

# On the Classical Lorenz System

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**Abstract** The classical Lorenz system is considered. For many years, this system has been the subject of study by numerous authors. However, until now the structure of the Lorenz attractor is not clear completely yet, and the most important question at present is to understand the bifurcation scenario of chaos transition in this system. Using some numerical results and our bifurcational geometric approach, we present a new scenario of chaos transition in the classical Lorenz system.

**Keywords** Lorenz system · Bifurcation · Singular point · Limit cycle · Chaos

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## 1 Introduction

We consider a three-dimensional dynamical system

$$\dot{x} = \sigma(y - x), \quad \dot{y} = x(r - z) - y, \quad \dot{z} = xy - bz \quad (1)$$

known as the Lorenz system. Historically, (1) was the first dynamical system for which the existence of an irregular attractor (chaos) was proved for  $\sigma = 10$ ,  $b = 8/3$ , and  $24,06 < r < 28$ . For many years, the Lorenz system has been the subject of study by numerous authors; see, e.g., [1–8]. However, until now the structure of the Lorenz attractor is not clear completely yet, and the most important question at present is to understand the bifurcation scenario of chaos transition in system (1).

In Sect. 2 of this paper, we recall a relatively new scenario of chaos transition in the Lorenz system (1) proposed by N.A. Magnitskii and S.V. Sidorov [6]. In Sect. 3,

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we revise this scenario and present a different bifurcation scenario of chaos transition in system (1), where  $\sigma = 10$ ,  $b = 8/3$ , and  $r > 0$ , using numerical results of [6] and our bifurcational geometric approach to the global qualitative analysis of three-dimensional dynamical systems which we applied earlier in the planar case [9–15].

## 2 The Magnitskii–Sidorov Scenario

There exists a contemporary point of view on the structure of the Lorenz attractor and chaos transition in system (1); see [1–8]. However, in [6], it is shown that absolutely another scenario of chaos transition is realized in the Lorenz system (1). It turns out that all cycles from infinite family of unstable cycles, generating the Lorenz attractor [6], have crossing with an one-dimensional unstable not invariant manifold  $V^u$  of the origin of system (1) (do not confuse with the invariant unstable manifold  $W^u$  of this point). This result follows from the theory of dynamical chaos stated in [6]. After the derivation of analytic formulas for the manifold  $V^u$ , it becomes possible to reduce the problem of establishing and proving the existence of unstable cycles in the Lorenz system to the one-dimensional case, namely, to finding stable points of the one-dimensional first return mapping defined on the unstable manifold [6]. By this method, it is shown that some items of the classical scenario of chaos transition in the Lorenz system (1) are invalid, while other require a more detailed investigation. The Magnitskii–Sidorov scenario is the following.

1. The Lorenz system (1) is dissipative and symmetric with respect to the  $z$ -axis. The origin  $O(0, 0, 0)$  is a singular point of system (1) for any  $\sigma$ ,  $b$ , and  $r$ . It is a stable node for  $r < 1$ . For  $r = 1$ , the origin becomes a triple singular point, and then, for  $r > 1$ , there are two more singular points in the system:  $O_1(\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1)$  and  $O_2(-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1)$  which are stable up to the parameter value  $r_a = \sigma(\sigma + b + 3)/(\sigma - b - 1)$  ( $r_a \approx 24,74$  for  $\sigma = 10$  and  $b = 8/3$ ). For all  $r > 1$ , the point  $O$  is a saddle-node. It has a two-dimensional stable manifold  $W^s$  and a one-dimensional unstable manifold  $W^u$ . If  $1 < r < r_1 \approx 13,9$ , then separatrices  $\Gamma_1$  and  $\Gamma_2$  issuing from the point  $O$  along its one-dimensional unstable manifold  $W^u$  are attracted by their nearest stable points  $O_1$  and  $O_2$ , respectively.

2. If  $r = r_1 \approx 13,9$ , then the separatrices  $\Gamma_1$  and  $\Gamma_2$  do not form two separate homoclinic loops. Here we have a bifurcation with the generation of a single closed contour surrounding both stationary points  $O_1$  and  $O_2$ ; the end of the separatrix  $\Gamma_1$  enters the beginning of the separatrix  $\Gamma_2$ , and vice versa, the end of  $\Gamma_2$  enters the beginning of  $\Gamma_1$ . As  $r$  grows, from this contour, a closed cycle  $C_0$  appears there first. It is an eight-shaped figure surrounding both points  $O_1$  and  $O_2$ .

3. If  $r_1 < r < r_2 \approx 24,06$ , then cycles  $L_1$  and  $L_2$  surrounding the points  $O_1$  and  $O_2$ , respectively, do not appear; but with further growth of  $r$ , pairs of cycles  $C_n^+$ ,  $C_n^-$ ,  $n = 0, 1, \dots$ , are successively generated. They determine the generation of the Lorenz attractor. The cycle  $C_n^+$  makes  $n$  complete rotations in the half-space containing the point  $O_1$  and one incomplete rotation around the point  $O_2$ .

Conversely, the cycle  $C_n^-$  makes  $n$  complete rotations around the point  $O_2$  and one incomplete rotation around the point  $O_1$ .

For each  $r$ ,  $r_1 < r < r_2$ , there exists the number  $n(r)$  ( $n(r) \rightarrow \infty$  as  $r \rightarrow r_2$ ) such that in the phase-space of (1), there are unstable cycles  $C_0, C_k^+, C_k^-, k = 0, \dots, n$ , and cycles  $C_{km}^+, C_{km}^-, k, m < n$ , which make  $k$  rotations around the point  $O_1$  and  $m$  rotations around the point  $O_2$  and are various combinations of the cycles  $C_n^+$  and  $C_n^-$ , and many other cycles generated by bifurcations of the cycles  $C_n^+$  and  $C_n^-$  [6]. Points of intersection of all these cycles with the manifold  $V_u$  have the following arrangement on the curve  $V_u$  for  $0 \leq z_{min} \leq z \leq z_{max} < r - 1$ . The point  $z_{min}$  corresponds to the right large single loop of the cycle  $C_n^-$ . This loop is the larger face of the right truncated cone of the set  $S$ . Further, the trajectory of the cycle passes into the left half-plane and makes  $n$  clockwise rotations around the point  $O_2$ . The smallest first loop around the point  $O_2$  is the smaller face of the truncated cone of the set  $S$ . The point  $z_{max}$  corresponds to the smallest loop of the cycle  $C_n^+$  around the point  $O_1$ . This loop is the smaller face of the right truncated cone. Further, the trajectory of this cycle makes  $n$  rotations around the point  $O_1$  clockwise, passes into the left half-plane, and makes one large rotation around the point  $O_2$ . This rotation is the larger face of the left truncated cone. Between the points  $z_{min}$  and  $z_{max}$  there is a point  $z_0$  corresponding to the main cycle  $C_0$ .

Boundaries of the attraction domains of the stable points  $O_1$  and  $O_2$  are given by the smallest loops of the cycles  $C_n^+$  and  $C_n^-$ , whose size decay as  $r$  grows. Therefore, for some  $r = r_m$ , the attraction domain of the set  $B$  no longer intersects the attraction domains of points  $O_1$  and  $O_2$ , and the set  $B$  becomes an attractor. Therefore, in the Lorenz system ( $a = 10, b = 8/3$ ), metastable chaos exists only in the interval  $r_1 < r < r_m$ , and in the interval  $r_m < r < r_2$ , the system has three stable limit sets, namely,  $O_1$  and  $O_2$  and the Lorenz attractor.

If  $r \rightarrow r_2$ , then the eye size decreases as the number of rotations of the cycles  $C_n^+$  and  $C_n^-$  around the points  $O_1$  and  $O_2$ , respectively, grows. The value  $z_{max}$  grows, and  $z_{min}$  decays; moreover,  $z_{min} \rightarrow 0$  as  $r \rightarrow r_2$ . The lengths of generatrices of truncated cones grow, since additional rotations are added to the cone vertex and diminish the size of the smaller face. Conversely, the larger face grows. If  $r = r_2$ , then  $z_{min} = 0$ , but  $z_{max} < r - 1$ ; thus, the larger face of each cone achieves its maximal size, while the smaller face is not contracted into a point, the cone vertex. The following bifurcation takes place. In the limit as  $n \rightarrow \infty$ , each set of cycles  $C_n^+$  (respectively,  $C_n^-$ ) forms a point-cycle heteroclinic structure consisting of two separatrix contours of the point  $O$ . The first contour consists of a separatrix issuing from the point  $O$  along its unstable manifold and spinning on the appearing (only for  $r = r_2$ ) saddle cycle  $L_1$  (respectively,  $L_2$ ) of the point  $O_1$  (respectively,  $O_1$ ). The second contour consists of the separatrix spinning out from the saddle cycle  $L_1$  (respectively,  $L_2$ ) and entering the point  $O$  along its stable manifold.

As mentioned above, the described bifurcation does not lead to generation of the Lorenz attractor for  $r = r_2$ . It is more correct to say that it is only a prerequisite of destruction of the attractor as  $r$  decays. The attractor itself, existing in the system for  $r = r_2$ , is formed from finitely many stable cycles  $C_k^\pm, k = 0, \dots, l$ , for  $r < 313$ . It

contains neither separatrices  $\Gamma_1$  and  $\Gamma_2$  of the point  $O$  nor infinitely many unstable cycles  $C_n^\pm$  existing in the neighborhood of the point-cycle heteroclinic structure.

If  $r_2 < r < r_3 = r_a$ , then points  $O_1$  and  $O_2$  are still stable, and their attraction domains are bound by the appearing limit cycles  $L_1$  and  $L_2$  contracting to points as  $r \rightarrow r_3$ . But the Lorenz attractor  $B$  is not a set of integral curves going from  $L_1$  to  $L_2$  and back, and separatrices  $\Gamma_1$  and  $\Gamma_2$  of the saddle point  $O$  do not belong to the attractor. Cycles  $L_1$  and  $L_2$  have already made their job at  $r = r_2$  and no longer have anything to do with the attractor. If  $r_2 < r < r_3$ , then, just as in the case of  $r_1 < r < r_2$ , the cycles  $C_n^+$  and  $C_n^-$  appear again from separatrix contours. The attractor is determined by finitely many such cycles [6].

4. For  $r = r_3 = r_a \approx 24,74$ , the saddle cycles  $L_1$  and  $L_2$  disappear. In the system, there is a unique limit set, namely, the Lorenz attractor.

5. There exist one more important value of the parameter  $r$  which affects the formation of the Lorenz attractor. This is a point  $r_4 \approx 30,485$ . If  $r$  grows from  $r_3$  to  $r_4$ , then the number of rotations of the cycles  $C_n^+$  and  $C_n^-$  first rapidly decays, then grows again. In this case, eyes by separatrices of the point  $O$  are much smaller than attractor eyes and begin to grow as  $r$  increases. Therefore, almost heteroclinic and almost homoclinic contours exist in system (1) at the point  $r_4$ .

The process of generation of the Lorenz attractor in system (1) as  $r$  decays from the value 313 up to  $r_4$  is referred to as the incomplete double homoclinic cascade [6]. The complete cascade occurs if the  $r$ -axis passes exactly through the point of existence of two homoclinic contours. Note that in systems with a single homoclinic contour, there can be a simple complete or incomplete homoclinic cascade of bifurcations of transition to chaos, and in [6], a detailed description of transition to chaos through the double homoclinic (complete or incomplete) cascade of bifurcations is given. Just as in item 6 of the classical scenario, if  $r > 313$ , then in the system, there exists a unique stable limit cycle  $C_0$  surrounding both points. If  $r \approx 313$ , then the cycle  $C_0$  becomes unstable and generates two stable cycles  $C_0^+$  and  $C_0^-$  which also surround the points  $O_1$  and  $O_2$  but have deflections in the direction of corresponding halves of the unstable manifold  $V^u$  of the point  $O$ . This is the point where the double homoclinic cascade of bifurcations really begins. In case of an incomplete cascade, it consists of finitely many stages of appearance of stable cycles  $C_k^\pm$ ,  $k = 0, \dots, l$ , and their infinitely many further bifurcations. But in case of a complete cascade, the number of stages is infinite, and at the limit of  $l \rightarrow \infty$ , cycles tend to homoclinic contours of the points  $O_1$  and  $O_2$ , respectively. At the  $k$ -th stage of the cascade, originally stable cycles  $C_k^\pm$  undergo a subharmonic cascade of bifurcations and form two band-form attractors that consist of infinitely many unstable limit cycles intersecting the respective domains of the unstable manifold  $V^u$  of the point  $O$ . Then these two bands merge and form a single attractor surrounding both the points  $O_1$  and  $O_2$ , after which there is a cascade of bifurcations of cycles generated as a result of the merger and making rotations separately around the points  $O_1$  and  $O_2$  and simultaneously around both the points. The last cascade of bifurcations has the property of self-organization, since it is characterized by simplification of the structure of cycles and the generation of new stable cycles with a smaller number of rotations around the points  $O_1$  and  $O_2$  as  $r$  decays. Each cycle of the cascade

of self-organization bifurcations undergoes its own subharmonic cascade of bifurcations, after which all cycles formed during infinitely many bifurcations of all subharmonic cascades and cascades of self-organization bifurcations of cycles become unstable and form some set  $B_k$ . After an incomplete homoclinic cascade of bifurcations, we obtain a set  $B = \bigcup B_k$  consisting of infinitely many possible unstable cycles appearing at all stages of the cascade. These cycles generate an incomplete double homoclinic attractor, that is the classical Lorenz attractor.

6. If  $r > 313$ , then the unique stable limit cycle is an attractor in system (1).

### 3 The Bifurcational Geometric Scenario

Revising the above scenario, we present a new scenario of chaos transition in the Lorenz system (1) for  $\sigma = 10$ ,  $b = 8/3$ , and  $r > 0$ .

1. If  $r < 1$ , the unique singular point  $O$  of system (1) is a stable node. For  $r = 1$ , it becomes a triple singular point, and then, for  $r > 1$ , there are two more singular points in the system:  $O_1$  and  $O_2$  which are stable up to the parameter value  $r_a \approx 24,74$ . For all  $r > 1$ , the point  $O$  is a saddle-node. It has a two-dimensional stable manifold  $W^s$  and an one-dimensional unstable manifold  $W^u$ . If  $1 < r < r_l = r_1 \approx 13,9$ , then the separatrices  $\Gamma_1$  and  $\Gamma_2$  issuing from the point  $O$  along its one-dimensional unstable manifold  $W^u$  are attracted by their nearest stable points  $O_1$  and  $O_2$ , respectively.

2. If  $r = r_l$ , then each of the separatrices  $\Gamma_1$  and  $\Gamma_2$  becomes a closed homoclinic loop. In this case, two unstable homoclinic loops,  $C_0^+$  and  $C_0^-$ , are formed around the points  $O_1$  and  $O_2$ , respectively. They are tangent to each other and the  $z$ -axis at the point  $O$  and form together a homoclinic butterfly.

3. If  $r_l < r < r_a \approx 24,74$ , then, unfortunately, neither the classical scenario nor the Magnitskii–Sidorov scenario can be realized. The reason is that, in both cases, trajectories of system (1) should intersect the two-dimensional stable manifold  $W^s$  of the point  $O$ . Since this is impossible, the only way to overcome the contradiction is to suppose that a cascade of period-doubling bifurcations [6] will begin immediately in each of the half-spaces with respect to the manifold  $W^s$ , when  $r > r_l$ . In this case, each of the homoclinic loops  $C_0^+$  and  $C_0^-$  generates an unstable limit cycle of period 2 which makes one rotation around the point  $O_1$  and one rotation around the point  $O_2$  but in the corresponding half-spaces containing the points  $O_1$  and  $O_2$ , respectively, and a stable limit cycle of period 1 lying between the coils of the cycle of period 2. With further growth of  $r$ , each of the cycles of period 2 generates an unstable limit cycle of period 4 with a stable limit cycle of period 3 inside of it and each of the cycles of period 1 generates a stable limit cycle of period 2 with an unstable limit cycle of period 1 inside of it. Then, after next doubling, we will have in each of the half-spaces an unstable limit cycle of period 8 with an inserted stable limit cycle of period 7 and a stable limit cycle of period 6 with an inserted unstable limit cycle of period 5, and a stable limit cycle of period 4 with an inserted unstable limit cycle of period 3, and an unstable limit cycle of period 2 with

an inserted stable limit cycle of period 1. Continuing this process further, we will obtain limit cycles of all periods from one to infinity, and the space between these cycles will be filled by spirals issuing from unstable limit cycles and tending to stable limit cycles as  $t \rightarrow +\infty$ . These cycles are inserted into each other, they make various combinations of rotation around the points  $O_1$  and  $O_2$  in the corresponding half-spaces containing these points and form geometric constructions (limit periodic sets) which look globally like very flat truncated cones described in item 3 of the Magnitskii–Sidorov scenario [6].

4. For  $r = r_a \approx 24,74$ , the biggest unstable limit cycles of infinite period disappear through the Andronov–Shilnikov bifurcation [4, 5] in each of the half-spaces containing the points  $O_1$  and  $O_2$  (the cone vertices are at these points), and these points become unstable saddle-foci.

5. If  $r_a < r < +\infty$ , then a cascade of period-halving bifurcations [6] occurs in each of the half-spaces with respect to the manifold  $W^s$ . We have got again two symmetric with respect to the  $z$ -axis limit periodic sets consisting of limit cycles of all periods which are inserted into each other and make various combinations of rotation around the points  $O_1$  and  $O_2$  in the corresponding half-spaces containing these points, and the space between the cycles is filled by spirals issuing from unstable limit cycles and tending to stable limit cycles as  $t \rightarrow +\infty$ . The biggest limit cycles of these sets are stable now, and with further growth of  $r$ , the period-halving process makes them and the whole limit periodic sets more and more flat. The obtained geometric constructions are the only stable limit sets of system (1). The spirals of the unstable saddle-foci  $O_1$  and  $O_2$  and the trajectories issuing from infinity tend to these limit periodic sets (more precisely, to their stable limit cycles) as  $t \rightarrow +\infty$ . Just these stable limit periodic sets form two symmetric parts of the so-called Lorenz attractor, and this really looks very chaotic.

6. If  $r \rightarrow +\infty$  (numerically, when  $r > 313$ ), then the period-halving process will be finishing and system (1) will have two stable limit cycles in two phase half-spaces containing the unstable saddle-foci  $O_1$  and  $O_2$  of (1). This completes our scenario of chaos transition in the Lorenz system (1).

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