Examples of the Usage of Infinities and Infinitesimals in Numerical Computations

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Abstract. It is well known that traditional computers work numerically with finite numbers only and situations where a use of infinite or infinitesimal quantities is required are studied mainly theoretically by human beings. In this paper, a recently introduced computational methodology that has been proposed with the intention to change this differentiation is discussed. It is based on the principle 'The part is less than the whole' applied to all quantities (finite, infinite, and infinitesimal) and to all sets and processes (finite and infinite). The methodology uses as a computational device the Infinity Computer (patented in USA, EU, and Russian Federation) working numerically with infinite and infinitesimal numbers that can be written using a numeral positional system with an infinite base. On a number of examples it is shown that it becomes possible both to execute computations of a new type and to simplify computations where infinity and/or infinitesimals are required.

Keywords: Numerical infinities and infinitesimals · Infinity computer · Numeral systems

1 Introduction

Even though there exist codes allowing one to work *symbolically* with ∞ and other symbols related to the concepts of infinity and infinitesimals, traditional computers work *numerically* only with finite numbers and situations where the usage of infinite or infinitesimal quantities is required are studied mainly theoretically (see [2,3,6,8,9,12,13,18,19,38] and references given therein). Many among the approaches developed for this purpose are rather old: ancient Greeks following Aristotle distinguished the potential infinity from the actual infinity; John Wallis (see [38]) credited as the person who has introduced the infinity symbol, ∞, has published his work *Arithmetica infinitorum* in 1655; the foundations of analysis we use nowadays have been developed more than 200 years; Georg Cantor (see [2]) has introduced his cardinals and ordinals more than 100 years ago, as well.

C. Pizzuti and G. Spezzano (Eds.): WIVACE 2014, CCIS 445, pp. 190–200, 2014.
DOI: 10.1007/978-3-319-12745-3_15

The fact that numerical manipulations with infinities and infinitesimals have not been implemented so far on computers can be explained by several difficulties. Obviously, among them we can mention the fact that arithmetics developed for this purpose are quite different with respect to the way of computing we use when we deal with finite quantities. For instance, there exist undetermined operations ($\infty - \infty$, $\frac{\infty}{\infty}$, etc.) that are absent when we work with finite numbers. There exist also practical difficulties that preclude an implementation of numerical computations with infinity and infinitesimals. For example, it is not clear how to store an infinite quantity in a finite computer memory.

A new computational methodology introduced recently (see [23,29,33]) allows one to look at infinities and infinitesimals in a new way and to execute *numerical* computations with infinities and infinitesimals on the Infinity Computer patented in USA (see [27]) and other countries. Nowadays there exists a rapidly growing international scientific community that has developed a number of interesting theoretical and applied results using the new methodology in several research areas.

Among them it is worthy to mention studies linking the new approach to the historical panorama of ideas dealing with infinity and infinitesimals (see [14–16,35]). Then, the new methodology has been applied for studying Euclidean and hyperbolic geometry (see [17,20]), percolation (see [10,11,37]), fractals (see [22,24,32,37]), numerical differentiation and optimization (see [4,25,30,40]), infinite series and the Riemann zeta function (see [26,31,39]), the first Hilbert problem and Turing machines (see [28,35,36]), cellular automata (see [5]), ordinary differential equations (see [34]), etc.

In this paper, we briefly describe the new methodology and provide a number of examples showing how it can be used in different situations where infinities and infinitesimals are required. An interested reader is invited to have a look at surveys [23,29,33] and the book [21] written in a popular way.

2 A New Standpoint on Infinity and a New Numeral System

In order to start, let us remind that there exists a distinction (being very important for the new methodology) between *numbers* and *numerals*. A *numeral* is a symbol (or a group of symbols) that represents a *number*. A *number* is a concept that a *numeral* expresses. The same number can be represented by different numerals. For example, the symbols ‘9’, ‘nine’, ‘IIIIIIIII’, and ‘IX’ are different numerals, but they all represent the same number. Rules used to write down numerals together with algorithms for executing arithmetical operations form a *numeral system*.

It is necessary to remind also that different numeral systems can express different sets of numbers and they can be more or less suitable for executing arithmetical operations. Even the powerful positional system is not able to express, e.g., the number π by a finite number of symbols (the finiteness is essential for executing numerical computations) and this special numeral, π, is deliberately introduced to express the desired quantity. There exist many numeral systems that are weaker than the positional one. For instance, Roman numeral system is not able to express zero and negative numbers and such an expression as III–V is an indeterminate form in this numeral system. As a result, before appearing the positional numeral system and inventing zero mathematicians were not able to create theorems involving zero and negative numbers and to execute computations with them. Thus, developing new (more powerful than existing ones) numeral systems can help a lot both in theory and practice of computations.

There exist very weak numeral systems allowing their users to express a very limited quantity of numbers and one of them will be illuminating for our study. This numeral systems is used by a primitive tribe, Pirahã, living in Amazonia nowadays. A study published in *Science* in 2004 (see [7]) describes that these people use an extremely simple numeral system for counting: one, two, many. For Pirahã, all quantities larger than two are just 'many' and such operations as $2+2$ and $2+1$ give the same result, i.e., 'many'. Using their weak numeral system Pirahã are not able to see, for instance, numbers 3, 4, and 5, to execute arithmetical operations with them, and, in general, to say anything about these numbers because in their language there are neither words nor concepts for that. It is worthy to mention that the result 'many' is not wrong. It is just inaccurate. The introduction of a numeral system having numerals for expressing numbers 3 and 4 leads to a higher accuracy of computations and allows one to distinguish results of operations $2+1$ and $2+2$.

The weakness of the numeral system of Pirahã leads also to the following results

$$\text{'many'} + 1 = \text{'many'}, \qquad \text{'many'} + 2 = \text{'many'}, \qquad \text{'many'} + \text{'many'} = \text{'many'}$$

that are crucial for changing our outlook on infinity. In fact, by changing in these relations 'many' with ∞ we get relations used to work with infinity in the traditional calculus

$$\infty + 1 = \infty, \qquad \infty + 2 = \infty, \qquad \infty + \infty = \infty.$$

This comparison suggests that our difficulty in working with infinity is not connected to the nature of infinity but is a result of inadequate numeral systems used to express infinite numbers. In order to increase the accuracy of computations with infinities, the computational methodology developed in [21,23,29] proposes a new numeral system that avoids situations similar to 'many'$+1 =$ 'many' providing results ensuring that if a is a numeral written in this numeral system then for any a (i.e., a can be finite, infinite, or infinitesimal) it follows $a+1 > a$.

The new numeral system works as follows. A new infinite unit of measure expressed by the numeral ① called *grossone* is introduced as the number of elements of the set, $\mathbb{N}$, of natural numbers. Concurrently with the introduction of grossone in the mathematical language all other symbols (like ∞, Cantor's ω, $\aleph_0, \aleph_1, ...$, etc.) traditionally used to deal with infinities and infinitesimals are excluded from the language because ① and other numbers constructed with its help not only can be used instead of all of them but can be used with a higher accuracy. Grossone is introduced by describing its properties postulated by the Infinite Unit Axiom (see [23,29]) added to axioms for real numbers (similarly, in order to pass from the set, $\mathbb{N}$, of natural numbers to the set, $\mathbb{Z}$, of integers a new element – zero expressed by the numeral 0 – is introduced by describing its properties).

The new numeral ① allows us to construct different numerals expressing different infinite and infinitesimal numbers and to execute computations with all of them. As a result, instead of the usual symbol ∞ different infinite and/or infinitesimal numerals can be used. Indeterminate forms are not present and, for example, the following relations hold for infinite numbers ①, $①^2$ and $①^{-1}$, $①^{-2}$ (that are infinitesimals), as for any other (finite, infinite, or infinitesimal) number expressible in the new numeral system

$$0 \cdot ① = ① \cdot 0 = 0, \quad ① - ① = 0, \quad \frac{①}{①} = 1, \quad ①^0 = 1, \quad 1^{①} = 1, \quad 0^{①} = 0,$$

$$0 \cdot ①^{-1} = ①^{-1} \cdot 0 = 0, \quad ①^{-1} > 0, \quad ①^{-2} > 0, \quad ①^{-1} - ①^{-1} = 0,$$

$$\frac{①^{-1}}{①^{-1}} = 1, \quad (①^{-1})^0 = 1, \quad ① \cdot ①^{-1} = 1, \quad ① \cdot ①^{-2} = ①^{-1},$$

$$\frac{①^{-2}}{①^{-2}} = 1, \quad \frac{①^2}{①} = ①, \quad \frac{①^{-1}}{①^{-2}} = ①, \quad ①^2 \cdot ①^{-1} = ①, \quad ①^2 \cdot ①^{-2} = 1.$$

The introduction of the numeral ① allows us to represent more infinite and infinitesimal numbers in a unique framework. For this purpose a new numeral system similar to traditional positional numeral systems was introduced in [21,23]. To construct a number C in the numeral positional system with base ①, we subdivide C into groups corresponding to powers of ①:

$$C = c_{p_m} ①^{p_m} + \ldots + c_{p_1} ①^{p_1} + c_{p_0} ①^{p_0} + c_{p_{-1}} ①^{p_{-1}} + \ldots + c_{p_{-k}} ①^{p_{-k}}. \quad (1)$$

Then, the record

$$C = c_{p_m} ①^{p_m} \ldots c_{p_1} ①^{p_1} c_{p_0} ①^{p_0} c_{p_{-1}} ①^{p_{-1}} \ldots c_{p_{-k}} ①^{p_{-k}} \quad (2)$$

represents the number C, where all numerals $c_i \neq 0$, they belong to a traditional numeral system and are called *grossdigits*. They express finite positive or negative numbers and show how many corresponding units $①^{p_i}$ should be added or subtracted in order to form the number C. Note that in order to have a possibility to store C in the computer memory, values k and m should be finite.

Numbers p_i in (2) are sorted in the decreasing order with $p_0 = 0$

$$p_m > p_{m-1} > \ldots > p_1 > p_0 > p_{-1} > \ldots p_{-(k-1)} > p_{-k}.$$

They are called *grosspowers* and they themselves can be written in the form (2). In the record (2), we write $①^{p_i}$ explicitly because in the new numeral positional system the number i in general is not equal to the grosspower p_i. This gives the possibility to write down numerals without indicating grossdigits equal to zero.

The term having $p_0 = 0$ represents the finite part of C since $c_0①^0 = c_0$. Terms having finite positive grosspowers represent the simplest infinite parts of C. Analogously, terms having negative finite grosspowers represent the simplest infinitesimal parts of C. For instance, the number $①^{-1} = \frac{1}{①}$ mentioned above is infinitesimal. Note that all infinitesimals are not equal to zero. In particular, $\frac{1}{①} > 0$ since it is a result of division of two positive numbers.

A number represented by a numeral in the form (2) is called *purely finite* if it has neither infinite not infinitesimals parts. For instance, 4 is purely finite and $4 + 3.5①^{-1}$ is not. All grossdigits c_i are supposed to be purely finite. Purely finite numbers are used on traditional computers and for obvious reasons have a special importance for applications.

All of the numbers introduced above can be grosspowers, as well, giving thus a possibility to have various combinations of quantities and to construct terms having a more complex structure. However, in this paper we consider mainly purely finite grosspowers.

Before we conclude this section let us mention that the new numeral system, as all numeral systems, cannot express all numbers and give answers to all questions. Let us consider, for instance, the set of *extended natural numbers* indicated as $\widehat{\mathbb{N}}$ and including $\mathbb{N}$ as a proper subset

$$\widehat{\mathbb{N}} = \{\underbrace{1, 2, \ldots, ① - 1, ①}_{\text{Natural numbers}}, ① + 1, ① + 2, \ldots, ①^2 - 1, ①^2, ①^2 + 1, \ldots\}. \qquad (3)$$

What can we say with respect to the number of elements of the set $\widehat{\mathbb{N}}$? The introduced numeral system based on ① is too weak to give answers to this question. It is necessary to introduce in a way a more powerful numeral system by defining new numerals (for instance, ②, ③, etc.).

We finish this section by emphasizing that different numeral systems, if they have different accuracies, cannot be used together. For instance, the usage of '*many*' from the language of Pirahã in the record 4 + '*many*' has no any sense because for Pirahã it is not clear what 4 is and for people knowing what 4 is the accuracy of the answer 'many' is too low. Analogously, the records of the type $①+\omega$, $①-\aleph_0$, $①/\infty$, etc. have no sense because they include numerals developed under different methodological assumptions, in different mathematical contests, for different purposes, and, finally, numeral systems these numerals belong to have different accuracies.

3 Examples of Computations with Infinities and Infinitesimals

Let us start by giving an example of multiplication of two infinite numbers A and B written in the numeral system (1), (2) (for a comprehensive description on arithmetical operations see [21,23]).

Example 1. Let us consider numbers A and B, where

$$A = 2.4①^{45.3}5.8①^{-7.2}, \qquad B = 6.3①^{5.8}7.1①^{0}9①^{-4.3}.$$

The number A has one infinite part, $2.4①^{45.3}$, and one infinitesimal part equal to $5.8①^{-7.2}$. The number B has one infinite part, $6.3①^{5.8}$, the finite part, 7.1 (remind that $①^0 = 1$), and the infinitesimal one, $9①^{-4.3}$. Their product C is equal to

$$C = B \cdot A = 15.12①^{51.1}17.04①^{45.3}21.6①^{41}36.54①^{-1.4}41.18①^{-7.2}52.2①^{-11.5}.$$

It has three infinite parts and three infinitesimal ones. □

The new approach gives the possibility to develop a new Analysis (see [26]) where functions assuming not only finite values but also infinite and infinitesimal ones can be studied. For all of them it becomes possible to introduce a new notion of continuity that is closer to our modern physical knowledge. Functions assuming finite and infinite values can be differentiated and integrated.

Example 2. The function $f(x) = x^{2.5}$ has the first derivative $f'(x) = 2.5x^{1.5}$ and both $f(x)$ and $f'(x)$ can be evaluated at infinite and infinitesimal x. Thus, for infinite $x = ①$ we obtain infinite values

$$f(①) = ①^{2.5}, \qquad f'(①) = 2.5①^{1.5}$$

and for infinitesimal $x = 9①^{-1.5}$ we have

$$f(9①^{-1.5}) = 243①^{-3.75}, \qquad f'(9①^{-1.5}) = 67.5①^{-2.25}.$$

Both values, $f(9①^{-1.5})$ and $f'(9①^{-1.5})$, are infinitesimal. □

We can also work with functions defined by formulae including infinite and infinitesimal numbers.

Example 3. The function $f(x) = \frac{1}{①}x^2 + ①x$ has a quadratic term infinitesimal and the linear one infinite. It has the first derivative $f'(x) = \frac{2}{①}x + ①$. For infinite $x = 4①$ we obtain infinite values

$$f(①) = 4①^2 + 16①, \qquad f'(①) = ① + 8$$

and for infinitesimal $x = ①^{-1}$ we have

$$f(①^{-1}) = 1 + ①^{-3}, \qquad f'(①^{-1}) = ① + 2①^{-2}.$$ □

By using the new numeral system it becomes possible to measure certain infinite sets. As we have seen above, relations of the type 'many' + 1 = 'many' are consequences of the weakness of numeral systems applied to express numbers. Thus, one of the principles of the new computational methodology consists of adopting the principle 'The part is less than the whole' to all numbers (finite, infinite, and infinitesimal) and to all sets and processes (finite and infinite).

Example 4. Grossone has been introduced in such a way that (see [14,28,29] for a detailed discussion) the sets of even and odd numbers have $①/2$ elements each. The set, $\mathbb{Z}$, of integers has $2①+1$ elements ($①$ positive elements, $①$ negative elements, and zero). Within the countable sets and sets having cardinality of the continuum it becomes possible to distinguish infinite sets having different number of elements expressible in the numeral system using grossone and to see that, for instance,

$$\frac{①}{2} < ① - 1 < ① < ① + 1 < 2① + 1 < 2①^2 - 1 < 2①^2 < 2①^2 + 1 <$$

$$2①^2 + 2 < 2^{①} - 1 < 2^{①} < 2^{①} + 1 < 10^{①} < ①^{①} - 1 < ①^{①} < ①^{①} + 1. \quad \square$$

In order to see how the principle 'The part is less than the whole' agrees with our traditional views on infinite sets, let us consider the following two examples. The first of them is related to the one-to-one correspondence and takes its origins in studies of Galileo Galilei.

Example 5. The traditional point of view: even numbers can be put in a one-to-one correspondence with all natural numbers in spite of the fact that they are a part of them:

$$\begin{array}{ll} \text{even numbers:} & 2,\ 4,\ 6,\ 8,\ 10,\ 12,\ \dots \\ & \updownarrow\ \ \updownarrow\ \ \updownarrow\ \ \updownarrow\ \ \ \updownarrow\ \ \ \updownarrow \\ \text{natural numbers:} & 1,\ 2,\ 3,\ 4\ \ 5,\ \ 6,\ \dots \end{array} \qquad (4)$$

The usual conclusion is that both sets are countable and they have the same cardinality $\aleph_0$. However, now we know that when one executes the operation of counting the accuracy of the result depends on the numeral system used for counting. Since for cardinal numbers it follows

$$\aleph_0 + 1 = \aleph_0, \qquad \aleph_0 + 2 = \aleph_0, \qquad \aleph_0 + \aleph_0 = \aleph_0,$$

these relations suggest that the accuracy of the cardinal numeral system of Alephs is not sufficiently high to see the difference with respect to the number of elements of the two sets.

In order to look at the record (4) using the new numeral system we need the following fact from [21]: ① is even. It is also necessary to remind that numbers that are larger than ① are not natural, they are extended natural numbers. For instance, $① + 2$ is even but not natural, it is extended natural, see (3). Since the number of elements of the set of even numbers is equal to $\frac{①}{2}$, we can write down not only initial (as it is usually done traditionally) but also the final part of (4)

$$\begin{array}{l} 2,\ 4,\ 6,\ 8,\ 10,\ 12,\ \dots\ ① - 4,\ \ ① - 2,\ \ ① \\ \updownarrow\ \ \updownarrow\ \ \updownarrow\ \ \updownarrow\ \ \ \updownarrow\ \ \ \updownarrow\ \qquad\quad \updownarrow \qquad\quad \updownarrow \qquad \updownarrow \\ 1,\ 2,\ 3,\ 4\ \ 5,\ \ 6,\ \dots\ \frac{①}{2} - 2,\ \frac{①}{2} - 1,\ \frac{①}{2} \end{array} \qquad (5)$$

concluding so (4) in a complete accordance with the principle 'The part is less than the whole'. Both records, (4) and (5), are correct but (5) is more accurate, since it allows us to observe the final part of the correspondence that is invisible if (4) is used. □

In order to become more familiar with natural and extended natural numbers we provide one more example.

Example 6. We consider the set of natural numbers, $\mathbb{N}$, and multiply each of its elements by 2. We would like to study the resulting set, that will be called $\mathbb{E}^2$ hereinafter, to calculate the number of its elements and to specify which among its elements are natural and which ones are extended natural numbers and how many they are.

The introduction of the new numeral system allows us to write down the set, $\mathbb{N}$, of natural numbers in the form

$$\mathbb{N} = \{1, 2, \ \dots \ \frac{①}{2} - 2, \frac{①}{2} - 1, \frac{①}{2}, \frac{①}{2} + 1, \frac{①}{2} + 2, \ \dots \ ① - 2, \ ① - 1, \ ①\}.$$

By definition, the number of elements of $\mathbb{N}$ is equal to ①. Thus, after multiplication of each of the elements of $\mathbb{N}$ by 2, the resulting set, $\mathbb{E}^2$, will also have grossone elements. In particular, the number $\frac{①}{2}$ multiplied by 2 gives us ① and $\frac{①}{2} + 1$ multiplied by 2 gives us $① + 2$ that is even extended natural number. Analogously, the last element of $\mathbb{N}$, i.e., ①, multiplied by 2 gives us $2①$. Thus, the set of even numbers $\mathbb{E}^2$ can be written as follows

$$\mathbb{E}^2 = \{2, 4, 6, \ \dots \ ① - 4, ① - 2, ①, ① + 2, ① + 4, \ \dots \ 2① - 4, 2① - 2, 2①\},$$

where numbers $\{2, 4, 6, \ \dots \ ① - 4, ① - 2, ①\}$ are even and natural (they are $\frac{①}{2}$) and numbers $\{① + 2, ① + 4, \ \dots \ 2① - 4, 2① - 2, 2①\}$ are even and extended natural, they also are $\frac{①}{2}$. □

The last example is taken from [4]. It is related to the field of nonlinear constrained optimization being an important class of problems with a broad range of scientific and engineering applications. In the literature, there exists a number of algorithms proposed for solving this kind of problems (see, e.g., [1] and references given therein). The authors of [4] concentrate their attention on optimization methods using penalty functions to reduce the original constrained problem to an unconstrained one. Traditionally, this kind of methods requires to solve a sequence of unconstrained minimization problems for increasing values of a penalty parameter and it is necessary to understand what is the point the sequence of solutions converges to. The following example shows that there exist situations where the usage of ① as a penalty coefficient allows us to avoid the necessity to solve such a sequence of problems.

Example 7. The authors of [4] consider the following quadratic two-dimensional optimization problem with a single linear constraint

$$\min_{x} \ \frac{1}{2}x_1^2 + \frac{1}{6}x_2^2,$$

$$\text{subject to } \ x_1 + x_2 = 1.$$

The pair $(\overline{x}, \overline{\pi})$ is a Karush-Kuhn-Tucker point where $\overline{x} = \begin{bmatrix} 1/4 \\ 3/4 \end{bmatrix}$ and $\overline{\pi} = -\frac{1}{4}$. Then the corresponding unconstrained optimization problem can be constructed using a penalty coefficient P as follows

$$\min_{x} \ \frac{1}{2}x_1^2 + \frac{1}{6}x_2^2 + \frac{P}{2}(1 - x_1 - x_2)^2.$$

For instance, suppose that we have taken $P = 20$ then the first order optimality conditions can be written as follows

$$\begin{cases} x_1 - 20(1 - x_1 - x_2) = 0, \\ \frac{1}{3}x_2 - 20(1 - x_1 - x_2) = 0. \end{cases}$$

Solution to this system of linear equations is the stationary point of the unconstraint problem, namely, it is

$$x_1^*(20) = \frac{20}{81}, \qquad x_2^*(20) = \frac{60}{81}$$

and it is not clear how to obtain the exact pair $(\overline{x}, \overline{\pi})$ from the pair $(x_1^*(20), x_2^*(20))$. Very often people take a sequence $\{p_k\}$ of increasing values of P, solve the problem again and again, and try to understand where the sequence of points $(x_1^*(p_k), x_2^*(p_k))$ converges.

To avoid the necessity to solve a sequence of problems, the authors of [4] propose to use ① as the penalty coefficient P, i.e., to construct the following unconstrained problem

$$\min_{x} \ \frac{1}{2}x_1^2 + \frac{1}{6}x_2^2 + \frac{①}{2}(1 - x_1 - x_2)^2.$$

The first order optimality conditions then are

$$\begin{cases} x_1 - ①(1 - x_1 - x_2) = 0 \\ \frac{1}{3}x_2 - ①(1 - x_1 - x_2) = 0 \end{cases}$$

and the solution is

$$x_1^*(①) = \frac{①}{4①+1}, \qquad x_2^*(①) = \frac{3①}{4①+1}.$$

After division we have

$$x_1^*(①) = \frac{1}{4} - ①^{-1}(\frac{1}{16} - \frac{1}{64}①^{-1} + \ldots), \quad x_2^*(①) = \frac{3}{4} - ①^{-1}(\frac{3}{16} - \frac{3}{64}①^{-1} + \ldots).$$

This means that the finite parts of $x_1^*(①)$ and $x_2^*(①)$ give us the exact solution to the original constrained problem. Moreover,

$$-①(1 - x_1^*(①) - x_2^*(①)) = -\frac{1}{4} + \frac{4}{64}①^{-1} + \ldots$$

i.e., we have $\overline{\pi} = -\frac{1}{4}$. □

Thus, the main issue in this example consists of the fact that finite parts of the results can be easily separated from infinitesimals ones. In contrast, in the traditional approaches this is impossible since both the original problem and parameters work with finite values only. Thus, results $(x_1^*(p_k), x_2^*(p_k))$ provided by traditional methods are finite numbers and, therefore, inside the results corresponding to the unconstrained problem one is not able to see the impact of the parameters (perturbations) on the solution of the original problem.

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