voisin C avec une force BI composée de la pesanteur BK, & de l'effort BH qu'il a reçû de la Clef A.

Couplet repeats the same construction twice for the voussoirs C and D and finds DP, the force exerted by the arch against its pier.

In his second corollary, he decomposes this force into a vertical component and a horizontal component. He says that the vertical component doesn't contribute to overturn the pier; on the contrary, it contributes to reinforce its stability.

> If the joint Sσ, which is not horizontal,[40] is the last that we consider in the vault, it will be necessary to decompose the pressure DQ acting on this point into two forces Dπ, Dδ, one Dπ being vertical, and the other Dδ horizontal.
>
> Then the vertical force Dπ will make no force to overthrow the right pier on the contrary it will all be used to strengthen it, rather than the horizontal force Dδ will all be used to overturn it.[41]

The horizontal component is totally dedicated to overturning the pier. This component is the thrust. The triangle law of composition enables him than to demonstrate that:

> [o]ne will find the weight of half the vault, if from the bottom and from the top of the key, two horizontal lines are drawn, that is to say, a tangent to the intrados, and a tangent to the extrados, both drawn to meet the last lower joint of the extended Voussoir, and half the sum of these two tangents multiplied by the height of the key which is the first Voussoir, will give an area equal to the section of the arch, and therefore will give the strength of the arch, and consequently its weight.[42]

Couplet demonstrates this in the following way: He traces two tangents, one to the top of the keystone and the second to its bottom. He draws lines along the sides of the voussoirs going from those tangents to the centre C of the dome. In this

Or cette pesanteur BK, & cet effort BI du Voussoir B contre le Voussoir C seront facile à déterminer, si l'on fait attention que l'effort composé BI doit être perpendiculaire sur le joint, & que la force BH que ce Voussoir a reçû de la Clef A, est donnée = AE avec sa direction AEBH.

Donc si l'on fait BI perpendiculaire sur le joint, & si par le point donné H, l'on tire la verticale HI, & que du point I, où elle rencontre BI, l'on tire IK, parallele à BH ; l'on aura un parallelogramme BHIK, dont le côté vertical BK exprimera la pesanteur du Voussoir B, le côté BH, la force que ce Voussoir a reçû de la Clef A, & la diagonale BI, qui est perpendiculaire au joint, exprimera l'effort composé que le Voussoir B fait contre le Voussoir C (Couplet 1729, pp. 81–83).

[40] To avoid the problem of an infinite force.

[41] *Si le joint Sσ, qui n'est point horizontal, est le dernier que l'on considere dans la Voûte, il faudra décomposer la pression DQ sur ce point en deux forces Dπ, Dδ, dont l'une Dπ est verticale, & l'autre Dδ est horizontale.*

Alors la force verticale Dπ ne fera point effort pour renverser le pied-droit, mais au contraire elle sera toute employée à l'affermir, au lieu que la force horizontale Dδ sera toute employée à faire effort pour le renverser (Couplet 1729, p. 84).

[42] *L'on aura la pesanteur de la moitié de la Voûte, si du dessous & du dessus de la Clef l'on tire deux lignes horizontales, c'est-à-dire, une tangente à l'intrados, & une tangente à l'extrados, toutes deux menées jusqu'à la rencontre du dernier joint inférieur du Voussoir prolongé ; & la moitié de la somme de ces deux tangentes étant multipliée par la hauteur de la Clef qui est le premier Voussoir, donnera une surface égale à la coupe de la Voûte, & donnera par conséquent la solidité de la Voûte, & partant sa pesanteur* (Couplet 1729, pp. 92–93).

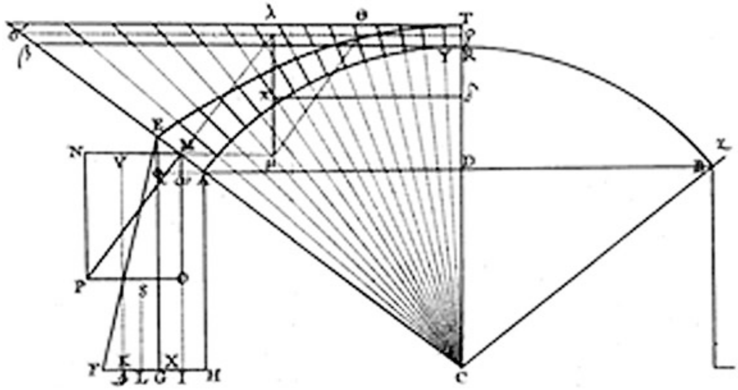

Fig. 18 Couplet's computation of the weights of the voussoirs

way he forms two series of triangles having all their sides perpendicular to forces of the parallelogram of figure II. The sides on the tangents are perpendicular to the weights and the others to the components of the weights. The triangles of both series are in the same proportion as their bases lying on one or the other tangent, thus in the proportion of their weights. Thus the series of the differences of each pair of triangles, or the parallelograms between tangents are also in the same proportion. Their surface is in the same proportion as their weights. This is a very simple and clear relation that every stonecutter understands and finds useful (Fig. 18).

This law is a particular case of what Parent explained in his article read to the Academy (1704a). Parent considered a general curve as the intrados, in contrast to Couplet who considers a circular one. Of course, Parent's general case is not that easy to explain to a stonecutter.

At the beginning of his second book dedicated to domes in his famous *Science des ingénieurs dans la conduite des travaux de fortification et d'architecture* (Bélidor 1729), published in the same year as Couplet's first article, Bernard Forest de Bélidor (1698–1761) set out a mechanical principle (Fig. 19).

This principle is the parallelogram law and Couplet shows its equivalence with La Hire's triangle rule:

> Principle drawn from mechanics
>
> 2. It is shown in mechanics that three powers P, Q, R, pulling or pushing around a point A in directions AP, AQ, AR, will be balanced among themselves, if after having made the parallelogram ABCD, the power P is expressed by the side AB, power Q, by the side AD, and power R, by the diagonal CA: or what is the same, if each power is expressed by one of the sides of the triangle ABC, because instead of AD, BC may taken, that is equal;

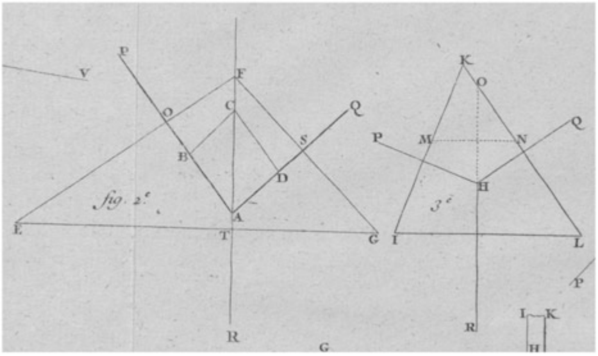

Fig. 19 Bélidor's parallelogram law

supposing that one is well informed of that truth, here is a basic proposition that can be drawn from it.[43]

Nevertheless, he doesn't use the parallelogram law and doesn't show the transmission of the force through the dome. In order to find the width of the piers, he returns to La Hire's explanation, with the three pieces of the dome—a keystone and two piers—and uses the equilibrium of a lever having its fulcrum at the extremity of the pier as shown in Fig. 20.

One lever arm is OP, where the weight of half the keystone or the voussoir CFGD acts, there are two other lever arms, one is PT, half of PS the thickness of the pier where the weight (n^2) of the pier acts, the other is PR, where the weight of the voussoir CFGD acts, its centre of gravity located in Q, vertically aligned with R. Where y is the thickness of the pier, n^2 = surface (or the weight) of each voussoir CFGD or CFEB; f = MP—BV; g = PS—PR; d = ZP the height of the pier. Bélidor writes the equilibrium of the lever and gets:

$$n^2 f - n^2 y = \frac{dy^2}{2} - n^2 y - n^2 g.$$

He then solves the equation, which La Hire refused to do, and gets "Bélidor 's Formula":

$$\sqrt{\frac{2n^2 f + 2n^2 g}{d} + \frac{4n^2}{d^2}} - \frac{2n^2}{d} = y,$$

[43] *Principe tiré de la mécanique*

2. Il est démontré dans la mécanique que trois puissances P, Q, R, qui tirent ou poussent autour d'un point A, selon des directions AP, AQ, AR, seront en équilibre entr'elles, si après avoir fait le parallelogramme ABCD, la puissance P, est exprimée par le côté AB, la puissance Q, par le côté AD, & la puissance R, par la diagonale CA : ou ce qui revient au meme, si chaque puissance est exprimée par un des côtés du triangle ABC, parce qu'à la place de AD, l'on pourra prendre BC, qui lui est égal; suposant donc qu'on soit bien prévenu de cette vérité, voici une proposition fondamentale qu'on peut en tirer (Bélidor 1729, Book II, pp. 6–7).

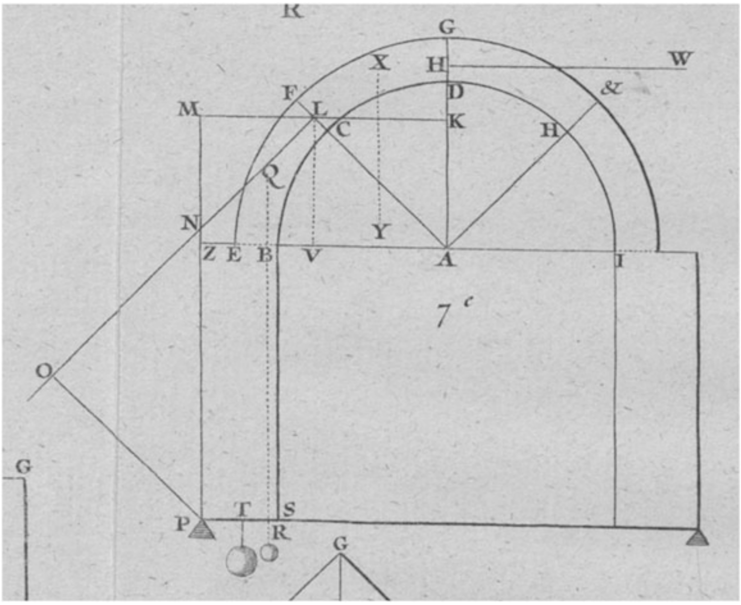

Fig. 20 The rule of Bélidor and La Hire

The next year, in the second part of his memoir on the thrust of arches (Couplet 1730), Couplet goes on using the parallelogram law. It allows him (Problem II) to evaluate the thinnest possible thickness of an arch. This was a really new problem that paved the way to Soufflot's idea of lightness and is in full opposition with Derand's over-evaluated rule.

In Problem III, he goes on with his reasoning about the force exerted by the arch against its pier and finds an equation giving its measure in a particular case. Namely, he considers a half of the arch as a one piece:

> Now since the weight of the half-arch is collected in its centre of gravity P, if through this centre of gravity P, one draws the vertical LR, and that through point S, middle of AM, the horizontal SL is drawn and that from the point L where it meets the vertical LR, one draws LX in the middle of the springer, and that from point X, one draws XR parallel to LT, and we make RT parallel to LX, we'll get a parallelogram TX, with a diagonal LR expressing the weight of the half-arch AN, the line LT will express the force the half vault AN exerts horizontally to resist the similar force of the other half-vault, and the line LX will express the force this same half arch AN exerts in the direction LX against the springer.[44]

---

[44]*Maintenant puisque la pesanteur de la demi-voûte est réunie à son centre de gravité P ; si par ce centre de gravité P, l'on tire la verticale LR, & que par le point S, milieu de AM, l'on tire l'horizontale SL, & que du point L, où elle rencontre la verticale LR, l'on tire LX au milieu du coussinet, & que du point X, l'on tire XR parallele à LT, & que l'on fasse RT parallele à LX, l'on aura un parallelogramme TX, dont la diagonale LR exprimant la pesanteur de la demi-voûte AN, la ligne LT exprimera l'effort que cette demi-voûte AN fait horizontalement pour résister à l'effort*

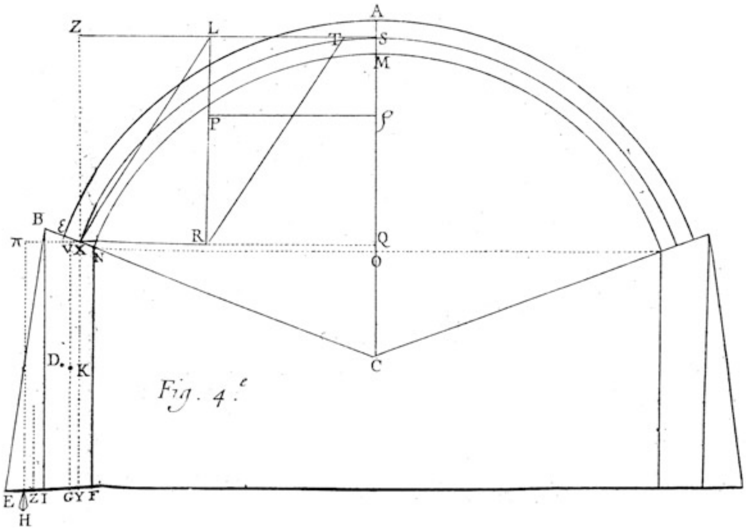

Fig. 21 Couplet's use of the parallelogram law for the vault

He decomposes the weight of half of the arch into one horizontal component LT applied in S, which is equilibrated by the reaction of the other half of the arch, and another component LX that acts on the pier. Then he decomposes LX into its component LR, which equals the weight of half of the arch, and RX, the thrust (Fig. 21).

Couplet calculates then the thrust RX thanks exclusively to proportions:

$$RX = \frac{2arm + amm}{2dr} \times \sqrt{2ar - dd} - \frac{6mrr - 6rmm - 2m^3}{6r + 3m},$$

where r = radius of the intrados, m = thickness of the arch, a = arc MN of the intrados, d = height MO of the intrados.

A particular case of this formula gives the value of the thrust. Instead La Hire had found the thickness to give to the pier, with help of the law of the lever. This is Couplet's last problem.

> When the voussoirs can not slide over each other, to find the base EF of the pier, such that the force composed by the weight of the vault, the horizontal thrust and gravity of that pier is pointing towards any given point H of said base EF.[45]

semblable de l'autre demi-voûte, & la ligne LX exprimera l'effort que cette même demi-voûte AN fait suivant cette direction LX contre le coussinet (Couplet 1730, p. 133).

[45] *Losque les voussoirs ne sçauroient glisser les uns sur les autres trouver la base EF du pied-droit, telle que l'effort composé de la pesanteur de la voûte, de la poussée horisontale, & de la pesanteur dudit pied-droit soit dirigé vers un point donné quelconque H de ladite base EF* (Couplet 1730, p. 135).

His condition is thus that the resultant of the weight of the arch, of the thrust, and of the weight of the pier passes through the base EF of the pier.

I won't go into the details of the article by Pierre Bouguer (1698–1758), "Sur les lignes courbes qui sont propres à former les voûtes en dome" (1734), published in the *Mémoires* of 1736 (see Radelet-de Grave 1994, 1999). Not only does Bouguer be perfectly trained in the use of the parallelogram law but also in the use of differential an integral calculus. We are with his text as well as with Jacob Bernoulli's manuscript written in 1705 in a completely different world.

In his three-volume *La théorie et la pratique de la coupe des pierres* ... (Frézier 1737–1739), Amédée François Frézier (1682–1773) gives a very good survey of the works of his predecessors in the long chapter XII "Appendice concernant le dispositif de la construction des voutes. Premièrement, de la poussée des voutes" (Frézier 1737–1739 (vol. III), pp. 345–410). His aim is clearly to impose La Hire's idea. He claims proudly

> For the second way, which is building without calculation, with ruler & compass, which is more convenient and more within the reach of workmen, we give that of Mr. La Hire, which Gautier viewed ... as unintelligible, and we will see that it is no more difficult to perform than many of the lines of stonecutting.[46]

This implies that he understood it, but although he often mentions Couplet, he clearly doesn't understand what Couplet had done:

> Although this first hypothesis [that of La Hire] provides a very safe way to practice, Mr. Couplet, in the same academy, judged that one could find more precisely the force of the thrust of vaults particularly considering each voussoir as a wedge making a force to spread its effects, and because these wedges can be considered to be either polished bodies, or granular and rough, he studied the result of each of these assumptions, to determine the thickness of the piers.[47]

It is true that Couplet considers those various modelisations of voussoirs, but that is not the important point of his *Mémoire*, as we have emphasized. In fact, Frézier didn't grasp the importance of the parallelogram law. Even when he reproduces the figure of the problem and of its demonstration, he never speaks of either a parallelogram or of its components. His presentation is purely geometric; his only concession to mechanics is to call P the centre of gravity, but he only considers the triangles forming the parallelogram (Fig. 22).[48]

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[46]*Pour la seconde voie, qui est celle de la construction sans calcul, avec la règle & le compas, qui est plus commode & plus à la portée des ouviers, nous donnerons celle de M. de la Hire, que Gautier a regardé (page 6) comme inintelligible, & l'on verra qu'elle n'est pas d'une exécution plus difficile qu'un grand nombre des traits de la coupe des pierres* (Frézier 1739 (vol. III), p. 349).

[47]*Quoique cette premiere hypothese* [de La Hire] *fournisse une solution très-sûre pour la pratique, M. Couplet, de la même académie, a jugé qu'on pouvoit trouver avec plus de précision l'effort de la poussée des voûtes, en considérant en particulier chaque voussoir comme un coin qui fesoit effort pour écarter ses collatéraux ; & parce que ces coins peuvent être considérés comme des corps polis, ou comme grenus & raboteux, il a exminé le résultat de chacune de ces suppositions, pour déterminer l'épaisseur des piédroits* (Frézier 1739 (vol. III), pp. 344–345).

[48]For the curious reader, the proof occupies (Frézier 1739 (vol. III), pp. 378–380).

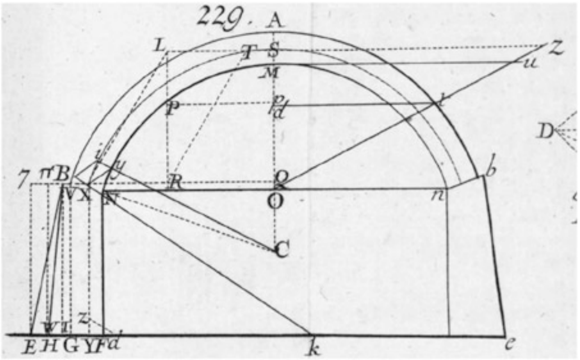

Fig. 22 Frézier's reproduction of Couplet's figure 4

Here I ask the reader to allow me to insert a piece of a story that is related but not crucial for our purpose. Frézier tells us that after having read without understanding Couplet because *le calcul en est long & fort composé*, he decided to go and ask for an explanation to Johann I Bernoulli (1667–1748):

> I thought that I would render a public service if I proposed a solution easier & cleaner to practice. In this idea, rightly considering Mr. Bernoulli as one of Europe's geometers most capable of finding it, I asked him to give it a few hours of his time, which he has kindly given to me, although he was unwell, thus giving me a sign of friendship for which I am very grateful.[49]

Poor Frézier didn't realize the difficulty of the way he had chosen. He continues:

> But since this solution requires the knowledge of the rule of energy by virtual velocities, he was kind enough to give me a letter he wrote to Mr. Varignon in [26 February] 1715, concerning this rule, of which I will give an excerpt before entering into matter; he begins establishing the principle that in each equilibrium there is an equality of the energy of absolute forces times virtual velocities.[50]

Frézier then reproduces a part of Bernoulli's famous letter published by Varignon in his *Nouvelle Méchanique*. May I ask the reader which he finds easier: the concept of force and the parallelogram law, or the concepts of energy and work involved in the principle of virtual work?

At the end of 1742, many cracks appeared in the cupola of St. Peter's in Rome, leading to several publications regarding its repair. Three renowned scientists

[49] *J'ai cru que je rendrois service au public si je lui proposois une solution plus simple & plus propre à la pratique. Dans cette idée, considérant avec raison M. Bernoulli comme un des géometres de l'Europe les plus capables de la trouver, je l'ai prié d'y donner quelques heures de son tems, qu'il a bien voulu m'accorder, quoiqu'il fut incommodé ; en quoi il m'a donné une marque d'amitié dont je suis très-reconnoissant* (Frézier 1739 (vol. III), p. 361).

[50] *Mais comme cette solution suppose une connoissance de sa regle d'énergie par les vîtesse virtuelles, il a eu la bonté de me faire part d'une lettre qu'il écrivit à M. Varignon en* [26/2/] *1715, touchant cette regle, dont je vais faire un extrait avant que d'entrer en matiere ; il commence par établir ce principe, que dans chaque équilibre il y a une égalité d'énergie de forces absolues par les vitesses virtuelles* (Frézier 1739 (vol. III), p. 361).

proposed one solution in 1743. Two of them, Thomas Le Seur (1703–1770) and Francis Jacquier (1711–1788), were both Minims who published the fourth edition of Newton's *Principia* with lengthy comments. The third author, Ruggiero Boscovich (1711–1787), was a Jesuit who had taught mathematics and philosophy at the Collegio Romano. Strangely, their work, *Parere di tre mattematici sopra i danni che si sono trovati nella cupola di S. Pietro sul fine dell'anno 1742* (1743), is not technical; it gives a careful description of the cracks and of the state of the *cupola vaticana* and only mentions La Hire and Couplet.

Of a much more technical nature is the memoir by Giovanni Poleni (1683–1761), *Memorie istoriche della gran cupola del Tempio Vaticano* (1748), a very important text that surely had a profound influence on architecture and perhaps on the polemic we will examine below. After having given an overview of the history of the construction of the dome, in chap. VIII "concerning the shape of arches, and in general on the true regular shape of cupolas,"[51] he studies in general the form of arches and domes, calling this "the most difficult of the particular part of mechanics that concerns architecture.[52] He writes of a particular mechanics adapted to architecture, and says that "Architecture participated to advantage in the progress of the mechanical sciences".[53] In the summary of the principal facts of the story that concerns us here, Poleni starts with Derand, Blondel and Fontana, whose rules, he says, are not of the kind of those which link pure geometry with material mechanics together with architecture:

> Moreover these Rules are not even of that kind in which pure geometry is so well adapted to the Mechanics of materials of buildings, so that geometry and mechanics are united, their union benefitting Architecture.[54]

In his opinion, the first rules of that kind are those of La Hire and Couplet. He also mentions Joseph Dulacq (1706–1756) and Fontenelle as the anonymous author of the *Histoire de l'Académie de Paris* before going on with the story of the discovery of the use of the catenary in architecture (see Radelet-de Grave 1994). Finally Poleni arrives at his main point, namely the composition of forces, which he considers to be one of the three principal points, along with shape and equilibrium.

He gives the following definition of equilibrium concerning the stability of arches or domes:

> It must be observed, that the parts of any arch or vault all tend to fall, but it is necessary that none of them fall. Thus it is necessary, that all parts tend to fall in such a way that the actions of their forces are equally eliminated by opposition, and reaction of the other parts

[51] *Della figura degli archi, ed in universale della vera regolar figura delle cupole* (Poleni 1748, p. 30).

[52] *... la parte più difficile nella Meccanica particolare dell'Architettura* (Poleni 1748, p. 30).

[53] *Architettura partecipato abbia ne' vantaggi de' progressi della Meccanica Scienza* (Poleni 1748, p. 31).

[54] *Del resto esse Regole non sono già di quelle, in cui la pura Geometria è alla Meccanica materiale delle Fabbriche adattata cosi, che quella e questa ben s'uniscono per giovar con l'union loro all'Architettura* (Poleni 1748, col. 32).

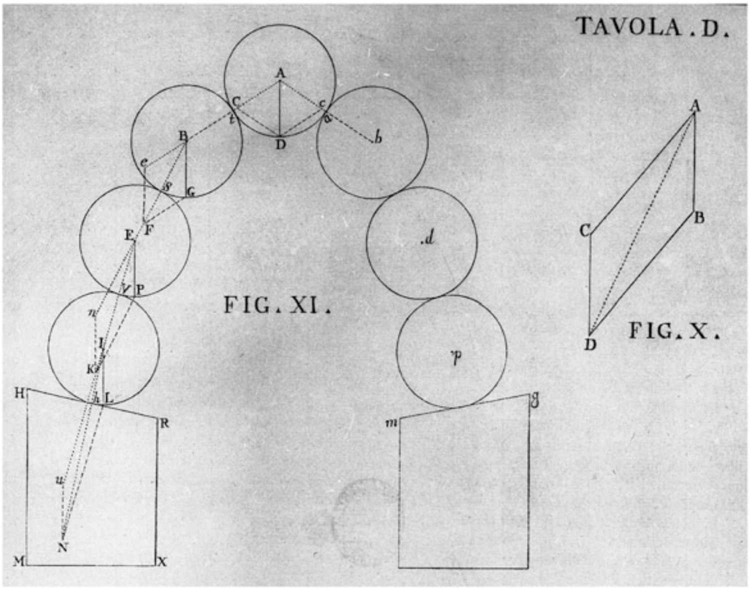

Fig. 23 Poleni on about the use of the parallelogram law for the vault (1748, Tav. D, Figs. X and XI)

> against whom they act: otherwise, the parts that would not find a resistance equal to their force to fall, would press down (when some other cause does not stop them) and would sag, elevating in consequence in some other place the arch or vault, the whole of which would suffer from those forces in excess.[55]

As for what concerns the parallelogram law, he reprises Newton's law saying (Fig. 23):

> If a body, in a given time, with the only force M acting in place [Tab D. Fig X] A, would go from A to B with a uniform motion, and with the power N acting in the same place, would go from A to C which completes the parallelogram ACDB, that body with both forces, in the same given time would go through the diagonal from A to D.[56]

[55] *Per rispetto agli Equilibrii, egli è da osservarsi, che le parti di qualunque Arco, o Volta tendono tutte a cadere ; ma bisogna, che nessuna cada. Quindi è di necessità, qu'esse parti a cader tendano tutte in maniera, che le azioni de' loro sforzi siano ugualmente distrutte da' contrasti, e dalle riazioni delle altre parti, contro cui quelle agiscono : altrimenti le parti, che non ritrovassero una resistenza uguale al loro conato per cadere, si sforzerebbero verso l'in giù (quando qualche altra causa non le impedisse) e si abbasserebbero, elevando per cosequenza, in qualche sito l'Arco, o la volta, che tutta si risentirebbe di que' sforzi eccedenti* (Poleni 1748, cols. 35–36).

[56] *Se un corpo, in un dato tempo, con la sola forza M nel luogo* [Tab. D. Fig. X] *A impressa ; potesse con un moto uniforme andare da A a B ; e con la sola forza N nello stesso luogo impressa, potesse andare da A a C ; si compisca il parallelogrammo ACDB, e quel corpo con ambedue le forze, nello stesso dato tempo anderà per la diagonale da A a D* (Poleni 1748, p. 36).

In fact, this law is not a law of composition of forces but of motions. Poleni knows this, and goes on saying that it is also valid for *sforzi al moto*, for forces giving rise to motion. With this law, we can go to the spheres of James Stirling (1692–1770). The figure is well known and the idea is generally attributed to Poleni himself, but Stirling had published it before Poleni (Radelet-de Grave 2012). However, even that attribution is still wrong because the idea was given to Stirling by Newton in a letter (Gregory and Newton 1694, p. 345). Poleni then quotes Stirling's explanation, which is close to that of Couplet, despite the fact that Couplet was dealing with voussoirs.

With the striking example of Newton-Stirling-Poleni, the parallelogram law of forces not only entered into architecture but became fundamental to it. This last point we owe to Poleni.

6 The Polemic as a Quarrel Between Tradition and New Ideas

In the middle of the century, there were two opposite tendencies into architecture, as we have seen: a traditional one that ignored or underestimated the importance of the parallelogram law, and a modern one that wanted to make use of this law and avoid the excessive sizes due to the rules of Derand and others. The moderns aimed at lightness, as did Soufflot. Poleni is a good example of the modern tendency, while the success of the second and third editions of Derand's *Architecture des voûtes* (1743), published in 1743 and in 1755 respectively, provide evidence of the vivacity of the traditional tendency. In that spirit we can now go and read between the lines of the defenders and opponents of Soufflot's project for the Panthéon, to see who they were and what they published.

A certain Desbœufs, a student of the Académie royale d'Architecture in Lyon, criticized Soufflot's project as early as 1765, but his criticism (Desbœufs 1765) exclusively concerned questions of taste and was not taken very seriously. In any case, here I shall not relate political or aesthetic problems, but will instead concentrate on the problems of stability.

6.1 First Act of the Polemic, Its Actors and Publications

6.1.1 Historical Overview

The initiator of the polemic was Pierre Patte (1723–1814) an architect better known for his polemical character than for his very few architectural realisations. He was close to Blondel. He even was co-author of the third part of Blondel's *Cours d'architecture* (Patte 1777), dedicated to the *construction des bastiments*.

His Memoir on the construction of the dome designed to crown the church of Ste. Geneviève (Patte 1770a) was written and circulated a year before its actual publication. Patte sent it first to Abel-François Poisson de Vandières, Marquis de Marigny (1727–1781) (brother of Madame de Pompadour), who had been *directeur général des Bâtiments, Arts, Jardins et Manufactures* of the King since 1751. It was Marigny who had commissioned Soufflot to build the *Eglise Sainte-Geneviève*. Patte also sent his text directly to Soufflot, who showed it to Rondelet and probably charged him to reply. There were three repercussions following the circulation of Patte's Memoir:

1. Rondelet's first reaction to Patte's pamphlet was to write his comments in the margins of the memoire. I shall discuss both Patte's text and Rondelet's reactions more carefully after this general description. Transcriptions of both texts are published in a separate chapter in this present volume (Radelet-de Grave 2015). When writing his *Marginalia*, Rondelet was planning to publish an article based on those ideas, but such an article appears never to have been published. The title he intended to give it was:

   > Refutation of a memoir on the construction of the dome designed to crown the new church of Ste. Geneviève, aimed at proving that the piers already executed and destined to support this Cupola have the necessary dimensions desired to sustain a similar work with enough strength and to give thereon the solution of several problems useful for the construction of buildings . . . .[57]

   In spite of their not being published, Rondelet's comments laid the foundation for two other publications, both published anonymously in 1770. The first, attributed to Rondelet by the nineteenth-century historian Joseph-Marie Quérard, is a "Letter of an engraver in architecture to M. Patte, his learned friend, on the occasion of his Memoir on the church of Ste. Geneviève (Rondelet 1770b). The other one was entitled "Reasonable doubts of a churchwarden of S.-Étienne-du-Mont on M. Patte's problem . . . concerning the building of the dome of the church of Ste. Geneviève"; the author of this will be revealed in a moment.
2. Almost simultaneously, as far as one can judge, two letters from Perronet to Soufflot, written on 22 and 26 of January 1770 (Perronet 1770a, b), appeared in the *Mercure de France* in April 1770. In the June issue of the *Mercure* one finds an anonymous review of Patte's *Mémoire sur la construction de la coupole* (Anon. 1770). It could be by Perronet.

   Perronet was a self-made man who initiated what would become the École des Ponts et Chaussées. In 1763, he became *Premier ingénieur* of the King; two years before the beginning of our polemic, he started to build the Pont de Neuilly

[57] *Réfutation d'un mémoire sur la construction de la coupole projettée pour couronner la nouvelle Eglise de Ste Geneviève, où il est question de prouver que le piliers déjà exécutés et destinés à porter cette Coupole, ont les dimensions nécessaires pour espérer d'y élever un semblable ouvrage avec solidité et à lui donner à ce sujet la solution de plusieurs problèmes utiles à la construction des édifices* . . . (Rondelet 1770a; see also Middleton and Baudouin-Matuszek 2007, pp. 299–301; Radelet-de Grave 2015).

in Paris, considered revolutionary for the time, during which he experienced difficulties similar to those that Soufflot encountered with Patte's *Mémoire*. This explains why he became involved in the polemic.

3. On 1 May, after Patte's *Mémoire* had become publicly known, Marigny wrote to Soufflot to beg him to reply to Patte (Marigny 1770). Soufflot answered four days later (Soufflot 1770a). His answer is conventional. He talks about Perronet's support and about the large edifices he had made in Lyon and in Paris, "The very large arches of the stock exchange and the Theatre in Lyon, whose daring gave concern, are existing evidence of the manner in which I direct forces back to resistances".[58] Then, in a second letter to Marigny written 12 May, he proposed a kind of challenge to Patte: He said he would deposit the sum of 12,000 pounds with a lawyer, which is to be paid to Patte "If I do not show that the equation, which is printed in his memoir, is false concerning the dome of Ste. Geneviève, and how it will be built";[59] Patte will get the money.

   It is important to note that Soufflot does not say that the formula is false—he would be wrong. What he in fact says is that it does not apply in the case of his church. This most important fact has never been noted.

   A slightly revised version of the letter dated 12 May was published in the July issue of the *Mercure de France* (Soufflot 1770b). In the following issue of *Mercure*, August 1770, Patte reacted to Soufflot's answer to Marigny. The title of Patte's answer is *Fragment d'une réponse de M. Patte à M. le Marquis de Marigny, relativement au défi proposé par M. Soufflot* (Patte 1770b).

   Charles-Nicolas Cochin (1715–1790), a well known engraver, friend of Madame de Pompadour, author of the frontispiece of the *Encyclopédie* of Diderot and d'Alembert and historiographer of the Académie Royale de Peinture et de Sculpture, announced in a *Lettre à l'auteur du Mercure de France* in October (Cochin 1770b), that he was the author of the *Doutes raisonnables d'un marguillier* (Cochin 1770a). This was followed that same month by a letter by Pierre Patte published in the *Mercure de France* (Patte 1770c).

This third group of events contains very few scientific arguments, so that I shall not consider it from that point of view.

6.1.2 Scientific Point of View

Patte explained the aim of his Memoir more than once, but the best formulation is found in his answer to Cochin in October 1770: "The aim of my thesis is to prove that, whatever may be the projected dome; piers already built are not able to bear

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[58] *Les voûtes très considérables de la Bourse et du Théâtre à Lyon, dont la hardiesse donnait de l'inquiétude, sont des preuves existantes de la manière dont j'ai su renvoyer les efforts sur les résistances* (Soufflot 1770a).

[59] *Si je ne démontre pas que l'équation, qui est dans son mémoire imprimé, est fausse vis-à-vis du dome de Sainte Genevièvre, & de la manière dont il sera construit* (Soufflot 1770b).

and brace these vaults with solidity".[60] The generality of his demonstration catches the attention of the scientific reader who knows that the thrust of a dome varies much with its form: Patte must have some simplification in mind.

After an interesting discussion on the role of mathematics in architecture (a discussion of which would lead us too far from our subject), Patte invokes Parent's memoir read at the Academy in 1704:

> Mr. Parent has shown, in the *Memoirs of the Academy of Sciences* for 1704, what the thrust of an arch is, how its various voussoirs act relative to their position, the key (*clef*) against the voussoirs on either side of it (*contre-clefs*) and those on adjacent voussoirs, and so on until they rest on the piers, and finally he determined the ratio of the thrust of an arch to the weight of the entire vault.[61]

Patte speaks next of La Hire:

> … the same academician [La Hire] later solved, in the Memoirs of the Academy of Sciences, of 1712, the problem of mechanics of vaults in its full extent, and gave specific rules to find on all occasions the force the piers or supporting walls of a vault must have to resist the thrust.[62]

He finally quotes Frézier and the articles of Fontenelle in the *Mémoires de Paris* and adds that many other authors found, using different ways, the same result as La Hire thus, "there can be no doubt about the certainty of the principles used to determine the thickness of the piers of the vaults they are mathematical truths".[63]

He then cites La Hire's theory and announces that he will use that theory to scrutinize the execution of the dome of Ste. Geneviève and to compare the dimensions of its supports to its thrust and to its weight.

To do this he plans to explain first the essence of the construction of a dome on pendentives, and then to confirm all that he says with examples. Finally he will compare those examples with the piers already built in the centre of Ste. Geneviève.

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[60] *Le but de mon mémoire est de prouver que, quelle que puisse être la coupole projettée, les piliers déjà élevés ne sont point en état de porter et contreventer ses voûtes avec solidité* (Patte 1770c, p. 169).

[61] *M. Parent a fait voir, dans les Mémoires de l'Académie des Sciences, de 1704, ce que c'est que la poussée d'une voûte, comment ses différents voussoirs agissent relativement à leur position; la clef sur les contre-clefs; les contre-clefs sur les voussoirs adjacents, & ainsi des autres jusqu'à leur retombée sur les piédroits, & enfin il a déterminé queI rapport a la poussée d'une voûte, eu égard au poids de la voûte entiere* (Patte 1770a, p. 6).

[62] *…ce même Académicien [la Hire] a résolu depuis, dans les Mémoires de l'Académie des Sciences, de 1712, le problème de la méchanique des voûtes dans toute son étendue, & a donné des régIes précises pour trouver en toutes occasions la force que doivent avoir les piédroits, ou les murs de soutennement d'une voute pour résister à la poussée* (Patte 1770a, p. 6).

[63] *…ainsi il ne sauroit y avoir de doute sur la certitude des principes qui servent à établir les épaisseurs des piédroits des voûtes ; ce sont des vérités Mathématiques* (Patte 1770a, p. 6).

6.1.3 Patte's Article Premier: On the Construction of Domes on Pendentives

Architects use various methods to built their domes, sometimes containing the outward forces of the domes by walls of uniform thickness, and sometimes building buttresses all around the tower that sustains the dome in order to transmit to them all the weight and the thrust of the dome. This sounds important, but Patte immediately shows that he is not comfortable with a global explanation of mechanics, saying that he only intends to look for the dimensions capable of sustaining a dome in a simple case, making use of "principals of whose certitude there is no doubt.[64] Thus he refers to the rule given by Fontana in Book V, ch. XXIV:

> Rule of the author in order to build drums, and simple domes with lanterns.
> We talk half the diameter of the space under the dome with large cornices above soccle with large arches, and exterior decoration of the drum A, the height of which is divided into four, three of which assigned to the ornament of the piers, foundation, architrave, frieze, cornice, as in B, and the fourth to the pedestal C, when the height of the roof allows this, which has no rule.[65]

While this is still purely geometrical and certainly not at the level of La Hire's rule, it is carefully explained by Patte. Rondelet's reaction is in the margin: "We will determine using more sure principles where one should place the iron rings".[66] On the next page, Rondelet adds: "We have nothing to say about the dimensions given by Fontana, they are more related to taste than to strength . . . .[67]

Patte gives then some measures and concludes that for a simple dome on pendentives one may give the sustaining walls a thickness equal to one tenth of the diameter. At this point Rondelet intervenes:

> Mr. Bélidor's formula can only be applied to barrel vaults, so it would have been necessary, before applying it to a spherical vault or spheroid of the same curve, to determine the true ratio of the thrust of a spherical vault to that of a barrel vault of the same diameter and thickness.

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[64] *Comme notre intention n'est pas de donner ici un traité de la Méchanique de toutes les espéces de voûtes mais seulement de mettre chacun à portée d'apprécier l'insuffisance des piliers de l'Eglise de Sainte Génevieve pour porter un dôme dans le cas le plus favorable, nous nous borneront à démontrer quelles doivent être les dimensions des supports d'une coupole simple, en nous servant de principes de la certitude desquels on ne puisse douter* (Patte 1770a, p. 8).

[65] *Regola dell'Autore per construire li Tamburi, e Cupole semplici con Lanterne. Destinato, che sara il Vano della Cupola, e Cornicioni sopra gl'Arconi con Zocolo, la metà della linea Diametrale constituirà l'Ornato esteriore del Tamburro A ; la di cui altezza si dividerà in parti quatro ; trè de quali s'assegnaranno all'Ornato de'Pilastri, Base, Architrave, Fregio, e Cornice, come in B ; e la quarta al Piedestallo C, quando però lo permetteranno l'elevazione de' Tetti, il che non hà Regola* (Fontana 1694, Bk. V, ch. XXIV, p. 362).

[66] *Nous allons determiner d'apres des principes plus certains les endroits où l'on doit placer les cercles de fer* (Rondelet *Marginalia* on Patte 1770a, p. 9).

[67] *Nous n'avons rien à dire sur les dimensions indiquées par Fontana elles sont plus relatives au gout qu'à la solidité, mais quant à l'épaisseur, elle* . . . (Rondelet *Marginalia* on Patte 1770a, p. 10).

Mr. Frézier in the third volume of stone cutting says about this that spherical vaults "push less than half as much as the simple barrel vaults of the same curve, diameter and thickness or load, and therefore that giving their piers only half the thickness of an equivalent barrel vault they will be even stronger than is necessary to bring them into balance with the thrust".

Thus, according to Mr Frézier and after the result of the computation of M. Patte, if for a barrel vault of 63 pieds in diameter, a thickness of 8 pieds 10°–11 lignes 1/3 are necessary for the pier then 4 pieds 5°-5 lignes 2/3 would be more than sufficient for a spheroid vault of the same form and thickness.

However, as Mr. Frézier makes no distinction between the ways these two species of vaults act, we will try to set more accurately this report and communicate all the advantage of spherical vaults with respect to barrel vaults, comparing a circular barrel as long as wide with a spherical vault of the same diameter and thickness. Profile or section of one and the other of these vaults will be represented by figure I. That said, the experience and the principles of mathematics prove that in any sort of barrel vault, the lower parts, up to a certain height, tend to fall inward, and that the upper parts only maintain themselves acting in the opposite direction with a force that seeks to overturn the lower parts and the piers that support them.[68]

He concludes after a quite long evaluation:

Reducing by computation these different forces in order to compare them, one finds that the pressure of a spherical vault is only one-sixth that of a barrel vault of the same diameter and thickness with a length equal to the width.[69]

[68] *La formule de M. Belidor ne pouvant s'appliquer qu'aux voutes en berceau, il aurait falu avant d'en faire l'application à une voute sphérique, ou sphéroïde de même ceintre déterminer le véritable rapport de la poussée d'une voute sphérique a celle d'une voute en berceau de même diametre et epaisseur.*

M. Frézier au 3e tome de la coupe des pierres dit à ce sujet que les voutes sphériques «poussent plus de la moitié moins que les berceaux simples de même cintre, diametre et epaisseur ou charge, et par conséquent qu'en ne donnant a leurs piedroits que la moitié de celle des berceaux conditionnés de même, ils seront encore plus forts qu'il n'est nécessaire pour les mettre en équilibre avec la poussée » .

Ainsi selon Mr Frézier et d'après le résultat du calcul de M Patte, si pour une voute en berceau de 63 pieds de diamètre il faut 8 pieds 10°-11 lignes 1/3 d'épaisseur de piedroits 4 pieds 5°-5 lignes 2/3 seroient plus que sufisans pour une voute sphéroïde de même ceintre et epaisseur.

Cependant comme M. Frézier n'a pas eu égard à la difference dont ces deux especes de voutes agissent, nous allons tacher de fixer plus exactement ce rapport et faire connoitre tout l'avantage des voutes sphériques sur les voutes en berceau, en comparant un berceau circulaire aussi long que large avec une voute sphérique de même diamètre et épaisseur. Le profil ou coupe de l'une et l'autre de ces voutes sera representé par la figure I. Cela posé, l'expérience et les principes de mathématique prouvent que dans toute sorte de voute en berceau, les parties inférieures jusqu'à une certaine hauteur tendent a tomber en dedans et que les parties superieures ne se soutiennent qu'en agissant en sens contraire, avec un effort qui tend à renverser les parties inferieures et les piedroits qui les soutiennent (Rondelet *Marginalia* on Patte 1770a, p. 10).

[69] *En reduisant par le calcul ces differens efforts pour les comparer on trouvera que la poussée d'une voute sphérique n'est que la sixième partie de celle d'une voute en berceau de même diamètre et epaisseur dont la longueur est égale a la largeur* (Rondelet *Marginalia* on Patte 1770a, p. 11).

After having given facts, as he says, or the measurements taken from existing domes but copied from Fontana, Patte says that if one passes from the proofs taken from facts to proofs taken from mechanics, one observes that the thrust of many domes is around half that of a barrel vault, but he does not go into mathematical details. He then compares those principles, which he did not explain fully, to Fontana's rule and concludes that practically speaking, the thickness given by mechanics is not enough and that one should add something to the thickness.

In the margin, Rondelet continues with his comparison of the barrel vaults with the spherical vaults and proves by action and reaction that the spherical vaults are much more solid:

> Thrust will decrease as we combine more slices together, and when the number of slices forms about half of the arch, it will support itself on its own, regardless of the number of parts it is composed of.[70]

Patte also considers the possibility of using buttresses:

> One establishes the thickness of the piers, calculating the action that the corresponding portion of the vault may exercise against each of them, with respect, as usual, to the nature of the curve of the dome, its diameter, its thickness, and to the load that it can support. All weights and the thrust of a dome are rejected by this process to the main points of support.[71]

The term *rejeter* (literally, "reject"), which he uses here, as he did at the beginning of his memoir, is the only one that might reflect a use of the parallelogram law, but we must confess that it is very far from it. The Gothic architects must have at least had that last idea. But it still corresponds to the pyramidal concept of stability, since the main points of support being outside the edifice, they enlarge the bearing surface.

Patte goes to say that Soufflot should have reinforced the footing of the walls sustaining the dome, and Rondelet correctly replies that this is generally done to avoid irregular settling of the ground. We have already seen that Soufflot had done this with his foundations. Rondelet adds that it is as useless to build buttresses as footings, since the thrust of spherical domes is very low:

> The buttresses are as useless as the footings since we have proved that the spherical vaults have so little thrust that by giving the walls that support them only the same thickness as that of the vault, they will have greater strength than barrel vaults having piers twice as thick as given by the formula.[72]

[70] *La poussée diminuera a mesure que l'on combinera plus de tranches ensemble et lorsque le nombre de tranches formera a peu près la moitié de la voûte, elle se soutiendra toute seule de quelque nombre de parties qu'elle soit composée* (Rondelet *Marginalia* on Patte 1770a, p. 13).

[71] *On établit l'épaisseur des contre-forts, en calculant l'action que la portion de voûte correspondante peut exercer contre chacun, en ayant égard, comme de coutume, à la nature de la courbe du dôme, à son diamétre, à son épaisseur, & au fardeau dont il peut être chargé. Tout le poids & la poussée d'une coupole étant par ce procédé rejettés vers des points d'appui principaux* (Patte 1770a, p. 13).

[72] *Les arc boutans sont aussi inutiles que les empatemens puisque nous avons prouvé que les voutes sphériques ont si peu de poussée qu'en ne donnant aux murs qui les supportent que la même epaisseur qu'a la voute elles auront plus de solidité que des voutes en berceaux aux piedroits*

His *marginalia* go on to say that the footings used in Patte's examples served principally to give a pyramdial form to (*piramider*) the building. *Piramider* is a term that Patte uses in the second part of his memoir, but was surely considered old-fashioned by scientists, because it reflects ignorance of the parallelogram law. The idea is the one of Fontana's figure where he shows the triangles that assure stability (see Fig. 9). After having given many measurements of large domes, Patte finally goes to his second article.

6.2 Patte's Article Second: Evidence of the Disproportion of the Piers of the Church of Ste. Geneviève, and of the Impossibility of Erecting there a Dome with Solidity[73]

The second article principally gives comparisons of measurements from different well-known domes. The list is completed with a rule of thumb: "assuming the height to be double the width, a proportion often used in similar circumstances".[74] Patte then formulates a remarks on the "singularity" of the design of Ste. Geneviève, the first of which is:

> ... Is this coronation admissible for terminating a Temple? Is there not any conventional form for its intended use, established by usage of all times and of all countries, from which one cannot deviate?[75]

Patte has invented the dome he thinks Soufflot had planned, and criticises it as not conforming to tradition. The rest of the text is full of little remarks such as "what one ordinarily observes" (Patte 1770a, p. 23).

I will quote two final sentences, because they show how deeply engrained in Patte's mind is the stability of the pyramid: "In accordance with the precepts of the art of building, which require what is supported to be set back and above what is

desquelles on donneroit une epaisseur double de ce que donneroit la formule (Rondelet *Marginalia* on Patte 1770a, p. 15).

[73] *Article second. Preuves de la disproportion des piliers de l'Eglise de Sainte Génevieve, & du peu d'apparence d'y pouvoir élever une coupole avec solidité* (Patte 1770a, p. 20).

[74] *... de sorte qu'en leur supposant une hauteur double de leur largeur, proportion assez usitée en pareille circonstance* (Patte 1770a, p. 21).

[75] *... Ce couronnement est-il admissible pour terminer un Temple? n'y a-t'il pas une forme analogue à sa destination et consacrée par l'usage de tous les tems & de tous les Pays, dont on ne peut gueres s'écarter* (Patte 1770a, p. 22).

carrying it.[76] Regarding the stability of the cube, Rondelet remarks, "Only cubic masses could counterbalance this load".[77]

Rondelet finally tires of fighting windmills, and writes:

> All this can only relate to the dome that Mr Patte has imagined, and believes impossible, because Mr Soufflot's dome will not be executed at all as he believes here. Indeed, nothing he says can be applied to Mr Soufflot's dome. That is why we dispense with countering endless contradictions that prove that the author would be very embarrassed if he were to execute such a work.[78]

Patte closes his second article enumerating a list of dangerous overhangs he imagines in the plan of Ste. Geneviève and Rondelet notes in margin that "all those overhangs exist only in Mr Patte's imagination and they prove not only his ignorance but also his insincerity".[79] In my opinion, it proves more his ignorance than his insincerity. Patte had very well understood the clever way in which Soufflot had laid the foundations, distributing the pressure on a larger surface. In the statics of the pyramid it is clear that the broader the base is the best. What he does not understand is the way an architect is able to draw the forces into places where they will find the best resistance.

Finally we arrive at Patte's conclusion:

> Thus, in whatever way we want to consider the implementation of the promised dome in the centre of the Church of Ste. Geneviève, it would be difficult to justify; practice and theory, parallel examples, and mathematical proofs agree in showing that the piers already erected are too manifestly disproportionate to bear it; they would collapse on all sides beneath it because of its weight and thrust, instead of forming footings, and in rising leaving large setbacks with respect to the bottom of the tower, as required by solidity; in addition, the cubic mass which they lack can not be compensated by the surrounding parts, since they are too weak, as evidenced by their height or by their overhang.[80]

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76 *...conformément aux préceptes de l'art de bâtir, qui exigent que ce qui est porté s'éléve en retraite au-dessus de ce qui porte* (Patte 1770a, p. 26).

77 *...ce ne sont que des massifs cubiques qui puissent contrebalancer ce fardeau* (Rondelet *Marginalia* on Patte 1770a, note 18, p. 29).

78 *... tout ceci ne peut concerner que le dôme que Mr Patte a imaginé, et qu'il a cru impossible car le dôme de Mr Souflot ne sera pas du tout executé comme il le croit ici. Ainsi tout ce qu'il dit ne peut être appliqué au dôme de Mr Souflot. C'est pourquoi on se dispense de relever une infinité de contradictions qui prouvent que l'auteur serait fort embarrassé s'il lui falloit faire executer un pareil ouvrage* (Rondelet *Marginalia* on Patte 1770a, p. 26).

79 *...tout ces porte a faux n'existent que dans l'imagination de M. Patte ce qui prouve autant d'ignorance que de mauvaise foi* (Rondelet *Marginalia* on Patte 1770a, p. 32).

80 *AINSI, de quelque façon que l'on veuille considérer l'exécution de la coupole promise au centre de l'Eglise de Sainte Génevieve, il seroit dificile de la justifier; la pratique & la théorie, les exemples mis en parallele, & les démonstrations Mathématiques s'accordent à prouver que les piliers déjà élevés sont d'une disproportion trop manifeste pour la porter; qu'ils se déroberont de tous côtés dans le bas à son poids & à sa poussée, au lieu de former des empattemens, & de s'élever en laissant de bonnes retraites au pied de la tour, comme la solidité le requiert ; qu'en outre la masse cubique qui leur manque ne sauroit être suppléée par les parties environnantes, vû qu'elles sont trop foibles, toutes évidées dans leur hauteur ou en porte-à-faux* (Patte 1770a, p. 35).

Patte's conclusion is a good summary of all his errors and shows plainly that he can only think in terms of the pyramidal or triangular statics, which is certain but not at all light.

At the end of his memoire, Patte calculates the thickness that the walls of Ste. Geneviève would have to have to support a dome with the particular measurements he proposes. To do so, he uses following formula:

$$\sqrt{\frac{2bgnn}{af} - \frac{2dnn}{f} + \frac{4nn^2}{ff} - \frac{2nn}{f}} = y.$$

This appears to be close to those of La Hire and Bélidor. Rondelet affirms that Patte uses Bélidor's formula. Actually, the formula Patte gives is conform to neither that of La Hire:

$$\frac{1}{2}bfy^2 + \frac{1}{2}fyv^2 + fhv^2 = s^2eg - s^2fy - s^2fa$$

nor that of Bélidor:

$$\sqrt{\frac{2n^2f + 2n^2g}{d} + \frac{4n^2}{d^2} - \frac{2n^2}{d}} = y.$$

However, it is difficult to explain the differences because Patte does not explain his notations. In the text he says that he used Bélidor's formula, but Rondelet says in his introduction that he finds 2 pieds 5° 2 lignes instead of 3 pieds 9° with the same formula.

6.2.1 Cochin's Doutes raisonnables d'un Marguillier

Cochin's "reasonable doubts of a churchwarden" (Cochin 1770a) is sometimes attributed to Rondelet and we shall not find new ideas in it, but their expression is sometimes clearer, as in the quotations that follow. With their help I will make a kind of summary of the arguments of Soufflot and Rondelet. Cochin concentrates principally on Patte's traditional views:

> In addition, are the laws laid down by Fontana so inviolable that we cannot deviate without sinning? Hippocrates said yes, but Galen said no, this is highly questionable: yet they are the great proofs that Mr. Patte believes so obvious that he does not imagine that they could be called into doubt.[81]

[81] *De plus, les loix posées par Fontana, sont-elles si inviolables, qu'on ne puisse s'en écarter sans pécher ? Hipocrate dit oui, mais Galien dit non ; tout cela est fort douteux : cependant ce sont les grandes preuves que M. Patte croit si évidentes, qu'il n'imagine pas qu'on puisse les révoquer en doute* (Cochin 1770a, p. 137).

He underlines the pyramid as criteria of stability:

> Where did Mr. Patte get what he announces as an indisputable principle, that everything should be erected starting from low foundations to ridge, with footings set back or sloped? If it were true, none of the Gothic churches would have survived, which experience contradicts: he should observe our admirable parish.[82]

A second point we emphasized is the excessive dimensions given by ancient rules, which lead of course to stability but also to excessive costs:

> But it remains to be seen whether since Fontana, and even before him, one had not found the secret to produce the same effects by bolder, more ingenious, less expensive & equally strong ways.[83]

As we already said, the parallelogram law is never mentioned, but some expressions reflect it, such as:

> It is not exactly true that these piers are isolated ... because in the place that should suffer some thrust, they are leaned against by four large arches, whose reciprocal effects eliminate each other, and are reduced to zero between them, though the same is not true of the robustness they give to these piers.[84]

Another example is:

> Why should we not assume that Mr. Soufflot has so combined its thrusts and the resistances that the thing would be in perfect balance.[85]

Using irony, Cochin emphasizes the difficulty of the language that architects have to use in order to be understood by workmen on the site:

> I persist even more so in this idea, because it is also the opinion of Mr. Moëllonnet, my master mason, who built me my pretty little house on Rue Coupeau, where we eat such good salads on Sundays after mass. I rely more on his practice, than all algebraic theories, which I do not understand.[86]

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[82] *Où M. Patte a-t-il pris ce qu'il nous annonce comme un principe incontestable, que tout doit être élevé, depuis les basses fondamentions jusqu'au faîte, avec empattement en retraite ou en talut ? S'il étoit vrai, aucune des églises gothiques n'auroit subsisté ; ce que l'expérience dément : qu'il observe notre admirable paroisse* (Cochin 1770a, p. 145).

[83] *Mais il reste à savoir si depuis Fontana, & même avant lui, on n'avoit point trouvé le secret de produire les mêmes effets par des moyens plus hardis, plus ingénieux, moins dispendieux & tout aussi solides* (Cochin 1770a, pp. 137–138).

[84] *Il n'est pas exactement vrai que ces piliers ... soient isolés, puisque à l'endroit qui doit souffrir quelque poussée, ils sont accôtés par quatre grands arcs, dont les effets réciproques se détruisent les uns les autres, & se réduisent à zero entr'eux, mais non pas quant à la solidité qu'ils donnent à ces piliers* (Cochin 1770a, pp. 132–133).

[85] *Pourquoi ne supposerions-nous pas que M. Souflot auroit si bien combiné ses poussées avec les résistances, que la chose seroit dans un parfait équilibre* (Cochin 1770a, p. 146).

[86] *Je persiste d'autant plus dans cette idée, que c'est l'opinion de M. Moëllonnet, mon maître maçon, qui m'a bâti ma jolie petite maison de la rue Coupeau, où nous mangeons de si bonnes salades les dimanches après l'office. Je m'en rapporte bien plus à sa pratique, qu'à toutes les théories algébriques, où je ne comprends rien* (Cochin 1770a, pp. 146–147).

Let us listen Cochin's conclusion, which is not far from our own, in the words of an invented nephew:

> I see you are astonished to see me reasoning about architecture, but you must know that I have my nephew, the son of my sister in Rue des Bourdonnois, whose is a bit of an architect. He did not studied much of architecture, but he has engraved a number of plans and elevations; ... he was not happy with the memoir of Mr. Patte, I was delighted, because I gave to him to read as a test ... the smart fellow has very well discovered that the algebraic equation is not true regarding the construction and shape of the dome of the church of Ste. Geneviève. ... Do not be surprised, Mr. Patte is not producing algebra or architecture, he is only a seller of it.[87]

At the end of his pamphlet, Cochin suggests that the reader read both the letters by Perronet published in the *Mercure de France* and we follow his advice.

With these letters, we enter into a completely different sphere. Not only are both Perronet and Soufflot architects, but they also share the same ambition of leaving the old traditional rules behind and using the modern parallelogram law, which while not mentioned explicitly is nevertheless present implicitly. In his first letter, dated 22 January, Perronet writes:

> It is true that Mr. Patte wanted to consult me about this. His main objection was that the dome would overhang the vaults: I told him that we could establish it as firmly on the double arches of the vaults whose thrust was well maintained (as it has to be in the church of Ste. Geneviève) on pendentives and centrings of the arcades of the crossing of the church that must bear the greater part of the dome of Ste. Geneviève, as is done in other churches.[88]

Four days later, on 26 January, after having studied Soufflot's plans for Ste. Geneviève, Perronet writes again:

> I compared all [Soufflot's plan] with the designs of similar buildings already constructed, both in the massive kind of antique architecture, and in that of the lightest Gothic. I recognized that giving your vertical supports and lateral thrust bearings enough strength to ensure your dome all suitable strength, you have taken an intermediate way that is as wise as economical between the two types of construction I just talked about.[89]

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[87] *Je vous vois fort étonné de me voir raisonner architecture ; mais il faut que vous sachiez que j'ai mon neveu, le fils de ma sœur de la rue des Bourdonnois, qui est un peu architecte. Il n'a pas beaucoup étudié l'architecture, mais il a gravé quantité de plans & d'élévations ; ... il n'a pas été content du mémoire de M. Patte ; j'en ai été charmé, car je ne le lui faisois lire que pour l'éprouver, ... Le petit drôle a fort bien découvert que l'équation algébrique est fausse relativement à la construction & à la forme du dome de l'église de Sainte-Geneviève. ... Ne vous en étonnez pas, M. ; Patte n'est pas fabriquant d'algèbre ni d'architecture, il n'en est que marchand* (Cochin 1770a, pp. 155–157).

[88] *Il est bien vrai que M. Patte a voulu me consulter à ce sujet. Sa principale objection étoit pour lors que ce dôme porteroit à faux sur les voûtes : je lui ai dit que l'on pouvoit l'établir aussi solidement sur les arcs doubleaux des voûtes dont la poussée étoit bien retenue (ainsi qu'elle doit l'être dans l'église Sainte Geneviève) que sur les panaches et les cintres des arcades de la croisée de l'église qui doivent porter la plus grande partie du dôme de Sainte Geneviève, comme cela se pratique aux autres églises* (Perronet 1770a, p. 195).

[89] *J'ai comparé le tout avec les desseins de pareils monumens qui sont construits, soit dans le genre massif de l'architecture antique, soit dans celui du plus léger gothique. J'ai reconnu qu'en donnant à vos points d'appuis verticaux, & aux buttées latérales assez de force pour assurer à votre dôme*

He then alludes to architects who do not know the laws of equilibrium:

> Architects who know less about the laws of equilibrium and the art of light constructions than those who have made such edifices, might believe they would make them stronger by increasing the volume of materials.[90]

This is the conclusion that comes out of the rules of the pyramidal statics. However, he goes on:

> But if the arches that serve as acting and destructive forces are fortified with greater proportions than walls and piers that are to resist their thrust, the building will be less strong; thus it is more the relationship of the acting powers to those which must resist them on which must depend the solidity of a building, than of the size of piers or walls to the disproportionately thick vaults that tend to overturn them.[91]

6.3 *Second Act of the Polemic: Scientists Come into Play*

An architect first and then engineer, having studied at the *Ecole des Ponts et Chaussées* at the time it was directed by Perronet, Emiland-Marie Gauthey was a friend of Soufflot's. Gauthey intervenes in the quarrel in 1771 with a booklet which shows his scientific point of view (Gauthey 1771). After an introduction stressing the importance of mathematical laws in architecture, Gauthey gives a summary of the evolution of the computation of the thickness to give to piers sustaining an arch.

In a note (2), Gauthey computes that thickness following La Hire and Bélidor but he insists on the importance of the direction of forces. He justifies the model used as follows: First, if the vault breaks, it breaks at 45° of the arch. Gauthey justifies this claim, saying:

> We prove, at least for barrel vaults, that when they split at this point [in the middle of the haunch], they push more than if they were broken elsewhere: therefore, calculating according to this assumption we took the more disadvantageous case that can happen in the break, and we took the most powerful ways to prevent these accidents.[92]

toute la solidité convenable, vous avez pris un parti moyen également sage & économique entre les deux genres de construction dont je viens de parler (Perronet 1770b, p. 196).

[90]*Des architectes qui connoîtraient moins bien les loix de l'équilibre & l'art des constructions légères que ceux qui ont fait de pareils édifices, pourroient croire qu'ils rendroient les leurs plus solides, en augmentant le volume des matériaux* (Perronet 1770b, pp. 196–197).

[91]*Mais si les voûtes qui tiennent lieu de puissances agissantes & destructives sont fortifiées en plus grande raison que les murs & les piliers butans qui doivent résister à leur poussée, l'édifice sera moins solide ; c'est donc encore plus du rapport des puissances agissantes à celles qui doivent leur résister que doit dépendre la solidité d'un édifice, que de la grosseur des piliers ou des murs, & de l'épaisseur disproportionnée des voûtes qui tendent à les renverser* (Perronet 1770b, p. 197).

[92]*On démontrera, du moins pour les voûtes en plein ceintre, que lorsqu'elles se fendent en cet endroit* [au milieu des reins]*, elles poussent davantage que si elles se fussent rompues partout ailleurs : par conséquent, en calculant d'après cette supposition on a pris le cas le plus désavantageux qui puisse arriver dans la rupture, & l'on a pris les moyens les plus puissans pour prévenir ces accidens* (Gauthey 1771, p. 7).

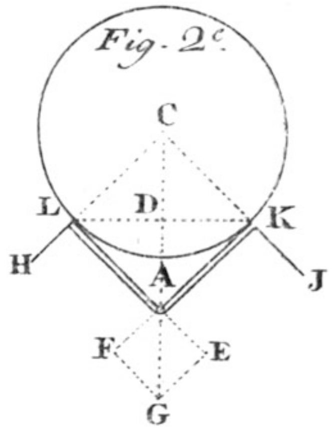

Fig. 24 Gauthey's parallelogram law

But there is more: in Bélidor's computation, the point of application of the voussoir's push is their centre of gravity. This is wrong, says Gauthey; the correct point of application is the middle of the joint between the voussoirs:

> Mr. Bélidor, following whom I did the calculation of thrust, assumes that centre pressure [centre of gravity] is located in the middle of the length of the voussoirs, it is also in this way that M. Patte regarded it, and it is certain that this may well be so before the vault is completely split, but provided that the top goes down, all the pressure is done on the edge of the voussoir that remains attached to the pier: then the lever arm of the active power will be less than was assumed for the calculation.[93]

The consequence is that, once again, La Hire and Bélidor's results are overestimated.

Another simplification is generally introduced in the model: the voussoirs are presumed to slide one upon the other without friction. The idea is that if the system is stable in that case, it will surely be stable when there is friction. Nevertheless, Gauthey gives, in his note (5), a computation of that overestimation. Strangely enough, there—in a note—he uses the parallelogram law (Fig. 24).

Having discussed every point of the model, Gauthey goes on to the "Refutation of objections against the proposed construction of the dome of the Church of Ste. Geneviève".[94]

[93]*M. Belidor d'après qui j'ai fait le calcul de la poussée, suppose que le centre d'impression se trouve sur le milieu de la longueur des voussoirs [au centre de gravité], c'est aussi de cette manière que M. Patte l'a considéré, & il est certain que cela peut bien arriver ainsi avant que la voûte se soit entièrement fendue ; mais pour peu que la partie supérieure descende, toute l'impression se fait sur l'arête du voussoir qui reste joint au piedroits : alors le bras de levier de la puissance agissante sera moins grand qu'on ne l'a supposé pour faire le calcul* (Gauthey 1771, pp. 7–8).

[94]*Réfutation des objections proposées contre la construction de la Coupole de l'Eglise de Sainte Genevieve* (Gauthey 1771, p. 11).

I shall not go in every detail of Gauthey's refutation. It is a typical criticism of Patte's pyramidal statics:

> He [Patte] objects, to support his assertion that the unengaged part of the columns cannot serve as buttresses, because of the large gap between the columns, and we can infer from his reasoning, that he believes the width of the pier of a vault is zero when it is not uniform in height, but he must know that it is a principle, regardless of either the shape of the piers or to the voids they contain, that their resistance depends only on their weight and on the distance of the direction from their centre of gravity to the fulcrum: everyone can see that the buttresses that are usually built to support the arches of churches, although pierced with openings, are nevertheless capable of great resistance.[95]

A good way to explain this would be to use the parallelogram law. The fact that Gauthey does not use it—despite the fact that he knows it and that he senses the importance of the direction of the forces—is symptomatic of the fact that he still underestimates its importance and generality. A few pages later he writes:

> Nor in creating in the spaces between their columns *lunettes en berceau* (barrel vaults) to reject the weight on one side against the large arches, and on the other against the pendantives, there would be no more problem in this construction, than there would be to pierce a wide opening in the abutment of a bridge that would be counter-buttresses by quay walls, and that the relieving arches, which there would be pierced, would be continued far enough to serve as an abutment to the haunch of the vault.[96]

This is certainly an allusion to the problems that Perronet had encountered with his project for the Pont de Neuilly. Rondelet had asked to Perronet to help him defend Soufflot's project precisely because he knew that he had those problems; it was the similarity between the problems they encountered that drove Gauthey into the quarrel. Perronet, Soufflot and Gauthey knew how to use the parallelogram law, but they knew that it would be difficult to use in a polemical publication.

Another example of the same phenomena is the following quotation, where we again find pyramidal statics:

> Everyone knows that the flat arch is the vault least able to bear considerable weight, because it is never relieved, and great care is taken to build relieving arches over them. A surbased arch is less strong than a round arch, and the latter has less force, in proportion, than a stilted

[95] *Il [Patte] objecte pour appuyer son assertion, que la partie non engagée des colonnes ne pourra servir de piedroits, à cause du grand vuide de l'entre-colonnement, & l'on peut inférer de ses raisonnemens, qu'il croit que la largeur du piédroit d'une voûte est nulle, lorsque elle n'est pas uniforme dans sa hauteur ; cependant il doit savoir qu'il est un principe que, sans avoir égard à la figure des piliers ni aux vuides qu'ils renferment, leur résistance dépend uniquement de leur poids & de la distance de la direction de leur centre de gravité au point d'appui : tout le monde peut voir que les arcboutants que l'on construit ordinairement pour soutenir les voûtes des Eglises, quoique percés à jour, n'en sont pas moins capables d'une grande résistance* (Gauthey 1771, pp. 14–15).

[96] *Ni de former dans leur entre-colonnements des lunettes en berceau, pour rejeter le poids, d'un côté contre les grands arcs, & de l'autre contre les pendantifs ; il n'y auroit dans cette construction pas plus d'inconvénient, qu'il n'y en auroit à percer d'une large ouverture la culée d'un pont qui seroit contre-butée par des murs de quai et que la lunette que l'on y perceroit seroit continuée assez loin pour servir de butée aux reins de la voûte* (Gauthey 1771, p. 21).

> arch, so that the best way to allow a vault to carry a considerable weight, is to raise it a lot, or better still, to terminate it in a pyramid.[97]

The title of Gauthey's fourth section, "Application of the principles of the theory to calculate the thrust of the vaults of the dome and of the nave domes of the church of Ste. Geneviève",[98] of course piques our curiosity, but is a great disappointment. Gauthey turns, just as all his predecessors did, to the rules of La Hire and Bélidor, Frézier, etc. and to the law of the lever.

One might conclude that our three architects—Rondelet, Perronet, and Gauthey—and certainly Soufflot, though but we have no proof until now, knew the parallelogram law but did not have the mathematics they needed to use it:

> He [Patte] then gives a long list of the overhangs he claims to have noticed and which he considers to be so many fundamental flaws, as if it were not generally known to all Architects and Builders, that those critical items cease to be overhangs when, lacking direct powers, they are counter-buttressed by equivalent oblique powers, and there is not one of those he criticises that is not counter-buttressed in that way.[99]

Gauthey is an architect who is definitely oriented towards theory, but his point of view nevertheless contrasts with that of Charles Bossut (1730–1814), author of a paper entitled *Recherches sur l'équilibre des voûtes* (Bossut 1774), read to the Académie des Sciences on 12 July 1770. However, the written text is said to have been delivered in 1777, and the published volume only appeared in 1778. Bossut's text, like that of Bouguer, takes place in another world. He not only uses the parallelogram law as a normal tool, but also uses the osculatory circle, the ancestor of the ray of curvature, which not only implies differential calculus but even second-order differential calculus. However, we have to mention it because its final section is dedicated to the defence of the stability of the dome of Ste. Geneviève: "Let us make an application of the whole theory. I take for example, the dome of the church of Ste. Geneviève in Paris, built by Mr. Soufflot.[100] His conclusions are, "[h]ence

[97] *Tous les gens savent que la plate-bande est la voûte la moins capable de porter un poids considérable, aussi ne la chargent-ils jamais, & ils ont une grande attention de faire au-dessus des arcs en décharge. Une voûte surbaissées est moins forte qu'une voûte en plein ceintre, & celle-ci a moins de force, à proportion, qu'une voûte surhaussée ; de sorte que la meilleure manière de faire porter un poids considérable à une voûte, est de la surhausser beaucoup, ou mieux encore de la terminer en pyramide* (Gauthey 1771, p. 22).

[98] *IV. Application des principes de la théorie, au calcul de la poussée des voûtes du dôme & des nefs de l'Eglise de Sainte Genevieve* (Gauthey 1771, p. 40).

[99] *Il [Patte] fait ensuite une longue énumération des porte-à-faux, qu'il prétend avoir remarqués & qu'il regarde comme autant de défauts essentiels, comme s'il n'étoit pas généralement connu de tous les Architectes & de tous les Constucteurs, que les objets qu'il critique cessent d'être des porte-à-faux, lorsqu'à défaut de puissances directes, ils sont contre-butés par des puissances obliques équivalentes, & il n'y en a pas un seul de ceux qu'il critique qui ne soit ainsi contre-buté* (Gauthey 1771, p. 24).

[100] *XII. Faisons une application de toute cette théorie. Je prends pour exemple, le dôme de l'église de Sainte Genviève de Paris, construite par M. Soufflot* (Bossut 1774, p. 564).

we see that the piers will have more than enough strength to support the thrust of the dome, and therefore there is no doubt this vault will be very strong.[101]

I shall not go further into Bossut's demonstration, not only because it is based on figures that are missing in the volume but also and principally because the mathematical methods used by Bossut are no longer similar to the ones we have encountered up to this point. It does, however, give an idea of how far Soufflot must have been ahead of his time, even if he did not use differential calculus.

6.4 *Third and Final Act of this Polemic*

Between 1776 and 1778 new cracks appeared in the piers of Ste. Geneviève, allowing Pierre Patte to rekindle the polemic with a new "Letter of Mr Patte about the weakness of the piers planned to support the dome of the new church of Ste. Geneviève " (1779), published a year before Soufflot's death.

About the same time, Rondelet wrote a letter to the Comte d'Angiviller (the document is not dated but was probably written between 1779 and 1780), in which he finally gives a mathematical proof of the equilibrium of a spherical vault (Rondelet 1780). In his demonstration he uses the parallelogram law of forces to show how the various forces are equilibrated in a spherical vault, which he considers as composed of spheres. He introduces his computations with these words:

A long time ago, I discovered this property of spherical vaults, but as I had reached it only by calculations that were long and very difficult for me, I had abandoned it, and it is because I was recently able to demonstrate it by simple principles, accessible to all those who know the first elements of geometry, that I will propose it, for then it may be useful.[102]

The fact that he uses spheres denotes the influence of Poleni, who used such spheres in his *Memorie istoriche della gran cupola del Tempio Vaticano* (1748) (see Fig. 22). Rondelet had two exemplars of that famous book in his library. Rondelet explains (Fig. 25):

> The force that the ball F makes, acting as a wedge to move apart the balls from the lower ring is expressed by the line gt, this force is composed of two other lines expressed by ti, ig, so that ig represents the weight of the ball F, which tends to act vertically and ti represents the horizontal force that the ball F exerts to move apart the balls of the lower ring. But,

[101] *D'où l'on voit que les pieds-droits auront une résistance plus que suffisante pour soutenir la poussée du dôme, & que par conséquent il n'y a pas à douter que cette voûte ne soit très-solide* (Bossut 1774, p. 564).

[102] *Il y a très long tems que j'ai decouver cette proprieté des voutes spheriques, mais comme je n'y etoit parvenu que par des calculs long et fort difficiles pour moi, je l'avois abandonnée et c'est parce que je suis venu a bout de la demontrer par des principes simples et a la portée de tous ceux qui ont les premiers elemens de geometrie que je vais la proposer, par ce qu'alors elle poura etre utile* (Rondelet, quoted in Middleton and Baudouin-Matuszek 2007, p. 304).

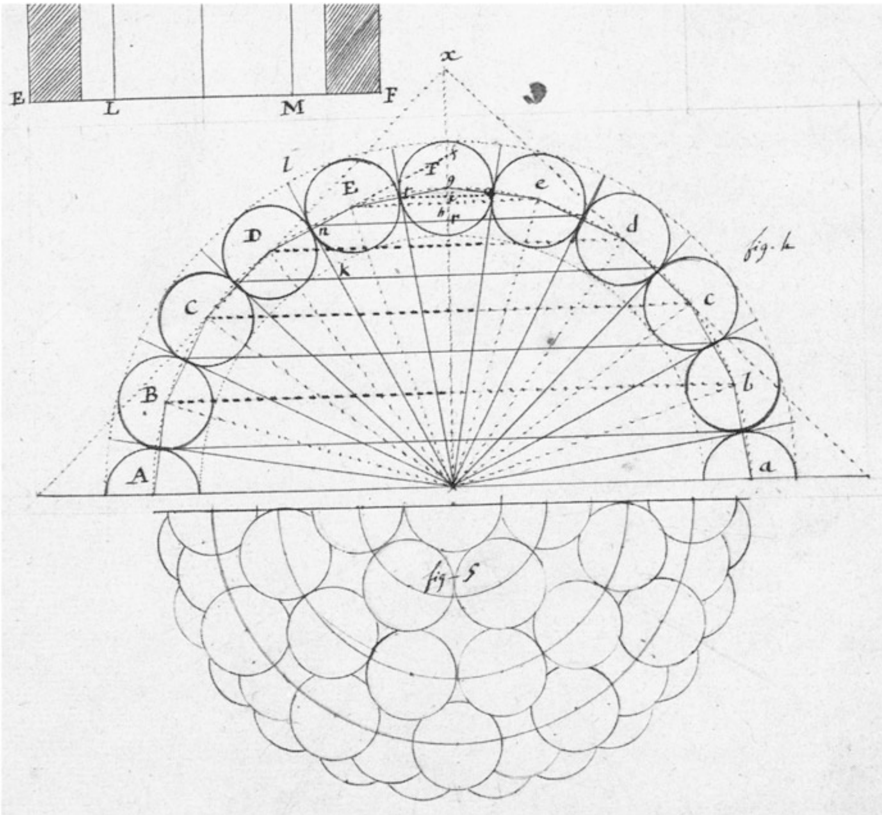

Fig. 25 Rondelet's figure in his letter to d'Angiviller

according to the laws of mechanics, it is necessary, in order to have equilibrium, that the weight of the ball E be to that of the lower ring as gi is to ti.[103]

And a bit later he continues:

Now consider the row Dd with respect to the above ring Ee joined to the F key, we will consider it as a wedge that makes an effort to move apart the balls from the ring Dd, acting in the direction fn, which is composed of two forces expressed by the lines nr and rf, one of which represents the weight of the upper part and the other, nr, the horizontal thrust that must be equal to the resistance of the ring Dd, but because of the similarity of the triangles

[103]*L'effort que fait la boule F, en agissant comme un coin pour ecarter les boules de la couronne inferieure sera exprimé par la ligne gt, cet effort se decompose en deux autres exprimés par les lignes ti, ig, ensorte que ig represente le poid de la boule F, qui tend a agir verticalement, et ti, represente l'effort horizontal selon lequel cette boule F, tache d'écarter les boules de la couronne inferieure. Or il faut, suivant les loix des mechaniques, pour qu'il y aye equilibre, que le poid de la boule E soit a celui de la couronne inferieure comme gi est a ti* ... (Rondelet, quoted in Middleton and Baudouin-Matuszek 2007, p. 304).

Fig. 26 Project for a more stable Panthéon by De Wailly (1787)

fnr, nro, we will have fr : rn :: nr : ro as the sine of the angle nor, which is 27 degrees, is to its cosine, that is to say, as 4599 is to 89.101.[104]

Rondelet's figure is completely different from that of Poleni. Poleni shows a vertical section of the dome. In contrast, Rondelet gives two superposed horizontal sections in order to count the number of spheres that share the weight of one sphere of the upper ring. However, both authors use their figure to show how to use the parallelogram law.

Seven years later, on 20 December 1787, Charles De Wailly (1730–1798) proposed some "Views on the French Pantheon and means of remedying the frightening damage that is manifested" (De Wailly 1787). The new design he proposes in order to save the *Panthéon* from collapse shows how deeply embedded the idea of pyramidal stability was in the mind of the architects of the time (Fig. 26).

[104] *Examinons maintenant le rang Dd, par raport au rang superieur Ee, joint a la clef F, que nous regarderons comme un coin qui fait effort pour ecarter les boules du rang Dd, en agissant selon la direction fn, qui se decompose en deux efforts exprimés par les lignes nr, rf, dont une fr, designe le poid de la partie superieure et l'autre nr, la poussée horizontale qui doit etre egale a la resistance du rang Dd; mais a cause des triangles semblables fnr, nro, on aura fr : rn :: nr : ro comme le sinus de l'angle nor, qui est de 27 degrès est à son cosinus, c'est a dire comme 4599 est a 89,101* (Rondelet, quoted in Middleton and Baudouin-Matuszek 2007, p. 304).

7 Conclusion

Patte's *Mémoire* and Rondelet's *marginalia* show perfectly how the clever law of La Hire, so well adapted to the study of the equilibrium of a vault, prevented architects working on the site from accepting the more abstract but much more powerful parallelogram law, or even to speak outright about it. In contrast, the drawings of Couplet, before those of Poleni and Bossut, show how this law allows architects to see how the forces "flow" into the stones and how they can best be directed. Soufflot and Rondelet, even though they did not say it, had understood that, as did those who I call "scientists":

> The architects of the churches of Ste. Geneviève and the Madeleine, who were the first to abandon the form of arches used by the moderns in the construction of almost all major churches, will without a doubt create a new epoque in the history of taste in Architecture: they will be the subject of criticism, but in all periods those who deviated from accepted usage experienced the same fate.[105]

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[105] *Les architectes des Eglises de Sainte-Genevieve & de la Magdeleine, qui les premiers ont abandonné la forme des arcades employées par les modernes dans la construction de presque toutes les grandes Eglises, feront sans doute époque dans l'histoire du goût de l'Architecture : ils sont en butte à la critique mais de tous les temps ceux qui se sont écartés des usages reçus, ont éprouvé le même sort* (Gauthey 1771, p. 67). It should be noted that Gauthey's use of the term "moderns" here refers to the Gothic architects.

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Transcription of Patte's 1770 *Mémoire* on the Panthéon's Stability Together with Rondelet's *Marginalia*

Patricia Radelet-de Grave

Abstract Conceived and begun by architect Jacques-Germain Soufflot in 1755, the construction of the Church of Ste. Geneviève (later the Panthéon) in Paris was continued after his death by Jean Baptiste Rondelet. The polemic regarding the Panthéon's stability that was initiated by Pierre Patte in 1770 was amply commented in the separate chapter in this same volume (p. 127–185). In order to complete the picture of that episode of history, we transcribe here Patte's "Memoir on the construction of the cupola designed to crown the new church of Sainte Genevieve" (Patte 1770a) from an exemplar found in Ms. BHVP 3459 conserved in the Bibliothèque historique de la ville de Paris (Patte 1770b).

As one sees on the title page (Fig. 1), the document contains Patte's Memoir together with the notes that Rondelet wrote in the margin. Rondelet planned to write an answer to Patte, which he provisionally entitled "Refutation of a Memoir on the construction of the cupola designed to crown the new church of Sainte Genevieve in Paris", but that answer was never published.

In the transcription that follows, Patte's illustrations, originally published all together on two plates, are reproduced here inserted where they belong and the complete plates are also reproduced, one at the beginning and the other at the end of the article. Patte's Memoir had some footnotes that I indicate with: Patte's footnote. Other footnotes are due to Rondelet's son, they are indicated by (*Rondelet fils*) and a few editor's notes of my own have been added where they are helpful.

English translations of excerpts from the text and Marginalia, with comments, appear in the previous chapter (Radelet-de Grave 2015).

Keywords Panthéon (Paris) • Pierre Patte • History of mechanics • Dome construction • Jean Baptiste Rondelet • Jacques-Germain Soufflot

I am grateful to the persons of that library who helped me to find the manuscript.

In Middleton, Robin; Baudouin-Matuszek, Marie-Noelle, Jean Rondelet, Yale University Press, 2007, p. 299–301, one finds the text of the *Marginalia* alone.

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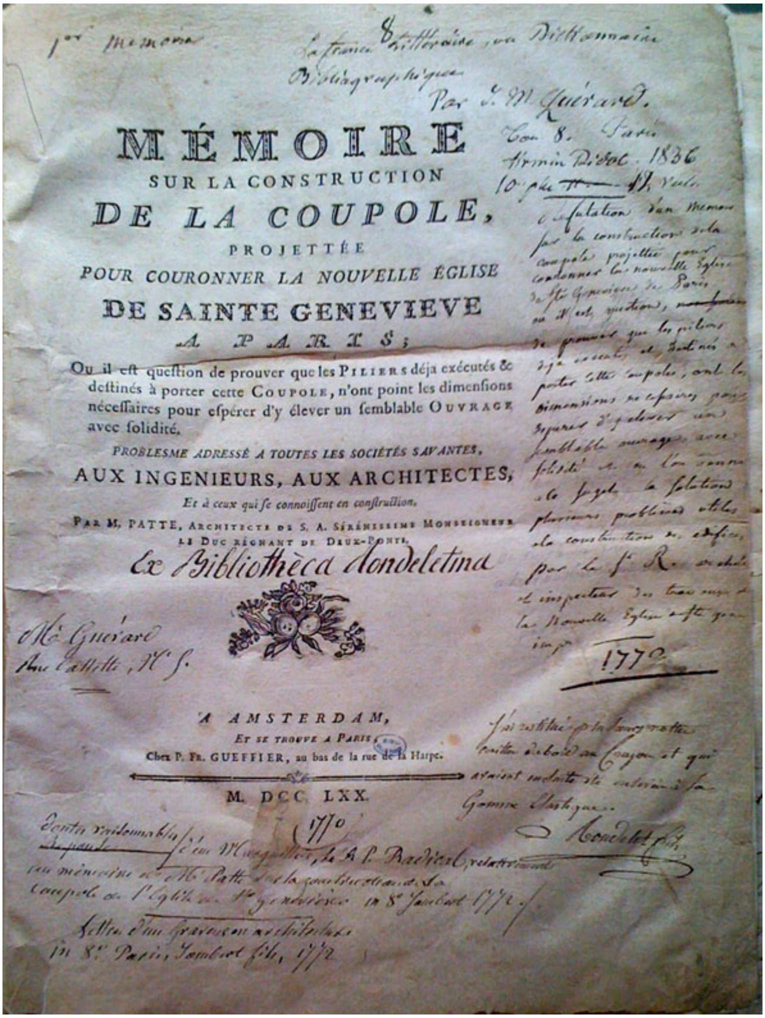


Fig. 1 Title page of Patte's 1770 *Mémoire*, with Rondelet's *Marginalia*

Patte's Title (Figs. 1 and 2)

Mémoire sur la construction de la coupole projettée pour couronner la nouvelle église de Sainte Geneviève à Paris;

Ou il est question de prouver que les Piliers déjà exécutés & destinés à porter cette Coupole, n'ont point les dimensions nécessaires pour espérer d'y élever un semblable Ouvrage avec solidité.

Problème adressé a toutes les sociétés savantes,

Aux ingénieurs, aux architectes, et à ceux qui se connoissent en construction.

PAR M. PATTE, ARCHITECTE DE S.A. SÉRÉNISSIME MONSEIGNEUR LE DUE RÉGNANT DE DEUX-PONTS

A Amsterdam, et se trouve à Paris chez P. Fr. Gueffier, au bas de la rue de la Harpe

M. DCC. LXX

Rondelet's Title

Réfutation d'un

Mémoire sur la construction de la coupole projettée pour couronner la nouvelle église de Ste Geneviève de Paris,

où il est question de prouver que les piliers déjà executés, et destinés à porter cette coupole ont les dimensions nécessaires pour espérer d'y élever un semblable ouvrage avec solidité

et où l'on donne a ce sujet la solution [à] plusieurs problèmes utiles a la construction des édifices,

PAR LE SR. [JEAN] R[ONDELET], ARCHITECTE ET INSPECTEUR DES TRAVAUX DE LA NOUVELLE EGLISE STE GENEVIÈVE

impr. 1770.[1]

---

[1] J'ai restitué plusieurs nottes écrittes d'abord au crayon et qui avaient ensuite été enlevées à la Gomme élastique. (*Rondelet fils*).

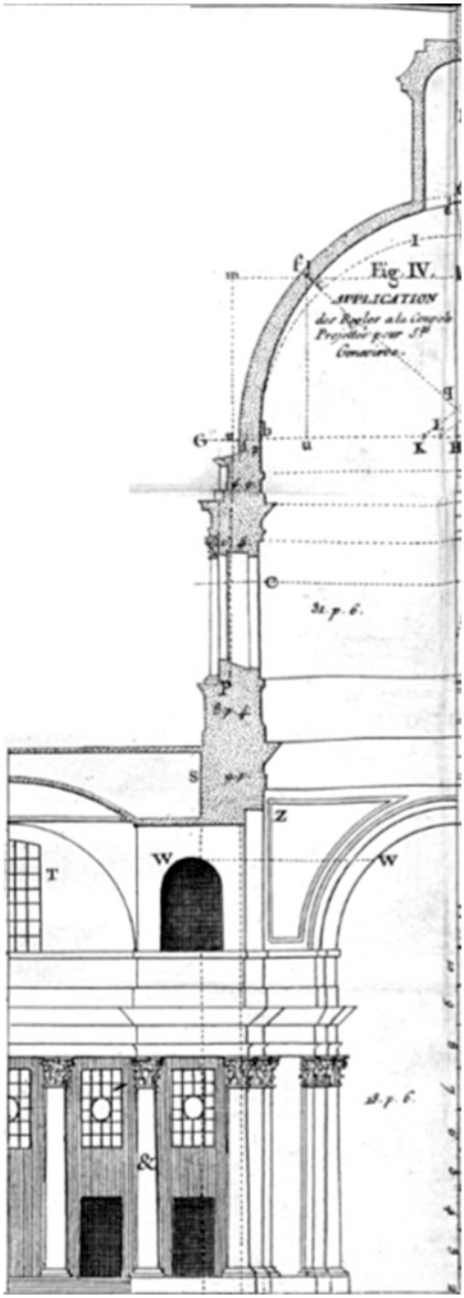
Other notes from Rondelet on the title page:

1er memoire: La France littéraire, ou Dictionnaire Bibliographique par L.M. Guérard, Tom. 8, Paris, Firmin Didot, 1836, 10 plus 12 vols. M. Guérard rue Cassette, n°. 5.

[2em memoire:] [Réponse, crossed out and replaced by] Doutes raisonnables d'un marguillier, le R.P. Radical, relativement au mémoire de Mr. Patte sur la construction de la coupole de l'église de Ste. Geneviève in 8° Jombert 1772.

[3em memoire:] Lettres d'un graveur en architecture in 8°, Paris, Jombert fils, 1772.

Fig. 2 Planche I, Fig. IV



AVANT-PROPOS

[p. iii] Quelque peu vraisemblable qu'il soit que l'on ait entrepris une Coupole aussi importante que celle de la nouvelle Eglise de Sainte Geneviève, sans avoir donné à ses principaux supports les proportions convenables pour assurer sa solidité, c'est ce dont il ne sera gueres permis de douter après la lecture de ce Mémoire. Si quelqu'un venoit sérieusement proposer d'élever sur un mur isolé de trois pieds neuf pouces d'épaisseur, & de quatre-vingt pieds d'élévation, un autre mur de plus de huit pieds d'épaisseur par le bas, & de quarante pieds de haut, avec l'obligation de faire encore soutenir à l'extrêmité de ce dernier la poussée de deux grandes voûtes, il ne pourroit à coup sûr y avoir qu'une voix pour condamner l'exécution d'un pareil ouvrage. Voilà cependant, dans toute sa simplicité, le sujet de notre problême. Le mur isolé de trois pieds neuf pouces, & de quatre-vingt pieds d'élévation, est la proportion des piliers de l'Eglise Sainte-Geneviève, déjà exécutés & destinés à porter son dôme : le mur, de plus de huit pieds d'épaisseur, est celui que les principes établis pour la poussée des voûtes, joint aux exemples de construction, necessitent de donner pour contreventer une Coupole de Soixante-

[Marginalia de Rondelet]

Quelque peu vraisemblable qu'il soit qu'un architecte ait entrepris de critiquer un ouvrage aussi important que la nouvelle Eglise de Ste Genevieve sans être en état de determiner lui-même les dimensions necessaires des points d'appui d'un pareil edifice, c'est ce dont il ne sera gueres permis de douter après la lecture des nottes faites en marge de ce mémoire.

Si un architecte, par envie et pour decrier un monument public, s'avisoit de supposer qu'il faut bâtir un mur de plus de 8 pieds d'épaisseur tandis que réellement un mur de 3 pieds 9 pouces est plus que sufisant, il n'y auroit certainement qu'une voix pour condamner la mechanceté ou l'ignorance d'un pareil architecte.

Voilà cependant dans toute sa simplicité le sujet qui me fait refuter ce memoire. Le mur isolé de 3 pieds 9° est une tour circulaire qui doit être soutenue par quatre piliers triangulaires déjà exécutés d'ensemble 535 pieds de superficie tandis que d'après les principes de mécanique et en se servant de la même formule que Mr Patte il ne faudroit que 2 pieds 5° 2 lignes au mur de cette tour pour resister à la poussée de la voute en coupole telle quelle est exprimée par le profil de Mr Patte fig. IV (Fig. 2).

[Suite de l'AVANT-PROPOS]

[p. iv] trois pieds de diamètre, comme celle en question. Est-il vrai ; en effet, qu'on ne puisse se dispenser de donner au moins huit pieds d'épaisseur au bas de la tour du dôme qu'il s'agit d'élever au centre de l'Eglise de Sainte Geneviève ? Telle est la question que nous allons développer, en nous appuyant sur des faits simples, & dont on ne puisse contester la vérité.

[Marginalia de Rondelet]

ainsi le résultat de ces nottes est de prouver qu'on peut se dispenser de donner plus de 3 pieds 9 pouces d'épaisseur en bas de la tour du dôme qu'il s'agit d'élever au centre de la nouvelle Eglise Ste Genevieve, telle est la question que nous nous proposons de développer dans ces nottes en nous appuyant sur des faits simples dont on ne puisse contester la vérité .

[p. 5] Mémoire sur la construction de la coupole projettée pour couronner la nouvelle église de Sainte Geneviève à Paris

C'est une des obligations que nous avons au progrès des Sciences, que de nous avoir mis en état de pouvoir apprécier d'avance une infinité d'opérations dans lesquelles on ne savoit se conduire précédemment qu'en tâtonnant, ou qu'en laissant le hasard I 'arbitre du succès. Entre tous les arts, il y en a peu d'aussi propre à être éclairé que celui de la construction. Comme il y est sans cesse question d'élever des corps les uns au-dessus des autres, de faire porter des fardeaux, de contreventer des poussees de voûtes, ainsi que de soutenir, soit des plans inclinés, soit des surfaces horisontales ou perpendiculaires, il s'ensuit que tous ces objets étant susceptibles de rapports ou de considérations relatives aux loix de l'équilibre & de la pesanteur, appartiennent de toute nécessité aux Mathématiques, & principalement à la Méchanique, c'est-à-dire peuvent être appréciés par ses régles.

Plusieurs de nos Savans ont plus d'une fois traité ces importantes

[Marginalia de Rondelet]

Il est très certain que les principes de mécanique appliqués à des faits et non des hipotheses, peuvent nous guider et nous faire juger d'avance du résultat de nos opérations surtout dans la construction des édifices. Pour y réussir, il faut non seulement être geometre, mais encore habile constructeur parce qu'il faut connoitre parfaitement tous les procédés dont les ouvriers se servent et les ressources ingénieuses que des praticiens dont on ne fait pas assez de cas, on[t] imaginés dans des cas extraordinaires. Il ne faut pas croire qu'une spéculation vague puisse servir de base pour raisonner consequemment sur

[Mémoire de Patte]

[p. 6] matieres. M. Parent a fait voir, dans les *Mémoires de l'Académie des Sciences*, de 1704, ce que c'est que la poussée d'une voûte, comment ses différents voussoirs agissent relativement à leur position; la clef sur les contre-clefs; les contre-clefs sur les voussoirs adjacents, & ainsi des autres jusqu'à leur retombée sur les piédroits, & enfin il a déterminé queI rapport a la poussée d'une voûte, eu égard au poids de la voûte entiere.

M. de la Hyre, dans son *Traité de Méchanique*, avoit déja démontré la disposition que l'on pouvoit donner aux voussoirs d'une voûte pour la rendre durable, & ce même Académicien a résolu depuis, dans les Mémoires de l'Académie des Sciences, de 1712, le problème de la méchanique des voûtes dans toute son étendue, & a donné des régIes précises pour trouver en toutes occasions la force que doivent avoir les piédroits, ou les murs de soutennement d'une voute pour résister à la poussée.

La plupart de ceux qui ont écrit sur la Méchanique, ont depuis examiné les mêmes questions sur la poussée des voûtes & sur les puissances en equilibre qu'il convient de lui opposer suivant les circonstances, & ont trouvé des résultats semblables, bien qu'ils se fussent servis de procédés différents pour y parvenir: ainsi il ne sauroit y avoir de doute sur la certitude des principes qui servent à établir les épaisseurs des piédroits des voûtes; ce sont des vérités Mathématiques.

Voici en général comme les Géométres s'y sont pris pour déterminer l'épaisseur du support d'une voûte ou la résistance en équilibre avec sa poussée. Ils ont considéré la demi-voûte où se fait d'ordinaire la rupture, comme un seul voussoir agissant a cause de sa forme de coin contre sa moitié inférieure jointe à tout le piédroit pour le renverser; & par la comparaison des rapports de la surface de ce voussoir avec le diamétre de la voûte la nature de sa courbe, la longueur de sa clef, la hauteur du piédroit, & même des différents poids dont ce piédroit pouvoit être chargé suivant les circonstances, ils ont trouvé par les régles de la Méchanique, les expressions algébriques des puissances qu'il convient d'opposer, dans tous les cas, à ces différents efforts pour être en équilibre avec eux.[2]

[Marginalia de Rondelet]

des objets qu'elle ne fait qu'apercevoir. Ainsi, les savants pour rendre leur formule plus generale, ayant fait abstraction des propriétés des matériaux et des procédés qu'on employe pour les mettre en oeuvre, le résultat de leur formule ne convient point du tout à la construction des édifices. Ce que j'avance est si vrai, qu'ayant appliqué ces formules à la plupart de nos édifices, j'en ai point trouvé qui n'ayent leur points d'appuis beaucoup moindre que le resultat trouvé par ces formules pour l'état d'équilibre seulement.

Ainsi quant ces édifices seront détruits, si ces formules existent encore, les Pattes de ce tems pourront persuader à la postérité, que Notre Dame, St Roch, St Sulpice n'ont jamais existé. L'édifice le plus hardi que j'aye trouvé est la petite église de Cluny près la Sorbonne dont les piedroits n'ont que le quart de ce que donneroit la formule de Mr Belidor employée par Mr Patte. Une si grande différence vient de ce que les geometres ont appliqués des principes certains à de fausses hippothèses, ils ont considéré les pierres dont on construit les voutes non seulement comme desunies entre elles, mais encore comme des corps parfaitement polis qui peuvent glisser librement à la moindre inclinaison et être mûs par la moindre puissance: on voit que ces propriétés ne peuvent convenir ***qu'à des boules exactement rondes et polies.***

[Mémoire de Patte]

[p. 7] C'est donc en nous appuyant sur les principes reconnus universellement, touchant la poussée des voûtes, & en les faisant marcher en parallele avec les proportions des meilleures constructions de même genre, que nous nous proposons, suivant la promesse que nous en avons faite page 185 de nos Mémoires sur les objets les plus importants de l'Architecture,[3] d'examiner l'exécution du dôme projetté pour couronner l'Eglise de Sainte Geneviève, & de comparer les dimensions des supports qui lui sont destinés relativement à la poussée & à son poids.

En conséquence, nous allons expliquer d'abord l'essence de la construction d'une coupole élevée sur des pendentifs, & quel doit être le rapport ou l'enchaînement de ses différentes parties pour avoir la solidité requise; & afin de confirmer par des exemples ce que nous dirons à ce sujet, nous en ferons remarquer l'application dans la construction des ouvrages les plus estimés en ce genre.

Ensuite nous examinerons comparativement aux principes reconnus & aux exemples proposés, si l'on peut espérer d'élever un dôme ou une voûte, soit sphérique, soit sphéroïde, avec une apparence de succès, au centre de l'Eglise de Sainte Geneviève, sur les piliers déjà exécutés.

[2]Note (1) de Patte : Dans les *Mémoires de l'Académie des Sciences années* 1704, 1712, 1726, 1727, 1728, 1729, 1730, dans le troisième tome du traité de la coupe des pierres, de M. Frezier, chp. XII, ainsi que dans *la Science des Ingénieurs*, de M . Belidor, l. II, on trouve développées toutes les circonstances de la méchanique des voûtes.

[3]Note (2) de Patte : Cet ouvrage se vend à Paris chez. Rozet, Libraire, rue S. Séverin, ainsi que, les *Monuments élevés en France à la gloire de Louis XV*, du même Auteur.

Article premier. De la Construction des Coupoles élevées sur des Pendentifs.

Une Coupole placée sur des pendentifs à la rencontre des bras de la croix d'une Eglise, est par son plan un cercle inscrit dans un quarré ou un octogone presque toujours irrégulier, dont le tambour ou la tour ne porte que sur quatre points, soit au milieu des côtés du quarré, soit au milieu des grands côtés de l'octogone. Or ces grands côtés étant d'ordinaire ouverts par des arcades, il résulte qu'une coupole, vû sa position, se trouve alors soutenue au-dessus des voûtes d'une Eglise précisement sur la clef des arcs formant là réunion des bras de la croix, & par des encorbellements au droit des angles, si c'est un quarré ; & au droit des petits côtés, si c'est un octogone irrégulier.

Avant de déterminer les dimensions des piliers du rez-de-chaussée

[Marginalia de Rondelet]

*Cette position des murs d'un dôme sur des encorbellements est la plus avantageuse qu'ils puissent avoir pour résister à l'effort latéral des voutes qui tendrait à les renverser, parce que la force avec laquelle ils tendent au centre de la tour, pour se soutenir, est capable de balancer la plus grande poussée (*Notte au crayon mise à l'encre par [Antoine] Rondelet[, le] fils [de Jean].*).*

[Suite de l'article premier]

[p. 8] d'une Eglise, destinés à porter une coupole, il faut auparavant décider le diamétre de sa tour, sa décoration, son épaisseur, la hauteur de ses piédroits, & la courbe de sa voûte ou de ses voûtes, si l'on en veut admettre deux: car il est évident que ce n'est que par la connoissance de ce qui sera porté que l'on peut parvenir à établir sûrement la solidité des supports: en user autrement, ce seroit opérer au hasard.

Les Architectes employent différents procédés pour terminer les coupoles. Les uns les exécutent avec une seule voûte : les autres les font à double voûte : D'autres ajoutent quelquefois au-dessus un dôme de pente pour couronner leur extérieur. On exécute aussi différemment les tambours; tantôt on contient l'effort des voûtes, en construisant leur mur d'une épaisseur uniforme, tantôt au contraire on distribue au pourtour de la tour, des contreforts vers lesquels on rejette tout le poids & la poussée: ce sont des raisons de construction qui occasionnent ces variétés. Comme notre intention n'est pas de donner ici un traité de la Méchanique de toutes les espéces de voûtes mais seulement de mettre chacun à portée d'apprécier l'insuffisance des piliers de l'Eglise de Sainte Génevieve pour porter un dôme dans le cas le plus favorable, nous nous borneront à démontrer quelles doivent être les dimensions des supports d'une coupole simple, en nous servant de principes de la certitude desquels on ne puisse douter.

Une coupole étant d'ordinaire un morceau de décoration destiné à faire l'ornement d'une Ville, & à annoncer de loin sa magnificence, la forme n'en sauroit être absolument arbitraire ; & ce n'est qu'autant que l'on parvient à lui donner un aspect gracieux sans sortir du caractere convenable à sa destination, c'est-à-dire à un Temple, que l'on réussit. On a déjà tant construit de ces ouvrages qu'on est en quelque sorte d'accord sur ce qui constitue leur vraie beauté. Si l'on fait la voûte d'une coupole plein ceintre ou surbaissée, il est d'expérience qu'elle aura l'air écrasé & sans agrément, si au contraire on la tient d'une courbe très allongée, alors elle dégénerera en un large clocher, ou un espéce de pyramide de mauvais goût & sans proportion. Entre ces deux extrêmes, & pour éviter l'inconvénient du pesant ou du mesquin, il y a sans doute un milieu à saisir.

Fontana, savant Architecte du siècle dernier a donné dans son ouvrage intitulé: *Descriptione del Tempio Vaticano, l. V. Ch. XXIV,* des régIes sures pour trouver les proportions les plus agréables des coupoles simple [p. 9] afin de produire à la fois un bon effet en dedans & en dehors ; comme il seroit difficile de rien ajouter à ce qu'il a dit à ce sujet, d'après les meilleurs modèles d'Italie, nous ne pouvons mieux faire que de le rapporter.

Après avoir établi le diamétre CC, *planche 1, figure I* (Fig. 3) , du tambour d'une coupole, la grande corniche A, & la proportion du piédestal Intérieur ou socle B, il faut prendre pour hauteur de la décoration du dehors du tambour, la longueur de son demi-diamétre total, c'est-à-dire, y compris son épaisseur de mur ou de piédroit, que l'on trouvera comme il sera dit ci-apres; & en divisant cette hauteur trouvée DE en quatre parties égales, les trois supérieures DD donneront l'élévation des colonnes ou pilastres avec leur entablement, & la quatrieme restante DE sera pour celle du piédestal, à moins que les toits de l'Eglise n'y mettent obstacle. Le demi-diamétre du dedans-Œuvre de la coupole donnera au contraire la proportion de la décoration intérieure du tambour : en plaçant cette moitié sur le piédestal B, on obtiendra sa hauteur totale, dans laquelle sera comprise le petit piédestal F destiné à recevoir la voûte.

On parviendra ensuite à déterminer la courbe de la Coupole que l'on construit d'ordinaire en briques, en divisant le diamétre du dedans du tambour en 12 parties égales, & en portant une de ces parties au-dessus du petit piédestal F, en G, la droite CG coupant à angles droits l'axe de la Coupole, sera la ligne diamétrale où l'on établira ses différents centres. Pour les trouver, du point d'intersection H, il faut tracer un demi- cercle I I, puis porter de part & d'autre du point H, & du sommet de ce demi-cercle une des 12 parties en question, & en tirant les lignes KI, K′I,

[Marginalia de Rondelet]

Nous allons determiner d'apres des principes plus certains les endroits où l'on doit placer les cercles de fer.

[Suite de l'article premier]

on aura la largeur de l'œil de la lanterne vers le haut de la voûte: après cela, en partageant H K en deux également, on aura quatre points K, L, L', K′, dont les deux K, K′ serviront à tracer chaque côté de la courbe intérieure M de la voûte, & les deux autres serviront à tracer la courbe extérieure N, dont la naissance fera une retraite convenable en dehors

[Marginalia de Rondelet]

du mur intérieur où commence la naissance de la voute, on élèvera une tangente indéfinie à la courbe de la voute, et une autre de l'angle de l'ouverture de la lanterne. Ces deux tangentes se rencontreront en un point, de ce point on tirera une perpendiculaire à la courbe qui rencontrera l'extérieur de la voute, à l'endroit où se fait le plus grand effort, et où l'on

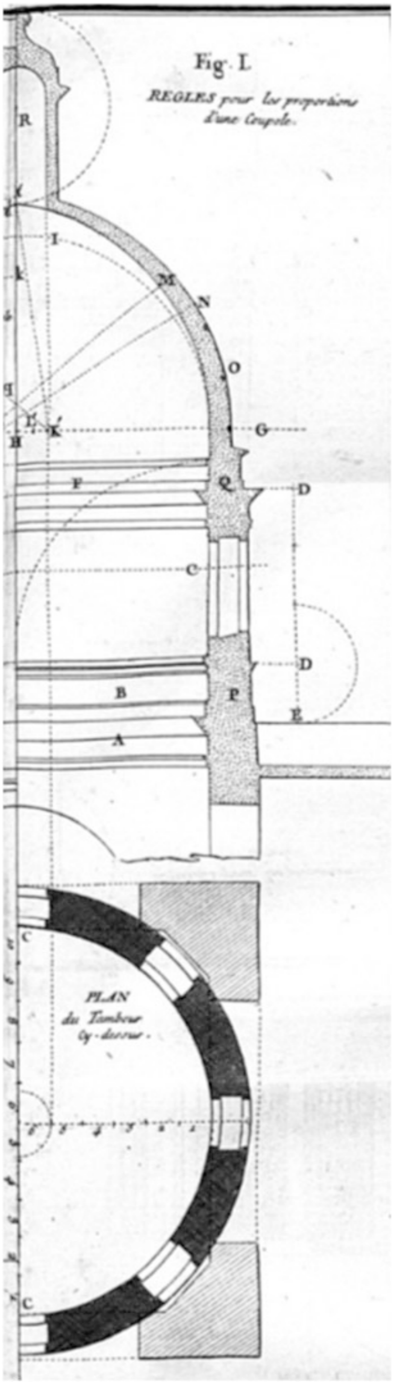


Fig. 3 Planche I, Fig. I

*doit placer le principal cercle de fer. Si l'on en veut placer d'autre, ce doit être aux points où les tangentes rencontrent l'extrados de la voute. (*notte au crayon mise à l'encre par [Antoine] Rondelet[, le] fils [de Jean])

[Suite de l'article premier]

sur le piédestal qui la reçoit,[4] laquelle courbe N se divisera en trois parties, dont l'inférieure indiquera la place d'un des cercles de fer O. Quant à la proportion de la lanterne R, on la fixera en lui donnant pour [p. 10] hauteur, sans comprendre la boule & la croix, le quart du diamètre intérieur C C.

Enfin on aura l'épaisseur P du tambour, nécessaire pour contreventer la coupole, en partageant le diamétre C C de son dedans-œuvre en 10 parties égales, & en donnant à son mur uniformément une de ces parties; & pour ce qui est de l'épaisseur du piédestal Q ou F, qui porte la naissance de la voûte, il suffira de lui donner les trois quarts d'une des 10 parties en question.

[Marginalia de Rondelet]

nous n'avons rien à dire sur les dimensions indiquées par Fontana elles sont plus relatives au gout qu'à la solidité, mais quant à l'épaisseur, elle ... n'est pas terminée. (R[ondelet] f[ils].)

[Suite de l'article premier]

Pour appuyer l'épaisseur qu'il assigne au mur pourtour d'une coupole, Fontana cite plusieurs dômes de Rome où ces régIes se trouvent en quelque sorte observées, tels que la coupole de S. André della Valle, qui a intérieurement 74 palmes & demie de diamétre,[5] & 7 palmes & demie d'épaisseur de mur, c'est-à-dire, un peu plus de la dixiéme partie de son diamétre.[6]

[Marginalia de Rondelet]

51 pieds 5° 6 lignes, l'épaisseur est de 5 pieds ° 9 lignes

[Suite de l'article premier]

La coupole di *S. Carlo à Catinari*, qui a 72 palmes de diamétre, & des murs épais de 7 palmes un quart.[7]

[Marginalia de Rondelet]

La coupole de St Carlo a Catinari a 46 pieds 6° de diamètre le mur de la tour a 4 pieds 6° d'épaisseur

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[4]Note (3) de Patte : Il est à observer que Fontana ne parle pas positivement de l'épaisseur de la voute, mais que dans son dessein il lui donne par le bas, la moitié de l'épaisseur du mur de la tour, allant toujours en diminuant jusqu'au col de la lanterne, où cette épaisseur se trouve réduite presqu'au quart de celle du mur de la tour : qui nous paroît ne pouvoir s'écarter de la vérité, eu égard à l'isolement de la voûte, au poids de la lanterne qu'il fait porter sur son son sommet, & à tous les exemples de voûtes exposées aux injures du temps.

[5]Note de Radelet-de Grave : Rondelet a barré 74 palmes & demie de diamétre pour le remplacer par ce qui est dans la marge : 51 pieds 5° 6 lignes.

[6]Note (4) de Patte : La palme a huit pouces, 3 lignes, 6 points du pied de Roi.

[7]Note de Radelet-de Grave: Dans le texte Rondelet a barré 7 palmes un quart pour le remplacer par 6 palmes ¾.

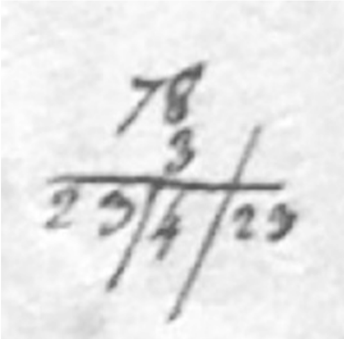

Fig. 4 Calcul de Rondelet

[Dans la marge de droite Rondelet a fait un petit calcul] (Fig. 4):
[Il a aussi ajouté dans la même marge après chaque item ou mesure]
1/10 ; 1/10 ; 1/10 ; 1/10 ; 1/19

[Suite de l'article premier]
La coupole della *Madona de Miracoli*, qui a 78 palmes trois quarts de diamétre; & 7 palmes deux tiers d'épaisseur.

La coupole de l'Eglise de Jésus, qui a 78 palmes de diamétre, avec des murs de 7 palmes trois quarts d'épaisseur.

La coupole di *Santa Margarita in monte Fiascone*, qui a 115 palmes de diamétre, avec des murs construits de tuf, lesquels ont 13 palmes un quart d'épaisseur, c'est-à-dire, environ la neuviéme partie de son diamétre.[8] D'après ces proportions fondées sur l'expérience, cet Architecte conclud qu'on ne saurait se dispenser de donner au moins pour épaisseur aux murs destinés à porter les coupoles simpIes sur pendentifs, la dixiéme par- [p. 11] tie de leur diamétre intérieur, pourvû toutefois, dit-il, qu'ils soient construits de bons matériaux bien durs; que quand on voudra les bâtir en pierres legéres, il sera à propos de leur donner au moins la neuviéme partie

[Marginalia de Rondelet]
S'il suffit de donner aux Murs des Dômes la 10^{e} partie de leur diamètre étant construit en blocage, comme on l'a fait a Rome quant on voudra les bâtir en pierre, il suffira de leur donner la douzième partie pour ceux en pierre de taille dont la pesanteur et la solidité

[8]Note (5) de Patte : On peut ajouter à ces exemples, que le Dôme de Sainte Marie des Fleurs à Florence, qui est un octogone portant uniformément sur ces gros piliers, a 182 palmes de diamétre, & 24 palmes d'épaisseur de mur pourtour, ce qui en fait environ la septiéme parttie ; & que le dôme du Panthéon à Rome, dont le diamétre est, 193 palmes 2 tiers, à des murs pourtour de 30 palmes d'épaisseur, c'est-à-dire, de plus du sisième de son diamétre.

Dans le cours de nos voyages en Italie, en Angleterre, en Hollande dans une partie de la France &. de l'Allemagne, nous avons beaucoup examiné la construction de la plupart des coupoles & des voûtes sphériques ou sphéroïdes, élevées dans ces différents pays, & nous n'en avons pas remarqué de quelque étendue dont l'épaisseur des murs, lorsqu'elle est uniforme, ne fût à peu près le dixiéme de leur diamétre.

est bien plus considérable. Il ne faut pas avoir une bien grande idée en construction pour concevoir qu'un mur en pierre de taille, est beaucoup plus solide qu'un mur en brique ou en moellons, et qu'ainsi il faut plutôt diminuer qu'augmenter l'épaisseur de ces murs. (Mis à l'encre par moi R[ondelet] f[ils].)

[Suite de l'article premier]

de leur diamétre; & qu'enfin, pour contenir des coupoles à double voûtes, il faudra encore donner davantage d'épaisseur à leurs murs.

Si, de ces preuves de fait, on passe aux préceptes que fournit la Mécanique pour déterminer l'épaisseur du mur qui doit porter à rez-de-chaussée une voûte sphérique ou sphéroïde dont on connoît le diamétre, la courbe de la voûte, le poids dont elle peut être chargée à son sommet, & la hauteur des piédroits, on trouve que ces sortes de voûtes poussent environ la moitié moins que celles en berceau simple, de même nature, diamétre, épaisseur ou charge, & que par conséquent, en ne donnant à leurs murs ou piédroits, que la moitié de l'épaisseur des voûtes en berceau, conditionnées de même ils auront toute la force nécessaire pour être en équilibre avec la poussée. C'est de cette maniere que M. Frezier, dans le troisiéme Tome de son *Traité de la coupe des pierres*, *Chapitre XII*, considére la poussée des voûtes sphériques & sphéroïdes : il la rapporte à l'action qu'exerce contre ses supports, une voûte en arc de cloître dont le plan seroit composé d'une infinité de côtés devenus si petits qu'ils seroient sensiblement confondus avec le cercle dans lequel le polygone serait inscrit.

Faisons l'application de ces principes à la voûte de Fontana, pour découvrir le rapport de sa régle pratique avec la Théorie, & en quoi l'épaisseur de son piédroit différe de celle assignée pour l'équilibre: on y parviendra en fixant le diamétre de cette voûte, son épaisseur & la hauteur de ses piédroits. Soit le diamétre 63 pieds la courbe de la voûte surmontée d'un douziéme, la hauteur des piédroits 36 pieds, & l'épaisseur réduite pour la demi-voûte 24 pouces, ce qui sera à peu près 18 à 20 pouces vers le col de la lanterne ; dimension qui est démonstrative relativement aux circonstances. En effet, l'épaisseur d'une voûte isolée sur ses supports & exposée par sa grande élévation à toutes les injures de l'air, doit être nécessairement bien différente de celle d'une voûte à couvert sous un toit de charpente, & dont les reins peuvent être fortifiés de toutes parts : elle se doit régler encore par son étendue, & par la considération du fardeau qu'elle sera contrainte de porter sur son sommet : il faut qu'elle soit en état de soutenir une couverture de plomb, la lanterne qui la couronnera, la neige qu'elle sera quelquefois obligée de recevoir; en un mot qu'elle soit capable de résister à la violence des ouragans & aux autres causes physiques qui peuvent concourir à sa destruction. Il est évident que toutes

[Marginalia de Rondelet]

La formule de M. Belidor ne pouvant s'appliquer qu'aux voutes en berceau, il aurait falu avant d'en faire l'application à une voute sphérique, ou sphéroïde de même ceintre déterminer le véritable rapport de la poussée d'une voute sphérique a celle d'une voute en berceau de même diametre et epaisseur.

M. Frezier au 3e tome de la coupe des pierres dit à ce sujet que les voutes sphériques «poussent plus de la moitié moins que les berceaux simples de même cintre, diametre et epaisseur ou charge, et par conséquent qu'en ne donnant a leurs piedroits que la moitié de *celle des berceaux conditionnés de même, ils seront encore plus forts qu'il n'est nécessaire pour les mettre en équilibre avec la poussée ».*

ainsi selon Mr Frezier et d'après le résultat du calcul de M Patte, si pour une voute en berceau de 63 pieds de diamètre, s'il faut 8 pieds 10°-11 lignes 1/3 d'épaisseur de piedroits 4 pieds 5°-5 lignes 2/3 seroient plus que sufisans pour une voute sphéroïde de même ceintre et epaisseur.

Cependant comme M. Frezier n'a pas eu égard à la difference dont ces deux especes de voutes agissent, nous allons tacher de fixer plus exactement ce rapport et faire connoitre tout l'avantage des voutes sphériques sur les voutes en berceau, en comparant un berceau circulaire aussi long que large avec une voute sphérique de même diamètre et épaisseur. Le profil ou coupe de l'une et l'autre de ces voutes sera representé par la figure I (Fig. 3). *Cela posé, l'expérience et les principes de mathématique prouvent que dans toute sorte de voute en berceau, les parties inférieures jusqu'à une certaine hauteur*

[Suite de l'article premier]

[p. 12] ces considérations exigent de tenir une voûte extérieure beaucoup plus épaisse que lorsqu'elle est à l'abri, & qu'elle n'a point d'inconvéniens à prévenir.[9] En faisant les calculs de cette voûte, comme si elle étoit seulement en berceau surmonté, on trouvera, par l'application de la formule donnée par Belidor dans la *Science des Ingénieurs, Liv. II.* 9 pieds d'épaisseur de piédroit pour puissance en équilibre, & en prenant la moitié de cette mesure, attendu que la voûte qui fait l'objet de notre examen est spheroïde, on aura pour l'épaisseur cherchée, du piédroit, 4 pieds 6 pouces.[10]

Mais, comme l'on sait que dans la pratique, l'épaisseur indiquée par la Méchanique ne suffit pas, il sera à propos d'ajouter en sus, afin que la puissance résistante soit supérieure à celle qui doit agir. Quand les voûtes ont à peu près 7 ou 8 toises de diamétre, il est d'usage d'augmenter cette épaisseur d'environ un pied, & de donner davantage, à proportion que la grandeur de la voûte s'accroît, ou que des circonstances locales paroissent le demander, comme pourroit être le cas d'une coupole portée en l'air sur 4 points quelquefois à plus de 100 pieds : car il n'est pas douteux que son exécution ne demande bien d'autres considérations, à cause de sa position extraordinaire, que si elle étoit placée à rez-de-chaussee, sans sujétion quelconque. En se contentant d'augmenter lentement d'un pied l'épaisseur du piédroit 4 pieds 6 pouces, pour se mettre en force au-dessus de l'équilibre; & en ajoutant encore 6 pouces au moins tant à cause de la position de la voûte sur pendentifs & du poids de la lanterne que nous n'avons pas fait entrer dans les calculs, lequel en agissant sur son sommet, éloignera son centre de gravité, qu'à cause des impressions de l'humidité, qui, en s'incorporant aisément dans les

[9] Note (6) de Patte : M. Couplet dans les *Mémoires de l'Académie des Sciences* de 1729, a essayé de déterminer par une formule la moindre longueur des voussoirs d'une, voûte en pierre: pour qu'elle puisse se soutenir en équilibre par l'énergie seule de ses parties, & a fait voir qu'une voûte de 18 pieds de diamétre, éxigoit 17 pouces 10 lignes 1/4 d'épaisseur ; c'est pourquoi en nous bornant à donner à peu près la même épaisseur au sommet d'une voûte sphéroïde, on ne sauroit nous soupçonner d'exagerer.

[10] Note (7) de Patte : A la fin de ce Mémoire on trouvera la solution de ce probléme.

pores d'une voûte extérieure, la surcharge souvent d'un poids très-considérable, & augmente en conséquence l'effort contre ses piédroits, il s'ensuit que l'épaisseur cherchée de la tour ne sauroit s'écarter de 6 pieds, & que la régIe de Fontana, en indiquant de donner 6 pieds 3 ou 4 pouces dans

[Marginalia de Rondelet]

tendent a tomber en dedans et que les parties superieures ne se soutiennent qu'en agissant en sens contraire, avec un effort qui tend à renverser les parties inferieures et les piedroits qui les soutiennent. Mr Belidor suppose dans sa formule que la partie superieure qui cause la poussée designée dans la figure 1[11] *par DEFHI est précisement la moitié de la voute en berceau et par conséquent que les deux parties inférieures qui resistent en partie à l'effort de la poussée font ensemble l'autre moitié. Ainsi en comparant une voute spherique a une voute en berceau de même diamètre et dont la longueur serait egale a la largeur, la projection de la partie qui cause la poussée représentée pour les deux voutes par le profil D E F H I, sera exprimée en plan, pour la voute en berceau, par le rectangle e.f.g.h. fig. 2, et pour la voute spherique, par le cercle i, k, m, n et comme ces parties sont de même epaisseur elle seront entre elles comme leur superficie moyenne c'est à dire comme 7 est à 3 de manière que si l'on supose que la partie qui cause la poussée d'une voute en berceau est 7, celle d'une voute sphérique de même diamètre et epaisseur ne sera que 3, de plus les parties inferieures qui resistent a la poussée d'une voute en berceau étant 7 celles qui resistent dans une voute spherique seront environ II ainsi dans la voute en berceau la partie qui cause l'effort superieur etant 7 est balancé par deux parties qui valent ensemble 7 au lieu que dans la voute sphérique la partie qui cause l'effort superieur n'étant que 3 se trouve balancée par une masse egal à II. De plus l'effort d'une voute en berceau aussi longue que large se fait contre deux murs isolés dont la longueur prise ensemble n'est que les 7/II du mur circulaire qui soutiendrait une voute spherique de même diamètre. En reduisant par le calcul ces differens efforts pour les comparer on trouvera que la poussée d'une voute sphérique n'est que la sixième partie de celle d'une voute en berceau de même diamètre et epaisseur dont la longueur est égale a la largeur.*[12]

[Suite de l'article premier]

[p. 13] le cas actuel, c'est-à-dire le dixiéme du diamétre, 63 pieds, doit être regardée comme une pratique excellente, d'accord avec la théorie, & qu'il ne pourrait être que dangereux de restraindre.[13]

Nous n'avons parlé jusqu'ici que de l'épaisseur uniforme qu'il convient de donner aux murs pourtours d'une coupole élevée sur pendentifs, pour résister également à son action; mais ce procédé n'a gueres lieu que pour des dômes d'un diamétre peu considérable ; car, lorsqu'ils deviennent d'une certaine grandeur, ou qu'ils doivent être à double-voûte, on trouve beaucoup plus d'avantage d'y repartir des contre-forts, vers lesqueIs on rejette par des lunettes ou des arcs en décharge

[11] Note de Radelet-de Grave : Rondelet ne semble pas utiliser les figures de Patte ou y a introduit ses propres notations. Malheureusement ses figures ne sont pas être restées jointes à l'article reproduit.

[12] Note de Radelet-de Grave : Cette Marginalia de Rondelet se poursuit au haut de la page suivante.

[13] Note (8) de Patte : Les Italiens estiment que la rupture ne se fait pas toujours au milieu de la demi - voûte lorsque le piédroit est trop foible, mais qu'elle s'opére aussi quelquefois vers son tiers inférieur, & que parconséquent il est plus sûr de la considérer depuis cet endroit. Relativement à cette observation qui nous a été faite par le Pere Boscovich, Correspondant de l'Académie des Sciences; il résulte que la surface du voussoir qui agit contre le piédroit devenant plus considérable, on trouve une épaisseur plus grande suivant cette maniere de calculer.

tout l'effort, & que l'on considére alors comme autant de tranches perpendiculaires comprenant toutes la clef, & allant correspondre aux côtés opposés du diamétre. La tour portant presque de toutes parts à faux sur le plan inférieur de l'Eglise, on parvient par le moyen des contre-forts à alléger le fardeau; & au lieu de le laisser porter au hasard sur les encorbellements, on se rend maître de le diriger à volonté vers les endroits les plus solides ou les plus capables de résister, tels que les arcs des bras

[Marginalia de Rondelet]

La poussée diminuera a mesure que l'on combinera plus de tranches ensemble et lorsque le nombre de tranches formera a peu près la moitié de la voûte, elle se soutiendra toute seule de quelque nombre de parties qu'elle soit composée (mise à l'encre par [Antoine] R[ondelet, le] f[il]s.)

[Suite de l'article premier]

de la croix, & les massifs des gros piliers, sans compter que par leur secours on vient à bout de lier le mur de la tour avec les piliers, ce qui n'est pas possible autrement. On établit l'épaisseur des contre-forts, en calculant l'action que la portion de voûte correspondante peut exercer contre chacun, en ayant égard, comme de coutume, à la nature de la courbe du dôme, à son diamétre, à son épaisseur, & au fardeau dont il peut être chargé.

Tout le poids & la poussée d'une coupole étant par ce procédé rejettés vers des points d'appui principaux, il est manifeste que leur épaisseur doit être plus grande que ne seroit celle des murs, si la voûte y était soutenue également, & que même cette épaisseur doit varier suivant les circonstances parriculieres de leur largeur ou de leur espacement. C'est pourquoi, comme il ne sauroit y avoir de régIes précises à cet égard, nous nous bornerons à remarquer que lorsqu'un plan est circulaire, il faut bien se garder de trop espacer les contre-forts, de crainte qu'à cause de

[Marginalia de Rondelet]

Si l'on divise une voute en berceau en plusieurs tranches perpendiculaires à l'axe, la poussée de chacune de ces tranches désignée par L, M, N, 0, P, Q, (fig. 2) agira separément selon des directions parallèles, en sorte que l'effort total ne sera ni augmenté ni diminué, mais si l'on divise une voute sphérique en un même nombre de parties par des lignes qui se croisent à l'axe, la direction de la poussée changera pour chaque tranche triangulaire en sorte que les efforts des parties eloignées d'un quart de la circonférence seront perpendiculaires entre elles au lieu d'être parallèles comme dans la voute en berceau; et les directions de celles qui sont eloignées d'une demie circonférence sont directement opposées. D'où il resulte que ces efforts, au lieu d'être independants comme dans les tranches d'une voute en berceau, se détruisent en partie.

C'est pour cette raison qu'on peut couper une voute sphérique en deux parties egales sur quel sens on veut et que les deux demis voutes en niche qui en resultent peuvent se soutenir independament l'une de l'autre et par consequent sans aucun effort d'une moitié contre l'autre, D'où il resuIte que les voutes sphériques n'ont presque point de poussée et qu'en ne donnant aux murs circulaires qui doivent les soutenir que la même epaisseur que les voutes, ils auront encore une epaisseur plus que sufisante pour soutenir leur poussée. Enfin, si l'on compare le peu de solidité d'une voute en berceau dont les joints peuvent se desunir dans toute la longueur et dont les murs seroient isolés, avec une voute sphérique dont chaque rang de voussoir forme autant de couronnes dont les points se relient les uns sur les autres, en sorte que pour qu'il se fasse un effort dans une voute spherique, il faut qu'il se fasse un dechirement dans la voute et les piedroits qui la soutiennent. On ne peut pas s'empecher *de conclure de ce rapport que les voutes spheriques ont beaucoup plus de solidité que les voutes en berceau.*

[Suite de l'article premier]
[p. 14] son effort excentrique ; la voûte ne souffle dans leur intervalle;

[Marginalia de Rondelet, marge de droite]
la partie de la circonférence qu'il occupe

[Suite de l'article premier]
que, quand on les espace du double de leur largeur, leur épaisseur est à peu près semblable à celle donnée par les formules pour les voûtes en berceau de même nature, diamétre, épaisseur & charge, & qu'enfin dans tous les exemples de construction de ce genre, jamais l'épaisseur des contre-fort n'est au-dessous du huitiéme de leur diamétre intérieur, ainsi qu'on le verra par la suite.

Pour ce qui est des murs compris dans leur intervalle, ce sont des considérations particuliéres qui déterminent leur épaisseur, telles que la saillie des pendentifs, ou l'obligation dans laquelle ils se trouvent de soutenir des platebandes, des saillies de corniche, &c.

L'épaisseur uniforme des piédroits, ou celle des contre-forts nécessaires pour soutenir la poussée d'une coupole, étant décidée, on ajoute au-delà les décorations d'Architecture, & l'on place vers le bas de la tour un espéce de soubassement ou grand piedestal qui lui sert particuliérement de fondation, & qui comprend tant en dedans qu'en dehors toutes les saillies. A l'aide de cet arrangement on vient à bout, malgré la différence des plans supérieurs & inférieurs, de réunir toutes les parties de la tour, & de les empêcher d'agir séparément sur les encorbellemens & sur les gros piliers de l'Eglise.

Mais, de même que dans le bas de toutes les fondations ordinaires, on met de bons empatemens ou des corps avancés pour fortifier leur assiéte sur le sol, il n'est pas moins essentiel d'en placer au bas de la tour d'un dôme sur pendentifs à sa jonction avec les voûtes des bras de la croix, dans l'intervalle d'un pilier à l'autre, & c'est proprement ce qu'on appelle contreventer.[14] On emploie pour cela quatre

[14]Note (9) de Patte : On sentira l'obligation de fortifier la fondation de la tour d'un dôme sur pendentifs, en faisant attention que dans toutes les voûtes la poussée agit de préférence contre le pied extérieur de son piédroit ou de ses contreforts ; endroit que les Mechaniciens ont appellé pour cela l'*Hypomochlion* ou le point d'appui de la puissance. On lit dans la *Science des ingénieurs de Belidor Liv. II*, page 29, un exemple qui prouve combien il est dangereux de ne pas fortifier le bas des Piédroits d'une voûte, & qu'il a été témoin de la chûte d'un magasin à poudre bien construit & avec des murs d'une bonne épaisseur pour résister à la poussée, uniquement parce qu'on avoit négligé la précaution d'y former des empatemens.

[Marginalia de Rondelet, marge de gauche]

A propos de cette note, Rondelet place deux notes : Marginalia de Rondelet, marge de gauche : La voute dont parle Mr. Belidor a pu etre mal faite, decintrée trop tôt ou peut etre cela aura pu provenir d'un tassement inegal du sol qui aura pu faire deverser un des murs plus que l'autre & puisqu'il existe une infinité d'edifices qui n'ont pas a beaucoup pres leurs piedroits aussi forts en proportion que les Magazins a poudre citté.

[Marginalia de Rondelet, marge de droite]

c'est qu'elle etait mal faite.

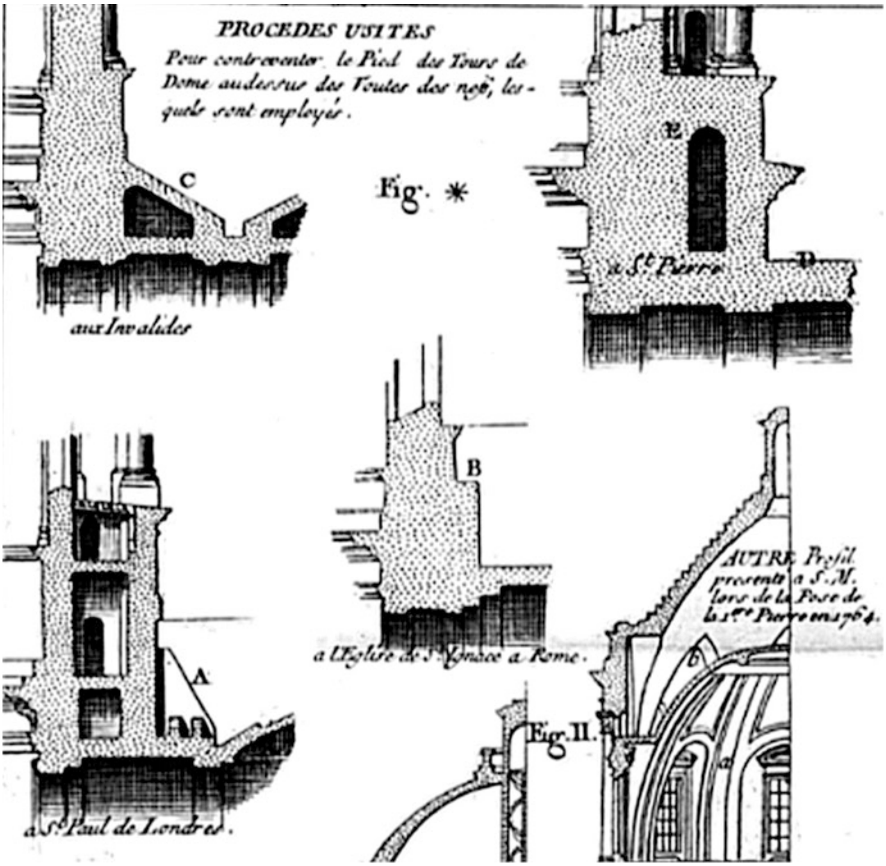

Fig. 5 Planche I, fig. * A-D

différens procédés : le premier consiste à distribuer de distance en distance autour du piédestal des éperons dirigés vers l'axe du dôme que l'on lie quelquefois entre eux par des arcs renversés ; c'est ainsi qu'on en a usé à S. Paul de Londres, dont on voit le profil du bas de la tour A *figure *, planche I* (Fig. 5).

[Marginalia de Rondelet, marge de gauche]

D'un autre coté la forme en plan de chaque moitié de mur circulaire qui supporte une voute spherique, a une resistance bien superieure a celle d'un mur droit de meme epaisseur. Car le bras de levier de resistance d'un mur droit n'est egal qu'à la moitié de son epaisseur telle longueur qu'il aye, au lieu que le bras de levier de la resistance d'un mur circulaire augmente en raison de sa longueur, en sorte que pour une demie circonference le bras de levier est egal a toute l'epaisseur du mur plus le tiers du Diamètre environ ainsi le bras de levier du mur de 8 pieds 10 pouces 11 lignes 4 points d'epaisseur trouvé par la formule de Mr Belidor pour une voute en berceau de 63 pieds de diametre, ne seroit que de 4 pieds 5 pouces 5 lignes 2/3 tandis que celui d'un mur circulaire de même epaisseur qui formeroit une demi circonference de cercle auroit 29 pieds 10° 11 lignes 1/3 de maniere que sa résistance seroit six fois plus forte que celle d'un mur droit; donc pour resister a un même *effort il faut a un mur circulaire qui comprendroit une demi circonference six fois moins d'epaisseur qu'à un mur droit.*

Pour prouver si l'experience se trouvait d'acord avec cette théorie j'ai fait faire deux modèles de voute savoir l'une sphérique et l'autre en berceau de chacune 9 pouces de diametre. La voute en berceaux divisée en 4 parties se soutient à peine par des piedroits de 22 lignes d'epaisseur tandis que la voute sphérique divisé en 12 parties se soutient fort bien sur 4 colonnes isolées dont le diametre est égal à l'epaisseur de la voute.

Cependant ces 8 colonnes ensemble ne font que la 9e partie du cube des deux piedroits et comme elles sont isolées le bras de levier de leur resistance n'est que les 3/4 de celui des murs de la voute en berceau. Ainsi l'expérience d'acord avec la théorie precedente prouve que les voutes sphériques n'ont presque point de poussée et que c'est un abus de faire leur piedroits plus forts que l'epaisseur par le bas de la voute qu'ils ont a soutenir. Toutes les augmentations que propose Mr Patte ne sont que des pretextes spécieux pour soutenir le paradoxe qu'il a osé avancer.

[Suite de l'article premier]

*figure *, plan-* [p. 15] *che I* (Fig. 5): le second à laisser uniformément tout au pourtour un large empatement,

on employe les empatemens et les retraites au rez de chaussée d'un edifice pour obvier au tassement inegal du sol mais lorsqu'il s'agit de s'eriger sur des murs en pierre de taille qui doivent etre incompressible lorsqu'ils sont construit avec soin ces empatemens deviennent abusif et inutile et Mr Patte ne les propose que pour soutenir son paradoxe et si l'on avait des retraites et des empatements à pratiquer ce devrait plutot etre pour elegir et non pour augmenter le poid.

[Suite de l'article premier]

tel que celui B, *fig.* * (Fig. 6), lequel a lieu au bas de la coupole de l'Eglise de S. Ignace à Rome: le troisiéme à rejetter par des arcs-boutans

[Marginalia de Rondelet]

Les arc boutans sont aussi inutiles que les empatemens puisque nous avons prouvé que les voutes sphériques ont si peu de poussée qu'en ne donnant aux murs qui les supportent que la même epaisseur qu'a la voute elles auront plus de solidité que des voutes en berceaux aux piedroits desquelles on donneroit une epaisseur double de ce que donneroit la formule.

Les exemples cittés par Mr Patte ne prouvent point ce qu'il pretend mais seulement qu'on s est servi de ces procédés plutot par rapport a la decoration extérieure et pour faire piramider l'edifice que par une necessité de contreventer la tour qui par sa position sur 4 pendentif n'a pas besoin qu'on augmente la force avec laquelle ils tendent a s'approcher de l'axe de la tour. Cet effort même pouroit être balancé avantageusement par la poussé des voutes qui servent a la couronner

[Suite de l'article premier]

toujours dirigés vers le centre, l'effort du bas de la tour contre les massifs voisins, comme on l'a pratiqué au dôme des Invalides C, fig. * (Fig. 7): enfin le quatriéme procédé qui est le plus ordinaire, & dont on a fait usage à S. Pierre de Rome D, *fig.* * (Fig. 8) consiste à contreventer le bas du piedestal par une sorte de voûte en berceau, placée le long des bras de la croix, & appuyée vers l'extrémité opposée par un bon mur; laquelle voûte fait alors précisément l'office d'un contrefort placé horisontalement, & embrassant le pourtour du ventre du piedestal entre les piliers. Il ne subsiste pas d'exemples où l'on n'ait contreventé le pied des coupoles de l'une de

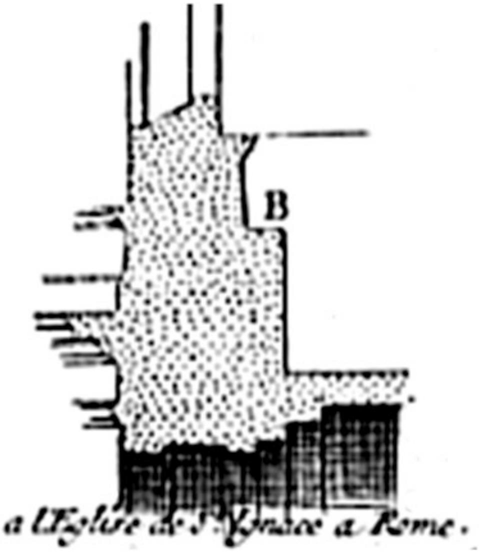

Fig. 6 Planche I, fig. * B

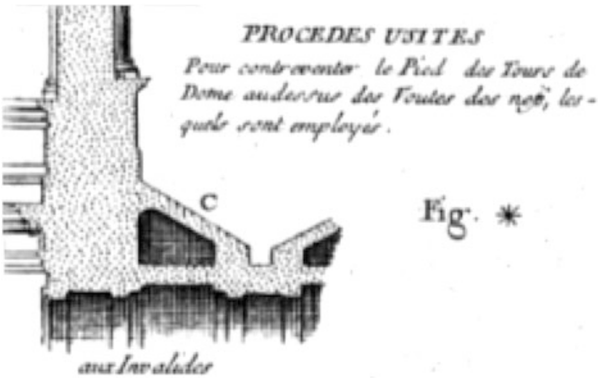

Fig. 7 Planche I, fig. *, C (aux Invalides)

ces maniéres ; & l'on ne sauroit s'en dispenser que dans le cas où, à raison du grand relief des décorations d'Architecture au-dela de l'épaisseur nécessaire des piédroits, le piedestal auroit par lui-même une saillie suffisante pour opérer cet effet.

Après avoir fixé les proportions du tambour d'une coupole, & être convenu de la maniére de le contreventer, il faut disposer les gros piliers d'une Eglise, qui sont d'ordinaire cantonnés dans les angles de la croix, de maniere à pouvoir porter ce fardeau considérable avec solidité. Nous avons dit qu'un dôme sur pendentifs était soutenu sur la clef des arcs formant la réunion des bras de la croix, & dans les angles par des encorbellements. Cette position, en indiquant comment doit agir le dôme contre ses piliers, fait voir conséquemment comment se peuvent considérer

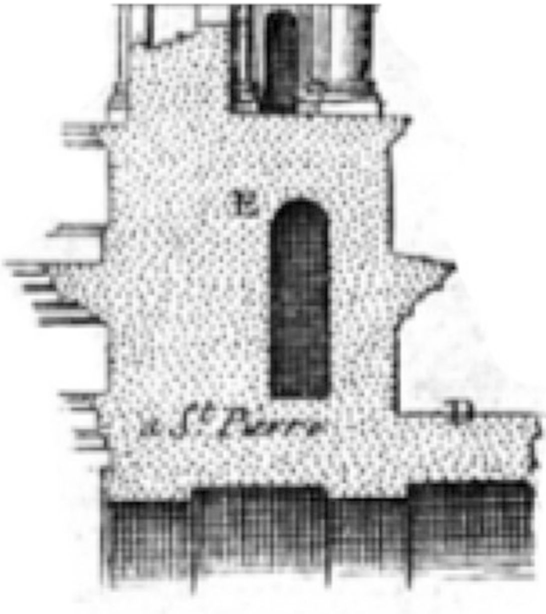

Fig. 8 Planche I, fig. *, D (St. Pierre)

leurs dimensions pour résister à ses efforts. Puisqu'une partie du dôme doit être portée sur la clef des arcs, il est donc essentiel de construire ces arcs en pierre dure, de même que leurs piédroits, & d'une proportion relative au fardeau : leur largeur est toujours donnée par l'épaisseur du bas du piédestal qui éléve la tour ; la hauteur de clef par le poids qu'ils doivent recevoir; & l'épaisseur desdits piédroits convenable pour contenir des arcs aussi chargés, se trouve par les calculs : il ne peut y avoir de doute à ce sujet ; toujours dans un édifice les parties inférieures & supérieures doivent avoir une certaine correspondance qui garantisse leur solidité.

Quant à l'action des pendentifs contre le pilier du côté de l'intérieur du dôme, il n'est pas moins évident que la partie la partie de la tour portant à faux au-dessus de cet endroit & agissant du côté du pan-coupé, ne fasse [p. 16] aussi de très-grands efforts pour le renverser, & qu'ainsi il ne lui faille uniformément dans cette direction une épaisseur suffisante pour résister.

[Marginalia de Rondelet]

a quoi je repond que lorsque les pendentifs sont bien faits, ils doivent s'appuyer sur les reins des grandes arcades qui forment l'ouverture des nefs et alors la plus grande partie de leur effort sert a detruire celui des grandes arcades; De même que les grandes arcades suportant une partie du poid de la tour doivent agir en sens contraire pour detruire l'effort des pendentifs: Car il faut les considerer comme des parties de voutes sphériques dont chaque rang d'assise forme une couronne incomplette qui dirige tout son effort contre les reins des grandes arcades qui les terminent.

C'est pour procurer plus parfaitement cet effort qu'il faudroit que les voussoirs qui terminent chaque couronne incomplette fussent communs a ceux des arcades.

Mais bien plus la forme circulaire de la tour empeche cet effet des pendentifs dans toute sa hauteur; car cet effêt tendant a l'axe de la tour il faudroit que les murs s'ecrasent verticalement pour qu'il eut lieu et de plus si cet effort avait lieu il serviroit a detruire la poussée des voutes superieures si elles en avaient. L'effort par lequel elles tendent au centre étant bien superieur a celui d'une voute meme en berceau. Il resulte donc que dans ces sortes d'ouvrages tous les efforts tendant à l'axe Lorsque les murs sont sur un plan circulaire tous les efforts contribuent a les rendre plus solides en concourant a en former un seul tout. Donc les procedés cités par Mr Patte et ceux qu'il propose sont abusifs et imaginé par la crainte et l'ignorance, car en détruisant ces efforts il rendent l'edifice moins solide. C Q F D.

[Suite de l'article premier]

Dans les exemples on remarque que cette épaisseur est toujours deux & trois fois plus considérable que la saillie de l'encorbellement; & par les calculs on trouve qu'il la faut plus d'une fois & demie : mais nous ne nous arrêterons pas à cette spéculation, parce qu'elle devient d'ordinaire inutile, quand on a donné une largeur suffisante à chaque côté du pilier en retour des bras de la croix, vû qu'en prolongeant cette épaisseur de façon à se rencontrer à angle droit derriere le pan-coupé du pilier, on obtient les dimensions dont on a besoin pour contenir l'action d'une coupole dans tous les sens.[15]

Tels sont les principes généraux qui constituent la bonne & solide construction des supports d'un dôme sur pendentifs, & leur relation avec sa tour. Ils sont absolument les mêmes que pour l'exécution des bâtimens ordinaires, ou le fort doit toujours porter le faible, ou toutes parties supérieures doivent être soutenues sur les inférieures sans porter à faux; où tout en un mot doit être élevé depuis les plus basses fondations jusqu'au faîte avec empatement, en retraite ou en talud, & de maniere que ce qui porte ait sans cesse un caractere de solidité apparente par rapport à ce qui est porté. Les piédroits d'une coupole sont nécessités à avoir une épaisseur proportionnée à l'étendue de la voûte & à sa poussée : le bas des piédroits demande à être acôté à cause du changement de plan, comme le sont toutes les fondations. Les quatre points d'appui, c'est-à-dire la clef des grands arcs des bras de la croix, sont obligés d'avoir une hauteur & une largeur correspondantes avec le poids du dôme & l'épaisseur de ses piédroits ou contreforts : Enfin chacun des gros piliers de l'Eglise ne peut se dispenser d'avoir d'abord une largeur positive le long de la nef au moins égale à l'arc destiné à porter la tour sur son sommet, c'est comme une jambe sous poutre qui doit monter du fond sans interruption ; & ensuite d'avoir une

[15] Note (10) de Patte : Il faut bien se garder de confondre les proportions d'un pilier destiné à porter un cul de four ou une calotte sur pendentifs avec celle d'un pilier destiné à porter une tour de dôme de même diamétre aussi sur pendentifs . Dans le premier cas, en construisant les voûtes des bras de la croix en berceau & d'une certaine épaisseur, on peut par ce moyen rejetter tout l'effort de la poussée du cul de four contre les murs des extrémités de l'Eglise, comme il a été pratiqué à s. Roch, à S. Sulpice & ailleurs, alors le pilier ne faisant plus que la fonction de porter n'a pas besoin d'autant de force que dans le second où il est d'obligation de porter & de contrebuter à la fois le poids du dôme: c'est pourquoi il ne saurait y avoir de comparaison.

épaisseur cubique ou [p. 17] massive, capable de s'opposer, non-seulement au poids extraordinaire ; placé directement sur l'arc qu'il soutient, mais encore à celui placé en encorbellement au droit du petit côté de l'octogone. Ce sont là les régles que l'on trouve mises en œuvre, dans les ouvrages des Michel Ange, des Fontana, des Wren, des Mansards & des plus habiles constructeurs : il ne sauroit y en avoir d'autres qui ne dérogent au bon sens & à la solidité.[16] On en va juger par l'exposé des rapports de construction des coupoles les plus estimées que nous considérerons, en faisant abstraction de la nature de leurs voûtes, pour généraliser nos principes.

[Marginalia de Rondelet]

Ce ne peut etre que par mechanceté ou par un ignorance extreme que Mr Patte confond un pendentif avec un encorbellement qui est réellement un porte à faux sans aucun soutient a l'exterieur au lieu qu'un pendentif est une voute en trompe qui est appuyée de tout côté en sorte qu'il ne peut agir d'aucune maniere etant retenu a droite et a gauche par des arcs tres solides, et au dessus par la forme circulaire de la tour

[Suite de l'article premier]

Le dôme de S. Pierre de Rome a 117 pieds de diamétre, avec 16 contre-forts de 21 pieds d'épaisseur, sans leurs décorations d'Architecture, lesquels sont distants l'un de l'autre d'environ quatre fois leur largeur ; l'épaisseur du mur, entre ces points d'appui capitaux, est égale à la saillie des pendentifs, de forte que les contre-forts au-dessus de ces endroits portent à plomb du massif des piliers, ce qui lie la tour du dôme avec eux. La tour est élevée sur un grand piédestal de 17 pieds d'épaisseur embrassant tous les murs & les saillies quelconques, tant intérieures qu'extérieures, dont le pied est contre-venté le long des bras de la croix par une voûte D, fig. * pl. I (Fig. 8). de près de 7 pieds d'épaisseur vers la clef: l'arc de la rencontre des bras de la croix a en retour, ainsique son piédroit, 29 pieds de largeur, sur près de 57 pieds d'épaisseur. On peut voir *pl. II, fig. XI* (Fig. 9). les rapports des parties supérieures & inférieures de cet ouvrage.[17]

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[16](Note (11) de Patte : Pere Guarini, Architecte Italien, ayant entrepris au commencement de ce siécle d'élever une vaste. coupole pour couronner l'Eglise de S. phillippe de Néri à Turin, et ayant négligé de donner à ses supports les dimensions nécessaires, l'on sçait que sa voûte tomba le 30 Mars 1715, peu après son exécution & entraîna la chûte de toute l'Eglise.

[17]Note (12) de Patte : Ce fut Bramante qui donna le premier dessein de S. Pierre, & l'on peut dire qu'il entreprit ce Monument sans avoir étudié les dimensions qui convenoient à ses supports pour porter & contreventer sa coupole : aussi les quatre arcs des bras de la croix furent-ils à peine déceintrés qu'ils menacerent de renverser les piliers : il fallut doubler leur proportions; & ce fut Michel Ange qui, plus éclairé que ses contemporains, .eut enfin la gloire de donner à ce premier édifice de l'Univers la solidité qui lui manquoit. Lorsque ce dernier fut chargé de cet important ouvrage, il fit en 15 jours un modèle qui ne coûta, dit-on que 25 écus Romains, lequel fût trouvé si bien entendu pour la construction ; que bien que la coupole n'ait été exécutée que plus de 30 ans après sa mort, on ne crut pas devoir y rien changer. Voyez les descriptions de S. Pierre par Fontana & Bonani.

[Marginalia de Rondelet]

former un seul Tambour dont les procédés de St Pierre de Rome avec les autres ainsi que leur propose par Mr Patte sont abusif et n'ont été imaginé que par l'Ignorance. [Etait illisible R f.]

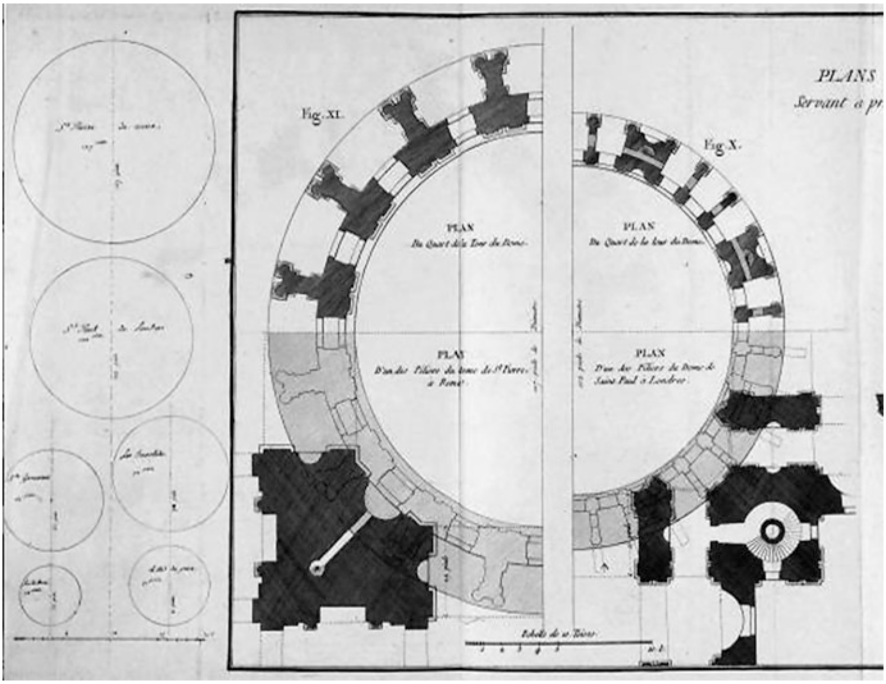

Fig. 9 Planche II, Figs. X et XI

Le dôme de S. Paul de Londres a 102 pieds de diamétre : comme le plan de la rencontre des bras de la croix est un octogone régulier, la saillie des pendentifs est peu considérable dans les angles ; les contre-forts de la tour sont au nombre de 24, y compris 8 corps de maçonnerie cantonnés à plomb des gros piliers, & ont 14 pieds d'épaisseur, [p. 18] sans compter trente-deux éperons, d'environ 8 pieds de saillie sur 4 pieds d'épaisseur servant à contreventer le bas de la tour, & qui sont liés de l'un à l'autre par deux arcs renversés : on peut remarquer leur disposition sur le plan en A, *pl. II, fig. X* (Fig. 9), & leur élévation en A, *pl. I, .fig.* * (Fig. 10).

[Marginalia de Rondelet]

tous ces exemples ne prouvent rien parce que tout ce qu'il en pourroit résulter peut être détruit par la comparaison de beaucoup d'autres edifices plus hardis et plus legers que ne sera le dôme de la nouvelle eglise de Ste genevieve tels que St Laurent et le Saint Suaire de Turin, St Augustin de plaisance, le Dôme de Milan, St Charles Catinari et St Charles du Cours à Rome, St Laurent et le St Esprit a Florence, St Philipe de Neri a Naples St Joseph et St Dominique le grand a Palerme, les Bernardins de Dijon etc.

[Suite de l'article premier]

La largeur de l'arc & celle des piliers du dôme le long des bras de la croix, est 26 pieds. L'épaisseur de ces piliers est percée très-industrieusement par les bas-côtés qui se réunissent de part & d'autres dans chaque angle par un cul-de-four en voussure, qui rejette tout effort de la partie de la coupole correspondante contre un

Fig. 10 Planche I, fig. * A (St. Paul de Londres)

gros massif qui est addossé. Il est à observer qu'on n'a donné une aussi grande largeur le long de la nef à ces piliers, qu'afin de les mettre en etat de porter & de contreventer eux seuls tout le dôme, vu que les voutes des bras de la croix étant construites à la légére & non en berceau, ne pouvaient être d'aucun secours à cet égard.[18]

[18]Note (13) de Patte : Nous avons entrepris exprès le voyage d'Angleterre pour étudier cette admirable construction, dont nous espérons faire part des détails au Public. L'Auteur de ce Monument fut le Chevalier Wren, à la fois grand Géométre & habile Architect deux qualités très compatibles, bien qu'elles se trouvent rarement réunies. Les Anglais, pour honorer le mérite de cet homme célébre, lui ont accordé le Privilége exclusif, ainsi qu'à sa famille, d'être inhumé dans ce Temple. Son Tombeau consiste en une tombe avec son seul nom, auprès de laquelle on lit l'inscription suivante qui est d'une simplicité vraiment sublime :

Subtùs conditur/Hujus Ecclesiae & urbis conditor / Christophorus Wren / qui vixit annos ultrà nonaginta ; / non fibi, sed bono publico : Lector, si Monumentum requiris / CIRCUMSPICE / Obiit XXV Feb. Anno MDCCXXIII.

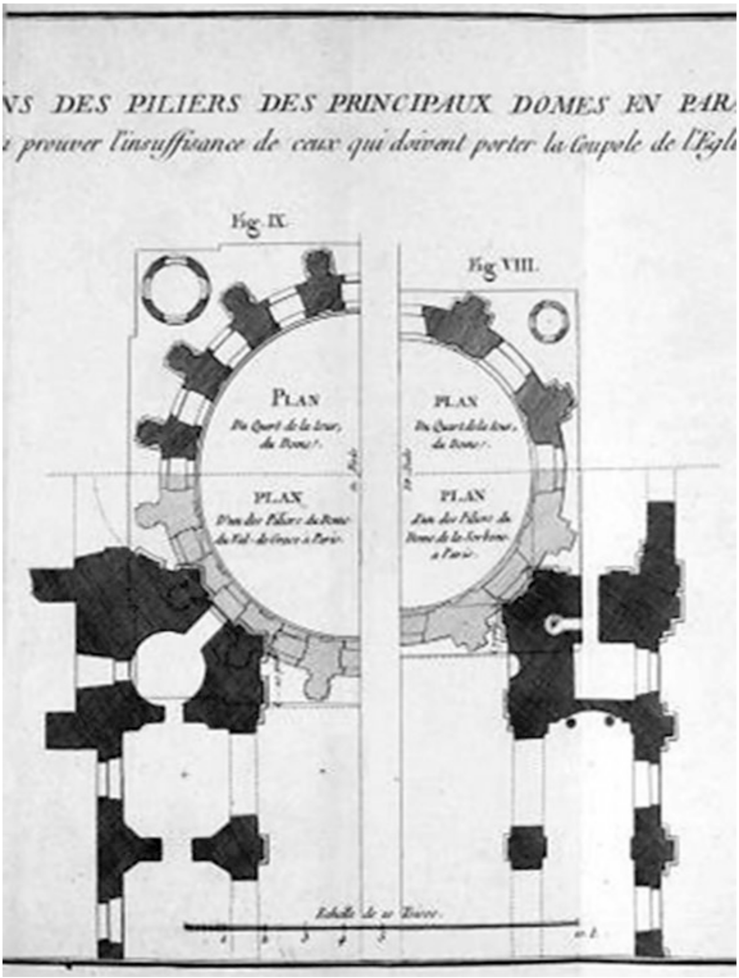

Fig. 11 Planche II, Figs. VIII et IX

Le dôme de la Sorbonne à Paris, *fig. VIII, pl. II* (Fig. 11), a 38 pieds de diamétre: il a 8 contre-forts de 6 pieds d'épaisseur, portés sur les grand arcs seulement, & non sur les pendentifs : ils sont distans l'un de l'autre d'environ trois fois leur largeur : l'épaisseur du mur de la tour du dôme est à peu près égale à la saillie des pendentifs : le bas de son piédestal, est fortifié par une voûte folide : l'arc a 6 pieds & demi, & la largeur des piédroits peut être considérée comme ayant au-dela de 7 pieds.

Le dôme du VaI-de-Grace, *fig. IX, pl. II* (Fig. 11), a 51 pieds de diamétre : il a 8 contre-forts de plus de 8 pieds d'épaisseur, espacés l'un de l'autre de deux fois & demi leur largeur : l'épaisseur du mur de la tour est encore égale à la saillie des pendentifs, de sorte que les Contre-forts placés au droit des piliers portent, comme à S. Pierre, immédiatement sur leur massif : le bas du piédestal est contreventé par

Table de comparaison entre les dimensions des ouvrages précédents

	Diamétres des Dômes	Hauteurs sous Les Coupoles	Largeur des pilliers	Epaisseur des pilliers	Epaisseur des Contre-forts
S. Pierre de Rome	127 *pi*	310 *pi*	29 *pi*	56	21 *pi*
S. Paul de Londres	102	235	26	26	14
La Sorbonne	38	104	7	12	6
Le Val-de-Grace	51	124	10	14	8
Les Invalides	73	182	11	15	10

les voûtes des bras de la croix : l'arc a environ 9 pieds, & la largeur des piliers, est de 10 pieds.

[p. 19] Le dôme des Invalides a 73 pieds de diamétre, *fig VII, pl. II* (Fig. 12); ses contre-forts ont 10 pieds, & sont placés à plomb des gros piliers : ses pendentifs n'ont gueres que 4 pieds de saillie : à plomb de la clef des arcs, il y a un piédroit dont l'épaisseur est environ le dixiéme du diamétre : la tour est élevée sur un soubassement de 10 pieds d'épaisseur, dont le bas est contreventré par des arc-boutants au droit du vuide des bras de la croix, & en outre accoté par des voûtes solides : on voit en C, *pl. l, fig.* * (Fig. 7) cet arrangement. Enfin la largeur des piliers de l'Eglise est d'environ onze pieds.

Nous n'avons point parlé de l'épaisseur de la plupart des piliers, parce qu'à l'exception de ceux de S. Pierre, l'épaisseur des autres n'est pas bien distincte dans le bas de l'Eglise, & se confond soit avec les murs de séparation des chapelles, soit avec d'autres parties, adjacentes entre lesquelles elle se trouve comme enclavée. Cependant, sans beaucoup d'erreur, les piliers du Val-de-Grace peuvent être considérés comme ayant plus de 14 pieds d'épaisseur; ceux de la Sorbonne au moins 12 pieds ; ceux des Invalides environ 15 pieds ; & ceux de Saint Paul de Londres, bien qu'ils n'ayent en apparence que 10 pieds, peuvent être appréciés avoir plus de 26 pieds d'épaisseur, en les prenant à l'endroit de leur massif, c'est-à-dire, au-dessus des arcades qui leur sont adossée, & qui les lient avec les grands corps de maçonnerie où sont pratiqués des escaliers.

Il résulte de tout ce que nous avons dit sur les proportions des supports des coupoles, que l'épaisseur des murs du tambour doit être au moins le dixiéme de leur diamétre intérieur ; que l'épaisseur des contre-forts, lorsqu'on en admet, roule du sixiéme au huitiéme de leur diamétre; & qu'enfin la largeur des piliers destinés à porter un dôme [p. 20] sur pendentifs, est, suivant les exemples, depuis le quatriéme de leur diamétre jusqu'au septiéme inclusivement, quelle que soit la nature de la voûte. Nous ne connoissons point de voûtes sphériques ou sphéroïdes de quelque étendue, dont les supports dérogent à ces régles générales : & nous ne croyons pas même qu'ils puissent y en avoir de contraires, parce qu'elles seroient opposées aux principes établis de l'équilibre & de la pesanteur, sur lesquels est fondée la solidité des bâtiments.

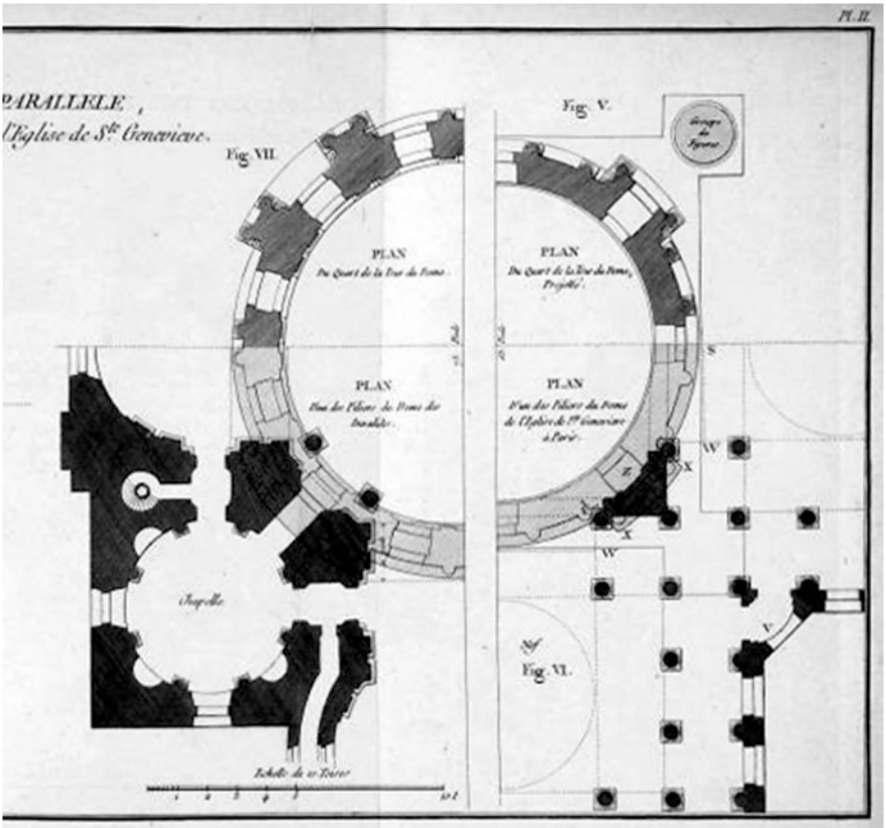

Fig. 12 Planche II, Figs. V à VII

[Marginalia de Rondelet dans la marge de gauche de la page 20, se poursuivant au bas de la page suivante.]

D'après la notte precedente, les dimensions que doivent avoir les piliers pour porter le dôme de l'église de Ste Genevieve, ne peuvent etre determinées que relativement au poid de cet edifice. Car nous avons prouvé qu'il ne peut se faire sur les piliers qu'un simple effort de pression.

Tout ce que M. Patte dit a ce sujet marque la plus mauvaise [foi] et le plus grand acharnement a nuire; il voudrait par des subtilités, faire croire que les piliers n'ont pas la superficie qu'ils ont réelement et qui est une chose de fait. Mais quoiqu'il puisse dire a ce sujet rien ne peut empecher que les piedroits ayent réelement chacun 130 pieds de superficie a leur endroit le plus faible ce qui fait pour les quatre 520 pieds.

Le dôme depuis le sommet de la croix, jusqu'a la hauteur du plan qui repond a cette superficie c est a dire, a l'endroit ou les piliers rachetent les voutes et les pendentifs. Le dôme, dis je toisé et calculé geometriquement doit peser environ 20 milions, en suposant toutes les voutes du Dôme en pierre et en y comprenant tout ce qui peut rendre cet edifice achevé. Or les 20 milions divisés par 520 pieds de superficie que contiennent les piedroits donneroient pour la charge d'un pied 38,461. Pour pouvoir citter quelqu'exemple executé j'ai calculé le clocher de St Merry bâti en 1612 dont le quart porte sur un pilier de 15 pieds *de superficie, et j'ay trouvé que chaque pied superficiel de ce pilier portait 63,325 ll sans y comprendre le beffroi ni les cloches, et l'effet que peut produire leur mouvement. A Angers dans l'eglise de St Toussaint une colonne de pierre de 95 pouces de superficie porte 58,20 ce qui feroit pour un pied 89,280. La proportion des piliers qui soutiennent le dôme de St Augustin de plaisance, par rapport au poid est encore plus hardie, mais rien n'est au dessus de ceux qui portent le Dôme de Milan qui est un chef d'oeuvre de construction. Il resulte de tous ces exemples que les piliers destinés a porter le dôme de Mr Souflot sont une fois plus fort que ceux que nous venons de citer qui existent cependant depuis plus de deux siecles. Monsr Souflot pour se rendre raison jusqu'à quel point on pouvait pousser la hardiesse de la construction a fait des experiences sur la force qu il faudroit pour ecraser les pierres ; il vient de faire faire une machine à cet effet et de toutes ses experiences faites en presence de plusieurs architectes du Roi et ingénieurs, il est résulté que la pierre dont sont construit les piliers de Ste Genevieve porteront un milion pour chaque pied superficiel avant de se rompre.*

Article second. Preuves de la disproportion des piliers de l'Eglise de Sainte Génevieve, & du peu d'apparence d'y pouvoir élever une coupole avec solidité

Nous avons établi dans l'article précédent, les régIes qui constituent la solidité des coupoles élevées sur pendentifs, & prouvé que le succès de leur construction dépend de la relation de leurs diverses parties, & surtout de la correspondance entre la largeur des piliers du rez-de chaussée de l'Eglise avec l'épaisseur des contre-forts ou piédroits du tambour ; présentement nous allons nous attacher à apprécier, d'apres ces principes, la construction de la coupole qu'il s'agit d'élever au centre de la nouvelle Eglise de Sainte Génevieve, & quelles doivent être les dimensions qui conviennent à ses supports.

Avant de procéder à cet examen, il est bon de prévenir le Lecteur qu'il ne doit point être étonné de ce que nous entreprenons de prononcer sur ce dôme, avant qu'il soit exécuté, ou que M. Soufflot, chargé de cet important ouvrage, ait jugé à propos de produire ses moyens de construction : car en y faisant attention, on s'apercevra qu'il n'est besoin, pour estimer d'avance l'exécution d'un pareil morceau, que d'avoir sous les yeux assez de déterminations, à l'aide desquelles on puisse parvenir à apprécier celles qu'on ignore. Il en est de ce problême comme de tous ceux de Géométrie, ce qui est connu combiné avec ses rapports, sert à déterminer la valeur de l'inconnue, & à manifester si cette derniere est réelle ou imaginaire : or le diamétre du dôme, & les dimensions des piliers qui doivent le porter, étant décidés, on va voir que ces données peuvent, sans erreur sensible, servir à déterminer ce qui reste à construire.

[p. 21] Chacun des piliers destinés à porter le dôme de Sainte Génevieve, n'a véritablement, dans le plan de l'Eglise, que 3 pieds, 9 pouces de largeur en retour sur les bras de la croix;

[Marginalia de Rondelet, marge de droite]

quoi qu'il y a reelement 5 pieds 6° cependant cet epaisseur de 3 p. 9° malgré le raisonnement est plus que suffisante pour suffire

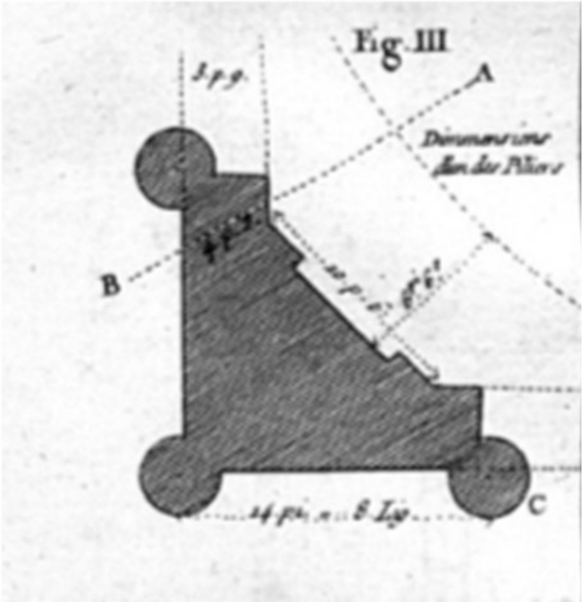

Fig. 13 Planche I, fig. III

[Suite de l'article second]
& vers les angles extérieurs de chaque pilier, on a engagé une colonne de 3 pieds, 6 pouces de diamétre par de bas, dont le centre saille en dehors de 3 pouces et demi, de sorte que l'engagement de chaque colonne est de près de 18 pouces.

Outre cette largeur, chaque pilier a d'épaisseur la distance qu'il y a d'un axe de colonne à l'autre, c'est-à-dire, 14 pieds, 8 lignes, avec un pan-coupé du côté du dedans du dôme, de 10 pieds & demi, *Fig. III* (Fig. 13).

Si l'on considére encore ces piliers au regard de la force qu'ils peuvent opposer aux pendentifs, on remarquera qu'ils ont dix pieds d'épaisseur, depuis le milieu du pan-coupé jusqu'à la pointe opposée ; mais qu'à mesure qu'on approche des deux colonnes en retour des bras de croix, vers le pli du pilastre, chaque pilier n'à, suivant la direction A B, *fig. III* (Fig. 13), que 4 pieds, 3 ou 4 pouces d'épaisseur.

[Marginalia de Rondelet, marge de droite]

c'est bien chercher du soin tandis que peut servir a soutenir son ser de mure (?)

[Suite de l'article second]
Les espacements des 4 piliers, & leur forme, font aussi connoître que le dôme projetté sera élevé sur des pendentifs saillants de plus de 6 pieqs et demi, & aura 63 pieds de diamétre dans le haut, & que les arcades de la rencontre des bras de la croix auront 37 pieds 3 pouces d'ouverture ; de sorte qu'en leur supposant une hauteur double de leur largeur, proportion assez usitée en pareille circonstance, il

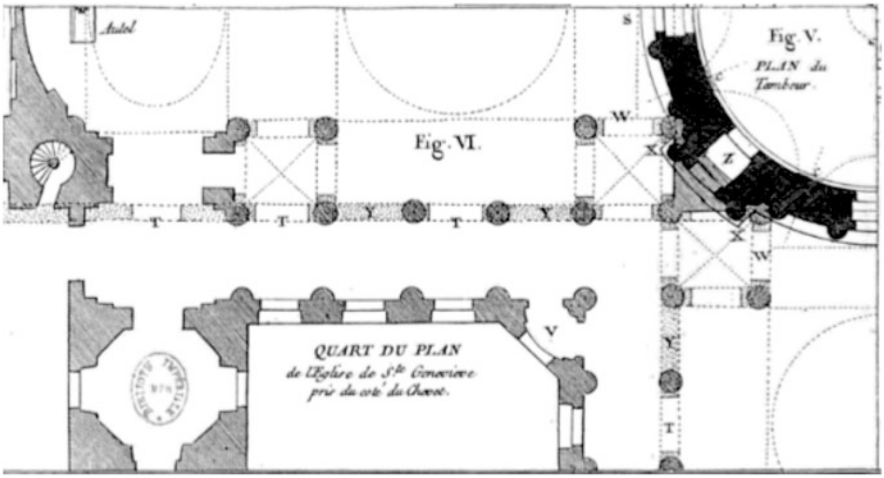

Fig. 14 Planche I, Fig. IV et VI

s’ensuit que la coupole sera élevée sur des arcs ayant sous clef 77 pieds[19] d’élévation environ.[20]

Il est également aisé de décrire quelle sera la coupole. Tous les desseins, modèles, gravures, médailles, & les descriptions qui ont paru dans le public, ont toujours annoncé qu’il y auroit une tour de dôme, décorée en dehors de colonnes Corinthiennes, de 3 pieds de diamétre environ, distribuées, dans son pourtour, comme il est représenté en plan, *fig. V* (Fig. 12), avec des avant-corps couronnés de fronton au-dessus des pendentifs, & qu’il n’entreroit point de charpente dans toute sa construction : cela étant bien confiant, on ne pourra gueres donner d’élévation jusqu’à la derniere voûte de couronnement, moins de 2 diamétres 2/3, pour avoir de la grace à l’extérieur : or, dans cette supposition, que les gens éclairés ne pourront disconvenir, devoir être plutôt en deçà [p. 22] de la vérité qu’au de-là, la decoration extérieure du tambour aura plus de 40 pieds d’élévation sans le piédestal. Les *fig. IV & V* (Fig. 14), expriment en profil cet arrangement.

[Marginalia de Rondelet]

ainsi les piliers du Dôme ayant 520 pieds porteraient 520 milions ce qui est 26 fois le poid du Dôme.

[19] Note di Radelet-de Grave : Rondelet a barré 77 pour le remplacer par 72. A la suite d’un petit calcul fait dans la marge.

[20] Note (14) de Patte : Il n’est pas inutile d’observer que, relativement à la disposition du plan des bras de la croix, *fig. IV*, & au peu d’épaisseur des murs pour-tours de l’Eglise, les voûtes ne pourront être exécutées autrement qu’à la légére, & que par conséquent elles seront inutiles pour contreventer la tour du dôme.

[Suite de l'article second]

La courbe extérieure de la voûte du dôme seroit difficile à déterminer, si elle pouvoit être véritablement arbitraire. Dans une Gravure que l'Architecte a publié de son projet en mil sept cent cinquante-sept, lorsqu'il à mis la main à l'oeuvre, on remarque deux voûtes demi-sphériques, dont la supérieure porte sur son sommet un piédestal, & l'inférieure un escalier. En jettant les yeux sur la *fig. II* (Fig. 3), on s'appercevra que cet amortissement ne pouvoit manquer de paroitre très-bas & très-écrasé en exécution. Ce fut vraisemblablement cette considération qui engagea l'Architecte à produire en 1764, une autre forme de couronnement tout opposee au-dessus de la tour, lequel se trouve consacré par la médaille frappée à l'occasion de la pose de la première pierre de ce monument. Il consiste aussi en deux voûtes, mais dont la supérieure est une ogive très surmontée, formée par deux arcs au moins de 60 degrés, dont l'extrados est terminé par un plan droit couvert de gradins, au sommet desquels sera élevé une lanterne ou un grand piedestal en pierre, avec cinq figures colossales.[21] Avant de passer à l'examen de cette construction qui a été annoncée pour être le vrai projet de l'Architecte; ne pourroit-on pas faire plusieurs réflexions préliminaires, sur sa singularité.

1° Ce couronnement est-il admissible pour terminer un Temple? n'y a-t'il pas une forme analogue à sa destination et consacrée par l'usage de tous les tems & de tous les Pays, dont on ne peut gueres s'écarter?

2° Est-il vrai que l'on puisse se promettre en exécution quelque succès du racourci produit successivement par la perspective de tous ces gradins, dont la largeur dérobera sans cesse à la vue la hauteur?

3° Puisque l'on fait tant que d'affecter en de-hors une tour de dôme, d'où vient ne la pas répéter en dedans, à l'exemple de tout ce qui est approuvé en ce genre pour donner de la grace & de la majesté à l'intérieur de cette Eglise, d'autant que ce défaut seroit facile à corriger en mettant à profit le vuide immense laissé sans aucun sujet, entre les deux voûtes?

4° En un mot, dans un morceau de décoration d'aussi grande importance, & fait pour annoncer aux siécles à venir la gloire de nos arts, est-ce là le cas de [p. 23] sacrifier la beauté de son ensemble intérieur & extérieur à un arrangement bisare qui n'a vraisemblablement été imaginé après coup, que dans l'espérance de favoriser l'insuffisance des piliers pour porter une coupole comme à l'ordinaire? Mais, sans nous arrèter dans des discussions qui par leur nature, seroient capables d'occasionner des grands changemens dans cet ouvrage, bornons-nous a parler de son exécution.

Il est d'expérience

[21] Note (15) de Patte : Dans le *VI Tome* de la derniere édition *des curiosités de Paris*, par Pigagnol, on trouve l'élévation de cet édifice donnée par l'Auteur. De plus on sait qu'on en a montré publiquement pendant quelque tems un grand modèle en bois : &. tous ceux qui, connoissent l'Architecte, en ont vu des desseins exposés chez lui.

[Marginalia de Rondelet]

l'experience et les principes de mathematique prouvent le contraire et cela par deux raisons bien simples. La premiere est que dans toute sorte de voute plus la partie inferieure est considerable plus elle a de force pour contre balancer l'effort de la partie superieure. La seconde c'est que plus les voussoirs du milieu ont d'inclinaison par rapport a la verticale moins elles ont de poussée en sorte qu'il faudroit un poid considerable sur le milieu pour que la poussée fut aussi considerable que dans les voutes circulaires extradossée d'égale epaisseur

[Suite de l'article second]

qu'une voûte ogive isolée, & formée par des arcs très-surmontés, dont l'extrados doit être terminée par un plan droit, n'est point capable de porter sur son sommet un poids considérable, attendu que la courbe par son grand allongement, fait alors dégénérer les arcs en des murs inclinés l'un vers l'autre, qui, outre qu'ils dirigent défectueusement l'action de la pesanteur du fardeau vers les supports, forment encore un ventre vicieux

[Marginalia de Rondelet]

Ce ventre vitieux ou que Mr Patte croit vitieux est une preuve de ce qu il ne connaît point du tout la maniere dont les voutes agissent car tout le monde conviendra qu'une voute en berceau circulaire aura infiniment plus de force et de solidité que deux murs inclinés a 45 degrés qui se joindroient pour couvrir un espace. Cependant la plus grande solidité de la voute en berceau ne serait occasionné que par la courbure que Mr Patte apelle un ventre vitieux.

[Suite de l'article second]

à cause de son inégalité dépaisseur : aussi prend-on toujours en pareil cas le parti de substituer pour la solidité, un vrai cône ou une pyramide dont les côtés portent uniformément dans toute leur hauteur : les exemples sont formels à cet égard. Ce n'est pas tout, la force des supports de cette prétendue coupole ne paroît pas plus aisé à justifier que celle de sa courbe. Dans la *fig. II* (Fig. 15), on voit en élévation des arcs doubleaux *a* rachetant des lunettes *b* dans la voûte supérieure, à dessein sans doute de leur faire porter seuls tout le poids du couronnement, & de soulager en conséquence les parties de mur comprises dans leur intervalle, & par leur disposition *c* en plan *figure V* (Fig. 14), on s'apperçoit que ces points d'appui capitaux auront à peine la même épaisseur du mur, sans aucune saillie en dehors comme de coutume, & ne correspondront ni vis-à-vis des colones, ni vis-à-vis des corps avancés X qui pouvoient les fortifier; c'est-à-dire que ce seront des arcs doubleaux qui feront la fonction de contre-forts sans en avoir la proportion, & qui, bien loin de servir à fortifier la tour, se trouveront au contraire plus faibles que le reste de son mur. Cet arrangement paroit si contradictoire à ce qui s'observe d'ordinaire, que nous croyons inutile de nous y arrèter davantage.

C'est pourquoi, au défaut de projets connus, véritablement raisonnés pour la décoration & la construction de cette coupole, nous ne pouvons dans notre examen nous dispenser de nous en tenir à la maniere usitée de terminer, ces sortes d'ouvrages : & dans l'intention de considérer la poussée sous le point de vue le plus avantageux, bien que ce soit un fait constant que le projet a toujours été jusqu'ici de couronner ce Mo- [p. 24]nument par une double voûte,

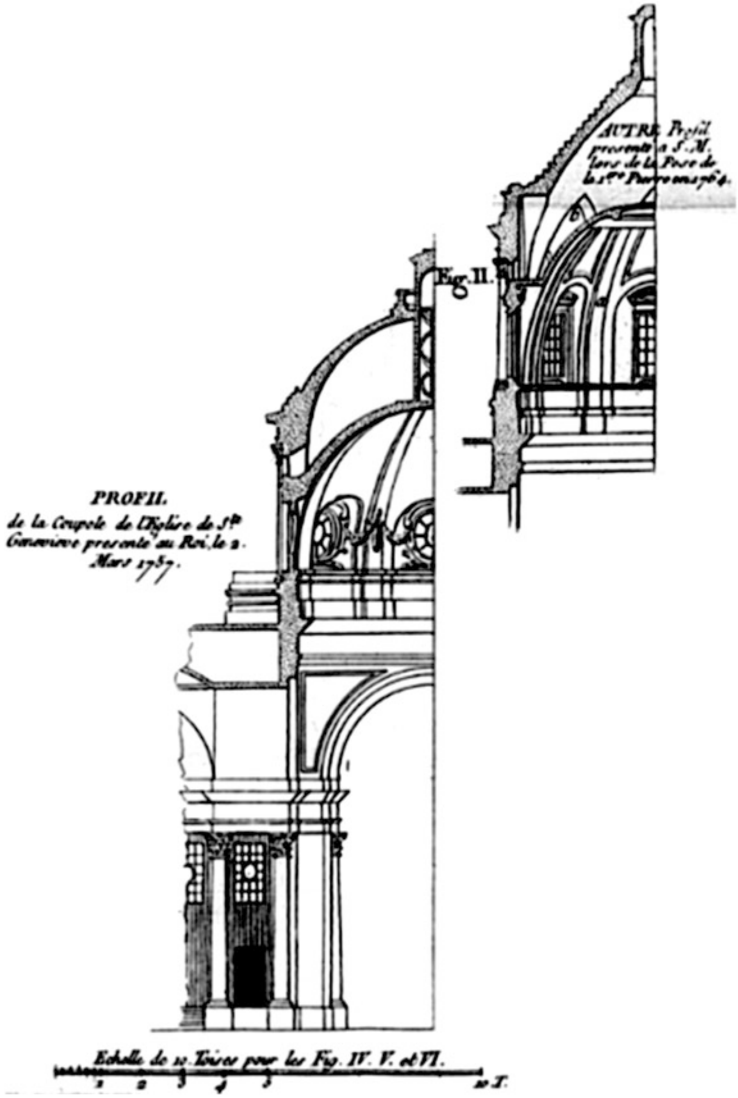

Fig. 15 Planche I, Fig. II

[Marginalia de Rondelet]

Ceci prouve que Mr Patte ne serait pas en état d'exécuter le dôme qu'il propose et dont les moindres constructeurs se chargeront

[Suite de l'article second]

nous supposerons qu'il n'y en aura qu'une seule extérieure, construite en briques, d'une courbe approuvée capable de faire un bon effet au-dessus de la tour en dedans & en dehors, en un mot, approchante de celle proposée par Fontana d'après les meilleurs modèles, avec deux pieds d'épaisseur réduite au droit de la demie voûte, ce qui ne seroit qu'environ 18 pouces vers le col du piédestal ou de la lanterne. Nous avons rendu raison, dans l'article précédent, de cette épaisseur qui est démonstrative par la pratique & la théorie.

[Marginalia de Rondelet]

C'est donc le dôme de Mr Patte qui est impossible selon lui même, et par son ignorance

[Suite de l'article second]

Ainsi le diamétre du dôme étant connu, la hauteur de sa tour, l'épaisseur & la forme d'une de ses voûtes par approximation, il sera aisé, à l'aide de ce qui a été dit dans l'article précédent, de trouver les puissances en équilibre avec la poussée, & ensuite de décider si les gros piliers de l'Eglise ont les dimensions requises pour porter sûrement cet ouvrage.

Attachons-nous d'abord à trouver les résistances qu'il conviendra d'opposer à l'action de la coupole. Fera-t'on les murs de la tour uniformément épais dans tout leur pourtour, ou bien y distribuera t'on des contre-forts vers lesqueIs on rejettera son poids & sa poussée?

Dans le premier cas, nous avons vu qu'il y avoit deux façons de déterminer cette épaisseur. Si l'on adopte la régIe de Fontana fondée sur tous les exemples, laquelle indique de donner le dixiéme du diamétre intérieur à l'épaisseur de la tour, il faudra 6 pieds, 4 pouces, attendu qu'il est de 63 pieds. Si l'on se sert au contraire des principes établis pour la poussée, dont il a été fait précédemment l'application à la même voûte,

[Marginalia de Rondelet]

par ce que nous avons dit precedemment, nous avons prouvé qu'un mur uniforme de 3 pieds d'épaisseur seroit plus que sufisant pour contrebalancer l'effort de la poussée puisque leur resistance seroit double de la poussée. Ainsi tout ce que Mr Patte ajoute ici n'est qu'un raisonnement captieux et illusoire pour soutenir un paradoxe dont je suis bien sur qu'il connoît toute la fausseté.

[Suite de l'article second]

nous avons prouvé qu'il falloit au moins 6 pieds d'épaisseur de mur; lesquelles mesures données par la pratique & la théorie étant approchant les mêmes, il s'ensuit qu'en se servant réciproquement de preuves, elles manifestent bien clairement qu'on ne sauroit donner, pour obtenir une solidisé suffisante, moins de 6 pieds, 3 ou 4 pouces aux piédroits de la tour du dôme de Sainte Geneviève.

Mais ce ne sera pas assez de donner une épaisseur convenable à la tour,

[Marginalia de Rondelet]

par ce que la critique tomberait

[Suite de l'article second]

elIe ne pourra se soutenir qu'autant que l'on contreventera son pied à cause du changement du plan circulaire en celui d'un octogone irrégulier; & comme il n'est pas possible de s'aider pour cela des voûtes des bras de la croix, vu qu'elles ne peuvent être exécutées qu'à la légére, il conviendra d'employer, soit des contre-forts, soit des arcs-boutans, soit de bons empatemens. En supposant, comme une circonstance très-fa- [p. 25] vorable que les colones destinées à décorer le dehors du tambour seront à demi engagées, & que les 2 pieds dont elles augmenteront, y compris la saillie de leur base; l'épaisseur du piédestal au-delà des 6 pieds 4 pouces, seront suffisants pour cela, il est évident que l'épaisseur vers le pied sera d'environ 8 pieds 4 pouces.

[Marginalia d'Antoine Rondelet]

Cela n'est evident que pour Mr Patte qui croit par la pouvoir soutenir le paradoxe. Il en résulte que ce n'est pas le dôme de Mr Soufflot mais le sien qui serait impossible et qui aurait tous les deffauts dont il parle.

Mon pauvre Patte c'est que la tienne n'est pas possible.

Pauvre sot, si tu savais que par l'avantage de la forme circulaire de la tour, le poids porté en bascule et qui te fait trembler serait ce qui doit rassurer ceux qui en savent plus que toi parce que comme cet effet se fait en sens contraire de la poussée que tu crains, il doit donc détruire ou diminuer la poussée de la voute que tu crains, contre laquelle tu as l'impudence et la maladresse de parler. R[ondelet] f[ils].

[Suite de l'article second]

Nous disons comme une circonstance très favorable, parce que 2 pieds d'empatement ne doivent pas paraître assez pour accoter les fondations du piédroit d'une voûte de 10 toises ½ de diamétre. Personne n'ignore que dans tous les bâtimens, on donne d'ordinaire de largeur, sans même y avoir de voûte la moitié en sus de l'épaisseur du mur hors de terre: or, suivant cette régle, le mur qui soutiendra la coupole en question devant être de plus de 6 pieds, il lui faudroit au-delà de 9 pieds d'épaisseur dans le bas & non 8 pieds 4 pouces.

Il est à observer que l'épaisseur du mur que nous venons de fixer, ne regarde que les parties du dôme correspondantes aux grands arcs, & qu'à plomb des pendentifs qui auront plus de 6 pieds ½ de saillie, le projet est d'y joindre des corps avancés, tant pour, contre-balancer le poids extraordinaire porté en bascule à cet endroit, que pour lier la tour avec les gros piliers. On peut voir dans la *fig. V, pl. I & II* (Figs. 12 et 14), la distribution extérieure de ce plan qui est d'accord dans son pour-tour avec la décoration annoncée par les desseins & la médaille, & qui, à l'épaisseur des murs près que nous avons fixé au dixiéme du diamétre suivant la regle, ne sauroit s'écarter du plan de l'Architecte.

Dans le second cas, où l'on desireroit des contre-forts distribués autour du tambour, procédé que nous estimons préférable au précédent, à cause de l'étendue du dôme & des raisons déduites dans l'article I ; pour parvenir à connoître leur épaisseur, il serait à propos de calculer l'effort que la portion de voûte correspondante peut opérer contre chacun : mais comme la disposition extérieure du plan porte à croire que ce n'est pas l'intention de l'Architecte,

[Marginalia de Rondelet]

Nous avons deja fait voir que cette disposition etait plutot avantageuse que nuisible autrement les ... qui pourroit co ... les ... on .... sont opposée et ... par ... de la[22] ...

[Suite de l'article second]

& qu'il préférera des murs d'une épaisseur uniforme au-dessus du vuide des bras de la croix, à dessein de favoriser le peu de largeur de ses piliers, nous nous bornerons à envisager cette partie de la tour sous ce point de vue dans tout le reste de ce Mémoire.

Passons maintenant à l'examen des dimensions des piliers qui doivent porter la coupole. Le piédestal de la tour devant être soutenu par la clef des arcs formant la réunion des bras de la croix & sur les pendentifs le long des petits côtés de l'octogone, il est constant qu'on ne peut se [p. 26] dispenser de donner à ces points d'appui une largeur & une épaisseur en rapport avec le fardeau. De crainte de confondre, examinons d'abord quelles doivent être les proportions des gros piliers en retour le long des bras de la croix, pour porter le dôme, & ensuite nous verrons celles qui leur sont nécessaires vis-à-vis des pendentifs.

[Marginalia de Rondelet]

tout ceci ne peut concerner que le dôme que Mr Patte a imaginé, et qu'il a cru impossible car le dôme de Mr Soufflot ne sera pas du tout executé comme il le croit ici. Ainsi tout ce qu'il dit ne peut être appliqué au dôme de Mr Soufflot. C'est pourquoi on se dispense de relever une infinité de contradictions qui prouvent que l'auteur serait fort embarrassé s'il lui falloit faire executer un pareil ouvrage.

[Suite de l'article second]

Premierement, la partie de la tour correspondante à la clef des arcs ayant: 8 pieds 4 pouces, il faudra donc construire ces arcs en bonne pierre dure à cause du grand fardeau qu'ils porteront, & leur donner à chacun environ 9 pieds de largeur, de même qu'aux retours des bras de la croix destinés à leur servir de piédroits, conformément aux préceptes de l'art de bâtir, qui exigent que ce qui est porté s'éléve en retraite au-dessus de ce qui porte. Mais ce ne seroit pas assez que ces piédroits eussent une largeur suffisante, il leur faudra encore une épaisseur qui les fortifie, de maniere à contreventer sûrement ce qu'ils auront à soutenir, &, vû l'immensité de la charge qui sera de plusieurs millions de livres pesant, il est à propos que cette épaisseur soit pleine, & sans aucun vuide capable d'altérer sa force. Les piedroits ayant 57 pieds d'élévation, l'arc étant un plein ceintre de 38 pied de diamètre avec trois pieds de hauteur de clef, laquelle hauteur est assurément la moindre que l'on puisse donner en semblable rencontre, on trouve par les calculs 8 pieds 9 pouces pour l'équilibre, & qu'en faisant entrer en considération l'épaisseur des reins de cet arc, qui contribuera

[22] Note de Radelet-de Grave : Il s'agit de la lecture lacunaire par Antoine Rondelet d'un texte au crayon aujourd'hui disparu de Rondelet.

à le fortifier, & le poids de la partie de la tour portant sur sa clef, ce pilier aura besoin d'une épaisseur uniforme de 12 à 13 pieds : or on voit par l'examen du plan des piliers de cette Eglise, que leur largeur est seulement de 3 pieds 9 pouces; par conséquent elle n'est donc point faite pour recevoir un arc de 9 pieds de largeur, dont on ne peut se passer pour soutien de la tour du dôme, & il lui manque 5 pieds 3 pouces, sur 13 pieds d'épaisseur, pour remplir cet objet. Quant à la colonne engagée dans l'angle que nous n'avons pas considéré comme faisant partie du pilier, il est évident qu'elle ne lui procure qu'une apparence factice de largeur au delà des 3 pieds 9 pouces ; par la raison que la largeur du piédroit d'une voûte est nulle, à moins qu'il n'ait en même tems dans toute sa hauteur une épaisseur suffisante pour résister à la poussée; & comme il n'y pas d'épaisseur de mur derriere la partie de la colonne qui n'est pas engagée, il résulte qu'elle ne sauroit contreventer une portion d'arc correspondante, & que la saillie de la colonne n'est qu'un masque incapable d'augmenter la force du pilier.

[p. 27] Secondement, si l'on compare l'épaisseur de ce pilier, du côté de l'intérieur du dôme, avec la saillie des pendentifs, on s'appercevra par le plan que, comme ils auront plus de 6 pieds & demi d'encorbellement le long du petit-côté de l'octogone, ils seront obligés de porter au moins 14 pieds de longueur de la tour sur 40 pieds de hauteur sans la voûte.

[Marginalia de Rondelet]

Si les voutes sphériques avaient autant de poussée que Mr Patte veut le faire croire ce sont ces pendentifs qu'il appelle encorbellement qui seroient plus que sufisant pour la detruire.

Mais de malice et de ...

Tout ce qui suit, est encore une preuve de sa malice, et de son ignorance

[Suite de l'article second]

Il n'est pas douteux que ce poids énorme tendra également de son côté à renverser le pilier, à moins qu'il ne soit d'une force convenable. Dans les exemples, cette épaisseur se trouve souvent 2 à 3 fois plus considérable que la saillie, des pendentifs respectifs ; c'est pourquoi, dans le cas actuel, en supposant l'épaisseur du pilier une fois & demie la saillie des pendentifs, c'est-à-dire 10 pieds, nous ne pouvons être suspect d'exagération : or le pilier en question a véritablement au droit du sommet du triangle à peu près l'épaisseur désignée, mais à mesure qu'on approche de ces deux autres angles, cette épaisseur diminue au point que vers le pli du pilastre A B, *fig. III* (Fig. 13), à peine se trouve-t-il 4 pieds 3 pouces. On demeurera de plus en plus convaincu que cette dimension n'a point de rapport avec la force qu'exige un encorbellement de plus de 6 pieds & demi, chargé d'un gros mur, en faifant attention comment porte le plan de la tour du dôme ; car on s'appercevra qu'au-dessus de ces piliers, il y aura des espéces de contre-forts X, *fig. V* (Fig. 14), qui, à cause de leur manque de proportion, excéderont leur massif, & que comme, entre ces contre-forts, il y aura une croisée Z, l'effort de la tour sur les pendentifs agira uniquement sur les parties foibIes du pilier, parce que le Vuide répondant à sa plus grande épaisseur, celle-ci deviendra inutile pour sa solidité. Il arrivera delà que l'effort de la masse de Ia tour sur les pendentifs, à raison de leur appareil, tendra de part & d'autre à écarter les naissances des arcs, lesquels étant supposés assez solides, détruiront l'action dans ce sens; mais il en restera un autre qui tendra à faire glisser la partie du pendentif excentriquement, & à renverser le piédroit suivant cette direction, à cause des joints verticaux dont les coupes tendront par le plan vers l'axe du dôme. Cette derniere action fera donc son effort sur les parties foibles du piédroit A B, *fig. III* (Fig. 13), comme si elle était appliquée à un levier plus grand que la hauteur du piIier dont la longueur se détermineroit par l'intersection d'une perpendiculaire qui lui serait menée du bas extérieur dudit pilier; or il est évident qu'un poids de 500 milliers, qui sera au moins celui de la partie du dôme qui répondra à ces endroits foibles, agissant en bassecule sur un levier [p. 28] de plus de 60 pieds, ne trouvera pas une résistance assez grande dans l'épaisseur 4 pieds 3 pouces du pilier, puisque nous avons vu ci-devant, qu'une partie de voûte de 38 pieds d'ouverture & de 3 pieds d'épaisseur de clef, agissant de la même maniere sur des piédroits de 57 pieds de hauteur, a donné 8 pieds 9 pouces pour l'équilibre; donc il faudroit plus de 8 pieds 9 pouces vers le pli du pilastre pour résister ; & il n'y a réellement que 4 pieds 3 pouces. Cette reflexion, jointe à la précédente, prouve clairement, l'insuffisance de ces supports pour porter un dôme, & qu'ils se déroberont de tous côtés à son effort, au lieu de s'y opposer, comme il seroit nécessaire.

Pour faire sentir, indépendamment de tous les raisonnements, le peu d'apparence de pouvoir faire soutenir une coupole sur ces piliers, quelle que soit la nature de sa voûte ou de ses voûtes, il n'y a qu'à, d'une part, placer sur leur plan celui de la tour de l'Eglise en question, avec les épaisseurs de mur & la saillie du piédestal, faisant ensemble 8 pieds 4 pouces, *fig. V* (Fig. 14); & d'une autre part appliquez aussi le plan de la tour de tous les dômes que l'on jugera à propos, sur celui de leur rez- de- chaussée ; alors on demeurera persuadé qu'autant il y a de correspondance dans tout ce qui est exécuté en ce genre entre leurs supports respectifs, pour former de bons empattements capables de favoriser la solidité autant le dôme de Sainte Génevieve, dont le plan de la tour excédera de toutes parts les piliers sera arrangé contradictoirement à ces ouvrages.[23] Ce qui frappera surtout, sera de voir que les largeurs des piliers des dômes élevés sur pendentifs ont toutes un certain rapport avec leur diamétre, lequel rapport s'étend, comme il a été dit dans l'article précédent, depuis le quart jusqu'au septiéme au plus, tandis que la largeur du pilier de l'Eglise dont il s'agit, n'est que le seiziéme de son diamétre.

[23]Note non numérotée (16) de Patte : Lucrece Liv . IV. v. 516, a décrit en beaux Vers que l'on diroit, faits exprès pour le sujet que nous traitons, & qui peuvent tres-bien lui convenir pour Epigraphe, ce que l'on doit attendre d'une semblable construction. ...... ln *Fabricâ, si prava est regula prima, Normaque si fallax rectis regionibus exit ; / Et libella aliquâ, si ex claudicas hilum, / Omnia mendose fieri, atque obstipa necessum est ; / Prava, cubantia, prona, supina, atque obsona recta, / Jam ruere, ut quaedam videantur velli / Prodita judiciis fallacibus omnia primis.*

[Marginalia de Antoine Rondelet]

Cet etonnement n'aura venir qu'a M. Patte qui ne peut jamais concevoir qu'on puisse faire autrement que par ce qu'il voit, sans etre en etat de juger et qui en apparence le trompent. Rf.

[Suite de l'article second]

En effet le dôme de Saint Pierre a 127 pieds de diamétre, avec des piliers de 26 pieds de largeur..

Le dôme de Saint Paul de Londres a 102 pieds, & des piliers de 26 pieds de largeur.

Le dôme du Val-de-Grace à Paris a 51 pieds, & des piliers au moins de 10 pieds de largeur.

[p. 29] Le dôme de la Sorbonne a 38 pieds, & des piliers de 7 pieds de largeur.

Le dôme des Invalides a 73 pieds, et des piliers de 11 pieds de largeur[24]: &, par proportion à ces modèles & à ceux qu'il plaira y joindre, le dôme de Sainte Génevieve, qui sera de 63 pieds de diamétre, n'aura que des piliers de 3 pieds 9 pouces. On peut voir dans les *figures VI, VII, VIII, IX, X & XI. Pl. II*, (Fig. 17), les rapports des plans des tours de dô de ces Edifices, avec ceux des gros piliers destinés à les porter.

Ajoutez à ces comparaisons que les piliers de tous ces exemples sont accotés par les murs des bras de la croix es bas-côtés & des séparations des Chapelles, au lieu que ceux de l'Eglise en question sont isolés au milieu de colonnades, & sans pouvoir, tirer de secours directe de toutes les parties adjacentes : il est nécessaire, qu'à cause de leur position, ils fassent à la fois fonction de piles & de culées, & qu'ils se suffisent absolument à eux-mêmes, soit pour porter, soit pour contre-buter la voûte du dôme : circonstance tellement désavantageuses qu'elles autoient dû engager à augmenter les dimensions qu'on a coutume de donner à ces piliers, plutôt que de les tenir deux & trois fois moins considérables.

Voilà donc le pilier bien prouvé n'avoir par lui-même aucun rapport avec l'action que le dôme doit opérer contre lui, soit au droit des arcs, soit au droit des pendentifs, c'est pourquoi il ne nous reste plus qu'à faire voir si, au défaut de force réelle, on peut se flatter du moins de lui procurer des secours véritables de quelqu'une de ses parties adjacentes.

Espérera-t-on tirer des forces du pan-coupé V, *fig. VI* (Fig. 14), du mur extérieur, & y établir des arc-boutans suffisants[25] pour contenir l'effort de la tour au droit du

[24]Note (17) de Patte: Voici encore quelques proportions de piliers portant des coupoles que nous nous rappellons.

La coupole de l'Eglise de Saint Georges à Venîse, exécutée par Palladio, a environ 36 pieds de diamétre, & des piliers de 7 pieds en quarré pour la porter.

La coupole de l'Eglise de Saint Nicolas de Tolentin à Rome: a 31 pieds & 10 pieds de largeur de pilier.

La coupole de l'Eglise neuve des Père de l'Oratoire, aussi à Rome, a 47 pieds de diamétre, & des piliers de 10 pieds de largeur.

[25]Note (18) de Patte : Des Arc-boutans placés ainsi derriere l'encorbellement, seroient d'ailleurs d'un bien foible secours, attendu qu'il n'est point question ici de contenir seulement une poussée

petit-côté de l'octogone ? mais cela seroit [p. 30] contraire à une bonne construction. Quelle solidité en effet pourroit-on attendre d'un mur ouvert de toutes parts dans une hauteur de 36 pieds, d'abord au rez-de-chaussée par un large corridor qui réduit l'épaisseur du mur de dehors à environ 2 pieds, puis au-dessus par 3 grandes croisées qui décomposent sa force dans d'autres sens, comme on en peut juger par le plan? A-t-on jamais vu jusqu'ici le piédroit d'un contre-fort ou arc-boutant, destiné, nous ne disons pas à contreventer l'action d'un dôme,

[Marginalia de Rondelet]

l'experience prouve tout les jours le contraire il n y a pour cela qu'a examiner les constructions gothiques

[Suite de l'article second]

mais d'une voûte quelconque, évidé comme un coffre ? Son essence n'est-elle pas d'être un massif, un véritable cube en un mot, une vraie jambe sous-poutre que rien ne doit altérer? Se permettroit-on seulement de poser la moindre ferme de charpente dans un bâtiment ordinaire à plomb du vuide d'une croisée? & n'affecte-t-on pas toujours au contraire de la placer sur un trumeau[26]? Donc à plus forte raison un arc-boutant de l'importance dont il s'agit demande-t-il d'être élevé sur un plein? Il n'y a personne qui ne puisse décider la nullité d'un pareil point d'appui.

Envain prétendroit-on encore augmenter la largeur du pilier par le haut en proposant d'avancer jusqu'à la colonne d'angle C, *fig. III* (Fig. 13), les murs, *fig. VI* (Fig. 14), placés au-dessus de la file des colonnes, à dessein de leur faire contrebuter une largeur de 18 pouces de l'arc au-delà des 3 pieds 9 pouces; car, indépendamment de ce que le pilier se trouveroit alors plus épais par le haut que par le bas, ce qui serait contraire à la construction ordinaire de tous les bâtiments, laquelle s'éléve toujours en retraite, il est aisé de s'appercevoir que par leur nature, ces murs n'auront pas la solidité convenable pour remplir cet objet. En effet, ces murs, devant être percés dans leur hauteur par les croisées T, T, T qui éclaireront les voûtes des bras de la croix, ainsi que par les entrées des tribunes, & de plus étant guindés à 40 pieds de terre sur des plate-bandes, il n'est pas concevable comment dans cet état de foiblesse, ils pourroient faire l'office de contre-forts; & si c'étoit l'intention de l'architecte, il auroit, fallu du moins assurer toutes les extrémités de chacun d'eux par de bons piédroits, & n'en pas excepter : car, si l'on suit le plan des deux files de colonnes qui aboutissent au chevet de l'Eglise, on s'apercevra que vers ce chevet, il ne se trouve au rez-de-chaussée aucun mur dans leur direction pour les recevoir, c'est-à-dire que bien-loin de pouvoir contenir une partie de l'arc, à l'aide de ces deux murs, à peine seront-ils en état de se soutenir eux-mêmes vers leurs extrémités.

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de voûte ordinaire, mais de soutenir à la fois l'effort d'un corps de Maçonnerie immense, agissant de tout son poids perpendiculairement & en bassecule : ce ne sont que des massifs cubiques directs qui puissent contrebalancer ce fardeau, & opérer un effet efficace dans cette circonstance et non un arc-boutant. Il ne faudroit que faire un profil au droit du pli du pilastre pour s'en convaincre.

[26]Note de Radelet-de Grave : Linteau.

[p. 31] Mais, quand bien même on supposeroit, contre toute apparence de solidité, que ce mur pût servir à contreventer 18 pouces de l'arc au-delà de la largeur du pilier, comme nous avons fait voir que cet arc seroit d'obligation d'avoir 9 pieds, il est manifeste qu'il y aura toujours une partie d'environ 3 pieds 9 pouces, qui débordera la colonne sur 14 pieds, dont le support ne sauroit être justifié.

Peut-être alléguera-t-on, que pour obvier au défaut de proportion des piliers, on construira le grand arc en pierre de toute la largeur de l'entre-colonnement, au lieu de le borner à l'épaisseur de la tour; mais la grande difficulté sera de faire porter les reins de cet arc; & si, dans l'impossibilité manifeste de les faire asseoir directement sur le plafond de l'entre-colonne, on s'avisoit par le moyen de quelque lunette W, *fig. IV & VI* (Figs. 13 et 14) faisant une espéce de voûte d'arrête ou d'arceau en décharge, d'ouvrir lesdits reins pour rejetter le poids à droite & à gauche sur les colonnes, alors il n'y auroit plus de solidité. Car c'est l'épaisseur seule des reins qui peut faire en cette circonstance toute la force de cet arc, & qui le rendra capable de résister à la charge immense qu'il aura à soutenir.

Ce seroit une erreur de croire pouvoir traiter un arc de cette conséquence comme une voûte légére qui ne doit rien porter, & où l'on peut se permettre toutes sortes de percés. Ne désaprouveroit-on pas le projet de pratiquer de larges ouvertures dans le flanc de la culée d'un pont, comme devant lui ôter sa résistance ; or notre cas est tout semblable ; l'arc joint au pilier est, vû sa position, une véritable culée destinée à soutenir & contre-buter tout le poids du dôme. Proposer de l'évider, ce seroit vouloir lui ôter toute sa force & le mettre hors d'état de remplir l'objet proposé :

[Marginalia de Rondelet]

au pont de Neuilly les buttées sont faites par des arcades qui font des ouvertures

[Suite de l'article second]

aussi ne subsiste-t-il aucun exemple de construction où l'on en ait usé de cette maniere. On a reproché à Bramante d'avoir pratiqué de petits escaliers au milieu des gros piliers de l'Eglise de Saint Pierre : les premiers Mathématiciens & Architectes d'Italie assemblés en 1743, pour aviser aux moyens de remédier aux lézardes qui s'étoient ouvertes de toutes parts dans la tour du dôme de Saint Pierre, les attribuerent unanimement au petit corridor E, *fig.* pl. I* (Fig. 8), pratiqué dans le grand piédestal qui éléve la tour. Mais, suivant l'arrangement que nous discutons, la circonstance seroit bien autrement grave, puisque c'est la voûte même de l'arc qui doit porter sur sa clef le dôme de Sainte Génevieve, dont il s'agirait d'ouvrir ou de supprimer la plus grande partie des reins, &, cela précisément dans l'endroit W, où se fera la principale action de la [p. 31(b)] poussée. Cette réflexion seule peut convaincre qu'un pareil moyen, ne pouvant être admissible, il y a conséquemment impossibilité de faire porter l'arc d'une maniere solide, au-delà de la largeur du pilier sur l'entre-colonnement.[27]

[27]Note (19) de Patte : Nous ne parlerons pas des arcs placés en enfoncement le long des bras de la croix, bien qu'ils paroissent pouvoir contribuer à accôter dans ce sens les pillers par le haut, par la raison que la lunette W étant décidée inexécutable, leur utilité ne sauroit être réelle.

On pourroit ajouter, s'il en était besoin, la spéculation du peu d'ap- [p. 32] parence de pouvoir faire porter sur deux colonnes la plus grande partie de l'arc chargé du dôme. En effet, comment se persuader que des colonnes de pierre de 3 pieds de diamétre dans le haut, & surtout celle qui est tout à fait isolée, dont les tambours ne sont pas même entretenus entre eux par aucun axe de fer, comme de coutume, soient en état de soutenir & de contrebuter un tel fardeau, sans que les tambours s'écrasent, ou au moins se dérangent par l'effort même de la poussée qui

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Comme quelqu'un pourroit être porté à croire qu'à l'aide de cercles de fer placés autour d'une coupole, Il seroit aisé de diminuer sa poussée, & conséquemment, l'épaisseur de ses soutiens, il ne sera pas inutile de faire voir ce que l'on peut espérer à cet égard. Le fer ne doit jamais être employé dans un édifice destiné à passer à la postérité comme un agent principal, & pour suppléer à des empattements, à des contre-forts, à des épaisseurs de murs ; mais il doit toujours être employé au contraire comme un moyen précaire, un moyen de surérogation, & une surabondance de force. La raison en est que, par sa constitution, il n'est pas fait pour être de longue durée; la rouille l'altére au bout d'un tems, & passe même, en augmentant son volume, pour faire éclater la pierre où il se trouve renfermé. Les mouffles qui lient les tirans sont toujours un endroit très-foible ; ses pores se resserrent ou s'étendent suivant le chaud ou le froid. D'ailleurs il n'est pas vrai que le fer acquiert de la consistance a raison de sa grosseur : il s'en faut bien qu'un barreau double du volume d'un autre, ait pour cela le double de force. Les expériences prouvent à la vérité qu'un fil de fer, rond d'une ligne de diamétre, [(**) Voyez les expériences de Musembrock dans sa *Physique*, & celles de M. de Buffon, dans *l'Art du Serrurier*, publié par l'Académie Royale des Sciences.] bien étiré, peut soutenir un effort d'environ 490 liv. pesant; mais elles font voir aussi que dans un bareau de 18 lignes de gros, par exemple, chaque élément d'une ligne quarrée de fer ne résiste gueres en commun qu'à un effort de 40 livres pesant, & que cet effort diminue toujours à proportion que la grosseur du barreau augmente. Cette grande diminution de force provicnt de ce que plus les fers acquierent de diamétre, plus il devient difficile de les forger, & de condencer avec le marteau suffisamment leur intérieur.

Il n'y a pas de construction de dôme où l'on n'employe quelques cercles de fer : on en met au col de la coupole, vers son imposte, & quelquefois aussi vers son milieu, mais cela n'empêche: jamais qu'on ne place des murs ou contre-forts suffisants pour contenir la poussée, & qu'on ne donne surtout une bonne épaisseur au retour des piliers le long des bras de la croix : on ne connoît point d'exemples ou l'on en ait usé autrement. Lors de la construction de la coupole de Saint Pierre de Rome, il fut mis quatre forts cercles de fer indépendamment de ses 16 contre-forts de 21 pieds d'épaisseur & malgré ces précautions, on sait qu'elle s'est lézardée de toutes parts : autour du dôme de Saint Paul de Londres, il y en a plusieurs; mais cela n'a pas empêché qu'on ne l'ait contreventé par 14 contre-forts arc-boutés par des éperons : on a vu aussi que Fontana indique de mettre trois cercles de fer O dans la partie inférieure d'un dôme.

Les cercles de fer dont on environne une coupole, ont principalement pour objet de résister a la premiere impulsion de la poussée, lorsqu'on lâche la voûte de dessus les ceintres ; de donner le tems au mortier de bien faire sa prise, aini qu'à toutes les parties de la construction d'opérer sa compression, de se convenir reciproquement, de prendre peu-a-peu leur faix & leur direction vers les contre-forts, ou les points d'appui capitaux : ils ne peuvent avoir d'autre fonction. C'est la perfection de l'appareil des pierres, l'excellence du mortier, la bonne proportion des suppors, & leur relation avec la poussée des voûtes qui doit faire la force d'un édifice dont on veut éterniser la durée : en user autrement, la faire dépendre absolument d'une force artificielle, comme est celle d'un cercle de fer, ce serait sans doute compromettre sa solidité.

agit toujours à raison de la longueur des leviers, tandis que l'on voit de très-gros massifs être à peine suffisants pour remplir cet objet.[28]

Récapitulons le nombre des porte-à-faux qui ont passé en revue successivement, à l'occasion de l'insuffisance des piliers, pour soutenir une coupole.

[Marginalia de Rondelet]

tout ces porte a faux n'existent que dans l'imagination de M. Patte ce qui prouve autant d'ignorance que de mauvaise foi

[Suite de l'article second]

4 ... Porte-à-faux Z, *fig. V* (Fig. 14), occasionnés par le support de la tour en encorbellement au droit des petits-côtés de l'octogone, comme de coutume.

4 ... Porte-a-faux formés par le ventre S, *fig. V & VI* (Fig. 14), de la tour, qui ne peut être contre-venté en aucune mannière vis-à-vis des voûtes des bras de la croix à cause de leur c onstruction à la légère & du manque de largeur des piIiers.

8 ... Porte-à-faux au-delà du massif des piliers, occasionnés par chacun des angles X, *fig. V*, des corps avancés de la tour du dôme.

8 ... Porte-à-faux sur les entre-colonnements W, *fig. VI*, soit qu'on leur fasse porter la retombée des reins des arcs, soit qu'on supprime lesdits reins.

2 ... Porte-à-faux de la part des deux murs Y, *fig. VI*, placés sur des platebandes du côté du chevet de l'Eglise, lesquels n'ont dans leur direction, vers leurs extrémités, aucun corps pour les recevoir.

4 ... Porte-à-faux V, *fig. VI*, sous les piédroits des arcs-boutants que l'on pourrait croire indispensables pour contenir l'effort des pendentifs.

Total 30 Porte-à-faux.

[Marginalia de Rondelet]

la fausseté de ce que Mr Patte a eu l'impudence d avancer dans son avant propos, puisque le mur de 3 pieds 9° dont il parle est une chimere ainsi que le mur de plus de 8 pieds d'epaisseur. Ce trait de mauvaise foi ne peut être pallié qu'en supposant dans l'auteur une ignorance complette de l'art de bâtir ce que je suis bien éloigné de croire

[Suite de l'article second]

C'est présentement qu'on doit sentir la vérité de ce que nous avons [p. 33] avancé dans notre *Avant-propos*.

Le bas de la tour du dôme de Ste Génevieve ayant été prouvé ne pouvoir avoir moins de 8 pieds d'épaisseur, & la largeur du pilier, destinée à le porter, ne pouvant évidemment être secourue que par des porte-à-faux, il s'ensuit

[28]Note (20) de Patte : Il y auroit une réflexion très importante à faire sur la colonne d'anglo C, *fig. III* (Fig. 12), qui est engagée dans le pilier ; c'est qu'au lieu d'augmenter la force de l'arc commc on pourroit être porté à le croire, elle servira a l'affaiblir, à cause de sa position singuliere, car elle obligera de diviser son intradss Z, *fig. IV* (Fig. 13), & de le faire ressauter de plus de 2 pieds au droit de la clef qui est le point d'appui capital de la coupole, tandis que l'intérêt de la construction d'un arc aussi chargé demanderoit au contraire que tous ses voussoirs fussent réunis tant par le haut que par le bas, pour ne point décomposer sa force. Les constructeurs conviendront qu'un tel resaut ne sauroit être que préjudiciable en cette rencontre.

[Marginalia de Rondelet]

il s'ensuit que Patte a imaginé une chimere pour la combattre et que c'est la méchanceté la mauvaise foi et l'ignorance bien p .... les même qui sont exact des faits

[Suite de l'article second]

donc que le cas reste tout semblable à celui d'un mur isolé de 3 pieds, 9 pouces d'épaisseur, & de 80 pieds de haut, auquel nous l'avons comparé, sur lequel on proposeroit de consuuire un mur de plus de 8 pieds d'épaisseur & de 40 pied d'élévation, avec l'obligation de faire encore soutenir, à l'extrémité de ce dernier, la poussée d'une ou de deux grandes voûtes : la parité est la même.

Enfin, en voyant l'insuffisance démontrée des piliers élevés, peut - être croira-t'on, qu'en faisant le sacrifice de la coupole annoncée; ainsi que de la décoration de tout le morceau milieu, l'on pourroit y substituer une calotte ou un cul-de-four; mais, outre que ce seroit détruire la grace extérieure de cet édifice, & tromper l'attente du public à cet égard, il n'y auroit pas beaucoup à gagner dans ce nouvel arrangement. Toute la difference seroit que les arcs, au lieu de porter la tour du dôme, contreventeroient directement, au droit de leur clef, la poussée de la voûte du même dôme : cela ne pouvant être autrement, il faudroit donc que ces arcs, & par conséquent les retours des piliers qui les recevroient, eussent la largeur déterminée par les formules pour résister à cette action, c'est-à-dire au moins 7 pieds. Le seul moyen de se passer de cette largeur, seroit de faire ensorte de reporter l'effort de la poussée du cul-de-four le long des voûtes des bras de la croix, jusqu'aux murs qui les terminent ainsi qu'on le pratique d'ordinaire [29]; mais alors il seroit nécessaire de construire ces voûtes d'une certaine épaisseur dans toute leur longueur en forme de berceau, & non à la légére, comme on ne peut s'en dispenser.

Or cetre construction de voûte en berceau n'est pas praticable à Sainte Génevieve, tant à cause des colonnes qu'il faudroit ajouter dans les retours des bras de la croix, pour supporter le berceau, que par rapport à l'obligation où l'on se trouveroit de contenir son action par des piliers-butans appuyés contre les murs pourtours de l'Eglise, lesquels ne son aucunement disposés, non plus que leurs fondations, pour en recevoir. C'est pourquoi les voûtes des bras de la croix ne pouvant, à raison de la [p. 34] forme du plan, être construites d'un bout à l'autre d'une force suffisante pour résister à une poussée, & faire l'office d'un contrefort horizontal, on ne peut pas plus faire contreventer un cul-de-four par la largeur 3 pieds 9 pouces des piliers de l'Eglise dont il s'agit, que leur faire porter un dôme.

[29] Note (21) de Patte : Dans les Eglises de Saint Sulpice, de Saint Roch à Paris, & dans toutes celles où l'on a élevé des calottes ou des voûtes en cul-de-four au centre des bras de la croix, on peut remarquer que c'est de cette façon qu'on a contreventé la poussée, &. Cela ne sauroit être autrement, à moins de donner aux piliers une épaisseur suffisante.

Conclusion

AINSI, de quelque façon que l'on veuille considérer l'exécution de la coupole promise au centre de l'Eglise de Sainte Génevieve, il seroit difficile de la justifier; la pratique & la théorie, les exemples mis en parallele, & les démonstrations Mathématiques s'accordent à prouver que les piliers déjà élevés sont d'une disproportion trop manifeste pour la porter; qu'ils se déroberont de tous côtés dans le bas à son poids & à sa poussée, au lieu de former des empattemens, & de s'élever en laissant de bonnes retraites au pied de la tour, comme la solidité le requiert ; qu'en outre la masse cubique qui leur manque ne sauroit être suppléée par les parties environnantes, vû qu'elles sont trop foibles, toutes évidées dans leur hauteur ou en porte-à-faux; & qu'en un mot pour soutenir l'effort d'une coupole de 63 pieds de diamétre, placée sur pendentifs, il faudroit que les supports en question eussent tout au moins 9 pieds de largeur à la place de 3 pieds 9 pouces. Mais à combien plus forte raison sera-t-on persuadé de l'insuffisance de ces piliers, si l'on fait attention que nous n'avons considéré que la poussée d'une seule voûte, & qu'il doit y en avoir deux suivant le projet, agissantes contre les mêmes piédroits. Enfin, il faudrait condamner la construction de tous les ouvrages célébres exécutés en ce genre, pour approuver les proportions des supports de celui-ci, & dire que les Michel-Ange, les Fontana, les Wren, les Mansards, & tous les habiles constructeurs qui ont fait des coupoles, ne ce sont point douté de la force qu'il falloit pour les soutenir ; puisque, comparée à celle des piliers destinés à porter le dôme de l'Eglise de Sainte Génevieve, ces derniers se trouvent toujours proportionnellement deux & trois fois moins considérables que les autres C'est aux savants & aux Constructeurs instruits à prononcer sur cet objet important, qui intéresse trop la gloire de nos arts & la sureté publique, pour pouvoir être regardé avec indifférence.[30]

[30]Note (22) de Patte : nous invitons ceux qui entreprendroient de répondre à ce Mémoire, de ne point donner pour raison, des conjectures ou des opinions paticuliéres, mais de suivre à peu près notre marche, c'est-à-dire d'appuyer ce qu'ils avanceront, par des faits en paralelle [p. 36] joints à des démonstrations : car, en pareille matiere ce ne sont que ceux qui prouvent, qui méritent attention. On vient de nous assurer qu'il alloit paroître dans le Mercure une lettre d'un ami de M. Soufflot, où il est dit qu'il prendra pour modèle de l'exécution de son dôme les procédés gothiques : mais jamais les Goths dont on admire avec raison la legéreté de plusieurs de leurs ouvrages, n'ont connu les constructions des coupoles sur pendentifs; jamais ils n'ont rien entrepris de comparable pour la hardiesse à ces chefs-d'oeuvres d'industrie. Des fardeaux aussi considérables portés en l'air sur quatre points, demandoient trop d'études, de connoissances & de combinaisons, pour avoir été le fruit de siécles où l'on ne savait ni lire ni écrire, & encore moins calculer. L'on sait que c'est à la renaissance des Arts & des Sciences qu'est due la perfection de cette découverte faite sous Justinien. Cette remarque suffit pour faire voir le peu d'égard que méritera une Apologie sans vraisemblance, destituée de faits homogenes & de preuves démonstratives.

[p. 36] Problème

Une voûte en berceau de 63 pieds de diamètre, *Fig. IV* (Fig. 14), surmontée d'un 12° avec 24 pouces d'épaisseur réduite pour la partie supérieure *f d* de la voûte, & des piedroits de 36 pieds de hauteur, étant donnée, trouver par l'application de la formule

$$\sqrt{\frac{2bgnn}{af}-\frac{2dnn}{f}+\frac{4nn^2}{ff}}-\frac{2nn}{f}=y$$

quelle est la puissance en équilibre avec sa poussée.

Solution

Soit la voûte *b f t d* de 63 pieds de diamètre, son épaisseur réduite deux pieds, son rayon 36 pieds 9 pouces, & la hauteur *n p* du piedroit 36 pieds.

Il faut sçavoir que pour parvenir à l'équation $\sqrt{\frac{2bgnn}{af}-\frac{2dnn}{f}+\frac{4nn^2}{ff}}-\frac{2nn}{f}=y$ dont il s'agit de faire l'application :

1°. il a été tiré du centre K′ de l'un des arcs, les droites K′ d & K′ f ; l'une à l'intersection des deux arc, l'autre au point *c*, milieu de l'arc *b d* :
2°. que l'on a abaissé du point l, milieu de f c, la perpendiculaire l u, sur le diamètre b G :
3°. que par ce même point l, on a mené une autre perpendiculaire l k sur l'axe d H, de la voûte, dont le prolongement rencontre en m, celui n p du piedroit.

L'on a appellé l k *(a) ;* b u *(d) ;* n p *(f) ;* m p *(g),* & la surface f c d t qui tend à écarter le piedroit, *(nn)*. C'est donc en découvrant les valeurs de ces analytiques, au moyen des données de notre problême, que nous aurons l'expression cherchée de y.

K′d & K′H étant connues dans le triangle rectangle H K′ d, on aura par les méthodes, la valeur de l'angle H K′ d, qui dans ce cas est de [p. 37] 81 dégrés 47 m. 2/3 sa moitié, 40 dégrés 53 m. 4/3 sera celle de l'angle l K′ u. L'on connoîtra donc dans cet autre triangle l K′ u, les trois angles & l'hippotenuse ; ainsi il est facile de trouver par la Trigonométrie les côtés K u & l u. En retranchant K′ H de K′ u, on aura la valeur de Hu, & par conséquent de l k qui lui est égal : en ôtant aussi K′ u de K′ b, on aura la valeur de bu : enfin, par la comparaison des triangles semblables K′ u l & k l q, on obtiendra la valeur de k q.

On connaîtra la surface *n n* qui reste, en cherchant d'abord la surface de la couronne entiere, ensuite en retranchant le cercle qui auroit K′ c pour rayon, de celui dont le rayon serait K f ; & enfin faisant la proportion, 360 dégrés est à 40

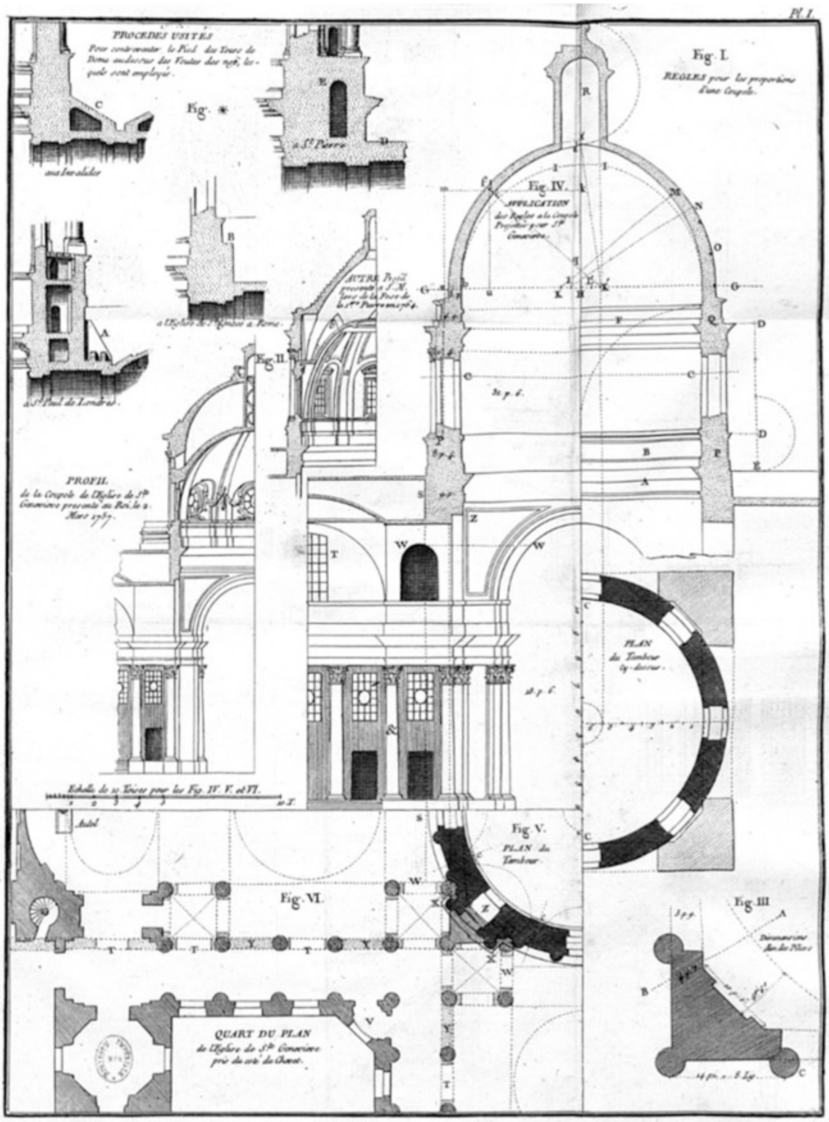

Fig. 16 Representation of the dome of Ste. Geneviève, Patte 1770, Pl. I, figs. I, IV

dégr. 53 m. 2/3 valeur de l'arc c d ou c b, comme la couronne est à la partie f c d t *(nn)* = 1 toise 2 pieds 11 pouces 9 lignes 10 points.

Ainsi par toutes ces opérations, on trouvera

que a = 3 toises 5 pieds 3 pouc. 5 lign. :

que b = 3 tois. 2. pi. 1 pouc. 11 lig. 4 p. :

que d = 1 t. 2 pi. 2 pouc. 5 lig. :

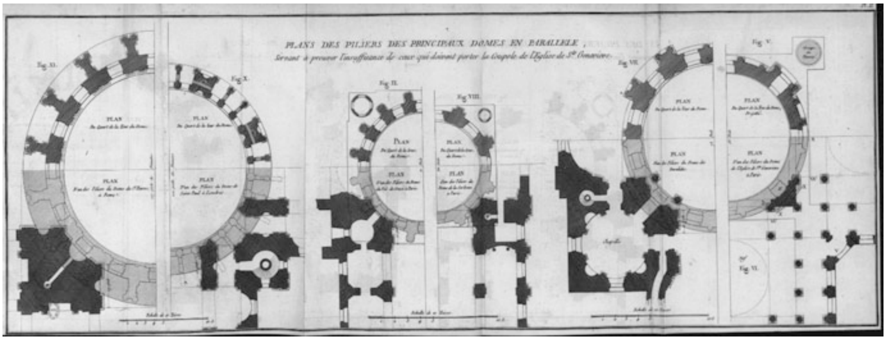

Fig. 17 Planche II

que $f = 6$ toises :
que $g = 10$ tois. 0 pi. 8 po. 6 li.: &
que $n\,n = 1$ toise 2 pieds 11 pouces 9 lignes 10 points.
En faisant les produits de l'équation, on découvrira que

$$\frac{2bgnn}{af} = 4^{toises}2^{pieds}2^{pouces}10^{lignes}2^{po\,\text{int}\,s}7$$
$$\frac{2dnn}{f} = 0^{toises}4^{pieds}1^{pouces}2^{lignes}5^{po\,\text{int}\,s}8$$
$$\frac{4nn^2}{ff} = 0^{toises}1^{pieds}5^{pouces}11^{lignes}3^{po\,\text{int}\,s}$$
$$\frac{2nn}{f} = 0^{toises}2^{pieds}11^{pouces}11^{lignes}3^{po\,\text{int}\,s}$$

Par conséquent

$$\sqrt{\frac{2bgnn}{af} - \frac{2dnn}{f} - \frac{4nn^2}{ff}} = \sqrt{3^{toises}5^{pieds}7^{pounces}6^{lignes}11^{po\,\text{int}\,s}9};$$

ou bien égale 1 toise 5 pieds 10 pouces 10 lignes dont il faut retrancher 0 toises 2 pieds 11 pouces 11 lignes 3 points, valeur $\sqrt{\frac{2nn}{f}}$; & le reste 1 toise 2 pieds 10 pouces 11 lignes 40 points sera l'épaisseur qu'il conviendra de donner pour l'équilibre aux piedroits de la voûte en berceau.

Mais comme l'on sçait que les voûtes sphériques dont l'arc droit est circulaire, ainsi que les voûtes sphéroïdes dont l'arc droit est elliptique surhaussé ou sur-baissé, poussent environ la moitié moins que celles en berceau de même nature diamètre, épaisseur, & en un mot qui sont con- [p. 38] ditionnées de même ; il s'ensuit que pour connoître l'épaisseur des piédroits de la voûte sphéroïde, qui est l'objet de notre examen, il faut prendre la moitié de 1 toise 2 pieds 10 pouces 11 lignes 4 points ; & l'on aura 4 pieds 5 pouces 5 lignes 8 points pour la puissance en équilibre avec la poussée de cette voûte ; ce qu'il falloit démontrer (Figs. 16 et 17).

FIN

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Notes on Limit and Nonlinear Elastic Analyses of Masonry Arches

Danila Aita, Riccardo Barsotti, and Stefano Bennati

Abstract This chapter is a critical presentation of studies over the last 10 years on the mechanical response of masonry arches and vaults. More precisely, we focus on a study of masonry arches conducted in parallel via both nonlinear elastic and limit analyses. The one-dimensional elastic model for masonry arches incorporates a simple but effective nonlinear constitutive law. In turn, collapse analysis is performed by the so-called 'method of stability areas', originally proposed by Durand-Claye in 1867. Rather than offering two alternative paths, the approaches may be considered complementary points of view on the same problem. A reasoned illustration of the concepts in question is furnished by the topics addressed: an analysis of the possible failure mechanisms of a pointed arch subject to its own weight; the search for explicit solutions to the equilibrium problem of a depressed arch subject to a uniformly distributed load; the study of arches of different shapes subject to their own weight and the weight of a superimposed wall.

Keywords Masonry • Arches • Walls • Elasticity • Nonlinear analysis • Durand-Claye method

1 Introduction

Determining the structural response of masonry arches, vaults and domes, in terms of both displacements and stresses, still represents a challenging task. In fact, a well-known property of masonry structural elements is that their response is typically characterized by a pronounced nonlinear trend even for load intensities that are quite low with respect to the collapse values.

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The need to take into account the nonlinear constitutive relations for “masonry” material represents a further challenge that must be faced, except in some special cases. For this reason, especially in the not-so-rare cases of complex building shapes, the solution is sought by recurring to large-dimension numerical analyses, even when geometric nonlinearities are not accounted for (Alfano et al. 2000; Lourenço 2005).

The present work aims to analyze the mechanical response of one of the most common structural elements in masonry constructions: the arch. The two theoretical models that will be used in this regard belong to different, but complementary frameworks, as we shall see: limit analysis, on the one hand, and nonlinear elasticity, on the other hand. Although the two models, which will be illustrated in the first two sections to follow, differ in many respects, they nonetheless share the goal of simplicity as one of their main features. In this regard, it should be noted that the use of simple schemes is motivated not only by reasons of an operational nature, but also by two further considerations regarding the analysis results. First, the simple models generally enable simpler, more concise verification of the results, as they focus attention on the main aspects of the phenomenon in question. Second, to be used effectively, the more complex models—which in principle would allow obtaining more accurate results in closer agreement with experiment—often require knowing a large number of mechanical and geometrical parameters, which are generally uncertain and difficult to determine experimentally. One example that effectively illustrates the many complexities involved in the analysis of masonry structures is offered by the basic, and apparently easy-to-solve, case of a masonry arch subjected to its own weight and the weight of a superimposed wall. Determining the mechanical response of such a system represents a challenging problem due to at least three aspects: the strong material nonlinearities typical of the complex mechanical behavior of masonry, the pronounced heterogeneity of the masonry, and the large number of geometrical and mechanical parameters involved, which include shape, span and thickness of the arch, the strength of the masonry (usually accounted for by considering some approximate or average values), the inclination of the wall extrados, the considered boundary conditions on the walls’ vertical sides and at the arch springings, as well as others. In particular, determining the actual load distribution transmitted from the wall to the arch is by no means a trivial matter, and approximate solutions obtained via finite-element computation codes or, alternatively, by means of elementary calculation schemes, are often considered acceptable in the literature (see, for instance, Cavicchi and Gambarotta 2005).

In this paper, the two simplified methods that will be used to obtain the solution to some equilibrium problems for a masonry arch will be indicated as the “stability areas method” and the “nonlinear elastic model”, respectively. The first solution technique is based on an expressly developed extension of the historical method of “stability areas” introduced by Alfred Durand-Claye (1867). It aims at determining the set of statically admissible solutions within the limits imposed by the ultimate compressive and tensile strengths and the limited shear capacity of the joints. When the extent of the stability area goes to zero, a limit equilibrium condition is attained for the whole system (Foce and Aita 2003; Aita et al. 2004, 2007). This methodology preserves the fundamental concepts of limit analysis and, at the same time, embodies some aspects of the nonlinear elastic analysis by imposing a restriction on the stress level.

The second solution technique focuses instead on the stress and strain fields generated in the arch, which is considered as a one-dimensional element made of a material offering poor resistance to tension. Such mechanical behavior can be modeled, as a first approximation, via a nonlinear elastic constitutive relation. The problem is tackled by studying and numerically integrating systems of nonlinear equations (Barsotti and Bennati 2002). The condition of incipient collapse is considered to be reached when the residual stiffness of the system falls below a predetermined fraction of its initial value. From a historical point of view, this approach has his roots in the contribution of Antonio Signorini, who first proposed direct-method studies of the mechanical behavior of elastic materials unable to bear tensile stresses (Signorini 1925a, b).

The two methods achieve complementary purposes: the stability area method allows for readily determining a collapse load value, while the nonlinear elastic analysis provides a helpful and, in some aspects, essential check of its mechanical significance by following the evolution of the displacement field and extension of the nonlinear regions where cracking and crushing phenomena arise as the load increases.

In what follows, the two theoretical frameworks introduced above are used in the search for the solution to three problems, which are described in the third and fourth section. All three cases presented, which are a selection of case studies examined over the past ten years, refer to plane arches subject to in-plane vertical loads.

In each of the two first applications the attention is focused on one of the two methods in order to highlight its main aspects. In particular, the first case illustrates application of the method of stability area to the collapse of pointed arches, while also taking into account the limited shear strength of the masonry. The second case shows how nonlinear elastic analysis enables obtaining the explicit solution in terms of displacements, stresses, strains and extension of the nonlinear zones in which cracks and damage are expected in the masonry in the case of depressed arches subjected to uniformly distributed vertical loads. Finally, the parallel use of both analysis methods is illustrated in the last case, in which we examine the mechanical system consisting of an arch and a superimposed wall. The problem, which commonly occurs in masonry buildings and bridges, is investigated for arches of different shapes: pointed, semicircular and elliptical.

2 The Durand-Claye Method

In 1867, Alfred Durand-Claye introduced a graphical procedure aimed at assessing whether the values of admissible thrust at the crown section of an arch could be determined for symmetric equilibrium problems. The object of the procedure, which is now commonly indicated as the Durand-Claye method, is to draw a particular

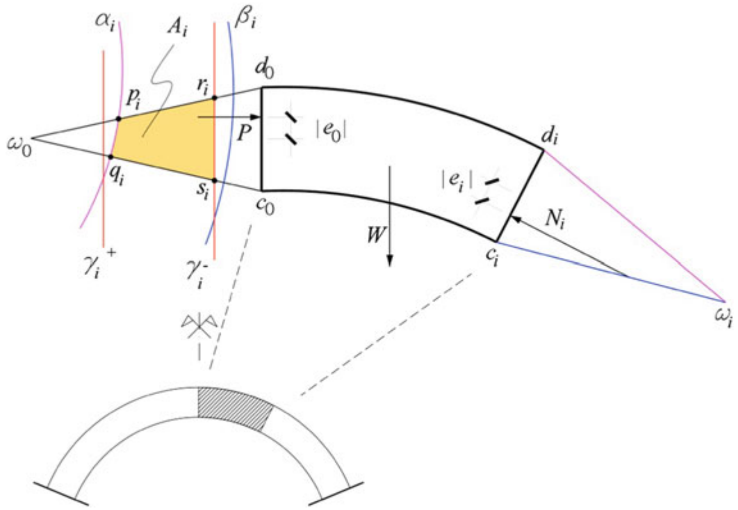

Fig. 1 The stability area ($\sigma_t = 0$)

plane figure called the stability area. Each point belonging to such area is the extreme of a vector representing an admissible value for the crown thrust, i.e. a thrust that allows for verifying the equilibrium of any given part of the arch and is at the same time compatible with the masonry strength. It is worth observing that here the method proposed by Durand-Claye has been suitably modified so as to account for a nonlinear stress distribution both in tension and compression. Here we will briefly describe the method thus modified (a more detailed description is given in Aita et al. 2004).

We consider a symmetric masonry arch, indicating with σ_c and σ_t the masonry compressive and tensile strength, respectively. We assume, for the sake of simplicity, that the arch is loaded by its own weight only.

We begin by examining the ideal *voussoir* comprised between the crown joint c_0d_0 and a generic joint c_id_i (Fig. 1). We indicate with W the weight of the *voussoir*, with N_i the normal force at joint c_id_i, with P the thrust at the crown section, and with e_0 and e_i the eccentricity of their corresponding application points.

If we set the value for the crown eccentricity, e_0, a first limitation for the thrust value at the crown section, P, can be obtained by requiring it to be compatible with the masonry compressive and tensile strength. Unlike the original version of the stability area method, here any given cross-section of the arch may be subdivided into three different parts: two parts where the normal stress is constant and equal to σ_c and σ_t, respectively, and a third part where the stress varies linearly between these two threshold values. By varying e_0, simple calculations allow for drawing the corresponding curve $c_0\omega_0d_0$, bounding the admissible region. Analogously, by considering the axial force N_i, transmitted across the joint c_id_i, the corresponding curve $c_i\omega_id_i$ is obtained.

By imposing rotational equilibrium on the voussoir, further limitations are introduced. In fact, it is an easy matter to show that, when the end of the vector representing the axial force N_i, moves along the curve $c_i\omega_i d_i$, the corresponding extreme of the thrust at the crown belongs to the hyperbolas α_i and β_i.

The last limitation follows directly from the voussoir translational equilibrium and adoption of Coulomb's friction law. The latter requires the magnitude of the shear force transmitted along joint $c_j d_j$ not to exceed the limit value $N_i \tan\phi$, where ϕ is the internal friction angle. In turn, from the translational equilibrium in the direction normal to the joint $c_i d_i$, it is straightforward to conclude that two limit values for the horizontal thrusts at the crown section can be assessed, corresponding to the limit inward and outward shear forces, respectively. The two vertical red lines drawn in Fig. 1, labeled γ_i^- and γ_i^+, correspond to these two thrust limit values.

The admissible thrusts P with respect to both the voussoir equilibrium and masonry strength are then represented by the horizontal vectors whose extremes are contained within the area $r_i s_i p_i q_i$, internal to the curve $c_0\omega_0 d_0$, and comprised between the hyperbolas α_i and β_i, and the straight lines γ_i^- and γ_i^+.

By repeating the foregoing for every joint i, the stability area, A, common to all the areas A_i can be determined. If A has a finite extension, there are infinite admissible thrust values (and infinite corresponding eccentricities), ranging within a suitable interval. On the contrary, if A is empty, no admissible thrust value exists. Finally, as we shall see later, when the stability area A shrinks to a point or to a segment, the limit condition is attained, and a unique admissible thrust value exists.

3 The Non-linear One-dimensional Elastic Model

In order to build a simple but effective model, we consider the masonry arch as a deformable curved beam. As usual in the theory of the bending of beams, the cross-section of an arch of height h and, for the sake of simplicity, unit width is assumed to remain plane and normal to the longitudinal fibers after bending, thus neglecting any shear strain. Moreover, the longitudinal normal stresses σ_θ are assumed to depend on the corresponding strains ε_θ according to the same relation holding in a uniaxial state of stress.

The longitudinal strain is linear over any given cross-section. Thus, the kinematics of the arch can be described by three functions of the curvilinear abscissa s along the line of axis, namely: the displacement components u and v of the points of the axis of the arch in the tangential and radial directions, and the rotation φ of the cross-section, positive if clock-wise (Fig. 2).

Simple calculations, omitted here for the sake of brevity, show that, in the case the arch is circular with radius R, the radial displacement $v(\theta)$ is a solution to the differential equation

$$v'' + v = -R^2\chi - R\varepsilon, \tag{1}$$

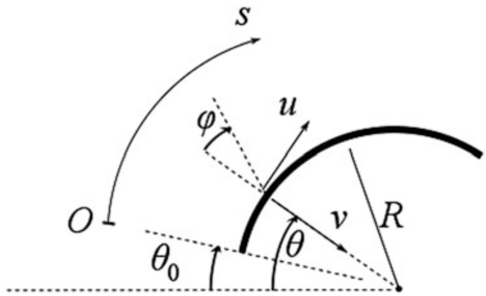

Fig. 2 Symbols and notations adopted in the one-dimensional model

where ε and χ are the axial strain and curvature of the line of axis, respectively, $\theta = s/R + \theta_0$, the prime denotes differentiation with respect to θ, and θ_0 is the starting angle of the circular arc. The integral of Eq. (1) is

$$v(\theta) = (R\varphi_0 - u_0)\sin(\theta - \theta_0) + v_0\cos(\theta - \theta_0) + \\ - \int_{\theta_0}^{\theta} \sin(\theta - t)\left(R^2\chi(t) + R\varepsilon(t)\right)dt. \tag{2}$$

In turn,

$$\varphi(\theta) = \varphi_0 - \int_{\theta_0}^{\theta} R\chi(t)dt, \tag{3}$$

and

$$u(\theta) = -v' + R\varphi = u_0\cos(\theta - \theta_0) + v_0\sin(\theta - \theta_0) + \\ + R\varphi_0(1 - \cos(\theta - \theta_0)) + \int_{\theta_0}^{\theta}\left[\cos(\theta - t)\left(R^2\chi(t) + R\varepsilon(t)\right) - R^2\chi(t)\right]dt, \tag{4}$$

while u_0, v_0 and φ_0 are the tangential and radial components of the displacement and the rotation of the initial cross-section, at $\theta = \theta_0$.

Following Signorini's idea, we adopt for the masonry the piecewise-linear constitutive relation between the longitudinal strain, ε_θ, and stress, σ_θ, shown in Fig. 3.

The simple elastic stress–strain constitutive relation used here reduces to the basic linear case for stresses comprised within the threshold values σ_c and σ_t, denoting the material's resistance to compression and tension, respectively. Outside

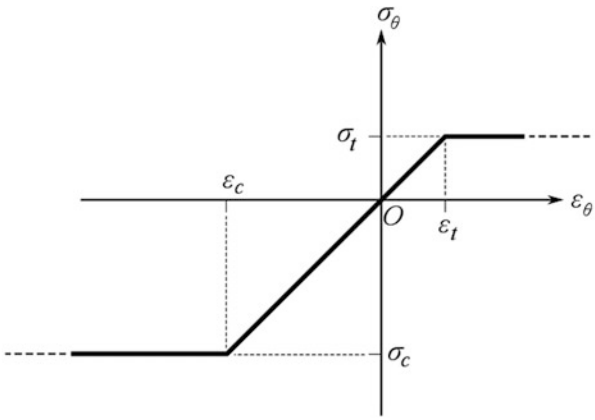

Fig. 3 The $\sigma_\theta - \varepsilon_\theta$ relation

this range, strain may grow at constant stress. Consequently, denoting by E the material's Young's modulus, we have

$$\sigma_\theta = \begin{cases} \sigma_c & \varepsilon_\theta \leq \varepsilon_c, \\ E\varepsilon_\theta & \varepsilon_c < \varepsilon_\theta < \varepsilon_t, \\ \sigma_t & \varepsilon_\theta \geq \varepsilon_t, \end{cases} \tag{5}$$

where $\varepsilon_t = \sigma_t / E$ and $\varepsilon_c = \sigma_c / E$. This simple nonlinear equation, already adopted by the authors in previous works (Barsotti and Bennati 2002; Aita et al. 2003a,b, 2009, 2012), enables accounting for masonry's weak tensile strength and bounded compressive strength. One drawback is that, in the case $\sigma_t \neq 0$, the material is unrealistically assumed to be able to transmit low tensile stresses, even in the presence of high strains.

From constitutive equation (5), and by using the kinematic assumption made on the cross-section, we can build the set of corresponding nonlinear constitutive equations that hold at the cross-sectional level between the kinematic parameters, axial strain ε and curvature χ, and the internal actions, axial force N and bending moment M. Figure 4 shows a plot of the elastic domain in the (n, m) plane, where $n = N/\sigma_c h$ and $m = -M/\sigma_c h^2$ indicate the dimensionless axial force and bending moment, and $t = \sigma_t / \sigma_c$ the masonry tensile to compressive strength ratio.

In each of the seven regions of the elastic domain, the axial strain ε and the curvature χ are known functions of the dimensionless axial force and bending moment, n and m. By way of example, the constitutive relations between generalized strains and dimensionless internal actions holding in the B^+ region are

$$e = \frac{\varepsilon}{\varepsilon_c} = 1 + \frac{8(n-1)^2\,(n-1+3m)}{9(n-1+2m)^2}, \quad c = \frac{\chi h}{\varepsilon_c} = \frac{8(n-1)^3}{9(n-1+2m)^2} \tag{6}$$

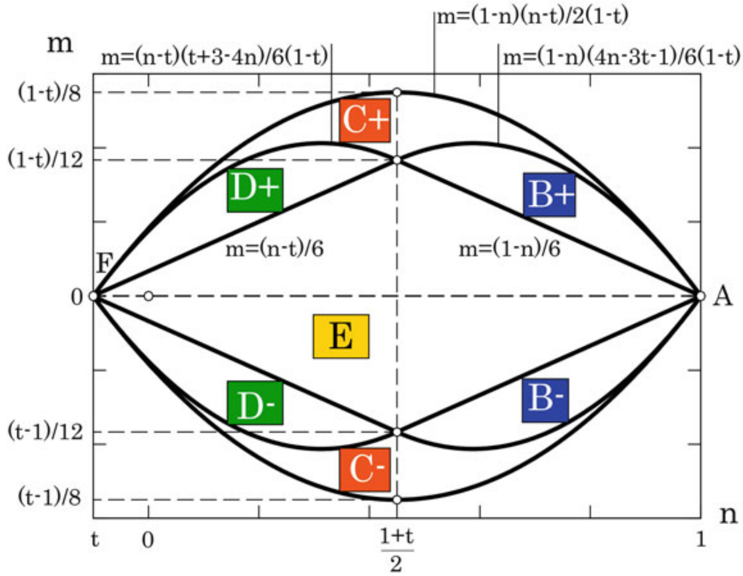

Fig. 4 The elastic domain for the arch cross-section

or, in inverse form,

$$n = \frac{e+1}{2} + \frac{c}{8} + \frac{(e-1)^2}{2c}, \quad m = \frac{1-e}{2} - \frac{c}{24} + \frac{(e-1)^3}{6c^2}. \tag{7}$$

Analogous relations, omitted here for the sake of brevity, hold in the other regions of the elastic domain.

Except for very special cases, the equilibrium problem for the arch is statically indeterminate. Thus, the sequence of linear and nonlinear regions along the arch is not known a priori, and the set formed by the nonlinear kinematic equations (2)–(4), the constitutive relations (6) and the equilibrium equations is usually solved numerically through an iterative procedure (Bennati and Barsotti 1999, 2001). However, for simple cases, such as uniformly loaded flat or depressed arches, an explicit solution can be found, as will be shown in one of the following examples (Aita et al. 2003a,b, 2004).

4 Some Case Studies

The two different solution methods described in the previous sections, namely, the method of stability areas, on the one hand, and nonlinear elastic analysis, on the other, are used to solve the three equilibrium problems illustrated in section.

The applications clearly show that the two methods, though independent, prove to be fruitful when applied complementarily to each other. On the one hand, thanks to its simplicity of use, the method of stability areas allows for relatively quick determination of the value of the collapse load, while nonlinear elastic analysis allows for following the evolution of displacements, stresses, and extension of nonlinear regions, and hence evaluating the actual conditions of an arch that is subjected to an assigned distribution of loads.

Some of the main distinctive features of each of the two analysis methods are examined in the first two applications. In particular, an illustrative example application of the stability areas method is shown in the first case, which addresses pointed arches. The method allows for taking into account the influence of masonry's limited shear strength on arch collapse. The case dealt with, in which the arch is subject to its own weight alone, reveals the influence of the thickness and the friction coefficient on arch collapse.

The second case examines the nonlinear elastic analysis of masonry arches, modeled as one-dimensional elements. This example aims to demonstrate that it is possible to obtain an explicit solution in terms of displacements, stresses and extension of the regions in which the behavior is nonlinear, where the onset and progression of cracking and material damage would be expected in the case of greatly depressed arches subject to uniformly distributed vertical loads. The analysis is conducted by assuming an increasing loading process, so as to describe the evolution of the solution up to collapse.

Lastly, the parallel use of both analysis methods is illustrated in the last of the three cases, in which the mechanical system formed by an arch and an overlying wall is considered. The problem, which occurs commonly in masonry buildings and bridges, has been examined for arches of different shapes: pointed, circular and elliptical. The main objective is to determine the stress levels as a function of the main geometrical and mechanical parameters, and thereby assess the safety margin under conditions of incipient collapse, as well as the actual mechanism by which such collapse would occur (Aita et al. 2012).

4.1 Pointed Arches Under Their Own Weight

The stability areas method, suitably modified to account for masonry's limited compressive strength, allows for easily determining the limit values for the thrust corresponding to the attainment of some limit condition along a generic joint $c_i d_i$ by considering the corresponding stability area, A_i. As already described in Sect. 2, when the extreme point of the crown thrust vector is on the border of A_i, a limit condition is attained on joint $c_i d_i$ (see Fig. 1). If the bending moment at joint $c_i d_i$ reaches the limit value (which in turn depends on the eccentricity and magnitude of the axial force), the corresponding limit condition concerns the rotational equilibrium of the voussoir between the crown and the joint in question. If, instead, the shear force at joint $c_i d_i$ reaches the limit value, a limit condition for the translational equilibrium of the same voussoir ensues. By scanning every joint along the arch, we determine the arch's overall stability area A as the intersection of all A_i areas.

If A is empty, equilibrium of the arch is not possible. In other words, the internal forces are not statically admissible. If the area A is greater than zero, there are (infinite) values of the eccentricity and magnitude of P that correspond to statically admissible internal force distributions. Lastly, the limit case is when A shrinks to zero: as we shall see later, when A becomes a segment, only one statically admissible value of the magnitude of P can be found, while in the case that A becomes a single point, only one statically admissible value of both the eccentricity and magnitude can be found. In these latter cases, the arch is in a limit condition.

By varying both the arch thickness, h, and friction coefficient, μ, the Durand-Claye method enables finding the pairs (h, μ) that correspond to a limit condition for the arch. In each of these cases, the limit value of the eccentricity and/or magnitude of P is determined. It is worth noting that, although masonry is considered here as a non-standard material, we can nevertheless find the collapse mechanism that may actually arise in correspondence to each limit thrust value (some further considerations on this can be found in Sinopoli et al. 2007). For any given collapse mechanism, each portion of the arch between any two consecutive joints that attain a limit condition will undergo rigid motion. We classify the different types of rigid motion, together with the corresponding symbols adopted for the crown thrust, in the following Tables 1 and 2, where the centers of rotation and the application points of the thrust at the crown are defined assuming infinite compressive strength, σ_c; in the case of limited σ_c, the terms "intrados", "extrados" are to be interpreted as "near the intrados" and "near the extrados", while a limit condition is reached at the crown or at joint $c_i d_i$.

By setting $l = R = 10$ m, $\gamma = 20$ kN/m^3, $\sigma_c = –20$ MPa, $\sigma_t = 0$, and varying the thickness h and friction coefficient μ, different collapse modes for the pointed arch

Table 1 Notation for limit thrust values corresponding to hinging mechanisms

Centre of rotation at joint $c_i d_i$	Application point of thrust at crown	Thrust value
Intrados	Extrados	$P^{r,e}_{\min}(\theta_i)$
	Intrados	$P^{r,i}_{\min}(\theta_i)$
	Internal	$P^{r,*}_{\min}(\theta_i)$
Extrados	Extrados	$P^{r,e}_{\max}(\theta_i)$
	Intrados	$P^{r,i}_{\max}(\theta_i)$
	Internal	$P^{r,*}_{\max}(\theta_i)$

Table 2 Notation for limit thrust values corresponding to sliding mechanisms

	Direction	Thrust value
Relative sliding	Downwards	$P^{s}_{\min}(\theta_i)$
	Upwards	$P^{s}_{\max}(\theta_i)$

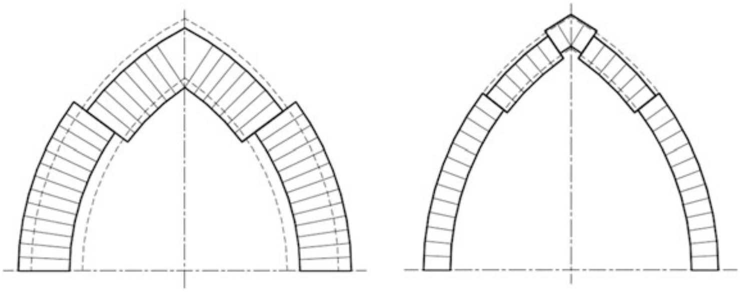

Fig. 5 Sliding collapse mechanisms. (**a**, *left*) mode 5; (**b**, *right*) mode

can be found. These have been identified and classified according to the well-known eight collapse modes of a symmetric arch illustrated by Pierre-Félix Michon (1857). The results, reported in the following, are taken from (Aita et al. 2004; 2007).

4.1.1 Sliding Collapse (Modes 5 and 6)

When the stability area reduces to a vertical line, the range of admissible thrusts shrinks to a single value and collapse can occur by mode 5 or mode 6 (Fig. 5).

In particular, it is an easy matter to verify that mode 5 (Fig. 5a) can occur when the condition

$$\begin{aligned} P = \max P^{s}_{\min}(\theta) = P^{s}_{\min}(\theta_i) = \\ = \min P^{s}_{\max}(\theta) = P^{s}_{\max}(90°), \end{aligned} \tag{8}$$

for the crown thrust is fulfilled. The angle value θ_i, corresponding to the joint between the crown and the abutment where the shear force attains a limit condition, depends on the particular values chosen for h and μ. Condition (8) is verified for:

$$h > 128 \text{ cm} \quad \text{and} \quad \mu \leq 0.19.$$

Analogously, Mode 6 (Fig. 5b) can occur when:

$$\begin{aligned} P = \min P^{s}_{\max}(\theta) = P^{s}_{\max}(\theta_i) = \\ = \max P^{s}_{\min}(\theta) = P^{s}_{\min}\left(\theta_j\right), \quad \text{with} \quad \theta_i < \theta_j, \end{aligned} \tag{9}$$

Once again, the angle values θ_i and θ_j, corresponding to the joints where the shear force attains a limit condition, depend on the particular values chosen for h and μ. Condition (9) is verified for

$$86 \text{ cm} < h < 128 \text{ cm} \quad \text{and} \quad \mu = \mu_6(h),$$

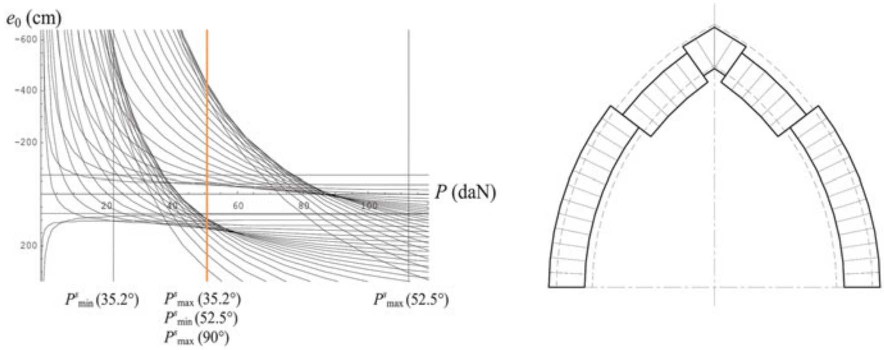

Fig. 6 Transitional collapse 5–6. (**a**, *left*) stability area in the (P, e_0) plane; (**b**, *right*) scheme of the collapse mechanism

where μ_6 takes values in the range of $0.19 < \mu_6 < 0.21$.

The transition between modes 5 and 6 takes place for $\mu = 0.19$ and $h = 128$ cm (Fig. 6b). Such limit condition corresponds to

$$P = P^s_{\max}\left(35.2°\right) = P^s_{\min}\left(52.5°\right) = P^s_{\max}\left(90°\right) = 0.5 \text{ kN}.$$

In the (P, e_0) plane, the corresponding stability area is the vertical orange line shown in Fig. 6a.

4.1.2 Mixed-Mode Collapse (Mode 4*)

For any given thickness value in the range of 40 cm < h < 86 cm, the stability area reduces to a single point if the friction coefficient equals a limit threshold μ_4, whose values depend on h and fall within the range of $0.21 < \mu_4 < 0.33$. In such cases, by referring to the (P, e_0) plane, the straight line corresponding to the limit thrust for the translational equilibrium at some joint $c_i d_i$ and the two curves corresponding to limit rotational equilibrium at two other joints $c_j d_j$ and $c_k d_k$ intersect each other at the same point (Fig. 7a), the mixed sliding-hinging collapse mode schematized in Fig. 7b can occur, and the limit thrust condition is:

$$P = \min P^s_{\max}(\theta) = P^s_{\max}(\theta_i) = P^{r,*}_{\max}\left(\theta_j\right) = P^{r,*}_{\min}(\theta_k), \quad \text{with } \theta_i < \theta_j < \theta_k. \tag{10}$$

We indicate this collapse mode as Mode 4*, because it is somewhat similar to Michon's mode 4, according to which, however, the limit shear joint should be located between the two joints where a rotational limit condition is reached.

When the value of the thickness is $h = 86$ cm, it can be seen that for $\mu = 0.21$ the two curves corresponding to limit rotational equilibrium at joint $\theta = 37.4°$ and $\theta = 69.5°$ and the straight line corresponding to the limit thrust for the translational

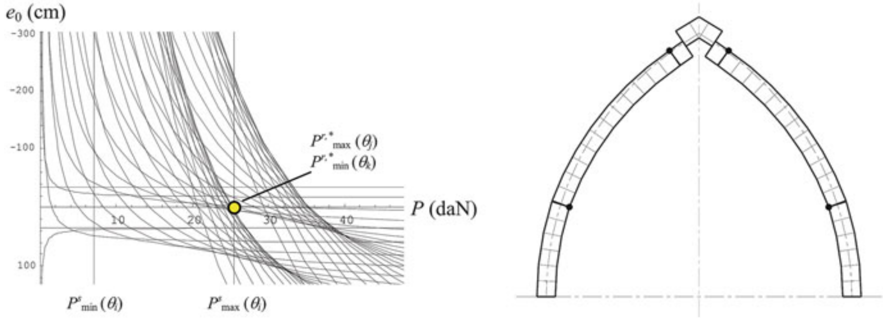

Fig. 7 Mixed sliding-hinging collapse mode 4*; (**a**, *left*) stability area in the (P, e_0) plane; (**b**, *right*) scheme of the collapse mechanism

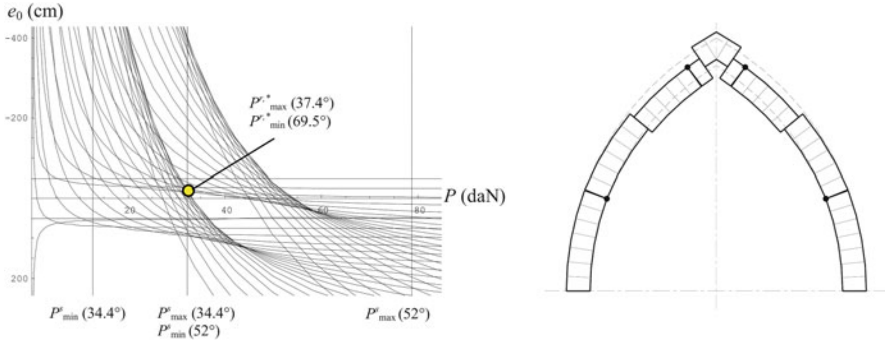

Fig. 8 Transitional mixed-mode collapse $6-4^*$. (**a**, *left*) stability area in the (P, e_0) plane; (**b**, *right*) scheme of the collapse mechanism

equilibrium (Mode 6, with $\theta_i = 34.4°$ and $\theta_j = 52°$) all intersect each other at the same point (the yellow circle in Fig. 8a). In this case, the stability area shrinks to that point, and the transitional mixed sliding-hinging collapse mode illustrated in Fig. 8b may occur. Such limit condition corresponds to

$$P = P^s_{\max}\left(34.4°\right) = P^{r,*}_{\max}\left(37.4°\right) = P^s_{\min}\left(52°\right) = P^{r,*}_{\min}\left(69.5°\right) = 0.32 \text{ kN}.$$

4.1.3 Hinging Collapse (Mode 2)

For $h = 40$ cm and $\mu > 0.33$ the stability area reduces to a single point (Fig. 9a), and the hinging collapse mode represented in Fig. 9b can occur. The corresponding limit condition becomes

$$\begin{aligned} P &= \min P^{r,i}_{\max}(\theta) = P^{r,i}_{\max}(39.6°) = \\ &= \max P^{r,i}_{\min}(\theta) = P^{r,i}_{\min}(69.2°) = 0.19 \text{ kN}. \end{aligned} \tag{11}$$

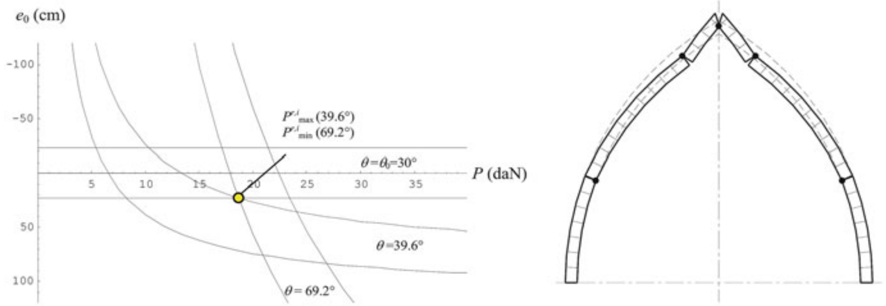

Fig. 9 Hinging collapse (mode 2). (**a**, *left*) stability area in the (P, e_0) plane; (**b**, *right*) scheme of the collapse mechanism

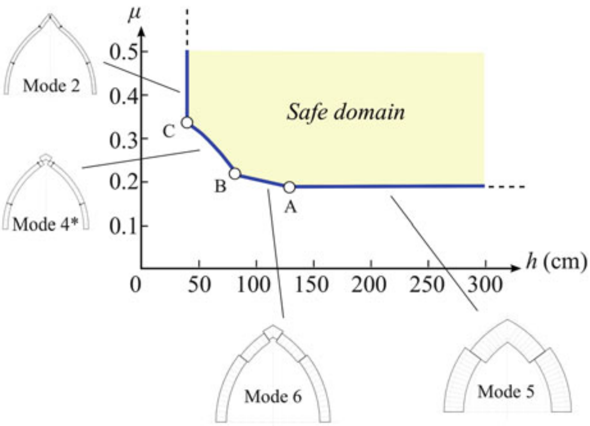

Fig. 10 Collapse modes in the (h, μ) plane

For $h = 40$ cm and $\mu = 0.33$ the transitional mixed-mode collapse 2–4* can arise. The corresponding limit condition is:

$$\begin{aligned} P &= \min P^{s}_{\max}(\theta) = P^{s}_{\max}(33.6°) = \\ &= \min P^{r,i}_{\max}(\theta) = P^{r,i}_{\max}(39.6°) = \\ &= \max P^{r,i}_{\min}(\theta) = P^{r,i}_{\min}(69.2°) = 0.19\ \text{kN} \end{aligned}$$

The foregoing results are summarized in Fig. 10 in terms of thickness and friction coefficient. Each segment along the 'safe' domain border corresponds to one of the collapse modes illustrated above; the transitional collapse mechanisms correspond to points A, B, C.

The nonlinear elastic model of the arch can be used to deal with the same problem already studied using the Durand-Claye method. Although comparison of the two solutions is necessarily incomplete, as the elastic model in its current version neglects shear deformability, it is worthwhile underlining that the results of

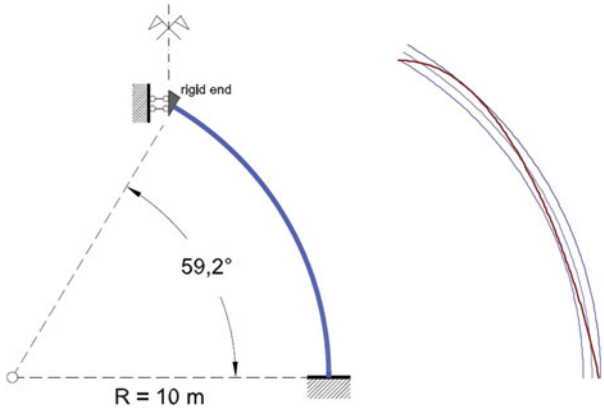

Fig. 11 (**a**, *left*) Structural scheme; (**b**, *right*) line of thrust

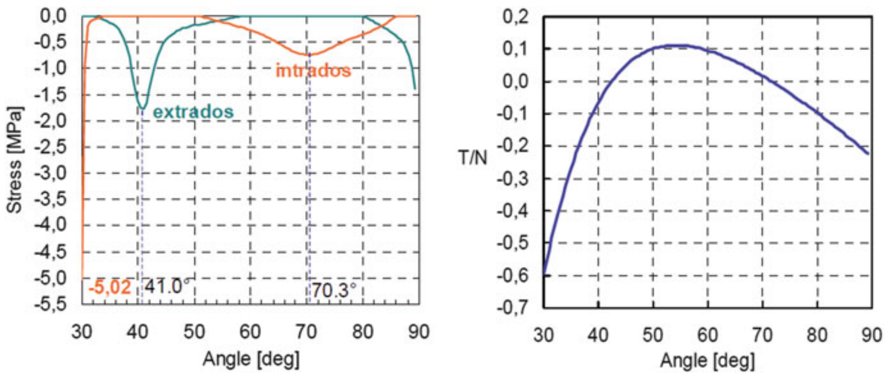

Fig. 12 (**a**, *left*) Normal stress distribution; (**b**, *right*) shear over axial force ratio ($h = 45$ cm)

the nonlinear elastic analysis are in good agreement with those obtained with the stability areas method. This is particularly evident for the limit case corresponding to a pure flexural collapse mechanism (very thin arch). Some correspondences are also evident between the two solution methods in the other limit case (very thick arch), in which a pure shear collapse mechanism takes place.

The nonlinear elastic model has been applied to the structural scheme shown in Fig. 11a, and the equilibrium problem solved for different thickness values, ranging from 45 cm to 150 cm. For a thin arch, the position of the line of thrust (Fig. 11b) reveals that the arch attains a limit condition that can be considered near collapse. Such a situation is confirmed by the diagram of the normal stresses at the arch's extrados and intrados (Fig. 12a), characterized by high compressions. It should be noted that the hinge positions for collapse mode 2 (Fig. 9) are fully compatible with the line of thrust in Fig. 11.

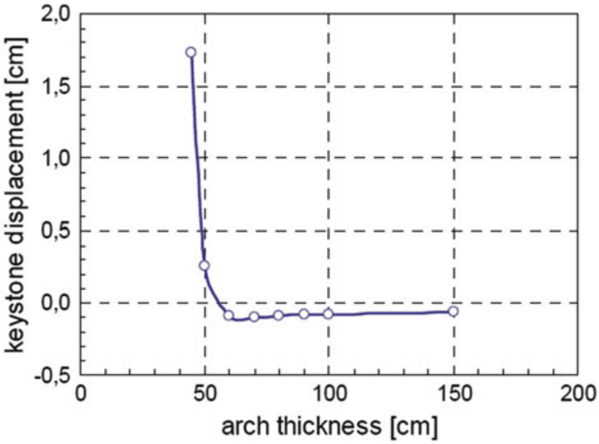

Fig. 13 Keystone vertical displacement as a function of arch thickness

Figure 13 shows the vertical displacement values at the keystone plotted against the arch thickness. The rapid increase in displacement as the thickness approaches 45 cm once again confirms that in this case the arch is close to a hinging collapse mechanism. It should also be noted that the sign of the displacement for the two cases corresponding to 45 cm and 150 cm are consistent with the mechanisms determined via the stability areas method.

4.2 *A Depressed Circular Arch Subjected to a Vertical Load: Some Explicit Solutions*

Let us consider the problem for the depressed circular arch showed in Fig. 14a, for which we indicate with l, R and 2α, the clear span, the radius of the line of axis and the center angle of the arch, respectively, and assume by hypothesis that the *ratio l/R is small* with respect to unity (Aita et al. 2003a, b). Here we will show that tackling a simplified version of the problem enables finding the analytical expressions for the arch's displacements and rotations by making some simple but reasonable hypotheses.

As already pointed out in the foregoing, the equilibrium problem for the arch in question is anything but simple to solve because, among other reasons, in general the load distribution that the wall transmits to the arch, as well as the constraints imposed on the arch end sections are all unknown.

We represent the masonry arch as a deformable curved beam. For the sake of simplicity, a uniformly distributed vertical load will model the actions exerted on the arch by the overlying wall. Moreover, we will assume that at both ends the horizontal thrust, P, and the couple of moment M_B are assigned, while the vertical

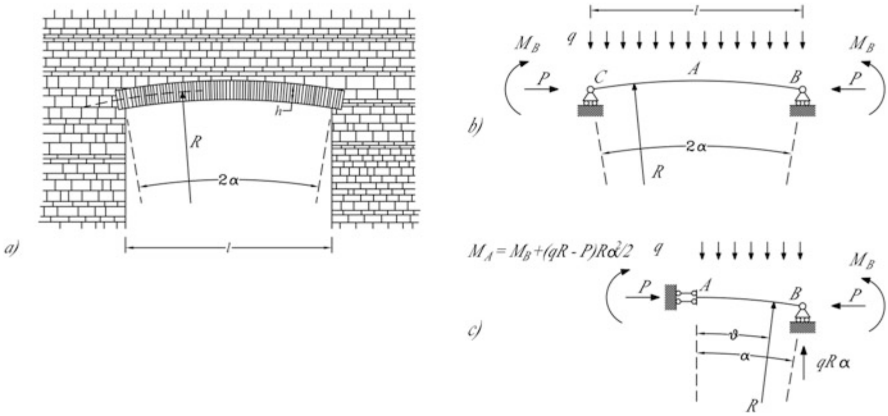

Fig. 14 (**a**) Depressed circular arch loaded by a superimposed wall; (**b**) the corresponding statically determinate problem; and (**c**) the mechanical scheme

component of the displacement is fully restrained (Fig. 14b). It is worth noting that such a statically determinate problem could be considered a first rough scheme somehow related to the case in which the structures supporting the arch (i.e. piers and abutments) have attained a limit condition. In other words, the end actions, P and M_B, could be considered limit values for the reactive forces exerted by the piers and abutments.

Symmetry enables limiting our analysis to the right half of the arch (Fig. 14c), to which end we indicate θ (with $0 \leq \theta \leq \alpha$) as the angle formed between any given cross-section and the keystone. In order to obtain explicit expressions for the displacements and rotations of the cross-sections, it is useful to make some simplifications. Firstly, since the values of θ are sufficiently small with respect to unity, the expressions for both the axial force and the bending moment will be simplified according to a Taylor series up to the second order. Moreover, by considering that the axial force can, for a sufficiently thin arch, be approximated as a constant and that the thrust is of the same order of magnitude as the resultant of the load, we assume that:

$$n(\theta) = n_0, \quad m(\theta) = m_A + \frac{(n_0\eta - p)}{2\eta^2}\theta^2, \tag{12}$$

where we have set $n_0 = -P/\sigma_c h$, $m_A = -M_A/\sigma_c h^2$, $p = -q/\sigma_c$ and $\eta = h/R$.

Since the arch is a statically determinate structure, the internal forces are known, and Eq. (12) enable establishing the mechanical response (linear elastic, nonlinear in tension, etc.) within each segment of the line of axis. Therefore, at each cross-section we can choose the suitable constitutive relation from those listed in set (6) and obtain the explicit expressions for the cross-section's rotation and displacement by integrating (2)–(4). To this end, we express rotations φ and displacements, u and v, as

$$\varphi(\theta) = \varphi_0 + F(\theta), \tag{13}$$

$$v(\theta) = (R\varphi_0 - u_0)\,\theta + v_0\left(1 - \frac{\theta^2}{2}\right) + G(\theta) - \theta\, H(\theta), \tag{14}$$

$$u(\theta) = u_0\left(1 - \frac{\theta^2}{2}\right) + v_0\theta + \frac{R\varphi_0\theta^2}{2} - \frac{\theta^2}{2}H(\theta) + \theta\, G(\theta) + L(\theta), \tag{15}$$

in which $\theta_0 = 0$ and

$$F(\theta) = -\int_0^\theta R\chi dt, \tag{16}$$

$$G(\theta) = \int_0^\theta t\left(R^2\chi + R\varepsilon\right) dt, \tag{17}$$

$$H(\theta) = \int_0^\theta \left(R^2\chi + R\varepsilon\right) dt, \tag{18}$$

$$L(\theta) = \int_0^\theta \left[R\varepsilon - \frac{t^2}{2}\left(R^2\chi + R\varepsilon\right)\right] dt. \tag{19}$$

The distributions of the curvature and axial strain, $\chi(\theta)$ and $\varepsilon(\theta)$, can be determined from that of the internal forces, $n(\theta)$ and $m(\theta)$, and (16)–(19) can be integrated, thus yielding the explicit expressions for $F(\theta)$, $G(\theta)$, $H(\theta)$ e $L(\theta)$. Such integrals are easily assessed by subdividing the line of axis into a finite number of segments, each characterized by a different mechanical response (linear elastic, nonlinear in tension, etc.). Lastly, constants u_0, v_0 and φ_0 are determined by imposing the boundary conditions at the arch end sections. Further details will be provided in a forthcoming paper.

As an example, let us now consider a 10 m span, 100 cm thick depressed arch. The line of axis is a circular arc with radius $R = 50$ m and central angle 2α of 0.2 rad. For the sake of simplicity, we assume that the arch is of unit width (1 m) in the transverse direction. A horizontal thrust of $P = 2{,}000$ kN and a bending moment $M_B = -100$ kNm are imposed on both the arch's end sections. A vertical load per unit length of the horizontal projection of the line of axis, $q = 120$ kN/m, is uniformly distributed throughout the arch (Fig. 15).

Lastly, we assume that $E = 7$ GPa is the masonry Young's modulus and that the masonry tensile and compressive strengths are equal to $\sigma_t = 0.3$ MPa

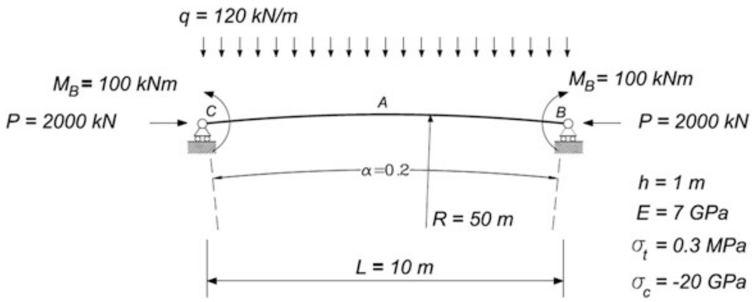

Fig. 15 Statically determinate example: mechanical scheme

Table 3 Dimensionless parameter values

Dimensionless parameters	
$\eta = h/R = 0.02$	$m_{\mathrm{A}} = -M_{\mathrm{A}}/\sigma_c h^2 = 0.045$
$t = \sigma_t/\sigma_c = -0.015$	$n_0 = -P/\sigma_c h = 0.1$
$\varepsilon_c = \sigma_c/\mathrm{E} = -2.9\times 10^{-3}$	$p = -q/\sigma_c = 0.006$
$\varepsilon_t = t\varepsilon_c = 4.4\times 10^{-5}$	

and $\sigma_c = –20$ MPa, respectively. The corresponding values for the dimensionless parameters appearing in relations (16)–(19) are listed in Table 3.

As before, due to symmetry we can limit the analysis to the arch's right side alone. Ideally, the arch's line of axis may be subdivided into two parts: the first, which starts at the keystone, is in a mechanical regime that is nonlinear under tension (region D+; see Fig. 4), while the response in the second part is linear elastic (region E). The angle θ_1 corresponding to the point separating the arch's D+ and E segments can be obtained via Eq. (12). By imposing that $m(\theta) = m_I$,we obtain $\theta_1 = 0.072$ rad. By suitably adapting Eqs. (13)–(15) to the present case, the following relations are easily obtained:

$$\varphi(\theta) = \begin{cases} F^{D+}(\theta), & 0 \le \theta \le \theta_1, \\ F^{E}(\theta) + F_1, & \theta_1 \le \theta \le \alpha, \end{cases} \tag{20}$$

$$v(\theta) = v_0\left(1 - \frac{\theta^2}{2}\right) + \begin{cases} G^{D+}(\theta) - \theta H^{D+}(\theta), & 0 \le \theta \le \theta_1, \\ G^{E}(\theta) + G_1 - \theta\left[H^{D+}(\theta) + H_1\right], & \theta_1 \le \theta \le \alpha, \end{cases} \tag{21}$$

$$u(\theta) = v_0\theta + \begin{cases} L^{D+}(\theta) - \dfrac{\theta^2 H^{D+}(\theta)}{2} + \theta G^{D+}(\theta), & 0 \le \theta \le \theta_1, \\ L^{E}(\theta) + L_1 - \dfrac{\theta^2\left[H^{E}(\theta) + H_1\right]}{2} + \theta\left[G^{E}(\theta) + G_1\right], & \theta_1 \le \theta \le \alpha, \end{cases} \tag{22}$$

in which

Table 4 Explicit expressions of the integrals appearing in Eqs. (20)–(22)

Region D+	
$a = \frac{n_0\eta - p}{\eta^2(t - n_0 + 2m_A)} = 400$	$b = -\frac{3(n_0\eta - p)}{2\eta^2((t-n_0)(\eta-1)+3m_A)} = 60.56$
$c = t - n_0 + 3m_A = 0.02$	$d = \frac{3(n_0\eta - p)}{2\eta^2} = -5$
$k_1 = -\frac{4\varepsilon_c(t-n_0)^3}{9\eta(t-n_0+2m_A)^2} = -0.155$	$k_2 = -\frac{8R\varepsilon_c(t-n_0)^2[(t-n_0)(\eta-1)+3m_A]}{9\eta(t-n_0+2m_A)^2} = 0.266$
$k_3 = -\frac{8R\varepsilon_c(t-n_0)^2}{9(t-n_0+2m_A)^2} = 2687$	
$F^{D+}(\theta) = k_1\left(\frac{\arctan\theta\sqrt{a}}{\sqrt{a}} + \frac{\theta}{1+a\theta^2}\right)$	$G^{D+}(\theta) = k_2\left(\frac{b}{2a^2}\ln\left(1+a\theta^2\right) + \frac{(a-b)\theta^2}{2a(1+a\theta^2)}\right) + Rt\varepsilon_c\frac{\theta^2}{2}$
$H^{D+}(\theta) = k_2\left(\frac{(a+b)\arctan\theta\sqrt{a}}{2a\sqrt{a}} + \frac{\theta(a-b)}{2a(1+a\theta^2)}\right) + Rt\varepsilon_c\theta$	
$L^{D+}(\theta) = \left(\frac{k_3(ac+d)}{2a^{3/2}} - \frac{k_2(a-3b)}{4a^{5/2}}\right)\arctan\theta\sqrt{a} + \left(\frac{k_3(ac-d)}{2a} - \frac{k_2(b-a)}{4a^2}\right)\frac{\theta}{1+a\theta^2} + $ $+ \left(Rt\varepsilon_c - \frac{k_2 b}{2a^2}\right)\theta - \frac{Rt\varepsilon_c}{6}\theta^3$	
Region E	
$i = -R\varepsilon_c\left(\frac{12m_A}{\eta} - n_0\right) = 3,843$ $j = -\frac{R\varepsilon_c(n_0\eta - p)}{\eta^3} = -71,429$	$k = R\varepsilon_c\left(\frac{2m_A}{\eta} - \frac{n_0}{6}\right) = -641$
$F^E(\theta) = \frac{12\varepsilon_c}{\eta}\left[m_A\theta + \frac{(n_0\eta - p)}{6\eta^2}\theta^3\right]$	$G^E(\theta) = \frac{i}{2}\theta^2 + \frac{3j}{2}\theta^4$
$H^E(\theta) = i\theta + 2j\theta^3$	$L^E(\theta)_A = R\varepsilon_c n_0\theta + k\theta^3 - \frac{3j}{5}\theta^5$

$$F_1 = F^{D+}(\theta_1) - F^E(\theta_1), \quad G_1 = G^{D+}(\theta_1) - G^E(\theta_1),$$
$$H_1 = H^{D+}(\theta_1) - H^E(\theta_1), \quad L_1 = L^{D+}(\theta_1) - L^E(\theta_1).$$

The analytical expressions appearing in Eqs. (20)–(22), all functions of angle θ, are listed in Table 4. The constant v_0 is determined by imposing

$$u(\alpha)\sin\alpha + v(\alpha)\cos\alpha = 0, \tag{23}$$

as the constraint condition at the springing, which yields $v_0 = 42.8$ mm.

The diagrams of the rotations and displacements of the line of axis are plotted in Fig. 16, together with the arch's deformed shape. In the case under examination it is a straightforward matter to verify that the nonlinear response predicted by the model illustrated here differs from that which would be obtained by assuming linear elastic behavior (Fig. 16a, b, d). In particular, the presence of an arch segment where the behavior is nonlinear modifies the arch's mechanical response so that the displacements and rotations are tripled with respect to the linear elastic solution. Thus, the simple nonlinear model proposed here is in agreement, at least qualitatively, with the experimentally reported arch response.

The evolution of the reactive forces and the distribution of the arch regions that behave non-linearly can be followed for increasing loads. Moreover, a conventional value of the limit load based on the magnitude of the displacement at the keystone

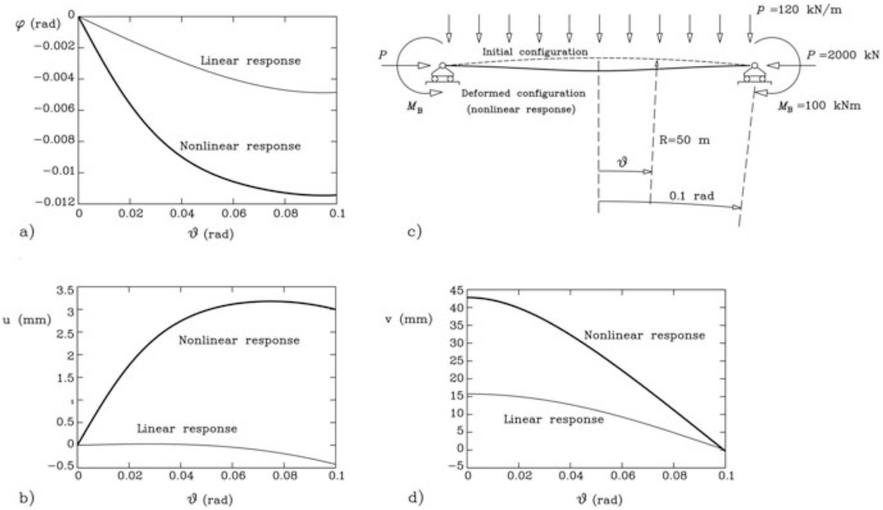

Fig. 16 Rotation (**a**) and axial and transverse displacements (**b**, **d**) of the right side of the line of axis; (**c**) deformed shape, displacement magnified ten times

can be proposed. The proposed solution method described here can be readily extended to statically indeterminate problems. Such an extension will be illustrated in a forthcoming paper by the authors.

4.3 Pointed, Circular and Elliptical Masonry Arches Bearing Vertical Walls

Let us consider a masonry arch subjected to its own weight and the weight of a superimposed wall (Fig. 17). Some interesting approximate solutions to this equilibrium problem can be obtained by making some simplifying assumptions and limiting the treatment to some cases deemed particularly significant. In particular, we assume that the superimposed wall can be divided into vertical strips, the weight of each strip being sustained directly by the underlying arch element. Such a hypothesis is commonly adopted in technical practice and can be considered conservative in terms of safety.

The first aspect we focus on concerns the influence of the masonry's limited compression strength on the arch's bearing capacity. The "classic" results of limit analysis under the Heyman hypotheses (1966) can in effect be easily compared with those obtained by applying the Durand-Claye method (1867), modified as described in Sect. 2 in order to account for a nonlinear stress distribution both in tension and compression. For the sake of simplicity, in this first application of the Durand-Claye method, we assume the arch shear strength to be unbounded. Since the arches

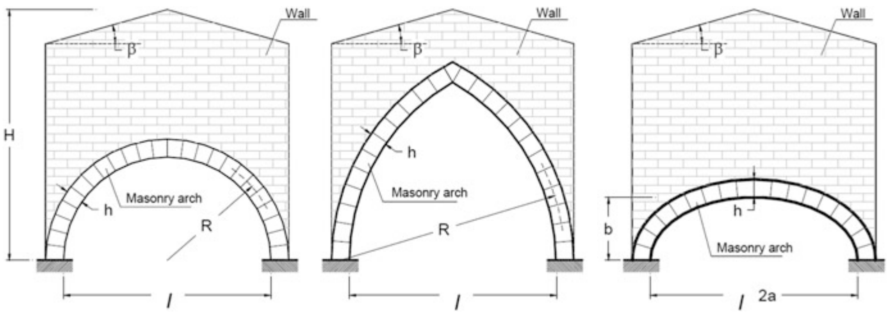

Fig. 17 Scheme of the arch-wall system for circular (*left*), pointed (*center*) and elliptical (*right*) arches

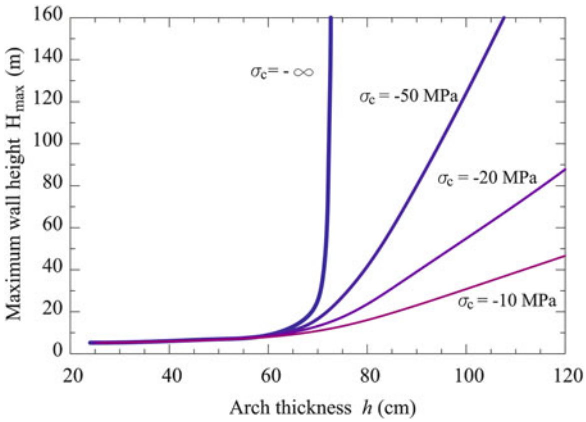

Fig. 18 Maximum height of the wall vs. arch thickness for different values of the masonry compressive strength (circular arch, $l = 10$ m)

considered here are sufficiently thin, we assume that the influence of shear forces on the solution may be disregarded.

The stability area method allows for assessing the maximum height, H_{max}, of the overlying vertical wall that a masonry arch, assumed here to be circular, could sustain under equilibrium conditions as a function of both the arch thickness, h, and the masonry compressive strength, σ_c. Figure 18 shows the results obtained in terms of H_{max} values as a function of the arch thickness for different constant σ_c values. The effect of the limited compressive strength is evident and, what is more, turns out to be clearly non-negligible even for values of arch thickness and masonry compressive strength that are likely to be encountered in historical masonry structures. As expected, by taking σ_c to be infinite, we revert to the well-known results obtained by Heyman through limit analysis. Analogous sets of results, omitted here for the sake of brevity, are obtained for pointed and elliptical arches as well.

The second aspect under examination concerns the mechanical response of the masonry arch to growing loads. The stability area method can then be applied to studying various arch-wall systems with different shapes. In all cases examined, some parameters were maintained constant: the clear span $l = 10$ m; the compressive strength $\sigma_c = -20$ MPa; the tensile strength $\sigma_t = 0$. These values have been chosen from among the many possible alternatives in order to represent masonry of medium strength (good brick or sandstone masonry).

Once the problem has been defined in terms of its geometrical and mechanical aspects, it is a relatively simple matter to determine the maximum value of the wall height measured from the springings, H_{max}, corresponding to a limit condition for the arch-wall system. The results illustrated in Figs. 19 and 20 have been obtained by means of an in-house, expressly developed algorithm implemented in Mathematica.

Figure 19 sums up the most significant results obtained through the analyses. It shows a plot of the maximum height of the overlying wall for semicircular, elliptical and pointed arches. It is noteworthy that for any arch thickness, h, pointed arches allow for much greater wall heights than circular and elliptical ones. From an architectural point of view, such a result finds evident corroboration in the characteristically great heights attained by slender Gothic structures, in which pointed arches are widely employed.

It is also rather interesting to note that for many values of h commonly used in construction, elliptical flat arches also afford higher load-bearing capacities than circular arches of the same thickness. Such outstanding performance of elliptical flat arches can easily be attributed to their shape, which, in the presence of compressive stresses compatible with the limited material strength, evidentially manages to maintain the line of thrust within the arch thickness for values of H_{max} that are decidedly higher than those attainable with circular and elliptical arches.

The analyses conducted via the stability area method were aimed at evaluating the maximum height of the wall beyond which no equilibrium is possible for the arch in its initial configuration, under the hypothesis of limited masonry compressive strength. It cannot however be excluded that, as the situation approaches

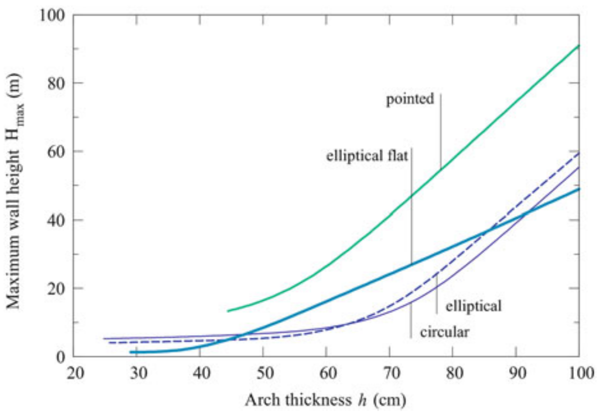

Fig. 19 Maximum height of the wall vs. arch thickness for circular, elliptical and pointed arches ($l = 10$ m, $\sigma_c = -20$ MPa)

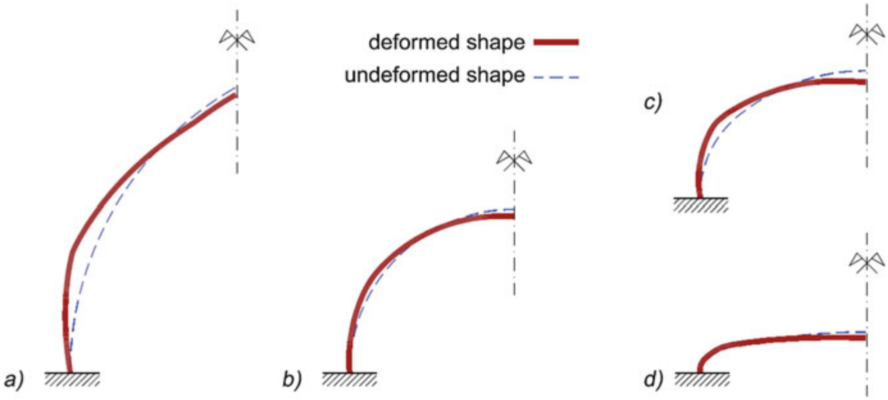

Fig. 20 Deformed configurations of the arches close to collapse: (**a**) pointed; (**b**) circular; (**c**) elliptical; (**d**) elliptical flat

impending collapse and the structure's stiffness falls significantly, displacements will grow steadily, often accompanied by the emergence of extensive cracking. Thus, the geometric and constitutive nonlinearities could considerably degrade the actual ultimate load. Accounting for such aspects requires estimates of both the displacement and strain fields.

The evolution of the displacement, strain and stress fields with increasing external loads has been followed by making use of the simple one-dimensional nonlinear elastic model described in Aita et al. (2012). The nonlinear elastic analysis has been applied to study the four arch-wall systems previously investigated via the stability area method. The same main parameter values as before have been adopted. The clear span l was maintained constant at 10 m, compressive strength σ_c at −20 MPa, and tensile strength σ_t at 0; lastly, a mean value of 4 GPa was chosen for Young's modulus, E. The maximum attainable value of the wall height, H_{max}, under equilibrium conditions, once again measured from the springings and clearly dependent on the arch thickness, h, is defined as the maximum wall height value for which the numerical procedure results to be convergent (that is, in correspondence to which the error associated with the last iteration falls below the pre-set threshold within the maximum permitted number of iterations, here set equal to 1,000).

In order to compare the results of the nonlinear elastic analysis with those obtained through application of the Durand-Claye method, the springings of all the arches studied are assumed to be perfectly clamped. We moreover exclude the presence of any possible initial self-equilibrated stress fields, such as those caused, for instance, by settling at the abutments. Without attempting to delve further into the associated issues, we simply recall here the widely known fact that any settling or initial self-equilibrated stress field would not in any event influence the value of the collapse load.

Figure 20 shows the deformed lines of axis of the four different arches corresponding to the maximum wall heights (because of symmetry, only half arches are

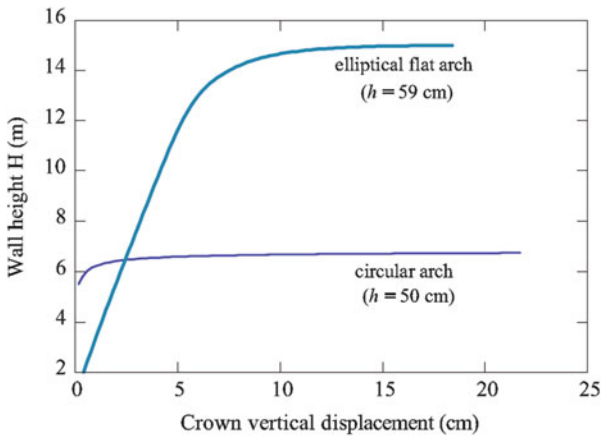

Fig. 21 Wall height vs. crown vertical displacement (circular and elliptical flat arches, $l = 10$ m, $\sigma_c = -20$ MPa)

shown). The displacements near collapse may be so great as to cast serious doubts on the judiciousness of the commonly adopted choice of writing the equilibrium equations in the undeformed configuration.

As a consequence, when seeking to accurately evaluate the residual stiffness of the arch as it approaches collapse conditions, the geometrical nonlinearities stemming from the large displacements should probably be taken into careful consideration.

As the height of the wall grows, the nonlinear regions, under both tension and compression, become larger and larger. This will lower the overall arch stiffness, which could thus become very small relative to its initial value. The decrease in the stiffness of the arch is evident in Fig. 21, where the vertical displacements measured at the crown section are plotted against the wall height values, as measured from the springings. In particular, a 50 cm thick circular arch and a 59 cm thick elliptic flat arch are considered. The rapidly increasing growth in the displacements as the wall height reaches its limit value is noteworthy.

The behavior of the elliptical flat arch exhibits a wide linear elastic range, consistent with expectations that, as the height of the wall increases, the distribution of the nonlinear regions in tension remains nearly unchanged, and nonlinear behavior would emerge under compression only when the normal compressive strains become very large. Moreover, when linear behavior ceases, the arch is still capable of sustaining considerable increases in the height of the overlying wall. Instead, the circular arch rapidly loses its linear behavior and the subsequent collapse is not heralded by any perceptible nonlinear increases in load. Very good agreement is observed between the limit load values determined through the nonlinear elastic analysis and the modified Durand-Claye method.

5 Concluding Remarks

We have addressed some equilibrium problems for masonry arches subject to vertical in-plane loads. The solutions are pursued by means of suitably simplified schemes developed within two different theoretical frameworks: more precisely, a parallel study of masonry arches is performed via both limit and nonlinear elastic analyses.

The first solution technique is based on an expressly developed extension of the historical method of stability areas introduced by Durand-Claye in 1867. Our aim in modifying the method is to determine the set of statically admissible solutions within the limits imposed by the ultimate compressive and tensile strengths and the limited shear capacity of the joints. The second solution technique, instead, focuses on the stress and strain fields generated in the arch. Masonry's mechanical behavior, usually characterized by very low resistance to tension, is represented, as a first approximation, via a nonlinear elastic constitutive relation incorporated into a one-dimensional model.

In the present work, the two different solution techniques have been applied to three problems. Each of the first two applications has focused on one of the two solution methods to highlight its main features. In particular, the first case describes an example application of the stability areas method to the collapse of pointed arches by accounting for masonry's limited shear strength. Among other things, the method has enabled determining the range of values of the arch thickness and the masonry friction coefficient that correspond to each different sliding collapse mode.

The second case demonstrates that nonlinear elastic analysis enables obtaining the explicit solution, in terms of displacements, stresses and extension of nonlinear regions, where masonry cracking and damage are to be expected. It regards some particular cases, such as that of depressed arches subjected to uniformly distributed vertical loads, and calls for making some reasonable approximations. However, the results obtained suggest that the search for the solution to more general cases could be much more demanding, and is therefore likely to necessitate recourse to suitable numerical techniques.

The last problem, concerning a mechanical system composed of an arch and overlying wall, serves to illustrate combined application of the two methods. The problem considered—actually quite a common one in masonry buildings and bridges—has been solved for arches of different shapes, namely: pointed, circular and elliptical. The results obtained confirmed the well-known fact that shape strongly affects the load capacity of a masonry arch and clearly indicate that pointed arches are expected to exhibit better performances than circular and elliptical ones.

The solved examples highlight the fact that the two methods perform complementary functions: the stability area method allows for readily determining a collapse load value, while the nonlinear elastic analysis provides a helpful and, in some aspects, essential check of its mechanical significance by following the evolution of the displacement field and extension of the nonlinear regions where cracking and crushing phenomena arise as the load increases.

In conclusion, the results reported in the present paper suggest that by simultaneously using the two solution methods it is possible, on the one hand, to readily determine the collapse load by accounting for both the joint's limited shear strength and masonry's limited compressive strength, and, on the other, to reconstruct a concise, yet often comprehensive, description of the evolution of the displacement and stress fields within an arch subject to in-plane increasing loads. In particular, the models presented here seem able to provide useful indications in all those cases in which the actual conditions or the conservation state of existing masonry arches should be assessed, or when planning restoration operations.

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Some Aspects on the Statics of Masonry Arches

Elio Sacco

Abstract The paper deals with the analysis of masonry arches. The differential equation defining the line of thrust is reviewed. Then, the equilibrium equations for the arch subjected to a distribution of point-wise forces acting on nodes lying on the line of thrust are written. The equilibrated force distributions for given curves of thrust are deduced and several numerical examples for circular and parabolic arches are presented. The problem of the determination of the line of thrust for prescribed arch geometry and loading distribution is approached by formulating a suitable nonlinear constrained minimization problem. The proposed numerical technique is used to derive the profile of the line of thrust for circular and parabolic arches subjected to uniform and non-uniform loading distributions. Then, the problem of the elastic arch is formulated and solved making use of the method of consistent deformations (the force method). The effects of the horizontal settlement of the impost of the arch, which induces a modification of the position of the line of thrust, and of the increasing horizontal distributed load, which can lead to the collapse of the arch, are investigated.

Keywords Masonry • Arch • Statics • Stress approach • Force method • Line of thrust • Method of consistent deformations

1 Introduction

Arches are extraordinary and essential elements of masonry constructions; their discovery and first use can be dated to more than 6000 years ago in Mesopotamia. It may be that arches were discovered on the basis of the observation of the nature, as many arches can be found in nature around the world.

Egyptians used arches to build tombs; Greeks adopted arches for simple constructions. The arch was used by Etruscans in several kind of monuments, such as

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the Porta Augusta, where the arch is combined with the Greek architectural style. Romans exported the arch and the ability to design and to build arches throughout the Mediterranean area; in fact, many classical round arches built by the Romans can be admired and are still used today in Europe and in the North Africa. The Romans used arches for all kinds of construction, from monumental buildings to more utilitarian structures, such as bridges and aqueducts. Several shapes for the arch have been developed and adopted during the centuries from the different cultures, such as the pointed arch in Gothic architecture or ogival arches in Islamic architecture.

Arches are still today fascinating structural elements, which are indeed included in concrete or steel structures, because of their static effectiveness and appreciable esthetical value. The structural behavior of the arch has been object of many researches and studies form engineers, architects and also mathematicians during the centuries. Many people were involved in the research on the statics and kinematics of masonry arches, and the history of the studies on the analysis of masonry arches can be found in several very interesting books and papers.

In 1981 Edoardo Benvenuto presented his treatise on the history of structural mechanics in Italian (Benvenuto 1981); in Chap. 9 he presented a complete, in-depth and detailed history of the scientific development of the stability of the arches, vaults and domes. He later published the text in English as well (Benvenuto 1991). Jacques Heyman (1998) presented the main results of the structural analysis following a historical description; in particular, in Chap. 5 he described the principal contributions on the scientific researches on masonry arches. The history of masonry constructions, with an interesting interplay between architecture and statics, can be found in Sparacio (1999), where several parts are dedicated to the statics of the arch. Recently, Mario Como wrote a noteworthy book (2010, 2013), where the basic theorems of the limit analysis for so-called no-tension material are reported and the models for the evaluation of stability for different structural elements of the masonry constructions are illustrated. Moreover, a history of the developments in the statics of masonry arches is illustrated. The book (Paradiso et al. 2007) illustrates the developments in the research of the statics and kinematics of arches and includes also a computer code for the analysis of masonry arches. A critical history of the researches on the statics of masonry arches, paying attention even to the development of the elastic theory can be found in Sinopoli et al. (1995, 1997), in Becchi and Foce (2002) and in Foce (2005). Santiago Huerta has presented a historic analysis of the researches on the stability of arches, from the "geometrical" approach of the old master builders to the "scientific" theory including the elastic and the limit analysis approaches (Huerta 2001). He concludes that the safety of a masonry structure is a matter of geometry and that the "equilibrium" is the best approach for the analysis and design of masonry structures. Huerta also discussed the research developed by Galileo on the stability of masonry buildings (Huerta 2006).

From the books and papers cited, it can be deduced that, regarding the main developments in the study of masonry arches, Vitruvius (late first century B.C.)

was perhaps the first to recognize the presence of the drift force for the arch; Leonardo da Vinci (fifteenth century) deeply understood the structural behaviour of the arch, stating that the arch consists of “two weaknesses [that] combine to form one strength” (Leonardo 2011, vol. II, p. 86). Robert Hooke (1635–1703) realized that the analysis of a masonry arch can be developed considering the system of overturn chain. In the seventeenth century Johann I Bernoulli, Gottfried Leibnitz and Christiaan Huygens provided a fundamental contribution to the derivation and solution of the catenary equation. In the eighteenth century Philippe de La Hire introduced the so-called funicular polygon, while Charles-Augustin de Coulomb investigated the collapse mechanisms of the arch taking into account even the shear failure. In the nineteenth century Henry Moseley proposed the “principle of least pressure”. Emil Winkler considered the elastic approach as the best option for the analysis of masonry arches, and also remarked the possible difficulties in the determination of the position of the line of thrust, which was to account for the deformation during the construction, the settlement of the buttresses and the effect of changes of temperature.

Masonry is a heterogeneous material obtained as an aggregate of stones, bricks or rubble in dry contact or joined by mortar. In old masonry constructions, the mortar, when it exists, is characterized by very weak tensile response; in fact, it can reasonably be assumed that the interaction between the different elements constituting the masonry is of compressive stresses. Moreover, the existing masonry structures are characterized by elements with large cross sections leading to very low values of the compressive stresses. Thus, the possibility of failure for crushing of compressed masonry elements is generally very remote and, hence, the compressive strength of the masonry is not of interest in the calculations. As a consequence, a simple and effective model for the masonry considers a constitutive law characterized by linear elastic stress–strain relationship in compression with no tensile strength. This model is often referred in the literature as a no-tension material (NTM). Jacques Heyman in his memorable works (1966, 1969) presented a history of the main scientific developments on the statics of masonry structures, and stated the main properties and features of the no-tension model. He then extended the classic theorems of plastic design, developed originally for steel frames, to the masonry constructions, on the basis of an initial idea proposed in Kooharian (1952). Then, Heyman (1966) applied the limit analysis theorems for evaluating the stability of Gothic cathedrals. Moreover, he proposed a “geometrical” factor of safety of masonry arches subjected also to travelling forces; the proposed approach has been applied to the analysis of two masonry bridges (Heyman 1969).

The no-tension material model has received and still receives great attention by many researchers (mainly in Italy) to study the behavior of old masonry structures. Indeed, several studies were developed regarding NTMs from a mechanical (Romano and Sacco 1984; Como and Grimaldi 1985; Del Piero 1989), mathematical (Giaquinta and Giusti 1988) and computational point of view, developing displacement (Sacco 1990; Lucchesi et al. 1994; Alfano et al. 2000), as well as stress and mixed variational formulations (Luciano and Sacco 1994; Cuomo and Ventura 2000). Computational procedures based on the kinematic theorem of the limit analysis applied to NTMs have been also developed, for instance in Cavicchi and Gambarotta (2005), which proposed a finite element approach for the study of masonry bridges taking into account arch-fill interaction.

The extension of the NTM model to the case of limited compressive strength is presented in Lucchesi et al. (1996), assuming a hyper-elastic stress–strain relationship. The case of inelastic strain in compression has been treated in Marfia and Sacco (2005).

This work presents two methodologies for the analysis of masonry arches. The first approach is derived considering the equilibrium equations for the arch subjected to a distribution of point-wise forces acting on the nodes lying on the line of thrust. The method is able to evaluate the equilibrated force distribution for a given line of thrust. Then, in the same framework, a nonlinear constrained minimization problem is formulated for determining the line of thrust for arches subjected to prescribed loading distributions. As second approach, the elasticity equations for the masonry arch modeled using the NTM constitutive relationship are written and solved making use of the method of consistent deformations (the force method).

Numerical applications concerning circular and parabolic arches subjected to uniform and non-uniform loading distributions and to possible horizontal settlements of the impost are investigated, remarking the possibility of the structural collapse.

2 The Catenary Equation

It is assumed that, in a masonry arch, only compressive stresses can be transmitted from bricks, that is, a unilateral effect is considered. The line of thrust is the imaginary curve through which the resultant thrust (the axial forces) acts in the masonry arch. When the masonry arch is subjected only to its own weight or to a uniformly distributed load, the line of thrust describes a very special curve which is known as a "catenary". The first idea about this special curve is due to Galileo, who stated that the catenary is a parabola. Joachim Jungius (1587–1657) proved that the catenary is not a parabola. Applications of the concept of the catenary to the construction of arches were due to Hooke; many scientists in the seventeenth and eighteenth centuries contributed in deriving and solving the equations governing the catenary, including Leibniz, Huygens and Johann I Bernoulli.

With reference to Fig. 1, the equilibrium equations for the typical infinitesimal part of the catenary can be written as:

$$\begin{aligned} &\tfrac{d}{ds}\,(N\ \cos\theta) = 0 \\ &\tfrac{d}{ds}\,(N\ \sin\theta) = f \end{aligned} \tag{1}$$

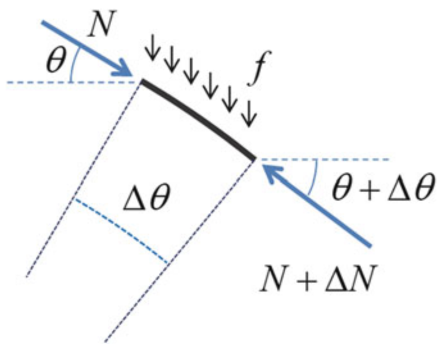


Fig. 1 Equilibrium of the typical infinitesimal part of the catenary

According to the Eq. (1)$_1$ the projection of the axial force in the horizontal direction is constant along the whole arch and it is, in fact, the drift force H (the horizontal reaction of the constraint), so that it is:

$$N = \frac{H}{\cos\theta} \tag{2}$$

which, substituted in Eq. (1)$_2$, gives:

$$f = \frac{d}{ds}(H \ \tan\theta) = -H\,\frac{dy'}{ds} = -H\,\frac{dy'}{dx}\frac{dx}{ds} = -H\,y''\frac{dx}{ds} \tag{3}$$

where $\tan\theta = -y'$, with y' and y'' the first and second derivatives of the function $y(x)$ with respect to x, respectively, according to the cartesian coordinate system(x,y) represented in Fig. 2. Recalling that:

$$ds = \sqrt{1 + (y')^2}dx \tag{4}$$

Eq. (3) leads to:

$$y'' + \frac{f}{H}\sqrt{1 + (y')^2} = 0 \tag{5}$$

which is the differential equation governing the catenary. The solution of Eq. (5) is:

$$y = -\frac{H}{f}\cosh\left(\frac{f}{H}x + c_1\right) + c_2 \tag{6}$$

where c_1 and c_2 are constants of integration. As a solution which is symmetric with respect to the y axis is desired, it is $c_1 = 0$, while it results $c_2 = (hf + H)/f$ with h the rise of the arch. Finally, the catenary is defined by the function:

$$y = \frac{H}{f}\left[1 - \cosh\left(\frac{f}{H}x\right)\right] + h. \tag{7}$$

The cases of non-uniform distributed vertical loading or of the presence of horizontal distributed loads are more complex, so that the analytical solution of the governing differential equations could not be determined. In such cases, different (simplified, approximated and numerical) approaches can be developed, such as the one proposed in what follows.

3 Equilibrium of a Masonry Arch Subjected to Point-Wise Forces

The arch is considered to be subject to a distribution of point-wise forces acting on the n nodes introduced on the line of thrust. In such a way, the arch is divided in $t = n - 1$ pieces. The position of the nodes of the line of thrust of the arch are determined adopting a polar coordinate system. Thus, the k-th node is defined by the angle θ_k and the radius R_k and (with reference to Fig. 2a) its coordinates are:

$$\begin{aligned} x_k &= R_k \sin \theta_k \\ y_k &= R_k \cos \theta_k \end{aligned} \tag{8}$$

The components of the force applied on the k-th node ($k = 2, \ \ldots, n - 1$), along the horizontal and vertical directions, are denoted by S_k and F_k, respectively. The forces S_k and F_k represent the resultant of the external horizontal and vertical distributed loads acting in the part of arch whose middle section corresponds to the k-th node.

Note that the first and the last node are subjected to reactive forces; in particular, the horizontal reactive forces H_1 and H_n are the drift forces acting on the arch. It can be remarked that when all the horizontal forces are zero (i.e., $S_k = 0$ with $k = 2$, $\ldots, n - 1$), the reactive forces H_1 and H_n are equal and opposite.

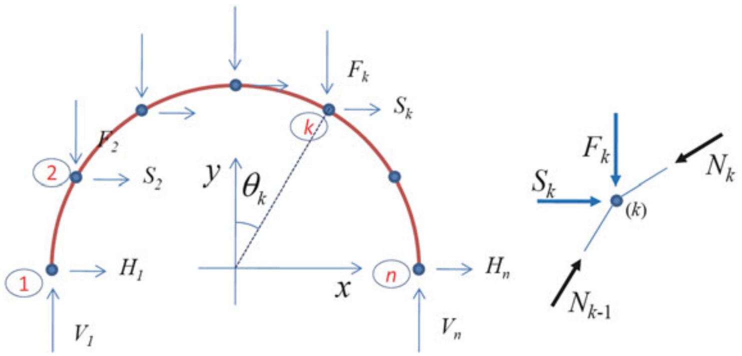


Fig. 2 (**a**, *left*) Line of thrust of the arch subjected to point-wise forces; (**b**, *right*) equilibrium of the axial force along the line of thrust

The equilibrium equation along the horizontal direction of the first node of the line of thrust of the arch is:

$$-N_1\alpha_1 + H_1 = 0 \tag{9}$$

while the equilibrium equations at the typical k-th node of the line of thrust of the arch (with the aid of Fig. 2b) are:

$$\begin{aligned} N_{k-1}\alpha_{k-1} - N_k\alpha_k + S_k &= 0 \\ N_{k-1}\beta_{k-1} - N_k\beta_k - F_k &= 0 \end{aligned} \qquad k = 2, 3, .., n-1 \tag{10}$$

where α_k and β_k are the components of the unit vector along the direction defined by the nodes k and $k+1$:

$$\alpha_k = \frac{x_{k+1} - x_k}{\Delta_k} \qquad \beta_k = \frac{y_{k+1} - y_k}{\Delta_k} \tag{11}$$

with Δ_k the distance between the nodes k and $k+1$.

Note that only axial forces are applied on the typical node of the line of thrust, so that only translation equilibrium equations are written. Thus, from a mechanical point of view, each node of the line of thrust can be considered as a structural hinge.

Considering the equilibrium conditions given by Eqs. (9) and (10), it is possible to write $2n-3$ equations, while the unknowns are the $n-1$ axial forces $N_1, N_2, \ldots, N_{n-1}$ and the drift force of the arch H_1 at the node 1; that is, there are more equations than unknowns. In fact, it is well known that the considered structural scheme is hypostatic, so that it might be unsolvable, depending on the values of the applied forces. Thus, for a given structural scheme it is possible to find a distribution of applied forces which makes resoluble the system of the nodal equilibrium equations. Hence, the following problem can be formulated: given a hypostatic structural system, find the distribution of applied forces, which satisfies the horizontal equilibrium equations of the nodes for 1 to $n-1$:

$$\mathbf{A}\ \mathbf{N} + \mathbf{S} = 0 \tag{12}$$

where it is set:

$$\mathbf{A} = \begin{bmatrix} -\alpha_1 & & & & & & \\ \alpha_1 & -\alpha_2 & & & & & \\ & \alpha_2 & -\alpha_3 & & & & \\ & & .. & .. & & & \\ & & & \alpha_k & -\alpha_{k+1} & & \\ & & & & \alpha_{k+1} & & \\ & & & & .. & .. & \\ & & & & & \alpha_{t-1} & -\alpha_t \end{bmatrix} \quad \mathbf{N} = \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \\ .. \\ N_k \\ N_{k+1} \\ .. \\ N_t \end{Bmatrix} \quad \mathbf{S} = \begin{Bmatrix} H_1 \\ S_2 \\ S_3 \\ .. \\ S_k \\ S_{k+1} \\ .. \\ S_{n-1} \end{Bmatrix} \tag{13}$$

The matrix **A**, with dimensions $t \times t$, is invertible, so that the vector of the axial forces **N** can be computed as:

$$\mathbf{N} = -\mathbf{B}\ \mathbf{S} \qquad \mathbf{B} = \mathbf{A}^{-1} \tag{14}$$

From formula (14) the k-th component of the vector **N** is obtained as:

$$N_k = \frac{1}{\alpha_k}\left[H_1 + \sum_{j=2}^{k} S_k\right] \tag{15}$$

Then, the equilibrium equations in the vertical direction of the nodes for 2 to $n-1$ can be written:

$$\mathbf{D}\ \mathbf{N} = \mathbf{F} \tag{16}$$

where:

$$\mathbf{D} = \begin{bmatrix} \beta_1 & -\beta_2 & & & & & \\ & \beta_2 & -\beta_3 & & & & \\ & & .. & .. & & & \\ & & & \beta_k & -\beta_{k+1} & & \\ & & & & \beta_{k+1} & & \\ & & & & .. & .. & \\ & & & & & \beta_{t-1} & -\beta_t \end{bmatrix} \qquad \mathbf{F} = \begin{Bmatrix} F_2 \\ F_3 \\ .. \\ F_k \\ F_{k+1} \\ .. \\ F_{n-1} \end{Bmatrix} \tag{17}$$

The matrix **D** is rectangular, with dimensions $(t\text{–}1) \times t$. When formula (14) is substituted into Eq. (16), the result is:

$$\mathbf{F} = -\mathbf{D}\ \mathbf{B}\ \mathbf{S} \tag{18}$$

so that the typical component of the force vector **F** can be written as:

$$F_k = \frac{\beta_{k-1}}{\alpha_{k-1}}\left(H_1 + \sum_{j=2}^{k-1} S_j\right) - \frac{\beta_k}{\alpha_k}\left(H_1 + \sum_{j=2}^{k} S_j\right) \qquad k = 2, 3, .., n-1 \tag{19}$$

Hence, assigning the value of the horizontal component of the external forces $S_2, S_3, \ldots, S_{n\text{–}1}$ and, moreover, assigning as well the value of the left drift force of the arch H_1, it is possible to determine the unique set of vertical components of the external force $F_2, F_3, \ldots, F_{n\text{–}1}$ which equilibrates the hypostatic structural system.

The formula (19) assumes a special form when the horizontal forces are taken proportional to the vertical forces (i.e., $S_k = \lambda F_k$ for $k = 2,3,\ldots,n-1$). In fact, in this case it results:

$$F_k = \frac{\alpha_k}{\alpha_k - \lambda\ \beta_k}\left[\lambda \sum_{j=2}^{k-1} F_j \left(\frac{\beta_k}{\alpha_k} - \frac{\beta_{k-1}}{\alpha_{k-1}}\right) - \alpha_1 H_1 \left(\frac{\beta_k}{\alpha_k} - \frac{\beta_{k-1}}{\alpha_{k-1}}\right)\right] \qquad k = 2, 3, .., n-1 \tag{20}$$

In particular, when $\lambda = 0$, formula (20) takes the following simplified form:

$$F_k = \alpha_1 H_1 \left(\frac{\beta_{k-1}}{\alpha_{k-1}} - \frac{\beta_k}{\alpha_k}\right) \qquad k = 2, 3, .., n-1 \tag{21}$$

4 Equilibrated Forces for a Four Hinges Arch

Initially, the very simple case illustrated in Fig. 3 is considered; it is a four-node (and hence four-hinge) arch characterized by the geometrical and loading data specified in Table 1.

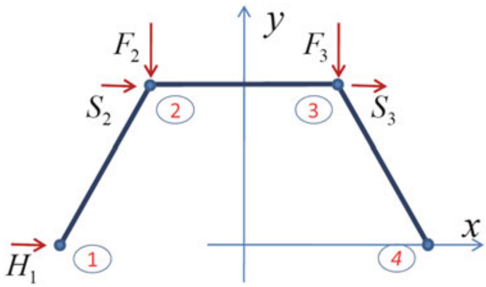


Fig. 3 Structural scheme of a four hinges arch

Table 1 Geometrical and loading data for the four hinges arch illustrated in Fig. 3

	R [mm]	θ	x [mm]	y [mm]	S [kN]	F [kN]
1	1,000	$-\pi/2$	$-1{,}000$	0		
2	1,000	$-\pi/6$	-500	$-500\sqrt{3}$	10	?
3	1,000	$\pi/6$	500	$500\sqrt{3}$	10	?
4	1,000	$\pi/2$	1,000	0		

The drift force at the node 1 is set $H_1 = 30$ kN. From Eqs. (13) and (14), the result is:

$$\mathbf{N} = \{60\ 40\ 100\}^T \tag{22}$$

and Eq. (16) gives the values of the vertical applied forces which equilibrate the hypostatic structure:

$$\mathbf{F} = \sqrt{3}\{30\ 50\}^T \tag{23}$$

Setting a different value for the drift force, the solution changes, that is, the internal axial forces and the external vertical forces assume different values. Thus, for instance, setting $H_1 = 60$ kN and leaving again $S_2 = S_3 = 10$ kN, the result is:

$$\mathbf{N} = \{120\ 70\ 160\}^T \qquad \mathbf{F} = \sqrt{3}\{60\ 80\}^T \tag{24}$$

5 Equilibrated Forces for a Circular Line of Thrust

The proposed approach can be adopted to determine the value of the vertical forces corresponding to a given profile of the thrust line. In fact, let a circular shape be considered for the thrust line, characterized by the initial angle $\theta_i = -4\pi/5$ and the final angle $\theta_f = 4\pi/5$. Calculations are performed assuming $\mathbf{S} = \mathbf{0}$ $H_1 = 1{,}000$ kN and setting the radius of the line of thrust R = 3,000 mm. The arch is analyzed considering the number of total nodes equal to $n = 41$, so that node 21 is located at the key section of the arch.

In Fig. 4 the line of thrust, with the vertical loading distribution which equilibrates a pure axial stress state in the arch, is reported. Variations of the drift

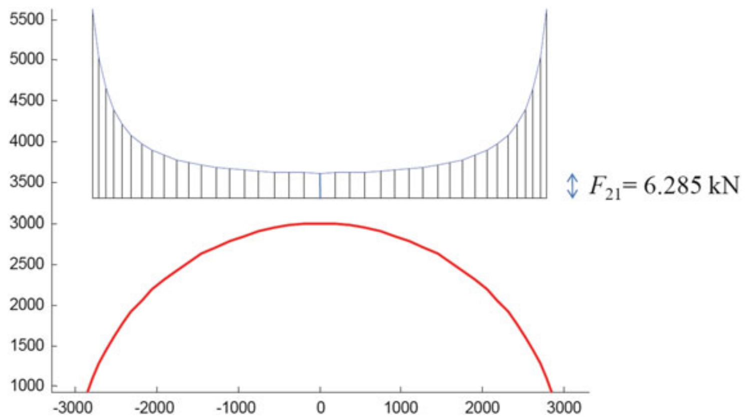


Fig. 4 Equilibrated vertical loading distribution on a circular line of thrust of the arch

horizontal force H_1 induce proportional variations of the values of the vertical distributed forces.

It is clear that when a round arch is considered, the equilibrated forces increase for increasing values of $|\theta|$. In particular, in the limit as $\theta_i \rightarrow -\pi/2$ and $\theta_f \rightarrow \pi/2$, the forces F_1 and F_n tend to infinity.

6 Equilibrated Forces for a Special Line of Thrust

Then, a very special shape is considered for the arch; in fact, the line of thrust is defined by the function:

$$y = a + h + \cosh\left[\frac{2\,x}{L}\cosh^{-1}\frac{a+h}{a}\right] \tag{25}$$

where $h = 2{,}000$ mm is the rise and $L = 5{,}000$ mm is the span of the arch. The line of thrust defined by Eq. (25) was determined by Charles Edward Inglis as the solution of a voussoir arch supporting a level roadway, with the fill supposed to be of uniform density and to exercise a purely vertical load upon the arch, with $a = 1{,}000$ mm the distance between the crown of the arch and the roadway.

In Fig. 5, the inverse solution of the Inglis problem is reported setting $n = 61$; it can be remarked that the loading distribution equilibrating the assigned line of thrust is in perfect agreement with the shape of the arch.

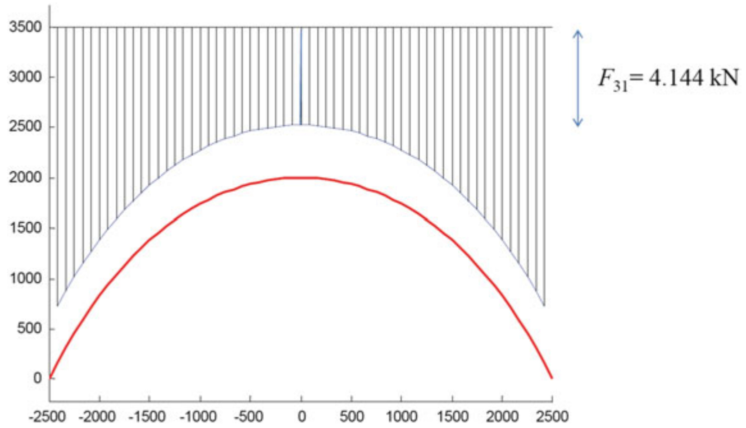


Fig. 5 Solution determined by Inglis for a voussoir arch supporting a level roadway

7 Equilibrated Forces for a Parabolic Line of Thrust

Next, a parabolic curve for the line of thrust is considered. The geometry is characterized by the rise h and the span L. The development of the arch is parameterized adopting a polar coordinate system, in which the radius is function of the angle θ according to the expression:

$$R = \frac{\sqrt{L^2(\cos\theta)^2 + 16\,h\,(y_0 + h)\,(\sin\theta)^2} - L\,\cos\theta}{8\,h\sin\theta} L \tag{26}$$

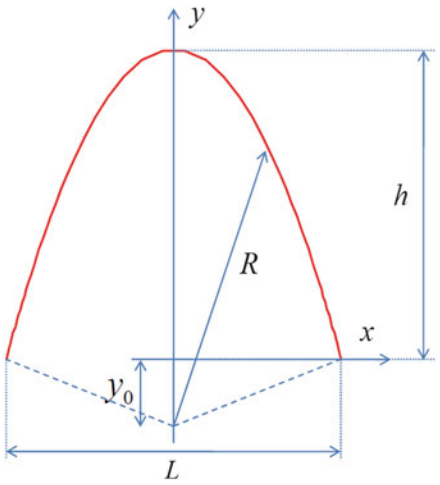
where y_0 is the vertical distance between the center of the initial radius and the base of the arch, as illustrated in Fig. 6.

Calculations are performed setting $H_1 = 1{,}000$ kN and considering the number of total nodes equal to $n = 62$.

In the structural scheme of the arch, with the vertical loading distribution which equilibrate a pure axial stress state in the line of thrust is reported for $L = 6{,}000$ mm and considering different values for the rise h. In Fig. 7, the distribution of the equilibrating forces to apply on the nodes is reported for all the cases in the same scale.

We can see in Fig. 7 that, as well known, when the rise of the arch is increased, the loading distribution, corresponding to given value of the drift force, also increases. In other words, a higher arch submitted to a fixed vertical load distribution present a lower value of the drift force.

Fig. 6 Geometrical scheme of a parabolic arch



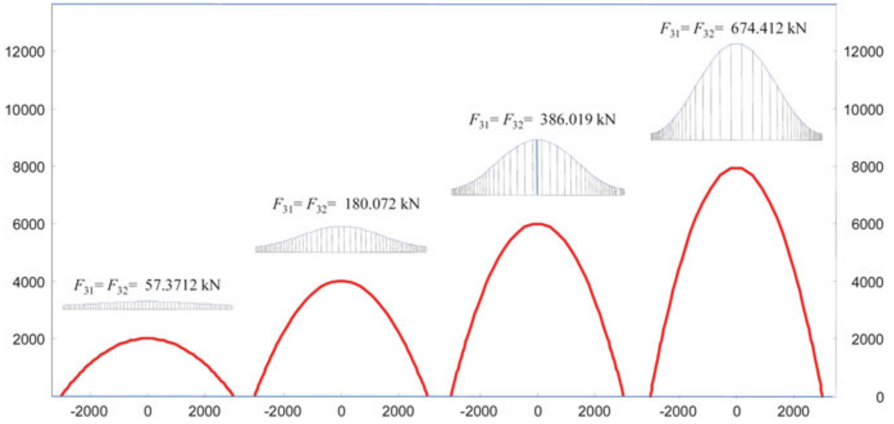


Fig. 7 Equilibrated vertical loading distribution on parabolic line of thrust of arches characterized by different rises: $h = 2{,}000$ mm, $h = 4{,}000$ mm, $h = 6{,}000$ mm and $h = 8{,}000$ mm

8 Determination of the Line of Thrust

The inverse problem, which clearly is more interesting, consists in the determination of the shape of the line of thrust into the arch, which is able to equilibrate a given loading distribution.

It can be remarked that, once the horizontal forces S_k ($k = 2, \ldots, n-1$) are assigned, because of the formula (19), the vertical forces F_k with $k = 2,3,\ldots,$ $n-1$ are function of the drift force H_1 and, moreover, depend on the value of the components of the unit vectors α_e, β_e ($e = 1,2,\ldots,n-1$). Taking into account the expressions (8) and (11), the components of the unit vectors are functions of the radii R_j with $j = 1,2,\ldots,n$, which define the line of thrust. Thus, it can be deduced that the values of the nodal forces are function of H_1 and of the radii R_j ($j = 1,2,\ldots,n$) (i.e., $F_k = F_k(H_1,R_1,R_2,\ldots,R_n)$ with $k = 2,\ldots,n-1$).

Considering a set of prescribed external vertical forces $\widehat{F}_k$ with $k = 2,\ldots,n-1$, the following conditions would be enforced:

$$F_k\,(H_1, R_1, R_2, .., R_n) = \widehat{F}_k \qquad k = 2, .., n-1 \tag{27}$$

Equations (27) are rewritten in the residual form as:

$$e_k\,(H_1, R_1, R_2, .., R_n) = \left(F_k\,(H_1, R_1, R_2, .., R_n) - \widehat{F}_k\right)^2 = 0 \qquad k = 2, .., n-1 \tag{28}$$

and the total residual is introduced:

$$\Gamma\,(H_1, R_1, R_2, .., R_n) = \sum_{k=2}^{n-1} e_k\,(H_1, R_1, R_2, .., R_n) = \sum_{k=2}^{n-1} \left(F_k\,(H_1, R_1, R_2, .., R_n) - \widehat{F}_k\right)^2 \tag{29}$$

As the arch has physical dimensions, admissibility conditions for the line of thrust have to be considered; in fact, the line of thrust cannot be outside of the arch, that is, it has to be almost contained in the thickness of the arch, defined in each cross-section by the internal and the external radius, respectively R_{int} and R_{ext}.

Finally, the problem of the determination of the shape of the line of thrust into the arch, which is able to equilibrate a set of external vertical forces $\widehat{F}_k$ with $k = 2,3, \ldots,$ $n-1$, can be formulated in the evaluation of the set of radii R_j with $j = 1,2, \ldots, n$, which satisfy the following set of equations:

$$\Gamma\left(H_1, R_1, R_2, .., R_n\right) = 0 \quad \Big| R_{int} \leq R_j \leq R_{ext} \qquad j = 1, 2, \ldots, n \tag{30}$$

Of course, problem (30) can have no solution and, because of the strong nonlinearity of the first of Eq. (30), depending on the distribution of the applied forces. Moreover, when the solution exists, it is not simple to determine. Thus, an approximated solution for the problem (30) can be evaluated, which can be obtained determining the set of radii R_j with $j = 1,2, \ldots, n$, and the value of H_1 which minimizes the expression of the residual (29), under the constraint that R_j ($j = 1,2, \ldots, n$) define points inside of the physical arch, that is, the following problem is recovered:

$$\min_{H_1, R_j} \left\{ \Gamma \Big| R_{int} \leq R_j \leq R_{ext} \;\; j = 1, 2, .., n \right\} \tag{31}$$

The problem (31) corresponds to a nonlinear optimization problem, which is not trivial to solve. It can be remarked that, contrarily to the problem (30), problem (31) always admits solution, as the residual $\Gamma(H_1, R_1, R_2, \ldots, R_n)$ is not enforced to be zero and, as consequence, Eq. (27) could not be exactly verified.

Furthermore, the minimization problem (31) cannot admit a unique solution, but it gives a relation between all the radii R_j with $j = 1,2, \ldots, n$ and H_1, which can be written in the form:

$$g\left(H_1, R_1, R_2, \ldots, R_n\right) = 0 \tag{32}$$

In order to define a unique position of the line of thrust inside the thickness of the arch, further requirements have to be introduced. A way to recover a unique position of the line of thrust can be obtained following the suggestion by Moseley, who introduced his (well-known) principle, which states that the line of thrust solution of the problem is determined in correspondence of the minimal pressure distribution. It is assumed that the minimal pressure distribution is attained when the drift force H_1 takes a minimum value. On the other hand, the choice to minimize the drift force is also motivated by the circumstance that if the arch is subjected to an abutment settlement, the drift force decreases, until reaching a minimum value.

The unique line of thrust is determined minimizing the value of the drift force H_1 which corresponds to the solution of the problem (31), that is, under the constraint (32):

$$\min_{H_1,R_j}\left\{(H_1)^2\middle|g\left(H_1,R_1,R_2,\ldots,R_n\right)=0\right\} \tag{33}$$

The problem (34) can be reformulated as a unique minimization statement:

$$\min_{H_1,R_j}\left\{\widehat{\Gamma}\left(H_1,R_1,R_2,..,R_n\right)\middle|R_{int}\leq R_j\leq R_{ext}\ \ j=1,2,..,n\right\} \tag{34}$$

with

$$\widehat{\Gamma}\left(H_1,R_1,R_2,..,R_n\right)=\Gamma\left(H_1,R_1,R_2,..,R_n\right)+\omega\ \ (H_1)^2 \tag{35}$$

where ω is a weight multiplier. In the following, the problem (34) is solved using a powerful routine based on the sequential quadratic programming (SQP) algorithm.

9 Masonry Frame

The ability of the proposed procedure to recover a geometrical position of the line of thrust and the values of the equilibrated forces applied on the structure is investigated considering a very simple example. It is the masonry frame represented in Fig. 8; it is obtained considering only four nodes for a round arch with internal and external radii $R_{int}=4{,}000$ mm and $R_{ext}=5{,}000$ mm, respectively. The arch is subjected to external forces applied at nodes 2 and 3.

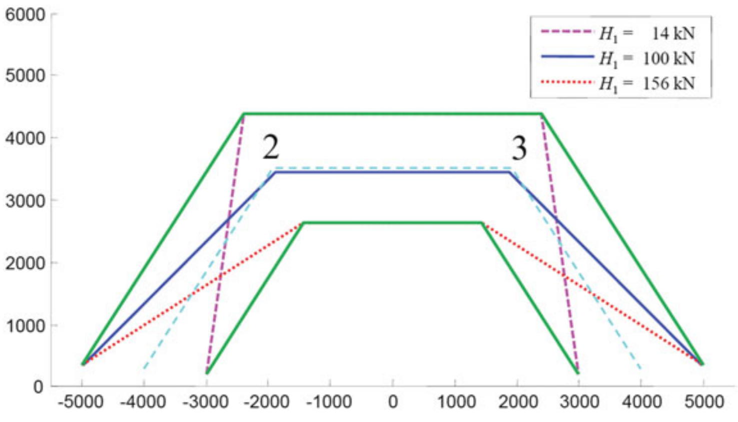


Fig. 8 Lines of thrust for a four node arch corresponding to three different values of the drift force (i.e., $H_1=14$ kN, $H_1=100$ kN, $H_1=155$ kN)

Table 2 Ratios ρ and dimensionless axial forces in the masonry frame

	Case I	Case II	Case III
ρ	0.07	0.50	0.78
N_1/V	0.51	0.71	0.92
N_2/V	0.07	0.50	0.78
N_3/V	0.51	0.71	0.93

Setting the values of the external forces as $\widehat{F}_2 = \widehat{F}_3 = 100$ kN and solving the problem defined by the minimization (31), three different solutions are determined, characterized by three geometries of the thrust line and three values of the drift forces.

In Fig. 8, the three lines of thrust corresponding to the three different values of the drift force are plotted. Three different values of the drift force correspond to the three thrust lines. A value of the drift force $H_1 = 14$ kN, $H_1 = 100$ kN and $H_1 = 155$ kN, corresponds to the line of thrust reported with dashed, continuous and dotted curve, respectively.

It can be noted that, in the first case, when $H_1 = 14$ kN, the line of thrust touches the extrados surface in the middle part of the frame and also the internal surface at the basis; thus, values of the drift force lower than $H_1 = 14$ kN are not allowed to guarantee the equilibrium with the external forces. Analogously, in the third case, when $H_1 = 155$ kN, the line of thrust touch the intrados surface of the frame; thus, values of the drift force greater than $H_1 = 155$ kN are not allowed. Summarizing, for $\widehat{F}_2 = \widehat{F}_3 = 100$ kN, the drift force can vary from 15 kN to 155 kN, depending on the position of the line of thrust.

In Table 2, the values of the ratios $\rho = H_1/V$, N_1/V, N_2/V and N_3/V, with $V = \widehat{F}_2 + \widehat{F}_3$, are reported for the three analyzed cases, with N_1, N_2 and N_3 the axial forces arising in the elements of the structure. It can be remarked that the case I can be considered the most effective for the arch, as it is characterized by values of the drift force and of the axial forces N_1, N_2 and N_3 which are the lowest possible with respect to the applied external forces $\widehat{F}_2, \widehat{F}_3$. In fact, the solution determined for the case I is characterized by the minimal value for the ratio ρ, that is, it corresponds to the minimal pressure solution.

The solution corresponding to $H_1 = 14$ kN can be obtained solving the problem (34). In fact, minimizing the residual, defined by Eq. (35), the solution for the considered example coincides with the line of thrust reported with dashed curve in Fig. 8.

10 Round Arch

The determination of the line of thrust of minimal pressure for a round masonry arch characterized by a given geometry is performed. The arch has the internal and the external radii $R_{int} = 4{,}000$ mm and $R_{ext} = 5{,}000$ mm, respectively. Calculations are performed considering the number of total nodes equal to $n = 29$.

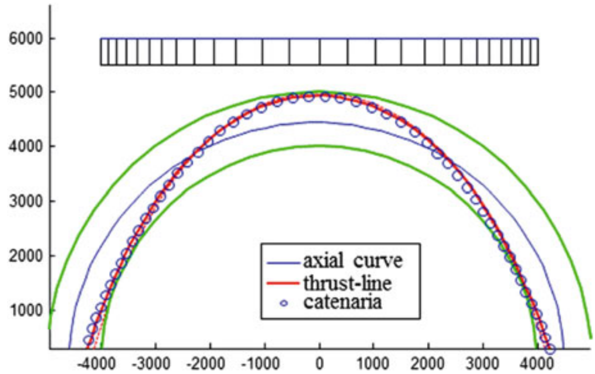


Fig. 9 Round arch subjected to uniform vertical force distribution: line of thrust (*continuous line*) and catenaria (*circles*)

Table 3 Values of the drift force for a round arch subjected to non-uniform loading conditions

η	1	0.75	0.50	0.25	0.18
H_1 [kN]	45,500	39,500	34,400	30,100	29,000

In Fig. 9, the line of thrust for the arch subjected to a uniform distribution of vertical forces is plotted, with $\widehat{F}_k = 10,000$ kN $(k = 2,3, \ldots, n-1)$. The line of thrust is determined solving the problem (34) and the corresponding value of the minimal drift force is about $H_1 = 45{,}500$ kN. Together with the line of thrust determined using the proposed numerical procedure, in the same Fig. 9 the catenary curve corresponding to the loading condition is reported. It can be remarked the perfect agreement of the two determined curves, showing the effectiveness of the simple proposed algorithm in the determination of the line of thrust.

Then, the case of non-uniform distribution of vertical forces acting on the arch is investigated. In particular, a piecewise uniform loading condition is considered for the arch; in fact, the span of the arch is divided in two parts: in the first part, where $x \leq 0$, a uniform distribution of vertical forces is considered, in the second part, where $x > 0$, the vertical forces are obtained multiplying the value of the forces applied for $x \leq 0$ by a factor η. Five cases are considered setting $\eta = 1.00$, $\eta = 0.75$, $\eta = 0.5$, $\eta = 0.25$ and $\eta = 0.18$. Note that the solution given in Fig. 9 corresponds to the case $\eta = 1.00$. The values of the drift force for different factors η are reported in Table 3.

In Fig. 10, the lines of thrust for the arch subjected to the non-uniform vertical forces distributions are reported. It can be noted that the line of thrust significantly changes for the different loading conditions; decreasing the ratio η the line of thrust becomes more and more unsymmetrical.

Lower values than $\eta = 0.18$ are not admissible as the line of thrust should be positioned outside of the physical arch. In fact, for $\eta = 0.18$ the curve of the pressure touches the surfaces of the arch in four points, where four hinges take place, two at the intrados and two at the extrados, leading to a hypostatic structural scheme, as illustrated in Fig. 10d. In other words the loading condition characterized by $\eta = 0.18$ represents a limit equilibrium state for the arch, that is, a collapse condition.

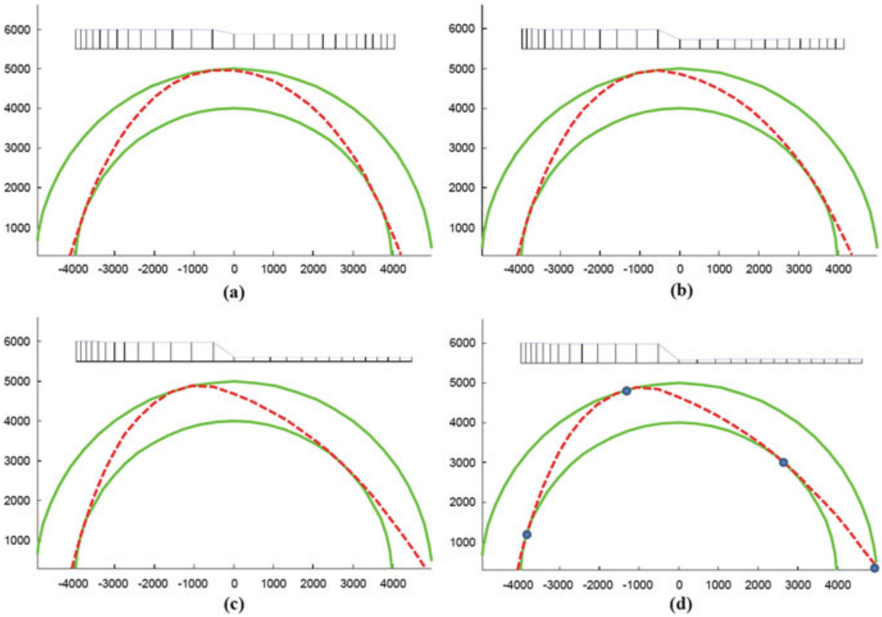


Fig. 10 Round arch subjected to non-uniform vertical forces distribution: (**a**) $\eta = 0.75$, (**b**) $\eta = 0.50$, (**c**) $\eta = 0.25$, (**d**) $\eta = 0.18$

11 Parabolic Arch

The determination of the line of thrust of minimal pressure for an arch characterized by a given geometry and subjected to an assigned loading distribution is performed. A parabolic masonry arch is considered, with internal and external span:

$$L_{int} = 4,600 \text{ mm} \quad L_{ext} = 6,000 \text{ mm} \tag{36}$$

respectively, while different values for the internal and external rise are assumed. In particular, four different parabolic arches are investigated, characterized by increasing values of the internal and external rises: case I, $h_{int} = 1{,}700$ mm and $h_{ext} = 2{,}000$ mm; case II, $h_{int} = 2{,}700$ mm and $h_{ext} = 3{,}000$ mm; case III, $h_{int} = 3{,}700$ mm and $h_{ext} = 4{,}000$ mm; case IV $h_{int} = 4{,}700$ mm and $h_{ext} = 5{,}000$ mm.

Computations are performed setting the number of total nodes $n = 30$. At each node of the arch a prescribed force $\widehat{F}_k = 10,000$ kN is considered. The values of the drift forces corresponding to the four cases are reported in Table 4.

In Fig. 11, the lines of thrust of minimal pressure, obtained solving problem (34), for the four parabolic arches subjected to uniform loading distributions are plotted; in Fig. 11, the values of the forces equilibrating the structure for the prescribed drift force are reported. It can be noted that, as expected, all the solutions are symmetric

Table 4 Values of the drift force for parabolic arches subjected to uniform loading condition

Case	I	II	III	IV
H_1 [kN]	65,650	37,500	24,000	16,600

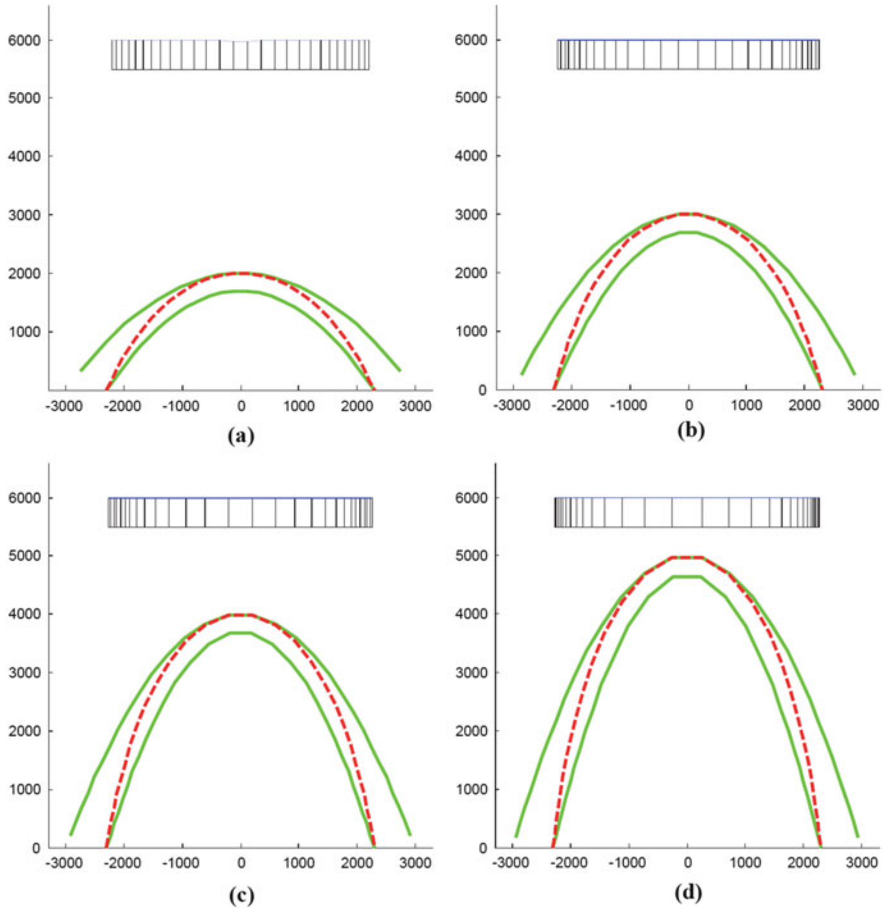


Fig. 11 Line of thrusts for parabolic arches with different rises: (**a**) $h_{int} = 1{,}700$ mm and $h_{ext} = 2{,}000$ mm, (**b**) $h_{int} = 2{,}700$ mm and $h_{ext} = 3{,}000$ mm, (**c**) $h_{int} = 3{,}700$ mm and $h_{ext} = 4{,}000$ mm, (**d**) $h_{int} = 4{,}700$ mm and $h_{ext} = 5{,}000$ mm

and the value of the equilibrated vertical forces increase with the rise of the arch. Moreover, the lines of thrust are very close to the internal surface in three sections, leading to almost statically determined schemes.

Then, the parabolic arch defined as case III is subjected to a piecewise uniform vertical load, as for $x > 0$ the vertical forces are obtained multiplying by 0.30 the value of the forces acting in the part of the arch defined by $x \leq 0$. The drift force in this case results to be equal to $H_1 = 19{,}760$ kN.

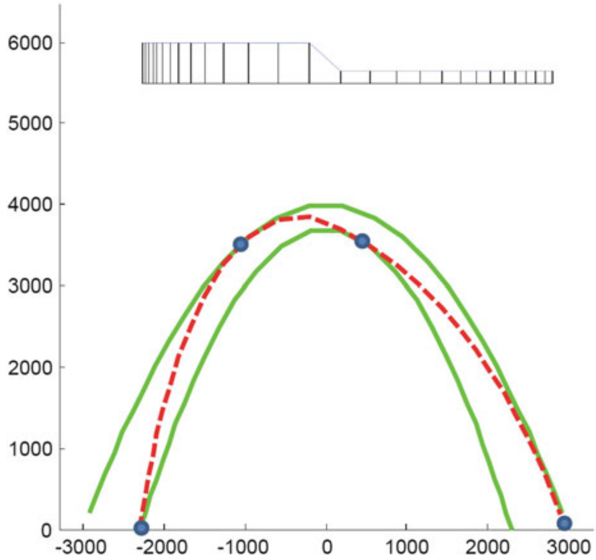


Fig. 12 Parabolic arch subjected to a nonuniform loading distribution

In Fig. 12, the geometry of the arch, the line of thrust and the forces distribution is illustrated. It can be remarked the unsymmetrical shape of the line of thrust, which touches the external surfaces of the arch in four sections, two at the extrados and two at the intrados, leading to a limit equilibrium state, that is, to a collapse mechanism.

12 Elastic Arch

The solution of the masonry arch can be derived by making use of the elasticity theory, that is, by writing the equilibrium, the constitutive and the compatibility equations.

Regarding the constitutive law, Heyman's no-tension model is introduced; it is based on the three well-known hypotheses concerning masonry behavior:

1. masonry has no tensile strength;
2. masonry has infinite compressive strength;
3. sliding failure cannot occur,

which lead to the following stress–strain relationship:

$$\sigma = h(\varepsilon) E\varepsilon \qquad \text{with } h(\varepsilon) = \begin{cases} 1 \text{ if } \varepsilon \leq 0 \\ 0 \text{ if } \varepsilon > 0 \end{cases} \tag{37}$$

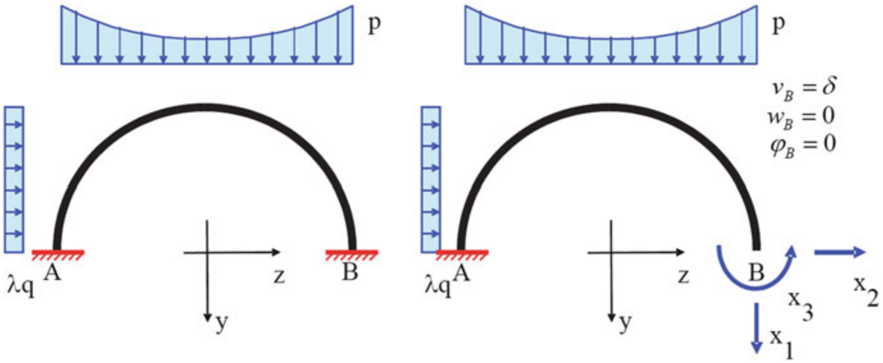


Fig. 13 Clamped arch and statically equivalent arch

with E the Young's modulus of the masonry. As a consequence of the constitutive equations (37) the typical cross section A of the arch is split in two parts: A_{nt}, the no-tension part of A, where $\varepsilon > 0$ and $\sigma = 0$; and A_e, the elastic part of A, where $\varepsilon \leq 0$ and $\sigma = E\varepsilon$. The parts A_{nt} and A_e are not known a priori, as they are defined by the position of the neutral axis, in correspondence of which the strain is equal to zero.

The structural problem can be solved adopting a displacement formulation within the finite element method as proposed in Lucchesi et al. (1997) and Cancelliere et al. (2010).

In the following, the structural problem is solved adopting the method of consistent deformations, sometimes referred to as the force or flexibility method (Marfia et al. 2008). In fact, with reference to Fig. 13, the statically undetermined clamped arch is transformed into a statically determined structure subjected to the unknown reactive forces x_1, x_2, and x_3. The unknown forces are computed, according to Müller-Breslau's principle, enforcing the compatibility conditions $v_B = \delta$, $w_B = 0$ and $\varphi_B = 0$, which state that, at the section B of the arch, the horizontal displacement is equal to the possible settlement δ, the vertical displacement is zero and rotation is zero.

The compatibility equations can be written in vector form as:

$$\mathbf{s}_p + \lambda \mathbf{s}_q + \sum_{i=1}^{h} x_i \mathbf{s}_i = \mathbf{0} \tag{38}$$

where $\mathbf{s}_p$, $\mathbf{s}_q$ and $\mathbf{s}_i$ are vectors of three components which assume the physical meaning of the displacements associated to the permanent and to the variable loadings $\mathbf{p}$ and $\mathbf{q}$, respectively, and to the stress resultants corresponding to the statically undetermined forces, when they assume the unit value. Moreover, λ represents the multiplier of the variable distributed load $\mathbf{q}$.

It can be remarked that, because of the considered elastic nonlinear constitutive law (37), the vectors $\mathbf{s}_p$, $\mathbf{s}_q$ and $\mathbf{s}_i$ depend on the solution, as they are functions of the partition of the cross section of the masonry into the no tension part and the elastic part. In fact, it is:

$$\mathbf{s}_p = \frac{1}{E}\int_{\theta_i}^{\theta_f} \mathbf{H}\,\widehat{\mathbf{c}}_p \cdot \widehat{\mathbf{c}}_j \, R \, d\theta \quad \mathbf{s}_q = \frac{1}{E}\int_{\theta_i}^{\theta_f} \mathbf{H}\widehat{\mathbf{c}}_q \cdot \widehat{\mathbf{c}}_j \, R \, d\theta \quad \mathbf{s}_i = \frac{1}{E}\int_{\theta_i}^{\theta_f} \mathbf{H}\widehat{\mathbf{c}}_i \cdot \widehat{\mathbf{c}}_j \, R \, d\theta \qquad (39)$$

with the 2×2 matrix $\mathbf{H}$ defined as:

$$\mathbf{H} = \begin{bmatrix} A_e & S_e \\ S_e & I_e \end{bmatrix}^{-1} \qquad (40)$$

where A_e, S_e and I_e are respectively the area, the moment of area and the moment of inertia of the elastic part of the cross section. The two components of the vectors $\hat{\mathbf{c}}_p$, $\hat{\mathbf{c}}_q$ and $\hat{\mathbf{c}}_j$ represent the axial force and the bending moment at a typical cross section of the arch, equilibrated with the external loads $\mathbf{p}$ and $\mathbf{q}$ and with the unknown reactive force x_j, respectively.

A numerical procedure able to solve the nonlinear problem (38) is developed considering the loading applied in several steps. In such a way, it is possible to evaluate the limit load for the structure, that is, the load multiplier λ that induces the collapse of the arch. In order to evaluate the whole nonlinear structural response of the arch and to compute the limit load, an arc-length technique, based on the stress formulation, is developed. Details on the computational procedure can be found in Marfia et al. (2008).

13 Arch Subjected to Vertical Load and Settlement

The elastic analysis of the arch considered in section 10 is performed, considering different values for the settlement δ. In Fig. 14 the position of the line of thrust is plotted for the arch subjected to an uniform distributed load and to the settlement of the right impost. It can be remarked that for higher values of δ the position of the line of thrust tends to the one determined in section 10, where the drift is minimized. In Fig. 14 the effective part, that is, the elastic part A_e, of the cross sections of the arch is shown.

14 Arch Subjected to Vertical and Horizontal Loads

We next consider the round arch defined by the external and the internal radii $R_{ext} = 5{,}500$ mm and $R_{int} = 4{,}500$ mm, respectively, and by the width $b = 300$ mm. It is subject to a vertical distributed load $p = -10$ N/mm and to a horizontal

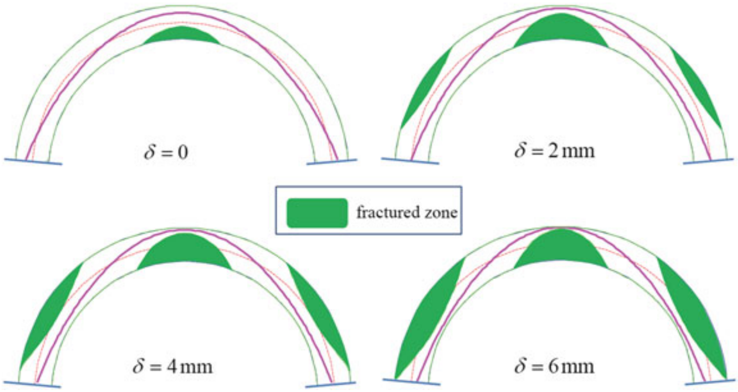


Fig. 14 Circular arch subjected to an uniformly distributed vertical load and to a settlement of the right impost

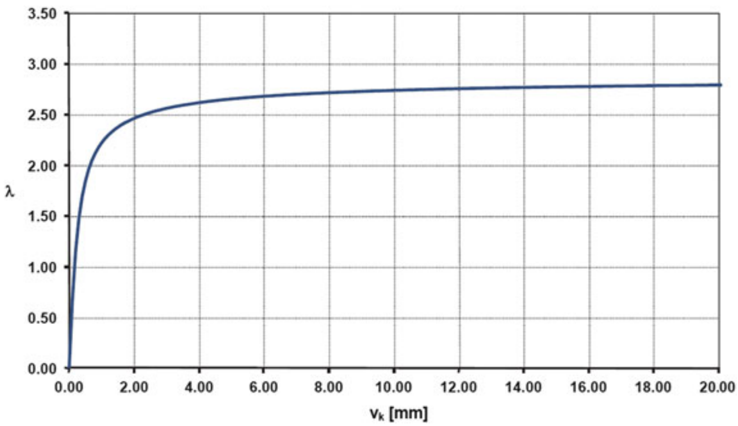


Fig. 15 Mechanical response of the arch in terms of horizontal displacement of the key section vs horizontal load multiplier

distributed load $q = 1$ N/mm amplified by the multiplier λ, simulating the effect of a seismic action. The Young's modulus is set $E = 15{,}000$ MPa. In Fig. 15, the value of the multiplier λ of the distributed horizontal load is plotted versus the horizontal displacement v_k computed at the key of the arch.

It can be remarked as the proposed computational procedure is able to determine the behavior of the arch under increasing values of the multiplier λ, to define the whole response during the loading history and to evaluate the limit load multiplier, for which the horizontal displacement of the key stone tends to infinity.

Finally, in Fig. 16 the line of thrust and the fractured zones are reported for the arch when it is close to the collapse condition.

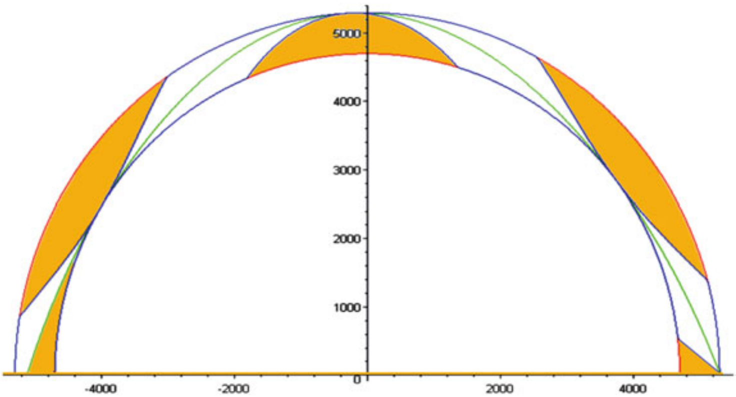


Fig. 16 Line of thrust and fractured zones of the arch subjected to vertical and horizontal loads in correspondence of the collapse

15 Conclusions

The arch is an essential structural element of the masonry constructions. The studies on the statics and kinematics of masonry arch involved many researchers over the centuries, leading to important advancements in all of structural mechanics.

Statics is able to provide important information about the stability of the arch. In fact, the classical equilibrium equations, written for a discrete number of cross sections of the arch, lead to algebraic equations which can be used to determine the distribution of external forces corresponding to a given line of thrust. Moreover, solving a nonlinear minimization problem makes it possible to define, if it exists, the line of thrust for an arch subjected to a prescribed distribution of forces. The several examples reported in the paper illustrate the ability of the simple numerical procedure, based only on equilibrium conditions, to evaluate the position of the line of thrust and, hence, the safety state of the arch.

The elastic solution of arches can be determined considering the no-tension model for the masonry and solving the structural scheme adopting the method of consistent deformations (the force method). The method is simple and, for a clamped-clamped arch, leads to a nonlinear algebraic system of three equations in the three unknown reactive forces. The nonlinear problem is solved using a Newton–Raphson method, conjugated with the arc-length procedure, developed for the stress formulation of the problem. The elasticity approach is used to investigate the effect of the horizontal settlement of an impost of the arch, which induces a modification of the position of the line of thrust. In particular, greater values of the settlement lead to a reduction of drift force; thus, increasing the horizontal settlement of the impost the minimal pressure line of thrust is recovered. Moreover, the proposed numerical procedure for the elastic arch is able to evaluate the limit load of the arch due to the presence of a dead load and to an increasing variable load.

Finally, although the arch is an ancient structural element and many studies have developed during the centuries, the research in this field is still active for at least two reasons. On one hand, the research is motivated by historical intentions, as it is aimed to better capture the insights, the ideas and the results of the greatest scientists of the past that have addressed the issue of stability and safety of the arches and vaults. On the other hand, the research is devoted to the use and development of innovative techniques of structural analysis for the evaluation of the safety of masonry constructions and for the design of interventions aimed at restoration and strengthening.

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A Direct Approach to Membrane Reinforced Bodies

Massimiliano Lucchesi, Miroslav Šilhavý, and Nicola Zani

Abstract The paper deals with membrane reinforced bodies with the membrane treated as a two dimensional surface with concentrated material properties. The membrane response is linearized so that it depends linearly on the surface strain tensor. The response of the matrix is treated separately in three cases: (a) as a nonlinear material, (b) as a linear material and finally (c) as a no-tension material. For the general nonlinear material, the principle of minimum energy and complementary energy are proved. For the linearly elastic matrix the surface Korn inequality is used to prove the existence of equilibrium state under general loads. Finally, for the no-tension material a theorem stating that the total energy of the system is bounded from below on the space of admissible displacements if and only if the loads are equilibrated by a statically admissible stress that is negative semidefinite. An example presenting an admissible stress solution is given for a rectangular panel with membrane occupying the main diagonal plane.

Keywords Equilibrium of forces • Membranes in the bulk matrix • No-tension body

1 Introduction

The present paper outlines an approach to membrane-reinforced bodies in which the membrane is treated as an ideal two dimensional surface with concentrated material properties. We refer to Lucchesi et al. (2013, 2014) for details.

We consider static situations. The system of forces is described by the bulk stress tensor describing the situation in the matrix and of the surface stress tensor describing the situation on the membranes. The interaction of the matrix with the membranes is implicitly contained in the virtual power principle which postulates

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effectively the equilibrium. A localization procedure leads to the strong form of the equilibrium equations which contain some new forms, see Eq. (3) (below).

The constitutive theory starts with large deformation approach. However, within the present approach, which is ultimately oriented on masonry (no-tension) materials, we soon move to the small strain for both the bulk and membrane response. Thus for the bulk we use the familiar small strain tensor but the response is assumed linear only in Sect. 4, where we treat membrane in a linearly elastic matrix. The rest deals with a nonlinear response of the matrix. As for the membrane, we linearize the response in the reference configuration to deal with the linearly elastic membrane; the surface strain tensor plays essential role.

In Sect. 3 we deal with a nonlinear matrix reinforced by linearly elastic membrane. We introduce the equilibrium states and state, for a material with the convex energy function, the principle of minimum potential energy at equilibrium. Under appropriate invertibility condition on the stress strain relation we prove the principle of minimum complementary energy.

In Sect. 4 we deal with the linearly elastic matrix with the linearly elastic reinforcement. The main goal is to prove the existence of the equilibrium solution for arbitrary loads. For this we need an appropriate version of the surface Korn inequality which we take from Ciarlet (2000, Theorem 2.7-1). The form of the inequality motivates the definition of the space of admissible displacements which is essentially $W^{1,2}$ for the bulk and $W^{1,2}$ for the tangential part of the displacement of $\mathcal{M}$ and which is L^2 for the normal component of the displacement. The existence of the equilibrium displacement is then proved in Lucchesi et al. (2014) by an application of the Lax Milgram lemma.

Section 5 deals with the matrix made of no-tension material. The existence theory is beyond the scope of the present paper. Instead, we treat simpler topics like the principle of minimum complementary energy which does not follow from the aforementioned proof as the stress strain relation is not invertible. The main result is the necessary and sufficient condition for the boundedness of the total energy from below: this occurs if and only if the loads can be balanced by a square integrable stressfield consisting of the negative semidefinite bulk stress and square integrable surface stress in the membrane. In Lucchesi et al. (2010) it is argued that the collapse of the masonry body occurs exactly at the point of the loading process at which the total energy ceases to be bounded from below. Here a simple considerations show that the reinforcement always improves (or at least does not worsen) the situation, the collapse of the body with the reinforcement occurs later in the loading process.

Throughout we use the conventions for vectors and second order tensors identical with those in Gurtin (1981). Thus Lin denotes the set of all second order tensors on $\mathbb{R}^3$, i.e., linear transformations from $\mathbb{R}^3$ into itself, Sym is the subspace of symmetric tensors, Skw is the subspace of skew (antisymmetric) tensors, Sym_+ the set of all positive semidefinite elements of Sym; additionally, Sym_- is the set of all negative semidefinite elements of Sym . The scalar product of $\mathbf{A}, \mathbf{B} \in \mathrm{Lin}$ is defined by $\mathbf{A} \cdot \mathbf{B} = \mathrm{tr}(\mathbf{A}\mathbf{B}^{\mathrm{T}})$ and $|\cdot|$ denotes the associated Euclidean norm on Lin . We denote by $\mathbf{1} \in \mathrm{Lin}$ the unit tensor. If $\mathbf{A}, \mathbf{B} \in \mathrm{Sym}$, we write $\mathbf{A} \leq \mathbf{B}$ if $\mathbf{B} - \mathbf{A} \in \mathrm{Sym}_+$.

2 Equilibrium of Forces

To describe the forces in the reinforced body, we consider the stress system which consists of the bulk stress in the matrix and the surface stress in the membrane. The equilibrium is postulated in the form of the virtual power principle which allows us to effectively introduce the force interactions between the matrix and the membrane. Next we postulate the constitutive equations of the given system. Here the main measures of deformation are the bulk and surface deformation gradients. We treat the basic properties of the constitutive equations like the principle of objectivity and the symmetry group. Then in the last section we introduce the linearization of the membrane response whose results will be used throughout the paper. The correct form of the linearization has been given in the paper by Gurtin and Murdoch (1975).

2.1 The System of Forces

We identify the body with its reference configuration Ω which is a bounded open subset of $\mathbb{R}^3$ with sufficiently smooth boundary $\partial\Omega$. We assume that within the bulk body Ω there is a collection of nonintersecting surfaces whose union we denote by $\mathcal{M}$, which represents the membranes in the body with different material properties. We denote by $\partial\mathcal{M}$ the collection of boundaries of the membranes and consider the general situation when part of $\partial\mathcal{M}$ is contained in $\partial\Omega$ and part in Ω itself. We denote the general material point in Ω by $\mathbf{x}$ and below we postulate different properties for $\mathbf{x} \in \Omega \setminus \mathcal{M}$ and for $\mathbf{x} \in \mathcal{M}$. We assume that $\mathcal{M}$ is a two dimensional manifold so that the tangent cone reduces to a two dimensional tangent space $\mathrm{Tan}(\mathcal{M}, \mathbf{x})$ for every point $\mathbf{x} \in \mathcal{M}$. We call a relative normal to $\partial\mathcal{M}$ at $\mathbf{a} \in \mathcal{M}$ the normal to $\partial\mathcal{M}$ which lies in the tangent space to $\mathcal{M}$ at $\mathbf{a}$. We use the same terminology for relative normals to the boundary of a subregion of $\mathcal{M}$.

A system of forces for the body with membranes consists of the bulk stress tensor $\mathbf{T}$, the bulk body force $\mathbf{b}$, the surface stress tensor $\mathbb{T}$ on membranes, and the surface body force $\mathbb{b}$. Here $\mathbf{T}$ and $\mathbf{b}$ are defined on Ω with values in the set Sym of symmetric second order tensors, and $\mathbb{R}^3$, respectively. For every $\mathbf{x} \in \mathcal{M}$, $\mathbb{T}(\mathbf{x})$ is an element of Sym which is superficial in that $\mathbb{T}\mathbb{n} = \mathbf{0}$ where $\mathbb{n}$ is the unit normal to $\mathcal{M}$ and finally $\mathbb{b}$ is defined on $\mathcal{M}$ with values in $\mathbb{R}^3$.

The system of forces is in internal equilibrium and in equilibrium with the environment if the principle of virtual power holds:

$$\int_\Omega \mathbf{T} \cdot \nabla \mathbf{v} - \mathbf{b} \cdot \mathbf{v} \, d\mathcal{L}^3 + \int_{\mathcal{M}} \mathbb{T} \cdot \nabla \mathbf{v} - \mathbb{b} \cdot \mathbf{v} \, d\mathcal{H}^1 = \int_{\mathcal{S}} \mathbf{s} \cdot \mathbf{v} \, d\mathcal{H}^2 + \int_{\partial\mathcal{M} \cap \mathcal{S}} \mathbb{s} \cdot \mathbf{v} \, d\mathcal{H}^1 \tag{1}$$

for every virtual velocity field $\mathbf{v}$ on $\mathrm{cl}\,\Omega$ such that $\mathbf{v} = \mathbf{0}$ on $\mathcal{D}$.

To state the strong form of the balance of forces, we introduce the following notation:

$$[\mathbf{T}]_{\mathcal{M}}\mathbb{m}(\mathbf{x}) := \lim_{\rho\to 0}\big(\mathbf{T}(\mathbf{x}+\rho\mathbb{m}(\mathbf{x}))\mathbb{m}(\mathbf{x}) - \mathbf{T}(\mathbf{x}-\rho\mathbb{m}(\mathbf{x}))\mathbb{m}(\mathbf{x})\big)$$

for each $\mathbf{x} \in \mathcal{M}$ and

$$\langle\mathbf{T}\rangle_{\mathcal{M}}(\mathbf{a}) := \lim_{\rho\to 0}\rho\int_{\{\mathbf{e}\in\mathrm{Norm}(\partial\mathcal{M},\mathbf{a}):\mathbf{e}\cdot\mathbb{t}(\mathbf{a})\geq 0,|\mathbf{e}|=1\}}\mathbf{T}(\mathbf{a}+\rho\mathbf{e})\mathbf{e}\,d\mathcal{H}^1(\mathbf{e}) \tag{2}$$

for each $\mathbf{a} \in \partial\mathcal{M}\cap\Omega$. The existence of the limits is an assumption. The integration range in (2) is the unit hemicircle normal to $\partial\mathcal{M}$ at $\mathbf{a}$ and the integration variable is $\mathbf{e}$. We obtain the following system of equilibrium equations:

$$\left.\begin{array}{ll} \mathrm{div}\,\mathbf{T}+\mathbf{b}=\mathbf{0} & \mathcal{L}^3 \text{ a.e. on } \Omega \\ \mathbb{div}\,\mathbb{T}+[\mathbf{T}]_{\mathcal{M}}\mathbb{m}+\mathbb{b}=\mathbf{0} & \mathcal{H}^2 \text{ a.e. on } \mathcal{M} \\ \mathbb{T}\mathbb{t}-\langle\mathbf{T}\rangle_{\mathcal{M}}=\mathbf{0} & \mathcal{H}^1 \text{ a.e. on } \partial\mathcal{M}\cap\Omega \\ \mathbf{T}\mathbf{n}=\mathbf{s} & \mathcal{H}^2 \text{ a.e. on } \mathcal{S} \\ \mathbb{T}\mathbb{t}=\mathbb{s} & \mathcal{H}^1 \text{ a.e. on } \partial\mathcal{M}\cap\mathcal{S} \end{array}\right\} \tag{3}$$

where $\mathbb{t}$ is the relative normal to $\partial\mathcal{M}$ and $\mathbf{n}$ is the normal to $\partial\Omega$.

2.2 Constitutive Equations

The bulk response is determined by the response functions

$$\hat{f}:\mathrm{Lin}_+\to\mathbb{R},\quad \hat{\mathbf{T}}:\mathrm{Lin}_+\to\mathrm{Lin} \tag{4}$$

giving the referential volume density of stored energy and stress in terms of the deformation gradient $\mathbf{F}$ where Lin_+ is the set of second order tensors with positive determinant. The surface response is determined by giving, for every $\mathbf{x} \in \mathcal{S}$, the response functions

$$\hat{\mathbb{f}}_{\mathbf{x}}:\mathrm{Lin}^{\mathbf{x}}_+\to\mathbb{R},\quad \hat{\mathbb{T}}_{\mathbf{x}}:\mathrm{Lin}^{\mathbf{x}}_+\to\mathrm{Lin}^{\mathbf{x}}, \tag{5}$$

delivering the referential surface density of stored energy and the surface stress in terms of the surface deformation gradient. Here $\mathrm{Lin}^{\mathbf{x}}$ is the set of all $\mathbf{A} \in \mathrm{Lin}$ such that $\mathbf{A}\mathbb{P}(\mathbf{x}) = \mathbf{A}$ where $\mathbb{P}(\mathbf{x})$ is the projection from $\mathbb{R}^3$ onto the tangent space of $\mathcal{M}$ at $\mathbf{x} \in \mathcal{M}$. We have attached the subscript $\mathbf{x}$ to the response functions since the domain of $\hat{\mathbb{f}}$, $\hat{\mathbb{T}}$ is different for every $\mathbf{x} \in \mathcal{S}$; however, for reasons of notational simplicity we often omit the subscript and write $\hat{\mathbb{f}}$ and $\hat{\mathbb{T}}$ in place of $\hat{\mathbb{f}}_{\mathbf{x}}$ and $\hat{\mathbb{T}}_{\mathbf{x}}$. The stress relations read (are postulated here)

$$\hat{\mathbf{T}} = \mathrm{D}\hat{f}, \quad \hat{\mathbb{T}}_{\mathbf{x}} = \mathrm{D}\hat{\mathbb{f}}_{\mathbf{x}}$$

where D denotes the differentiation of a function with respect to its argument, which is the deformation gradient and the surface deformation gradient, respectively, keeping $\mathbf{x}$ fixed. The constitutive equations then say that the stress corresponding to the deformation $\mathbf{y} : \Omega \to \mathbb{R}^3$ is given by

$$\left.\begin{aligned} \mathbf{T}(\mathbf{x}) &= \hat{\mathbf{T}}(\mathbf{F}(\mathbf{x})), \qquad \mathbf{x} \in \Omega \setminus \mathcal{S}, \\ \mathbb{T}(\mathbf{x}) &= \hat{\mathbb{T}}_{\mathbf{x}}(\mathbb{F}(\mathbf{x})), \qquad \mathbf{x} \in \mathcal{S}. \end{aligned}\right\} \tag{6}$$

Here $\mathbf{F}$ and $\mathbb{F}$ are the bulk and surface deformation gradients, given by

$$\begin{aligned} \mathbf{F} &= \nabla \mathbf{y}, \qquad \text{on } \Omega \setminus \mathcal{M}, \\ \mathbb{F} &= \nabla\!\!\!\!\nabla \mathbb{y} \qquad \text{on } \mathcal{M}, \end{aligned}$$

where $\mathbb{y}$ is the restriction of $\mathbf{y}$ to $\mathcal{M}$ and $\nabla\!\!\!\!\nabla$ is the surface gradient on $\mathcal{M}$, in the specific form defined in Šilhavý (2011, Appendix A and B). The definition makes the surface gradient at $\mathbf{x} \in \mathcal{M}$ of a function h defined on $\mathcal{M}$ and with values in a finite dimensional vectorspace V a linear transformation from $\mathbb{R}^3$ to V and not just the linear transformation from the tangent space $\mathrm{Tan}(\mathcal{M}, \mathbf{x})$ at $\mathbf{x}$ to V. This differs from the definition employed in Gurtin and Murdoch (1975), which is just the restriction of the present ∇h to $\mathrm{Tan}(\mathcal{M}, \mathbf{x})$. Thus in particular $\mathbb{F}(\mathbf{x})$ is an element of Lin, in fact of $\mathrm{Lin}^{\mathbf{x}}$, for any $\mathbf{x} \in \mathcal{M}$.

We define the bulk and surface energy f and $\mathbb{f}$ corresponding to the deformation $\mathbf{y} : \Omega \to \mathbb{R}^3$ by

$$\left.\begin{aligned} f(\mathbf{x}) &= \hat{f}(\mathbf{F}(\mathbf{x})), \qquad \mathbf{x} \in \Omega \setminus \mathcal{S}, \\ \mathbb{f}(\mathbf{x}) &= \hat{\mathbb{f}}_{\mathbf{x}}(\mathbb{F}(\mathbf{x})), \qquad \mathbf{x} \in \mathcal{S}. \end{aligned}\right\}$$

2.3 Linearization of the Membrane Response

Since we deal with small deformations here, we use the bulk and surface small strain tensors to be introduced here.

Define the displacement $\mathbf{u}$ corresponding to the deformation process $\mathbf{y}$ by $\mathbf{u}(\mathbf{x}) = \mathbf{y}(\mathbf{x}) - \mathbf{x}$ for $\mathbf{x} \in \Omega$, and let $\mathbb{u}$ be the restriction of $\mathbf{u}$ to $\mathcal{M}$. Define the bulk and surface small strain tensors by

$$\left.\begin{aligned} \mathbf{E} &= \tfrac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^{\mathrm{T}}) \qquad \text{in } \Omega \setminus \mathcal{M}, \\ \mathbb{E}_{\mathbf{x}} &= \tfrac{1}{2}\mathbb{P}(\nabla\!\!\!\!\nabla \mathbb{u} + \nabla\!\!\!\!\nabla \mathbb{u}^{\mathrm{T}})\mathbb{P} \qquad \text{on } \mathcal{M}; \end{aligned}\right\} \tag{7}$$

We have

$$\mathbf{E} = \tfrac{1}{2}(\mathbf{F} + \mathbf{F}^{\mathrm{T}} - 2\mathbf{1}), \quad \mathbb{E} = \tfrac{1}{2}\mathbb{P}(\mathbb{F} + \mathbb{F}^{\mathrm{T}} - 2\mathbb{P})\mathbb{P},$$

$$\mathbb{E} = \mathbb{P}\mathbf{E}\mathbb{P}.$$

Proposition 2.1 *Consider the response specified by the response functions* $\hat{f}, \hat{\mathfrak{f}}$ *as in* $(4)_1$ *and* $(5)_1$ *with the stress response given by* $(4)_2$ *and* $(5)_2$*; assume that the response is objective and that the reference configuration is stress free, i.e.,*

$$\hat{\mathbf{T}}(\mathbf{1}) = \mathbf{0} \quad \text{and} \quad \hat{\mathbb{T}}_{\mathbf{x}}(\mathbb{P}(\mathbf{x})) = \mathbf{0} \ \text{ for every } \mathbf{x} \in \mathcal{M}.$$

Define the bulk and surface elasticity tensors C *and* $\mathfrak{C}_{\mathrm{x}}$ *by*

$$\mathsf{C} = \mathrm{D}\,\hat{\mathbf{T}}(\mathbf{1}) = \mathrm{D}^2 \hat{f}(\mathbf{1}),$$

$$\mathfrak{C}_{\mathrm{x}} = \mathrm{D}\,\hat{\mathbb{T}}_{\mathbf{x}}(\mathbb{P}(\mathbf{x})) = \mathrm{D}^2\,\hat{\mathfrak{f}}_{\mathbf{x}}(\mathbb{P}(\mathbf{x})), \quad \mathbf{x} \in \mathcal{M}.$$

Let $\mathbf{x} \in \mathcal{M}$ *be fixed and write* $\mathfrak{C} \equiv \mathfrak{C}_{\mathrm{x}}$*. Then*

(i) C *and* $\mathfrak{C}$ *have major symmetry;*
(ii) C *and* $\mathfrak{C}$ *map* Lin *into* Sym *and* Skw *into* $\{\mathbf{0}\}$;
(iii) $\mathfrak{C}$ *is superficial in the sense that* $\mathfrak{C}[\mathbf{A}] = \mathfrak{C}[\mathbb{P}\mathbf{A}\mathbb{P}] = \mathbb{P}\mathfrak{C}[\mathbf{A}]\mathbb{P}$ *for each* $\mathbf{A} \in \mathrm{Lin}$.

Hence

$$\mathsf{C}[\mathbf{H}] = \mathsf{C}[\mathbf{E}], \quad \mathfrak{C}[\mathbb{H}] = \mathfrak{C}[\mathbb{E}]$$

for each $\mathbf{H}, \mathbb{H} \in \mathrm{Lin}$ *where*

$$\mathbf{E} = \tfrac{1}{2}(\mathbf{H} + \mathbf{H}^{\mathrm{T}}), \quad \mathbb{E} = \tfrac{1}{2}\mathbb{P}(\mathbb{H} + \mathbb{H}^{\mathrm{T}})\mathbb{P}.$$

We denoted the value of the linear transformations C and $\mathfrak{C}_{\mathrm{x}}$ on their respective arguments by $\mathsf{C}[\mathbf{E}]$ and $\mathfrak{C}_{\mathrm{x}}[\mathbb{E}]$ to emphasize that they are fourth order tensors. However, we often simplify the notation and write $\mathsf{C}\mathbf{E}$ and $\mathfrak{C}_{\mathrm{x}}\mathbb{E}$. We also often omit the subscript $\mathbf{x}$ and write $\mathfrak{C}$ for $\mathfrak{C}_{\mathrm{x}}$ for notational convenience.

Let $\mathrm{Sym}^{\mathrm{x}} = \mathrm{Sym} \cap \mathrm{Lin}^{\mathrm{x}}$. Assume that we are given for every $\mathbf{x} \in \mathcal{M}$ a fourth order tensor $\mathfrak{C}_{\mathrm{x}}$ such that Assertions (i)–(iii) of Proposition 2.1 hold. If $\mathbf{u} : \Omega \times \mathbb{R} \to \mathbb{R}^3$ is a displacement field, we define the superficial stress in the linear response by

$$\mathbb{T}(\mathbf{x}) = \mathfrak{C}_{\mathrm{x}}\mathbb{E}(\mathbf{x}) \quad \mathbf{x} \in \mathcal{M} \tag{8}$$

in place of $(6)_2$, where $\mathbb{E}$ is given by (7). In the same situation, we define the linearized free energy $\mathfrak{f}$ by

$$\mathfrak{f}(\mathbf{x}) = \tfrac{1}{2}\mathbb{E}(\mathbf{x}) \cdot \mathfrak{C}_{\mathbf{x}}\mathbb{E}(\mathbf{x}), \qquad \mathbf{x} \in \mathcal{M}. \tag{9}$$

If the membrane is isotropic then (8) read

$$\mathbb{T} = \mathfrak{l}(\operatorname{tr}\mathbb{E})\mathbb{P} + 2\mathfrak{m}\mathbb{E};$$

with $\mathfrak{l}$ and $\mathfrak{m}$ the Lamé coefficients.

About the bulk response we assume generally that the free energy is expressed as a function of the small strain tensor $\mathbf{E}$, by a possibly nonlinear function, i.e.,

$$f(\mathbf{x}) = \hat{f}(\hat{\mathbf{E}}(\mathbf{x})), \quad \mathbf{x} \in \Omega \setminus \mathcal{M}, \tag{10}$$

where $\hat{f}$ is a given response function and

$$\hat{\mathbf{E}}(\mathbf{u}) = \tfrac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{T}})$$

with

$$\mathbf{u}(\mathbf{x}) = \mathbf{y}(\mathbf{x}) - \mathbf{x}$$

the displacement. In Sect. 3 we assume that $\hat{f}$ is a general nonlinear (convex) function of quadratic growth. In Sect. 4 we shall assume that $\hat{f}$ is quadratic in $\mathbf{E}$, (linear elastic material), and in Sect. 5 (below) we shall deal with $\hat{f}$ the response function of a no-tension material, in which case $\hat{f}$ is genuinely nonlinear, but does not satisfy all the hypotheses of Sect. 3. The stress relation gives the symmetric stress $\hat{\mathbf{T}}$ as the derivative of $\hat{f}$,

$$\hat{\mathbf{T}} = \mathrm{D}\hat{f}.$$

This gives the constitutive equation

$$\mathbf{T} = \hat{\mathbf{T}}(\hat{\mathbf{E}}(\mathbf{u})). \tag{11}$$

3 Admissible Displacements: Energy

In this section we assume that the symmetric bulk stress is a generally nonlinear function of the small strain tensor with a convex store energy function of quadratic growth while the membrane response is linear as outline in Sect. 2.3. We define the space of admissible displacements in (12) below which is based on the Sobolev spaces of square integrable displacements with the square integrable gradients both in the bulk and on the membrane. We consider the total energy and under the invertibility assumption on the stress strain relation also the complementary energy. The minima of these corresponding to the equilibrium states are proved.

3.1 Basic Hypotheses: Admissible Displacements

Throughout the section V is a finite dimensional real vectorspace. Let $\mathcal{M}$ be a surface of dimension 2 in $\mathbb{R}^3$. Throughout this section we assume that $\mathcal{M}$ is of class ≥ 1. We denote by $W^{1,2}(\mathcal{M}, V)$ the set of all $\mathbf{u} : \mathcal{M} \to V$ such that both $|\mathbf{u}|$ and its derivative are square integrable on $\mathcal{M}$. Thus each $\mathbf{u} \in W^{1,2}(\mathcal{M}, V)$ has a well defined weak surface derivative $\nabla \mathbf{u} \in L^2(\mathcal{M}, \mathrm{Lin}(\mathbb{R}^3, V))$. We define a norm $|\cdot|_{W^{1,2}(\mathcal{M},V)}$ on $W^{1,2}(\mathcal{M}, V)$ by

$$|\mathbf{u}|^2_{W^{1,2}(\mathcal{M},V)} = |\mathbf{u}|^2_{L^2(\mathcal{M},V)} + |\nabla \mathbf{u}|^2_{L^2(\mathcal{M},\mathrm{Lin}(\mathbb{R}^3,V))}$$

for every $\mathbf{u} \in W^{1,2}(\mathcal{M}, V)$. It is easy to see that $W^{1,2}(\mathcal{M}, V)$ is a reflexive Banach space.

We assume that we are given the objects $\hat{f}$, $\mathbb{f}$, $\mathcal{D}$, $\mathcal{S}$, $\mathbf{b}$, $\mathbb{b}$, $\mathbf{s}$, and $\mathbb{s}$ as in the preceding sections. About these objects, and about the objects derived thereof we stipulate the following hypotheses:

H1 The function $\hat{f}$ is continuously differentiable and convex and the function $\hat{\mathbf{T}}$ satisfies the growth condition

$$|\hat{\mathbf{T}}(\mathbf{E})| \leq c(1 + |\mathbf{E}|)$$

for each $\mathbf{E} \in \mathrm{Sym}$ and some c.

H2 For $\mathcal{H}^2$ a.e. $\mathbf{x} \in \mathcal{M}$ the tensor $\mathfrak{C}_\mathbf{x}$ satisfies

$$\mathfrak{C}_\mathbf{x}\mathbf{E} = \mathfrak{C}_\mathbf{x}(\mathbb{P}(\mathbf{x})\mathbf{E}\mathbb{P}(\mathbf{x}))$$

for every $\mathbf{E} \in \mathrm{Sym}$,

$$\mathbb{E}_1 \cdot \mathfrak{C}_\mathbf{x}\mathbb{E}_2 = \mathfrak{C}_\mathbf{x}\mathbb{E}_1 \cdot \mathbb{E}_2$$

for every $\mathbb{E}_1, \mathbb{E}_2 \in \mathrm{Sym}$ such that

$$\mathbb{E}_i = \mathbb{P}(\mathbf{x})\mathbb{E}_i\mathbb{P}(\mathbf{x}), \quad i = 1, 2,$$

and

$$d|\mathbb{E}|^2 \geq \mathbb{E} \cdot \mathfrak{C}_\mathbf{x}\mathbb{E} \geq c|\mathbb{E}|^2$$

for each $\mathbb{E} \in \mathrm{Sym}$ such that

$$\mathbb{E} = \mathbb{P}(\mathbf{x})\mathbb{E}\mathbb{P}(\mathbf{x}),$$

where $c > 0$, and d are constants independent of $\mathbf{x}$ and $\mathbb{E}$.

H3 The map $\mathbf{x} \mapsto \mathfrak{C}_{\mathbf{x}}$ is $\mathcal{H}^2$ measurable on $\mathcal{M}$.

H4 We have $\mathbf{b} \in L^2(\Omega, \mathbb{R}^3)$, $\mathbb{b} \in L^2(\mathcal{M}, \mathbb{R}^3)$, $\mathbf{s} \in L^2(\mathcal{S}, \mathbb{R}^3)$, $\mathbb{s} \in L^2(\mathcal{S} \cap \partial\mathcal{M}, \mathbb{R}^3)$. Moreover, $\mathbb{P}(\mathbf{x})\mathbb{s}(\mathbf{x}) = \mathbb{s}(\mathbf{x})$ for $\mathcal{H}^1$ a.e. $\mathbf{x} \in \mathcal{S} \cap \partial\mathcal{M}$.

H5 $\mathcal{H}^2(\mathcal{D}) > 0$.

H6 $\mathcal{M}$ is an oriented admissible surface with Lipschitz boundary; moreover, $\mathcal{M}$ is a union of finitely many connected components whose closures are pairwise disjoint.

The last condition in H6 is a mild condition guaranteeing that every $\mathbf{u} \in W^{1,2}(\Omega, \mathbb{R}^3)$ has a well defined trace $\tau\mathbf{u}$ on $\mathcal{M}$ which is an element of $L^2(\mathcal{M}, \mathbb{R}^3)$. We often write $\mathbf{u}$ for $\tau\mathbf{u}$ when there is no danger of confusion. We put

$$H := \{\mathbf{u} \in W^{1,2}(\Omega, \mathbb{R}^3) : \mathbb{u} := \tau\mathbf{u},\ \mathbb{u}^{\|} := \mathbb{P} \quad \in W^{1,2}(\mathcal{M}, \mathbb{R}^3)\} \tag{12}$$

and

$$|\mathbf{u}|_H = |\mathbf{u}|_{W^{1,2}(\Omega, \mathbb{R}^3)} + |\mathbb{P} \quad |_{W^{1,2}(\mathcal{M}, \mathbb{R}^3)}.$$

It is easy to see that under $|\cdot|_H$, the space H is a Hilbert space. We furthermore put

$$U = \{\mathbf{u} \in H : \mathbf{u} = \mathbf{0} \text{ on } \mathcal{D}\}.$$

We call the elements $\mathbf{u} \in U$ the kinematically admissible displacements.

3.2 Energy and Complementary Energy

We define the total energy of the displacement $\mathbf{u} \in U$ by

$$\mathsf{F}(\mathbf{u}) = \mathsf{E}(\mathbf{u}) - \mathsf{W}(\mathbf{u}), \tag{13}$$

where E is the internal energy, given by

$$\mathsf{E}(\mathbf{u}) := \int_\Omega f \, d\mathcal{L}^3 + \int_{\mathcal{M}} \mathbb{f} \, d\mathcal{H}^2 \tag{14}$$

where f and $\mathbb{f}$ are given by the constitutive equations (10) and (9). Furthermore, the potential energy W of the loads is

$$\mathsf{W}(\mathbf{u}) := \int_{\Omega} \mathbf{b} \cdot \mathbf{u} \, d\mathcal{L}^3 + \int_{\mathcal{M}} \mathbb{b} \cdot \mathbb{u} \, d\mathcal{H}^2 + \int_{\mathcal{S}} \mathbf{s} \cdot \mathbf{u} \, d\mathcal{H}^2 + \int_{\mathcal{S} \cap \partial \mathcal{M}} \mathbb{s} \cdot \beta \mathbb{u}^{\parallel} \, d\mathcal{H}^1 \quad (15)$$

where it will be recalled that $\mathbb{u}^{\parallel} = \mathbb{P}$ and where we denote by $\beta \mathbb{u}^{\parallel}$ the trace of $\mathbb{u}^{\parallel}$ on $\partial \mathcal{M}$, see Lucchesi et al. (2013, Appendix C). We here note that $\mathbb{u}^{\parallel}$ is an element of $W^{1,2}(\mathcal{M}, \mathbb{R}^3)$, whereas $\mathbb{u}$ is generally not in $W^{1,2}(\mathcal{M}, \mathbb{R}^3)$. This is reflected in the definitions of the spaces H and U.

For the given objects as above the equilibrium boundary value problem seeks a displacement $\mathbf{u} \in U$ which satisfies the constitutive equations (11) and (8) and the equilibrium equations (3) in the weak form (1).

Proposition 3.1 *Assume H1–H6. The displacement* $\mathbf{u} \in U$ *solves the equilibrium problem (1) if and only if the first variation of the total energy* F *vanishes. Moreover, any solution* $\mathbf{u}$ *of the equilibrium problem is also a minimizer of the total energy among all kinematically admissible displacements, i.e.,*

$$\mathsf{F}(\mathbf{v}) \geq \mathsf{F}(\mathbf{u})$$

for all $\mathbf{v}$ *satisfying* $\mathbf{v} = \mathbf{0}$ *on* $\mathcal{D}$.

This is standard in the absence of the inner membrane. The case with the membrane is treated in a way similar to Gurtin and Murdoch (1975, Proof of Theorem 9.4).

Assume further the following.

H7 The equation

$$\hat{\mathbf{T}}(\mathbf{E}) = \mathbf{T}$$

has exactly one solution $\mathbf{E}$ for every $\mathbf{T} \in \mathrm{Sym}$.

We let $\tilde{\mathbf{E}} : \mathrm{Sym} \to \mathrm{Sym}$ denote the inverse of $\hat{\mathbf{T}}$, i.e., $\tilde{\mathbf{E}}$ satisfies

$$\hat{\mathbf{T}}(\tilde{\mathbf{E}}(\mathbf{T})) = \mathbf{T}$$

for every $\mathbf{T} \in \mathrm{Sym}$. Define the density of the complementary energy $\tilde{f}$ as the Legendre transformation of $\hat{f}$, i.e., by

$$\tilde{f}(\mathbf{T}) = \tilde{\mathbf{E}}(\mathbf{T}) \cdot \mathbf{T} - \hat{f}(\tilde{\mathbf{E}}(\mathbf{T}))$$

for any $\mathbf{T} \in \mathrm{Sym}$. Then under H1 the energy $\tilde{f}$ is a convex function of $\mathbf{T}$ and

$$\tilde{\mathbf{E}} = D\tilde{f}.$$

Assuming H2, we denote, for every $\mathbf{x} \in \mathcal{M}$, by $\mathfrak{C}_{\mathbf{x}}^{-1}$: Sym $\to$ Sym the pseudoinverse of $\mathfrak{C}_{\mathbf{x}}$, i.e., a linear transformation such that (a) $\mathfrak{C}[\mathbb{H}] = \mathbb{P}\mathfrak{C}[\mathbb{P}\quad]$ for every $\mathbf{H} \in$ Sym and (b)

$$\mathfrak{C}_{\mathbf{x}}^{-1}[\mathfrak{C}_{\mathbf{x}}[\mathbb{E}]] = \mathbb{E}$$

for all $\mathbb{E} \in$ Sym such that $\mathbb{P}(\mathbf{x})\mathbb{E}\mathbb{P}(\mathbf{x}) = \mathbb{E}$.

Let $(\mathbf{T}, \mathbb{T})$ be a pair of functions with $\mathbf{T} \in L^2(\Omega, \mathrm{Sym})$ and $\mathbb{T} \in L^2(\mathcal{M}, \mathrm{Sym})$ such that $\mathbb{T}\mathbb{n} = \mathbf{0}$ everywhere on $\mathcal{M}$. We say that $(\mathbf{T}, \mathbb{T})$ is a statically admissible stressfield if it satisfies (1) for every virtual velocity field $\mathbf{v} \in U$. For every statically admissible stressfield $(\mathbf{T}, \mathbb{T})$, define the complementary energy $\mathsf{G}(\mathbf{T}, \mathbb{T})$ by

$$\mathsf{G}(\mathbf{T}, \mathbb{T}) = \int_{\Omega} \tilde{f}(\mathbf{T})\, d\mathcal{L}^n + \tfrac{1}{2} \int_{\mathcal{M}} \mathfrak{C}^{-1}\mathbb{T} \cdot \mathbb{T}\, d\mathcal{H}^{n-1}.$$

Note that if $\mathbf{u}$ is the solution of the equilibrium equations then the pair $(\mathbf{T}, \mathbb{T})$ of functions given by $\mathbf{T} = \hat{\mathbf{T}}(\hat{\mathbf{E}}(\mathbf{u}))$ and $\mathbb{T} = \mathfrak{C}\hat{\mathbb{E}}(\mathbb{u})$ is a statically admissible stressfield. We have the following principle of minimum complementary energy:

Proposition 3.2 *Under H1–H7, the stressfield* $(\mathbf{T}, \mathbb{T})$ *corresponding to an equilibrium solution gives the minimum complementary energy among all statically admissible stresses, i.e.,*

$$\mathsf{G}(\mathbf{T}, \mathbb{T}) \leq \mathsf{G}(\mathbf{S}, \mathbb{S})$$

for all statically admissible stressfields $(\mathbf{S}, \mathbb{S})$.

4 Linearly Elastic Matrix

In this section we consider the linear response both in the matrix and in the membrane. Under the positive definiteness of the bulk and membrane tensor of elasticities we prove the existence of the equilibrium state of minimum energy. The proof of the positive definite character of the total energy, which is the main step to the proof of existence of the equilibrium state, requires the bulk and membrane Korn's inequalities; of these the membrane Korn inequality is less known.

4.1 Linearization of the Bulk Response

In this section it is assumed that not only the surface response, but also the bulk response, is linear.

Assume that we are given a fourth order tensor C such that Assertions (i)–(iii) of Proposition 2.1 hold. If $\mathbf{u} : \Omega \times \mathbb{R} \to \mathbb{R}^n$ is a displacement field, we define the bulk stresses in the linear response by

$$\mathbf{T}(\mathbf{x}) = \mathsf{C}\mathbf{E}(\mathbf{x}), \qquad \mathbf{x} \in \Omega \setminus \mathcal{M},$$

in place of $(6)_1$, where $\mathbf{E}$ is given by $(7)_1$. In the same situation, we define the linearized free energy f by

$$f(\mathbf{x}) = \tfrac{1}{2}\mathbf{E}(x) \cdot \mathsf{C}\mathbf{E}(\mathbf{x}).$$

4.2 Korn's Inequality for Membranes

The proof of the existence of the solution to the equilibrium problem requires the coercivity of the total energy. The Korn inequalities for the membrane and for the matrix are required. These reads as follows.

Theorem 4.1 *If $\mathcal{M}$ is an admissible surface then there exists a $c > 0$ such that*

$$c|\nabla \mathbb{u}^{\parallel}|_{L^2(\mathcal{M},\mathbb{R}^3)} \leq |\hat{\mathbb{E}}(\mathbb{u})|_{L^2(\mathcal{M},\mathrm{Lin})} + |\mathbb{u}|_{L^2(\mathcal{M},\mathbb{R}^3)}$$

for each $\mathbb{u} \in S$.

This is essentially the second inequality of Korn's type "without boundary conditions" on a general surface, Ciarlet (2000, Theorem 2.7-1) according to the terminology of Ciarlet (2000).

Recall also the classical Korn inequality for the bulk strain tensor.

Theorem 4.2 *Let Ω be a bounded open subset of $\mathbb{R}^3$ with Lipschitz boundary and let $\mathcal{D} \subset \partial\Omega$ be a $\mathcal{H}^2$ measurable set with $\mathcal{H}^2(\mathcal{D}) > 0$. Then there exists a constant $c > 0$ such that*

$$c|\nabla \mathbf{u}|_{L^2(\Omega,\mathrm{Lin})} \leq |\hat{\mathbf{E}}(\mathbf{u})|_{L^2(\Omega,\mathrm{Sym})}$$

for all $\mathbf{u} \in W^{1,2}(\Omega, \mathbb{R}^3)$ such that $\mathbf{u} = \mathbf{0}$ on $\mathcal{D}$ in the sense of trace, where

$$\hat{\mathbf{E}}(\mathbf{u}) = \tfrac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^{\mathrm{T}})$$

is the small strain tensor.

4.3 Existence of Solutions for Membranes in a Linearly Elastic Matrix

Assume that we are given a bulk elasticity tensor $\mathsf{C} \in \mathrm{Lin}(\mathrm{Sym}, \mathrm{Sym})$ and for each $\mathbf{x} \in \mathcal{M}$ the surface elasticity tensor $\mathfrak{C}_{\mathbf{x}} \in \mathrm{Lin}(\mathrm{Sym}, \mathrm{Sym})$, and the associated objects, with $f(\mathbf{E}) = \frac{1}{2}\mathbf{E} \cdot \mathsf{C}\mathbf{E}$ for each $\mathbf{E} \in \mathrm{Sym}$. The proof of the existence of the solution to the equilibrium problem is based on appropriate conditions of the positivity of the tensors of elastic constants. When combined with the Korn inequalities, it leads to the coercivity of the total energy mentioned above. We recall the positivity of the membrane elasticity tensor embodied in Hypothesis H2. For the bulk elasticity tensor we assume the following.

H8 The tensor C satisfies

$$\mathbf{E}_1 \cdot \mathsf{C}\mathbf{E}_2 = \mathsf{C}\mathbf{E}_1 \cdot \mathbf{E}_2$$

for all $\mathbf{E}_1, \mathbf{E}_2 \in \mathrm{Sym}$ and there exist constant $c > 0$ such that

$$\mathbf{E} \cdot \mathsf{C}\mathbf{E} \geq c|\mathbf{E}|^2$$

for each $\mathbf{E} \in \mathrm{Sym}$.

It is immediate that H8 implies H1 and H7.
Recall the definition

$$U = \{\mathbf{u} \in H : \mathbf{u} = \mathbf{0} \text{ on } \mathcal{D}\}. \tag{16}$$

We define the potential energy F of the displacement $\mathbf{u} \in U$ by (13). Here the internal energy E is given by (14) where $\mathbb{u} = \tau\mathbf{u}$ is the trace of $\mathbf{u}$ on $\mathcal{M}$ and f, $\mathbb{f}$ are given by

$$\left.\begin{aligned} f(\mathbf{x}) &= \tfrac{1}{2}\hat{\mathbf{E}}(\mathbf{u})(x) \cdot \mathsf{C}\hat{\mathbf{E}}(\mathbf{u})(\mathbf{x}), && \mathbf{x} \in \Omega \setminus \mathcal{M}, \\ \mathbb{f}(\mathbf{x}) &= \tfrac{1}{2}\hat{\mathbb{E}}(\mathbb{u})(\mathbf{x}) \cdot \mathfrak{C}_{\mathbf{x}}\hat{\mathbb{E}}(\mathbb{u})(\mathbf{x}), && \mathbf{x} \in \mathcal{M}. \end{aligned}\right\} \tag{17}$$

Theorem 4.3 *Assume that Hypotheses H8 and H2–H6 hold. Then* F *has a unique minimum relative to U at some point* $\mathbf{u} \in U$.

5 Matrix Made of No-Tension Material

In this section we assume that the matrix response is that of a no-tension material, i.e., an elastic material with the stress constrained to be negative semidefinite for all values of strain. Constitutive equations of the no-tension material are introduced. Then the principle of minimum complementary energy is stated: this version of the principle is different from that stated in Proposition 3.2 as the invertibility hypothesis H7 is not satisfied by the no-tension material. The existence theory for the no tension matrix is out of the scope of the present paper. We recall that without the reinforcement the proof requires the introduction of the space of displacements of bounded deformation Anzellotti (1985) and Giaquinta and Giusti (1985). We prove a simpler result saying that the total energy is bounded from below if and only if the loads can be equilibrated by a stressfield that is square integrable and negative semidefinite in its bulk part.

5.1 Constitutive Equations of No-Tension Materials

In this section we assume that the bulk body is made of a no-tension material to be introduced below while we continue to assume that the membrane response is linear in the sense of Sect. 2.3. The purpose of the present section is to introduce the response functions of no-tension materials. The stress $\mathbf{T}$ depends on the small deformation tensor $\mathbf{E} = \hat{\mathbf{E}}(\mathbf{u})$,

$$\mathbf{T} = \hat{\mathbf{T}}(\mathbf{E}),$$

where $\hat{\mathbf{T}}$ is given by the constitutive equation of a masonry material defined in (18) (below).

Proposition 5.1 *Assume H8. If* $\mathbf{E} \in \mathrm{Sym}$, *there exists a unique triplet* $(\mathbf{T}, \mathbf{E}^{\mathrm{e}}, \mathbf{E}^{\mathrm{f}})$ *of elements of* Sym *such that*

$$\left.\begin{array}{c} \mathbf{E} = \mathbf{E}^{\mathrm{e}} + \mathbf{E}^{\mathrm{f}}, \\ \mathbf{T} = \mathsf{C}\mathbf{E}^{\mathrm{e}}, \\ \mathbf{T} \in \mathrm{Sym}_{-}, \quad \mathbf{E}^{\mathrm{f}} \in \mathrm{Sym}_{+}, \\ \mathbf{T} \cdot \mathbf{E}^{\mathrm{f}} = 0. \end{array}\right\}$$

We refer to Anzellotti (1985), Giaquinta and Giusti (1985) and Del Piero (1989) for various forms of the above statement and the proof.

We define the elastic stress $\hat{\mathbf{T}} : \mathrm{Sym} \to \mathrm{Sym}$ and stored energy $\hat{f} : \mathrm{Sym} \to \mathbb{R}$ of a masonry material by

$$\hat{\mathbf{T}}(\mathbf{E}) = \mathbf{T}, \quad \hat{f}(\mathbf{E}) = \tfrac{1}{2}\hat{\mathbf{T}}(\mathbf{E}) \cdot \mathbf{E} \tag{18}$$

for any $\mathbf{E} \in \mathrm{Sym}$ where $(\mathbf{T}, \mathbf{E}^{\mathrm{e}}, \mathbf{E}^{\mathrm{f}})$ is the triplet associated with $\mathbf{E}$ as in Proposition 5.1; $\mathbf{E}^{\mathrm{e}}$ and $\mathbf{E}^{\mathrm{f}}$ are called the elastic and fracture parts of the deformation $\mathbf{E}$. The explicit form of the response function $\hat{\mathbf{T}}$ and its further analysis have been given in Lucchesi et al. (1994, 1996, 2008) in dimensions 2 and 3, respectively, in case C is isotropic.

5.2 Equilibrium Displacements

We assume the partition of $\partial\Omega$ into the two complementary sets $\mathcal{D}$ and $\mathcal{S}$ and assume that we are given the forces $\mathbf{b}$, $\mathbb{b}$, $\mathbf{s}$, $\mathbb{s}$ as in Sect. 2.1. We assume that $\mathcal{M}$ is of class 1 and well placed in Ω. This allows us to define the space U as in (16), and the total energy $\mathsf{F} : U \to \mathbb{R}$ by (13). Here, for the given displacement $\mathbf{u} \in U$, the internal energy $\mathsf{E}(\mathbf{u})$ is given by (14) where $\mathbb{u} = \tau\mathbf{u}$ is the trace of $\mathbf{u}$ on $\mathcal{M}$ and where f is given by the constitutive equation

$$f(\mathbf{x}) = \hat{f}(\mathbf{E}(\mathbf{x}))$$

with $\mathbf{E}$ the small strain tensor of $\mathbf{u}$ and $\hat{f}$ the energy function of a no-tension material, and $\mathbb{f}$ is given by $(17)_2$. We note that $\hat{f}$ satisfies H1 but not H7. The potential energy W of the loads is given by (15) where it will be recalled that $\mathbb{u}^{\|} = \mathbb{P}$ and where we denote by $\beta\mathbb{u}^{\|}$ the trace of $\mathbb{u}^{\|}$ on $\partial\mathcal{M}$.

The notion of the equilibrium state is that defined in Sect. 3 and Proposition 3.1 about the variation of energy and minimum energy holds true for no-tension body, i.e., $\mathbf{u}$ is an equilibrium state if and only if the fields $\mathbf{T} = \hat{\mathbf{T}} \circ \hat{\mathbf{E}}(\mathbf{u})$ and $\mathbb{T} = \mathfrak{C}[\hat{\mathbb{E}}(\mathbf{u})]$ satisfy

$$\int_\Omega \mathbf{T} \cdot \hat{\mathbf{E}}(\mathbf{v})\, d\mathcal{L}^3 + \int_{\mathcal{M}} \mathbb{T} \cdot \hat{\mathbb{E}}(\mathbf{v})\, d\mathcal{H}^2 = \mathsf{W}(\mathbf{v})$$

for every $\mathbf{v} \in U$ where U is given by (16) and W is given by (15).

Let $(\mathbf{T}, \mathbb{T})$ be a pair of functions with $\mathbf{T} \in L^2(\Omega, \mathrm{Sym})$ and $\mathbb{T} \in L^2(\mathcal{M}, \mathrm{Sym})$ such that we have $\mathbb{T}\mathbb{n} = \mathbf{0}$ over $\mathcal{M}$. We say that the pair $(\mathbf{T}, \mathbb{T})$ is doubly admissible if it is statically admissible in the sense of definition is Sect. 3 and moreover $\mathbf{T} \leq \mathbf{0}$ for almost every point of Ω. For each stressfield $(\mathbf{T}, \mathbb{T})$ which is doubly admissible we define the complementary energy $\mathsf{G}(\mathbf{T}, \mathbb{T})$ by

$$\mathsf{G}(\mathbf{T}, \mathbb{T}) = \tfrac{1}{2} \int_\Omega \mathbb{C}^{-1}\mathbf{T} \cdot \mathbf{T}\, d\mathcal{L}^n + \tfrac{1}{2} \int_{\mathcal{M}} \mathfrak{C}^{-1}\mathbb{T} \cdot \mathbb{T}\, d\mathcal{H}^{n-1}$$

where $\mathfrak{C}^{-1}$ is the pseudoinverse of $\mathfrak{C}$ defined in Sect. 2.3. We have the following principle of minimum complementary energy:

Proposition 5.2 *Let H2–H6 and H8 hold. Let* $\mathbf{u}$ *be an equilibrium state of the system and define the pair* $(\mathbf{T}, \mathbb{T})$ *by*

$$\mathbf{T} = \hat{\mathbf{T}}(\hat{\mathbf{E}}(\mathbf{u})), \quad \mathbb{T} = \mathfrak{C}\hat{\mathbb{E}}(\mathbb{u}).$$

Then the pair $(\mathbf{T}, \mathbb{T})$ *has the minimum complementary energy among all doubly admissible stressfields equilibrating the loads, i.e.,*

$$\mathsf{G}(\mathbf{T}, \mathbb{T}) \leq \mathsf{G}(\mathbf{S}, \mathbb{S})$$

for any doubly admissible equilibrating stressfield $(\mathbf{S}, \mathbb{S})$.

This is identical in form with Proposition 3.2; however, that proposition does not apply as Hypothesis H7 is not satisfied. We refer to Anzellotti (1985) and Giaquinta and Giusti (1985) for the proof for a no-tension body without the reinforcement.

We note that the existence theory of the equilibrium states based on the minimization of the total energy in U does not work as the total energy functions is not generally coercive. Even to obtain the weaker property than the coercivity, one has to require that the loads be compatible in the sense of Sect. 5.3 (below). The existence theory in the absence of the membrane has to be build in the space $BD(\mathbb{R}^3)$ of the displacements of bounded deformation Anzellotti (1985) and Giaquinta and Giusti (1985), but even in this case the loads have to be safe in the sense that they can be equilibrated by an uniformly negative stressfield. In the presence of the membrane, the condition easily generalizes but unfortunately it does not lead to the coercivity of energy as the trace on $\mathcal{M}$ of displacements from $BD(\mathbb{R}^3)$ is not a continuous map.

5.3 *Lower Bound on Energy*

According to the limit analysis proposed in Lucchesi et al. (2010) the collapse occurs for the given loads if and only if the total energy is not bounded from below. Furthermore Lucchesi et al. (2010) shows that the total energy is bounded from below if and only if the loads are compatible in the sense of the existence of an admissible equilibrating stressfield. We here extend this equivalence to the case of the presence of membranes.

Assume that the partition $\mathcal{D}$ and $\mathcal{S}$ of $\partial\Omega$ is in Sect. 2.1 is given and let $\mathbf{b}, \mathbf{s}, \mathbb{b}, \mathbb{s}$ be the loads as in that section. We say that the loads are compatible if there exists a doubly admissible stressfield equilibrating them.

Proposition 5.3 *Assume that H2–H6 and H8 hold. Then the total energy functional* F *is bounded from below if and only if the loads are compatible.*

5.4 *Fiber Reinforced Panel*

In this subsection we demonstrate the construction of the admissible stressfield equilibrating the loads considered in Sect. 5.3.

Let us consider the square panel $\Omega = (0, 1) \times (0, 1)$ with the origin of the coordinate system $\mathbf{o}$ in the upper right corner of the panel, the x axis pointing to the left and the y axis pointing downwards. Let us denote by $\mathbf{i}$ and $\mathbf{j}$ the unit vectors corresponding to the x and y axis, respectively. The panel, made of masonry-like material, is fixed at its base $(0, 1) \times \{1\}$ and, in absence of gravity, undergoes a vertical load $p > 0$ and a tangential load $q < 0$ that are uniformly distributed on its top $(0, 1) \times \{0\}$. The panel is reinforced by an elastic fiber that is applied on the diagonal $\{\mathbf{r} = (x, y) \in \Omega : x = y\}$ and it is fixed at the point $(1, 1)$. The fiber divides the panel into the triangular regions

$$\Omega_+ = \{\mathbf{r} = (x, y) \in \Omega : x > y\} \quad \text{and} \quad \Omega_- = \{\mathbf{r} = (x, y) \in \Omega : x < y\}.$$

We want to construct a negative semidefinite and equilibrated stress field $\mathbf{T}$ that is defined in $\Omega_+ \cup \Omega_-$. Note that in the absence of the fibre there is no such a stressfield. For Ω_+ we take

$$\mathbf{T}_+ = \begin{cases} -q^2/p\mathbf{i} \otimes \mathbf{i} - q(\mathbf{i} \otimes \mathbf{j} + \mathbf{j} \otimes \mathbf{i}) - p\mathbf{j} \otimes \mathbf{j} & \text{if } \mathbf{r} \in \Omega_+^1, \\ \mathbf{0} & \text{if } \mathbf{r} \in \Omega_+ \setminus \Omega_+^1 \end{cases} \tag{19}$$

where $\Omega_+^1 = \{\mathbf{r} \in \Omega_+ : x < 1 + qy/p\}$ is the triangular region delimited by the top of the panel, the fiber and the isostatic line of equation $y = p(x-1)/q$ that starts at the upper left corner of the panel $\mathbf{q} \equiv (1, 0)$ and meets the fiber at the point $\mathbf{r}_0 \equiv (p/(p-q), p/(p-q))$. For Ω_- we are looking for a negative semidefinite stress field

$$\mathbf{T}_- = \begin{cases} \sigma\kappa^2\mathbf{i} \otimes \mathbf{i} + \sigma\kappa(\mathbf{i} \otimes \mathbf{j} + \mathbf{j} \otimes \mathbf{i}) + \sigma\mathbf{j} \otimes \mathbf{j} & \text{if } \mathbf{r} \in \Omega_-^1, \\ \mathbf{0} & \text{if } \mathbf{r} \in \Omega_- \setminus \Omega_-^1, \end{cases} \tag{20}$$

with $\sigma < 0$. Here κ is the cotangent of the angle between the active isostatic lines and the x axis and $\Omega_-^1 \subset \Omega_-$ is a region that is delimited by the fiber for $0 < x < p/(p-q)$ and will be specified below.

Let us denote by $s = s(x)$ the normal force in the fiber and by

$$\delta_{11}\mathbf{i} \otimes \mathbf{i} + \delta_{12}(\mathbf{i} \otimes \mathbf{j} + \mathbf{j} \otimes \mathbf{i}) + \delta_{22}\mathbf{j} \otimes \mathbf{j}$$

the jump of the stress field $[\mathbf{T}] = \mathbf{T}_+ - \mathbf{T}_-$ across the fiber. From Lucchesi et al. (2006, p. 518), as in this case $\omega(x) = x$ and $J = \sqrt{2}$, s has to satisfy the jump conditions

$$\left.\begin{aligned} s'/\sqrt{2} &= -\delta_{11} + \delta_{12}, \\ s'/\sqrt{2} &= \delta_{22} - \delta_{12}, \end{aligned}\right\} \tag{21}$$

where s' is the derivative of s with respect to x. Let us denote by σ_0 the value of σ in $(20)_1$ for $x = y$. In view of (19) and (20) we have

$$\left.\begin{aligned} \delta_{11} &= -q^2/p - \sigma_0\kappa^2, \\ \delta_{12} &= -q - \sigma_0\kappa, \\ \delta_{22} &= -p - \sigma_0 \end{aligned}\right\} \tag{22}$$

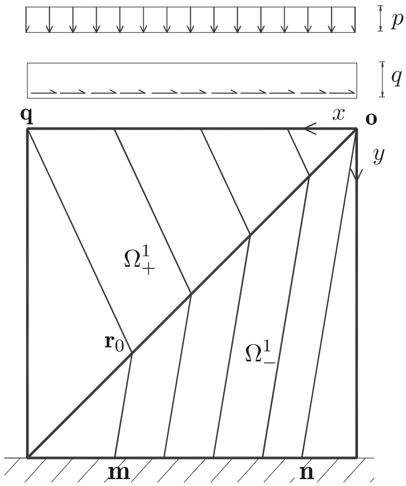

Fig. 1 Fiber-reinforced masonry panel. Isostatic curves of the stress field (25)

for $0 < x < p/(p-q)$ and

$$\delta_{11} = \delta_{12} = \delta_{22} = 0$$

for $p/(p-q) < x < 1$. By subtracting $(21)_2$ from $(21)_1$ we obtain $\delta_{11}+\delta_{22}-2\delta_{12} = 0$, which, in view of (22), implies

$$\sigma_0 = -\frac{(p-q)^2}{p(1-\kappa)^2}. \tag{23}$$

Then, from $(21)_2$ and (23) we obtain

$$s'/\sqrt{2} = -p + q - \sigma_0(1-\kappa) \tag{24}$$

for $0 \leq x \leq p/(p-q)$ and $s' = 0$, for $p/(p-q) < x < 1$. From (24) we deduce that in order to have $s' \geq 0$ we need $q/p < \kappa < 1$.

For region Ω_- we will construct two different kinds of stress fields. Firstly, we consider a stress field that is constant in Ω^1_-. In this case Ω^1_- is the region that is delimited by the fiber, the base of the panel and the two isostatic lines

$$y = \frac{1}{\kappa}\left(x - \frac{p}{p-q}\right) + \frac{p}{p-q} \quad \text{and} \quad y = x/\kappa$$

starting from $\mathbf{r}_0$ and the origin $\mathbf{o}$, respectively (Fig. 1). If we denote by $\mathbf{m}$ and $\mathbf{n}$ the points where these isostatic lines meet the base of the panel, Ω^1_- is the quadrilateral region $\mathbf{mnor}_0$. Moreover, in order to avoid that the isostatic lines meet the right hand side of the panel we require $0 \leq \kappa < 1$. Thus, we find

$$\mathbf{T} = \begin{cases} -q^2/p\mathbf{i}\otimes\mathbf{i} - q(\mathbf{i}\otimes\mathbf{j}+\mathbf{j}\otimes\mathbf{i}) - p\mathbf{j}\otimes\mathbf{j} & \text{if } \mathbf{r}\in\Omega^1_+, \\ -\dfrac{(p-q)^2}{p(1-\kappa)^2}(\kappa^2\mathbf{i}\otimes\mathbf{i} + \kappa(\mathbf{i}\otimes\mathbf{j}+\mathbf{j}\otimes\mathbf{i}) + \mathbf{j}\otimes\mathbf{j}) & \text{if } \mathbf{r}\in\Omega^1_-, \\ \mathbf{0} & \text{otherwise.} \end{cases} \tag{25}$$

From (23) and (24) we deduce that the normal force in the fiber is

$$s(x) = \begin{cases} \frac{\sqrt{2}(p-q)(\kappa p-q)}{p(1-\kappa)}x & \text{if } 0 \leq x \leq p/(p-q), \\ \frac{\sqrt{2}(\kappa p-q)}{(1-\kappa)} & \text{if } p/(p-q) < x \leq 1 \end{cases} \tag{26}$$

and then the reaction at the fixed end of the fiber is

$$\frac{(\kappa p-q)}{(1-\kappa)}(\mathbf{i},\mathbf{j}).$$

The other stress field that we consider for region Ω_- is different from zero in the wedge

$$\Omega^1_- = \{\mathbf{r} = (x,y) \in \Omega_- : p(y-1)/q < x < 0\}$$

having its apex at the lower right corner of the panel $\mathbf{a} \equiv (0,1)$ and delimited by the isostatic curve joining the points $\mathbf{a}$ and $\mathbf{r}_0$, the right lateral side of the panel and the fiber (Fig. 2). We suppose that in the wedge Ω^1_- all isostatic curves intersect at $\mathbf{a}$, so that their equation is $y - 1 = x/\kappa$ and then

$$\kappa = \frac{x}{(y-1)} \tag{27}$$

with $p/q < \kappa < 0$. Moreover, from (23) and (27) we deduce

$$\sigma_0 = -\frac{(p-q)^2}{p}(x-1)^2.$$

We assume that in Ω^1_- the stress field $\mathbf{T}_-$ has the same form as in $(20)_1$. Then, in view of (27) σ has to satisfy the equilibrium equation Lucchesi and Zani (2003)

$$\frac{x}{y-1}\sigma_{,x} + \sigma_{,y} = -\frac{1}{y-1}\sigma \tag{28}$$

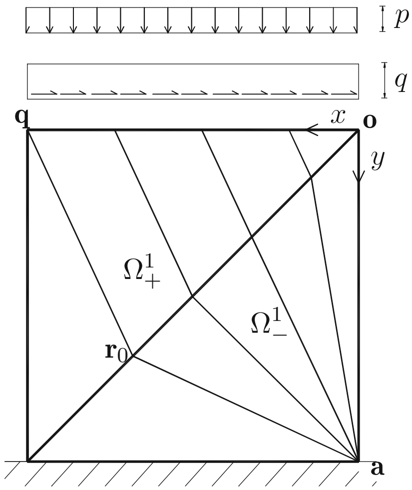

Fig. 2 Fiber-reinforced masonry panel. Isostatic curves of the stress field (30)

with the condition $\sigma = \sigma_0$ for $x = y$. The linear PDE (28) can be explicitly solved to obtain

$$\sigma = -\frac{(p-q)^2}{p}\frac{(y-1)^2}{(x-y+1)^3}. \tag{29}$$

Finally we have

$$\mathbf{T}_- = \begin{cases} \sigma\kappa^2 \mathbf{i}\otimes\mathbf{i} + \sigma\kappa(\mathbf{i}\otimes\mathbf{j}+\mathbf{j}\otimes\mathbf{i}) + \sigma\mathbf{j}\otimes\mathbf{j} & \textit{if } \mathbf{r}\in\Omega^1_-, \\ \mathbf{0} & \textit{if } \mathbf{r}\in\Omega_-\setminus\Omega^1_-, \end{cases} \tag{30}$$

with κ and σ given by (27) and (29), respectively. It is an easy matter to verify that in this case $\mathbf{T}_-$ is an integrable but not a square integrable function. For the fiber, from $(21)_2$ we deduce

$$s'/\sqrt{2} = q - p + \frac{(p-q)^2}{p(1-\kappa)} \tag{31}$$

for $0 \le x \le p/(p-q)$ and $s' = 0$ for $p/(p-q) < x < 1$. In order to have $s' > 0$ we need $-q > p$ by (31) and then stress field (30) can be used only when this condition is satisfied. In this case from (31) we deduce

$$s(x) = \begin{cases} -\frac{\sqrt{2}q(p-q)}{p}x - \frac{\sqrt{2}(p-q)^2}{2p}x^2 & \textit{if } 0 \le x \le p/(p-q), \\ -\sqrt{2}(\frac{1}{2}p+q) & \textit{if } p/(p-q) < x \le 1. \end{cases}$$

The density of the complementary energy in regions Ω_+ and Ω_- is given by

$$e_c = \frac{1}{4\mu}\mathbf{T}_0 \cdot \mathbf{T}_0 + \frac{1}{8(\mu+\lambda)}\,\mathrm{tr}(\mathbf{T})^2 \tag{32}$$

where λ and μ are the Lamé constants of the masonry. The density of the complementary energy of the fiber is

$$e_{cf} = \frac{1}{\alpha}s^2 \tag{33}$$

where α is the extensional rigidity of the fiber. For the stress field defined in (25) we want to study the behavior of the complementary energy as a function of κ, and because the density in the region Ω_+^1 is a constant, we limit ourselves to considering the complementary energies of region Ω_-^1 and the fiber. For region Ω_-^1 we have

$$\mathrm{tr}(\mathbf{T}) = -\frac{(p-q)^2}{p(1-\kappa)^2}(1+\kappa^2),$$

$$\mathbf{T}_0 = -\frac{(p-q)^2}{p(1-\kappa)^2}\Big(\tfrac{1}{2}(\kappa^2-1)\mathbf{i}\otimes\mathbf{i} + \kappa(\mathbf{i}\otimes\mathbf{j}+\mathbf{j}\otimes\mathbf{i}) + \tfrac{1}{2}(1-\kappa^2)\mathbf{j}\otimes\mathbf{j}\Big)$$

by (30) and then, from (32) we obtain

$$e_c = \frac{2\mu+\lambda}{8\mu(\mu+\lambda)}\frac{(p-q)^4}{p^2}\frac{(1+\kappa^2)^2}{(1-\kappa)^4}. \tag{34}$$

Because the area of region Ω_-^1 is $p(1-\kappa)(p-2q)/2(p-q)^2$, $\lambda = \nu E/(1+\nu)(1-2\nu)$ and $\mu = E/2(1+\nu)$, where E and ν are the Young modulus and the Poisson ratio of the masonry, respectively, from (34) we obtain the complementary energy of region Ω_-,

$$\mathcal{E}_c(\Omega_-) = \frac{1-\nu^2}{4E}\frac{(p-q)^2(p-2q)}{p}\frac{(1+\kappa^2)^2}{(1-\kappa)^3}.$$

Moreover, from (33) and (26) we deduce the complementary energy of the fiber

$$\mathcal{E}_{cf} = \frac{1}{2\alpha}\int_0^{p/(p-q)}\left(\frac{\sqrt{2}(p-q)(\kappa p-q)}{p(1-\kappa)}\right)^2 x^2dx + \frac{-q}{2\alpha(p-q)}\left(\frac{\sqrt{2}(\kappa p-q)}{1-\kappa}\right)^2$$
$$= \frac{1}{3\alpha}\frac{(p-3q)}{(p-q)}\left(\frac{\kappa p-q}{1-\kappa}\right)^2.$$

As for $0 \le \kappa < 1$ the derivatives with respect to κ of both $\mathcal{E}_c(\Omega_-)$ and $\mathcal{E}_{cf}$ are positive functions we conclude that the minimum of the total complementary energy

$\mathcal{E}_c(\Omega) + \mathcal{E}_{cf}$ is attained for $\kappa = 0$. Moreover, we note that $d\sigma/d\kappa = 2(p-q)^2/p(1-\kappa)^3 > 0$, by (23) and $ds'/d\kappa = \sqrt{2}(p-q)^2/p(1-\kappa)^2 > 0$ by (24). Then, for $\kappa = 0$ we have the minimum compressive stress in Ω_- and the minimum value of the net shear stress acting on the fiber. Finally, we note that the complementary energy of region Ω_- corresponding to the stress field defined in (30) is equal to $+\infty$, because in this case $\mathbf{T}_-$ is not a square integrable function.

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The Thrust of an Elastic Soil of Variable Density Against a Rigid Wall

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Abstract The theory of the equilibrium of a wall retaining earth masses was formulated by Coulomb in 1773. This is still applied by engineers in order to design walls. However, while Coulomb's theory assumes that the material is earthy, like sand of soft clay, in actual fact the soil often behaves elastically, and thus the stress state inside the mass and the associated pressures on the retaining walls are different. Here we study a boundary value problem of linear elasticity in order to determine the stresses exerted by a heavy elastic mass on a rigid vertical plane, or approximately, on a rigid half-plane.

Keywords Soil mechanics • Complex variable method in elasticity

1 Introduction

Charles-Augustin de Coulomb, in a celebrated memoir entitled "Essai sur une application des règles de maximis & minimis à quelques problémes de statique relatifs à l'architecture" (1773), proposed an elementary model in order to estimate the thrust of a soil. The mechanics of earth masses was studied long before Coulomb. Pierre Bullet, in his *L'Architecture practique* (1691) appears to have been the first to apply the principles of mechanics to earth pressure. The theory was successively improved by men such as Pierre Couplet, Bernard Forest de Bélidor and Alessandro Papacino d'Antoni, but according to Jacques Heyman (1997, p. 130), Coulomb was the first to introduce the cohesion of soil in his theory and the 'enormous advance' of not assuming a priori the angle of slip of soil at incipient failure. Coulomb's theory, gradually and slightly modified, is currently applied by practicing engineers in designing walls, and appears in handbooks of civil engineers (see, for example, Jenne 1967).

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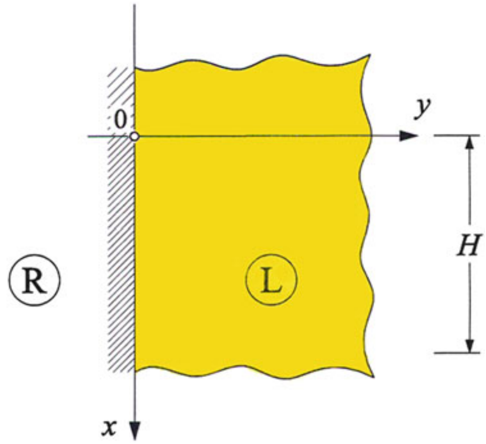

Fig. 1 The elastic region is the half-plane

However, Coulomb's theory is perfectly valid for sandy or clayey materials, but not for relatively consistent materials, loaded much below their limits of rupture. Thus the question arises of how to analyse the elastic stress state in a heavy medium in contact with a rigid plane, and how to determine the stress distribution at the interface. In particular, we consider an elastic half-plane loaded by a vertical straight-line (Fig. 1). In the Cartesian system of axes chosen in Fig. 1, the medium occupies the region $y \geq 0$.

The medium is regarded as infinitely extended in the direction perpendicular to the x,y-plane, so that the stress state is plane in the deformation. The body forces due to gravity act in the positive x-direction and can be constant or linearly increasing, as happens in a medium of variable density where the deeper layers are heavier.

There exists a large body of literature on the problem of the heavy elastic wedge. Figure 1 shows a particular example, where the angle at the vertex O is straight. The first solution to the problem was obtained by Paul Fillunger in 1912 (Poulos and David 1974, p. 226). Other solutions are recorded in the article by Worch (1967).

Our treatment differs from those cited above in two respects. The first is that the solution is obtained in terms of complex variables, a technique particularly effective in the presence of mixed boundary data. In particular, we have adopted Louis Milne-Thomson's notation (1960). The second, more original, novelty is that we can consider the case in which the density of the medium is variable and not constant, as the classic solutions assume. The extension has been rendered possible by applying an important, but unjustly ignored, method proposed by Milne-Thomson (1960, pp. 21–23) capable of recovering the displacement state associated with these variable body forces, provided that they derive from a potential.

2 Constant Body Forces

We first treat the case in which the specific weight of the soil is a constant γ. Since the problem is plane, we can consider a slice of material of unit thickness parallel to the x-y-plane, so that the dimensions of V are, say, $\frac{\mathrm{kg}}{\mathrm{m}^2}$.

The body force is represented in complex form by a vector, where, in the present case, $b_x = \gamma, b_y = 0$. Its components derive from a potential V such that

$$b_x - ib_y = \frac{\partial V}{\partial x} - i\frac{\partial V}{\partial y} = 2\frac{\partial V}{\partial z}, \tag{1}$$

where $z = x + iy$. Here $\mathrm{V} = \gamma x$. Following Milne-Thomson's procedure, stresses and displacements induced by V are defined by a function Q such that

$$\nabla^2 V = (\text{in terms of complex variables}) = 4\frac{\partial^2 Q}{\partial z \partial \bar{z}} = V. \tag{2}$$

Milne-Thomson (1960, p. 22) acutely observes that it is always possible to determine Q by quadrature since only a particular solution of (2) is required. In our case we have

$$4\frac{\partial^2 Q}{\partial z \partial \bar{z}} = V = \gamma x = \gamma \frac{z + \bar{z}}{2}, \tag{3}$$

whence we derive

$$Q = \frac{\gamma}{16}\left(z^2\bar{z} + z\bar{z}^2\right). \tag{4}$$

Then, omitting the details recorded in Milne-Thomson's book (Sects. 2.10 and 2.12), it is easy to show that the stress state deriving from the function Q is

$$\Theta_0 = \sigma_x^0 - \sigma_y^0 = -(8 - 4\nu)\frac{\partial^2 Q}{\partial z \partial \bar{z}} = -(2 - \nu)\gamma\frac{z - \bar{z}}{2}, \tag{5}$$

$$\Phi_0 = \sigma_y^0 - \sigma_x^0 + 2i\,\tau_{xy}^0 = -4\nu\frac{\partial^2 Q}{\partial z^2} = -\gamma\frac{1}{2}\nu\bar{z}, \tag{6}$$

where ν is an elastic constant defined by the ratio

$$\nu = 2(k - 1)/(k + 1), \tag{7}$$

with $k = 3 - 4\eta$, η being Poisson's ratio. The stress state (5), (6) is called the 'fundamental state'. In terms of Cartesian x,y-coordinates the single stress components are

$$\sigma_x^0 = -\left(1 - \frac{\nu}{4}\right)\gamma x, \sigma_y^0 = -\left(1 - \frac{3}{4}\nu\right)\gamma x, \tau_{xy}^0 = -\left(\frac{\nu}{4}\right)\gamma x, \tag{8}$$

which clearly satisfy the equilibrium equations

$$\begin{aligned} \sigma_{x,x}^0 + \tau_{xy,y}^0 + \gamma = 0, \\ \tau_{xy,y}^0 + \sigma_{y,y}^0 = 0. \end{aligned} \tag{9}$$

The displacement field too, expressed in terms of Q, is given by

$$4\mu v_0 = 4\mu\left(u_0 + iv_0\right) = 4\nu\frac{\partial Q}{\partial \bar{z}} + 4\mu\left(\alpha + i\beta + i\gamma z\right),$$

where μ is a Lamé's modulus and α, β, γ parameters corresponding to a small rigid motion that can be neglected. Hence, the 'fundamental' displacement, written in x,y-coordinates, becomes

$$\begin{aligned} 4\mu v_0 = 4\mu\left(u_0 + iv_0\right) &= \tfrac{\gamma}{4}v\left(z^2 + z\bar{z}\right) \\ &= -\tfrac{\gamma}{4}v\left(3x^2 + y^2 + 2ixy\right). \end{aligned} \tag{10}$$

The fundamental state, however, does not satisfy the effective boundary condition of the problem, which requires the edge $y = 0$ to be fixed. In fact, from (10) we have

$$4\mu\left(u_0 + iv_0\right)|_{y=0} = -\frac{3}{4}\gamma v x^2,$$

or,

$$4\mu\frac{\partial}{\partial x}\left(u_0 + iv_0\right)\bigg|_{y=0} = -\frac{3}{2}\gamma v x. \tag{11}$$

It is thus necessary to superimpose an 'additional' state of $\sigma_x^*, \sigma_y^*, \tau_{x,y}^*,\ u_*, v_*$, such that

$$4\mu\left(u_* + iv_*\right)|_{y=0} = -\frac{3}{4}\gamma v x^2$$

or, equivalently,

$$4\mu\frac{\partial}{\partial x}\left(u_* + iv_*\right)\bigg|_{y=0} = -\frac{3}{2}\gamma v x. \tag{12}$$

The additional state is defined by two analytic functions $W_*(z), w_*(z)$ such that

$$\Theta_* = \sigma_x^* - \sigma_y^* = W_*(z) + \overline{W}_*(\overline{z}), \tag{13a}$$

$$\Phi_* = \sigma_y^* - \sigma_x^* + zi\,\tau_{xy}^* = \overline{z}W'_*(z) + w_*(z), \tag{13b}$$

where the bar indicates complex conjugation and the apex $'$ differentiation. Written in terms of W_*, w_*, the boundary condition (12) becomes (by Milne-Thomson 1960, Sect. 4.30)

$$4\mu\left(u'_* + iv'_*\right)\Big|_{y=0} = \kappa W_*(z) - \overline{W}_*(\overline{z}) - z\overline{W}'_*(\overline{z}) - \overline{w}_*(\overline{z})\Big|_{y=0} = \frac{3}{2}\gamma v x \tag{14}$$

From (14) it is possible to determine both $W_*(z)$ and $w_*(z)$ through a classical procedure introduced by Nikoloz Muskhelishvili in the third decade of the twentieth century and diffused in Western countries by the treatise *Some Basic Problems of the Mathematical Theory of Elasticity* published in 1953 (in the translation by J.R.M. Radok) (1953). Later Muskhelishvili's method was rephrased in a more elegant form by Milne-Thomson (1960) in terms of the two functions $W_*(z)$ and $w_*(z)$. Equation (14) is a particular example of how to formulate the second boundary values problem in plane elasticity. Its solution is obtained in two steps. We first define $W_*(z)$ as the sectionally holomorphic function

$$W_*(z) \equiv \begin{cases} W_*^L(z) = \frac{3}{4}\frac{\gamma\mu z}{\kappa} & (z \text{ in } L) \\ W_*^R(z) = \frac{3}{4}\gamma\nu z & (z \text{ in } R) \end{cases} \tag{15}$$

Then we construct $w_*(z)$ by 'reflection':

$$\begin{aligned} w_*(z) &= -\overline{W}_*^R(z) - W_*^L(z) - \left(W_*^L(z)\right)' z \\ &= \tfrac{3}{4}\gamma\nu z - 2\cdot\tfrac{3}{4}\tfrac{\gamma\nu z}{\kappa} = \tfrac{3}{4}\gamma\nu z\left(1 - \tfrac{2}{\kappa}\right). \end{aligned} \tag{16}$$

Once we have the complex stresses $W_*(z), w_*(z)$ we derive from (13a, 13b) the stress combinations

$$\Theta_*(z) = \frac{3}{4}\frac{\gamma\nu}{\kappa}(z + \overline{z}) \qquad (z \text{ in } L), \tag{17a}$$

$$\Phi_*(z) = \frac{3}{4}\frac{\gamma\nu}{\kappa}\overline{z} + \frac{3}{4}\gamma\nu\left(1 - \frac{2}{\kappa}\right)z \qquad (z \text{ in } L), \tag{17b}$$

and hence the stress components

$$2\left(\sigma_y^* + i\tau_{xy}^*\right) = \frac{3}{4}\gamma\nu\left[\left(1 - \frac{1}{\kappa}\right)z + \frac{1}{\kappa}\overline{z}\right], \tag{18a}$$

$$2\left(\sigma_x^* + i\tau_{xy}^*\right) = \frac{3}{4}\gamma\nu\left[\left(-1+\frac{3}{\kappa}\right)z + \frac{1}{\kappa}\bar{z}\right]. \tag{18b}$$

Thus the effective stress is constituted by the superposition of the fundamental state (4), (5) with the additional state (18a, 18b). In particular, at the interface we have

$$\begin{aligned}\sigma_y^e + i\tau_{xy}^e\big|_{y=0} &= \left(\sigma_y^0 + it_{xy}^0\right) + \left(\sigma_y^* + it_{xy}^*\right)\big|_{y=0}\\ &= -\left(1-\tfrac{3}{4}\nu\right)\gamma x + \tfrac{3}{8}\gamma\nu x\\ &= -\left(1-\tfrac{9}{8}\nu\right)\gamma x.\end{aligned} \tag{19}$$

The consequence is that the effective stress exerted by the medium on the well is a pure normal stress, without shear, linearly increasing with the depth x. This stress can be tensile or compressive according to the value of μ. More precisely, from (7) we can write ν in terms of Poisson ratio: $\nu = 2\frac{\kappa-1}{\kappa+1} = \frac{1-2\eta}{1+\eta}$. Since $-1 < \eta \le \frac{1}{2}$, it follows that $\sigma_y^e(x,0)$ is positive (traction) for $-1 < \eta < \frac{1}{10}$, and negative (compression) for $\frac{1}{10} \le \eta \le \frac{1}{2}$.

The solution is valid under the assumption that the medium occupies the whole half-plane $y \ge 0$, but a technically more interesting problem is that of finding the pressures on the vertical x-axis whenever the medium occupies the quarter plane $x \ge 0, y \ge 0$. In this case the vertical x-edge is still clamped, and the horizontal y-edge is free. But, according to the present solution, the y-edge is loaded by the stresses (obtained from (8) and (18b)):

$$\begin{aligned}\sigma_y^e + i\tau_{xy}^e\big|_{x=0} &= \left(\sigma_x^0 + it_{xy}^0\right) + \left(\sigma_x^* + it_{xy}^*\right)\big|_{x=0}\\ &= i\tfrac{\gamma\nu}{4}y - +i\tfrac{3}{8}\left(-1+\tfrac{4}{\kappa}\right)\gamma\nu x\\ &= -i\left(\tfrac{1}{8}-\tfrac{4}{\kappa}\right)\gamma\nu y.\end{aligned} \tag{20}$$

This means that the upper edge $x = 0$, though free from normal stress σ_x^e, is loaded by a residual tangential stress, which is a linear function of y. However, considering that this stress is small in proximity to the origin and hence does not noticeably influence the stress state near the wall calculated according to (20), we may conclude that the solution (20)—exact for the half-plane—represents an acceptable approximation for the elastic quadrant whose vertical edge is fixed and whose horizontal edge is free.

3 Linearly Variable Body Forces

The advantage of having introduced the potential function Q for the body forces is that it allows the treatment of cases of media endowed with variable specific weight.

We consider in particular the case in which the specific weight of the medium is linearly increasing with the depth of x, so that $\gamma(x) = c_0 x$, where c_0 is a positive constant. In this case, equation (3) becomes

$$4\frac{\partial^2 Q}{\partial z \partial \bar{z}} = V = \frac{-c_0 x^2}{2} = -\frac{c_0 (z+\bar{z})^2}{8}, \tag{21}$$

and hence

$$Q = -\frac{c_0}{32}\left(\frac{1}{3}z^3\bar{z} + \frac{1}{2}z^2\bar{z}^2 + \frac{1}{3}z\bar{z}^3\right). \tag{22}$$

Therefore, according to (5), (6), the fundamental stress combinations have the forms:

$$\Theta_0 = \sigma_x^0 - \sigma_y^0 = (8-4\nu)\frac{\partial^2 Q}{\partial z \partial \bar{t}} = -(2-\nu)\frac{c_0}{8}(z+\bar{z})^2, \tag{23}$$

$$\Phi_0 = \sigma_y^0 - \sigma_x^0 + zi\,\tau_{xy}^0 = -4\nu\frac{\partial^2 Q}{\partial z^2} = -\frac{c_0 \upsilon}{8}\bar{z}\,(2z+\bar{z})\,, \tag{24}$$

where v is again given by (7). The single stress components are then

$$\sigma_x^0 = \frac{1}{2}\Theta_0 - \frac{1}{4}\left(\Phi_0 + \overline{\Phi}_0\right) = -(2-\upsilon)\frac{c_0}{16}(z+\bar{z})^2 + \frac{c_0 \upsilon}{32}\left(z^2 + 4z\bar{z} + \bar{z}^2\right), \tag{25a}$$

$$\sigma_y^0 = \frac{1}{2}\Theta_0 + \frac{1}{4}\left(\Phi_0 + \overline{\Phi}_0\right) = -(2-\upsilon)\frac{c_0}{16}(z+\bar{z})^2 - \frac{c_0 \upsilon}{32}\left(z^2 + 4z\bar{z} + \bar{z}^2\right), \tag{25b}$$

$$\tau_{xy}^0 = -i\frac{1}{4}\left(\Phi_0 - \overline{\Phi}_0\right) = i\frac{c_0 \upsilon}{32}\left(\bar{z}^2 - z^2\right). \tag{25c}$$

The fundamental stress state also determines a displacement state still expressed by (10). Disregarding the rigid motion, the substitution of (22) into (10) yields

$$4\mu\nu_0 = 4\mu\,(u_0 + iv_0) = 4\upsilon\frac{\partial Q}{\partial z} = \frac{-c_0 \upsilon}{24}\left(3z^2\bar{z} + 2z\bar{z}^2 + \bar{z}^3\right). \tag{26}$$

Along the x-axis, for $y = 0$, this displacement is

$$4\mu\,(u_0 + iv_0)|_{y=0} = -\frac{c_0 \upsilon}{6}x^3,$$

and its tangential derivative has the form

$$4\mu\left(u_0' + iv_0'\right)\big|_{y=0} = -\frac{c_0 \upsilon}{2}x^2. \tag{27}$$

Therefore, in order to ensure the fixity of the vertical edge $y = 0$, we must superimpose an additional state, still denoted by $W_*(z), w_*(z)$, such that (see Eq. 14)

$$4\mu \left(u'_* + iv'_*\right)\big|_{y=0} = \kappa W_*(z) - \overline{W}_*(\overline{z}) - z\overline{W}'_*\overline{z}\Big) - \overline{w}_*(\overline{z})\Big|_{y=0} \\ = \tfrac{c_0 v}{2}x^2. \tag{28}$$

This equation can be solved by applying again the method of prolongation sketched in the previous Sect. 2. Omitting the details, we arrive at the result

$$W_*(z) = \begin{cases} W_*^L(z) = \frac{c_0 v}{4\kappa}z^2 & (z \text{ in } L) \\ W_*^R(z) = -\frac{c_0 v}{4\kappa}z^2 & (z \text{ in } R)\,, \end{cases} \tag{29}$$

and hence, by reflection (formula 16), we obtain $w_*(z)$:

$$w_*(z) = -\overline{W}_*^R(z) - W_*^L(z) - \left(W_*^L(z)\right)' z \\ = \tfrac{c_0 v}{4}z^2 - 3\tfrac{c_0 v}{4\kappa}z^2 = \tfrac{1}{4}c_0 v z^2\left(1 - \tfrac{3}{\kappa}\right). \tag{30}$$

From the expressions of $W_*(z), w_*(z)$, just derived, we can determine the single stress components of the additional state. In particular we have

$$\Theta_*(z) = \frac{c_0 v}{4\kappa}\left(z + \overline{z}^2\right) \qquad (z \text{ in } L)\,, \tag{31a}$$

$$\Phi_*(z) = 2\frac{c_0 v}{4\kappa}z\overline{z} + \frac{c_0 v}{4}z^2\left(1 - \frac{3}{\kappa}\right)z \qquad (z \text{ in } L)\,, \tag{31b}$$

whence

$$2\left(\sigma_y^* + i\tau_{xy}^*\right) = \frac{c_0 v}{4}\left[\left(1 - \frac{2}{\kappa}\right)z^2 + \frac{2}{\kappa}z\overline{z} + \frac{\overline{z}^2}{\kappa}\right], \tag{32a}$$

$$2\left(\sigma_x^* - i\tau_{xy}^*\right) = \frac{c_0 v}{4}\left[\left(1 - \frac{2}{\kappa}\right)z^2 - \frac{2}{\kappa}z\overline{z} - \frac{\overline{z}^2}{\kappa}\right]. \tag{32b}$$

Thus the effective stress acting on the rigid wall $y = 0$ is obtained by combining (18a) with (32a, 32b):

$$\sigma_y^e + i\tau_{xy}^e\big|_{y=0} = \left(\sigma_y^0 + it_{xy}^0\right)\big|_{y=0} + \left(\sigma_y^* + it_{xy}^*\right)\big|_{y=0} \\ = -\tfrac{c_0}{16}\left(8 - 4\nu\right)x^2 + \tfrac{3c_0 v}{16}x^2 + \tfrac{c_0 v}{8}\left(1 + \tfrac{1}{\kappa}\right)x^2 \\ = -\tfrac{c_0}{16}\left[8 - v\left(3 + \tfrac{2}{\kappa}\right)\right]x^2. \tag{33}$$

Here again the question arises whether this solution, valid when the medium is clamped along the entire x-axis, may be accepted as an approximation for the stress state in the first quadrant whose edge $x \geq 0$ is clamped and whose edge $y \geq 0$ is free. For $x = 0$ the effective stresses are

$$\begin{aligned}\sigma_x^e + i\tau_{xy}^e\big|_{x=0} &= \left(\sigma_x^0 + it_{xy}^0\right)\big|_{x=0} + \left(\sigma_x^* + it_{xy}^*\right)\big|_{x=0}\\ &= -\tfrac{c_0 v}{16}y^2 - \tfrac{c_0 v}{8}\left(1+\tfrac{1}{\kappa}\right)y^2\\ &= -\tfrac{c_0 v}{16}\left(1+\tfrac{2}{\kappa}\right)y^2.\end{aligned} \tag{34}$$

Note that, unlike the solution (20), the stress on the edge $x = 0$ is not tangential but normal. However, this stress is a quadratic function of y, vanishing for $y = 0$. Hence its influence is relatively small near the wall, and can therefore be neglected, so that the formula (33) represents a reasonable approximation for the effective stress state in a heavy quadrant.

4 Conclusions

Spontaneous curiosity leads to comparing the outcome of the elastic theory with that of Coulomb's theory in determining the thrust of soil against a vertical wall. The comparison is necessarily possible only in the case of constant density γ, since the simple Coulomb's theory stems from this assumption.

Considering a piece of wall $0 \leq x \leq H$ (Fig. 1) the distribution of pressures predicted by both theories is linear. Then the resultant, say P, of these pressures acts at the distance $x_0 = \frac{2}{3}H$ from the origin of the x-axis. According to Coulomb's theory, the value of P is given by the classical formula

$$P_{Coul} = \frac{1}{2}\gamma H^2 tg^2\left(\frac{\pi}{4} - \frac{\varphi}{2}\right), \tag{35}$$

where φ is the angle of friction of the medium, measured in radians. Coulomb's theory holds for incoherent materials like sand or moist clay. For an elastic medium the thrust, obtained by integration of (19) for $0 \leq x \leq H$ is

$$P_{el} = \frac{1}{2}\gamma H^2\left(1 - \frac{9}{8}v\right) = \frac{1}{2}\gamma H^2\left(1 - \frac{9}{8}\frac{1-2\eta}{1-\eta}\right). \tag{36}$$

For $0 \leq \varphi \leq \frac{\pi}{2}$, P_{Coul} ranges in the interval $0 \leq P_{Coul} \leq \frac{1}{2}\gamma H^2$, while, for $-1 < \eta \leq \frac{1}{2}$, P_{el} is such that $-\frac{11}{32}\gamma H^2 \leq P_{el} \leq \frac{1}{2}\gamma H^2$. As observed at the end of Sect. 2, the thrust exerted by an elastic medium on a semi-infinite rigid wall is not necessarily a force tending to overturn the wall towards the exterior. In some cases, namely when Poisson's ratio of the medium is less than 1/10, the wall is pulled by the material situated behind it.

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In Memory of Piero Villaggio

Stefano Bennati

Piero Villaggio passed away on 4 January 2014. Professor Emeritus of the Mechanics of Solids and Structures at the University of Pisa and member of the celebrated *Accademia dei Lincei*, Piero Villaggio graduated in 1957 from the University of Genoa. In 1966 he was appointed Professor of Mechanics of Solids and Structures at the Engineering Faculty of the University of Pisa, a role that he held until his retirement in 2008. He was also Professor of Continuum Mechanics and Fluid Dynamics at the *Scuola Normale Superiore* of Pisa and, in consideration of his long, impassioned experience in rock climbing, a member of the prestigious CAAI (Italian Academic Alpine Club).

Right from the beginning of his academic career he manifested a keen research interest in a wide range of subjects—from Applied mathematics and engineering, to the philosophy of science. He delved into a number of basic issues, always aiming at the simplest possible solutions, always highlighting the fundamental physical aspects involved in the problem. From this perspective, it is not surprising that he chose the classical theory of elasticity—a theory whose elegance is matched only by its effectiveness—as a key instrument, one which he preferentially applied to a long list of mechanical applications: from the study of masonry elements and structures, to that of tunnels, avalanche and landslides; from the solution to contact problems, to joint design and the analysis of stress concentrations, to name only a few.

These few short lines, however, are by no means intended to commemorate, even summarily, his long-standing scientific and teaching achievements: a detailed presentation is instead contained in the article by Roger Fosdick and Gianni Royer (2014), recently published in the *Journal of Elasticity*, whose reading I strongly recommend to those who knew and loved Piero Villaggio. Here, as Director of the

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Ph.D. School of Engineering "Leonardo da Vinci" in Pisa, I only wish to recall the course that Piero Villaggio gave regularly since 1982 in differential equations of mathematical physics, firstly for our research doctorate in structural engineering, then for our doctoral school. I find it hard to imagine just how many students must surely recall his boundless love of knowledge, let alone his lessons—disarming in their simplicity, sometimes even elusive in their subtlety—his ability to delineate the mechanical significance of a problem with just a few chalk strokes on the blackboard, as well as the veiled elegance of his analytical solutions.

Those friends, colleagues and students who knew him will certainly remember him for his passionate, selfless love of knowledge, for his unbounded devotion to his work, as well as his moral rectitude and integrity. From these perspectives, Piero Villaggio represented and will always represent a role model for all those who have had the good fortune to work with him, or even to have had the opportunity to meet and exchange some thoughts with him, or simply to have watched him at work, seated at his desk, surrounded by his beloved books.

There are some individuals in this world who, like Piero Villaggio, are set 'apart' from all others by the intensity, the determination and constancy with which they immerse themselves in their personal pursuits. Such individuals certainly leave a great emptiness, but they also leave countless lasting traces of themselves in those who have known and admired them in many different ways: in this sense, Piero and his singular personality live on in all who loved him and have, if only in part, understood his profound message.

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Index

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