Recent Progress on Spheroidal Monogenic Functions

Hung Manh Nguyen

Abstract Monogenic function theories are considered as generalizations of the holomorphic function theory in the complex plane to higher dimensions and are refinements of the harmonic analysis based on the Laplace operator's factorizations. The construction of spherical monogenic functions has been studied for decades with different methods. Recently, orthogonal monogenic bases are developed for spheroidal reference domains, first by J. Morais and later by others. This survey will go through the construction of spheroidal monogenic functions and discuss up-to-date results.

Keywords Harmonic functions · Monogenic functions · Quaternion analysis · Spheroidal functions

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1 Introduction

The theory of harmonic functions plays an important role in many fields, both in pure and applied aspects. It can be seen, for example, in gravitational potential problems or approximation of the Earth's gravity and magnetic fields (see [18, 23, 31]). Spherical harmonic functions are used frequently because of their simple form and easy calculation. It is preferred for (almost) symmetric geometries. For asymmetric cases, it is inappropriate as shown in [31]. Simple generalizations of spherical domains are ellipsoidal domains. Garabedian introduced in [14] sets of orthogonal harmonic polynomials over prolate and oblate spheroids taken in several different norms. It is the root of the construction of orthogonal spheroidal monogenic functions, since monogenic functions can be obtained by applying the hypercomplex derivative to harmonic functions. The construction of Green's function for the Laplace equation on an ellipsoid of revolution has been studied by means of

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ellipsoidal harmonic functions in several articles (see [12, 22]). As refinements of harmonic functions, these spheroidal monogenic functions could play a role to solve some Dirichlet problems in spheroids.

There are several methods to construct a complete system of monogenic functions. Fueter variables $z_i = x_i - x_0\mathbf{e_i}$ $(i = 1, \ldots, n)$, named after R. Fueter [13], were introduced as an idea to construct bases of homogeneous monogenic polynomials (see [6, 15, 21]). The construction is completely independent of endowed inner products. In general, the obtained sets of monogenic functions are not orthogonal. For spherical domains, one can obtain a complete orthogonal system of monogenic functions by the Gelfand–Tsetlin procedure which calculates functions by induction and then it costs time and memory. More information about the method can be found in [5, 20]. The harmonic function approach was developed based on factorizations of the Laplace operator in terms of Cauchy–Riemann or Dirac operators. To the best of our knowledge, I. Cação firstly used it to construct orthogonal bases for L_2-spaces of reduced quaternion ($\mathcal{A}$)—or quaternion ($\mathbb{H}$)-valued monogenic functions which are solutions of Riesz or Moisil–Theodorescu systems on the unit ball (cf. [7, 9]). The hypercomplex derivative and the monogenic primitive are also studied in [8, 10, 11]. Later on, S. Bock modified the $\mathbb{H}$-valued elements of the basis with respect to the Riesz system to obtain the Appell property. This property was introduced by Appell [1] by generalizing $\frac{d}{dx}x^n = nx^{n-1}$ to more general polynomial systems. Also, S. Bock proved recurrence formulae, an explicit representation formula for polynomials [2–4] and applied it to solve a boundary value problem for the equations of linear elasticity in spherical domains. $\mathcal{A}$-valued solutions of the Riesz system were also researched by J. Morais in the quaternionic setting in a similar way. Properties were investigated such as real part theorems, Bohr's type theorem and local mapping properties by means of spherical monogenic functions (cf. [16, 17, 24]).

The aim of this paper is to give a brief survey about the construction of complete orthogonal monogenic systems on spheroidal domains. In Sect. 3, inner prolate and oblate spheroidal monogenic functions will be revisited. The recurrence formulae and the explicit presentation will be discussed therein. In applications, we also need information on the exterior domain. That is the reason why in Sect. 4, outer spheroidal monogenic functions in the exterior domain of a prolate spheroid are described. Conclusions will be given in the last section.

2 Preliminaries

Let $\mathbb{H}$ be the algebra of real quaternions generated by the basis $\{\mathbf{1}, \mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ subjected to the multiplication rules

$$\mathbf{e_i}\mathbf{e_j} + \mathbf{e_j}\mathbf{e_i} = -2\delta_{ij}, \quad i, j = 1, 2, 3; \qquad \mathbf{e_1}\mathbf{e_2} = \mathbf{e_3}.$$

Each quaternion can be represented in the form $q = q_0 + q_1\mathbf{e_1} + q_2\mathbf{e_2} + q_3\mathbf{e_3}$ where q_j $(j = 0, \ldots, 3)$ are real numbers. Like in the complex case, the conjugate of q is $\overline{q} = q_0 - q_1\mathbf{e_1} - q_2\mathbf{e_2} - q_3\mathbf{e_3}$ and the norm $|q|$ of q is defined by $|q|^2 = q\overline{q} = \overline{q}q = \sum_{j=0}^{3}(q_j)^2$. The real vector space $\mathbb{R}^3$ will be embedded in $\mathbb{H}$ by identifying the element $x = (x_0, x_1, x_2) \in \mathbb{R}^3$ with the reduced quaternion $x := x_0 + x_1\mathbf{e_1} + x_2\mathbf{e_2}$. Denote by $\mathcal{A}$ the real space of all reduced quaternions. The operator $\overline{\partial} = \frac{\partial}{\partial x_0} + \mathbf{e_1}\frac{\partial}{\partial x_1} + \mathbf{e_2}\frac{\partial}{\partial x_2}$ is called generalized Cauchy–Riemann (C–R) operator. Given a domain Ω in $\mathbb{R}^3$, a function $\mathbf{f}$ is called monogenic in Ω if satisfying $\overline{\partial}\mathbf{f}(x) = 0$ for all $x \in \Omega$. The hypercomplex derivative is simply denoted by $\frac{1}{2}\partial$, where ∂ is the conjugate C–R operator. $\mathcal{M}(\Omega, \mathcal{A})$ and $\mathcal{M}(\Omega, \mathbb{H})$ stand for the Hilbert spaces of square integrable $\mathcal{A}$—or $\mathbb{H}$-valued monogenics in Ω respectively, endowed with the inner products

$$\begin{aligned} \langle \mathbf{f}, \mathbf{g} \rangle_{L^2(\Omega;\mathbb{R})} &= \int_\Omega \mathrm{Sc}(\bar{\mathbf{f}}\mathbf{g})dV, \\ \langle \mathbf{f}, \mathbf{g} \rangle_{L^2(\Omega;\mathbb{H})} &= \int_\Omega \bar{\mathbf{f}}\mathbf{g}dV. \end{aligned} \tag{2.1}$$

The induced norm is in both cases $\|\mathbf{f}\|_{L^2(\Omega)} = \langle \mathbf{f}, \mathbf{f} \rangle_{L^2(\Omega)}^{\frac{1}{2}}$. In this paper, let Γ be a spheroid with x_0-axis as the symmetry axis. The equation of Γ is given by

$$\frac{x_0^2}{a^2} + \frac{x_1^2 + x_2^2}{b^2} = 1,$$

where $a = c\cosh\mu_0$, $b = c\sinh\mu_0$ (prolate spheroid) or $a = c\sinh\mu_0$, $b = c\cosh\mu_0$ (oblate spheroid). For the sake of simplicity, it is assumed that $c = 1$. We adopt the notations Ω^+ and Ω^- for the interior and exterior domains of Γ, respectively. In particular, $x \in \Omega^-$ can be given by the spheroidal coordinate

$$x_0 = \cosh\mu\cos\theta, \qquad x_1 = \sinh\mu\sin\theta\cos\varphi, \qquad x_2 = \sinh\mu\sin\theta\sin\varphi,$$

for prolate cases or

$$x_0 = \sinh\mu\cos\theta, \qquad x_1 = \cosh\mu\sin\theta\cos\varphi, \qquad x_2 = \cosh\mu\sin\theta\sin\varphi,$$

for oblate cases, with $\mu \in (\mu_0, \infty)$, $\theta \in [0, \pi)$, $\varphi \in [0, 2\pi)$.

3 Inner Spheroidal Monogenics Revisited

Since 2010, J. Morais has intensively investigated sets of prolate spheroidal monogenic functions which play a role for constructing bases in L_2-spaces of monogenic functions in a prolate spheroid over $\mathbb{R}$ and $\mathbb{H}$ in [25, 26] with applications in [19, 27, 28]. An analogous monogenic system can be constructed for oblate spheroids as shown in [30]. In general, spheroidal monogenic functions have the structure as follows:

$$
\begin{aligned}
X_{n,m} &= \frac{n+m+1}{2} A_{n,m}(\mu,\theta)\cos(m\varphi) \\
&\quad + \frac{\delta}{4(n-m+1)} A_{n,m+1}(\mu,\theta)\big\{\cos\big[(m+1)\varphi\big]\mathbf{e_1} + \sin\big[(m+1)\varphi\big]\mathbf{e_2}\big\} \\
&\quad - \frac{\delta(n+m+1)(n+m)(n-m+2)}{4} A_{n,m-1}(\mu,\theta) \\
&\quad \times \big\{\cos\big[(m-1)\varphi\big]\mathbf{e_1} - \sin\big[(m-1)\varphi\big]\mathbf{e_2}\big\} \\
Y_{n,m} &= \frac{(n+m+1)}{2} A_{n,m}(\mu,\theta)\sin(m\varphi) \\
&\quad + \frac{\delta}{4(n-m+1)} A_{n,m+1}(\mu,\theta)\big\{\sin\big[(m+1)\varphi\big]\mathbf{e_1} - \cos\big[(m+1)\varphi\big]\mathbf{e_2}\big\} \\
&\quad - \frac{\delta(n+m+1)(n+m)(n-m+2)}{4} A_{n,m-1}(\mu,\theta) \\
&\quad \times \big\{\sin\big[(m-1)\varphi\big]\mathbf{e_1} + \cos\big[(m-1)\varphi\big]\mathbf{e_2}\big\}
\end{aligned}
$$

where

$$
A_{n,m}(\mu,\theta) = \sum_{k=0}^{[(n-m)/2]} \delta^{k+1} \frac{(2n+1-4k)(n+m-2k+1)_{2k}}{(n-m-2k+1)_{2k+1}} U_{n-2k,m},
$$

with

$$
A_{n,-1}(\mu,\theta) := \begin{cases} -\frac{1}{n(n+1)^2(n+2)} A_{n,1}(\mu,\theta), & n = 1,2,\ldots \\ 0, & n = 0, \end{cases}
$$

$m = 0,\ldots,n+1$ and $(a)_r = a(a+1)(a+2)\ldots(a+r-1)$ with $(a)_0 = 1$, denotes the Pochhammer symbol. The notations δ and $U_{n-2k,m}$ take values $\delta = 1$, $U_{n-2k,m} = P^m_{n-2k}(\cosh\mu)P^m_{n-2k}(\cos\theta)$ in cases of prolate monogenic functions and $\delta = -1$, $U_{n-2k,m} = \mathbf{i}^{n-2k-m}P^m_{n-2k}(\mathbf{i}\sinh\mu)P^m_{n-2k}(\cos\theta)$ in cases of oblate monogenic functions. The first were studied in [25, 26] and the second were studied in [30]. Spheroidal monogenic functions $X_{n,m}$ and $Y_{n,m}$ are $\mathcal{A}$-valued and they form a complete orthogonal system in the space $\mathcal{M}(\Omega^+,\mathcal{A})$. A complete orthogonal system of the space $\mathcal{M}(\Omega^+,\mathbb{H})$ of $\mathbb{H}$-valued monogenic functions can be constructed by functions of the form

$$
\Phi^m_n := X_{n,m} - Y_{n,m}\mathbf{e_3},
$$

with $m = 0,\ldots,n$ and $n = 0,1,\ldots$ That is similar to the spherical case, investigated by I. Cação [8] and then by S. Bock [3].

A common property of those functions is that they are inhomogenous polynomials. That fact can be seen in [30] as well as in the underlying theorems. That makes it difficult to calculate them numerically. In [30], the authors found several recurrence formulae and their explicit representation in terms of spherical monogenic polynomials. Precisely, one has the following theorems.

Theorem 3.1 *The four-step recurrence formula for* Φ_n^m *is given by*

$$\begin{aligned}\Phi_{n+1}^m = &-\frac{2n+3}{2(n-m+2)(n-m+1)}\big[(2n+3)x+(2n+1)\overline{x}\big]\Phi_n^m\\&-\frac{(2n+3)(2n+1)(n+m+1)}{(n-m+2)(n-m+1)^2}\overline{x}x\Phi_{n-1}^m\\&+\frac{(2n+1)(n+m+1)}{2(n-m+2)(n-m+1)^2}\left[2n+3+\frac{(2m+1)^2}{2n-1}\right]\Phi_{n-1}^m\\&+\frac{(2n+3)(n+m+1)(n+m)}{2(n-m+2)(n-m+1)^2(n-m)}\big[(2n+1)\overline{x}+(2n-1)x\big]\Phi_{n-2}^m\\&-\frac{(2n+3)(n+m+1)(n+m)^2(n+m-1)}{(2n-1)(n-m+2)(n-m+1)^2(n-m)}\Phi_{n-3}^m.\end{aligned}$$

Theorem 3.2 *The relation between* $\{\Phi_n^m\}$ *and* $\{\widetilde{A}_n^m\}$ *can be described as follows*:

$$\Phi_{n+k}^n = (-1)^{k+1}\sum_{j=0}^{[k/2]}\frac{(2n+k-2j+1)!(2n+1)!!}{2^{n+j}\cdot(k+1)!j!(n+k-2j)!}a_{k,j}^n\widetilde{A}_{n+k-2j}^n,$$

where

$$a_{k,j}^n := \frac{(2n+2)_{2k-2j}}{2^{k-j}(n+1)_{k-j}}\cdot(2n+k+2-2j)_{2j},$$

and $\{\widetilde{A}_n^m\}$ *is the Appell system in* [3].

Notice that the Appell functions for spherical domains $\{\widetilde{A}_n^m\}$ are homogeneous polynomials and they satisfy the two step recurrence formula (cf. [3])

$$\widetilde{A}_{n+1}^l = \frac{n+1}{2(n-l+1)(n+l+2)}\Big[\big((2n+3)x+(2n+1)\overline{x}\big)\widetilde{A}_n^l - 2nx\overline{x}\widetilde{A}_{n-1}^l\Big],$$

with initial polynomials

$$\widetilde{A}_{l+1}^l = \frac{1}{4}\big[(2l+3)x+(2l+1)\overline{x}\big]\widetilde{A}_l^l; \quad \widetilde{A}_l^l = (x_1 - x_2\mathbf{e_3})^l.$$

Theorem 3.1 shows the analogy between spherical and spheroidal cases. The other terms in the formula express the asymmetry of oblate spheroids. These results help to reduce computational time for those functions. Especially, it is shown in [30] that there does not exist a complete system for spheroidal domains with respect to the standard inner product satisfying both orthogonal and Appell properties.

4 Outer Prolate Spheroidal Monogenics

Initially, inner spheroidal monogenic functions were described by means of associated Legendre functions of the first kind. With the help of associated Legendre functions of the second kind, J. Morais tried to construct a complete orthogonal system for the exterior domain of a prolate spheroid. This work becomes more complicated since the latter contains logarithmic functions so that a simple substitution is not enough. In [4], the Kelvin transform was applied for the construction of $\mathbb{H}$-valued outer spherical monogenic functions from inner spherical monogenic functions. That keeps properties such as orthogonality invariant. However for $\mathcal{A}$-valued functions in a spheroid, the Kelvin transform is not directly applicable. The method, based on the decomposition of a function space into subspaces of homogeneous functions to prove the completeness of a function system (see [2, 24]), fails because of the appearance of logarithmic functions. To this end, we firstly pay attention to the asymptotic behavior of the constructed functions compared with spherical cases. The extra term in the coefficient function is dealt with to prove the orthogonal property which will be discussed later. Finally, by using the harmonic extension to the outer domain of a function defined on the boundary of a prolate spheroid, one can prove the completeness of such a system. This research can be found in [29]. Here it is summarized briefly.

4.1 A System of Outer Prolate Spheroidal Monogenics

A system of outer prolate spheroidal monogenic functions is obtained by applying $\frac{1}{2}\partial$ to outer spheroidal harmonic functions

$$V_{n,l}(\mu,\theta)\cos(l\varphi), \qquad V_{n,l}(\mu,\theta)\sin(l\varphi),$$

where $V_{n,l}(\mu,\theta) := Q_n^l(\cosh\mu)P_n^l(\cos\theta)$, $(n=0,1,\ldots;\ l=0,\ldots,n)$. Denote $\widehat{\mathcal{E}_{n-1,l}} := \frac{1}{2}\partial[V_{n,l}(\mu,\theta)\cos(l\varphi)]$ and $\widehat{\mathcal{F}_{n-1,l}} := \frac{1}{2}\partial[V_{n,l}(\mu,\theta)\sin(l\varphi)]$, one gets

$$\widehat{\mathcal{E}_{-1,0}}(\mu,\theta,\varphi) := \frac{-\sinh\mu\cos\theta+\cosh\mu\sin\theta(\cos\varphi\mathbf{e_1}+\sin\varphi\mathbf{e_2})}{\sinh\mu(\sin^2\theta+\sinh^2\mu)}, \tag{4.1}$$

$$\begin{aligned}\widehat{\mathcal{E}_{0,0}}(\mu,\theta,\varphi) &:= \frac{1}{4}\ln\left(\frac{\cosh\mu+1}{\cosh\mu-1}\right)-\frac{1}{2}\frac{\cosh\mu}{\sin^2\theta+\sinh^2\mu}\\ &\quad+\frac{1}{2}\frac{\sin\theta\cos\theta}{\sinh\mu(\sin^2\theta+\sinh^2\mu)}(\cos\varphi\mathbf{e_1}+\sin\varphi\mathbf{e_2}),\end{aligned} \tag{4.2}$$

$$\begin{aligned}\widehat{\mathcal{E}_{n,l}}(\mu,\theta,\varphi) &:= \frac{(n+l+1)}{2}B_{n,l}(\mu,\theta)\cos(l\varphi)\\ &\quad+\frac{1}{4(n-l+1)}B_{n,l+1}(\mu,\theta)\big[\cos\big((l+1)\varphi\big)\mathbf{e_1}+\sin\big((l+1)\varphi\big)\mathbf{e_2}\big]\end{aligned}$$

$$+\frac{1}{4}(n+1+l)(n+l)(n-l+2)B_{n,l-1}(\mu,\theta)$$
$$\times\big[-\cos\big((l-1)\varphi\big)\mathbf{e_1}+\sin\big((l-1)\phi\big)\mathbf{e_2}\big], \tag{4.3}$$

$$\begin{aligned}\widehat{\mathcal{F}_{n,l}}(\mu,\theta,\varphi) :=\ & \frac{(n+l+1)}{2}B_{n,l}(\mu,\theta)\sin(l\varphi)\\ &+\frac{1}{4(n-l+1)}B_{n,l+1}(\mu,\theta)\big[\sin\big((l+1)\phi\big)\mathbf{e_1}-\cos\big((l+1)\varphi\big)\mathbf{e_2}\big]\\ &-\frac{1}{4}(n+1+l)(n+l)(n-l+2)B_{n,l-1}(\mu,\theta)\big[\sin\big((l-1)\varphi\big)\mathbf{e_1}\\ &+\cos\big((l-1)\varphi\big)\mathbf{e_2}\big],\end{aligned} \tag{4.4}$$

(for $l=0,\ldots,n;\ n=1,2,\ldots$)

$$\begin{aligned}\widehat{\mathcal{E}_{n,n+1}}(\mu,\theta,\varphi) :=\ & (n+1)B_{n,n+1}(\mu,\theta)\cos\big((n+1)\varphi\big)\\ &-\frac{\cosh\mu\, P_{n+2}^{n+2}(\cos\theta)\,Q_{n+1}^{n+2}(\cosh\mu)}{4(2n+3)(\sin^2\theta+\sinh^2\mu)}\\ &\times\big[\cos\big((n+2)\phi\big)\mathbf{e_1}+\sin\big((n+2)\varphi\big)\mathbf{e_2}\big]\\ &+\frac{(2n+2)(2n+1)}{4}B_{n,n}(\mu,\theta)\big[-\cos(n\varphi)\mathbf{e_1}+\sin(n\phi)\mathbf{e_2}\big],\end{aligned} \tag{4.5}$$

$$\begin{aligned}\widehat{\mathcal{F}_{n,n+1}}(\mu,\theta,\varphi) :=\ & (n+1)B_{n,n+1}(\mu,\theta)\sin\big((n+1)\varphi\big)\\ &-\frac{\cosh\mu\, P_{n+2}^{n+2}(\cos\theta)\,Q_{n+1}^{n+2}(\cosh\mu)}{4(2n+3)(\sin^2\theta+\sinh^2\mu)}\\ &\times\big[\sin\big((n+2)\phi\big)\mathbf{e_1}-\cos\big((n+2)\varphi\big)\mathbf{e_2}\big]\\ &-\frac{(2n+2)(2n+1)}{4}B_{n,n}(\mu,\theta)\big[\sin(n\varphi)\mathbf{e_1}+\cos(n\phi)\mathbf{e_2}\big],\end{aligned} \tag{4.6}$$

(for $n=0,1,\ldots$). The coefficients are given by

$$\begin{aligned}B_{n,l}(\mu,\theta) :=\ & \frac{1}{\sin^2\theta+\sinh^2\mu}\big[\cosh\mu\, P_n^l(\cos\theta)\,Q_{n+1}^l(\cosh\mu)\\ &-\cos\theta\, P_{n+1}^l(\cos\theta)\,Q_n^l(\cosh\mu)\big],\end{aligned} \tag{4.7}$$

where

$$B_{n,-1}(\mu,\theta) := -\frac{1}{n(n+1)^2(n+2)}B_{n,1}(\mu,\theta) \quad \text{for } n=1,2,\ldots$$

It can be proved that $B_{n,l}(\mu,\theta)$ has the explicit presentation

$$B_{n,l}(\mu,\theta) = \sum_{k=0}^{[\frac{n-l}{2}]-1} \frac{(2n+1-4k)(n+l-2k+1)_{2k}}{(n-l-2k+1)_{2k+1}} P_{n-2k}^{l}(\cos\theta) Q_{n-2k}^{l}(\cosh\mu) + \begin{cases} \frac{(2l+1)_{n-l}}{(n-l+1)!} B_{l,l}(\mu,\theta) & \text{if } n-l \text{ even} \\ \frac{2(2l+2)_{n-l-1}}{(n-l+1)!} B_{l+1,l}(\mu,\theta) & \text{if } n-l \text{ odd.} \end{cases}$$

Because the terms $Q_{n+1}^{l}(\cosh\mu)$ contain logarithmic functions, the question of their behavior at infinity arises and we will see that they are completely similar to the outer spherical monogenic functions.

4.2 Outer Spherical Monogenics Revisited

To compare, we firstly revisit the spherical case. The construction of outer spherical monogenic functions has been studied in parallel with the construction of inner spherical monogenics. In [6], they are constructed based on the Cauchy kernel function and its derivatives. Spherical monogenics can be obtained also by applying the Kelvin transform as in [4]. Different methods we apply, different representations we get. For $\mathcal{A}$-valued monogenic functions, these methods do not lead directly to what we need. Hence, the harmonic function approach is again used together with spherical harmonic functions. Let $\mathbb{B}(R)$ be a ball with radius $R > 0$. Denote by $\mathcal{H}(\mathbb{R}^3\backslash\overline{\mathbb{B}(R)}, -(n+1))$ the space of real-valued homogeneous harmonic functions of degree $-(n+1)$ in $\mathbb{R}^3\backslash\overline{\mathbb{B}(R)}$ with $n \geq 0$. A basis of $\mathcal{H}(\mathbb{R}^3\backslash\overline{\mathbb{B}(R)}, -(n+1))$ is given by

$$\left\{ \frac{1}{r^{n+1}} P_n(\cos\theta),\ \frac{1}{r^{n+1}} P_n^m(\cos\theta)\cos(m\varphi),\ \frac{1}{r^{n+1}} P_n^m(\cos\theta)\sin(m\varphi) \right\}$$

where $m = 1, \ldots, n$. By applying the hypercomplex derivative $\frac{1}{2}\partial$, one obtains a system of monogenic functions defined in $\mathbb{R}^3\backslash\overline{\mathbb{B}(R)}$ as follows:

$$X_{-(n+2)}^{0} = -\frac{n+1}{2}\frac{P_{n+1}(\cos\theta)}{r^{n+2}} - \frac{1}{2}\frac{P_{n+1}^{1}(\cos\theta)}{r^{n+2}}[\cos\varphi\mathbf{e_1} + \sin\varphi\mathbf{e_2}]$$

$$\begin{aligned} X_{-(n+2)}^{m} = &-\frac{n-m+1}{2}\frac{P_{n+1}^{m}(\cos\theta)}{r^{n+2}}\cos(m\varphi) \\ &- \frac{1}{4}\frac{P_{n+1}^{m+1}(\cos\theta)}{r^{n+2}}\big[\cos\big((m+1)\varphi\big)\mathbf{e_1} + \sin\big((m+1)\varphi\big)\mathbf{e_2}\big] \\ &+ \frac{(n-m+1)(n-m+2)}{4}\frac{P_{n+1}^{m-1}(\cos\theta)}{r^{n+2}} \\ &\times\big[\cos\big((m-1)\varphi\big)\mathbf{e_1} - \sin\big((m-1)\varphi\big)\mathbf{e_2}\big] \end{aligned}$$

$$
\begin{aligned}
Y^m_{-(n+2)} = &-\frac{n-m+1}{2}\frac{P^m_{n+1}(\cos\theta)}{r^{n+2}}\sin(m\varphi) \\
&-\frac{1}{4}\frac{P^{m+1}_{n+1}(\cos\theta)}{r^{n+2}}\big[\sin\big((m+1)\varphi\big)\mathbf{e_1}-\cos\big((m+1)\varphi\big)\mathbf{e_2}\big] \\
&+\frac{(n-m+1)(n-m+2)}{4}\frac{P^{m-1}_{n+1}(\cos\theta)}{r^{n+2}} \\
&\times\big[\sin\big((m-1)\varphi\big)\mathbf{e_1}+\cos\big((m-1)\varphi\big)\mathbf{e_2}\big].
\end{aligned}
$$

Note that $\frac{1}{2}\partial$ establishes an isomorphism between $\mathcal{H}(\mathbb{R}^3\backslash\overline{\mathbb{B}(R)}, -(n+1))$ and $\mathcal{M}(\mathbb{R}^3\backslash\overline{\mathbb{B}(R)}, \mathcal{A}, -(n+2))$. The latter consists of all homogeneous monogenic polynomials of degree $-(n+2)$. Due to the orthogonal decomposition

$$
\mathcal{M}\big(\mathbb{R}^3\backslash\overline{\mathbb{B}(R)}, \mathcal{A}\big) = \bigoplus_{n=0}^{\infty}\mathcal{M}\big(\mathbb{R}^3\backslash\overline{\mathbb{B}(R)}, \mathcal{A}, -(n+2)\big),
$$

the system $\{X^0_{-(n+2)}, X^m_{-(n+2)}, Y^m_{-(n+2)}\}_{n=0,1,\dots;\, m=1,\dots,n}$ forms an orthogonal basis of $\mathcal{M}(\mathbb{R}^3\backslash\overline{\mathbb{B}(R)}, \mathcal{A})$.

4.3 Asymptotic Behavior

The behavior at infinity of outer spheroidal monogenic functions is related closely to the behavior of $Q^l_n(z)$. When z tends to infinity

$$
Q^l_n(z) = \frac{(n+l)!}{(2n+1)!!}\frac{1}{z^{n+1}} + O\bigg(\frac{1}{z^{n+3}}\bigg).
$$

Now let $z = \cosh\mu \simeq \sinh\mu \simeq r = |x|$ if μ is large enough, it leads to

$$
B_{n,l}(\mu,\theta) = -\frac{(n-l+2)(n+l)!}{(2n+3)!!}\frac{P^l_{n+2}(\cos\theta)}{r^{n+3}} + O\bigg(\frac{1}{r^{n+5}}\bigg).
$$

As a result, we obtain the asymptotic behavior of $\widehat{\mathcal{E}_{n,l}}$ and $\widehat{\mathcal{F}_{n,l}}$ for $l = 0, \dots, n+1$; $n = 0, 1, \dots$

$$
\widehat{\mathcal{E}_{n,l}} = \frac{(n+l+1)!}{(2n+3)!!}X^l_{-(n+3)} + O\bigg(\frac{1}{|x|^{n+5}}\bigg),
$$

$$
\widehat{\mathcal{F}_{n,l}} = \frac{(n+l+1)!}{(2n+3)!!}Y^l_{-(n+3)} + O\bigg(\frac{1}{|x|^{n+5}}\bigg).
$$

Particularly, when $|x| \to \infty$

$$
\widehat{\mathcal{E}_{-1,0}} = -\frac{\overline{x}}{|x|^3} + O\bigg(\frac{1}{|x|^4}\bigg),
$$

and it behaves like *the Cauchy kernel* in a neighborhood of infinity.

4.4 Orthogonality

It could be easy to see that each following pair of functions is orthogonal with respect to the inner product (2.1) whenever $l_1 \neq l_2$

- $\{\widehat{\mathcal{E}_{n_1,l_1}}, \widehat{\mathcal{E}_{n_2,l_2}}\}$.
- $\{\widehat{\mathcal{F}_{n_1,l_1}}, \widehat{\mathcal{F}_{n_2,l_2}}\}$.
- $\{\widehat{\mathcal{E}_{n_1,l_1}}, \widehat{\mathcal{F}_{n_2,l_2}}\}$.

The assertion is based on the orthogonalities of $\sin(l\varphi)$ and $\cos(k\varphi)$ on $[0, 2\pi]$. In the other cases, one can decompose coefficient functions $B_{n,l}(\mu,\theta)$ into summands of the form

$$Q^l_{n-2k}(\cosh\mu)P^l_{n-2k}(\cos\theta), \tag{4.8}$$

except one extra term

$$\frac{\cosh\mu P^l_l(\cos\theta)Q^l_{l-1}(\cosh\mu)}{\sin^2\theta+\sinh^2\mu} \quad \text{or} \quad \frac{\cos\theta P^l_l(\cos\theta)Q^l_{l-1}(\cosh\mu)}{\sin^2\theta+\sinh^2\mu}. \tag{4.9}$$

Consequently, orthogonality holds for the terms of the form (4.8) according to equalities

$$\int_0^\pi P^l_n(\cos\theta)P^l_s(\cos\theta)\sin\theta d\theta = 0,$$

$$\int_0^\pi P^l_{n+1}(\cos\theta)\cos\theta P^l_s(\cos\theta)\sin\theta d\theta = 0$$

for $s < n$. Besides, we can prove by induction the following proposition.

Proposition 4.1 *Let $B_{n,l}(\mu,\theta)$ be functions as in* (4.7), *then with $l = 0, 1, \ldots$ the following equalities hold when n, k are equal to l or $l+1$*

$$\int_0^\pi B_{n,l}(\mu,\theta)P^l_k(\cos\theta)\sin\theta d\theta = 0.$$

The proposition is applied to deal with the extra term (4.9) in expansions of $B_{n,l}(\mu,\theta)$ and it results in the following theorem.

Theorem 4.2 *The constructed functions* (4.1)–(4.6) *form an orthogonal system in $\mathcal{M}(\Omega^-,\mathcal{A})$ with respect to the inner product* (2.1).

The proof can be found in [29].

4.5 Completeness

Any function $\mathbf{f} \in \mathcal{M}(\Omega^-, \mathcal{A})$ has a Fourier series expansion related to the function system (4.1)–(4.6). The question is whether the Fourier series expansion converges to $\mathbf{f}$ in L_2-norm. In order to find the answer, one needs the following result.

Theorem 4.3 *Let* $\mathbf{f}$ *be a function in* $\mathcal{M}(\Omega^-, \mathcal{A}) \cap C^1(\Omega^- \cup \Gamma)$. *Then the Fourier series expansion of* $\mathbf{f}$ *converges to* $\mathbf{f}$ *in the sense of the* $L^2(\Omega^-)$*-norm.*

Notice that Theorem 4.3 considers only the case of smooth functions in $\mathcal{M}(\Omega^-, \mathcal{A})$. For the L_2-case, the analogous result is obtained by applying the following corollary.

Corollary 4.4 *Any outer spherical monogenic functions*

$$\left\{X^0_{-(n+2)}, X^m_{-(n+2)}, Y^m_{-(n+2)}\right\}_{n\geq 0;\, m=1,\dots,n}$$

can be presented by its Fourier series expansion with respect to the system (4.1)–(4.6).

To this end, we give the completeness theorem.

Theorem 4.5 *The function system* (4.1)–(4.6) *forms a complete orthogonal system of the space* $\mathcal{M}(\Omega^-, \mathcal{A})$ *with respect to the inner product* (2.1) *in the exterior domain* Ω^-.

Details can be found in [29].

5 Conclusion

Ellipsoidal harmonic functions have attracted the attention of several researchers and shown their importance in many fields. By means of the hypercomplex derivative, ellipsoidal monogenic functions are currently being developed. In accordance with advantages of Clifford analysis, it will become a helpful tool for solving problems in ellipsoidal domains. Further applications of such systems hopefully will be announced in the near future.

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