Ergodic Theory, Boole Type Transformations, Dynamical Systems Theory

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Abstract The arithmetic properties of generalized one-dimensional ergodic Boole type transformations are studied in the framework of the operator-theoretic approach. Some invariant measure statements and ergodicity conjectures concerning generalized multi-dimensional Boole-type transformations are formulated.

Keywords Generalized Boole type transformations · Arithmetic properties · Ergodic dynamical systems · Invariant measures · Frobenius–Perron operator

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1 Introduction

With its origins, going back several centuries, discrete analysis becomes now an increasingly central methodology for many mathematical problems related to discrete dynamical systems and algorithms, widely applied in modern science. Our theme, being related with studying ergodic aspects and the related arithmetic properties of discrete Boole type dynamical systems [3, 7], is of deep interest in many branches of modern science and technology [6, 19], especially in discrete mathematics, numerical analysis, statistics and probability theory as well as in electrical and electronic engineering. But the important viewpoint is that this topic belongs to a much more general realm of mathematics, namely, to calculus, differential equations and differential geometry, because of the remarkable analogy of the subject especially to these branches of mathematics. Nonetheless, although the topic is discrete, our approach to treating ergodicity and the related arithmetic properties of the generalized Boole type discrete dynamical systems will be completely analytical.

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The generalized Boole transformation looks as

$$\mathbb{R} \ni x \to \varphi(x) := \alpha x + a - \sum_{j=1}^{N} \frac{\beta_j}{x - b_j} \in \mathbb{R}, \tag{1.1}$$

where a and $b_j \in \mathbb{R}$, $j = \overline{1, N}$, are some real and $\alpha, \beta_j \in \mathbb{R}_+$, $j = \overline{1, N}$, and was analyzed in [3, 15]. It generalizes that classical [7] Boole transformation

$$\mathbb{R} \ni x \to \phi(x) := x - 1/x \in \mathbb{R}, \tag{1.2}$$

which appeared to be ergodic [5] with respect to the invariant standard infinite Lebesgue measure on $\mathbb{R}$. This, in particular, means that the following Boole's [7] equalities

$$\int_{\mathbb{R}} f(x - 1/x)dx = \int_{\mathbb{R}} f(x)dx, \tag{1.3}$$

and

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n-1} f(\phi^n x)}{\sum_{k=0}^{n-1} g(\phi^n x)} = \frac{\int_{\mathbb{R}} f(x)dx}{\int_{\mathbb{R}} g(x)dx} \tag{1.4}$$

hold for any $f \in L_1(\mathbb{R}; \mathbb{R})$ and $g \in L_1(\mathbb{R}; \mathbb{R}_+)$. In the case $\alpha = 1$, $a = 0$, a similar ergodicity result was proved in [1–3] making use of a specially devised inner function method. The related spectral aspects of the mapping (1.1) were in part studied also in [3]. In spite of these results the case $\alpha \neq 1$ still persists to be challenging as the only relating result [4] concerns the following special case of (1.1):

$$\mathbb{R} \ni x \to \varphi(x) := \alpha x + a - \frac{\beta}{x - b} \in \mathbb{R} \tag{1.5}$$

for $0 < \alpha < 1$, and arbitrary $a, b \in \mathbb{R}$ and $\beta \in \mathbb{R}_+$. The ergodicity of the Boole type mapping (1.5) can be easily enough stated. Really, concerning a general nonsingular mapping $\varphi : \mathbb{R} \to \mathbb{R}$, the problem of constructing the measure preserving ergodic measures was analyzed [4, 12, 14] by means of studying the spectral properties of the adjoint Frobenius–Perron operator $\hat{T}_\varphi \rho : L_2(\mathbb{R}; \mathbb{R}) \to L_2(\mathbb{R}; \mathbb{R})$, where, by definition,

$$\hat{T}_\varphi \rho(x) := \sum_{y \in \{\varphi^{-1}(x)\}} \rho(y) J_\varphi^{-1}(y) \tag{1.6}$$

for any $\rho \in L_2(\mathbb{R}; \mathbb{R}_+)$ and $J_\varphi^{-1}(y) := |\frac{d\varphi(y)}{dy}|$, $y \in \mathbb{R}$. Then, if $\hat{T}_\varphi \rho = \rho$, $\rho \in L_2(\mathbb{R}; \mathbb{R}_+)$, then the expression $d\mu(x) := \rho(x)dx$, $x \in \mathbb{R}$, will be invariant, in general infinite, measure with respect to the mapping $\varphi : \mathbb{R} \to \mathbb{R}$. Another way to finding a general algorithm for finding such an invariant measure was devised in [15–17], making use of the generating measure function method. (1.5) at $\alpha = 1/2$ and $b = 2a \in \mathbb{R}$ appears to be measure preserving and ergodic. Namely, the following propositions [15] hold.

Proposition 1.1 *The Boole type transformation* (1.5) *at* $\alpha = 1/2$ *and* $b = 2a \in \mathbb{R}$ *is measure preserving and ergodic with respect to the measure*

$$d\mu(x) := \frac{|\gamma| dx}{\pi[(x-2a)^2 + \gamma^2]}, \tag{1.7}$$

where $x \in \mathbb{R}$ *and* $\gamma^2 = 2\beta \in \mathbb{R}_+$.

Proof (*Sketch*) A proof follows easily from the fact that the function

$$\rho(x) := \frac{\gamma}{\pi[(x-2a)^2 + \gamma^2]} \tag{1.8}$$

satisfies for all $x \in \mathbb{R}$ the determining condition (1.6):

$$\hat{T}_\varphi \rho(x) := \sum_I \rho(y_\pm) \left| y'_\pm(x) \right|, \tag{1.9}$$

where, by definition, $\varphi(y_\pm(x)) := x$ for any $x \in \mathbb{R}$. The relationship (1.9) is, evidently, equivalent to the next infinitesimal invariance condition

$$\sum_\pm d\mu\big(y_\pm(x), y_\pm(x) + dy\big) = d\mu(x) := \mu(x, x + dx) \tag{1.10}$$

for any infinitesimal subset $[x, x + dx) \subset \mathbb{R}$. □

Proposition 1.2 *The measure* (1.7) *is ergodic with respect to the Boole type transformation* (1.5) *at* $\alpha = 1/2$ *and* $b = 2a \in \mathbb{R}$ *as it is equivalent to the canonical ergodic mapping* $\mathbb{R}/\mathbb{Z} \ni s :\to \psi(s) := 2s \pmod{\mathbb{Z}} \in \mathbb{R}/\mathbb{Z}$ *with respect to the standard Lebesgue measure on* $\mathbb{R}/\mathbb{Z}$.

Proof (*Sketch*) Put, by definition, $\mathbb{R}/\mathbb{Z} \ni s :\to \xi(s) = y \in \mathbb{R}$, where

$$\xi(s) := \gamma \cot \pi s + 2a, \tag{1.11}$$

Then transformation (1.5) at $\alpha = 1/2$, $b = 2a \in \mathbb{R}$ and $\gamma^2 := 2\beta \in \mathbb{R}_+$, owing to the mapping (1.11), yields

$$\begin{aligned}
\varphi(y) = \varphi\big(\xi(s)\big) &= \frac{\gamma}{2} \cot \pi s + 2a - \frac{\gamma}{2} \tan \pi s \\
&= \frac{\gamma(\cos^2 \pi s - \sin^2 \pi s)}{2 \sin \pi s \cos \pi s} + 2a = \gamma \frac{\cos 2\pi s}{\sin 2\pi s} + 2a \\
&= \gamma \cot 2\pi s + 2a := \xi(2s)
\end{aligned} \tag{1.12}$$

for any $s \in \mathbb{R}/\mathbb{Z}$. The result (1.12) means that the transformation (1.5) is conjugated [3, 12] to the transformation

$$\mathbb{R}/\mathbb{Z} \ni s :\to \psi(s) = 2s \bmod 1 \in \mathbb{R}/\mathbb{Z}, \tag{1.13}$$

that is the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{R}/\mathbb{Z} & \stackrel{\psi}{\rightarrow} & \mathbb{R}/\mathbb{Z} \\ \xi\downarrow & & \downarrow\xi \\ \mathbb{R} & \stackrel{\varphi}{\rightarrow} & \mathbb{R} \end{array} \tag{1.14}$$

that is $\xi\cdot\psi = \varphi\cdot\xi$, where $\xi : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ is the conjugation mapping defined by (1.11). It is easy now to check that the measure (1.7) under the conjugation (1.11) transforms into the standard normalized Lebesgue measure on $\mathbb{R}/\mathbb{Z}$:

$$\begin{aligned} d\mu(x)|_{x=\gamma\cot\pi s+2a} &= \frac{ds\gamma^2|d(\cot\pi s)/ds|}{(\gamma^2\cot^2\pi s+\gamma^2)} \\ &= \frac{\sin^2\pi s\cdot\sin^{-2}\pi s ds}{\cos^2\pi s+\sin^2\pi s} = ds, \end{aligned} \tag{1.15}$$

where $s \in \mathbb{R}/\mathbb{Z}$. The infinitesimal measure ds on $\mathbb{R}/\mathbb{Z}$ as well as the infinitesimal measure (1.7) on $\mathbb{R}$ are normalized, being thus probabilistic. Now it is enough to make use of the fact that the measure ds on $\mathbb{R}/\mathbb{Z}$ on the interval $[0, 1) \simeq \mathbb{R}/\mathbb{Z}$ is ergodic [4, 12] with respect to the mapping $\psi : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$. □

It is important to mention that in the framework of the theory of inner functions in [4] there was stated that there exists an invariant measure $d\mu(x)$, $x \in \mathbb{R}$, on the axis $\mathbb{R}$, such that the generalized Boole type transformation (1.1) for any $N > 1$, $\alpha = 1$ and $a = 0$ is ergodic. If $\alpha = 1$ and $a \neq 0$, the transformation (1.1) appears to be not ergodic, being totally dissipative, that is the wandering set $\mathcal{D}(\varphi) := \bigcup \mathcal{W}(\varphi) = \mathbb{R}$, where $\mathcal{W}(\varphi) \subset \mathbb{R}$ are such subsets that all sets $\varphi^{-n}(\mathcal{W})$, $n \in \mathbb{Z}_+$, are disjoint. Similar the above statement can be also [1, 4] formulated for the mostly generalized Boole type transformation

$$\mathbb{R} \ni x \to \varphi(x) := \alpha x + a + \int_{\mathbb{R}} \frac{d\nu(s)}{s-x} \in \mathbb{R}, \tag{1.16}$$

where $a \in \mathbb{R}$, $\alpha \in \mathbb{R}_+$ and a measure ν on $\mathbb{R}$ has the compact support $\operatorname{supp}\nu \subset \mathbb{R}$, being such that the following natural conditions

$$\int_{\mathbb{R}} \frac{d\nu(s)}{1+s^2} = a, \qquad \int_{\mathbb{R}} d\nu(s) < \infty, \tag{1.17}$$

hold.

Below we will analyze the related arithmetic aspects of the generalized ergodic Boole type transformation (1.1), making use of the approaches recently initiated in [8, 11, 13, 20–22] concerning the old general Baudet arithmetic progression conjecture.

2 The Generalized Boole Type Ergodic Transformations and Their Arithmetic Properties

Consider the generalized Boole type transformation (1.1) and its right orbit $Or(\varphi; x_0) := \{\varphi^j(x_0) \in \mathbb{R} : x_0 \in \mathbb{R}, j \in \mathbb{Z}_+\}$ for an arbitrary $x_0 \in \mathbb{R}\backslash\{\alpha \in \mathbb{R} : \varphi(\alpha) = \alpha\}$. Owing to the ergodicity of the mapping (1.1), one easily obtains that the closure $\overline{Or(\varphi; x_0)} = \bar{\mathbb{R}}$. Thus, for any point $\alpha \in \mathbb{R}$ one can find a convergent subsequence $\{\varphi^{n_j}(x_0) \in \mathbb{R} : n_j := n_j(\alpha) \in \mathbb{Z}_+, j \in \mathbb{Z}_+\} \subset Or(\varphi; x_0)$, such that

$$\lim_{j\to\infty} \varphi^{n_j}(x_0) = \alpha. \tag{2.1}$$

The corresponding integer subsequence

$$A(\alpha) := \big\{n_1(\alpha), n_2(\alpha), \ldots, n_j(\alpha), \ldots\big\} \subset \mathbb{Z}_+, \tag{2.2}$$

owing to the condition (2.1), *a priori* possesses, owing to the Weil theorem [9, 12, 14] the following upper density property:

$$\bar{d}\big(A(\alpha)\big) := \overline{\lim_{m\to\infty}} \sharp\big(A(\alpha) \cap \{0, 1, 2, \ldots, m\}\big)/(m+1) > 0. \tag{2.3}$$

Consider now the left shift mapping

$$\theta : l_\infty(\mathbb{Z}_+; \mathbb{R}) \ni (c_0, c_1, \ldots, c_n, \ldots) \to (c_1, \ldots, c_n, \ldots) \in l_\infty(\mathbb{Z}_+; \mathbb{R}) \tag{2.4}$$

and put, by definition, the set

$$\mathcal{A}(\alpha) := \overline{\big\{\theta^n 1_{A(\alpha)} \in l_\infty(\mathbb{Z}_+; \mathbb{R}) : n \in \mathbb{Z}_+\big\}}, \tag{2.5}$$

the closure with respect to the weak σ^*-topology of $l_\infty(\mathbb{Z}_+; \mathbb{R})$. The constructed set (2.5) is, by definition, θ-invariant and σ^*-weakly compact in $l_\infty(\mathbb{Z}_+; \mathbb{R})$. Its subset

$$\mathcal{A}_0(\alpha) := \big\{(c_0, c_1, \ldots, c_n, \ldots) \in \mathcal{A}(\alpha) : c_0 := 1\big\}, \tag{2.6}$$

as well as its preimages $\theta^{-j}(\mathcal{A}_0(\alpha))$, $j \in \mathbb{N}$, are open-closed subsets of $\mathcal{A}(\alpha)$. It is easy to observe the following characteristic [8, 10] property of the set $\mathcal{A}_0(\alpha)$:

$$n \in A(\alpha) \quad \text{iff} \quad \theta^n 1_{A(\alpha)} \in \mathcal{A}_0(\alpha). \tag{2.7}$$

Following the classical Furstenberg scheme [10] one can construct an invariant probabilistic measure ν on the compact set $\mathcal{A}(\alpha)$. Namely, owing to the condition (2.3) one can chose an infinite subsequence $\{m_j \in \mathbb{Z}_+ : j \in \mathbb{Z}\}$, such that there exists the limit

$$\lim_{j\to\infty} \frac{1}{m_j + 1} \sum_{k=0}^{m_j} \delta_k\big(A(\alpha)\big) = \bar{d}\big(A(\alpha)\big). \tag{2.8}$$

Now making use of the property (2.8) one can define an infinite sequence of probability measures $\{\nu_j : j \in \mathbb{Z}_+\}$ on $\mathcal{A}(\alpha)$

$$\nu_j(\mathcal{B}) := \frac{1}{m_j+1}\sum_{k=0}^{m_j}\delta_{\{\theta^k 1_{A(\alpha)}\}}(\mathcal{B}) \tag{2.9}$$

for any Borel subsets of $\mathcal{A}(\alpha)$. In particular, one has

$$\nu_j\big(\mathcal{A}_0(\alpha)\big) := \frac{1}{m_j+1}\sum_{k=0}^{m_j}\delta_{\{\theta^k 1_{A(\alpha)}\}}(\mathcal{B}) \overset{j\to\infty}{\to} \bar{d}\big(A(\alpha)\big). \tag{2.10}$$

Based on the Banach–Alaoglu theorem and on the metrisability of the set $\mathcal{A}(\alpha) \subset l_\infty(\mathbb{Z}_+;\mathbb{R})$ with respect to the weak σ^*-topology one obtains that there exists a convergent subsequence of measures (2.10) to some probability measure ν on $\mathcal{A}(\alpha)$, which can be re-denoted as $\{\nu_j : j \in \mathbb{Z}_+\}$. It is important that the obtained above measure ν on $\mathcal{A}(\alpha)$ is θ-invariant:

$$\begin{aligned}
&\nu(\mathcal{B}) - \nu\big(\theta^{-1}\mathcal{B}\big)\\
&\quad= \lim_{j\to\infty}\frac{1}{m_j+1}\left(\sum_{k=0}^{m_j}\delta_{\{\theta^k 1_{A(\alpha)}\}}(\mathcal{B}) - \delta_{\{\theta^{k+1} 1_{A(\alpha)}\}}(\mathcal{B})\right)\\
&\quad= \lim_{j\to\infty}\frac{1}{m_j+1}\left(\sum_{k=0}^{m_j}\delta_{\{1_{A(\alpha)}\}}(\mathcal{B}) - \delta_{\{\theta^{m_j} 1_{A(\alpha)}\}}(\mathcal{B})\right) = 0
\end{aligned} \tag{2.11}$$

for any Borel subset of $\mathcal{B} \subset \mathcal{A}(\alpha)$, as $m_j \to \infty$ if $j \to \infty$. Thus, the constructed measure-theoretic dynamical system $(\mathcal{A}(\alpha), \theta; \nu)$ is characterized by the condition $\nu(\mathcal{A}_0(\alpha)) = \bar{d}(A(\alpha)) > 0$. Moreover, taking into account that the ergodic measures are extreme points [9, 12, 14] of the set of invariant measures on $\mathcal{A}(\alpha)$, one can choose this limiting invariant measure ν on the set $\mathcal{A} = \mathcal{A}(\alpha)$ to be ergodic.

Define now for the mapping (1.1) a linear operator $T_\varphi : L_2^{(\nu)}(\mathcal{A};\mathbb{R}) \to L_2^{(\nu)}(\mathcal{A};\mathbb{R})$, which satisfies for any $f \in L_2^{(\nu)}(\mathcal{A};\mathbb{R})$ the shift property $T_\theta f(c) := f(\theta(c))$, $c \in \mathcal{A}$. Based on the existence of the invariant and ergodic measure ν on $\mathcal{A}(\alpha)$, one can state the following characteristic proposition.

Proposition 2.1 *If the set $A(\alpha) \subset \mathbb{Z}_+$ possesses a positive upper density $\bar{d}(A(\alpha)) > 0$, then for any strongly positive function $f \in L_2^{(\nu)}(\mathcal{A};\mathbb{R}_+) \cap L_\infty^{(\nu)}(\mathcal{A};\mathbb{R}_+)$ and for arbitrary ergodic measure ν on the set $\mathcal{A} = \mathcal{A}(\alpha)$ there holds the following strong inequality*:

$$\overline{\lim}_{N\to\infty}\frac{1}{N+1}\sum_{n=0}^{N}\int_{\mathcal{A}}\left(\prod_{j=0}^{N}T_\theta^{n_j} f\right)d\nu > 0. \tag{2.12}$$

And conversely, if for a chosen subsequence $A = \{n_1, n_2, \ldots, n_j, \ldots\} \subset \mathbb{Z}_+$ *and any strongly positive function* $f \in L_2^{(\nu)}(\mathcal{A}; \mathbb{R}_+) \cap L_\infty^{(\nu)}(\mathcal{A}; \mathbb{R}_+)$ *on a compact set* $\mathcal{A}$ *there holds the strong inequality* (2.12), *then the upper density* $\bar{d}(A) > 0$ *and there exists such a point* $x_0 \in \mathbb{R}$ *and a real value* $\alpha \in \mathbb{R}$ *that there exists the limit* $\lim_{j\to\infty} \varphi^{n_j}(x_0) = \alpha$. *Moreover, the corresponding sets* $\mathcal{A}(\alpha)$ *and* $\mathcal{A}$ *coincide.*

Proof (*Sketch*) We can easily observe, taking into account (2.7) that the following one-to-one mapping holds between the sets $A(\alpha)$ and the subset $\{\theta^{n_1} 1_{A(\alpha)}, \theta^{n_2} 1_{A(\alpha)}, \ldots, \theta^{n_j} 1_{A(\alpha)}, \ldots\} \subset \mathcal{A}_0(\alpha)$. Thus, since each set $\theta^{-j}(\mathcal{A}_0(\alpha)) \subset \mathcal{A}(\alpha)$ is open and the points $\theta^n 1_{A(\alpha)}$, $n \in \mathbb{Z}_+$, are dense in $\mathcal{A}(\alpha)$, one finds that the set $A(\alpha)$ is in one-to-one correspondence to the condition that $\bigcap_{j\in\mathbb{Z}_+} \theta^{-n_j}(\mathcal{A}_0(\alpha)) \neq \emptyset$. The latter easily reduces to the relationship $\prod_{j=0}^{N} T_\theta^{n_j} 1_{\mathcal{A}_0(\alpha)} \neq 0$ for any $N \in \mathbb{Z}_+$, which allows to formulate a sufficient integral condition [10] in the form (2.12), thus proving the first part of the proposition. Having followed back by the reasonings above, one can state that for every ergodic measure ν on a chosen weakly σ^*-compact set $\mathcal{A} \subset l_\infty(\mathbb{Z}_+; \mathbb{R})$ one can find a subset of integers $A := \{n_j \in \mathbb{Z}_+ : j \in \mathbb{Z}_+\}$ with a nonzero upper density $\bar{d}(A) > 0$, for which there exists the standard representation

$$\mathcal{A} := \overline{\left\{\theta^n 1_A \in l_\infty(\mathbb{Z}_+; \mathbb{R}) : n \in \mathbb{Z}_+\right\}}. \tag{2.13}$$

Moreover, for the open subset

$$\mathcal{A}_0 := \left\{(c_0, c_1, \ldots, c_n, \ldots) \in \mathcal{A} : c_0 := 1\right\} \tag{2.14}$$

there holds the equality $\bar{d}(A) = \nu(\mathcal{A}_0)$. Now making use of the condition $\prod_{j=0}^{N} T_\theta^{n_j} 1_{\mathcal{A}_0} \neq 0$, $N \in \mathbb{Z}_+$, one can find such a point $x_0 \in \mathbb{R}$ that the corresponding iterations $\{\varphi^{n_j}(x_0) \in \mathbb{R} : n_j \in A, j \in \mathbb{Z}_+\}$ are convergent to some real value $\alpha := \lim_{j\to\infty} \varphi^{n_j}(x_0)$ and, simultaneously, the whole orbit $\{\varphi^n(x_0) \in \mathbb{R} : n \in \mathbb{Z}_+\}$ is dense on the axis $\mathbb{R}$. The letter proves the second part of the proposition. □

3 Conclusion

Recently in [18] there was proposed a set of multi-dimensional Boole type transformations $\varphi_{\sigma|\eta} : \mathbb{R}^n \to \mathbb{R}^n$, where

$$\begin{aligned} &\varphi_{\sigma|\eta}(x_1, x_2, \ldots, x_n) \\ &\quad := (x_{\eta(1)} - 1/x_{\sigma(1)}, x_{\eta(2)} - 1/x_{\sigma(2)}, \ldots, x_{\eta(n)} - 1/x_{\sigma(n)}) \end{aligned} \tag{3.1}$$

for any $n \in \mathbb{N}$ and arbitrary permutations σ and $\eta \in S_n$. For the case $n = 2$ one obtains the following two-dimensional Boole type mappings:

$$\varphi_{1|1}(x, y) := (x - 1/x, y - 1/y), \tag{3.2}$$

$$\varphi_{2|2}(x, y) := (y - 1/y, x - 1/x), \tag{3.3}$$

and

$$\varphi_{1|2}(x, y) := (x - 1/y, y - 1/x), \tag{3.4}$$

$$\varphi_{2|1}(x, y) := (y - 1/x, x - 1/y) \tag{3.5}$$

for all $(x, y) \in \mathbb{R}^2 \backslash \{0, 0\}$. It is easy to observe that the infinitesimal measure $d\mu(x, y) := dxdy$ on the plane $\mathbb{R}^2$ is, by the Fubini theorem, invariant subject to the mapping (3.2) as the tensor product of two one-dimensional measures dx and dy, every one of which is invariant with respect to the corresponding true Boole transformation. The latter entails right away that the generalized Boole type transformation (3.2) is also ergodic. In the case of the generalized two-dimensional transformation (3.4) the infinitesimal invariance property of the measure $d\mu(x, y) := dxdy$ holds owing to the following lemma, stated in [18].

Lemma 3.1 *The mapping (3.4) subject to the measure $d\mu(x, y)$ on $\mathbb{R}^2$ satisfies the following infinitesimal invariance property*:

$$\begin{aligned} &\mu\big(\psi_{1|2}^{-1}\big([x, x + dx] \times [y, y + dy]\big)\big) \\ &\quad = dxdy = \mu\big([x, x + dx] \times [y, y + dy]\big) \end{aligned} \tag{3.6}$$

for all $(x, y) \in \mathbb{R}^2 \backslash \{0, 0\}$.

It is easy to check that a similar statement and the infinitesimal invariance property like (3.6) hold also in the case of the two-dimensional Boole type transformations (3.3), (3.4) and (3.5). As the problem of ergodicity of the mappings (3.3)–(3.5) is of great interest, we formulate the following conjecture, generalizing that from [18].

The constructed above mappings (3.2)–(3.5) are ergodic with respect to the invariant infinitesimal measure $d\mu(x, y) := dxdy$ on $\mathbb{R}^2$. Moreover, for any $n \in \mathbb{N}$ the infinitesimal measure $d\mu(x_1, x_2, \dots, x_n) := \prod_{j=1}^n dx_j$ is invariant and ergodic with respect to generalized multi-dimensional Boole type transformations (3.1) for arbitrary chosen permutations σ and $\eta \in S_n$.

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