The Influence of Limited Kinematical Hardening on Shakedown of Materials and Structures

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Abstract One of the most important tasks in design for construction engineers is the determination of the load bearing capacity of the considered engineering structure. This can be particularly challenging when the applied thermo-mechanical loads vary with time and are high enough to exceed the material's elastic regime. In these cases, the lower bound shakedown analysis provides a convenient tool. Since accounting for the realistic material behavior is inevitable to achieve reliable results, it is highly relevant to consider limited kinematical hardening. Although there exist different formulations in the literature, in which limited kinematical hardening is incorporated, these usually do not take into account the underlying hardening law in an explicit manner. The most important question in that context is whether such formulations can cover both linear and nonlinear hardening laws. In consequence, the aim of this paper is to investigate the effect of nonlinearity of the hardening law by showing that in certain scenarios the introduction of an explicit hardening law as a subsidiary constraint is unavoidable.

1 Introduction

One of the most important tasks in design for construction engineers is the determination of the load bearing capacity of the considered engineering structure. This can be particularly challenging when the applied thermo-mechanical loads vary with time and are high enough to exceed the material's elastic regime. Then, the computation of the so-called shakedown loading factor is necessary, which is the maximum loading factor such that the system can be considered as 'safe', such that neither alternating plasticity nor spontaneous or incremental collapse occur.

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In most cases, conventional step-by-step computations are performed whenever the exact stress-strain distributions are needed. In contrast, if only the material's or structure's limit state is of interest, the load bearing capacity can be conveniently determined by means of limit or shakedown analysis. In general, there exist two different approaches to shakedown analysis, which complement each other: The lower bound approach by Melan [19, 20], which is formulated in statical quantities, and the upper bound approach of Koiter [15], which makes use of the kinematical ones. From these, the lower bound approach is adopted in this work, because the formulation in terms of stresses is particularly suited for the extension to kinematical hardening.

The majority of elasto-plastic materials exhibit kinematical hardening during the evolution of plastic deformations. Therefore, this phenomenon needs to be incorporated into the procedure in order to obtain realistic results. In its original formulation the statical shakedown theorem only holds for elastic-perfectly plastic continua as well as for unlimited kinematical hardening ones. Since the unlimited kinematical hardening case does not cover incremental collapse at all, taking into account limited (or bounded) kinematical hardening is inevitable. Hence, this issue has been addressed by several authors in the field of shakedown analysis [6, 8, 10, 18, 21–27, 30, 36–40].

From those, the first explicit formulation for limited kinematical hardening materials has been given by Weichert and Groß-Weege [40] (WGW), who introduced a two-surface model. Their formulation is based on the concept of generalized standard materials [11], and thus implies an associated hardening rule, together with the assumption of limited *linear* kinematical hardening. Almost at the same time, Stein et al. [37–39] have proposed another approach based on an overlay model. The formulation presented therein has been said to be valid for limited *general nonlinear* kinematical hardening with associated flow. Noteworthy, Heitzer [12] has investigated the relation between the two different formulations. He has stated that both theorems, even though formulated differently, lead to the same optimal value for the shakedown factor, and that the only difference might appear in the corresponding residual stress fields.

More recently Pham has presented an extension of the theorem proposed by Weichert and Groß-Weege for the generally nonlinear case [23]. He has claimed, that this theorem holds for any generally nonlinear hardening law as long as the hysteresis is positive for any closed cycle of plastic deformations.

In all of these works, it turns out that the shakedown load is independent of the underlying hardening law and the according stress-strain curve. The shakedown load seems to depend only on the magnitudes of the initial yield stress σ_Y and the ultimate stress σ_H.

In contrast, independently of each other, Staat and Heitzer [36] as well as Bouby et al. [4, 5] have presented results with significant differences in the shakedown limit load between the limited linear hardening and the limited nonlinear hardening. Thus, the aim of the present paper is to investigate the effect of different hardening rules on shakedown loads.

2 Lower Bound Shakedown Analysis Accounting for Limited Kinematical Hardening

In the following, an elastic-perfectly plastic body with volume V and surface A is considered, which is subjected to: temperature loads $T(\boldsymbol{x}, t)$ in V, body forces $\boldsymbol{f}_V(\boldsymbol{x}, t)$ in V, surface loads $\boldsymbol{f}_A(\boldsymbol{x}, t)$ on $A_f \subseteq A$, and prescribed displacements $\boldsymbol{u}(\boldsymbol{x}, t)$ on $A_u \subseteq A$, such that $A = A_f \cap A_u$ and $A_f \cup A_u = \emptyset$. Only time- and temperature-independent material behavior is taken into account, while material damage and geometrical nonlinearity are neglected. The existence of a convex yield function $f[\boldsymbol{\sigma}(\boldsymbol{x}, t)]$ is assumed. Then, the elastic limit is described by a yield surface in stress space $\mathbb{S}$ as closure of the convex domain $\mathscr{C}_Y \subseteq \mathbb{S}$ of admissible states of stress with the strict interior $\mathscr{C}_Y^i$:

$$\mathscr{C}_Y^i = \left\{ \boldsymbol{\sigma} \in \mathbb{S} \,\middle|\, f\left[\boldsymbol{\sigma}(\boldsymbol{x}, t)\right] < \sigma_Y^2(\boldsymbol{x}), \quad \forall \boldsymbol{x} \in V, \quad \forall t \right\} \tag{1}$$

2.1 *Melan's Statical Shakedown Theorem for Elastic-Perfectly Plastic Materials*

As already mentioned above, this work is based on the statical shakedown theorem by Melan [19, 20], which provides a lower bound to the shakedown loading factor. To apply Melan's theorem, the total stress $\boldsymbol{\sigma}(\boldsymbol{x}, t)$ in a point $\boldsymbol{x} \in V$ within the volume V of the considered body at time t is decomposed into an elastic reference stress $\boldsymbol{\sigma}^E(\boldsymbol{x}, t)$ and a residual stress $\boldsymbol{\rho}(\boldsymbol{x}, t)$ induced by the evolution of plastic strains.

$$\boldsymbol{\sigma}(\boldsymbol{x}, t) = \boldsymbol{\sigma}^E(\boldsymbol{x}, t) + \boldsymbol{\rho}(\boldsymbol{x}, t) \tag{2}$$

The fictitious stress state $\boldsymbol{\sigma}^E(\boldsymbol{x}, t)$ is the one which would occur in a purely elastic reference body under the same conditions and loadings as the original one. Both the elastic reference stresses and the residual stresses satisfy the equilibrium constraints as well as the statical boundary conditions.

$$\text{equilibrium:} \qquad \nabla \cdot \boldsymbol{\sigma}^E = -\boldsymbol{f}_V \qquad \nabla \cdot \boldsymbol{\rho} = \boldsymbol{0} \qquad \text{in } V \tag{3}$$

$$\text{statical bc:} \qquad \boldsymbol{n} \cdot \boldsymbol{\sigma}^E = \boldsymbol{f}_A \qquad \boldsymbol{n} \cdot \boldsymbol{\rho} = \boldsymbol{0} \qquad \text{on } A_f \tag{4}$$

Then, Melan's shakedown theorem for elastic-perfectly plastic materials can be formulated as follows:

If there exist a loading factor $\alpha > 1$ and a time-independent residual stress field $\bar{\boldsymbol{\rho}}(\boldsymbol{x})$, such that the yield condition is satisfied for any loading path within the considered loading domain Ω at any time t and in any point $\boldsymbol{x}$ of the structure, then the system will shake down.

$$f\left[\alpha\, \boldsymbol{\sigma}^E(\boldsymbol{x}, t) + \bar{\boldsymbol{\rho}}(\boldsymbol{x})\right] \leq \sigma_Y^2(\boldsymbol{x}), \;\; \forall \boldsymbol{x} \in V, \;\; \forall t \tag{5}$$

It is worth to mention that the numerical procedure allows for computing values $\alpha < 1$ as long as it is positive, $\alpha > 0$. Nevertheless, shakedown only can be guaranteed if $\alpha > 1$ holds, because only then the plastic dissipative energy is guaranteed to be bounded.

2.2 Two Surface Model for Limited Linear Kinematical Hardening by Weichert and Groß-Weege (WGW)

The first explicit formulation of the statical shakedown theorem accounting for limited kinematical hardening has been proposed by Weichert and Groß-Weege [40] in 1988. The formulation presented therein is based on the Generalized Standard Material Model (GSMM) introduced by Halphen and Nguyen [11]. Thus, it is implied that the normality rule holds, restricting the formulation to associated hardening laws. Moreover, the proof of the theorem makes use of the assumption of limited *linear* kinematical hardening.

The kinematical hardening is considered as a translational motion of the yield surface—described by $f = \sigma_Y^2$—in stress space without change of orientation, form or size. This motion is limited by the bounding surface, $f = \sigma_H^2$, which corresponds to the ultimate stress σ_H. Further, the motion is defined by the six-dimensional vector of back-stresses $\boldsymbol{\pi}$ representing the translation of the yield surface's center, see Fig. 1. Thereby, the total stresses $\boldsymbol{\sigma}(\boldsymbol{x}, t)$ are decomposed into the back stresses $\boldsymbol{\pi}$ and the so-called reduced stresses $\boldsymbol{\upsilon}$. The latter are responsible for the occurrence of plastic strains.

$$\boldsymbol{\sigma}(\boldsymbol{x}, t) = \boldsymbol{\pi}(\boldsymbol{x}, t) + \boldsymbol{\upsilon}(\boldsymbol{x}, t) \tag{6}$$

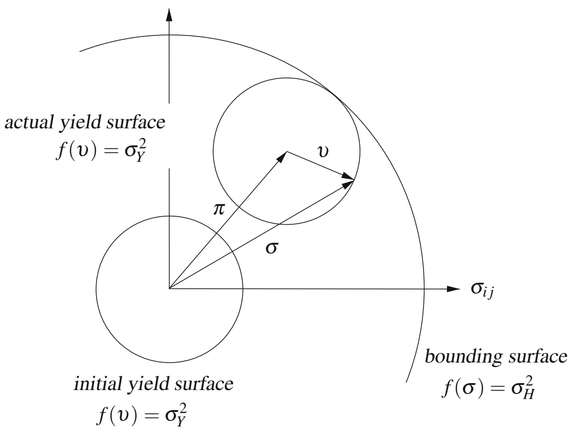

Fig. 1 Limited kinematical hardening considered as translation of the yield surface in stress space

Since the bounding surface is fixed in stress space, the back-stresses have to be time-independent, which will be indicated in the following by an overbar, $\boldsymbol{\pi} = \bar{\boldsymbol{\pi}}(\boldsymbol{x})$. Further, the decomposition of the total stresses into an elastic reference part and a residual one still holds. Consequently, the reduced stresses $\boldsymbol{v}(\boldsymbol{x}, t)$ can be expressed as follows:

$$\boldsymbol{v}(\boldsymbol{x}, t) = \boldsymbol{\sigma}(\boldsymbol{x}, t) - \bar{\boldsymbol{\pi}}(\boldsymbol{x}) = \boldsymbol{\sigma}^E(\boldsymbol{x}, t) + \bar{\boldsymbol{\rho}}(\boldsymbol{x}) - \bar{\boldsymbol{\pi}}(\boldsymbol{x}) \tag{7}$$

Finally, a shakedown theorem accounting for limited kinematical hardening can be written as:

If there exist a loading factor $\alpha > 1$, a time-independent self-equilibrated (residual) stress field $\bar{\boldsymbol{\rho}}$ and a time-independent field of back-stresses $\bar{\boldsymbol{\pi}}$,such that the yield condition and the bounding condition are satisfied for any loading path within the considered loading domain Ω at any time t and in any point $\boldsymbol{x}$ of the structure, then the system will shake down.

$$f\left[\alpha\, \boldsymbol{\sigma}^E(\boldsymbol{x}, t) + \bar{\boldsymbol{\rho}}(\boldsymbol{x}) - \bar{\boldsymbol{\pi}}(\boldsymbol{x})\right] \leq \sigma_Y^2(\boldsymbol{x}) \tag{8}$$

$$f\left[\alpha\, \boldsymbol{\sigma}^E(\boldsymbol{x}, t) + \bar{\boldsymbol{\rho}}(\boldsymbol{x})\right] \leq \sigma_H^2(\boldsymbol{x}) \tag{9}$$

It should be noted, that the consideration of specific hardening rules—such as the linear one—implies restrictions on the field of back-stresses $\bar{\boldsymbol{\pi}}$. This issue and its consequences will be discussed in Sect. 3.

2.3 Extension of WGW-Model for Limited General Nonlinear Kinematical Hardening by Pham

While the proof of the theorem given above is based on the assumption of limited linear kinematical hardening, the same formulation has been derived for generally-nonlinear hardening laws later by Pham [22, 23]. The only restriction Pham has postulated on the considered hardening is the *positive hysteresis postulate*, which states that for any closed cycle of plastic deformations ($t \in [0, \theta]$) the following condition has to hold:

$$\oint \boldsymbol{\pi} : d\boldsymbol{\varepsilon}_\pi^p = \int_0^\theta \boldsymbol{\pi} : \boldsymbol{\varepsilon}_\pi^p dt \geq 0 \qquad \left(\boldsymbol{\varepsilon}_\pi^p(0) = \boldsymbol{\varepsilon}_\pi^p(\theta)\right) \tag{10}$$

where $\boldsymbol{\pi}$ denotes the back-stresses, and $\boldsymbol{\varepsilon}_\pi^p$ denotes the corresponding plastic deformation. In the case of a simple loading-unloading closed plastic cycle, this restriction implicates that the hysteresis loop is followed in clockwise direction, but not anti-clockwise.

2.4 Overlay-Model for Limited Nonlinear Kinematical Hardening by Stein et al.

An alternative approach to formulate a shakedown theorem taking into account limited kinematical hardening has been presented by Stein et al. [37–39]. This formulation is based on the overlay model. As in the WGW-model, the normality condition is assumed to hold. Except of this restriction the authors have stated that the formulation is valid for limited *general nonlinear* kinematical hardening.

Interestingly, the formulation of Stein et al. is rather similar to the one by Weichert and Groß-Weege. However, there is one very important difference in the way the bounding condition is expressed. Instead of the condition (9), Stein et al. derive the following one:

$$f\,[\bar{\boldsymbol{\pi}}(\boldsymbol{x})] \leq [\sigma_H(\boldsymbol{x}) - \sigma_Y(\boldsymbol{x})]^2 \tag{11}$$

The most severe difference between (9) and (11) is the fact that the back-stresses $\bar{\boldsymbol{\pi}}$ appear only implicitly in the first one, whereas they show up in an explicit manner in the second one. It is worth to mention, that the relation between these two different formulations has been investigated in more detail by Heitzer [12]. As a result of these investigations, Heitzer has stated that both theorems lead to the same optimal value for the shakedown factor, even though the corresponding residual stress fields might differ. As will be shown within the example in Sect. 4, this turns out not to be true in certain scenarios.

2.5 Description of the Loading Domain

In the following, the loading histories $\mathscr{H}(\boldsymbol{x}, t)$ under consideration are assumed to be describable as superposition of a finite number NL of different loading sets $P_\ell(\boldsymbol{x}, t)$. The latter can be expressed in terms of load multipliers $\mu_\ell(t)$ for any loading case ℓ and the unity load $P_0(\boldsymbol{x})$.

$$\mathscr{H}(\boldsymbol{x}, t) = \sum_{\ell=1}^{NL} P_\ell(\boldsymbol{x}, t) = \sum_{\ell=1}^{NL} \mu_\ell(t)\, P_0(\boldsymbol{x}) \tag{12}$$

As shown by König [16], it is sufficient to consider only the convex hull of the loading history, which is polyhedral with $NC = 2^{NL}$ corners. These corners are defined in the loading space by introducing bounding values μ_ℓ^+ and μ_ℓ^- for each multiplier μ_ℓ. Doing so, the set $\mathscr{U}$ is defined, which contains all possible combinations of loading sets within these bounds through merging all loading multipliers to the vector $\boldsymbol{\mu} = \mu_\ell\, \boldsymbol{e}_\ell$.

$$\mathscr{U} = \left\{ \boldsymbol{\mu} \in \mathbb{R}^{NL} \;\middle|\; \mu_\ell^- \leq \mu_\ell \leq \mu_\ell^+ \,,\ \forall \ell \in [1, NL] \right\} \tag{13}$$

Then, the loading domain Ω is described as set of all possible loading histories contained within $\mathscr{U}$.

$$\Omega = \left\{ \mathscr{H}(\boldsymbol{x}, t) \;\middle|\; \mathscr{H}(\boldsymbol{x}, t) = \sum_{\ell=1}^{NL} \mu_\ell(t)\, P_0(\boldsymbol{x}) \,,\ \forall \boldsymbol{\mu} \in \mathscr{U} \right\} \tag{14}$$

Consequently, the elastic reference stresses are split in analogy to (12).

$$\boldsymbol{\sigma}^E(\boldsymbol{x}, t) = \sum_{\ell=1}^{NL} \mu_\ell(t)\, \boldsymbol{\sigma}_\ell^E(\boldsymbol{x}) \tag{15}$$

2.6 Discretization

Using the finite element method (FEM), the stresses are approximately represented by their values in the Gaussian points, which will be referred to by the index $r \in [1, NG]$. Here NG is the total number of Gaussian points in the system. Consequently, the fictitious elastic stresses $\boldsymbol{\sigma}_{r,\ell}^E$ can be computed for any loading case ℓ by purely elastic analysis.

$$\boldsymbol{\sigma}_r^E(t) = \sum_{\ell=1}^{NL} \mu_\ell(t)\, \boldsymbol{\sigma}_{r,\ell}^E \tag{16}$$

To ensure shakedown for all possible loading paths inside of the loading domain, only its corners need to be examined. Thus, the time-dependence of $\boldsymbol{\sigma}_r^E$ can be expressed through the stress states in the corners $j \in [1, NC]$ of the loading domain. For this, the matrix $\boldsymbol{U}_{NL} \in \mathbb{R}^{NC \times NL}$ with entries $U_{j\ell}$ is introduced, where $j \in [1, NC]$ and $\ell \in [1, NL]$.

$$\boldsymbol{\sigma}_r^{E,j} = \sum_{\ell=1}^{NL} U_{j\ell} \boldsymbol{\sigma}_{r,\ell}^E \tag{17}$$

Each row of this matrix $\boldsymbol{U}_{NL}$ represents the coordinates of one corner of the loading domain in the NL-dimensional loading space, which are defined by the factors μ_ℓ^- and μ_ℓ^+ as introduced in (13). The matrix can be defined in an automatic way for arbitrary numbers of loading cases NL, as shown in [35].

Since the elastic reference stress field $\boldsymbol{\sigma}^E$ is in equilibrium with the external loading, the residual stress field $\bar{\boldsymbol{\rho}}$ has to be self-equilibrated. This fact can be expressed by means of the principle of virtual work [9],

$$\int_V \delta\boldsymbol{\varepsilon} : \bar{\boldsymbol{\rho}}\, dV = 0 \tag{18}$$

where $\delta\boldsymbol{\varepsilon}$ denotes any virtual strain field which satisfies the kinematical boundary conditions.

Using the FEM with isoparametric elements, the displacements $\boldsymbol{u}$ are approximated by shape functions and nodal displacements $\boldsymbol{u}_K$. Thus, the virtual strain field can be expressed through the nodal displacements as well: $\delta\boldsymbol{\varepsilon}(\boldsymbol{x}) = \delta\boldsymbol{u}_K \cdot \boldsymbol{B}$, where $\boldsymbol{B}(\boldsymbol{x})$ is the differentiation matrix. Furthermore, the integration is carried out numerically. Thereby (18) is approximated by a system of linear equations for the residual stresses $\bar{\boldsymbol{\rho}}_r$ in the Gaussian points.

$$\int_V \boldsymbol{B}(\boldsymbol{x}) : \bar{\boldsymbol{\rho}}\, dV =: \sum_{r=1}^{NG} \mathbb{C}_r \cdot \bar{\boldsymbol{\rho}}_r = \boldsymbol{0} \tag{19}$$

The equilibrium matrices $\mathbb{C}_r$ depend only on the geometry of the system and the applied element type as well as the kinematical boundary conditions.

2.7 Resulting Nonlinear Optimization Problem

Based on the Eqs. (17) and (19), the extended Melan's theorem for limited kinematical hardening can be expressed in terms of an optimization problem for the loading factor $\alpha > 1$:

$$(\mathscr{P}^H) \qquad \alpha_{SD} = \max_{\bar{\boldsymbol{\rho}},\, \bar{\boldsymbol{\pi}}} \; \alpha$$

$$\sum_{r=1}^{NG} \mathbb{C}_r \cdot \bar{\boldsymbol{\rho}}_r = \boldsymbol{0} \tag{20a}$$

$$\forall j \in [1, NC],\ \forall r \in [1, NG] :$$

$$f\left(\alpha\, \boldsymbol{\sigma}_r^{E,j} + \bar{\boldsymbol{\rho}}_r - \bar{\boldsymbol{\pi}}_r\right) \leq \sigma_{Y,r}^2 \tag{20b}$$

$$f_H\left(\boldsymbol{\sigma}_r^j, \bar{\boldsymbol{\pi}}_r, \sigma_{H,r}\right) \leq 0 \tag{20c}$$

Depending on which formulation is used, the according bounding condition, (9) or (11), needs to be inserted:

$$\text{WGW \& Pham}: \quad f_H\left(\boldsymbol{\sigma}_r^j, \bar{\boldsymbol{\pi}}_r, \sigma_{H,r}\right) = f\left(\alpha\, \boldsymbol{\sigma}_r^{E,j} + \bar{\boldsymbol{\rho}}_r\right) - \sigma_{H,r}^2 \tag{21a}$$

$$\text{Stein}: \quad f_H\left(\boldsymbol{\sigma}_r^j, \bar{\boldsymbol{\pi}}_r, \sigma_{H,r}\right) = f\left(\bar{\boldsymbol{\pi}}_r\right) - \left(\sigma_{H,r} - \sigma_{Y,r}\right)^2 \tag{21b}$$

The solution procedure for this nonlinear convex optimization problem is not in the scope of the current paper. The interested reader is referred to [31–34].

3 Effect of the Underlying Kinematical Hardening Law

In this section, the effect of the underlying limited kinematical hardening model on the shakedown load is investigated. As already has been mentioned, it is frequently stated in literature that shakedown limits only depend on the initial yield stress σ_Y and the ultimate stress σ_H, but not at all on the hardening behavior in between, see e.g. [1, 13, 22–24, 37, 40]. Even so, in the following it will be shown that in principle there can exist cases, in which the shakedown load is in fact influenced by the applied hardening law.

To do so, the shakedown theorems presented in the Sects. 2.2–2.4 are rewritten in a more formal way. For this, the set of all time-independent and self-equilibrated stress fields is denoted by $\mathscr{R}$. Further, the set of all time-independent and permissible fields of back-stresses is denoted by $\mathscr{B}$. Here, the definition of a permissible stress field depends on the considered theorem:

WGW: A back-stress field is permissible if it can evolve under the given loading domain following the corresponding linear hardening law with an associated flow rule.

Stein: A back-stress field is permissible if it can evolve under the given loading domain following any hardening law with an associated flow rule.

Pham: A back-stress field is permissible if it satisfies the positive hysteresis assumption.

Then, the three theorems can be written in the following way:
If there exist a scalar $\alpha > 1$ and fields $\bar{\boldsymbol{\rho}}$ and $\bar{\boldsymbol{\pi}}$, such that the following conditions hold, then the system will shake down.

$$\bar{\boldsymbol{\pi}} \in \mathscr{B} \tag{22a}$$

$$\bar{\boldsymbol{\rho}} \in \mathscr{R} \tag{22b}$$

$$\forall j \in [1, NC],\ \ \forall r \in [1, NG] :$$

$$f\left(\alpha\, \boldsymbol{\sigma}_r^{E,j} + \bar{\boldsymbol{\rho}}_r - \bar{\boldsymbol{\pi}}_r\right) \leq \sigma_{Y,r}^2 \tag{22c}$$

$$f_H\left(\boldsymbol{\sigma}_r^j, \bar{\boldsymbol{\pi}}_r, \sigma_{H,r}\right) \leq 0 \tag{22d}$$

If one compares this set of conditions with the subsidiary constraints of the optimization problem formulated in Sect. 2.7, one can observe that not all of the conditions (22a)–(22d) are reflected. While the yield and the bounding conditions (22c) and (22d) are represented by (20b) and (20c), respectively, the equation (20a) ensures that $\bar{\boldsymbol{\rho}}$ is self-equilibrated (22b). In contrast, the condition (22a) is not incorporated anymore.

In fact, as long as a general hardening case is considered, in which the evolution of $\bar{\boldsymbol{\pi}}$ is not restricted by any specific kind of hardening law, the optimization problem

($\mathscr{P}^H$) is equivalent to (22a–22d) without loss of generality. In this case the back-stresses can be considered as unrestricted variables of the optimization problem. Therefore, the corresponding hardening law will be called *unrestricted hardening* in the following.

Since α is maximized over $\bar{\boldsymbol{\pi}}$, the solution of the optimization problem ($\mathscr{P}^H$) involves the one particular field $\bar{\boldsymbol{\pi}}^*$, which leads to the maximum value of α. Hence, the unrestricted hardening, which allows the back-stresses $\bar{\boldsymbol{\pi}}^*$ to evolve under the given loading domain, is the most advantageous one leading to the highest shakedown loading factors amongst all possible hardening rules. In other words, the computed shakedown factor might be higher than the one, which can be obtained if a specific hardening rule is applied.

Clearly, only the yield and bounding surface need to be defined by σ_Y and σ_H, respectively, when the unrestricted hardening is considered using the optimization problem ($\mathscr{P}^H$). This is in accordance with the above mentioned references. Even so, the shakedown limit may depend on the hardening behavior in between the initial yield state and the ultimate state. If a specific hardening law is to be considered, then the feasible set of back-stresses can be restricted, such that $\bar{\boldsymbol{\pi}} \in \mathscr{B}^\circ$ and $\mathscr{B}^\circ \subset \mathscr{B}$. To ensure that the solution is admissible, this restriction for the back-stresses has to be included into the optimization problem as a separate constraint. Otherwise, a non-admissible solution might be obtained, $\bar{\boldsymbol{\pi}}^* \notin \mathscr{B}^\circ$.

If, for example, the special case of limited *linear* kinematical hardening shall be investigated, the set of feasible back-stresses $\mathscr{B}^\circ$ has to be formulated such that the resulting solution $\bar{\boldsymbol{\pi}}^\circ$ can evolve under the given loading domain following the corresponding linear hardening law, $\dot{\boldsymbol{\pi}}^\circ = C\,\dot{\boldsymbol{\varepsilon}}^p$, where C is a material parameter. However, introducing such additional constraints directly into the optimization problem can be problematic, because kinematic variables (e.g. plastic strains $\boldsymbol{\varepsilon}^p$) would have to show up at least implicitly in the statical theorem, which is formulated in stresses. Nevertheless, the generally-nonlinear hardening can be directly incorporated into the procedure by defining the restrictions on the back-stresses.

4 Sample Under Constant Tension and Alternating Torsion

To illustrate the correlation between different hardening laws, an illustrative example is presented in this section. In particular, a specimen is considered, which is subjected to a constant tension $\bar{\sigma} > 0$ and alternating torsion τ with zero mean shear stress, such that $\tau_{\min} = -\tau_{\max}$. The according loading domain consists of only two points $(\bar{\sigma}, \tau_{\max})$ and $(\bar{\sigma}, \tau_{\min})$. Noticeably, such a system has been previously examined by e.g. Portier et al. [28], where ratcheting has been investigated experimentally as well as numerically. Furthermore, numerical and analytical results of shakedown analysis accounting for different types of kinematical hardening are presented in [7, 14, 17, 36] for the plane stress state, while the plane strain state has been investigated in [4, 5].

In the following, the plane stress state is considered:

$$\boldsymbol{\sigma} = \begin{pmatrix} \bar{\sigma} & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\pi} = \begin{pmatrix} X & Y & 0 \\ Y & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{23}$$

In this simple problem both the stress and the strain field are uniformly distributed. Therefore, the residual stresses $\bar{\boldsymbol{\rho}}$ need not to be considered. In fact, the back-stresses $\boldsymbol{\pi}$ play the role of residual stresses in this case.

In the following, the yield criterion is expressed by the von Mises yield condition, which reads:

$$f_Y\left(\boldsymbol{\sigma} - \boldsymbol{\pi}, \sigma_Y\right) = \left(\bar{\sigma} - X\right)^2 + 3\left(\tau - Y\right)^2 - \sigma_Y^2 = 0 \tag{24}$$

Since both maxima $(\bar{\sigma}, \tau_{\max})$ and $(\bar{\sigma}, -\tau_{\max})$ shall be located on the yield surface which is described by (24), the following two equations have to hold:

$$\left(\bar{\sigma} - X\right)^2 + 3\left(\tau_{\max} - Y\right)^2 - \sigma_Y^2 = 0 \tag{25a}$$

$$\left(\bar{\sigma} - X\right)^2 + 3\left(-\tau_{\max} - Y\right)^2 - \sigma_Y^2 = 0 \tag{25b}$$

The difference between these Eq. (25a), (25b), gives:

$$\left(\tau_{\max} - Y\right)^2 - \left(-\tau_{\max} - Y\right)^2 = -4\,\tau_{\max}\,Y = 0 \tag{26}$$

A non-trivial solution, $\tau_{\max} \neq 0$, can therefore only be obtained if $Y = 0$. Thus, to reach the shakedown state, the yield surface is moved in stress space only in the direction of σ. The yield condition simplifies to:

$$f_Y\left(\boldsymbol{\sigma} - \boldsymbol{\pi}, \sigma_Y\right) = \left(\bar{\sigma} - X\right)^2 + 3\,\tau^2 - \sigma_Y^2 = 0 \tag{27}$$

The positive solution of (27) is:

$$\sqrt{3}\,\tau = \sqrt{\sigma_Y^2 - \left(\bar{\sigma} - X\right)^2} \tag{28}$$

From (28) it can be observed that the shakedown load τ can in fact depend on the according back-stress X. However, this back-stress is restricted by the applied hardening law. To illustrate the influence of different kinematical hardening rules, the following types of plastic behavior are incorporated:

1. **Perfectly plastic behavior**:
   No hardening occurs and thus no back-stress evolves, $X = 0$, leading to the following maximum value of admissible shear stress:
   $\sqrt{3}\,\tau_0 = \sqrt{\sigma_Y^2 - \bar{\sigma}^2}$ (black dashed line in Fig. 2)
   Alternating plasticity only occurs in pure shear, $\bar{\sigma} = 0$, whereas the remaining shakedown domain represents incremental collapse.

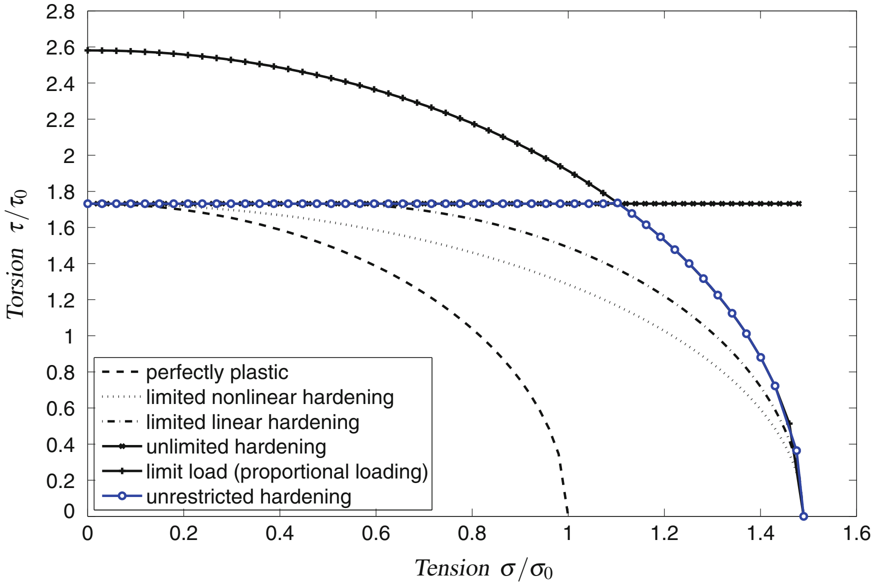

Fig. 2 Shakedown domains for a specimen under constant tension and alternating torsion with different hardening rules

2. **Limit load for proportional loading path**:
   For the proportional loading path, the limit domain is similar to the one in the perfectly plastic case (case 1). The only difference is that σ_Y needs to be substituted by σ_H. Hence, the following maximum value of admissible shear stress is obtained:
   $\sqrt{3}\,\tau_L = \sqrt{\sigma_H^2 - \bar{\sigma}^2}$ (black solid line with + in Fig. 2)
3. **Unlimited kinematical hardening**:
   The shakedown state is defined solely by alternating plasticity. It can be obtained by any hardening rule setting $\sigma_H \to \infty$. The evolution of back-stresses is not restricted at all, and consequently $\forall\bar{\sigma}\ :\ X = \bar{\sigma}$, leading to the following maximum value of admissible shear stress:
   $\sqrt{3}\,\tau_u = \sigma_Y$ (black solid line with × in Fig. 2)
4. **Limited unrestricted kinematical hardening**:
   The unrestricted kinematical hardening is obtained from the solution of the optimization problem ($\mathscr{P}^H$), in which no explicit restriction is formulated for the back-stresses, since no condition in terms of (22a) is accounted for. Since the back-stress X is not restricted, $\forall\bar{\sigma}\ :\ X = \bar{\sigma}$ holds. This leads to alternating plasticity in case of $\bar{\sigma} \le \bar{\sigma}^* = \sqrt{\sigma_H^2 - \sigma_Y^2}$, where the admissible shear stress is τ_u (see case 3).
   On the other hand, if $\bar{\sigma} > \bar{\sigma}^*$, the bounding condition enforces incremental collapse independently of the back-stresses. The according bounding condition

is the one given by WGW and Pham:

$$f_H(\boldsymbol{\sigma}, \sigma_H) = \bar{\sigma}^2 + 3\,\tau^2 - \sigma_H^2 = 0 \tag{29}$$

This leads to the following maximum value of shear stresses:

$$\sqrt{3}\,\tau = \sqrt{\sigma_H^2 - \bar{\sigma}^2} \tag{30}$$

As a result, the maximum value of admissible shear stress reads:

$$\sqrt{3}\,\tau_{un} = \begin{cases} \sqrt{3}\,\tau_u = \sigma_Y & \text{if } \bar{\sigma} \le \bar{\sigma}^* \\ \sqrt{3}\,\tau_L = \sqrt{\sigma_H^2 - \bar{\sigma}^2} & \text{if } \bar{\sigma} > \bar{\sigma}^* \end{cases} \quad \text{(blue solid line with} \circ \text{ in Fig. 2)}$$

It should be mentioned, that the yield surface is allowed to partly move outside of the bounding surface, as long as the considered stress points on the yield surface $(\bar{\sigma}, \tau_{\max})$ and $(\bar{\sigma}, \tau_{\min})$ stay inside.

5. **Limited linear kinematical hardening**:
For the limited linear kinematical hardening, the hardening rule of Prager [29] is applied: $\dot{\boldsymbol{\pi}} = C\,\dot{\boldsymbol{\varepsilon}}^p$, where C denotes the kinematical hardening modulus and $\dot{\boldsymbol{\varepsilon}}^p$ denotes the plastic strain rate. As shown in [14, 38], the back-stresses are restricted by:

$$f(\boldsymbol{\pi}, \sigma_H) = X^2 - (\sigma_H - \sigma_Y)^2 = 0 \tag{31}$$

Hence, the back-stress $X \le \bar{\sigma}^\circ$ cannot exceed the value $\bar{\sigma}^\circ = \sigma_H - \sigma_Y$. In consequence, alternating plasticity can only occur if $\bar{\sigma} \le \bar{\sigma}^\circ$, because then $X = \bar{\sigma}$ is possible.
On the contrary, for $\bar{\sigma} > \bar{\sigma}^\circ$, the restriction of the back-stresses leads to $X = \bar{\sigma}^\circ$, which enforces incremental collapse. The resulting maximum value of admissible shear stress for limited linear hardening is:

$$\sqrt{3}\,\tau_P = \begin{cases} \sqrt{3}\,\tau_u = \sigma_Y & \text{if } \bar{\sigma} \le \bar{\sigma}^\circ \\ \sqrt{\sigma_Y^2 - (\bar{\sigma} - \bar{\sigma}^\circ)^2} & \text{if } \bar{\sigma} > \bar{\sigma}^\circ \end{cases} \quad \text{(dotted line in Fig. 2)}$$

It is worth to mention, that this is in agreement with the solution presented in [3] for a specific nonlinear Prager's rule: $\dot{\boldsymbol{\pi}} = C\,\dot{\boldsymbol{\varepsilon}}^p - (\gamma/C)^2\,X_{eq}^2\,\dot{\boldsymbol{\varepsilon}}^p$. The back-stress corresponding to the stabilized cycle has only to be replaced by $\bar{\sigma}^\circ = C/\gamma$.
Further, it should be mentioned that this is exactly the result which can be obtained by using the theorem by Stein, because (31) obviously corresponds to (21b). This could be expected, since the considered hardening law is based on an associated flow rule.

6. **Limited nonlinear kinematical hardening**:
Finally, for the limited nonlinear kinematical hardening the hardening rule of Armstrong and Frederick [2] is used:

$$\dot{\boldsymbol{\pi}} = \frac{2}{3}\,C\,\dot{\boldsymbol{\varepsilon}}^p - C\,\frac{\boldsymbol{\pi}}{X_\infty}\,\dot{p} \tag{32}$$

where $\dot{p} = \sqrt{\frac{2}{3}\,\dot{\boldsymbol{\varepsilon}}^p : \dot{\boldsymbol{\varepsilon}}^p}$ and $X_\infty = \bar{\sigma}^\circ = \sigma_H - \sigma_Y$.

For this hardening law, the considered example has been intensively investigated by several authors. The first solution has been proposed by Lemaitre and Chaboche [17], followed by Saxcé and coworkers [7], where an analytical solution is derived and verified by an alternative theoretical calculation on the basis of the bipotential approach. In [4, 5] an analytical solution is presented for the plane strain state, which can be transferred to the plane stress state considered here simply by setting $\nu = 0$. Moreover, a numerical implementation is given, which is in perfect agreement with the analytical solution. Finally, Staat and Heitzer [36] obtained lower bound results using a finite element computation with basis reduction. In all these references, it turns out that the back-stresses are restricted even more than in the linear hardening case:

$$X = \bar{\sigma}\,\frac{\sigma_H - \sigma_Y}{\sigma_H} \tag{33}$$

Since $\sigma_H > \sigma_Y$, this leads to $\forall\bar{\sigma}\ :\ X < \bar{\sigma}$. Noteworthy, this restriction enforces incremental collapse in the whole loading domain, and alternating plasticity only can occur in case of pure shear, $\bar{\sigma} = 0$. Consequently, a significant influence of the restriction on the back-stresses can be observed, which contradicts some statements presented in literature, see e.g. [1]. The according maximum value of admissible shear stress reads:

$$\sqrt{3}\,\tau_{AF} = \frac{\sigma_Y}{\sigma_H}\sqrt{\sigma_H^2 - \bar{\sigma}^2} \qquad \text{(dash-dot line in Fig. 2)}$$

The results for all of the above mentioned cases are shown in Fig. 2 for an arbitrarily chosen value $\sigma_H/\sigma_Y = 3/2$, where both axes of the plot are scaled to the according shakedown value in the perfectly plastic case, σ_0 and τ_0, respectively.

Noteworthy, the limited linear hardening model [29] predicts a higher shakedown load than the limited nonlinear one [2]. Furthermore, the nonlinear model only predicts failure due to incremental collapse, whereas in the linear case two different regions exist, one of which represents the incremental collapse and the other one represents alternating plasticity. However, both models give a lower value for the shakedown load than the unrestricted one resulting from the optimization problem $(\mathscr{P}^H)$, as expected according to the discussion above.

5 Conclusions

Concluding, it could be shown that the lower bound shakedown analysis is well suited to determine the limit states of materials or structures. In particular, formulations have been investigated which addressed: perfectly plastic behavior, unlimited kinematical hardening, limited linear kinematical hardening, and limited nonlinear kinematical hardening. The most important result of these investigations is the fact that an unrestricted hardening law is implicitly applied whenever the back-stresses are not restricted by an according subsidiary constraint in the optimization problem.

Even more, the result of such unrestricted problems gives the highest shakedown factor, which can be obtained by any hardening law. Nonetheless, specific hardening laws can be incorporated as indicated above by formulating additional constraints in the optimization problem. This is still covered by the shakedown theorem and its proof. In that sense, the general character of the theorem itself is not curtailed by the discussion above.

References

1. Abdel-Karim M (2005) Shakedown of complex structures according to various hardening rules. Int J Press Vessel Pip 82(5):427–458
2. Armstrong P-J, Frederick C-O (1966) A mathematical representation of the multiaxial Bauschinger effect. Technical report, C.E.G.B. Report No. RD/B/N907
3. Bodovillé G, De Saxcé GD (2001) Plasticity with nonlinear kinematic hardening: modelling and shakedown analysis by the bipotential approach. Eur J Mech A/Solids 20:99–112
4. Bouby C, De Saxcé G, Tritsch J-B (2006) A comparison between analytical calculations of the shakedown load by the bipotential approach and step-by-step computations for elastoplastic materials with nonlinear kinematic hardening. Int J Solids Struct 43:2670–2692
5. Bouby C, De Saxcé G, Tritsch J-B (2009) Shakedown analysis: comparison between models with the linear unlimited, linear limited and non-linear kinematic hardening. Mech Res Commun 36:556–562
6. Corigliano A, Maier G, Pycko S (1995) Kinematic criteria of dynamic shakedown extended to nonassociate constitutive laws with saturation nonlinear hardening. Redic Accademia Lincei IX 6:55–64
7. De Saxcé GD, Tritsch J-B, Hjiaj M (2000) Shakedown of elastic-plastic structures with nonlinear kinematic hardening by the bipotential approach. In: Weichert D, Maier G (eds) Inelastic analysis of structures under variable loads. Kluwer Academic Publishers, Dordrecht
8. Fuschi P (1999) Structural shakedown for elastic-plastic materials with hardening saturation surface. Int J Solids Struct 36:219–240
9. Groß-Weege J (1997) On the numerical assessment of the safety factor of elastic-plastic structures under variable loading. Int J Mech Sci 39(4):417–433
10. Groß-Weege J, Weichert D (1992) Elastic-plastic shells under variable mechanical and thermal loads. Int J Mech Sci 34:863–880
11. Halphen B, Nguyen QS (1975) Sur les materiaux standards generalises. J Mech 14:39–63
12. Heitzer M (1999) Traglast- und Einspielanalyse zur Bewertung der Sicherheit passiver Komponenten, PhD thesis, Forschungszentrum Jülich, RWTH Aachen, Germany
13. Heitzer M, Pop G, Staat M (2000) Basis reduction for the shakedown problem for bounded kinematical hardening material. J Glob Optim 17:185–200
14. Heitzer M, Staat M, Reiners H, Schubert F (2003) Shakedown and ratcheting under tension-torsion loadings: analysis and experiments. Nucl Eng Des 225:11–26
15. Koiter WT (1960) General theorems for elastic-plastic solids. In: Sneddon IN, Hill R (eds) Progress in solid mechanics. North-Holland Publisher, Amsterdam, pp 165–221
16. König JA (1987) Shakedown of elastic-plastic structures. Elsevier, Amsterdam
17. Lemaitre J, Chaboche J-L (1990) Mechanics of solid materials. Cambridge University Press, Cambridge
18. Mandel J (1976) Adaptation d'une structure plastique ecrouissable et approximations. Mech Res Commun 3:483–488
19. Melan E (1938) Der Spannungszustand eines "Mises-Hencky'schen" Kontinuums bei veränderlicher Belastung. Sitzungsber Akad Wiss Wien, Math-Nat Kl, Abt IIa 147:73–87
20. Melan E (1938) Zur Plastizität des räumlichen Kontinuums. Ing-Arch 9:116–126

21. Nguyen Q-S (2003) On shakedown analysis in hardening plasticity. J Mech Phys Solids 51:101–125
22. Pham DC (2007) Shakedown theory for elastic plastic kinematic hardening bodies. Int J Plast 23:1240–1259
23. Pham DC (2008) On shakedown theory for elastic-plastic materials and extensions. J Mech Phys Solids 56:1905–1915
24. Pham DC, Weichert D (2001) Shakedown analysis for elastic-plastic bodies with limited kinematical hardening. Proc R Soc Lond A 457:1097–1110
25. Pham PT, Vu DK, Tran TN, Staat M (2010) An upper bound algorithm for shakedown analysis of elastic-plastic bounded linearly kinematic hardening bodies. In: Proceedings of the ECCM 2010
26. Polizzotto Castrenze (1986) A convergent bounding principle for a class of elastoplastic strain-hardening solids. Int J Plast 2(4):359–370
27. Polizzotto Castrenze (2010) Shakedown analysis for a class of strengthening materials within the framework of gradient plasticity. Int J Plast 26(7):1050–1069
28. Portier L, Challoch S, Marquis D, Geyer P (2000) Ratchetting under tension-torsion loadings: experiments and modelling. Int J Plast 16:303–335
29. Prager W (1959) An introduction to plasticity. Addison-Wesley, London
30. Pycko S, Maier G (1995) Shakedown theorems for some classes of nonassociative hardening elastic-plastic material models. Int J Plast 11(4):367–395
31. Simon J-W (2013) Direct evaluation of the limit states of engineering structures exhibiting limited, nonlinear kinematical hardening. Int J Plast 42:141–167
32. Simon J-W, Weichert D (2011) Numerical lower bound shakedown analysis of engineering structures. Comput Methods Appl Mech Eng 200:2828–2839
33. Simon J-W, Weichert D (2012) Interior-point method for lower bound shakedown analysis of von Mises-type materials. In: de Saxcé G, Oueslati A, Charkaluk E, Tritsch J-B (eds) Limit states of materials and structures—direct methods 2. Springer, Berlin, pp 103–128
34. Simon J-W, Weichert D (2012) Shakedown analysis of engineering structures with limited kinematical hardening. Int J Solids Struct 49(4):2177–2186
35. Simon J-W, Weichert D (2012) Shakedown analysis with multidimensional loading spaces. Comput Mech 49(4):477–485
36. Staat M, Heitzer M (2002) The restricted influence of kinematical hardening on shakedown loads. In: Proceedings of the WCCM V
37. Stein E, Zhang G, König JA (1992) Shakedown with nonlinear strain-hardening including structural computation using finite element method. Int J Plast 8(1):1–31
38. Stein E, Zhang G, Huang Y (1993) Modeling and computation of shakedown problems for nonlinear hardening materials. Comput Methods Appl Mech Eng 103(1–2):247–272
39. Stein E, Zhang G, Mahnken R, König JA (1990) Micromechanical modelling and computation of shakedown with nonlinear kinematic hardening including examples for 2-D problems. In: Axelard DR, Muschik W (eds) Recent developments of micromechanics. Springer, Berlin
40. Weichert D, Groß-Weege J (1988) The numerical assessment of elastic-plastic sheets under variable mechanical and thermal loads using a simplified two-surface yield condition. Int J Mech Sci 30(10):757–767