An Approach for Developing Fourier Convolutions and Applications

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Abstract Based on the papers published recently, this talk presents a concept of convolution so-called pair-convolution which is a generalization of known convolutions, and considers applications for solving integral equations.

Keywords Generalized convolution · Integral equation of convolution type · Banach algebra

Mathematics Subject Classification (2010) Primary 42A85 · 45E10 · Secondary 44A20 · 46J10

1 Introduction

It is well-known that the transform

$$(f \underset{\mathcal{F}}{*} g)(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} f(x-y)g(y)dy \tag{1.1}$$

is called the Fourier convolution of two functions g and f, and the following factorization identity holds

$$\mathcal{F}(f \underset{\mathcal{F}}{*} g)(x) = (\mathcal{F}f)(x)(\mathcal{F}g)(x).$$

The above-mentioned convolution was found most early, and nowadays it has been applying widely in both theoretical and practical problems.

We can say that many convolutions, generalized convolutions, and polyconvolutions of the well-known integral transforms as Fourier's, Hankel's, Mellin's,

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V.V. Mityushev, M.V. Ruzhansky (eds.), *Current Trends in Analysis and Its Applications*, Trends in Mathematics, DOI 10.1007/978-3-319-12577-0_57

Laplace's and their applications have been published. Loosely speaking, the theory of convolutions has been strongly developing and vigorously discussing in many research groups (see [1, 2, 4, 5, 7, 10, 11, 14, 15], and references therein).

In fact, convolutions are considered as a powerful tool in many fields of mathematics such as numerical computing, digital, image and signal processing, partial-differential equations, and other fields of mathematics (see [3, 6–16, 20]). Other reason for which the theory of convolutions attracts attention of many mathematicians is that each of convolutions is a new integral transform, therefore it could be a new object of study.

1.1 Present Studies of Convolution Operators

It is easy to show a long list of authors and their works concerning convolution operators such as: A. Böttcher, L.E. Britvina, Yu. Brychkov, L. Castro, I. Feldman, H.J. Glaeske, I. Gohberg, N. Krupnik, O.I. Marichev, S. Saitoh, B. Silbermann, H.M. Srivastava, V.K. Tuan, S.B. Yakubovich… Among those listed, there are many mathematicians leading the potential and strong groups in the worldwide, they have been creating significant discoveries, namely: A. Böttcher (Germany), L.E. Britvina (Ukraina), Yu. Brychkov (Russia), L. Castro and S. Saitoh (Aveiro-Portugal and Gunma-Japan), I. Gohberg (Israel), B. Silbermann (Germany), H.M. Srivastava (Canada), V.K. Tuan (USA), S.B. Yakubovich (Porto, Portugal).

2 An Approach to Developing Convolutions

The nice idea of convolution focuses on the factorization identity. We now deal with the concept of convolutions. Let U_1, U_2, U_3 be the linear spaces on the field of scalars $\mathcal{K}$, and let V be a commutative algebra on $\mathcal{K}$. Suppose that $K_1 \in L(U_1, V)$, $K_2 \in L(U_2, V)$, $K_3 \in L(U_3, V)$ are linear operators from U_1, U_2, U_3 to V respectively. Let δ denote an element in algebra V. We recall the definition of convolutions.

Definition 2.1 (see also [4]) A bilinear map $* : U_1 \times U_2 :\longrightarrow U_3$ is called a convolution associated with K_3, K_1, K_2 (in that order) if the following identity holds

$$K_3\big(*(f,g)\big) = \delta K_1(f) K_2(g),$$

for any $f \in U_1$, $g \in U_2$. Above identity is called the factorization identity of the convolution.

We now deal with several approaches to developing convolutions.

2.1 Using Eigenfunctions

Let Φ_α denote the Hermite function (see [14]).

Theorem 2.2 ([18, 19]) *The following transform defines a convolution*

$$(f \overset{\Phi_\alpha}{\underset{\mathcal{F}}{*}} g)(x) = \frac{i^{|\alpha|}}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(u) g(v) \Phi_\alpha(x-u-v) du dv.$$

Let $r_0 \in \{0, 1, 2, 3\}$ be given, and let

$$\Psi(x) = \sum_{|\alpha| = r_0 \pmod 4} a_\alpha \Phi_\alpha(x) \quad (a_\alpha \in \mathbb{C}) \tag{2.1}$$

be a finite linear combination of the Hermite functions ($|\alpha| \le N$ for some $N \in \mathbb{N}$). The following theorem is an immediate consequence of Theorem 2.2.

Theorem 2.3 *The following transform defines a convolution*

$$(f \overset{\Psi}{\underset{\mathcal{F}}{*}} g)(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(u) g(v) \Psi(x-u-v) du dv.$$

2.2 Trigonometric Weight Functions

Let T_c, T_s denote the Fourier-cosine and Fourier-sine integral transforms. Let $h \in \mathbb{R}^d$ be fixed. Put $\theta_1(x) = \cos xh := \cos(\langle x, h\rangle)$, $\theta_2(x) = \sin xh := \sin(\langle x, h\rangle)$ as there is no danger of confusion.

Theorem 2.4 (see [9, 18, 19]) *Each of the integral transforms* (2.2)–(2.5) *below defines a convolution*:

$$\begin{aligned}(f \overset{\theta_1}{\underset{T_c}{*}} g)(x) = \frac{1}{4(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \big[& f(x-u+h) + f(x-u-h) \\ & + f(x+u+h) + f(x+u-h)\big] g(u) du,\end{aligned} \tag{2.2}$$

$$\begin{aligned}(f \overset{\theta_1}{\underset{T_c, T_s, Ts}{*}} g)(x) = \frac{1}{4(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \big[& -f(x-u+h) - f(x-u-h) \\ & + f(x+u+h) + f(x+u-h)\big] g(u) du,\end{aligned} \tag{2.3}$$

$$\begin{aligned}(f \overset{\theta_2}{\underset{T_c, T_s, T_c}{*}} g)(x) = \frac{1}{4(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \big[& f(x-u+h) - f(x-u-h) \\ & + f(x+u+h) - f(x+u-h)\big] g(u) du,\end{aligned} \tag{2.4}$$

$$(f \underset{T_c,T_c,T_s}{\overset{\theta_2}{*}} g)(x) = \frac{1}{4(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \big[f(x-u+h) - f(x-u-h) - f(x+u+h) + f(x+u-h) \big] g(u)du. \quad (2.5)$$

Comparison 2.5 Different from the convolutions presented by other authors, what above does not require the invertibility of associated transforms. Indeed, according to our point of view as showed in Definition 2.1 the condition about the invertibility of transforms is not needed for constructing convolutions; namely, three operators K_1, K_2, K_3 may be un-injective. As we know that the Fourier-cosine and Fourier-sine transforms T_c and T_s are not injective, but there are still many infinitely many convolutions associated with them as presented in [9, 17–19]. In our point of view, that is a main reason why no convolution for un-invertible transforms appears until this moment. Most of convolution multiplications published are not commutative and not associative.

3 New Concept: Pair-Convolution

In this section we propose a new concept so-called *pair-convolution* which is a considerable generalization of convolution and generalized convolutions.

For any given multi-index $\alpha \in \mathbb{N}^d$, consider the transform

$$D_1(f,g)(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Phi_\alpha(x+u+v) f(u) g(v) du dv.$$

Using the convolutions in [18] we get

- *The case* $|\alpha| = 0 \pmod 4$:

$$T_c\big(D_1(f,g)\big)(x) = \Phi_\alpha(x)\big((T_c f)(x)(T_c g)(x) - (T_s f)(x)(T_s g)(x)\big),$$
$$T_s\big(D_1(f,g)\big)(x) = -\Phi_\alpha(x)\big((T_c f)(x)(T_s g)(x) + (T_s f)(x)(T_c g)(x)\big).$$

- *The case* $|\alpha| = 1 \pmod 4$:

$$T_c\big(D_1(f,g)\big)(x) = \Phi_\alpha(x)\big((T_c f)(x)(T_s g)(x) + (T_s f)(x)(T_c g)(x)\big),$$
$$T_s\big(D_1(f,g)\big)(x) = \Phi_\alpha(x)\big((T_c f)(x)(T_c g)(x) - (T_s f)(x)(T_s g)(x)\big).$$

- *The case* $|\alpha| = 2 \pmod 4$:

$$T_c\big(D_1(f,g)(x)\big) = \Phi_\alpha(x)\big((T_s f)(x)(T_s g)(x) - (T_c f)(x)(T_c g)(x)\big),$$
$$T_s\big(D_1(f,g)\big)(x) = \Phi_\alpha(x)\big((T_c f)(x)(T_s g)(x) + (T_s f)(x)(T_c g)(x)\big).$$

- *The case* $|\alpha| = 3 \pmod 4$:

$$T_c\big(D_1(f,g)\big)(x) = -\Phi_\alpha(x)\big((T_c f)(x)(T_s g)(x) + (T_s f)(x)(T_c g)(x)\big),$$
$$T_s\big(D_1(f,g)\big)(x) = \Phi_\alpha(x)\big((T_s f)(x)(T_s g)(x) - (T_c f)(x)(T_c g)(x)\big).$$

Motivated by the operational identities above, we introduce the following concept. Let U be a linear space, and let V be a commutative algebra on the complex field $\mathbb{C}$. Let $T_1, T_2 \in L(U, V)$ be the linear operators from U to V.

Definition 3.1 A bilinear map $* : U \times U :\longrightarrow U$ is called a *pair-convolution* associated with T_1, T_2, if there exist eight elements $\delta_k \in V$, $k = 1, \ldots, 8$ so that the following identities hold for any $f, g \in U$:

$$T_1\big(*(f, g)\big) = \delta_1 T_1 f T_1 g + \delta_2 T_1 f T_2 g + \delta_3 T_2 f T_1 g + \delta_4 T_2 f T_2 g,$$
$$T_2\big(*(f, g)\big) = \delta_5 T_1 f T_1 g + \delta_6 T_1 f T_2 g + \delta_7 T_2 f T_1 g + \delta_8 T_2 f T_2 g.$$

Example 3.2 The above-mentioned bilinear transform $D_1(\ ,\ )$ is the pair-convolution for T_c, T_s. Note that this transform is not the generalized convolution associated with T_c, T_s.

Example 3.3 Consider the transform

$$D_2(f, g)(x) := \frac{1}{4(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \big[a\Phi_\alpha(x + u + v) + b\Phi_\beta(x + u - v) + c\Phi_\gamma(x - u + v) + d\Phi_\delta(x - u - v)\big] f(u)g(v)dudv,$$

where $a, b, c, d \in \mathbb{C}$, and $\alpha, \beta, \gamma, \delta$ are the multi-indexes. As (2.2), $D_2(\ ,\ )$ is a pair-convolution associated with $\mathcal{F}, \mathcal{F}^{-1}$.

Example 3.4 Let Ψ be the Hermite-type function as defined by (2.1). Write

$$\Psi(x) := \sum_{|\alpha|=0 \pmod 4} a_\alpha \Phi_\alpha(x) + \sum_{|\alpha|=1 \pmod 4} a_\alpha \Phi_\alpha(x) + \sum_{|\alpha|=2 \pmod 4} a_\alpha \Phi_\alpha(x) + \sum_{|\alpha|=3 \pmod 4} a_\alpha \Phi_\alpha(x). \quad (3.1)$$

Using the operational identities of $D_1(\ ,\ )$, we can prove that the transform

$$D_3(f, g)(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Psi(x \pm u \pm v) f(u)g(v)dudv \quad (3.2)$$

is a pair-convolution associated with T_c, T_s.

Example 3.5 Suppose that a_1, a_2, a_3, a_4 are any complex numbers. By Theorem 2.4 we can prove that the transform

$$D_4(f, g)(x) := \frac{1}{4(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \big[a_1 f(x - u + h) + a_2 f(x - u - h) + a_3 f(x + u + h) + a_4 f(x + u - h)\big] g(u)du$$

defines a pair-convolution associated with the transforms T_c, T_s.

We now consider the general integral equation

$$\lambda\varphi(x) + \int_E K(x, y)\varphi(y)dy = f(x). \tag{3.3}$$

We suppose that, by way of decomposing the kernel as

$$K(x, y) = \sum_{n=1}^{N} k_n(x, y) \tag{3.4}$$

such that each one of the transforms

$$(K_n\varphi)(x) = \int_E k_n(x, y)\varphi(y)dy \tag{3.5}$$

is a pair-convolution for specific operators T_1, T_2, then (3.3) may be solved by convolution approach. The main key of this approach is that we can reduce integral equations to a linear algebraic system of functional equations, and then apply an inverse transform of the transform $aT_1 + bT_2$ for some $a, b \in \mathbb{C}$. Thanks to pair-convolutions this approach could be more flexible, and realizable for a larger class of equations.

4 Final Remarks

To summary Sect. 3, we can interpret in other words as: the generalized convolution transforms, and the pair-convolution transforms might be called the *factorisable integrals, and pair-factorisable integrals* respectively by means of two specific transforms.

Problems for Further Studying Construct more pair-convolutions for the well-known integral transforms such as Hilbert, Mellin, Laplace, ..., and look for their applications.

Finally, since the set of all Hermite functions is a normally orthogonal basic of $L^2(\mathbb{R}^d)$, and thanks to the infinitely many pair-convolutions concerning the Hermite functions as presented, we propose the following conjecture.

Conjecture 4.1 *For any function $k \in L^2(\mathbb{R}^d)$, there exists a function $f \in L^2(\mathbb{R}^d)$ sufficiently closed to k such that each one of the transforms*

$$\int_E \int_E k(x \pm u \pm v) f(u)g(v)dudv$$

is either convolution or pair-convolution for specific operators K_1, K_2.

If this fact would be proved, we would have an approximately solvable manner called convolution one which could be different from that of the Galerkin method for Fredholm integral equations.

Acknowledgements This talk is based on the works joint with P.K. Anh, L.P. Castro, B.T. Giang, N.T.T. Huyen, S. Saitoh, P.T. Thao, and P.D. Tuan. This work was supported partially by the Viet Nam National Foundation for Science and Technology Development.

References

1. F. Al-Musallam, V.K. Tuan, A class of convolution transforms. Fract. Calc. Appl. Anal. **3**, 303–314 (2000)
2. P.K. Anh, N.M. Tuan, P.D. Tuan, The finite Hartley new convolutions and solvability of the integral equations with Toeplitz plus Hankel kernels. J. Math. Anal. Appl. **397**, 537–549 (2013)
3. D. Bing, T. Ran, W. Yue, Convolution theorems for the linear canonical transform and their applications. Sci. China, Ser. F **49**(5), 592–603 (2006)
4. L.E. Britvina, Generalized convolutions for the Hankel transform and related integral operators. Math. Nachr. **280**, 962–970 (2007)
5. J.W. Brown, R.V. Churchill, *Fourier Series and Boundary Value Problems* (McGraw-Hill, New York, 2006)
6. L.P. Castro, S. Saitoh, N.M. Tuan, Convolutions, integral transforms and integral equations by means of the theory of reproducing kernels. Opusc. Math. **32**, 651–664 (2012)
7. V. Didenko, B. Silbermann, *Approximation of Additive Convolution-Like Operators. Real C^*-Algebra Approach*. Frontiers in Mathematics (Birkhäuser, Basel, 2008)
8. B.T. Giang, N.V. Mau, N.M. Tuan, Operational properties of two integral transforms of Fourier type and their convolutions. Integral Equ. Oper. Theory **65**, 363–386 (2009)
9. B.T. Giang, N.V. Mau, N.M. Tuan, Convolutions for the Fourier transforms with geometric variables and applications. Math. Nachr. **283**, 1758–1770 (2010)
10. H. Hochstadt, *Integral Equations* (Wiley, New York, 1973)
11. T. Matsuura, S. Saitoh, Analytical and numerical inversion formulas in the Gaussian convolution by using the Paley–Wiener spaces. Appl. Anal. **85**, 901–915 (2006)
12. M.A. Naimark, *Normed Rings* (1959). Groningen, Netherlands
13. K.J. Olejniczak, The Hartley transform, in *The Transforms and Applications Handbook*, ed. by A.D. Poularikas. The Electrical Engineering Handbook Series 2nd edn. (CRC Press/IEEE Press, Florida, 2000)
14. W. Rudin, *Functional Analysis* (McGraw-Hill, New York, 1991)
15. S. Saitoh, V.K. Tuan, M. Yamamoto, Reverse convolution inequalities and applications to inverse heat source problems. J. Inequal. Pure Appl. Math. **3** (2002). 11 pp. (electronic)
16. A.K. Singh, R. Saxena, On convolution and product theorems for FRFT. Wirel. Pers. Commun. **65**(1), 189–201 (2012)
17. N.M. Tuan, Generalized convolutions of the integral transform of Fourier type and applications, in *Progress in Analysis; Proc. of the 8th Congress of ISAAC, Moscow* (2011), pp. 331–338. ISBN 978-5-209-04582-3
18. N.M. Tuan, N.T.T. Huyen, The solvability and explicit solutions of two integral equations via generalized convolutions. J. Math. Anal. Appl. **369**, 712–718 (2010)
19. N.M. Tuan, N.T.T. Huyen, The Hermite functions related to infinite series of generalized convolutions and applications. Complex Anal. Oper. Theory **6**, 219–236 (2012)
20. D. Wei, Q. Ran, Y. Li, J. Ma, L. Tan, A convolution and product theorem for the linear canonical transform. IEEE Signal Process. Lett. **16**(10), 853–856 (2009)