Trends in Mathematics
Research Perspectives

Vladimir V. Mityushev
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Current Trends in Analysis and Its Applications

Proceedings of the 9th ISAAC Congress, Kraków 2013

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Preface

The *9th International ISAAC Congress* was held in the period 5–9 August 2013 in Krakow, Poland. The main organiser was the Krakow Pedagogical University. The congress continued the successful series of biennial meetings previously held in USA (1997), Japan (1999), Germany (2001), Canada (2003), Italy (2005), Turkey (2007), UK (2009) and Russia (2011). The total number of participants of the congress was 502 coming from 62 countries, including the special guests and the Organising Committee from Krakow and Rzeszow. There were 12 plenary speakers and 2 key speakers of the discussion group on applications. Totally, the congress had 23 sessions spanned over 4 working days, while 1 day was assigned to excursions. The congress was sponsored by industrial partners. When needed, the Polish consulates over Europe issued free visas for participants of the congress.

One of the main features of the congress was the invitation of industrial specialists, especially engineers working in industry, whose activity is related to applied mathematics (industrial mathematics) and computer sciences. The 9th International ISAAC Congress was an important scientific event during which mathematicians from different parts of the world had an opportunity to present new results and ideas. It was also a great possibility for young mathematicians to contact experts in a variety of fields.

The atmosphere during the Congress was warm and friendly at the sessions and at the social events: banquet, excursions, football match followed by a barbeque. It is impossible to give the final score of the football match because too many goals were scored.

The dates of the Congress, namely 5–9 August, provided for an extremely hot period in the history of Krakow that correlated well with hot and fruitful scientific discussions during the Congress.

It is already a well-established tradition to award one or several outstanding young researchers during the ISAAC congress. The ISAAC award of the 9th International ISAAC Congress was presented to

Jasson Vindas
(University of Ghent, Belgium)

for his outstanding results on the asymptotic behaviour of Schwartz distributions and their pointwise values.

At the ISAAC board meeting during the congress Professor Luigi Rodino of the University of Turin was elected as the new ISAAC President, and Professor Michael Reissig of Freiberg University as the new Vice-President. The former president, Professor Michael Ruzhansky of Imperial College London, UK, has finished his 4-year service. The election of the new vice-president and new rules concerning the structure and functioning of the interest groups within ISAAC completed the work on the new ISAAC constitution that took place in the year before the congress. The new constitution was approved by the general electronic vote by ISAAC members prior to the congress. Further information can be found on the ISAAC homepage

www.mathisaac.org

The ISAAC Board has decided the venue for the following 10th International ISAAC Congress in 2015 to be the University of Macau.

The plenary and key lectures given at the congress appear not here but in the independent volume:

Mityushev V., Ruzhansky M. (Eds.) *Analytic Methods in Interdisciplinary Applications*. Springer Proceedings in Mathematics & Statistics, Vol. 116, Springer, 2015.

The present volume contains the texts of a selection of talks delivered at the congress. As in the previous years, some of the sessions or interest groups decided to publish their own volumes of proceedings and are therefore excluded from the present collection. The work of the congress was spread over the following sessions:

- *Complex variables and potential theory*,
  organised by T. Aliyev, A. Golberg, M. Lanza de Cristoforis, S. Plaksa
- *Differential Equations: Complex and Functional Analytic Methods, Applications*,
  organised by H. Begehr
- *Complex-analytic methods for applied sciences*,
  organised by S.V. Rogosin
- *Clifford and Quaternion Analysis*,
  organised by S. Bernstein, U. Kähler, I. Sabadini and F. Sommen
- *Fixed Point Theory and Applications*,
  organised by E. Karapınar
- *Spaces of Differentiable Functions of Several Real Variables and Applications*,
  organised by V. Burenkov and S. Samko
- *Generalized Functions*,
  organised by M. Oberguggenberger and S. Pilipović
- *Qualitative Properties of Evolution Models*,
  organised by K. Yagdjian, F. Hirosawa and M. Reissig
- *Nonlinear Infinite Dimensional Evolutions and Control Theory with Applications*,
  organised by I. Lasiecka, J. Webster and G. Avalos
- *Nonlinear PDE*,
  organised by V. Georgiev and T. Ozawa
- *Topological and Geometrical Methods of Analysis*,
  organised by A. Prykarpatskyi, Yu. Zelinskyi and K. Soltanov

- *Didactical Approaches to Mathematical Thinking*,
  organised by E. Swoboda and V. Mityushev
- *Wavelet Theory And its Related Topics*,
  organised by K. Fujita and A. Morimoto
- *Integral Transforms and Reproducing Kernels*,
  organised by S. Saitoh and Ju. Rappoport
- *Pseudo differential operators*,
  organised by L. Rodino, J. Toft and M. W. Wong
- *Medical Mathematics and Computing*,
  organised by R. Gilbert, Ju. Rappoport and V. Yakushev
- *Toeplitz Operators and Their Applications*,
  organised by S. Grudsky and N. Vasilevski
- *Approximation Theory and Fourier Analysis*,
  organised by S. Tikhonov and E. Liflyand
- *Differential and Difference Equations with Applications*,
  organised by L. Berezansky, J. Diblik, A. Zafer and M. Zima
- *Analytic Methods in Complex Geometry*,
  organised by A. Schmitt
- *M-Frame Constructions*,
  organised by K. Rudol, H.G. Stark, A. Grybos and D. Onchis
- *Applications of Queueing Theory in Modelling and Performance Evaluation of Computer Networks*,
  organised by K. Grochla
- *Applied Mathematics*,
  organised by S. Bosiakov
- *Others* containing other talks,
  organised by the editors of this volume

We thank the organisers of all the sessions of the congress for their work. They spent an enormous amount of time inviting participants, arranging their sessions, providing chairmen and creating a familiar and workshop-like atmosphere within their meetings. The session organisers were also responsible for collecting contributions to this proceedings volume and for the refereeing process of the papers.

Krakow, Poland Vladimir Mityushev
London, UK Michael Ruzhansky

Contents

Part I
Tributes

A Tribute to the 70th Birthday of Prof Saburou Saitoh

Tsutomu Matsuura

Keywords ISAAC · Saburou Saitoh

Mathematics Subject Classification (2010) Primary 01A70 · Secondary 30DXX · 33E12 · 45EXX · 74A40 · 76D27 · 30E25

Professor Saburou Saitoh participated in the ISAAC congresses since the first congress was held at the University of Delaware in 1997. Every time he organized the session on reproducing kernels during the congress, publishing two volumes of

Dedicated to Prof. Saitoh's 70th anniversary.

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the proceedings with Kluwer Academic Publishers. He was also the Vice-President of ISAAC for 6 years.

Prof. Saburou Saitoh is a unique mathematician—he is always considering questions beyond mathematics: what is the meaning of our whole life and what is the meaning of mathematics. While his research group on the theory of reproducing kernels is not so large, he brings to it his deep love of reproducing kernels. He is teaching his students and colleagues about how fundamental and beautiful mathematics is, and it good impact on human beings.

This led him to a range of results involving the fundamental theory of linear transforms, generalizations of Pythagorean theorem, many beautiful norm inequalities, representations of non-linear simultaneous equations and implicit functions, some considerations of the Tikhonov regularization. A typical result of him is the success of the numerical and real inversion formula of the Laplace transform co-authored with Professors H. Fujiwara and T. Matsuura.

After his retirement from Gunma University at the age of 65 years, the University of Aveiro invited him for a visiting position for 5 years as a researcher. Thus, he got a very happy chance to realize his ideas further. He and his son published an essay book on the universal problems beyond mathematics. In mathematics, he worked on the development of the Aveiro discretization method, with colleagues Professors L.P. Castro, M.M. Rodrigues, and others, and he also presented Announcement 142: An Aveiro Dream in Mathematics. Roughly speaking, when we know some eigenfunctions of a linear operator, we can consider the related partial differential equation and we can solve the associated initial value problem; in this method, one considers the reproducing kernel forms and related integral transforms (linear mappings), and one can discuss the existence problem and the construction problem of the initial value problem. Furthermore, by using the theory of reproducing kernels one can consider further properties of solutions. From this general method, one can consider properties of many integral transforms and reproducing kernels in concrete forms from the known eigenfunctions.

Prof. Saitoh published over 100 papers and 7 books by himself or jointly with his colleagues. Now he is planning a publication of a new book on the theory of reproducing kernels with the young colleague Professor Yoshihiro Sawano–we are waiting for its publication. He is writing blogs on human beings and social problems, and his opinions are stated in Announcements of the Institute of Reproducing Kernels: at this moment, this consists of more than 150 items. We hope his ideas will be spread further and help making a beautiful world.

Sergei Rogosin: Achievements so Far and Further Plans

Maryna Dubatovskaya, Anna Koroleva, and Gennady Mishuris

Keywords ISAAC · Sergei Rogosin

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Dedicated to Prof. Rogosin's 60th anniversary.

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Professor Sergei Rogosin is an international expert in the field of mathematical analysis, complex variables and their applications. His main research interests are: boundary value problems for analytic functions, integral equations, wavelet analysis, nonlinear analysis, free boundary value problems, geometric functions theory, approximation theory, mechanics of continuum media, Hele-Show flows, fluid dynamics, theory of composite materials, fractional analysis, special functions and differential equations of fractional order. Sergei Rogosin is an author of many research publications including three monographs (one published jointly with V. Mityushev, another with F. Mainardi and the third one with R. Gorenflo, A. Kilbas, F. Mainardi). Clearly, his active international cooperation has contributed significantly to broadening his scientific interests. He is also known as an author of the history of mathematical study in the area of probability theory and rheology.

Sergei Rogosin is an outstanding representative of the scientific school created by academician F.D. Gakhov in the Former Soviet Union. The schools impact on the development of analytical methods for analyzing various boundary value problems has been widely recognized. As a pupil of F.D. Gakhov, Sergei Rogosin has made essential contributions to the schools achievements.

Since 2001, Sergei Rogosin has been an active participant in ISAAC Congresses. In 2009 he became a Member of the ISAAC International Advisory Board. He has organised special sessions at the 5th (July 25–30, 2005, Catania, Italy), 6th (August 13–18, 2007, Ankara, Turkey), 7th (July 12–19, 2009, London, UK), 8th (August 22–27, 2011, Moscow, Russia) and 9th (August 5–10, 2013, Krakow, Poland) ISAAC Congresses. Sergei Rogosin contributed extensively to form the scientific programs of these Congresses, to recruit new ISAAC members and to promote the Societies activities. He was an editor of the collection of scientific papers from participants of the 9th ISAAC Congress, accepted for publication at the publishing house Cambridge Scientific Publishers.

Another important activity by Professor Rogosin has been the organization of a series of AMADE international conferences in Minsk. It started in 1996, when professor Anatoly Kilbas brought forward the idea of holding a conference dedicated to the memory of academician F.D. Gakhov. Sergei Rogosin became a scientific secretary of the organizing committee for the international conference “Boundary Value problems, Special Functions and Fraction Calculus”.

Following the success of the previous conference, in 1999 the next conference was organised with the title “Analytic Methods of Analysis and Differential Equations” (AMADE) proposed by Prof. A.P. Prudnikov. Since then the title has remained the same while AMADE subject areas have consistently been extended at the conferences held in Minsk in 2001, 2003, 2006, 2009, 2011, 2012.

After the untimely demise of professor Anatoly Kilbas in 2010, Sergei undertook the main duties of organizing the conferences. AMADE-2011 was dedicated to the memory of professor A. Kilbas. The next conference AMADE-2015 will be organized in 2015 (September, 14–19) in Minsk as usual. All AMADE conferences were held under the guidance of ISAAC.

Along with his scientific activity, professor Rogosin has also been a dedicated teacher. He is an author of programs for courses in Real and Complex Analysis and has published several textbooks and lecture notes. His teaching experience includes a broad spectrum of problems and methodologies following from his scientific interests and lecturing activities in his own country and abroad. Over a long career at the Belarusian State University, Sergei Rogosin delivered lectures on Real and Complex Analysis, Optimization Methods, Optimal Control Theory, Mathematical and Economical Statistics, Econometrics as well as various aspects of teaching methodologies.

In 2009, Sergei Rogosin prepared and delivered lectures on modern problems of analysis and their applications for graduate and post-graduate students in Druskininkai, Lithuania (organized by the Institute of Mathematics and Informatics, Vilnius University). In January 2010, he became one of the organizers of the International Winter School "Modern problems of Mathematics and Mechanics" for young researchers and postgraduate students. Cycles of lectures were presented by international experts in the area from the UK, Germany, Lithuania, Poland, Belarus and Ukraine. The success of the school was confirmed by the publication of the lecture notes "Advances in Applied Analysis" in Birkhäuser Verlag.

Sergei Rogosin has a passion for working with the younger generation. He spares no efforts to help earlier stage researchers in formulating new problems, discussing their progress, and preparing their first research papers. His most successful postgraduate students were: M. Dubatovskaya (Minsk, BSU, 1997), S. Makaruk (Minsk, BSU, 2004), E. Pesetskaya (Minsk, BSU, 2006) and T. Vaitekhovich (Berlin, FU, 2008).

Professor Rogosin has created his own research groups in Belarusian State University and leads a number of projects within the framework of State Research Programs and the Belarusian Fund for Fundamental Research. He is also a coordinator of European international scientific programs at the Belarusian State University.

Sergei Rogosin is a member of the Editorial Board of the international journals: Mathematical Modelling and Analysis (Vilnius, Lithuania), Analysis (München, Germany), Mathematics in Engineering, Science and Aerospace (Cambridge, UK), Fractional Calculus and Applied Analysis (Sofia, Bulgaria) and the Siauliai Mathematical Seminar (Siauliai, Lithuania). His expertise and enormous scientific erudition is highly demanded by the colleagues asking him for proofreading, reviewing their books and papers.

And, yet, there are still so many exciting things for Sergei to do: he is intensively developing several new scientific ideas and awaiting the further papers, books and conference presentations to come. He is looking forward to meeting and communicating with his old and new colleagues, helping his new and previous pupils. Without a doubt, he will continue contributing to new and exciting directions of research and teaching.

Part II
Differential Equations: Complex and Functional Analytic Methods, Applications

Organizer: Heinrich Begehr

The Degenerate Second-Order Elliptic Oblique Derivative Problem in a Domain with a Conical Boundary Point

Mariusz Bodzioch and Mikhail Borsuk

Abstract We have investigated the behaviour of strong solutions to the degenerate oblique derivative problem for linear second-order elliptic equation in a neighborhood of a conical boundary point of an n-dimensional bounded domain ($n \geq 2$).

Keywords Elliptic equations · Oblique problem · Conical points

Mathematics Subject Classification (2010) 35J20 · 35J25 · 35J70

1 Introduction

This is a brief description of results of [1–3], which were presented at the 9th ISAAC Congress. In these articles we have investigated the behavior of strong solutions to the oblique degenerate derivative problem for the general second-order linear elliptic equation in a neighborhood of a conical boundary point of an n-dimensional bounded domain ($n \geq 2$). The degeneracy of the problem stems from the term in the boundary condition. We have found the precise exponent of the solution's decreasing rate under minimal assumptions on the coefficients of the problem.

Let G be an n-dimensional, $n \geq 2$, bounded domain with a boundary that is a smooth surface everywhere except at the origin $\mathcal{O} \in \partial G$ and near $\mathcal{O}$ it is a conical surface.

We consider the oblique derivative problem for the elliptic second-order linear equation:

$$\begin{cases} \mathcal{L}[u] \equiv a^{ij}(x)u_{x_i x_j} + a^i(x)u_{x_i} + a(x)u = f(x), & x \in G, \\ \mathcal{B}[u] \equiv \dfrac{\partial u}{\partial \vec{n}} + \chi(\omega)\dfrac{\partial u}{\partial r} + \dfrac{1}{|x|}\gamma(\omega)u = g(x), & x \in \partial G \backslash \mathcal{O}, \end{cases} \tag{L}$$

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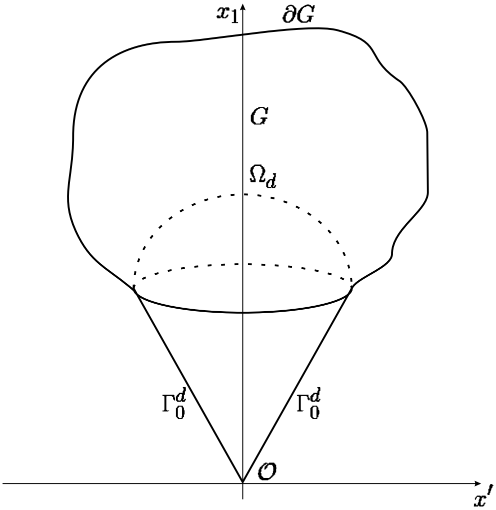

Fig. 1 n-dimensional bounded domain with a conical boundary point

where $\vec{n}$ denotes the unit exterior normal to $\partial G \backslash \mathcal{O}$, (r, ω) are spherical coordinates in $\mathbb{R}^n$ with pole $\mathcal{O}$; repeated indices are understood as summation from 1 to n.

2 Preliminaries

For a domain G we use the following notations:

- $\Omega := G \cap S^{n-1}$, where S^{n-1} denotes the unit sphere in $\mathbb{R}^n$;
- $\partial\Omega$: the boundary of Ω;
- $G_a^b := G \cap \{(r, \omega) : 0 \leq a < r < b, \omega \in \Omega\}$: a layer in $\mathbb{R}^n$;
- $\Gamma_a^b := \partial G \cap \{(r, \omega) : 0 \leq a < r < b, \omega \in \partial\Omega\}$: the lateral surface of G_a^b;
- $G_d := G \backslash G_0^d$, $\Gamma_d := \partial G \backslash \Gamma_0^d$, $\Omega_\varrho := \overline{G_0^d} \cap \{|x| = \varrho\}$, $\varrho \leq d$.

Without loss of generality, we assume, that there exists $d > 0$, such that G_0^d is a *rotational cone* (see Fig. 1) with vertex $\mathcal{O}$ and aperture $\omega_0 \in (0, \pi)$, thus

$$\Gamma_0^d = \left\{ (r, \omega) : \ x_1^2 = \cot^2 \frac{\omega_0}{2} \sum_{i=2}^{n} x_1^2, \ r \in (0, d), \ \omega_1 = \frac{\omega_0}{2}, \ \omega_0 \in (0, \pi) \right\}.$$

We use standard function spaces: $C^k(\overline{G})$ with the norm $|u|_{k,\overline{G}}$; $L_p(G)$ with the norm $\|u\|_{p,G}$, $p \geq 1$; $W^{k,p}(G)$ with the norm $\|u\|_{W^{k,p}(G)}$ for integer $k \geq 0$; the weighted Sobolev space $\overset{\circ}{W}{}_\alpha^k(G)$ for real α with the norm $\|u\|_{\overset{\circ}{W}{}_\alpha^k(G)} =$

$(\int_G \sum_{|\beta|=0}^{k} r^{\alpha+2(|\beta|-k)}|D^\beta u|^2 dx)^{\frac{1}{2}}$ and the space $\overset{\circ}{W}{}_{\alpha}^{k-\frac{1}{2}}(\partial G)$ denotes the space of traces on ∂G.

Definition 2.1 A function $u(x)$ is called a strong solution of problem (L) provided that for any $\varepsilon > 0$ function $u(x) \in W_{loc}^{2,n}(G) \cap W^{2,2}(G_\varepsilon) \cap C^0(\overline{G})$ and satisfies the equation $\mathcal{L}u = f$ for almost all $x \in G_\varepsilon$ as well as the boundary condition $\mathcal{B}u = g$ in the sense of traces on Γ_ε.

We assume that the following **conditions** are satisfied:

(a) the ellipticity condition

$$\nu|\xi|^2 \le \sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \le \mu|\xi|^2, \quad \forall \xi \in \mathbb{R}^n,\ x \in \overline{G}$$

with the ellipticity constants $\nu, \mu > 0$; $a^{ij}(x) = a^{ji}(x)$; $a^{ij}(0) = \delta_i^j$;

(b) $a^{ij}(x) \in C^0(\overline{G})$, $a^i(x) \in L^p(G)$, $p > n$; $a(x)$ and $f(x) \in L^n(G) \cap \overset{\circ}{W}{}_{4-n}^{0}(G)$, $g(x) \in \overset{\circ}{W}{}_{4-n}^{1/2}(\partial G)$; there exists a monotonically increasing nonnegative function $\mathcal{A}(r)$, continuous at zero, $\mathcal{A}(0) = 0$, such that for $x \in \overline{G}$

$$\left(\sum_{i,j=1}^{n} \left|a^{ij}(x) - \delta_i^j\right|^2\right)^{\frac{1}{2}} + |x|\left(\sum_{i=1}^{n}\left|a^i(x)\right|^2\right)^{\frac{1}{2}} + |x|^2\left|a(x)\right| \le \mathcal{A}\big(|x|\big);$$

(c) $\gamma(\omega), \chi(\omega) \in C^1(\overline{\Omega})$ and there exist numbers $\chi_0 \ge 0$, $\gamma_0 > 0$ such that $\gamma(\omega) \ge \gamma_0 > 0$, $0 \le \chi(\omega) \le \chi_0$;

(d) there exist numbers $f_1 \ge 0$, $g_1 \ge 0$, $g_0 \ge 0$, $s > 0$ such that

$$\left|f(x)\right| \le f_1|x|^{s-2}, \qquad \left|g(x)\right| \le g_1|x|^{s-1},$$
$$\int_{G_0^\varrho} r^{4-n}|\nabla g|^2 dx \le g_0^2\varrho^{2s}, \quad \varrho \in (0,1).$$

Let $\chi(\omega) \ge 0$, $\gamma(\omega) > 0$ be $C^0(\partial\Omega)$-functions and $\vec{\nu}$ be the unit exterior normal to G_0^1 at the points of $\partial\Omega$. We consider the **eigenvalue problem** for the Laplace–Beltrami operator Δ_ω on the unit sphere

$$\begin{cases} \Delta_\omega\psi + \lambda(\lambda+n-2)\psi(\omega) = 0, & \omega \in \Omega, \\ \dfrac{\partial\psi}{\partial\vec{\nu}} + \big\langle\lambda\chi(\omega) + \gamma(\omega)\big\rangle\psi(\omega) = 0, & \omega \in \partial\Omega, \end{cases} \qquad (EVP)$$

which consists of the determination of all values λ (eigenvalues), for which (EVP) has a non-zero weak solutions $\psi(\omega)$ (eigenfunctions).

Definition 2.2 A function ψ is called a weak solution of problem (*EVP*) provided that $\psi \in W^{1,2}(\Omega)$ and satisfies the integral identity

$$\int_{\Omega}\left(\frac{1}{q_i}\frac{\partial\psi}{\partial\omega_i}\frac{\partial\eta}{\partial\omega_i} - \lambda(\lambda+n-2)\psi\eta\right)d\Omega + \int_{\partial\Omega}\big\langle\lambda\chi(\omega)+\gamma(\omega)\big\rangle\psi\eta d\sigma = 0 \quad (2.1)$$

for all $\eta(x) \in W^{1,2}(\Omega)$.

Remark 2.3 It is easy to see, that $\lambda = 0$ is not an eigenvalue.

3 Main Results

Theorem 3.1 ([1–3]) *Let $u(x)$ be a strong solution of problem (L) and assumptions (a)–(d) are satisfied. Then there are $d \in (0,1)$ and constants $C_1, C_2, C_3 > 0$, $c_s > 0$ depending only on f_1, g_1, g_0, ν, μ, s, λ, γ_0, χ_0, $\|\gamma\|_{C^1(\partial G)}$, $\|\chi\|_{C^1(\partial G)}$,* diam G, meas G, *on the modulus of continuity of the leading coefficients and on the quantity $\int_0^1 \frac{\mathcal{A}(r)}{r}dr$, such that for all $x \in G_0^d$*

- $$\big|u(x)\big| \le C_1 \begin{cases} |x|^{\lambda}, & \text{if } s>\lambda, \\ |x|^{\lambda}\ln\dfrac{1}{|x|}, & \text{if } s=\lambda, \\ |x|^{s}, & \text{if } s<\lambda, \end{cases} \quad (3.1)$$

  if the function $\mathcal{A}(r)$ is Dini-continuous at zero;

- $$\big|u(x)\big| \le C \begin{cases} |x|^{\lambda-\varepsilon}, & \text{if } s \ge \lambda, \\ |x|^{s-\varepsilon}, & \text{if } s<\lambda, \end{cases} \quad (3.2)$$

  if the function $\mathcal{A}(r)$ is continuous at zero, but is not Dini continuous;

- $$\big|u(x)\big| \le C \ln^{c_s(\lambda)}\left(\frac{1}{|x|}\right) \cdot \begin{cases} |x|^{\lambda}, & \text{if } s \ge \lambda, \\ |x|^{s}, & \text{if } s<\lambda, \end{cases} \quad (3.3)$$

  if the function $\mathcal{A}(r) \sim \frac{1}{\ln\frac{1}{r}}$.

Theorem 3.2 ([1, 2]) *There exists the smallest positive eigenvalue λ of problem (EVP), which satisfies the following inequalities*

$$0 < \lambda < \sqrt{\left(\frac{\pi}{\omega_0}\right)^2 + \left(\frac{n-2}{2}\right)^2} - \frac{n-2}{2}$$

for $n \ge 3$.

4 The Ideas of the Proofs of Theorems 3.1 and 3.2

The idea of the proof of Theorem 3.1 is based on the deduction of a new inequality of Friedrichs–Wirtinger type (see Theorem 4.1 and Remark 4.2) with an exact constant as well as some integral-differential inequalities adapted to our problem. The precise exponent of the solution's decrease rate depends on this exact constant.

Theorem 4.1 *Let* λ *be the smallest positive eigenvalue of problem* (*EVP*) *and* $\chi(\omega) \geq 0$, $\gamma(\omega) > 0$ *be* $C^0(\partial\Omega)$*-functions. Then the inequality*

$$\int_\Omega u^2 d\Omega \leq \frac{1}{\lambda(\lambda+n-2)}\left[\int_\Omega |\nabla_\omega u|^2 d\Omega + \int_{\partial\Omega} \langle \lambda\chi(\omega)+\gamma(\omega)\rangle u^2 d\sigma\right] \quad (FW)$$

holds for any $u \in W^{1,2}(\Omega)$.

Remark 4.2 Inequality (*FW*) is the best possible, i.e. the estimating constant in this inequality is exact. This is easy to see if we put $u(\omega)$ as a solution of problem (*EVP*).

Theorem 3.1 for the Dirichlet and Robin problems was proved in [4]. In our case technical difficulties were related with the term $\chi(\omega)\frac{\partial u}{\partial r}$ in the boundary condition of (*L*).

The existence of the smallest positive eigenvalue of problem (*EVP*) for $n = 3$ was proved in [1]. The ideas of the proof of this theorem are based on the Legendre spherical harmonics (see [1]) and the Gegenbauer functions $\mathcal{C}^\alpha_\lambda$. We have proved the existence of the smallest positive solution λ of equation

$$(n-2)\sin\frac{\omega_0}{2}\cdot \mathcal{C}^{\frac{n}{2}}_{\lambda-1}\left(\cos\frac{\omega_0}{2}\right) = (\lambda\chi_0+\gamma_0)\cdot \mathcal{C}^{\frac{n-2}{2}}_{\lambda}\left(\cos\frac{\omega_0}{2}\right)$$

and that this solution is the smallest positive eigenvalue of (*EVP*).

5 Examples

We demonstrate our results by examples from [1] and [3].

Suppose $n = 2$ and the domain G lies inside the corner

$$G_0 = \left\{(r,\omega) : r > 0,\ \omega \in \left(-\frac{\omega_0}{2}, \frac{\omega_0}{2}\right),\ \omega_0 \in (0,\pi)\right\}, \quad \mathcal{O} \in \partial G$$

and in some neighborhood of $\mathcal{O}$ the boundary ∂G coincides with the sides of the corner $\omega = -\frac{\omega_0}{2}$ and $\omega = \frac{\omega_0}{2}$.

We denote $\Gamma_\pm = \{(r,\omega) \mid r > 0;\ \omega = \pm\frac{\omega_0}{2}\}$ and we put $\gamma(\omega)|_{\omega=\pm\frac{\omega_0}{2}} = \gamma_\pm = const > 0$; $\chi(\omega)|_{\omega=\pm\frac{\omega_0}{2}} = \chi_\pm = const \geq 0$.

In this case the eigenvalue problem (*EVP*) has the following form

$$\begin{cases} \psi''(\omega)+\lambda^2\psi(\omega)=0, \quad \omega\in\left(-\frac{\omega_0}{2},\frac{\omega_0}{2}\right),\\ \psi'\left(\frac{\omega_0}{2}\right)+(\lambda\chi_+ +\gamma_+)\psi\left(\frac{\omega_0}{2}\right)=0,\\ -\psi'\left(-\frac{\omega_0}{2}\right)+(\lambda\chi_- +\gamma_-)\psi\left(-\frac{\omega_0}{2}\right)=0. \end{cases} \tag{5.1}$$

The smallest positive eigenvalue λ of this problem satisfies the inequality $\frac{\pi}{2\omega_0}<\lambda<\frac{\pi}{\omega_0}$.

Example Let $\omega_0\in(0,\frac{\pi}{2})$ and $(\lambda,\psi(\omega))$ be the solution of (5.1). Therefore we have $\lambda>1$. The function $u(r,\omega)=r^\lambda(\ln\frac{1}{r})^{\frac{\lambda-1}{\lambda+1}}\cdot\psi(\omega)$ is the solution in the corner G_0 of the problem

$$\begin{cases} \sum_{i,j=1}^{2} a^{ij}(x)u_{x_ix_j}=0, \quad x\in G_0,\\ \left.\left(\frac{\partial u}{\partial\vec{n}}+\frac{1}{r}\gamma_\pm u+\chi_\pm\frac{\partial u}{\partial\vec{r}}\right)\right|_{\Gamma_\pm}=g(r,\omega)|_{\Gamma_\pm}, \end{cases}$$

where

$$a^{11}(x)=1-\frac{2}{\lambda+1}\cdot\frac{x_2^2}{r^2\ln 1/r},$$

$$a^{12}(x)=a^{21}(x)=\frac{2}{\lambda+1}\cdot\frac{x_1x_2}{r^2\ln 1/r},$$

$$a^{22}(x)=1-\frac{2}{\lambda+1}\cdot\frac{x_1^2}{r^2\ln 1/r},$$

$$g(r,\omega)|_{\Gamma_\pm}=-\frac{\lambda-1}{\lambda+1}\chi_\pm r^{\lambda-1}\left(\ln\frac{1}{r}\right)^{-\frac{2}{\lambda+1}}\cdot\psi\left(\pm\frac{\omega_0}{2}\right), \quad r>0.$$

If $d<e^{-2}$, then $\mu=1$ and $\nu=1-\frac{2}{\ln(1/d)}$. Furthermore, we have that the assumption (b) is fulfilled with $\mathcal{A}(r)=\frac{2}{\lambda+1}\cdot\frac{1}{\ln(\frac{1}{r})}$, but the function $\mathcal{A}(r)$ does not satisfy the Dini condition at zero. Moreover, $a^{ij}(x)$ are continuous at the point $\mathcal{O}$ and $g(r,\omega)|_{\Gamma_\pm}=O(r^{\lambda-1})$. This example shows that the assumption of Theorem 3.1 about the Dini-continuity of the leading coefficients of (L) and $s>\lambda$ are essential for $u(x)=O(|x|^\lambda)$. At the same time the example confirms the validity of (3.3) of Theorem 3.1.

Let the domain $G \subset \mathbb{R}^3$ lie inside the cone

$$G_0 = \left\{(r, \omega_1, \omega_2) : \ r > 0, \ \omega_1 \in \left(0, \frac{\omega_0}{2}\right), \ \omega_2 \in (0, 2\pi); \ \omega_0 \in (0, \pi)\right\},$$

$\mathcal{O} \in \partial G$ and in some neighborhood of $\mathcal{O}$ the boundary ∂G coincides with the lateral surface of the cone G_0. Denote

$$\Gamma_0 = \left\{(r, \omega_1, \omega_2) : \ r > 0, \ \omega_1 = \frac{\omega_0}{2}, \ \omega_2 \in (0, 2\pi); \ \omega_0 \in (0, \pi)\right\}.$$

Let $\chi \geq 0$, $\gamma > 0$ be constants.

Example The function

$$u(r, \omega_1, \omega_2) = r^\lambda \ln \frac{1}{r} \psi(\omega_1)$$

is the solution of the problem

$$\begin{cases} \Delta u = -(2\lambda + 1) r^{\lambda - 2} \psi(\omega_1), \quad x \in G_0, \\ \left(\dfrac{\partial u}{\partial \vec{n}} + \chi \dfrac{\partial u}{\partial r} + \dfrac{1}{r} \gamma u\right)\Bigg|_{\omega_1 = \frac{\omega_0}{2}} = -\chi r^{\lambda - 1} \psi\left(\dfrac{\omega_0}{2}\right). \end{cases}$$

Hence

$$f(x) = O\big(|x|^{\lambda - 2}\big), \qquad g(x) = O\big(|x|^{\lambda - 1}\big),$$

therefore in this case $s = \lambda$. Thus, this example confirms the validity of (3.1) of Theorem 3.1 for $s = \lambda$.

Example The function

$$u(r, \omega_1, \omega_2) = r^\lambda \ln \frac{1}{r} \psi(\omega_1)$$

is the solution of the problem

$$\begin{cases} \Delta u + \dfrac{2\lambda + 1}{r^2 \ln \frac{1}{r}} u = 0, \quad x \in G_0, \\ \left(\dfrac{\partial u}{\partial \vec{n}} + \chi \dfrac{\partial u}{\partial r} + \dfrac{1}{r} \gamma u\right)\Bigg|_{\omega_1 = \frac{\omega_0}{2}} = -\chi r^{\lambda - 1} \psi\left(\dfrac{\omega_0}{2}\right). \end{cases}$$

In this case

$$\mathcal{A}(r) = \frac{2\lambda + 1}{\ln \frac{1}{r}} \quad \Longrightarrow \quad \int_0^1 \frac{\mathcal{A}(r)}{r} dr = +\infty.$$

Therefore the assumption of Dini-continuity is not satisfied. Thus, this example confirms the validity of (3.3) of Theorem 3.1 for $s = \lambda$.

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On Locally Differentiable Solutions of the Stationary Schrödinger Equation with Discontinuous Coefficients

K.N. Ospanov

Abstract In this paper we study the binomial second order elliptic equation. The coefficient and the right-hand side of this equation belong to some space M of type F. We find necessary and sufficient conditions on M under which a generalized solution of this equation is continuously differentiable. We find necessary and sufficient conditions on M for continuous differentiability of the solution to the above equation, when M is a symmetric space, or one of the Lorentz spaces, a Sobolev or a Besov space.

Keywords Elliptic equation · Differentiability of solution · Symmetric space

Mathematics Subject Classification (2010) 35J10

Let $n \geq 3$ and Q be a bounded set in R^n with a smooth boundary and Ω be an arbitrary open subset of Q. Let $C^{(1)}(Q)$ be the space of continuously differentiable functions in Q and $W_p^s(\Omega)$ be the well-known Sobolev space.

We consider the following equation:

$$Lu \equiv -\Delta u + q(x)u = f(x), \tag{1}$$

where $x = (x_1, x_2, \ldots, x_n) \in \Omega$.

It is known that if $p > n$ and $q, f \in L_p(\Omega)$, then there is a generalized solution $u \in W_p^1(\Omega)$ of (1) such that it has continuous partial derivatives in Ω (see [1], Ch. 3, paragraph 15). This result can not be improved, since if at least one of q and f belongs to $L_n(\Omega)\backslash L_p(\Omega)$ $(p > n)$, then the solution u of (1) does not belong to $C_{loc}^{(1)}(\Omega)$ (For a Banach space $F(Q)$ of functions in Q, it is said that u belongs to $F_{loc}(\Omega)$, if $\psi(x)u(x) \in F(\Omega)$ for all $\psi(x) \in C_0^{\infty}(\Omega)$.) So the following question arises:

Question Are there other spaces such that the above precise result about differentiable solutions of (1) holds for any q, f in this spaces?

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The purpose of this work is to answer this question.

By a space of type F we mean a Banach space $F(Q)$ of functions in Q satisfying the following properties:

(a) $F(Q)$ is embedded in $L_1(Q)$, i.e., there is a constant $c > 0$ such that

$$\|f\|_{L_1(Q)} \leq c\|f\|_{F(Q)}, \quad \forall f \in F(Q);$$

(b) the set $C^\infty(Q)$ of infinitely differentiable functions in Q is dense in $F(Q)$;
(c) if $g \in F(Q)$ and $\psi \in C^\infty(Q)$, then $\psi \cdot g \in F(Q)$;
(d) if $g \in F(Q)$, then $|g| \in F(Q)$ and $\||g|\|_{F(Q)} \leq c_1\|g\|_{F(Q)}$.
For example, $L_p(Q)$ and the Lorentz space $L_{p,r}(Q)$ $(1 < p, r < \infty)$ are spaces of type F, where the norm of $L_{p,r}(Q)$ is given in the following form (see [2]):

$$\|u\|_{L_{p,r}(Q)} = \left(\int_0^{+\infty} \left\{\left[\mu\left(x \in Q : \left|f(x)\right| \geq t\right)\right]^{1/p} t\right\}^r \frac{dt}{t}\right)^{1/r}.$$

Here μ is the Lebesgue measure.

Definition 1 Let $M(Q)$ and $M_1(Q)$ be Banach spaces of functions in Q satisfying the above conditions (a), (b) and (c). Assume that $q(x), f(x) \in M(\Omega)$ and $u(x) \in M_1(\Omega)$. If there is a sequence $\{u_m(x)\}_{m\geq 1}$ of infinitely differentiable functions in Ω such that for any $\psi(x) \in C_0^\infty(\Omega)$

$$\|\psi u_m - \psi u\|_{M_1(\Omega)} \to 0, \qquad \|\psi L u_m - \psi f\|_{M(\Omega)} \to 0 \quad (m \to \infty),$$

then $u(x)$ is called a solution of (1).

Definition 2 Let $M(Q)$ be a Banach space of functions in Q satisfying the above conditions (a), (b) and (c). If for any $q, f \in M_{loc}(\Omega)$ the solution $u \in W^1_{p,loc}(\Omega)$ $(p > \frac{n}{2})$ of (1) belongs to $C^{(1)}_{loc}(\Omega)$, then $M(Q)$ is called "$\{-\Delta, C^{(1)}\}$ coordinated".

Remark 1 From this definition it follows that if $M(Q)$ is a "$\{-\Delta, C^{(1)}\}$ coordinated" space and $M_2(Q) \subseteq M(Q)$ is a Banach space of functions in Q satisfying the above conditions (a), (b) and (c), then $M_2(Q)$ is also a "$\{-\Delta, C^{(1)}\}$ coordinated" space.

In order to answer the above question, we need to find a "$\{-\Delta, C^{(1)}\}$ coordinated" space of functions in Q.

Let α be an integer and $0 \leq \alpha < n$. We denote by $\mathrm{P}_\alpha(Q)$ the completion of $(C_0^\infty(Q), \|.\|_{\mathrm{P}_\alpha(Q)})$, where

$$\|u\|_{\mathrm{P}_\alpha(Q)} = \sup_{x\in Q} \int_Q \frac{|u(y)|}{|x-y|^\alpha} dy.$$

It is easy to check that $\mathrm{P}_\alpha(Q)$ is a space of type F.

Theorem 1 *Let $M(Q)$ be a space of type F and $n \geq 3$. Then $M(Q)$ is "$\{-\Delta, C^{(1)}\}$ coordinated" if and only if*

$$M(\Omega) \subseteq \mathrm{P}_{n-1}(\Omega).$$

In particular, $\mathrm{P}_{n-1}(Q)$ is a "$\{-\Delta, C^{(1)}\}$ coordinated" space.

From Remark 1 and Theorem 1 it follows that $\mathrm{P}_{n-1}(Q)$ is the widest "$\{-\Delta, C^{(1)}\}$ coordinated" space of type F.

Theorem 2 *Let $M(Q)$ be a space of type F. Assume that there exits a constant $c_2 > 0$ such that for any $u \in M(Q)$ and any automorphism φ (one to one correspondence preserving the measure) in Q, one has that*

$$u\big(\varphi(x)\big) \in M(Q), \quad \big\|u(\varphi)\big\|_{M(Q)} \leq c_2 \|u\|_{M(Q)}.$$

Then $M(Q)$ is "$\{-\Delta, C^{(1)}\}$ coordinated" if and only if

$$M(Q) \subseteq L_{n,1}(Q). \tag{2}$$

A Banach space $M(Q)$ of functions in Q is called a symmetric space, if for any $u \in M(Q)$ and any automorphism φ in Q, one has that

$$u\big(\varphi(x)\big) \in M(Q), \quad \big\|u(\varphi)\big\|_{M(Q)} = \|u\|_{M(Q)}.$$

By Theorem 2 and Remark 1, we have that $L_{n,1}(Q)$ is the widest "$\{-\Delta, C^{(1)}\}$ coordinated" symmetric space satisfying the above conditions (a), (b) and (c). From Theorem 2 and Definition 2 we obtain the following result.

Corollary 1 *Let $1 \leq p, r < \infty$ and $q(x), f(x) \in L_{p,r,loc}(\Omega)$. Then the solution $u(x) \in W^1_{p,loc}(\Omega)$ $(p > \frac{n}{2})$ of (1) belongs to $C^{(1)}_{loc}(\Omega)$ if and only if one of the following conditions holds*

(i) $r = 1$, $p \geq n$,
(ii) $1 \leq r \leq \infty$, $p > n$.

By the inclusion (2) and Remark 1, we obtain the following results.

Corollary 2 *Let $1 \leq p < \infty$, $0 < s < \infty$ and $q(x), f(x) \in W^s_{p,loc}(\Omega)$. Then the solution $u(x) \in W^1_{p,loc}(\Omega)$ $(p > \frac{n}{2})$ of (1) belongs to $C^{(1)}_{loc}(\Omega)$ if and only if $s > \frac{n}{p} - 1$.*

Corollary 3 *Let $1 \leq p, \theta < \infty$, $0 < s < \infty$. If $q(x)$ and $f(x)$ belong to the Besov space $B^s_{p,\theta,loc}(\Omega)$, then the solution $u(x) \in W^1_{p,loc}(\Omega)$ $(p > \frac{n}{2})$ of (1) belongs to $C^{(1)}_{loc}(\Omega)$ if and only if one of the following conditions holds*

(i) $\theta = 1$, $s \geq \frac{n}{p} - 1$, $1 < p < n$,
(ii) $\theta \geq 1$, $s > \frac{n}{p} - 1$, $1 \leq p < \infty$.

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Riemann Problems for Single-Periodic Polyanalytic Functions II

Wang Yufeng and Wang Yanjin

Abstract In this article, we obtain the explicit expression of the solution and the conditions of solvability for the Riemann problem of single-periodic polyanalytic function.

Keywords Single-periodic polyanalytic function · Periodic poly-Cauchy integral · Riemann boundary-value problem

Mathematics Subject Classification (2010) Primary 30G30 · Secondary 45E05

1 Introduction

When a complex partial differential equation (PDE) couples with some boundary conditions and growth conditions, a boundary-value problem (BVP) takes shape. In recent years, the theory of BVPs for analytic functions has been generalized to those of different classes of functions, including polyanalytic functions [1–3], polyharmonic functions [4], metaanalytic functions [2], and even the family defined by other complex PDEs [5].

Let ω be a complex number different from 0. If the open set Ω on the complex plane $\mathbb{C}$ satisfies the condition $z \pm \omega \in \Omega$ for $\forall z \in \Omega$, then ω is called a period of the open set Ω. All the periods of the open set Ω form a set, denoted as $\mathcal{P}_\Omega$. In what follows, we assume that Ω is a periodic open set.

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Definition 1.1 Suppose f is a polyanalytic function [7] of order n on Ω, where Ω is a periodic open set. If the following two conditions are fulfilled: (1) There exists a complex constant $\omega \in \mathcal{P}_\Omega \setminus \{0\}$ such that

$$f(z+\omega) = f(z), \quad \forall z \in \Omega, \tag{1.1}$$

(2) If two complex constants $\omega_1, \omega_2 \in \mathcal{P}_\Omega \setminus \{0\}$ satisfy (1.1), $\frac{\omega_1}{\omega_2}$ is real, then we say that f is a single-periodic polyanalytic function of order n with period ω on Ω, or simply single-periodic polyanalytic function. The collection of all single-periodic polyanalytic functions on Ω is denoted as $PH_n(\Omega)$.

For a fixed number $\omega \in \mathcal{P}_\Omega$, we denote the subset of $PH_n(\Omega)$ as

$$PH_n(\Omega; \omega) = \left\{ f \in PH_n(\Omega) : \ f(z+\omega) = f(z) \text{ for } \forall z \in \Omega \right\}.$$

Obviously, $PH_n(\Omega; 0) = PH_n(\Omega)$ and $PH_n(\Omega; \omega) \subset PH_n(\Omega; m\omega)$ for all $m \in \mathbb{Z}$. The following decomposition of single-periodic polyanalytic functions is equivalent to Pokazeev's decomposition given in [8].

Theorem 1.2 (see [6]) *Let Ω be a periodic open set with period $\omega \neq 0$. Then*

$$PH_n(\Omega; \omega) = PH_1(\Omega; \omega) \oplus \left(\bar{z} - \frac{\bar{\omega}}{\omega} z\right) PH_1(\Omega; \omega) \oplus \cdots$$
$$\oplus \left(\bar{z} - \frac{\bar{\omega}}{\omega} z\right)^{n-1} PH_1(\Omega; \omega)$$

with $(\bar{z} - \frac{\bar{\omega}}{\omega} z)^j PH_1(\Omega; \omega) = \{(\bar{z} - \frac{\bar{\omega}}{\omega} z)^j f(z) : \ f \in PH_1(\Omega; \omega)\}$.

In [6], by the decomposition of single-periodic polyanalytic functions, the Riemann problem of single-periodic polyanalytic functions has been discussed according to two growth conditions of functions at infinity. Finally, we obtain the explicit expression of the solution and the conditions of solvability in three cases. In this article, we will discuss the Riemann BVP of single-periodic polyanalytic functions in other cases.

2 Preliminaries

Let $\omega \neq 0$ in the following. We define the family of straight lines

$$l_m : \ z_m(t) = \left(\frac{m}{2} + it\right)\omega, \quad -\infty < t < +\infty, \ m \in 2\mathbb{Z}+1, \tag{2.1}$$

which cut the complex plane $\mathbb{C}$ into countable open strips

$$\Xi_{\frac{m+1}{2}} = \left\{ z = \left(\frac{m}{2} + \xi + i\eta\right)\omega : \ 0 < \xi < 1, \ \eta \in \mathbb{R} \right\}. \tag{2.2}$$

The special open strip Ξ_0 defined in (2.2) is said to be the basic strip, and $\Xi_{\frac{m+1}{2}} = \frac{m+1}{2}\omega + \Xi_0$, $m \in 2\mathbb{Z}+1$. For $a, b \in \mathbb{R}$, we also define three subsets

$$\begin{cases} \Xi_{0,b}^{+} = \left\{ z = (\xi + i\eta)\omega : -\frac{1}{2} < \xi < \frac{1}{2},\ \eta > b \right\}, \\ \Xi_{0,a}^{-} = \left\{ z = (\xi + i\eta)\omega : -\frac{1}{2} < \xi < \frac{1}{2},\ \eta < a \right\} \end{cases} \tag{2.3}$$

and $\square_{a,b} = \Xi_{0,a}^{+} \cap \Xi_{0,b}^{-}$. We assume that $\square_{a,b} = \emptyset$ if $a \geq b$. Let

$$\Omega_b^{+} = \bigcup_{m\in\mathbb{Z}} \big(m\omega + \overline{\Xi_{0,b}^{+}}\big), \qquad \Omega_a^{-} = \bigcup_{m\in\mathbb{Z}} \big(m\omega + \overline{\Xi_{0,a}^{-}}\big), \tag{2.4}$$

where $\Xi_{0,b}^{+}$ and $\Xi_{0,a}^{-}$ are defined in (2.3).

Definition 2.1 (see [6]) Suppose $f \in PH_n(\Omega_R^{+}; \omega)$. If there exists an integer m such that $\limsup_{z\in\Xi_{0,R}^{+}, z\to\infty} |e^{i\frac{2m\pi z}{\omega}} f(z)| = \alpha$, $\alpha \in (0, +\infty)$, then we say that f possesses order m at $+\infty i\omega$, denoted as $\mathrm{Ord}(f, +\infty i\omega) = m$. If $\limsup_{z\in\Xi_{0,R}^{+}, z\to\infty} |e^{i\frac{2m\pi z}{\omega}} f(z)| = +\infty$ for any $m \in \mathbb{Z}$, then we say that f has order $+\infty$ at $+\infty i\omega$, denoted as $\mathrm{Ord}(f, +\infty i\omega) = +\infty$. We assume $\mathrm{Ord}(f, +\infty i\omega) = -\infty$ if and only if $f \equiv 0$. Similarly, the growth order at $-\infty i\omega$ is defined.

Theorem 2.2 (see [6]) (1) *If* $f \in PH_n(\Omega_R^{+}; \omega)$, *then*

$$\mathrm{Ord}(f, +\infty i\omega) = \max\big\{ \mathrm{Ord}(f_0, +\infty i\omega), \mathrm{Ord}(f_k, +\infty i\omega) + 1, \\ k = 1, 2, \ldots, n-1 \big\},$$

where f_k *is the kth-component of* f; (2) *If* $f \in PH_n(\Omega_R^{-}; \omega)$, *then*

$$\mathrm{Ord}(f, -\infty i\omega) = \max\big\{ \mathrm{Ord}(f_0, -\infty i\omega),\ \mathrm{Ord}(f_k, -\infty i\omega) + 1, \\ k = 1, 2, \ldots, n-1 \big\},$$

where f_k *is the kth-component.*

For $m, \ell \in \mathbb{Z}$, we introduce the family of periodic Laurent-type polynomials

$$\Pi_{\ell,m} = \begin{cases} \left\{ q_{\ell,m}(z) = \sum_{k=\ell}^{m} c_k e^{i\frac{2k\pi z}{\omega}} \ :\ c_k \in \mathbb{C},\ k = \ell, \ell+1, \ldots, m \right\}, & m \geq \ell, \\ \{0\}, & m < \ell. \end{cases} \tag{2.5}$$

Obviously $\Pi_{\ell,m} \subset PH_1(\mathbb{C};\omega)$. Let

$$\Pi^n_{\ell,m} = \left\{ q^0_{\ell,m}(z) + \sum_{j=1}^{n-1} \left(\bar{z} - \frac{\bar{\omega}}{\omega} z \right)^j q^j_{\ell+1,m-1}(z) :\ q^j_{\ell,m} \in \Pi_{\ell,m} \right\}. \tag{2.6}$$

In particular, $\Pi^n_{\ell,m} = \Pi_{\ell,m}$ if $n = 1$. Clearly,

$$\lim_{z\in \Xi^+_{0,R},\ z\to\infty} \tan\frac{\pi z}{\omega} = i \quad \text{and} \quad \lim_{z\in \Xi^-_{0,R},\ z\to\infty} \tan\frac{\pi z}{\omega} = -i. \tag{2.7}$$

Moreover, for every $j \in \mathbb{N}$, one has the Fourier expansions

$$\begin{cases} \tan^j \dfrac{\pi z}{\omega} = i^j + \displaystyle\sum_{k=1}^{+\infty} c^+_{j,k} \cdot e^{i\frac{2k\pi z}{\omega}}, & z \in \Omega^+_0, \\ \tan^j \dfrac{\pi z}{\omega} = (-i)^j + \displaystyle\sum_{k=1}^{+\infty} c^-_{j,k} \cdot e^{-i\frac{2k\pi z}{\omega}}, & z \in \Omega^-_0, \end{cases} \tag{2.8}$$

where $c^\pm_{j,k}$ is the Fourier coefficient.

Let $L_0 \subset \Xi_0$ be a piecewise-smooth closed Jordan curve, oriented counterclockwise. The basic open strip Ξ_0 is divided into two domains by the curve L_0, denoted as S^+_0 and S^-_0, respectively. Without loss of generality, we always assume $0 \in S^+_0$. Let $L_m = m\omega + L_0$ for $m \in \mathbb{Z}$, $S^+ = \bigcup_{m\in\mathbb{Z}}(m\omega + S^+_0)$, $S^- = \mathbb{C} \setminus \overline{S^+}$ and we assume that L_m has the same orientation as L_0 for every $m \in \mathbb{Z}$. We set $L = \bigcup_{m\in\mathbb{Z}} L_m$. Our problem is to find a function $\Phi \in PH_1(S^+ \cup S^-;\omega)$ satisfying a Riemann-type boundary condition and two growth conditions

$$\begin{cases} \Phi^+(t) = G(t)\Phi^-(t) + g(t), & t \in L, \\ \mathrm{Ord}(\Phi, +\infty i\omega) \le \ell, & \mathrm{Ord}(\Phi, -\infty i\omega) \le m, \end{cases} \tag{2.9}$$

where the boundary data G, g are Hölder continuous on every curve L_m and $G(t) \neq 0$, $t \in L$. Moreover, $G(t+\omega) = G(t)$ and $g(t+\omega) = g(t)$ for $t \in L$. The problem (2.9) is simply called $PR_{\ell,m}$ problem.

The index of the problem (2.9) is $\kappa = \frac{1}{2\pi}[G(t)]_{L_0}$. Introduce the periodic Cauchy-type integral operator

$$C_\omega[g](z) = \frac{1}{2\omega i} \int_{L_0} g(\tau) \cot\frac{\pi(\tau - z)}{\omega} \mathrm{d}\tau, \quad z \in S^+ \cup S^- \tag{2.10}$$

with $g \in H(L_0)$. As in [9], the canonical function is introduced as follows

$$X(z) = \begin{cases} e^{\Gamma(z)}, & z \in S^+, \\ \cot^\kappa \dfrac{\pi z}{\omega} \cdot e^{\Gamma(z)}, & z \in S^- \end{cases} \tag{2.11}$$

with $\Gamma(z) = \frac{1}{2\omega i}\int_{L_0} \log[\cot^{\kappa}\frac{\pi t}{\omega}\cdot G(t)]\cot\frac{\pi(t-z)}{\omega}\mathrm{d}t$, $z \in S^+ \cup S^-$, where the branch of the logarithm is arbitrarily chosen. In addition, the set Π_k^0 is defined by

$$\Pi_k^0 = \begin{cases} \{p_k(z) = d_k z^k + \cdots + d_1 z : \ d_j \in \mathbb{C}, \ j = 1, 2, \ldots, k\}, & k > 0, \\ \{0\}, & k \le 0. \end{cases} \tag{2.12}$$

At some neighborhood of $z = \frac{\omega}{2}$, the function $e^{i\frac{2k\pi z}{\omega}}$ possesses the Lagrange–Bürmann expansion [8]

$$e^{i\frac{2k\pi z}{\omega}} = (-1)^k + \sum_{j=1}^{\infty} \alpha_{j,k}\cot^j\frac{\pi z}{\omega}, \quad k \in \mathbb{Z}, \tag{2.13}$$

where $\alpha_{j,k}$ is a constant.

Theorem 2.3 (see [6]) *For the $PR_{\ell,m}$ problem (2.9), there are four cases:*

(1) *If $\ell \ge 0$ and $m \ge 0$, the solution of the problem (2.9) can be represented as $\Phi(z) = X(z)\cdot\{C_\omega[\frac{g}{X^+}](z) + q_{-\ell,m}(z) + p_\kappa(\tan\frac{\pi z}{\omega})\}$ with $p_\kappa = \sum_{j=1}^{\kappa} d_j z^j \in \Pi_\kappa^0$ and $q_{-\ell,m} = \sum_{j=-\ell}^{m} c_j e^{i\frac{2j\pi z}{\omega}} \in \Pi_{-\ell,m}$, where all the coefficients satisfy the system of algebraic equations*

$$\begin{cases} \displaystyle\sum_{k=-\ell}^{m}(-1)^k c_k - \frac{1}{2\omega i}\int_{L_0}\frac{g}{X^+}(t)\tan\frac{\pi t}{\omega}\mathrm{d}t = 0, \\ \displaystyle\sum_{k=-\ell}^{m}\alpha_{j,k}c_k - \frac{1}{2\omega i}\int_{L_0}\frac{g}{X^+}(t)\frac{\sin^{j-1}\frac{\pi t}{\omega}}{\cos^{j+1}\frac{\pi t}{\omega}}\mathrm{d}t = 0, \quad j = 1, \ldots, -\kappa - 1, \end{cases} \tag{2.14}$$

where $\alpha_{j,k}$ is determined by (2.13).

(2) *If $\ell \ge 0$ and $m < 0$, the solution of the problem (2.9) can be expressed as $\Phi(z) = X(z)\cdot\{C_\omega[\frac{g}{X^+}](z) + q_{-\ell,0}(z) + p_\kappa(\tan\frac{\pi z}{\omega})\}$ with $q_{-\ell,0}(z) = \sum_{j=-\ell}^{0} c_j e^{i\frac{2j\pi z}{\omega}} \in \Pi_{-\ell,0}$, $p_\kappa(z) = \sum_{j=1}^{\kappa} d_j z^j \in \Pi_\kappa^0$, where all the coefficients satisfy*

$$\begin{cases} \displaystyle c_0 + \sum_{j=1}^{\kappa}(-i)^j\cdot d_j = \frac{1}{2\omega}\int_{L_0}\frac{g}{X^+}(t)\mathrm{d}t, \\ \displaystyle c_k + \sum_{j=1}^{\kappa}c_{j,-k}^{-}\cdot d_j = \frac{1}{\omega}\int_{L_0}\frac{g}{X^+}(t)e^{-i\frac{2k\pi t}{\omega}}\mathrm{d}t, \quad k = -\ell, -\ell+1, \ldots, -1, \\ \displaystyle\sum_{j=1}^{\kappa}c_{j,-k}^{-}\cdot d_j = \frac{1}{\omega}\int_{L_0}\frac{g}{X^+}(t)e^{-i\frac{2k\pi t}{\omega}}\mathrm{d}t, \quad k = m+1, m+2, \ldots, -\ell-1 \end{cases} \tag{2.15}$$

and (2.14) with $m = 0$.

(3) *If $\ell < 0$ and $m \ge 0$, the solution of the problem (2.9) can be expressed as $\Phi(z) = X(z)\cdot\{C_\omega[\frac{g}{X^+}](z) + q_{0,m}(z) + p_\kappa(\tan\frac{\pi z}{\omega})\}$ with $q_{0,m}(z) = \sum_{j=0}^{m} c_j e^{i\frac{2j\pi z}{\omega}} \in$*

$\Pi_{0,m}$, $p_\kappa(z)=\sum_{j=1}^{\kappa} d_j z^j \in \Pi_\kappa^0$, where all the coefficients c_j and d_j satisfy (2.14) with $\ell=0$ and

$$\begin{cases} c_0+\sum_{j=1}^{\kappa} i^j\cdot d_j=-\frac{1}{2\omega}\int_{L_0}\frac{g}{X^+}(t)\mathrm{d}t, \\ c_k+\sum_{j=1}^{\kappa} c_{j,k}^+\cdot d_j=-\frac{1}{\omega}\int_{L_0}\frac{g}{X^+}(t)e^{-i\frac{2k\pi t}{\omega}}\mathrm{d}t, \quad k=1,2,\ldots,m, \\ \sum_{j=1}^{\kappa} c_{j,k}^+\cdot d_j=-\frac{1}{\omega}\int_{L_0}\frac{g}{X^+}(t)e^{-i\frac{2k\pi t}{\omega}}\mathrm{d}t, \qquad k=m+1,m+2,\ldots,-\ell-1. \end{cases} \tag{2.16}$$

(4) *If $\ell<0$ and $m<0$, the solution of the problem (2.9) can be written as $\Phi(z)=X(z)\cdot\{C_\omega[\frac{g}{X^+}](z)+d_0+p_\kappa(\tan\frac{\pi z}{\omega})\}$ with $d_0\in\mathbb{C}$ and $p_\kappa(z)=\sum_{j=1}^{\kappa} d_j z^j\in\Pi_\kappa^0$, where d_j, $j=0,1,\ldots,\kappa$ satisfy*

$$\begin{cases} d_0+\sum_{j=1}^{\kappa}(-i)^j\cdot d_j=\frac{1}{2\omega}\int_{L_0}\frac{g}{X^+}(t)\mathrm{d}t, \\ d_0+\sum_{j=1}^{\kappa} i^j\cdot d_j=-\frac{1}{2\omega}\int_{L_0}\frac{g}{X^+}(t)\mathrm{d}t, \\ \sum_{j=1}^{\kappa} c_{j,k}^+\cdot d_j=-\frac{1}{\omega}\int_{L_0}\frac{g}{X^+}(t)e^{-i\frac{2k\pi t}{\omega}}\mathrm{d}t, \quad k=1,2,\ldots,-\ell-1, \\ \sum_{j=1}^{\kappa} c_{j,-k}^-\cdot d_j=\frac{1}{\omega}\int_{L_0}\frac{g}{X^+}(t)e^{-i\frac{2k\pi t}{\omega}}\mathrm{d}t, \quad k=m+1,m+2,\ldots,-1. \end{cases} \tag{2.17}$$

3 Riemann BVP for Single-Periodic Polyanalytic Functions

Let L be a piecewise-smooth Jordan curves as in the preceding section. We will consider the following BVP: find a function $V\in PH_n(S^+\cup S^-;\omega)$ satisfying n Riemann-type boundary conditions and two growth conditions

$$\begin{cases} \left(\partial_{\bar z}^{j-1}V\right)^+(t)=G(t)\cdot\left(\partial_{\bar z}^{j-1}V\right)^-(t)+g_j(t), \quad t\in L,\ j=1,2,\ldots,n, \\ \mathrm{Ord}(V,+\infty i\omega)\le\ell, \qquad \mathrm{Ord}(V,-\infty i\omega)\le m, \end{cases} \tag{3.1}$$

where G and g_j, $j=1,2,\ldots,n$ are Hölder continuous on every curve L_m and $G(t)\neq 0$, $t\in L$. In addition, $G(t+\omega)=G(t)$, $g_j(t+\omega)=g_j(t)$, $t\in L$. The problem (3.1) is simply called $PPR_{\ell,m}$ problem.

Let the symbols be the same as before. By Theorem 1.2, let

$$V(z)=\sum_{j=0}^{n-1}\left(\bar z-\frac{\bar\omega}{\omega}z\right)^j V_{j+1}(z), \quad z\in S^+\cup S^-, \tag{3.2}$$

where V_{j+1} is a single-periodic analytic function for all j's. By Theorem 2.2, $PPR_{\ell,m}$ problem (3.1) is equivalent to n independent BVPs of single-periodic analytic functions discussed in the preceding section, respectively,

$$\begin{cases} V_1^+(t) = G(t)V_1^-(t) + \widetilde{g}_1(t), & t \in L, \\ \mathrm{Ord}(V_1, +\infty i\omega) \le \ell, & \mathrm{Ord}(V_1, -\infty i\omega) \le m, \end{cases} \tag{3.3}$$

$$\begin{cases} V_j^+(t) = G(t)V_j^-(t) + \widetilde{g}_j(t), & t \in L, \\ \mathrm{Ord}(V_j, +\infty i\omega) \le \ell - 1, & \mathrm{Ord}(V_j, -\infty i\omega) \le m - 1, \end{cases} \quad j = 2, 3, \ldots, n-1, \tag{3.4}$$

where $\widetilde{g}_j(t) = \sum_{k=j}^{n} \frac{(-1)^{k+j}}{(k-j)!(j-1)!}(\bar{t} - \frac{\bar{\omega}}{\omega}t)^{k-j} g_k(t)$ for $j = 1, 2, \ldots, n-1$.

In [6], when $\ell > 0$, the problem (3.1) has been discussed for $m > 0, = 0, < 0$. Here we will discuss the problem (3.1) in the other cases. As in [6], the single-periodic poly-Cauchy integral is defined by

$$W[f_0, \ldots, f_{n-1}](z) = \sum_{j=0}^{n-1} \frac{(-1)^j}{j!} \frac{1}{2\omega i} \cdot \int_{L_0} \left[\left(\overline{\tau} - \frac{\overline{\omega}}{\omega}\tau \right) - \left(\overline{z} - \frac{\overline{\omega}}{\omega} z \right) \right]^j \times \cot \frac{\pi(\tau - z)}{\omega} f_j(\tau) \mathrm{d}\tau, \tag{3.5}$$

where f_j is Hölder-continuous on L for all j's.

Theorem 3.1 *If $\ell = 0$ and $m > 0$, $PPR_{0,m}$ problem (3.1) is solvable and its solution can be expressed as*

$$V(z) = X(z) \left\{ W\left[\frac{g_1}{X^+}, \ldots, \frac{g_n}{X^+} \right](z) + \sum_{j=0}^{n-1} \left(\bar{z} - \frac{\bar{\omega}}{\omega} z \right)^j p_\kappa^j \left(\tan \frac{\pi z}{\omega} \right) + Q_{0,m}^n(z) \right\} \tag{3.6}$$

with $p_\kappa^j \in \Pi_\kappa^0$ and $Q_{0,m}^n(z) \in \Pi_{0,m}^n$, where all the coefficients and the free terms $g_j (1 \le j \le n)$ satisfy

$$\begin{cases} \displaystyle\sum_{k=0}^{m} (-1)^k c_k^0 - \frac{1}{2\omega i} \sum_{k=1}^{n} \frac{(-1)^{k+1}}{(k-1)!} \int_{L_0} \left(\bar{t} - \frac{\bar{\omega}}{\omega} t \right)^{k-1} \frac{g_k}{X^+}(t) \tan \frac{\pi t}{\omega} \mathrm{d}t = 0, \\ \displaystyle\sum_{k=0}^{m} \alpha_{s,k} c_k^0 - \frac{1}{2\omega i} \sum_{k=1}^{n} \frac{(-1)^{k+1}}{(k-1)!} \int_{L_0} \left(\bar{t} - \frac{\bar{\omega}}{\omega} t \right)^{k-1} \frac{g_k}{X^+}(t) \frac{\sin^{s-1} \frac{\pi t}{\omega}}{\cos^{s+1} \frac{\pi t}{\omega}} \mathrm{d}t = 0, \\ \quad s = 1, \ldots, -\kappa - 1 \end{cases} \tag{3.7}$$

and

$$\begin{cases} \sum_{k=0}^{m-1}(-1)^k c_k^{j-1} - \frac{1}{2\omega i}\sum_{k=j}^{n}\frac{(-1)^{k+j}}{(k-j)!(j-1)!} \\ \quad \times \int_{L_0}\left(\bar{t}-\frac{\bar{\omega}}{\omega}t\right)^{k-j}\frac{g_k}{X^+}(t)\tan\frac{\pi t}{\omega}\mathrm{d}t = 0, \\ \sum_{k=0}^{m-1}\alpha_{s,k}c_k^{j-1} - \frac{1}{2\omega i}\sum_{k=j}^{n}\frac{(-1)^{k+j}}{(k-j)!(j-1)!} \\ \quad \times \int_{L_0}\left(\bar{t}-\frac{\bar{\omega}}{\omega}t\right)^{k-j}\frac{g_k}{X^+}(t)\frac{\sin^{s-1}\frac{\pi t}{\omega}}{\cos^{s+1}\frac{\pi t}{\omega}}\mathrm{d}t = 0, \\ \quad s = 1,\ldots,-\kappa-1, \end{cases} \tag{3.8}$$

$$\begin{cases} c_0 + \sum_{k=1}^{\kappa} i^k \cdot d_k = -\frac{1}{2\omega}\sum_{k=j}^{n}\frac{(-1)^{k+j}}{(k-j)!(j-1)!}\int_{L_0}\left(\bar{t}-\frac{\bar{\omega}}{\omega}t\right)^{k-j}\frac{g_k}{X^+}(t)\mathrm{d}t, \\ c_s + \sum_{k=1}^{\kappa} c_{k,s}^+ \cdot d_k = -\frac{1}{\omega}\sum_{k=j}^{n}\frac{(-1)^{k+j}}{(k-j)!(j-1)!}\int_{L_0}\left(\bar{t}-\frac{\bar{\omega}}{\omega}t\right)^{k-j}\frac{g_k}{X^+}(t)e^{-i\frac{2s\pi t}{\omega}}\mathrm{d}t, \\ \quad s = 1,2,\ldots,m, \\ \sum_{k=1}^{\kappa} c_{k,s}^+ \cdot d_k = -\frac{1}{\omega}\sum_{k=j}^{n}\frac{(-1)^{k+j}}{(k-j)!(j-1)!}\int_{L_0}\left(\bar{t}-\frac{\bar{\omega}}{\omega}t\right)^{k-j}\frac{g_k}{X^+}(t)e^{-i\frac{2s\pi t}{\omega}}\mathrm{d}t, \\ \quad s = m+1,m+2,\ldots,-\ell-1, \end{cases} \tag{3.9}$$

for $j = 2,\ldots,n-1$.

Proof By case (1) in Theorem 2.2, the solution of $PR_{0,m}$ problem (3.3) can be represented as $V_1(z) = X(z)\{C_\omega[\frac{\widetilde{g}_1}{X^+}](z) + q_{0,m}^0(z) + p_\kappa^0(\tan\frac{\pi z}{\omega})\}$ with $q_{0,m}^0 = \sum_{k=0}^{m} c_k^0 e^{i\frac{2k\pi z}{\omega}} \in \Pi_{0,m}$, $p_\kappa^0(z) = \sum_{k=1}^{\kappa} d_k^0 z^k \in \Pi_\kappa^0$, where all the coefficients satisfy

$$\begin{cases} \sum_{k=0}^{m}(-1)^k c_k^0 - \frac{1}{2\omega i}\int_{L_0}\frac{\widetilde{g}_1}{X^+}(t)\tan\frac{\pi t}{\omega}\mathrm{d}t = 0, \\ \sum_{k=0}^{m}\alpha_{s,k}c_k^0 - \frac{1}{2\omega i}\int_{L_0}\frac{\widetilde{g}_1}{X^+}(t)\frac{\sin^{s-1}\frac{\pi t}{\omega}}{\cos^{s+1}\frac{\pi t}{\omega}}\mathrm{d}t = 0, \quad s = 1,\ldots,-\kappa-1. \end{cases} \tag{3.10}$$

Similarly, for $j = 2,3,\ldots,n-1$, from case (3) of Theorem 2.3, the solution of problem (3.4) is $V_j(z) = X(z)\{C_\omega[\frac{\widetilde{g}_j}{X^+}](z) + q_{0,m-1}^{j-1}(z) + p_\kappa^{j-1}(\tan\frac{\pi z}{\omega})\}$ with $q_{0,m-1}^{j-1} = \sum_{k=0}^{m-1} c_k^{j-1} e^{i\frac{2k\pi z}{\omega}} \in \Pi_{0,m-1}$, $p_\kappa^{j-1}(z) = \sum_{k=1}^{\kappa} d_k^{j-1} z^k \in \Pi_\kappa^0$, where all the coefficients satisfy

$$\begin{cases} \sum_{k=0}^{m-1}(-1)^k c_k^{j-1} - \frac{1}{2\omega i}\int_{L_0}\frac{\widetilde{g}_j}{X^+}(t)\tan\frac{\pi t}{\omega}\mathrm{d}t = 0, \\ \sum_{k=0}^{m-1}\alpha_{s,k}c_k^{j-1} - \frac{1}{2\omega i}\int_{L_0}\frac{\widetilde{g}_j}{X^+}(t)\frac{\sin^{s-1}\frac{\pi t}{\omega}}{\cos^{s+1}\frac{\pi t}{\omega}}\mathrm{d}t = 0, \quad s = 1, \ldots, -\kappa - 1 \end{cases} \tag{3.11}$$

and

$$\begin{cases} c_0 + \sum_{j=1}^{\kappa} i^j \cdot d_j = -\frac{1}{2\omega}\int_{L_0}\frac{\widetilde{g}_j}{X^+}(t)\mathrm{d}t, \\ c_k + \sum_{j=1}^{\kappa} c_{j,k}^+ \cdot d_j = -\frac{1}{\omega}\int_{L_0}\frac{\widetilde{g}_j}{X^+}(t)e^{-i\frac{2k\pi t}{\omega}}\mathrm{d}t, \quad k = 1, 2, \ldots, m, \\ \sum_{j=1}^{\kappa} c_{j,k}^+ \cdot d_j = -\frac{1}{\omega}\int_{L_0}\frac{\widetilde{g}_j}{X^+}(t)e^{-i\frac{2k\pi t}{\omega}}\mathrm{d}t, \quad k = m+1, m+2, \ldots, -\ell - 1. \end{cases} \tag{3.12}$$

Therefore, the solution of $PPR_{0,m}$ problem (3.1) can be written as

$$V(z) = X(z)\left\{\left[C_\omega\left[\frac{\widetilde{g}_1}{X^+}\right](z) + q_{0,m}^0(z) + p_\kappa^0\left(\tan\frac{\pi z}{\omega}\right)\right]\right.$$
$$\left. + \sum_{j=1}^{n-1}\left(\bar{z} - \frac{\bar{\omega}}{\omega}z\right)^j\left[C_\omega\left[\frac{\widetilde{g}_{j+1}}{X^+}\right](z) + q_{1,m-1}^j(z) + p_\kappa^j\left(\tan\frac{\pi z}{\omega}\right)\right]\right\},$$

where all the coefficients in $q_{0,m}^j$, $q_{1,m-1}^j$, p_κ^j satisfy (3.10), (3.11) and (3.12). Further, by simple calculation, the desired conclusion is true. □

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Riemann–Hilbert Problem for Multiply Connected Domains

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Abstract We discuss scalar Riemann–Hilbert problems for circular multiply connected domains considered by Mityushev (Functional Equations in Mathematical Analysis, pp. 599–632, 2012). The main attention is paid to the $\mathbb{R}$-linear and the Schwarz problems. Some details concerning applications of the metod of functional equation, outlined in Functional Equations in Mathematical Analysis, pp. 599–632, 2012 are extended in the present paper.

Keywords Riemann–Hilbert problem · Functional equations

Mathematics Subject Classification (2010) 30E25

1 Riemann–Hilbert Problem

Let D be a multiply connected domain on the complex plane whose boundary ∂D consists of n simple Jordan curves with positive orientation (∂D leaves D to the left). The linear Riemann–Hilbert problem for D is stated as follows. For given Hölder continuous functions $\lambda(t) \neq 0$ and $f(t)$ on ∂D, find a function $\phi(z)$ analytic in D, continuous in the closure of D with the boundary condition

$$\operatorname{Re}\big(\overline{\lambda(t)}\phi(t)\big) = f(t), \quad t \in \partial D. \tag{1.1}$$

The above condition can be also written in the following form

$$\phi(t) + G(t)\overline{\phi(t)} = g(t), \quad t \in \partial D, \tag{1.2}$$

where $|G(t)| = 1$. By the assumption that $\lambda(t) \neq 0$ we can rewrite (1.1) as follows

$$\phi(t) + \frac{\lambda(t)}{\overline{\lambda(t)}}\overline{\phi(t)} = 2\frac{f(t)}{\overline{\lambda(t)}}.$$

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Any multiply connected domain D can be conformally mapped onto a circular multiply connected domain. Hence, it is sufficient to solve the problem (1.1) for a circular domain and using conformal mapping again to transform the solvability conditions and the solution.

The problem (1.1) had been completely solved for multiply connected domains by Mityushev [1]. In the present paper, we follow [1] and extend explanations in details concerning the spectral theory.

1.1 $\mathbb{R}$-Linear Problem

Consider mutually disjoint disks

$$\mathbb{D}_k := \left\{z \in \mathbb{C} : |z - a_k| < r_k\right\}, \quad k = 1, 2, \ldots, n$$

in the complex plane $\mathbb{C}$. Let $\mathbb{D}$ be the complement of the closed disks $|z - a_k| \leq r_k$ to the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ i.e.

$$\mathbb{D} := \hat{\mathbb{C}} \setminus \bigcup_{k=1}^{n} (\mathbb{D}_k \cup \partial \mathbb{D}_k).$$

The circles $\mathbb{T}_k := \{t \in \mathbb{C} : |t - a_k| = r_k\}$ leaves $\mathbb{D}$ to the left.

Let D be a multiply connected domain described above. Let D_k $(k = 1, 2, \ldots, n)$ be simply connected domains complementing D to the extended complex plane. The $\mathbb{R}$-linear conjugation problem or simply $\mathbb{R}$-linear problem is stated as follows. For given Hölder continuous functions $a(t) \neq 0$, $b(t)$ and $c(t)$ on ∂D, find a function $\phi(z)$ analytic in $\bigcup_{k=1}^{n} D_k \cup D$, continuous in $D_k \cup \partial D_k$ and in $D \cup \partial D$ with the conjugation condition

$$\phi^+(t) = a(t)\phi^-(t) + b(t)\overline{\phi^-(t)} + c(t), \quad t \in \partial D. \tag{1.3}$$

Here, $\phi^+(t)$ is the limit value of $\phi(z)$ when $z \in D$ tends to $t \in \partial D$, $\phi^-(t)$ is the limit value of $\phi(z)$ when $z \in D_k$ tends to $t \in \partial D$.

In the case $|a(t)| \equiv |b(t)|$, the $\mathbb{R}$-linear problem is reduced to the Riemann–Hilbert problem (1.1). Since $|a(t)| \equiv |b(t)|$ we have $b(t) = a(t)e^{i\theta(t)}$, where $\theta(t)$ is a real function on ∂D

$$\phi^+(t) = a(t)\phi^-(t) + a(t)e^{i\theta(t)}\overline{\phi^-(t)} + c(t), \quad t \in \partial D.$$

This can be rewritten in the form

$$\mathrm{Re}\big(a(t)e^{-i\theta(t)/2}\phi^+(t)\big) = \mathrm{Im}\left(\frac{c(t)}{a(t)e^{i\theta(t)/2}}\right), \quad t \in \partial D,$$

i.e., as the Riemann–Hilbert problem (1.1) with $\overline{\lambda(t)} = a(t)e^{-i\theta(t)/2}$ and $f(t) = \mathrm{Im}(\frac{c(t)}{a(t)e^{i\theta(t)/2}})$.

Let κ_k be the index of $\lambda(t)$ along the curve $\mathbb{T}_k$ given by the formula

$$\kappa_k = \operatorname{wind}_{\mathbb{T}_k} \lambda(t) := \frac{1}{2\pi i} \int_{\mathbb{T}_k} d \ln \lambda(t).$$

The value $\kappa = \sum_{k=1}^{n} \kappa_k$ is called the index of the problem.

Theorem 1.1 *Let the coefficients of the problem (1.3) satisfy the inequality*

$$\left|b(t)\right| < \left|a(t)\right|. \tag{1.4}$$

If $\kappa = \operatorname{wind}_{\partial D} a(t) \geq 0$, the problem (1.3) is solvable and the homogeneous problem (1.3) ($f(t) = 0$) has 2κ $\mathbb{R}$-linearly independent solutions vanishing at infinity. If $\kappa = \operatorname{wind}_{\partial D} a(t) < 0$, the problem (1.3) has a unique solution if and only if $|2\kappa|$ $\mathbb{R}$-linearly independent conditions for $f(t)$ are fulfilled.

1.2 Riemann–Hilbert Problem with Constant Coefficients

The Riemann–Hilbert problem (1.1) is a partial case of the $\mathbb{R}$-linear problem.

Theorem 1.2 ([1]) *The problem*

$$\operatorname{Re} \overline{v_k} \phi(t) = c(t), \quad t \in \partial D, \tag{1.5}$$

is equivalent to the problem

$$\overline{v_k} \phi^+(t) = \phi^-(t) - \overline{\phi^-(t)} + c(t), \quad t \in \partial D, \tag{1.6}$$

i.e. the problem (1.5) is solvable if and only if (1.6) is solvable. If (1.5) has a solution $\phi(z)$, this function is analytic in D and can be constructed in D_k in such a way that (1.6) is fulfilled. It can be found from the following simple problem with respect to the function $2 \operatorname{Im} \phi^-(z)$ harmonic in D_k

$$2 \operatorname{Im} \phi^-(t) = \operatorname{Im} \overline{v_k} \phi^+(t) - c(t), \quad t \in \partial D. \tag{1.7}$$

The problem (1.7) has a unique solution up to an arbitrary additive real constant.

2 Functional Equations

2.1 Reflection with Respect to the Circle $\mathbb{T}_k$

Consider mutually disjoint disks $\mathbb{D}_k$ in the complex plane $\mathbb{C}$. Let

$$z_{(k)}^* = \frac{r_k^2}{\overline{z - a_k}} + a_k$$

be the reflection with respect to the circle $\mathbb{T}_k$. Note that if $\Phi(z)$ is analytic in the disk $|z-a_k|<r_k$ and continuous in its closure then $\overline{\Phi(z^*_{(k)})}$ is analytic in the disk $|z-a_k|>r_k$ and continuous in $|z-a_k|\geq r_k$. Introduce the compositions of successive reflections with respect to the circles $\mathbb{T}_{k_1}, \mathbb{T}_{k_2}, \dots, \mathbb{T}_{k_p}$

$$z^*_{(k_p k_{p-1}\cdots k_1)} := \left(z^*_{(k_{p-1}\cdots k_1)}\right)^*_{k_p}. \tag{2.1}$$

In the sequence $k_1, k_2, \dots, k_p$, no two neighbouring numbers are equal. The umber p is called the level of the mapping. When p is even, these are Möbius transformations. When p is odd, these are anti-Möbius transformations, i.e., Möbius transformations in $\overline{z}$. Thus this mapping can be written in the form

$$\gamma(z)=\frac{e_j z+b_j}{c_j z+d_j},\ p\in 2\mathbb{Z}, \qquad \gamma(\overline{z})=\frac{e_j \overline{z}+b_j}{c_j \overline{z}+d_j}, \quad p\in 2\mathbb{Z}+1. \tag{2.2}$$

Where $\gamma_0 := z$ (identical mapping with the level $p=0$), $\gamma_1(\overline{z}) := z^*_{(n)}$ (n simple reflections with level $p=1$), $\gamma_{n+1}(z) := z^*_{(12)}, \dots, \gamma_{n^2}(z) := z^*_{(n-1,n)}$ (n^2-n pairs on reflections with level $p=2$) and so on. The sets of the subscripts j of γ_j is ordered in such a way that the level p is increasing. The functions (2.2) generate a Schottky group $\mathcal{K}$. Thus each element of $\mathcal{K}$ is represented in the form of the composition of reflection (2.1) or in the form of linearly ordered functions (2.2).

2.2 Homogeneous Equation

Let G be a domain on the extended complex plane whose boundary ∂G consist of simply closed Jordan curves. Introduce the Banach space $\mathcal{C}(\partial G)$ of functions continuous on the curves of ∂G with the norm $\|f\| = \max_{1\leq k\leq n}\max_{\partial G}|f(t)|$. Consider the closed subspace $\mathcal{C}_{\mathcal{A}}(G)$ consisting of all function analytically continued into G. Introduce for brevity the designation $\mathcal{C}_{\mathcal{A}} = \mathcal{C}_{\mathcal{A}}(\sum_{k=1}^n \mathbb{D}_k)$. Let w be a fixed point from $\mathbb{D}\backslash\{\infty\}$.

Lemma 2.1 ([1]) *Let $v_k = exp(-i\mu_k)$ with $\mu_k\in\mathbb{R}$. Consider the system of functional equations with respect to the function $\phi_k(z)$ analytic in $\mathbb{D}_k$*

$$\begin{aligned}\phi_k(z) &= -v_k\sum_{m\neq k}\left[\overline{\phi_m\left(z^*_{(m)}\right)}-\overline{\phi_m\left(w^*_{(m)}\right)}\right],\\ |z-a_k| &\leq r_k, \quad k=1,2,\dots,n.\end{aligned} \tag{2.3}$$

This system has only the trivial solution.

Proof We now give the detailed proof of this lemma following [1]. Let $\phi_m(z)$ ($m=1,2,\dots,n$) be a solution of (2.3). Then the right-hand part of (2.3) implies that the function $\phi_k(z)$ is analytic in $|z - a_k| \le r_k$ $(k = 1, 2, \dots, n)$. Introduce the function

$$\psi(z) := -\sum_{m=1}^{n} \overline{v_m}\left[\overline{\phi_m\left(z^*_{(m)}\right)} - \overline{\phi_m\left(w^*_{(m)}\right)}\right],$$

analytic in the closure of $\mathbb{D}$. Then the functions ψ, ϕ_k satisfy the $\mathbb{R}$-linear boundary conditions

$$v_k\psi(t) = \phi_k(t) - \overline{\phi_k(t)} + \overline{\phi_k\left(w^*_{(k)}\right)}, \qquad |t - a_k| = r_k, \quad k = 1, \dots, n, \tag{2.4}$$

because

$$\begin{aligned} v_k\psi(t) &= -v_k \sum_{m=1}^{n} \overline{v_m}\left[\overline{\phi_m\left(t^*_{(m)}\right)} - \overline{\phi_m\left(w^*_{(m)}\right)}\right] \\ &= -v_k \sum_{m=1, m\neq k}^{n} \overline{v_m}\left[\overline{\phi_m\left(t^*_{(m)}\right)} - \overline{\phi_m\left(w^*_{(m)}\right)}\right] - v_k\overline{v_k}\overline{\phi_k(t)} + v_k\overline{v_k}\overline{\phi_k\left(w^*_{(k)}\right)} \\ &= \phi_k(t) - \overline{\phi_k(t)} + \overline{\phi_k\left(w^*_{(k)}\right)}, \end{aligned}$$

where

$$t^*_k = \frac{r_k^2}{\overline{t - a_k}} + a_k = \frac{r_k^2}{\overline{r_k e^{i\theta} + a_k - a_k}} + a_k = r_k e^{i\theta} + a_k = t.$$

Note that if $\overline{\phi_k(w^*_{(k)})} = c_k + i d_k$ then

$$\begin{aligned} v_k\psi(t) &= \operatorname{Re}\phi_k(t) + \operatorname{Im}\phi_k(t) - \operatorname{Re}\phi_k(t) + \operatorname{Im}\phi_k(t) + \overline{\phi_k\left(w^*_{(k)}\right)} \\ &= 2\operatorname{Im}\phi_k(t) + \overline{\phi_k\left(w^*_{(k)}\right)} = 2\operatorname{Im}\phi_k(t) + c_k + i d_k. \end{aligned}$$

Hence, we can also rewrite the relation in the following form

$$\operatorname{Re} v_k\psi(t) = c_k, \qquad |t - a_k| = r_k, \quad k = 1, \dots, n, \tag{2.5}$$

$$2\operatorname{Im}\phi_k(t) = \operatorname{Im} v_k\psi(t) + d_k, \qquad |t - a_k| = r_k, \quad k = 1, \dots, n. \tag{2.6}$$

One may consider equalities (2.5) as a boundary value problem with respect to the function $\psi(z)$ analytic in $\mathbb{D}$ and continuous in its closure, i.e., $\psi \in \mathcal{C}_{\mathcal{A}}(\mathbb{D})$. The real constants c_k have to be determined. The problem (2.5) has only constant solutions [1]. □

2.3 Non-homogeneous Equation

Below the following lemma from [1] is proved in detail concerning the spectral theory.

Lemma 2.2 *Let $h \in \mathcal{C}_{\mathcal{A}}$ and $v_k = exp(-i\mu_k)$ with $\mu_k \in \mathbb{R}$. Consider the system of functional equations*

$$\phi_k(z) = -v_k \sum_{m \neq k} \left[\overline{\phi_m\left(z^*_{(m)}\right)} - \overline{\phi_m\left(w^*_{(m)}\right)}\right] + h_k(z),$$
$$|z - a_k| \leq r_k, \quad k = 1, 2, \ldots, n. \tag{2.7}$$

It has a unique solution $\Phi \in \mathcal{C}_{\mathcal{A}}$, where

$$\Phi(z) := \phi_k(z), \qquad |z - a_k| \leq r_k, \quad k = 1, \ldots, n.$$

This solution can be found by the method of successive approximations. The approximations are converging in $\mathcal{C}_{\mathcal{A}}$.

Proof We can rewrite the system (2.7) on $\mathbb{T}_k$ in the form of the system of integral equations valid on $|t - a_k| = r_k$ ($k = 1, 2, \ldots, n$)

$$\phi_k(t) = -v_k \sum_{m \neq k} \overline{v_m \frac{1}{2\pi i} \int_{\mathbb{T}_m^-} \phi_m(\tau) \left(\frac{1}{\tau - t^*_{(m)}} - \frac{1}{\tau - w^*_{(m)}} \right) d\tau} + h_k(t), \tag{2.8}$$

The orientation on $\mathbb{T}_m^-$ leaves $\mathbb{D}_m$ to the left. The system (2.8) can be written as an equation in the space $\mathcal{C}(\bigcup_{k=1}^n \mathbb{T}_k)$

$$\Phi = A\Phi + h \tag{2.9}$$

The integral operators from (2.8) are compact in $\mathcal{C}(\mathbb{T}_k)$ [2], multiplication by $\overline{v_m}$ and complex conjugation are bounded operators in $\mathcal{C}$. Then A is a compact operator in $\mathcal{C}$. Since Φ is a solution of (2.9) in $\mathcal{C}$, we have $\Phi \in \mathcal{C}_{\mathcal{A}}$. This follows from the properties of Cauchy integrals and the condition $h \in \mathcal{C}_A$. Therefore, (2.9) in $\mathcal{C}_A$ and (2.7) in $\mathcal{C}_{\mathcal{A}}$ are equivalent when $h \in \mathcal{C}_{\mathcal{A}}$. By Lemma 2.1 the homogeneous equation $\Phi = A\Phi$ has only the trivial solution. Then the Fredholm theorems imply that (2.9) or the system (2.7) has a unique solution.

Let us show the convergence of the method of successive approximations. By virtue of The Successive Approximation Theorem [2] it is sufficient to prove the inequality $\rho(A) < 1$, where $\rho(A)$ is the spectral radius of the operator A, i.e. $\rho(A) = \sup_{x \in \sigma(A)} |x|$, where $\sigma(A)$ is the spectrum of A. Compactness of the operator A in the Banach space implies that for every g there exist a nontrivial solution equation $\Phi - A\Phi = g$. Hence, every non zero element of the spectrum of A is an eigenvalue of A. Moreover, the eigenvalues can only accumulate at 0. Hence, there are finitely many eigenvalues of A and $\rho(A) = \sup_{x \in \sigma(A)} |x| = \max_{x \in \sigma(A)} |x|$. If we have $\Phi = \lambda A\Phi$ for non-zero λ then $\frac{1}{\lambda}$ is an eigenvalues of A.

The inequality $\rho(A) < 1$ means that each eigenvalue is less than 1. Hence $\rho(A) < 1$ is satisfied if for all complex numbers λ such that $|\lambda| \leq 1$ equation

$$\Phi = \lambda A\Phi$$

has only the trivial solution, because then $\frac{1}{|\lambda|} > 1$, hence λ is not an eigenvalues of A.

Since $\Phi(z) := \phi_k(z)$, $|z - a_k| \leq r_k$, $k = 1, \ldots, n$, equation $\Phi = \lambda A \Phi$ can be rewritten in the form

$$\phi_k(z) = -\lambda v_k \sum_{m \neq k} \left[\overline{\phi_m\left(z^*_{(m)}\right)} - \overline{\phi_m\left(w^*_{(m)}\right)}\right], \qquad |z - a_k| \leq r_k, \quad k = 1, 2, \ldots, n. \tag{2.10}$$

Consider the case $|\lambda| < 1$. Introduce the function analytic in the closure of $\mathbb{D}$

$$\psi(z) = -\lambda \sum_{m=1}^{n} \overline{v_m}\left[\overline{\phi\left(z^*_{(m)}\right)} - \overline{\phi\left(w^*_{(m)}\right)}\right].$$

Then

$$\begin{aligned} v_k \psi(t) &= -\lambda v_k \sum_{m=1}^{n} \overline{v_m}\left[\overline{\phi\left(t^*_{(m)}\right)} - \overline{\phi\left(w^*_{(m)}\right)}\right] \\ &= \phi_k(t) - \lambda \overline{\phi_k\left(t^*_{(k)}\right)} + \lambda \overline{\phi_k\left(w^*_{(k)}\right)} = \phi_k(t) - \lambda \overline{\phi_k(t)} + \lambda \overline{\phi_k\left(w^*_{(k)}\right)}, \end{aligned}$$

where $|t - a_k| = r_k$, $k = 1, 2, \ldots, n$. Hence $\psi(z)$ and $\phi_k(z)$ satisfy the $\mathbb{R}$-linear problem

$$v_k \psi(t) = \phi_k(t) - \lambda \overline{\phi_k(t)} + \gamma_k, \qquad |t - a_k| = r_k, \quad k = 1, 2, \ldots, n$$

where $\gamma_k = \lambda \overline{\phi_k(w^*_{(k)})}$.

Let $\psi_0(z) = \psi(z) - \psi(\infty)$, then

$$v_k \psi(t) = v_k\left(\psi_0(t) + \psi(\infty)\right) = \phi_k(t) - \lambda \overline{\phi_k(t)} + \gamma_k$$

and

$$v_k \psi_0(t) = \phi_k(t) - \lambda \overline{\phi_k(t)} + \gamma_k - v_k \psi(\infty). \tag{2.11}$$

Theorem 1.1 implies that the problem (2.11) has the unique solution $\psi_0(z) \equiv 0$ and hence

$$v_k \psi_0(t) = \phi_k(t) - \lambda \overline{\phi_k(t)} + \gamma_k - v_k \psi(\infty).$$

This gives

$$\phi_k(z) = \frac{\gamma_k - v_k \psi(\infty) + \lambda \overline{(\gamma_k - v_k \psi(\infty))}}{|\lambda|^2 - 1},$$

hence $\phi_k(z) \equiv \textit{constant}$. Then (2.10) yields $\phi_k(z) \equiv 0$.

Consider the case $|\lambda| = 1$. Then, substituting $\omega_k(z) = \phi_k(z)/\sqrt{\lambda}$ we reduce the system (2.10) to the same system with $\lambda = 1$. It follows from Lemma 2.1 that $\omega_k(z) = \phi_k(z) = 0$. Hence, $\rho(A) < 1$.

This inequality proves the lemma. □

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The Schottky–Klein Prime Function

Inez Badecka

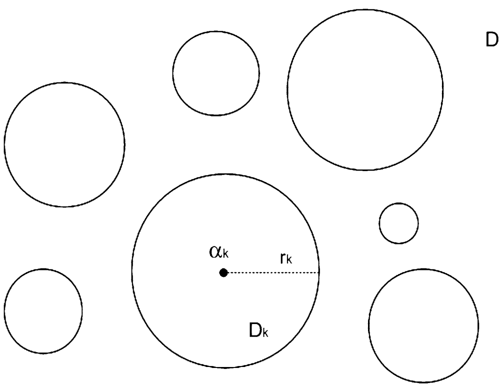
Abstract This paper describes the Schottky–Klein prime function. The classical Schottky groups and the Poincaré α-series are used to construct the Schottky–Klein prime function for arbitrary multiply connected circular domains.

Keywords Schottky groups · Schottky–Klein prime function · Riemann–Hilbert problem

Mathematics Subject Classification (2010) 30E25

1 Schottky Groups

Consider mutually disjoint disks $D_k = \{z \in \mathbb{C} : |z - \alpha_k| < r_k\}$ in the complex plane $\mathbb{C}$ and the multiply connected domain $D = \widehat{\mathbb{C}} \setminus \bigcup_{k=1}^{n}(D_k \cup \partial D_k)$ to the extended complex plane $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$.



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Consider the reflection with respect to the circle $|z-\alpha_k| = r_k$

$$z^*_{(k)} = \frac{r_k^2}{\overline{z-\alpha_k}} + \alpha_k.$$

Introduce the composition of successive reflections with respect to the circles:

$$z^*_{(k_p k_{p-1}\cdots k_1)} := \left(z^*_{(k_{p-1}\cdots k_1)}\right)^*_{k_p}. \tag{1.1}$$

These mappings can be written in the form:

$$\begin{aligned} \gamma_j(z) &= \frac{e_j z + b_j}{c_j z + d_j}, \quad p \in 2\mathbb{Z}, \\ \gamma_j(\overline{z}) &= \frac{e_j \overline{z} + b_j}{c_j \overline{z} + dj}, \quad p \in 2\mathbb{Z}+1, \end{aligned} \tag{1.2}$$

where $e_j d_j - b_j c_j = 1$, $j = 0, 1, 2, \ldots$ and p is called the level of the mapping.

Here, we introduce

$$\gamma_0(z) := z,$$

$$\text{(identical mapping, } p = 0\text{)}$$

$$\gamma_1(\overline{z}) := z^*_{(1)}, \quad \gamma_2(\overline{z}) := z^*_{(2)}, \quad \ldots, \quad \gamma_n(\overline{z}) := z^*_{(n)},$$

$$(n \text{ simple reflections, } p = 1)$$

$$\gamma_{n+1}(z) := z^*_{(1,2)}, \quad \gamma_{n+2}(z) := z^*_{(1,3)}, \quad \ldots, \quad \gamma_{n^2}(z) := z^*_{(n,n-1)},$$

$$\left(n^2 - n \text{ double reflections, } p = 2\right)$$

$$\gamma_{n^2+1}(\overline{z}) := z^*_{(1,2,1)}, \quad \gamma_{n^2+2}(\overline{z}) := z^*_{(1,2,2)}, \quad \ldots$$

and so on. The functions (1.2) generate a Schottky group $\mathcal{K}$. All elements γ_j of even level generate a subgroup $\mathcal{E}$. The set of the elements γ_j of odd $\mathcal{K} \setminus \mathcal{E}$ level is denoted by $\mathcal{O}$.

Let $H(z)$ be a rational function.

The following series

$$\theta_2(z) := \sum_{\gamma_j \in \mathcal{E}} \frac{H[\gamma_j(z)]}{(e_j z + d_j)^2} \tag{1.3}$$

is called the Poincaré θ_2-series associated with the subgroup $\mathcal{E}$.

Now, let us introduce the series:

$$\theta_2^{(1)}(z) = H(z) - \sum_{k=1}^{n} \overline{H\left[z^*_{(k)}\right]}\left(\overline{z^*_{(k)}}\right)' + \sum_{k=1}^{n} \sum_{k_1 \neq k} H\left[z^*_{(k_1,k)}\right]\left(z^*_{(k_1,k)}\right)'$$

$$
\begin{aligned}
&-\sum_{k=1}^{n}\sum_{k_1\neq k}\sum_{k_2\neq k_1}\overline{H\big[z^*_{(k_2,k_1,k)}\big]}\big(\overline{z^*_{(k_2,k_1,k)}}\big)'\\
&+\sum_{k=1}^{n}\sum_{k_1\neq k}\sum_{k_2\neq k_1}\sum_{k_3\neq k_2}H\big[z^*_{(k_3,k_2,k_1,k)}\big]\big(z^*_{(k_3,k_2,k_1,k)}\big)'-\cdots \qquad (1.4)
\end{aligned}
$$

$$
\begin{aligned}
\theta_2^{(2)}(z) = H(z) &+ \sum_{k=1}^{n}\overline{H\big[z^*_{(k)}\big]}\big(\overline{z^*_{(k)}}\big)' + \sum_{k=1}^{n}\sum_{k_1\neq k}H\big[z^*_{(k_1,k)}\big]\big(z^*_{(k_1,k)}\big)'\\
&+\sum_{k=1}^{n}\sum_{k_1\neq k}\sum_{k_2\neq k_1}\overline{H\big[z^*_{(k_2,k_1,k)}\big]}\big(\overline{z^*_{(k_2,k_1,k)}}\big)'\\
&+\sum_{k=1}^{n}\sum_{k_1\neq k}\sum_{k_2\neq k_1}\sum_{k_3\neq k_2}H\big[z^*_{(k_3,k_2,k_1,k)}\big]\big(z^*_{(k_3,k_2,k_1,k)}\big)'+\cdots \qquad (1.5)
\end{aligned}
$$

The Poincaré θ_2-series can be written in the form:

$$
\theta_2(z) = \frac{1}{2}\big(\theta_2^{(1)}(z)+\theta_2^{(2)}(z)\big). \qquad (1.6)
$$

Let α_k be real numbers from the range $[0, 2\pi)$. Introduce the multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and, following [7], also the series:

$$
\begin{aligned}
\theta_2^{(1)}(z,\alpha) = H(z) &- \sum_{k=1}^{n}e^{2i\alpha_k}\overline{H\big[z^*_{(k)}\big]}\big(\overline{z^*_{(k)}}\big)'\\
&+\sum_{k=1}^{n}\sum_{k_1\neq k}e^{2i(\alpha_k-\alpha_{k_1})}H\big[z^*_{(k_1,k)}\big]\big(z^*_{(k_1,k)}\big)'\\
&-\sum_{k=1}^{n}\sum_{k_1\neq k}\sum_{k_2\neq k_1}e^{2i(\alpha_k-\alpha_{k_1}+\alpha_{k_2})}\overline{H\big[z^*_{(k_2,k_1,k)}\big]}\big(\overline{z^*_{(k_2,k_1,k)}}\big)'\\
&+\sum_{k=1}^{n}\sum_{k_1\neq k}\sum_{k_2\neq k_1}\sum_{k_3\neq k_2}e^{2i(\alpha_k-\alpha_{k_1}+\alpha_{k_2}-\alpha_{k_3})}\\
&\times H\big[z^*_{(k_3,k_2,k_1,k)}\big]\big(z^*_{(k_3,k_2,k_1,k)}\big)'-\cdots \qquad (1.7)
\end{aligned}
$$

$$
\begin{aligned}
\theta_2^{(2)}(z,\alpha) = H(z) &+ \sum_{k=1}^{n}e^{2i\alpha_k}\overline{H\big[z^*_{(k)}\big]}\big(\overline{z^*_{(k)}}\big)'\\
&+\sum_{k=1}^{n}\sum_{k_1\neq k}e^{2i(\alpha_k-\alpha_{k_1})}H\big[z^*_{(k_1,k)}\big]\big(z^*_{(k_1,k)}\big)'
\end{aligned}
$$

$$+\sum_{k=1}^{n}\sum_{k_1\neq k}\sum_{k_2\neq k_1} e^{2i(\alpha_k-\alpha_{k_1}+\alpha_{k_2})}\overline{H\big[z^*_{(k_2,k_1,k)}\big]}\big(\overline{z^*_{(k_2,k_1,k)}}\big)'$$

$$+\sum_{k=1}^{n}\sum_{k_1\neq k}\sum_{k_2\neq k_1}\sum_{k_3\neq k_2} e^{2i(\alpha_k-\alpha_{k_1}+\alpha_{k_2}-\alpha_{k_3})}$$

$$\times H\big[z^*_{(k_3,k_2,k_1,k)}\big]\big(z^*_{(k_3,k_2,k_1,k)}\big)'+\cdots \tag{1.8}$$

$$\theta_2(z,\alpha)=\frac{1}{2}\big(\theta_2^{(1)}(z,\alpha)+\theta_2^{(2)}(z,\alpha)\big). \tag{1.9}$$

The series (1.8)–(1.10) are called the Poincaré α-series. The Poincaré θ_2-series and the Poincaré α-series are [7] uniformly convergent in every compact subset not containing the limit points of $\mathcal{K}$ and poles of $H[\gamma_j(z)]$. Moreover, the Poincare θ_2-series is an automorphic function of the weight (-2) [4]:

$$\theta_2(z)=\frac{\theta_2[\gamma_j(z)]}{(c_jz+d_j)^2}. \tag{1.10}$$

2 The Schottky–Klein Prime Function

Let $\mathcal{H}_A(\bigcup_{k=1}^n D_k)$ be the Banach space of functions analytic in $\bigcup_{k=1}^n D_k$ and with the norm $\|f\|=\max_{k=1,2,\dots,n}\max_{\partial D_k}|f(t)|$.

Theorem 2.1 ([1, 6]) *Given $f(z)\in\mathcal{H}_A(\bigcup_{k=1}^n D_k)$, the system of functional equations*:

$$\psi_k(z)=\sum_{m\neq k}\left(\frac{r_m}{z-a_m}\right)^2\overline{\psi_m\big(z^*_{(m)}\big)}+f(z),$$
$$|z-a_k|\leq r_k,\quad k=1,2,\dots,n \tag{2.1}$$

has a unique solution for any circular multiply connected domain D. This solution can be found by the method of successive approximations. The approximations converge in $\mathcal{H}_A(\bigcup_{k=1}^n D_k)$.

Let ζ and w be fixed points of $(D\cup\partial D)\setminus\{\infty\}$ and introduce [7]:

$$\omega(z)=-\sum_{m=1}^{n}\big[\overline{\varphi_m\big(z^*_{(m)}\big)}-\overline{\varphi_m\big(w^*_{(m)}\big)}\big]. \tag{2.2}$$

The function $\omega(z)$ belongs to $\mathcal{H}_A(\bigcup_{k=1}^n D_k)$ and vanishes at $z=w$.

Introduce the functions [7]:

$$\omega_0(z,\zeta,w) = \ln \prod_{j=1}^{\infty} \mu_j(z,\zeta,w), \tag{2.3}$$

where

$$\mu_j(z,\zeta,w) = \begin{cases} \dfrac{\zeta - \gamma_j(z)}{\zeta - \gamma_j(w)}, & \text{if } \gamma_j \in \mathcal{E}, \\[2ex] \dfrac{\overline{\zeta - \gamma_j(\overline{w})}}{\overline{\zeta - \gamma_j(\overline{z})}}, & \text{if } \gamma_j \in \mathcal{O}. \end{cases} \tag{2.4}$$

The infinite product converges uniformly in the variable z in every compact subset of $(D \cup \partial D) \setminus (\{\infty\}, \{\zeta\}, \{w\})$. The justification of these assertions is based on the application of Theorem 2.1 to the functional equations:

$$\begin{aligned} &\varphi_k(z) = -\sum_{m \neq k} \left[\overline{\varphi_m\left(z^*_{(m)}\right)} - \overline{\varphi_m(w^*_{(m)})} \right] + \ln \frac{z-\zeta}{w-\zeta}, \\ &|z - a_k| \le r_k, \quad k = 1, \dots, n. \end{aligned} \tag{2.5}$$

An application of the method of successive approximations to (2.5) yields the uniformly convergent series:

$$\begin{aligned} \varphi_k(z) = \ln \frac{z-\zeta}{w-\zeta} & \\ & - \sum_{k_1 \neq k} \ln \frac{\overline{\zeta - z^*_{(k_1)}}}{\overline{\zeta - w^*_{(k_1)}}} \quad \text{(1st approximations)} \\ & + \sum_{k_1 \neq k} \sum_{k_2 \neq k_1} \ln \frac{\zeta - z^*_{(k_2 k_1)}}{\zeta - w^*_{(k_2,k_1)}} \quad \text{(2nd approximations)} \\ & - \sum_{k_1 \neq k} \sum_{k_2 \neq k_1} \sum_{k_3 \neq k_2} \ln \frac{\overline{\zeta - z^*_{(k_3,k_2,k_1)}}}{\overline{\zeta - w^*_{(k_3,k_2,k_1)}}} \quad \text{(3rd approximations)} \\ & + \sum_{k_1 \neq k} \sum_{k_2 \neq k_1} \sum_{k_3 \neq k_2} \sum_{k_4 \neq k_3} \ln \frac{\zeta - z^*_{(k_4,k_3,k_2,k_1)}}{\zeta - w^*_{(k_4,k_3,k_2,k_1)}} + \cdots, \quad \text{(4th approximations)} \\ |z - a_k| \le r_k, \quad k = 1, \dots, n. & \end{aligned} \tag{2.6}$$

The function (2.2) can be written in the form on the uniformly convergent product:

$$\omega(z) = \ln\Biggl(\prod_{k=1}^{n} \frac{\overline{\zeta w^*_{(k)}}}{\overline{\zeta - z^*_{(k)}}} \prod_{k=1}^{n} \prod_{k_1 \neq k} \frac{\zeta z^*_{(k_1,k)}}{\zeta - w^*_{(k_1,k)}}$$

$$\times \prod_{k=1}^{n} \prod_{k_1 \neq k} \prod_{k_2 \neq k_1} \frac{\overline{\zeta - w^*_{(k_2,k_1,k)}}}{\overline{\zeta - z^*_{(k_2,k_1,k)}}}$$

$$\times \prod_{k=1}^{n} \prod_{k_1 \neq k} \prod_{k_2 \neq k_1} \prod_{k_3 \neq k_2} \frac{\zeta z^*_{(k_3,k_2,k_1,k)}}{\zeta - w^*_{(k_3,k_2,k_1,k)}} \cdots \Biggr).$$

Hence and [5] function $\omega_0(z)$ can be represented in the form (2.4). Now, instead of applying Theorem 2.1 to (2.6), we apply it to the following functional equations:

$$\varphi_k(z) = \sum_{m \neq k} \Bigl[\overline{\varphi_m\bigl(z^*_{(m)}\bigr)} - \overline{\varphi_m\bigl(w^*_{(m)}\bigr)}\Bigr] + \ln \frac{z - \zeta}{w - \zeta},$$
$$|z - a_k| \leq r_k, \quad k = 1, \ldots, n \tag{2.7}$$

and introduce the function:

$$\omega_1(z, \zeta, w) = \ln \prod_{j=1}^{\infty} v_j(z, \zeta, w), \tag{2.8}$$

where

$$v_j(z, \zeta, w) = \begin{cases} \dfrac{\zeta - \gamma_j(z)}{\zeta - \gamma_j(w)}, & \text{if } \gamma_j \in \mathcal{E}, \\[2ex] \dfrac{\overline{\zeta - \gamma_j(\overline{z})}}{\overline{\zeta - \gamma_j(\overline{w})}}, & \text{if } \gamma_j \in \mathcal{O}. \end{cases} \tag{2.9}$$

Similarly to (1.9), introduce the function:

$$\omega(z, \zeta, w) = \frac{1}{2}\bigl[\omega_0(z, \zeta, w) + \omega_1(z, \zeta, w)\bigr] = \frac{1}{2} \ln \prod_{j \in \mathcal{E} \setminus \{\gamma_0\}} \frac{\zeta - \gamma_j(z)}{\zeta - \gamma_j(w)} \tag{2.10}$$

Hence, the following product

$$\Omega(z, \zeta, w) = \ln \prod_{j \in \mathcal{E} \setminus \{\gamma_0\}} \frac{\zeta - \gamma_j(z)}{\zeta - \gamma_j(w)} \tag{2.11}$$

is correctly defined for $z \neq \zeta$.

We can introduce the function of two variables:

$$S(z, \zeta) = (\zeta - z)\Omega(\zeta, z, z)\Omega(z, \zeta, \zeta)$$
$$= (\zeta - z) \prod_{j \in \mathcal{E} \setminus \{\gamma_0\}} \frac{\zeta - \gamma_j(z)}{\zeta - \gamma_j(w)} \frac{\zeta - \gamma_j(z)}{\zeta - \gamma_j(w)}. \tag{2.12}$$

This function is called the Schottky–Klein function presented in the form of a uniformly convergent product. The uniform convergence is proved for $\Omega(\zeta, z, z)$ in the variable ζ in every compact subset of $(D \cup \partial D) \setminus (\{z\}, \{\infty\})$ and for $\Omega(z, \zeta, \zeta)$ in the variable z in every compact subset of $(D \cup \partial D) \setminus (\{\zeta\}, \{\infty\})$. A representation of the Schottky–Klein function in the form of an absolutely convergent product was given by Crowdy [2, 3] under geometrical restrictions on the disks D_k.

Similarly to (1.8)–(1.10) one can introduce the Schottky–Klein α–prime function:

$$S(z, \zeta, \alpha) = (\zeta - z) \prod_{j \in \mathcal{E} \setminus \{\gamma_0\}} e^{2is_j(\alpha)} \frac{\zeta - \gamma_j(z)}{\zeta - \gamma_j(w)} \frac{\zeta - \gamma_j(z)}{\zeta - \gamma_j(w)}, \tag{2.13}$$

where p is an odd number and $s_j(\alpha) = \alpha_k - \alpha_{k_1} + \cdots + \alpha_{k_{p-1}} - \alpha_{k_p}$.

The correspondence between j and $(k_p, k_{p-1}, \ldots, k_1, k)$ is established via the numeration of the elements of $\mathcal{E}$, i.e., via the relation $\gamma_j(z) = z^*_{(k_p k_{p-1} \cdots k_1)}$.

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Green Function of the Dirichlet Problem for the Laplacian and Inhomogeneous Boundary Value Problems for the Poisson Equation in a Punctured Domain

Baltabek Kanguzhin and Niyaz Tokmagambetov

Abstract The aim of this work is to present a new definition of the Green function of the Dirichlet problem for the Laplace equation prompted by the theory of ordinary differential equations and investigate correctly solvable boundary value problems for the Poisson equation in a punctured domain.

Keywords Green's function · Poisson equation · Dirichlet problem · Punctured domain · Non simply connected domain · Non-local boundary value problem

Mathematics Subject Classification (2010) Primary 35J08 · Secondary 35J05 · 35J25

1 Definition of the Green Function of the Dirichlet Problem for the Laplace Operator

From the work of M.A. Naimark [1], we introduce the definition of the Green function for ordinary differential operators. Let an operator L_1, generated by the ordinary differential expression with smooth coefficients on the interval (a, b)

$$l(y) = p_0(x)\frac{d^n u}{dx^n} + p_1(x)\frac{d^{n-1}u}{dx^{n-1}} + \cdots + p_n(x)u$$

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and some boundary conditions

$$U_\nu(u) = 0, \quad \nu = 1, 2, \ldots, n,$$

have an inverse L_1^{-1} the domain of which coincides with the range of values of the operator L_1. L_1^{-1} is an integral operator with a continuous kernel. This kernel is called Green function of the operator L_1. Let us formulate this definition more precisely.

Definition 1.1 If a function $\Gamma(x, \xi)$ satisfies the following conditions

(1) $\Gamma(\cdot, \xi) \in C^n([a, \xi) \cup (\xi, b])$;
(2) $l(\Gamma) = 0$ on intervals (a, ξ) and (ξ, b);
(3) $U_\nu(\Gamma) = 0$, $\nu = 1, 2, \ldots, n$;
(4) for $i = 1, 2, \ldots, n-2$

$$\lim_{\delta \to +0} \left[\frac{\partial^i}{\partial x^i} \Gamma(\xi + \delta, \xi) - \frac{\partial^i}{\partial x^i} \Gamma(\xi - \delta, \xi) \right] = 0$$

and

$$\lim_{\delta \to +0} \left[\frac{\partial^{n-1}}{\partial x^{n-1}} \Gamma(\xi + \delta, \xi) - \frac{\partial^{n-1}}{\partial x^{n-1}} \Gamma(\xi - \delta, \xi) \right] = \frac{1}{p_0(\xi)},$$

then the function $\Gamma(x, \xi)$ is called Green function of the operator L_1.

There can be found the proof of the following Green function's uniqueness theorem.

Theorem 1.2 *If the boundary value problem $L_1 y = 0$ has only the trivial solution, then the operator L_1 has a unique Green function.*

Let an operator L_2 be generated by the Laplacian $-\Delta$ and with the Dirichlet boundary condition. In the standard textbooks (for instance [2]), in a case of the Dirichlet problem for the Laplacian, a definition of the Green function is given as follows.

Definition 1.3 The Green function of the Dirichlet problem for the Laplace operator in the domain $\Omega \subset \mathbb{R}^2$ is the $\mathcal{G}(x, y)$, $x \in \overline{\Omega}$, $y \in \Omega$, which satisfies the properties:

(1) for all $y \in \Omega$

$$\mathcal{G}(x, y) = \frac{1}{2\pi} \ln |x - y| + g(x, y),$$

where $g(x, y)$ is harmonic in the domain Ω and continuous on $\overline{\Omega}$ respect to x;
(2) for any $y \in \Omega$

$$\mathcal{G}(x, y)|_{x \in \partial \Omega} = 0.$$

Conditions (1) and (2) imply that $\mathcal{G}(x, y)$ is a harmonic function in $\Omega \setminus \{y\}$ and continuous in $\overline{\Omega} \setminus \{y\}$ with respect to x, vanishes on $\partial\Omega$ and approaches to ∞ for $x \to y$.

In what follows, we present a new definition of the Green function of the Dirichlet problem for the Laplace equation prompted by the theory of ordinary differential equations. In the condition (1) of Definition 1.3 appears the fundamental solution of the Laplace operator, which is defined by the Dirac delta function. At the same time, in Definition 1.3 generalized functions are absent. It needs to have a definition of the Green function of the Dirichlet problem for the Poisson equation without involving generalized functions.

Consider the differential expression

$$\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

in $\Omega_0 := \Omega \setminus \{M_0\}$, where Ω is a bounded, simple-connected domain with sufficiently smooth boundary $\partial\Omega$ in $\mathbb{R}^2$, $M_0 = (x_0, y_0)$ is an inner fixed point of the domain Ω.

Let us denote

$$\Pi_\delta^0 = \big\{(x, y) : -\delta \le x - x_0 \le \delta,\ -\delta \le y - y_0 \le \delta\big\}.$$

Introduce a functional

$$\alpha(h) = \frac{1}{2} \lim_{\delta\to+0} \int_{x_0-\delta}^{x_0+\delta} \left[\frac{\partial h(\xi, y_0+\delta)}{\partial\eta} - \frac{\partial h(\xi, y_0-\delta)}{\partial\eta}\right] d\xi$$
$$+ \int_{y_0-\delta}^{y_0+\delta} \left[\frac{\partial h(x_0+\delta, \eta)}{\partial\xi} - \frac{\partial h(x_0-\delta, \eta)}{\partial\xi}\right] d\eta.$$

Note, the value of the functional $\alpha(\cdot)$ for continuously differentiable functions $h(\cdot)$ is equal to zero.

Definition 1.4 A function Γ defined on the set Ω_0 is called Green function in the classical sense of the Dirichlet problem for the Laplacian, if the following conditions are valid

(1) $\Gamma \in C^2(\Omega_0)$;
(2) $\Delta\Gamma = 0$ in the Ω_0;
(3) for every $(x_0, y_0) \in \Omega$

$$\Gamma(x, y, x_0, y_0)|_{(x,y)\in\partial\Omega} = 0;$$

(4) the functionals for Γ are equal to

$$\mathrm{I}_1(\Gamma) \equiv \lim_{\delta\to+0} \delta \int_{\partial\Pi_\delta^0} \big|\Gamma(x, y, x_0, y_0)\big| dS_{x,y} = 0,$$

$$\mathrm{I}_2(\Gamma) \equiv \lim_{\delta\to+0} \delta \int_{\partial\Pi_\delta^0} \left| \frac{\partial\Gamma(x, y, x_0, y_0)}{\partial n_{x,y}} \right| dS_{x,y} = 0,$$

and

$$\alpha(\Gamma) = 1.$$

Note, that the functional $\alpha(\cdot)$ first was introduced in the work [3].

Let a functional space $\widetilde{W}_2^2(\Omega_0)$ be a subspace of the Sobolev space $W_2^2(\Omega_0)$ and for elements $\widetilde{W}_2^2(\Omega_0)$ the functionals $\alpha(\cdot)$, $\mathrm{I}_1(\cdot)$, $\mathrm{I}_2(\cdot)$ exist, and moreover, the functionals $\mathrm{I}_1(\cdot)$ and $\mathrm{I}_2(\cdot)$ equal to zero.

The following theorem is the main result of this section.

Theorem 1.5 *In the space $\widetilde{W}_2^2(\Omega_0)$ the Green function in the classical sense of the Dirichlet problem for the Laplace operator exists and is uniquely determined.*

We note, that the existence of the Green function in the classical sense was proved in the work [4].

Denote by $\varepsilon(x, y, x_0, y_0)$ the function

$$\frac{1}{4\pi} \ln\big((x - x_0)^2 + (y - y_0)^2\big).$$

For this function we prove the following lemma.

Lemma 1.6 *For the function $\varepsilon(x, y, x_0, y_0)$ the relation*

$$\alpha(\varepsilon) = 1 \tag{1.1}$$

holds.

Proof As

$$\frac{\partial \ln((x - x_0)^2 + (y - y_0)^2)}{\partial x} = \frac{2(x - x_0)}{(x - x_0)^2 + (y - y_0)^2},$$

$$\frac{\partial \ln((x - x_0)^2 + (y - y_0)^2)}{\partial y} = \frac{2(y - y_0)}{(x - x_0)^2 + (y - y_0)^2}$$

by simple calculations, we have

$$\alpha(\varepsilon) = \frac{1}{4\pi} \lim_{\delta\to+0} \int_{x_0-\delta}^{x_0+\delta} \left(\frac{4\delta}{\delta^2 + (x_0 - t)^2} \right) dt + \frac{1}{4\pi} \lim_{\delta\to+0} \int_{y_0-\delta}^{y_0+\delta} \left(\frac{4\delta}{\delta^2 + (x_0 - t)^2} \right) dt = 1.$$

Lemma 1.6 is proved. □

Proof of Theorem 1.5 At first, we prove the existence of the Green function in the classical sense of the Dirichlet problem for the Laplace equation. Construct a function $P(x, y, x_0, y_0)$ of the form

$$P(x, y, x_0, y_0) = \varepsilon(x, y, x_0, y_0) + g(x, y),$$

where $g(x, y)$ is a sufficiently smooth function and

$$(-\Delta_{x,y})g(x, y) = 0, \quad (x, y) \in \Omega \times \Omega.$$

Also, we choose the function $g(x, y)$ such that the values of the functions $\varepsilon(x, y, x_0, y_0)$ and $-g(x, y)$ coincide on the boundary $\partial\Omega$. Since $g(x, y)$ is a continuously differentiable function, then $\alpha(g) = 0$. Hence, we get $\alpha(P) = 1$. The constructed function $P(x, y, x_0, y_0)$ has a singularity only at the point (x_0, y_0), and the singularity is logarithmic. Then the functionals $\mathrm{I}_1(\cdot)$ and $\mathrm{I}_2(\cdot)$ for P exist and are equal to zero. Therefore, it is established that the function $P(x, y, x_0, y_0)$ belongs to the space $\widetilde{W}_2^2(\Omega_0)$. In addition the function $P(x, y, x_0, y_0)$ satisfies all requirements of Definition 1.4. Hence, the existence is proved. □

Now it remains to prove the uniqueness of the Green function in the classical sense of the Dirichlet problem for the Laplace equation.

Problem 1 In the space $\widetilde{W}_2^2(\Omega_0)$ consider the Laplace equation

$$\Delta u(x) = 0, \quad x \in \Omega_0,$$

with the Dirichlet boundary condition on the exterior boundary of the domain Ω_0

$$u|_{\partial\Omega} = 0$$

and with a condition on the "inner" boundary

$$\alpha(u) = 0.$$

Our aim is to show that Problem 1 has only the trivial solution. Let $(x, y) \in \Omega_0$. By the Green formulae

$$\int_{\Omega\setminus\Pi_\delta^0}\int \big[\Delta u(\xi,\eta)G(x,y,\xi,\eta) - u(\xi,\eta)\Delta G(x,y,\xi,\eta)\big]d\xi d\eta$$
$$= \int_{\partial\Omega}\left[\frac{\partial u(\xi,\eta)}{\partial n_{\xi,\eta}}G(x,y,\xi,\eta) - u(\xi,\eta)\frac{\partial G(x,y,\xi,\eta)}{\partial n_{\xi,\eta}}\right]dS$$
$$- \int_{\partial\Pi_\delta^0}\left[\frac{\partial u(\xi,\eta)}{\partial n_{\xi,\eta}}G(x,y,\xi,\eta) - u(\xi,\eta)\frac{\partial G(x,y,\xi,\eta)}{\partial n_{\xi,\eta}}\right]dS,$$

where $\delta > 0$ is chosen such that $(x, y) \in \Omega \setminus \Pi_\delta^0$. From properties of the ε, the taken formulae can be written as

$$u(x, y) = -\int_{\partial \Pi_\delta^0} \left[\frac{\partial u(\xi, \eta)}{\partial n_{\xi,\eta}} \varepsilon(x, y, \xi, \eta) - u(\xi, \eta) \frac{\partial \varepsilon(x, y, \xi, \eta)}{\partial n_{\xi,\eta}} \right] dS \tag{1.2}$$

for any $(x, y) \in \Omega \setminus \Pi_\delta^0$. Let us show, that the second integral in equality (1.2) approaches to zero for $\delta \to 0$. Extend the function $u(x)$ at (x_0, y_0) continuously, i.e. get the function $u(x)$ at the point (x_0, y_0) equal to the value of the first integral of the right-hand side of equality (1.2). Hence, we get a harmonic function in Ω, which coincides with $u(x)$ in Ω_0.

Now, we show, that

$$\lim_{\delta \to 0} \int_{\partial \Pi_\delta^0} \left[\frac{\partial u(\xi, \eta)}{\partial n_{\xi,\eta}} \varepsilon(x, y, \xi, \eta) - u(\xi, \eta) \frac{\partial \varepsilon(x, y, \xi, \eta)}{\partial n_{\xi,\eta}} \right] dS = 0. \tag{1.3}$$

For this, we prove

$$\lim_{\delta \to 0} \int_{\partial \Pi_\delta^0} \frac{\partial u(\xi, \eta)}{\partial n_{\xi,\eta}} \varepsilon(x, y, \xi, \eta) dS = 0,$$

and the equality

$$\lim_{\delta \to 0} \int_{\partial \Pi_\delta^0} u(\xi, \eta) \frac{\partial \varepsilon(x, y, \xi, \eta)}{\partial n_{\xi,\eta}} dS = 0$$

follows from the same calculations. Thus

$$\begin{aligned}
&\lim_{\delta \to 0} \int_{\partial \Pi_\delta^0} \frac{\partial u(\xi, \eta)}{\partial n_{\xi,\eta}} \varepsilon(x, y, \xi, \eta) dS \\
&= \lim_{\delta \to 0} \int_{y_0 - \delta}^{y_0 + \delta} \left[\frac{\partial u(x_0 + \delta, \eta)}{\partial \xi} \varepsilon(x, y, x_0 + \delta, \eta) \right. \\
&\quad \left. - \frac{\partial u(x_0 - \delta, \eta)}{\partial \xi} \varepsilon(x, y, x_0 - \delta, \eta) \right] dS \\
&\quad + \lim_{\delta \to 0} \int_{x_0 - \delta}^{x_0 + \delta} \left[\frac{\partial u(\xi, y_0 + \delta)}{\partial \eta} \varepsilon(x, y, \xi, y_0 + \delta) \right. \\
&\quad \left. - \frac{\partial u(\xi, y_0 - \delta)}{\partial \eta} \varepsilon(x, y, \xi, y_0 - \delta) \right] dS.
\end{aligned}$$

By virtue that all derivatives of the function $\varepsilon(x, y, x_0, y_0)$ are bounded functions for $(x, y) \in \Pi_{\delta_1}^0 \setminus \Pi_\delta^0$ by the last two arguments, then we get

$$\lim_{\delta \to 0} \int_{\partial \Pi_\delta^0} \frac{\partial u(\xi, \eta)}{\partial n_{\xi,\eta}} \varepsilon(x, y, \xi, \eta) dS$$

$$= \lim_{\delta\to 0}\int_{y_0-\delta}^{y_0+\delta}\Bigg[\frac{\partial u(x_0+\delta,\eta)}{\partial\xi}\Bigg(\varepsilon(x,y,x_0,y_0)$$

$$+\,\delta\frac{\partial\varepsilon(x,y,x_0,y_0)}{\partial\xi}+(\eta-y_0)\frac{\partial\varepsilon(x,y,x_0,y_0)}{\partial\eta}+O\big(\delta^2\big)\Bigg)$$

$$-\,\frac{\partial u(x_0-\delta,\eta)}{\partial\xi}\Bigg(\varepsilon(x,y,x_0,y_0)-\delta\frac{\partial\varepsilon(x,y,x_0,y_0)}{\partial\xi}$$

$$+\,(\eta-y_0)\frac{\partial\varepsilon(x,y,x_0,y_0)}{\partial\eta}+O\big(\delta^2\big)\Bigg)\Bigg]dS$$

$$+\lim_{\delta\to 0}\int_{x_0-\delta}^{x_0+\delta}\Bigg[\frac{\partial u(\xi,y_0+\delta)}{\partial\eta}\Bigg(\varepsilon(x,y,x_0,y_0)$$

$$+\,(\xi-x_0)\frac{\partial\varepsilon(x,y,x_0,y_0)}{\partial\xi}+\delta\frac{\partial\varepsilon(x,y,x_0,y_0)}{\partial\eta}+O\big(\delta^2\big)\Bigg)$$

$$-\,\frac{\partial u(\xi,y_0-\delta)}{\partial\eta}\Bigg(\varepsilon(x,y,x_0,y_0)+(\xi-x_0)\frac{\partial\varepsilon(x,y,x_0,y_0)}{\partial\xi}$$

$$-\,\delta\frac{\partial\varepsilon(x,y,x_0,y_0)}{\partial\eta}+O\big(\delta^2\big)\Bigg)\Bigg]dS.$$

Since $u\in\widetilde{W}_2^2(\Omega_0)$, the functionals $I_1(\cdot)$ and $I_2(\cdot)$ for the function u are equal to zero. Finally, we have

$$\lim_{\delta\to 0}\int_{\partial\Pi_\delta^0}\frac{\partial u(\xi,\eta)}{\partial n_{\xi,\eta}}\varepsilon(x,y,\xi,\eta)dS=\alpha(u)\varepsilon(x,y,x_0,y_0)=0.$$

Whence limit (1.3) is valid. Thereby, it was established that the solution of Problem 1, i.e. the function u is a harmonic function in Ω. From the homogeneous Dirichlet condition the function u is identically equal to zero in the domain Ω. Hence, uniqueness of the Green function in the classical sense of the Dirichlet problem for the Laplacian is shown.

The proof of Theorem 1.5 is completed.

Proposition 1.7 *In the proof of the main theorem of this section was shown that the Green function in the classical sense coincides with the standard Green function.*

2 Inhomogeneous Boundary Value Problems for the Poisson Equation in a Punctured Domain

In the first part of this paper it was proved that for a fixed point $(x_0, y_0)\in\Omega$ the function $u(x,y)\equiv G(x,y,x_0,y_0)$ is a solution of the boundary value problem for the Laplace equation in the punctured domain $\Omega_0=\Omega\setminus\{M_0\}$, where $M_0=(x_0,y_0)$.

According to Theorem 1.5 the function $u(x, y)$ at first, belongs to the space $\widetilde{W}_2^2(\Omega_0)$ and, secondly satisfies the equation

$$-\Delta u(x, y) = 0, \quad (x, y) \in \Omega_0,$$

also the Dirichlet boundary condition

$$u\,|_{\partial\Omega} = 0$$

and the additional condition at the point M_0

$$\alpha(u) = 1.$$

The functional class $\widetilde{W}_2^2(\Omega_0)$ and the functional $\alpha(\cdot)$ were introduced in the first part of the work. Thus, the function $u(x, y) \equiv G(x, y, x_0, y_0)$ is a solution for the boundary value problem for the Laplace equation in the non simply connected domain Ω_0. In this section, we investigate correctly solvable boundary value problems for the Poisson equation in the punctured domain Ω_0.

Consider the Poisson equation

$$-\Delta w(x, y) = f(x, y), \quad (x, y) \in \Omega_0 \tag{2.1}$$

with the Dirichlet boundary condition

$$w\,|_{\partial\Omega} = 0. \tag{2.2}$$

We search for a solution w from the space $\widetilde{W}_2^2(\Omega_0)$. From the discussion of the first section the solution of the problem (2.1)–(2.2) in the functional class $\widetilde{W}_2^2(\Omega_0)$ is not unique, since for $f \equiv 0$ as w we can take the functions $u(x, y) \equiv G(x, y, x_0, y_0)$ and $w(x, y) \equiv 0$.

Raises the question: what additional conditions at the point M_0 must be imposed on the function $w(x, y)$ such that the problem (2.1)–(2.2) for all $f \in L_2(\Omega)$ has a unique solution.

Theorem 2.1 *In the space $\widetilde{W}_2^2(\Omega_0)$ the Poisson equation (2.1) with Dirichlet boundary condition (2.2) and with non-local condition at the point M_0*

$$\alpha(w) - \int_\Omega\int K(\xi, \eta)\big(-\Delta w(\xi, \eta)\big)d\xi d\eta = 0 \tag{2.3}$$

has a unique solution, where $K \in L_2(\Omega)$.

Proof Uniqueness of the solution of the problem (2.1)–(2.3) follows from the proof of Theorem 1.5. Let us consider the function

$$v(x, y) \equiv \int_\Omega\int G(x, y, \xi, \eta) f(\xi, \eta) d\xi d\eta$$

$$+ G(x, y, x_0, y_0) \int_{\Omega}\int K(\xi, \eta) f(\xi, \eta) d\xi d\eta.$$

It easy to see, that the introduced function v satisfies (2.1), the boundary condition (2.2) and the relation (2.3). Thereby, existence of the solution of the problem (2.1)–(2.3) is established.

Theorem 2.1 is proved. □

Denote by $\mathcal{L}_K$ an operator, which corresponds to the problem (2.1)–(2.3). Indeed, we get a class of operators. Every function K from $L_2(\Omega)$ generates new operator $\mathcal{L}_K$. For example, for a harmonic K in Ω, the condition (2.3) has the form

$$\alpha(w) + \int_{\partial\Omega} K(\xi, \eta) \frac{\partial w(\xi, \eta)}{\partial n_{\xi,\eta}} dS_{\xi,\eta} = 0,$$

i.e. a connection between the "inner" and "exterior" data.

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Boundary Value Problems and Method of Reflection for Quarter Ring and Half Hexagon

B. Shupeyeva

Abstract On basis of the reflection principle, the boundary value problems of Schwarz, Dirichlet and Neumann type are explicitly solved for two irregular domains.

Keywords Schwarz · Dirichlet · Neumann · Green function · Quarter ring · Half hexagon

Mathematics Subject Classification (2010) 30E20 · 30E25 · 31A20 · 31A25

1 Introduction

The basic boundary value problems for complex partial differential equations have been considered by many authors for different particular domains e.g. [1–14, 16, 17]. These problems for the inhomogeneous Cauchy–Riemann equation and the Poisson equation are solved explicitly in regular and irregular domains. The latter ones attract much attention since they entail the specific investigation of the behavior of solutions in the neighborhood of the corner points e.g. half disc and half ring [5], quarter disc [10], disc sectors [16], lenses and lunes [9]. For constructing the Schwarz kernel and the Green and Neumann functions, which are essential for solutions, a method described e.g in [7, 9] is used. The method uses reflection of the domain at all parts of the boundary and it is applicable if the whole complex plane can be covered by continuously repeated reflections. In this paper the method is shown in application to two irregular domains such as quarter ring and half hexagon. The boundary of the quarter ring consists of two straight segments and two circular arcs. Reflections at the segments gives the covering of a ring domain and continued reflections at the boundary circles of the ring produces a covering of the punctured complex plane. Similarly, reflection of the half hexagon to the whole hexagon leads to the whole covered plane. The reflection principle allows to use

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the Cauchy–Pompeiu representation formula to attain the Poisson kernel. It helps to solve the boundary value problems for the Cauchy–Riemann equation. The harmonic Green and ensuing Neumann functions obtained by this method form the Green and Neumann representation formulas. They are needed for solutions of the boundary value problems for the Poisson equation.

Here the main results for the domains mentioned above are presented in a short form. The necessary calculations as well as the proofs of the theorems and lemmas are fully described in [12, 13].

2 Some Results for the Quarter Ring Domain

The Cauchy–Pompeiu representation formula

$$w(z) = \frac{1}{2\pi i}\int_{\partial D} w(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_D w_{\overline{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta - z}, \quad z \in D \tag{2.1}$$

for any function $w \in C^1(D;\mathbb{C}) \cap C(\overline{D};\mathbb{C})$ in a bounded domain D of $\mathbb{C}$ with piecewise smooth boundary is complemented with the relation

$$0 = \frac{1}{2\pi i}\int_{\partial D} w(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_D w_{\overline{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta - z}, \quad z \in \mathbb{C}\backslash\overline{D}. \tag{2.2}$$

Let R^* be the upper right quarter ring domain in the complex plane $\mathbb{C}$

$$R^* = \{z \in \mathbb{C} : r < |z| < 1,\ \mathrm{Re}\, z > 0,\ \mathrm{Im}\, z > 0\}.$$

The boundary ∂R^* is piecewise smooth and contains four corner points r, 1, i, ir. A point $z \in R^*$ is reflected across the boundary parts. Continuing the reflection process, the new points

$$\begin{array}{cccc} \pm zr^{2n}, & \pm\overline{z}r^{2n}, & \pm\dfrac{z}{r^{2n}}, & \pm\dfrac{\overline{z}}{r^{2n}}, \\ \pm\dfrac{r^{2n}}{z}, & \pm\dfrac{r^{2n}}{\overline{z}}, & \pm\dfrac{1}{\overline{z}r^{2n}}, & \pm\dfrac{1}{zr^{2n}}, \end{array} \tag{2.3}$$

appear. Substituting them into formulas (2.1), (2.2) lead to the modified Cauchy–Pompeiu formula.

Theorem 2.1 *Any $w \in C^1(R^*;\mathbb{C}) \cap C(\overline{R^*},\mathbb{C})$ for the domain $R^* \subset \mathbb{C}$ can be represented as*

$$w(z) = \frac{1}{2\pi i}\int_{\partial R^*} w(\zeta)\left\{\frac{\zeta}{\zeta - z} + \sum_{n=1}^{\infty} r^{2n}\left[\frac{\zeta}{r^{2n}\zeta - z} + \frac{z}{\zeta - r^{2n}z}\right]\right\}\frac{d\zeta}{\zeta}$$

$$-\frac{1}{\pi}\int_{R^*} w_{\overline{\zeta}}(\zeta)\left\{\frac{1}{\zeta-z}+\sum_{n=1}^{\infty}r^{2n}\left[\frac{1}{r^{2n}\zeta-z}+\frac{z}{\zeta(\zeta-r^{2n}z)}\right]\right\}d\xi d\eta \quad (2.4)$$

and

$$\begin{aligned} w(z) =& \frac{1}{\pi i}\int_{\partial_1 R^*} \operatorname{Re} w(\zeta)\Lambda_1(\zeta,z)d\zeta - \frac{1}{\pi i}\int_{\partial_2 R^*} \operatorname{Re} w(\zeta)\Lambda_1(\zeta,z)d\zeta \\ &+ \frac{2}{\pi i}\int_{\partial_3 R^*} \operatorname{Re} w(\zeta)\Lambda_2(\zeta,z)d\zeta + \frac{2}{\pi i}\int_{\partial_4 R^*} \operatorname{Re} w(\zeta)\Lambda_2(\zeta,z)d\zeta \\ &+ \frac{2}{\pi}\int_{\partial_1 R^*} \operatorname{Im} w(\zeta)\frac{d\zeta}{\zeta} - \frac{2}{\pi}\int_{R^*}\left\{w_{\overline{\zeta}}(\zeta)\Lambda_2(\zeta,z) - \overline{w_{\overline{\zeta}}(\zeta)}\Lambda_2(\overline{\zeta},z)\right\}d\xi d\eta, \end{aligned} \quad (2.5)$$

where for $\zeta \in \partial R^*$

$$\begin{aligned} \Lambda_1(\zeta,z) =& \Bigg(\frac{\zeta^2+z^2}{\zeta^2-z^2} - \frac{\overline{\zeta}^2+z^2}{\overline{\zeta}^2-z^2} + 2\sum_{n=1}^{\infty} r^{4n}\Bigg[\frac{\zeta^2}{r^{4n}\zeta^2-z^2} - \frac{z^2}{r^{4n}z^2-\zeta^2} \\ &+ \frac{z^2}{r^{4n}z^2-\overline{\zeta}^2} - \frac{\zeta^2}{r^{4n}\overline{\zeta}^2-z^2}\Bigg]\Bigg)\frac{1}{\zeta}, \end{aligned} \quad (2.6)$$

$$\begin{aligned} \Lambda_2(\zeta,z) =& \Bigg(\frac{\zeta^2}{\zeta^2-z^2} - \frac{\zeta^2 z^2}{1-\zeta^2 z^2} + \sum_{n=1}^{\infty} r^{4n}\Bigg[\frac{\zeta^2}{r^{4n}\zeta^2-z^2} - \frac{z^2}{r^{4n}z^2-\zeta^2} \\ &+ \frac{z^2\zeta^2}{r^{4n}z^2\zeta^2-1} - \frac{1}{r^{4n}-\zeta^2 z^2}\Bigg]\Bigg)\frac{1}{\zeta} \end{aligned} \quad (2.7)$$

and the parts of the boundary are denoted as

$$\partial_1 R^* = \{|\zeta| = 1,\ \operatorname{Re}\zeta \geq 0,\ \operatorname{Im}\zeta \geq 0\};$$
$$\partial_2 R^* = \{|\zeta| = r,\ \operatorname{Re}\zeta \geq 0,\ \operatorname{Im}\zeta \geq 0\};$$
$$\partial_3 R^* = \{\zeta = t : r \leq t \leq 1\},$$
$$\partial_4 R^* = \{\zeta = it : r \leq t \leq 1\}.$$

To solve the related Schwarz problem, using Theorem 2.1, the boundary behavior of the boundary integral is studied, see [13]. It is also shown that continuity in the corner points holds.

Theorem 2.2 *The Schwarz problem*

$$w_{\overline{z}} = f \ \text{in}\ R^*,\ f \in L_p(R^*;\mathbb{C}),\ p > 2,\ \operatorname{Re} w = \gamma \ \text{on}\ \partial R^*, \quad (2.8)$$

$$\gamma \in C(\partial R^*;\mathbb{C}),\ \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \operatorname{Im} w(e^{i\varphi})d\varphi = c,\ c \in \mathbb{R} \quad (2.9)$$

is uniquely solvable by

$$\begin{aligned} w(z) &= \frac{1}{\pi i}\int_{\partial_1 R^*}\gamma(\zeta)\Lambda_1(\zeta,z)d\zeta - \frac{1}{\pi i}\int_{\partial_2 R^*}\gamma(\zeta)\Lambda_1(\zeta,z)d\zeta \\ &+ \frac{2}{\pi i}\int_{\partial_3 R^*}\gamma(\zeta)\Lambda_2(\zeta,z)d\zeta + \frac{2}{\pi i}\int_{\partial_4 R^*}\gamma(\zeta)\Lambda_2(\zeta,z)d\zeta + ic \\ &- \frac{2}{\pi}\int_{R^*}\big\{f(\zeta)\Lambda_2(\zeta,z) - \overline{f(\zeta)}\Lambda_2(\overline{\zeta},z)\big\}d\xi d\eta \end{aligned}$$

The Pompeiu-type operator [15] on the right-hand side provides a solution of (2.8) in a weak sense.

The modified Cauchy–Pompeiu formula and a solvability condition provide the solution of the Dirichlet and, hence, the Neumann problems, [13].

The harmonic Green function for R^* is obtained on the basis of the Green function for the ring R [14] and the upper half ring R^+ by observing the additional points appeared within reflection process. Thus, for the quarter ring domain R^* the Green function $G_1(z,\zeta)$ equals to

$$\begin{aligned} &\log\Bigg|\frac{(\overline{\zeta}^2 - z^2)(\overline{\zeta}^2 z^2 - 1)}{(\zeta^2 - z^2)(\zeta^2 z^2 - 1)}\prod_{n=1}^{\infty} \\ &\times \frac{(\overline{\zeta}^2 - r^{4n}z^2)}{(\zeta^2 - r^{4n}z^2)}\frac{(\overline{\zeta}^2 r^{4n} - z^2)(\overline{\zeta}^2 z^2 - r^{4n})(\overline{\zeta}^2 z^2 r^{4n} - 1)}{(\zeta^2 r^{4n} - z^2)(\zeta^2 z^2 - r^{4n})(\zeta^2 z^2 r^{4n} - 1)}\Bigg|^2 \end{aligned} \tag{2.10}$$

and as it is shown in [12, 13], it satisfies the properties of the harmonic Green function [2].

Theorem 2.3 [3] *Any* $w \in C^2(D;\mathbb{C}) \cap C^1(\overline{D};\mathbb{C})$ *can be represented as*

$$w(z) = -\frac{1}{4\pi}\int_{\partial D} w(\zeta)\partial_{\nu_\zeta} G_1(z,\zeta)ds_\zeta - \frac{1}{\pi}\int_D w_{\zeta\overline{\zeta}}(\zeta)G_1(z,\zeta)d\xi d\eta, \tag{2.11}$$

where s_ζ *is the arc length parameter on* ∂D *with respect to the variable* ζ *and* $G(z,\zeta) = \frac{1}{2}G_1(z,\zeta)$ *is the harmonic Green function for* D.

The Green representation formula provides a solution to the Dirichlet problem

$$\begin{aligned} &w_{z\overline{z}} = f \text{ in } R^*, \; w = \gamma \text{ on } \partial R^* \\ &\text{for } f \in L_2\big(R^*;\mathbb{C}\big) \cap C\big(R^*;\mathbb{C}\big), \; \gamma \in C\big(\partial R^*;\mathbb{C}\big). \end{aligned}$$

Solution is given in [12, 13]. From the Green function the harmonic Neumann function for R^* is obtained

$$N_1(z,\zeta) = \log\frac{|z\zeta|^8}{r^8} - \log\big|\big(\overline{\zeta}^2 - z^2\big)\big(\zeta^2 - z^2\big)\big(\overline{\zeta}^2 z^2 - 1\big)\big(\zeta^2 z^2 - 1\big)\big|^2$$

$$+\sum_{n=1}^{\infty}\Big(\log|z\zeta|^{16}-\log\big|\big(\overline{\zeta}^2-r^{4n}z^2\big)\big(\zeta^2-r^{4n}z^2\big)\big(\overline{\zeta}^2r^{4n}-z^2\big)$$
$$\times\big(\zeta^2r^{4n}-z^2\big)\big(\overline{\zeta}^2z^2-r^{4n}\big)\big(\zeta^2z^2-r^{4n}\big)$$
$$\times\big(\overline{\zeta}^2z^2r^{4n}-1\big)\big(\zeta^2z^2r^{4n}-1\big)\big|^2\Big) \tag{2.12}$$

The Neumann representation formula

Theorem 2.4 *Any $w \in C^2(R^*, C)\cap C^1(\overline{R^*}, C)$ can be represented by*

$$w(z) = -\frac{1}{4\pi}\int_{\partial R^*}\big[w(\zeta)\partial_{\nu_\zeta}N_1(z,\zeta)-\partial_{\nu_\zeta}w(\zeta)N_1(z,\zeta)\big]ds_\zeta$$
$$-\frac{1}{\pi}\int_{R^*}w_{\zeta\overline{\zeta}}(\zeta)N_1(z,\zeta)d\xi d\eta,$$

with $N_1 = 2N$, where N is the harmonic Neumann function for R^*.

provides a solution to the Neumann problem, see [12, 13]

$$w_{z\overline{z}} = f \text{ in } R^*,\ \partial_\nu w = \gamma \text{ on } \partial R^*,\ \frac{2}{\pi}\int_0^{\frac{\pi}{2}} w\big(re^{i\varphi}\big)d\varphi = c,$$
$$f\in L_2\big(R^*;\mathbb{C}\big)\cap C\big(R^*;\mathbb{C}\big),\ \gamma\in C\big(\partial R^*;\mathbb{C}\big),\ c\in\mathbb{C}$$

under condition

$$\frac{1}{4\pi}\int_{\partial R^*}\gamma(\zeta)ds_\zeta = \frac{1}{\pi}\int_{R^*}f(\zeta)d\xi d\eta.$$

3 Some Results for the Half Hexagon

The half hexagon P^+ contains 4 corner points: $\pm 2, \pm 1 + i\sqrt{3}$. A point $z \in P^+$ is reflected through the real axis, the entire set P^+ is reflected to the whole hexagon P. The points $z, \overline{z}$ are reflected across all the sides of P and the new reflection points appear: $-\frac{1}{2}(1+i\sqrt{3})\overline{z}+3+i\sqrt{3}$, $\overline{z}\pm 2i\sqrt{3}$, $-\frac{1}{2}(1-i\sqrt{3})\overline{z}-3+i\sqrt{3}$, $-\frac{1}{2}(1+i\sqrt{3})\overline{z}-3-i\sqrt{3}$, $-\frac{1}{2}(1-i\sqrt{3})\overline{z}+3-i\sqrt{3}$.

Adding the main period $\omega = 3m + i\sqrt{3}n$, $m+n\in 2\mathbb{Z}$, all the reflection points are described in general. Similarly, by substituting these reflection points into formulas (2.1), (2.2), different representation formulas for H^+ are obtained.

Theorem 3.1 *Any $w \in C^1(P^+;\mathbb{C})\cap C(\overline{P^+};\mathbb{C})$ for the half hexagon $P^+\subset\mathbb{C}$ can be represented as*

$$w(z) = \frac{1}{2\pi i}\int_{\partial P^+}w(\zeta)\frac{d\zeta}{\zeta - z}-\frac{1}{\pi}\int_{P^+}w_{\overline{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta - z},\quad z\in P^+, \tag{3.1}$$

and for $k = 1, 2, 3$

$$
\begin{aligned}
w(z) = \frac{1}{2\pi i}\Bigg\{ & \int_{\partial P^+} \operatorname{Re} w(\zeta) 2 \sum_{m+n\in 2\mathbb{Z}} \big[q^k_{mn}(\zeta, z) - q^k_{mn}(\zeta, 0)\big] d\zeta \\
& - \int_{\partial_1 P^+} \left[\operatorname{Re} w(\zeta) \frac{2(2\xi - 3)}{(2\xi - 3)^2 + 3} + \operatorname{Im} w(\zeta) \frac{2\sqrt{3}}{(2\xi - 3)^2 + 3}\right] ds_\zeta \\
& + \int_{\partial_2 P^+} \left[\operatorname{Re} w(\zeta) \frac{2\xi}{\xi^2 + 3} + \operatorname{Im} w(\zeta) \frac{2\sqrt{3}}{\xi^2 + 3}\right] ds_\zeta \\
& - \int_{\partial_3 P^+} \left[\operatorname{Re} w(\zeta) \frac{2(2\xi + 3)}{(2\xi + 3)^2 + 3} + \operatorname{Im} w(\zeta) \frac{2\sqrt{3}}{(2\xi + 3)^2 + 3}\right] ds_\zeta \\
& + \int_{\partial_4 P^+} \operatorname{Re} w(\zeta) \frac{2}{\xi} ds_\zeta \Bigg\} \\
& - \frac{1}{\pi} \int_{P^+} \Bigg\{ w_{\overline{\zeta}}(\zeta) \bigg(\sum_{m+n\in 2\mathbb{Z}} \big[q^k_{mn}(\zeta, z) - q^k_{mn}(\zeta, 0)\big] + \frac{1}{\zeta} \bigg) \\
& - \overline{w_{\overline{\zeta}}(\zeta)} \bigg(\sum_{m+n\in 2\mathbb{Z}} \big[q^k_{mn}(\overline{\zeta}, z) - q^k_{mn}(\overline{\zeta}, 0)\big] + \frac{1}{\overline{\zeta}} \bigg) \Bigg\} d\xi d\eta, \\
\xi = \operatorname{Re} \zeta, &
\end{aligned}
\tag{3.2}
$$

where $\partial_k P^+$, $k = 1, 2, 3, 4$, *are the four boundary segments of* P^+ *and*

$$
\begin{aligned}
q^1_{mn}(\zeta, z) &= \frac{3(\zeta - \omega_{mn} - 2)^2}{(\zeta - \omega_{mn} - 2)^3 - (z - 2)^3}, \\
q^3_{mn}(\zeta, z) &= \frac{3(\zeta - \omega_{mn} + 2)^2}{(\zeta - \omega_{mn} + 2)^3 - (z + 2)^3}, \\
q^2_{mn}(\zeta, z) &= \frac{3(\zeta - \omega_{mn} + 1 - i\sqrt{3})}{(\zeta - \omega_{mn} + 1 - i\sqrt{3})^3 - (z + 1 - i\sqrt{3})^3}.
\end{aligned}
$$

These formulas are equivalent and used for solving the Schwarz problem.

Theorem 3.2 *The Schwarz problem*

$$
\begin{aligned}
& w_{\overline{z}} = f \text{ in } P^+,\ f \in L_p\big(P^+; \mathbb{C}\big),\ p > 2, \\
& \operatorname{Re} w = \gamma \text{ on } \partial P^+,\ \gamma \in C\big(\partial P^+; \mathbb{C}\big), \\
& \gamma(\zeta) = 0 \text{ for } \zeta \in \{\pm 2, \pm 1 + i\sqrt{3}\},
\end{aligned}
\tag{3.3}
$$

$$
-\frac{1}{\pi i} \int_{\partial_1 P^+} \operatorname{Im} w(\zeta) \frac{\sqrt{3}}{(2\xi - 3)^2 + 3} ds_\zeta + \frac{1}{\pi i} \int_{\partial_2 P^+} \operatorname{Im} w(\zeta) \frac{\sqrt{3}}{\xi^2 + 3} ds_\zeta
$$

$$-\frac{1}{\pi i}\int_{\partial_3 P^+} \operatorname{Im} w(\zeta)\frac{\sqrt{3}}{(2\xi+3)^2+3}ds_\zeta = c \quad for\ c \in \mathbb{R} \tag{3.4}$$

is uniquely solvable by

$$\begin{aligned} w(z) = \frac{1}{\pi i}\Big\{\int_{\partial P^+}\gamma(\zeta)\sum_{m+n\in 2\mathbb{Z}} 2\big[q^k_{mn}(\zeta,z)-q^k_{mn}(\zeta,0)\big]d\zeta \\ -\int_{\partial_1 P^+}\gamma(\zeta)\frac{2\xi-3}{(2\xi-3)^2+3}ds_\zeta + \int_{\partial_2 P^+}\gamma(\zeta)\frac{\xi}{\xi^2+3}ds_\zeta \\ -\int_{\partial_3 P^+}\gamma(\zeta)\frac{2\xi+3}{(2\xi+3)^2+3}ds_\zeta + \int_{\partial_4 P^+}\gamma(\zeta)\frac{1}{\xi}ds_\zeta\Big\} + ic \\ -\frac{1}{\pi}\int_{P^+}\Big\{w_{\overline{\zeta}}(\zeta)\Big(\sum_{m+n\in 2\mathbb{Z}}\big[q^k_{mn}(\zeta,z)-q^k_{mn}(\zeta,0)\big]+\frac{1}{\zeta}\Big) \\ -\overline{w_{\overline{\zeta}}(\zeta)}\Big(\sum_{m+n\in 2\mathbb{Z}}\big[q_{mn}(\overline{\zeta},z)-q_{mn}(\overline{\zeta},0)\big]+\frac{1}{\overline{\zeta}}\Big)\Big\}d\xi d\eta \\ for\ k = 1,2,3. \end{aligned} \tag{3.5}$$

The proof and full explanation are given in [13].

The method allows to use the reflection points to construct the harmonic Green function. It turns out, that for the half hexagon it has three forms and used according to the boundary conditions.

$$G_1(z,\zeta) = \log\Big|\prod_{m+n\in 2\mathbb{Z}}\frac{(z-\omega_{mn}-2)^3-(\overline{\zeta}-2)^3}{(z-\omega_{mn}-2)^3-(\zeta-2)^3}\Big|^2 \text{ for the right-hand side.}$$

$$G_1(z,\zeta) = \log\Big|\prod_{m+n\in 2\mathbb{Z}}\frac{(z-\omega_{mn}+2)^3-(\overline{\zeta}+2)^3}{(z-\omega_{mn}+2)^3-(\zeta+2)^3}\Big|^2 \text{ for the left-hand side.}$$

For the upper boundary

$$G_1(z,\zeta) = \log\prod_{m+n\in 2\mathbb{Z}}\Big|\frac{(z-\omega_{mn}+1-i\sqrt{3})^3-(\overline{\zeta}+1+i\sqrt{3})^3}{(z-\omega_{mn}+1-i\sqrt{3})^3-(\zeta+1-\sqrt{3})^3}\Big|^2. \tag{3.6}$$

Applying the Green representation formula and observing the Poisson kernel, the related harmonic Dirichlet problem is solved explicitly, see [13].

Theorem 3.3 *The Dirichlet problem*

$$\begin{aligned} w_{z\overline{z}} = f\ in\ P^+,\ w = \gamma\ on\ \partial P^+\ for\ f \in L_p\big(P^+;\mathbb{C}\big), \\ 2 < p,\ \gamma \in C\big(\partial P^+;\mathbb{C}\big) \end{aligned} \tag{3.7}$$

is uniquely solvable in the space $W^{2,p}(P^+;\mathbb{C})\cap C(\overline{P^+};\mathbb{C})$ *by*

$$w(z) = -\frac{1}{4\pi}\int_{\partial P^+}\gamma(\zeta)\partial_{\nu_\zeta}G_1(z,\zeta)ds_\zeta - \frac{1}{\pi}\int_{P^+}f(\zeta)G_1(z,\zeta)d\xi d\eta \quad (3.8)$$

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Harmonic Dirichlet Problem in a Ring Sector

Ying Wang and Jinyuan Du

Abstract In this paper, we construct a harmonic Green function by reflection method in a general ring sector with angle $\theta = \frac{\pi}{\alpha}$ and $\alpha \geq \frac{1}{2}$, then the related harmonic Dirichlet problem for the Poisson equation is discussed explicitly.

Keywords Green function · Dirichlet problem

Mathematics Subject Classification (2010) 31A30 · 30E25

1 Introduction

Many kinds of concrete applied problems led to the investigation of boundary value problems for complex partial differential equations in different domains [1–6]. In [2], the authors studied harmonic boundary value problems in half disc and half ring. Also a harmonic Dirichlet problem was investigated in a quarter ring domain [3]. In this article, we extend some results to a general domain. Firstly we shall give a harmonic Green function based on conformal mapping in a ring sector with angle $\theta = \frac{\pi}{\alpha}$, $\alpha \geq \frac{1}{2}$ (when $\alpha = 1$ or $\alpha = 2$, it is the cases in [2, 3] respectively), and then discuss a related Dirichlet problem for the Poisson equation explicitly.

Let $\Omega = \{0 < r < |z| < 1,\ 0 < \arg z < \frac{\pi}{\alpha},\ \alpha \geq \frac{1}{2}\}$ be a ring sector with angle $\theta = \frac{\pi}{\alpha}$ $(\alpha \geq \frac{1}{2})$, and its boundary $\partial\Omega = [r, 1] \cup l_1 \cup [\varpi, \omega] \cup l_2$ is oriented counterclockwise, where r, 1, $\varpi = e^{i\theta}$, $\omega = re^{i\theta}$ are corner points and the oriented circular

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arc l_1, l_2 are parameterized by, respectively

$$l_1: \tau \longmapsto e^{i\tau}, \quad \tau \in \left[0, \frac{\pi}{\alpha}\right]; \qquad l_2: \tau \longmapsto re^{i\tau}, \quad \tau \in \left[\frac{\pi}{\alpha}, 0\right].$$

From [2], the harmonic Green function for the upper half ring is

$$G_1(z,\zeta) = \log\left|\frac{(1-z\overline{\zeta})(\overline{\zeta}-z)}{(\zeta-z)(1-z\zeta)}\prod_{n=1}^{\infty}\frac{(z-r^{2n}\overline{\zeta})(z\overline{\zeta}-r^{2n})(\overline{\zeta}-r^{2n}z)(1-r^{2n}z\overline{\zeta})}{(z-r^{2n}\zeta)(z\zeta-r^{2n})(\zeta-r^{2n}z)(1-r^{2n}z\zeta)}\right|^2$$

Then the harmonic Green function for Ω can be expressed by

$$G(z,\zeta) = \log\left|\frac{(1-z^\alpha\overline{\zeta}^\alpha)(\overline{\zeta}^\alpha - z^\alpha)}{(\zeta^\alpha - z^\alpha)(1-z^\alpha\zeta^\alpha)}\prod_{n=1}^{\infty}\frac{(r^{2n\alpha}\overline{\zeta}^\alpha - z^\alpha)(z^\alpha\overline{\zeta}^\alpha - r^{2n\alpha})}{(r^{2n\alpha}\zeta^\alpha - z^\alpha)(z^\alpha\zeta^\alpha - r^{2n\alpha})}\right.$$
$$\left.\times\prod_{n=1}^{\infty}\frac{(\overline{\zeta}^\alpha - r^{2n\alpha}z^\alpha)(1-r^{2n\alpha}z^\alpha\overline{\zeta}^\alpha)}{(\zeta^\alpha - r^{2n\alpha}z^\alpha)(1-r^{2n\alpha}z^\alpha\zeta^\alpha)}\right|^2. \tag{1.1}$$

Similarly in [3], the infinite product in (1.1) converges for $z, \zeta \in \Omega$. By simple computation, we know the harmonic Green function $G(z,\zeta)$ also satisfies $G(z,\zeta) = 0$ for $z \in \partial\Omega$.

The outward normal derivatives on the boundary $\partial\Omega$ are defined as

$$\partial_{\nu_z} = \begin{cases} z\partial_z + \overline{z}\partial_{\overline{z}}, & z \in l_1, \\ -\frac{1}{r}(z\partial_z + \overline{z}\partial_{\overline{z}}), & z \in l_2, \\ -i(\partial_z - \partial_{\overline{z}}), & z \in (r, 1), \\ i\left(e^{i\theta}\partial_z - e^{-i\theta}\partial_{\overline{z}}\right), & z \in (\varpi, \omega). \end{cases} \tag{1.2}$$

2 Harmonic Dirichlet Problem

Define a new function

$$\begin{aligned} H(z,\zeta) &= \frac{1}{\zeta^\alpha - z^\alpha} - \frac{1}{\zeta^\alpha - \overline{z^\alpha}} + \frac{\overline{z^\alpha}}{1-\overline{z^\alpha}\zeta^\alpha} - \frac{z^\alpha}{1-z^\alpha\zeta^\alpha} \\ &+ \sum_{n=1}^{\infty}\left[\frac{r^{2\alpha n}}{r^{2\alpha n}\zeta^\alpha - z^\alpha} - \frac{r^{2\alpha n}}{r^{2\alpha n}\zeta^\alpha - \overline{z^\alpha}} - \frac{z^\alpha}{r^{2\alpha n} - z^\alpha\zeta^\alpha} + \frac{\overline{z^\alpha}}{r^{2\alpha n} - \overline{z^\alpha}\zeta^\alpha}\right] \\ &- \sum_{n=1}^{\infty}\left[\frac{1}{r^{2\alpha n}z^\alpha - \zeta^\alpha} - \frac{1}{r^{2\alpha n}\overline{z^\alpha} - \zeta^\alpha} - \frac{r^{2\alpha n}z^\alpha}{r^{2\alpha n}z^\alpha\zeta^\alpha - 1} + \frac{r^{2\alpha n}\overline{z^\alpha}}{r^{2\alpha n}\overline{z^\alpha}\zeta^\alpha - 1}\right]. \end{aligned}$$

Then we have the following results.

Lemma 2.1 *If* $\gamma \in C(\partial\Omega;\mathbb{C})$, *then* $\lim_{z\in\Omega,z\to 1} \frac{\alpha}{2\pi i}\int_{\partial\Omega}\gamma(\zeta)H(z,\zeta)\zeta^{\alpha-1}\mathrm{d}\zeta = \gamma(1)$.

Proof By simple computation,

$$\begin{aligned}
&\lim_{z\in\Omega,\ z\to 1}\frac{\alpha}{2\pi i}\int_{l_1}\big[\gamma(\zeta)-\gamma(1)\big]H(z,\zeta)\zeta^{\alpha-1}\mathrm{d}\zeta\\
&=\lim_{z\in\Omega,z\to 1}\frac{\alpha}{2\pi i}\int_{l_1}\big[\gamma(\zeta)-\gamma(1)\big]\\
&\quad\times\left[\frac{\zeta^\alpha}{\zeta^\alpha-z^\alpha}-\frac{\zeta^\alpha}{\zeta^\alpha-\overline{z^\alpha}}+\frac{\overline{z^\alpha}\zeta^\alpha}{1-\overline{z^\alpha}\zeta^\alpha}-\frac{z^\alpha\zeta^\alpha}{1-z^\alpha\zeta^\alpha}\right]\frac{\mathrm{d}\zeta}{\zeta}\\
&=\lim_{z\in\Omega,z\to 1}\left\{\frac{1}{2\pi i}\int_{L_1}\big[\gamma\big(\zeta^{\frac{1}{\alpha}}\big)-\gamma(1)\big]\left[\frac{\zeta}{\zeta-z^\alpha}+\frac{\overline{\zeta}}{\overline{\zeta}-\overline{z^\alpha}}-1\right]\frac{\mathrm{d}\zeta}{\zeta}\right.\\
&\quad\left.+\frac{1}{2\pi i}\int_{\widetilde{L_1}}\big[\gamma(1)-\gamma\big(\overline{\zeta}^{\frac{1}{\alpha}}\big)\big]\left[\frac{\zeta}{\zeta-z^\alpha}+\frac{\overline{\zeta}}{\overline{\zeta}-\overline{z^\alpha}}-1\right]\frac{\mathrm{d}\zeta}{\zeta}\right\}\\
&=\lim_{z\in\Omega,z\to 1}\frac{1}{2\pi i}\int_{|\zeta|=1}\Gamma_1(\zeta,1)\left[\frac{\zeta}{\zeta-z^\alpha}+\frac{\overline{\zeta}}{\overline{\zeta}-\overline{z^\alpha}}-1\right]\frac{\mathrm{d}\zeta}{\zeta}\\
&=0,
\end{aligned}$$

where $L_1=\{|\tau|=1,\ \operatorname{Im}\tau>0\}$, $\widetilde{L_1}=\{\tau:\ |\tau|=1,\ \operatorname{Im}\tau<0\}$ are oriented counter-clockwise and

$$\Gamma_1(\zeta,z)=\begin{cases}\gamma\big(\zeta^{\frac{1}{\alpha}}\big)-\gamma(z), & \zeta\in L_1,\\ \gamma(z)-\gamma\big(\overline{\zeta}^{\frac{1}{\alpha}}\big), & \zeta\in\widetilde{L_1}.\end{cases}$$

The last equality of limitation is true from the continuity of $\Gamma_1(\zeta,1)$ at $\zeta=1$ and the property of Poisson operator on the unit circle. Similarly,

$$\begin{aligned}
&\lim_{z\in\Omega,z\to 1}\frac{\alpha}{2\pi i}\int_r^1\big[\gamma(\zeta)-\gamma(1)\big]H(z,\zeta)\zeta^{\alpha-1}\mathrm{d}\zeta\\
&=\lim_{z\in\Omega,z\to 1}\frac{\alpha}{2\pi i}\int_r^1\big[\gamma(\zeta)-\gamma(1)\big]\\
&\quad\times\left[\frac{\zeta^{\alpha-1}}{\zeta^\alpha-z^\alpha}-\frac{\zeta^{\alpha-1}}{\zeta^\alpha-\overline{z^\alpha}}+\frac{\overline{z^\alpha}\zeta^{\alpha-1}}{1-\overline{z^\alpha}\zeta^\alpha}-\frac{z^\alpha\zeta^{\alpha-1}}{1-z^\alpha\zeta^\alpha}\right]\mathrm{d}\zeta\\
&=\lim_{z\in\Omega,z\to 1}\frac{1}{2\pi i}\int_{r^\alpha}^{r^{-\alpha}}\Gamma_2(\zeta,1)\left[\frac{1}{\zeta-z^\alpha}-\frac{1}{\zeta-\overline{z^\alpha}}\right]\mathrm{d}\zeta=0
\end{aligned}$$

with the last equality above derives from the property of Poisson kernel and

$$\Gamma_2(\zeta, z) = \begin{cases} \gamma\big(\zeta^{\frac{1}{\alpha}}\big) - \gamma(z), & \zeta \in \big(r^\alpha, 1\big), \\ \gamma(z) - \gamma\big(\zeta^{-\frac{1}{\alpha}}\big), & \zeta \in \big(1, r^{-\alpha}\big). \end{cases}$$

Additionally, $H(z, \zeta) = 0$ for $(z, \zeta) \in \{1\} \times \{(\varpi, \omega) \cup l_2\}$, then

$$\lim_{z\in\Omega, z\to 1} \frac{\alpha}{2\pi i} \int_{(\varpi,\omega)\cup l_2} \big[\gamma(\zeta) - \gamma(1)\big] H(z, \zeta)\zeta^{\alpha-1} \mathrm{d}\zeta = 0.$$

From the above discussion, we obtain

$$\begin{aligned} &\lim_{z\in\Omega, z\to 1} \frac{\alpha}{2\pi i} \int_{\partial\Omega} \gamma(\zeta) H(z, \zeta)\zeta^{\alpha-1} \mathrm{d}\zeta \\ &= \lim_{z\in\Omega, z\to 1} \frac{\alpha}{2\pi i} \int_{\partial\Omega} \big\{\big[\gamma(\zeta) - \gamma(1)\big] + \gamma(1)\big\} H(z, \zeta)\zeta^{\alpha-1} \mathrm{d}\zeta \\ &= \gamma(1) \quad \left(\text{by } \frac{\alpha}{2\pi i} \int_{\partial\Omega} H(z, \zeta)\zeta^{\alpha-1} \mathrm{d}\zeta \equiv 1 \text{ when } z \in \Omega\right). \end{aligned}$$

Therefore, the lemma is true. □

Lemma 2.2 *With* $\gamma(\zeta) \in C(\partial\Omega; \mathbb{C})$,

$$\lim_{z\in\Omega,\ z\to\varpi} \frac{\alpha}{2\pi i} \int_{\partial\Omega} \gamma(\zeta) H(z, \zeta)\zeta^{\alpha-1} \mathrm{d}\zeta = \gamma(\varpi).$$

Proof Obviously,

$$\begin{aligned} &\lim_{z\in\Omega, z\to\varpi} \frac{\alpha}{2\pi i} \int_{l_1} \big[\gamma(\zeta) - \gamma(\varpi)\big] H(z, \zeta)\zeta^{\alpha-1} \mathrm{d}\zeta \\ &= \lim_{z\in\Omega, z\to\varpi} \frac{\alpha}{2\pi i} \int_{l_1} \big[\gamma(\zeta) - \gamma(\varpi)\big] \\ &\quad \times \left[\frac{\zeta^{\alpha-1}}{\zeta^\alpha - z^\alpha} - \frac{\zeta^{\alpha-1}}{\zeta^\alpha - \overline{z^\alpha}} + \frac{\overline{z^\alpha}\zeta^{\alpha-1}}{1 - \overline{z^\alpha}\zeta^\alpha} - \frac{z^\alpha\zeta^{\alpha-1}}{1 - z^\alpha\zeta^\alpha}\right] \mathrm{d}\zeta \\ &= \lim_{z\in\Omega, z\to\varpi} \frac{1}{2\pi i} \int_{|\zeta|=1} \Gamma_1(\zeta, \varpi) \left[\frac{\zeta}{\zeta - z^\alpha} + \frac{\overline{\zeta}}{\overline{\zeta} - \overline{z^\alpha}} - 1\right] \frac{\mathrm{d}\zeta}{\zeta} \\ &= 0. \end{aligned}$$

By the same way,

$$\lim_{z\in\Omega, z\to\varpi} \frac{\alpha}{2\pi i} \int_{\varpi}^{\omega} \big[\gamma(\zeta) - \gamma(\varpi)\big] H(z, \zeta)\zeta^{\alpha-1} \mathrm{d}\zeta$$

$$= \lim_{z\in\Omega, z\to\varpi} \frac{1}{2\pi i}\int_{-r^{-\alpha}}^{-r^{\alpha}} \Gamma_3(\zeta,\varpi)\left[\frac{1}{\zeta - z^{\alpha}} - \frac{1}{\zeta - \overline{z^{\alpha}}}\right]\mathrm{d}\zeta$$
$$= 0,$$

where

$$\Gamma_3(\zeta, z) = \begin{cases} \gamma\left(\zeta^{\frac{1}{\alpha}}\right) - \gamma(z), & \zeta \in \left(-1, -r^{\alpha}\right), \\ \gamma(z) - \gamma\left(\zeta^{-\frac{1}{\alpha}}\right), & \zeta \in \left(-r^{-\alpha}, -1\right). \end{cases}$$

Moreover, for $(z, \zeta) \in \{\varpi\} \times \{(r, 1) \cup l_2\}$, $H(z, \zeta) = 0$, then

$$\lim_{z\in\Omega, z\to\varpi} \frac{\alpha}{2\pi i}\int_{(r,1)\cup l_2} \left[\gamma(\zeta) - \gamma(\varpi)\right] H(z,\zeta)\zeta^{\alpha-1}\mathrm{d}\zeta = 0,$$

so,

$$\lim_{z\in\Omega,\ z\to\varpi} \frac{\alpha}{2\pi i}\int_{\partial\Omega} \gamma(\zeta)H(z,\zeta)\zeta^{\alpha-1}\mathrm{d}\zeta = \gamma(\varpi).$$

Now the proof is completed. □

Lemma 2.3 *With* $\gamma(\zeta) \in C(\partial\Omega; \mathbb{C})$,

$$\lim_{z\in\Omega,\ z\to\omega} \frac{\alpha}{2\pi i}\int_{\partial\Omega} \gamma(\zeta)H(z,\zeta)\zeta^{\alpha-1}\mathrm{d}\zeta = \gamma(\omega).$$

Proof We see

$$\lim_{z\in\Omega, z\to\omega} \frac{\alpha}{2\pi i}\int_{l_2} \left[\gamma(\zeta) - \gamma(\omega)\right] H(z,\zeta)\zeta^{\alpha-1}\mathrm{d}\zeta$$
$$= \lim_{z\in\Omega, z\to\omega} \frac{\alpha}{2\pi i}\int_{l_2} \left[\gamma(\zeta) - \gamma(\omega)\right]$$
$$\times \left[\frac{\zeta^{\alpha}}{\zeta^{\alpha} - z^{\alpha}} - \frac{\zeta^{\alpha}}{\zeta^{\alpha} - \overline{z^{\alpha}}} + \frac{r^{2\alpha}}{r^{2\alpha} - \overline{z^{\alpha}}\zeta^{\alpha}} - \frac{r^{2\alpha}}{r^{2\alpha} - z^{\alpha}\zeta^{\alpha}}\right]\frac{\mathrm{d}\zeta}{\zeta}$$
$$= -\lim_{z\in\Omega,\ z\to\omega} \frac{1}{2\pi i}\int_{|\zeta|=r^{\alpha}} \Gamma_4(\zeta,\omega)\left[\frac{\zeta}{\zeta - z^{\alpha}} + \frac{\overline{\zeta}}{\overline{\zeta} - \overline{z^{\alpha}}} - 1\right]\frac{\mathrm{d}\zeta}{\zeta}$$
$$= 0,$$

where $L_2 = \{z : |z| = r^{\alpha},\ \operatorname{Im} z > 0\}$, $\widetilde{L_2} = \{z : |z| = r^{\alpha},\ \operatorname{Im} z < 0\}$ are oriented counter-clockwise and

$$\Gamma_4(\zeta, z) = \begin{cases} \gamma\left(\zeta^{\frac{1}{\alpha}}\right) - \gamma(z), & \zeta \in L_2, \\ \gamma(z) - \gamma\left(\overline{\zeta}^{\frac{1}{\alpha}}\right), & \zeta \in \widetilde{L_2}. \end{cases}$$

Also, we have

$$
\begin{aligned}
&\lim_{z\in\Omega,\ z\to\omega}\frac{\alpha}{2\pi i}\int_{\varpi}^{\omega}\big[\gamma(\zeta)-\gamma(\omega)\big]H(z,\zeta)\zeta^{\alpha-1}\mathrm{d}\zeta\\
&=\lim_{z\in\Omega,\ z\to\omega}\frac{1}{2\pi i}\int_{-r^{-\alpha}}^{-r^{\alpha}}\Gamma_3(\zeta,\omega)\left[\frac{1}{\zeta-z^{\alpha}}-\frac{1}{\zeta-\overline{z^{\alpha}}}\right]\mathrm{d}\zeta\\
&=0.
\end{aligned}
$$

In addition, $H(z,\zeta)=0$ for $(z,\zeta)\in\{\omega\}\times\{(r,1)\cup l_1\}$, then

$$
\lim_{z\in\Omega, z\to\omega}\frac{\alpha}{2\pi i}\int_{(r,1)\cup l_1}\big[\gamma(\zeta)-\gamma(\omega)\big]H(z,\zeta)\zeta^{\alpha-1}\mathrm{d}\zeta=0.
$$

Similar as in Lemma 2.1, the result is true. □

Lemma 2.4 *If* $\gamma\in C(\partial\Omega;\mathbb{C})$, *then* $\lim_{z\in\Omega,z\to r}\frac{\alpha}{2\pi i}\int_{\partial\Omega}\gamma(\zeta)H(z,\zeta)\zeta^{\alpha-1}\mathrm{d}\zeta=\gamma(r)$.

Proof We have

$$
\begin{aligned}
&\lim_{z\in\Omega,z\to r}\frac{\alpha}{2\pi i}\int_{l_2}\big[\gamma(\zeta)-\gamma(r)\big]H(z,\zeta)\zeta^{\alpha-1}\mathrm{d}\zeta\\
&=-\lim_{z\in\Omega,z\to r}\frac{1}{2\pi i}\int_{|\zeta|=r^{\alpha}}\Gamma_4(\zeta,r)\left[\frac{\zeta}{\zeta-z^{\alpha}}+\frac{\overline{\zeta}}{\overline{\zeta}-\overline{z^{\alpha}}}-1\right]\frac{\mathrm{d}\zeta}{\zeta}\\
&=0,
\end{aligned}
$$

and

$$
\begin{aligned}
&\lim_{z\in\Omega,z\to r}\frac{\alpha}{2\pi i}\int_{r}^{1}\big[\gamma(\zeta)-\gamma(r)\big]H(z,\zeta)\zeta^{\alpha-1}\mathrm{d}\zeta\\
&=\lim_{z\in\Omega,z\to r}\frac{1}{2\pi i}\int_{r^{2\alpha}}^{1}\Gamma_5(\zeta,r)\left[\frac{1}{\zeta-z^{\alpha}}-\frac{1}{\zeta-\overline{z^{\alpha}}}\right]\mathrm{d}\zeta\\
&=0
\end{aligned}
$$

with

$$
\Gamma_5(\zeta,z)=\begin{cases}\gamma\big(\zeta^{\frac{1}{\alpha}}\big)-\gamma(z), & \zeta\in\big(r^{\alpha},1\big),\\ \gamma(z)-\gamma\big(r^2\zeta^{-\frac{1}{\alpha}}\big), & \zeta\in\big(r^{2\alpha},r^{\alpha}\big).\end{cases}
$$

Moreover, for $(z,\zeta)\in\{r\}\times\{(\varpi,\omega)\cup l_1\}$, $H(z,\zeta)=0$, then

$$
\lim_{z\in\Omega,\ z\to r}\frac{\alpha}{2\pi i}\int_{(\varpi,\omega)\cup l_1}\big[\gamma(\zeta)-\gamma(r)\big]H(z,\zeta)\zeta^{\alpha-1}\mathrm{d}\zeta=0.
$$

As the discussion in Lemma 2.1, the lemma is true. □

Lemma 2.5 *If* $\gamma \in C(\partial\Omega;\mathbb{C})$ *and* $t \in l_1\backslash\{\varpi, 1\}$, *then*

$$\lim_{z\in\Omega, z\to t} \frac{\alpha}{2\pi i}\int_{\partial\Omega} \gamma(\zeta)H(z,\zeta)\zeta^{\alpha-1}\mathrm{d}\zeta = \gamma(t).$$

Proof Let $\gamma^*(\zeta) = \gamma^*(\overline{\zeta})$ $(\zeta \in \widetilde{L_1})$ where $\gamma^*(\zeta) = \gamma(\zeta^{\frac{1}{\alpha}})$ $(\zeta \in L_1)$. Since $H(z,\zeta) = 0$ for $(z,\zeta) \in \{l_1\backslash\{\varpi, 1\}\} \times \{\partial\Omega\backslash l_1\}$, then

$$\begin{aligned}
&\lim_{z\in\Omega, z\to t} \frac{\alpha}{2\pi i}\int_{l_1} \gamma(\zeta)H(z,\zeta)\zeta^{\alpha-1}\mathrm{d}\zeta \\
&= \lim_{z\in\Omega, z\to t} \frac{1}{2\pi i}\int_{L_1} \gamma\big(\zeta^{\frac{1}{\alpha}}\big)\frac{1-|z|^{2\alpha}}{|\zeta - z^\alpha|^2}\frac{\mathrm{d}\zeta}{\zeta} \quad \text{(by simple computation)} \\
&= \lim_{z\in\Omega, z\to t} \frac{1}{2\pi i}\int_{|\zeta|=1} \gamma^*(\zeta)\left[\frac{\zeta}{\zeta - z^\alpha} + \frac{\overline{\zeta}}{\overline{\zeta} - \overline{z^\alpha}} - 1\right]\frac{\mathrm{d}\zeta}{\zeta} \\
&= \gamma(t) \quad \big(\text{by } \gamma^*\big(t^\alpha\big) = \gamma(t) \text{ for } t \in l_1\backslash\{\varpi, 1\}\big).
\end{aligned}$$

This proof is completed. □

Lemma 2.6 *If* $\gamma \in C(\partial\Omega;\mathbb{C})$ *and* $t \in l_2\backslash\{\omega, r\}$, *then*

$$\lim_{z\in\Omega,\ z\to t} \frac{\alpha}{2\pi i}\int_{\partial\Omega} \gamma(\zeta)H(z,\zeta)\zeta^{\alpha-1}\mathrm{d}\zeta = \gamma(t).$$

Proof By $H(z,\zeta) = 0$ for $(z,\zeta) \in \{l_2\backslash\{\omega, r\}\} \times \{\partial\Omega\backslash l_2\}$, then

$$\begin{aligned}
&\lim_{z\in\Omega, z\to t} \frac{\alpha}{2\pi i}\int_{l_2} \gamma(\zeta)H(z,\zeta)\zeta^{\alpha-1}\mathrm{d}\zeta \\
&= \lim_{z\in\Omega, z\to t} \frac{\alpha}{2\pi i}\int_{l_2} \gamma(\zeta)\frac{r^{2\alpha} - |z^\alpha|^2}{|\zeta^\alpha - z^\alpha|^2}\frac{\mathrm{d}\zeta}{\zeta} \\
&= -\lim_{z\in\Omega,\ z\to t} \frac{1}{2\pi i}\int_{L_2} \gamma\big(\zeta^{\frac{1}{\alpha}}\big)\frac{r^{2\alpha} - |z^\alpha|^2}{|\zeta - z^\alpha|^2}\frac{\mathrm{d}\zeta}{\zeta} \\
&= \gamma(t).
\end{aligned}$$

This proof is completed. □

Lemma 2.7 *If* $\gamma(\zeta) \in C(\partial\Omega;\mathbb{C})$ *and* $t \in (r, 1)$, *then*

$$\lim_{z\in\Omega, z\to t} \frac{\alpha}{2\pi i}\int_{\partial\Omega} \gamma(\zeta)H(z,\zeta)\zeta^{\alpha-1}\mathrm{d}\zeta = \gamma(t).$$

Proof Because $H(z,\zeta) = 0$ for $(z,\zeta) \in (r, 1) \times \{\partial\Omega\backslash(r, 1)\}$,

$$\lim_{z\in\Omega, z\to t} \frac{\alpha}{2\pi i}\int_{r}^{1} \gamma(\zeta)H(z,\zeta)\zeta^{\alpha-1}\mathrm{d}\zeta$$

$$
\begin{aligned}
&= \lim_{z\in\Omega, z\to t} \frac{\alpha}{2\pi i}\int_r^1 \gamma(\zeta)\frac{z^\alpha - \overline{z^\alpha}}{|\zeta^\alpha - z^\alpha|^2}\zeta^{\alpha-1}\mathrm{d}\zeta \\
&= \lim_{z\in\Omega, z\to t} \frac{1}{2\pi i}\int_{r^\alpha}^1 \gamma\big(\zeta^{\frac{1}{\alpha}}\big)\frac{z^\alpha - \overline{z^\alpha}}{|\zeta - z^\alpha|^2}\mathrm{d}\zeta \\
&= \gamma(t).
\end{aligned}
$$

This proof is completed. □

Lemma 2.8 *If $\gamma \in C(\partial\Omega; \mathbb{C})$ and $t \in (\varpi, \omega)$, then*

$$
\lim_{z\in\Omega,\ z\to t} \frac{\alpha}{2\pi i}\int_{\partial\Omega} \gamma(\zeta)H(z,\zeta)\zeta^{\alpha-1}\mathrm{d}\zeta = \gamma(t).
$$

Proof Since $H(z,\zeta) = 0$ for $(z,\zeta) \in (\varpi,\omega) \times \{\partial\Omega\backslash(\varpi,\omega)\}$,

$$
\begin{aligned}
&\lim_{z\in\Omega,\ z\to t} \frac{\alpha}{2\pi i}\int_\varpi^\omega \gamma(\zeta)H(z,\zeta)\zeta^{\alpha-1}\mathrm{d}\zeta \\
&= \lim_{z\in\Omega,\ z\to t} \frac{\alpha}{2\pi i}\int_\varpi^\omega \gamma(\zeta)\frac{z^\alpha - \overline{z^\alpha}}{|\zeta^\alpha - z^\alpha|^2}\zeta^{\alpha-1}\mathrm{d}\zeta \\
&= \lim_{z\in\Omega,\ z\to t} \frac{1}{2\pi i}\int_{-1}^{-r^\alpha} \gamma\big(\zeta^{\frac{1}{\alpha}}\big)\frac{z^\alpha - \overline{z^\alpha}}{|\zeta - z^\alpha|^2}\mathrm{d}\zeta \\
&= \gamma(t).
\end{aligned}
$$

This proof is completed. □

Theorem 2.9 [4, 6] *Any $w \in C^2(\Omega; \mathbb{C}) \cap C^1(\overline{\Omega}; \mathbb{C})$ can be represented as*

$$
w(z) = -\frac{1}{4\pi}\int_{\partial\Omega} w(\zeta)\partial_{\nu_\zeta} G(z,\zeta)\mathrm{d}s_\zeta - \frac{1}{\pi}\int_\Omega w_{\zeta\overline{\zeta}}(\zeta)G(z,\zeta)\mathrm{d}\xi\mathrm{d}\eta, \tag{2.1}
$$

where s_ζ is the arc length parameter on $\partial\Omega$ with respect to the variable ζ and $G(z,\zeta)$ is the harmonic Green function for Ω.

Theorem 2.10 *The Dirichlet problem*

$$
\partial_z\partial_{\overline{z}} w = f \ \text{in}\ \Omega,\ \ f \in L_p(\Omega; \mathbb{C}),\ \ p > 2,\ \ w = \gamma \ \text{on}\ \partial\Omega,\ \ \gamma \in C(\partial\Omega; \mathbb{C})
$$

is uniquely solvable by

$$
\begin{aligned}
w(z) = \frac{\alpha}{2\pi i}&\int_{\partial\Omega} \gamma(\zeta)H(z,\zeta)\zeta^{\alpha-1}\mathrm{d}\zeta \\
&- \frac{1}{\pi}\int_\Omega f(\zeta)G(z,\zeta)\mathrm{d}\xi\mathrm{d}\eta, \quad z \in \Omega,
\end{aligned} \tag{2.2}
$$

where G is given in (1.1).

Proof By (1.2), Theorem 2.9 and simple computation, the expression (2.1) is just (2.2) explicitly, thus we only need to verify that (2.2) is a solution. From (2.2) and the property of $G(z, \zeta)$, we easily obtain

$$\partial_z \partial_{\overline{z}} w(z) = -\partial_{\overline{z}} \left\{ \frac{\alpha}{\pi} \int_{\Omega} \frac{\zeta^{\alpha-1} f(\zeta)}{\zeta^{\alpha} - z^{\alpha}} \mathrm{d}\xi \mathrm{d}\eta \right\} = -\partial_{\overline{z}} \left\{ \frac{1}{\pi} \int_{\Omega} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\xi \mathrm{d}\eta \right\} = f(z).$$

In addition, by Lemmas 2.1–2.8 and $G(z, \zeta)$ vanishing for z on the boundary, we get $\lim_{z \to \zeta} w(z) = \gamma(\zeta)$ for $\zeta \in \partial\Omega$. This completes the proof. □

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The Parqueting-Reflection Principle

Heinrich Begehr

Abstract For certain plane domains with boundaries composed by arcs from circles and straight lines the parqueting-reflection principle is used to construct the Schwarz, Green, and Neumann kernels for solving the Schwarz, Dirichlet, and Neumann boundary value problems for the inhomogeneous Cauchy–Riemann and the Poisson equation, respectively.

Keywords Cauchy–Riemann equation · Poisson equation · Schwarz · Green · Neumann representations · Plane domains · Boundary value problems

Mathematics Subject Classification (2010) 35J08 · 31A10 · 31A25 · 30E25 · 35J25 · 35C15

1 Admissible Domains

A domain D in the complex plane $\mathbb{C}$ is admissible for the parqueting-reflection principle if its boundary consists of arcs from circles or straight lines, such that the continued reflections of the domain at the boundary parts lead to a parqueting of $\mathbb{C}$ or of several samples of $\mathbb{C}$.

Circles and straight lines are represented by equations of the form

$$\Gamma = \{\alpha z\overline{z} + \overline{a}z + a\overline{z} + \beta = 0,\ 0 < a\overline{a} - \alpha\beta,\ a \in \mathbb{C},\ \alpha, \beta \in \mathbb{R}\} \qquad (1.1)$$

Definition For $z \in \mathbb{C}$ the point z_r satisfying

$$\alpha z_r\overline{z} + \overline{a}z_r + a\overline{z} + \beta = 0, \qquad (1.2)$$

is called the reflection point of z at Γ.

Obviously a point from Γ is reflected onto itself while the reflection of a point not on Γ lies on the opposite side of Γ.

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Simple examples for admissible domains are discs and half planes where just one reflection provides a parqueting of $\mathbb{C}$, see [3, 7]. Half discs need two reflections, [4], cones of opening angle $\frac{\pi}{n}$, $n \in \mathbb{N}$, $2n$ reflections [1], disc sectors of opening angle $\frac{\pi}{n}$ need $2n+1$ reflections [17]. For rectangles and concentric rings [12–15] as well as for certain triangles [5, 6, 8, 17], ring sectors [12–14, 16, 18], half hexagons [12–14] countably many reflections have to be used. Reflecting cones with opening angle $\frac{m}{n}\pi$, $m < 2n$, $m, n \in \mathbb{N}$, [9], $2n$ times leads to an m-fold covering of $\mathbb{C}$.

In Sect. 3 a hyperbolic plane forming a certain lens [9, 10] is described. Four reflections serve for a parqueting of the plane $\mathbb{C}$. This is also true for the two complementary lunes of this lens.

2 The Principle

Having achieved a parqueting of $\mathbb{C}$ through continued reflections out of an admissible domain D, the Schwarz, Green, and Neumann representation formulas are attained as follows.

1. For the Schwarz formulas expressing an analytic function in D through the boundary values of its real part the Cauchy formula is used for a point $z \in D$ and its continued reflection points which are outside of D. Combining these Cauchy formulas in a proper way avoiding integrals with $\overline{z}$ involved by complex conjugation results in the representation formula aimed for. This method even works with the Cauchy–Pompeiu representation formula related to the inhomogeneous Cauchy–Riemann equation. The Schwarz kernel attained in this way may in cases countable many reflections are involved in the parqueting of $\mathbb{C}$ turn out as an infinite sum which has to be shown to converge.

2. In order to find the harmonic Green function the domains attained through the continued reflection process out of D are divided into two subclasses, the pole and the zero domains. D itself is determined to be a pole domain. Any direct reflection of a pole domain becomes a zero domain and any direct reflection of a zero domain is a pole domain. The point $z \in D$ is taken to be a simple pole of a meromorphic function P in the plane $\mathbb{C}$, the consecutive reflections of $z \in D$ lying in a zero domain is taken to be a simple zero of P, the ones in a pole domain are simple poles for P. Let the variable of P be ζ. P also depends on z as a parameter, $P(\zeta, z)$. If the reflection point of a point $\widehat{z}$ on an arc of a circle (1.1), $\alpha \neq 0$ is $\widehat{z}_r$ given by $\widehat{z}_r = -\frac{a\overline{\widehat{z}}+\beta}{\alpha\overline{\widehat{z}}+\overline{a}}$ then instead of $\zeta - \widehat{z}_r$ in the definition of $P(\zeta, z)$ the term $(\alpha\overline{\widehat{z}} + \overline{a})(\zeta - \widehat{z}_r) = (\alpha\overline{\widehat{z}} + \overline{a})\zeta + (a\overline{\widehat{z}} + \beta)$ is used. The reason is that the right-hand side of the last equation is symmetric in the sense

$$\overline{\alpha\overline{z}\zeta + \overline{a}\zeta + a\overline{z} + \beta} = \alpha z\overline{\zeta} + a\overline{\zeta} + \overline{a}z + \beta$$

for $\alpha, \beta \in \mathbb{R}$, $a, z, \zeta \in \mathbb{C}$. For the reflection on a straight line, i.e. $\alpha = 0$ in (1.1), $\zeta - \widehat{z}_r = \zeta + \frac{a}{\overline{a}}\overline{\widehat{z}} + \frac{\beta}{\overline{a}}$. $P(\zeta, z)$ thus is given as an eventually infinite product of appropriate quotients of linear polynomials in ζ with coefficients depending on the original point $z \in D$. Having asserted the absolute convergence of the product for $\zeta \in D$ then $\log |P(\zeta, z)|$ is the harmonic Green function for D. It may happen, see e.g. [15], that the repeated reflection of D does not perform a complete parqueting of $\mathbb{C}$ when single points are left out. In these cases proper additional factors have to be added to P in order to achieve a proper asymptotic behavior in these points.

3. The Neumann function is found through the same reflection points from $z \in D$. Here all reflected points are used as simple poles of a function meromorphic in the entire plane $\mathbb{C}$. The linear polynomial factors are just those from the Green function. In case of an infinite product here additional proper factors have to be added achieving convergence. These factors should be bounded and bounded away from zero, see e.g. [11, 15].

3 An Example: A Hyperbolic Half Plane

The lens $D = \mathbb{D} \cap D_m(r)$ is a hyperbolic half plane in the Poincaré disc $\mathbb{D} = \{|z| < 1\}$. It is the intersection of $\mathbb{D}$ with the disc $D_m(r) = \{|z - m| < r\}$, $0 < r < 1 < m$, $m^2 = 1 + r^2$. Its boundary ∂D consists of the two arcs $\partial_1 D = \partial\mathbb{D} \cap \overline{D_m(r)}$ and $= \partial D_m(r) \cap \overline{\mathbb{D}}$.

A reflections at a circle or a straight line is a combinations of a linear transformation with complex conjugation. Hence they preserve orthogonality and map circles and straight lines onto circles and straight lines. Therefore the reflection of D at $\partial_1 D$ gives the lune $L_r = D_m(r) \setminus D$ while the reflection of D at $\partial_r D$ results in the lune $L_1 = \mathbb{D} \setminus D$.

The point $z \in D$ is reflected at $\partial\mathbb{D}$ to $\frac{1}{\overline{z}} \in D_m(r) \setminus D$. Both these points $z, \frac{1}{\overline{z}} \in D_m(r)$ are reflected at $\partial D_m(r)$ to the points

$$m + \frac{r^2}{\overline{z} - m} = \frac{\overline{z}m - 1}{\overline{z} - m}, \qquad m + \frac{r^2}{\frac{1}{z} - m} = \frac{m - z}{1 - mz},$$

respectively. They are outside the closure $\overline{D_m(r)}$ of $D_m(r)$, in particular do not belong to D. The reflection of D onto L_r and the reflection of $D_m(r)$ achieves a parqueting of $\mathbb{C}$. The same points are attained if D is firstly reflected onto L_1 and then $\mathbb{D}$ onto its complement.

1. Applying the Cauchy–Pompeiu formula as mentioned in the preceding session gives the Schwarz representation formula [10].

Theorem *Any function $w \in C^1(D; \mathbb{C}) \cap C(\overline{D}; \mathbb{C})$ is representable as*

$$w(z) = \frac{1}{2\pi i} \int_{\partial D \cap \partial\mathbb{D}} \mathrm{Re}\, w(\zeta) \left[\frac{2\zeta}{\zeta - z} - 1 + \frac{2\zeta(1 - mz)}{\zeta(1 - mz) + z - m} - 1 \right] \frac{d\zeta}{\zeta}$$
$$+ \frac{1}{2\pi i} \int_{\partial D \cap \partial D_m(r)} \mathrm{Re}\, w(\zeta) \left[\frac{2(\zeta - m)}{\zeta - z} - 1 + \frac{2[\zeta - m)(1 - mz)}{(\zeta - m)(1 - mz) - zr^2} - 1 \right]$$

$$
\begin{aligned}
&\times \frac{d\zeta}{\zeta - m} \\
&+ \frac{1}{\pi} \int_{\partial D \cap \partial \mathbb{D}} \operatorname{Im} w(\zeta) \frac{d\zeta}{\zeta} + \frac{1}{\pi} \int_{\partial D \cap \partial D_m(r)} \operatorname{Im} w(\zeta) \frac{d\zeta}{\zeta - m} \\
&- \frac{1}{\pi} \int_D \left\{ w_{\overline{\zeta}}(\zeta) \left[\frac{1}{\zeta - z} + \frac{1 - mz}{z - m + \zeta(1 - mz)} \right] \right. \\
&\left. + \overline{w_{\overline{\zeta}}(\zeta)} \left[\frac{z}{1 - z\overline{\zeta}} - \frac{z - m}{\overline{\zeta}(z - m) + 1 - mz} \right] \right\} d\xi d\eta.
\end{aligned}
$$

Proof The Cauchy–Pompeiu formula

$$
\frac{1}{2\pi i} \int_{\partial D} w(\zeta) \frac{d\zeta}{\zeta - z} - \frac{1}{\pi} \int_D w_{\overline{\zeta}}(\zeta) \frac{d\xi d\eta}{\zeta - z} = \begin{cases} w(z), & z \in D, \\ 0, & z \notin \overline{D}, \end{cases}
$$

applied to $z \in D$, $\frac{1}{\overline{z}} \in D_m(r) \setminus \mathbb{D}$, $\frac{\overline{z}m-1}{\overline{z}-m}$, $\frac{m-z}{1-mz} \notin \overline{D_m(r)}$, gives for $z \in D$

$$
\begin{aligned}
\frac{1}{2\pi i} \int_{\partial D} w(\zeta) \frac{d\zeta}{\zeta - z} - \frac{1}{\pi} \int_D w_{\overline{\zeta}}(\zeta) \frac{d\xi d\eta}{\zeta - z} &= w(z), \\
\frac{1}{2\pi i} \int_{\partial D} w(\zeta) \frac{\overline{z} d\zeta}{1 - \overline{z}\zeta} - \frac{1}{\pi} \int_D w_{\overline{\zeta}}(\zeta) \frac{\overline{z} d\xi d\eta}{1 - \overline{z}\zeta} &= 0, \\
\frac{1}{2\pi i} \int_{\partial D} w(\zeta) \frac{(1 - mz) d\zeta}{\zeta(1 - mz) - m + z} - \frac{1}{\pi} \int_D w_{\overline{\zeta}}(\zeta) \frac{(1 - mz) d\xi d\eta}{\zeta(1 - mz) - m + z} &= 0, \\
\frac{1}{2\pi i} \int_{\partial D} w(\zeta) \frac{(\overline{z} - m) d\zeta}{\zeta(\overline{z} - m) + 1 - m\overline{z}} - \frac{1}{\pi} \int_D w_{\overline{\zeta}}(\zeta) \frac{(\overline{z} - m) d\xi d\eta}{\zeta(\overline{z} - m) + 1 - m\overline{z}} &= 0.
\end{aligned}
$$

Taking the complex conjugate of the second and forth formula, where $\overline{z}$ appears and adding the resulting four relations, leads to the claimed representation formula. □

This representation formula serves to solve the Schwarz boundary value problem for the inhomogeneous Cauchy–Riemann equation in the lens D.

2. The function $P(\zeta, z)$ mentioned in Sect. 2 for constructing the Green function for D is a rational function having simple zeroes at the points $\frac{1}{\overline{z}}$, $\frac{\overline{z}m-1}{\overline{z}-m}$ and simple poles at z, $\frac{m-z}{1-mz}$, see [9]. Hence the harmonic Green function for D is

$$
G(z, \zeta) = \log \left| \frac{1 - z\overline{\zeta}}{\zeta - z} \frac{m((\overline{\zeta} + z) - (1 + z\overline{\zeta}))}{\zeta + z - m(1 + z\zeta)} \right|.
$$

For the Poisson kernel on $|z| = 1$

$$
2z\partial_z G(z, \zeta) = \frac{z}{\zeta - z} - \frac{z\overline{\zeta}}{1 - z\overline{\zeta}} - \frac{z - mz\zeta}{\zeta + z - m(1 + z\zeta)} + \frac{z\overline{\zeta} - mz}{1 + z\overline{\zeta} - m(\overline{\zeta} + z)}
$$

gives for the part $\partial_1 D$ of ∂D

$$\partial_{\nu_z} G(z,\zeta) = \mathrm{Re}\left\{\frac{\zeta+z}{\zeta-z} + \frac{\zeta - z - m(1-z\zeta)}{\zeta + z - m(1+z\zeta)}\right\}.$$

Similarly, for $|z-m| = r$

$$\begin{aligned} &2(z-m)\partial_z G(z,\zeta) \\ &\quad = \frac{z-m}{\zeta - z} - \frac{(z-m)\overline{\zeta}}{1-z\overline{\zeta}} - \frac{z-m-m(z-m)\zeta}{\zeta+z-m(1+z\zeta)} + \frac{(z-m)\overline{\zeta} - m(z-m)}{1+z\overline{\zeta} - m(\overline{\zeta}+z)} \end{aligned}$$

shows for the boundary curve $\partial_r D$

$$\partial_{\nu_z} G(z,\zeta) = \mathrm{Re}\left\{\frac{\zeta+z-2m}{\zeta-z} - \frac{1+z\overline{\zeta}-2m\overline{\zeta}}{1-z\overline{\zeta}}\right\}.$$

This can be seen by using the relation

$$1 - mz = m\overline{z} - |z|^2.$$

With these expressions the Poisson representation formula

$$w(z) = -\frac{1}{2\pi}\int_{\partial D} w(\zeta)\partial_{\nu_\zeta} G(z,\zeta)ds_\zeta + \frac{2}{\pi}\int_D w_{\zeta\overline{\zeta}}(\zeta)G(z,\zeta)d\xi d\eta,$$

generally valid for bounded domains with piecewise smooth boundary, holds for D and proper functions w, see e.g. [2]. This representation formula serves to solve the Dirichlet problem for the Poisson equation in D.

3. The Neumann function for the lens D is

$$N(z,\zeta) = -\log\left|(\zeta-z)(1-z\overline{\zeta})\big(\zeta+z-m(1+z\zeta)\big)\big(1+z\overline{\zeta}-m(\overline{\zeta}+z)\big)\right|.$$

On its boundary part $\partial_1 D$ from $\partial\mathbb{D}$

$$\begin{aligned} \partial_{\nu_z} N(z,\zeta) &= 2\,\mathrm{Re}\big\{z\partial_z N(z,\zeta)\big\} \\ &= \mathrm{Re}\left\{\frac{z}{\zeta-z} + \frac{z\overline{\zeta}}{1-z\overline{\zeta}} - \frac{z(1-m\zeta)}{\zeta+z-m(1+z\zeta)} - \frac{z(\overline{\zeta}-m)}{1+z\overline{\zeta}-m(\overline{\zeta}+z)}\right\} \\ &= -2, \end{aligned}$$

for $\zeta \in D$, while for $\zeta \in \partial_1 D$

$$\partial_{\nu_z} N(z,\zeta) = \frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - 3.$$

Similarly, for the boundary part $\partial_r D$ on $\partial D_m(r)$

$$\begin{aligned}
&\partial_{\nu_z} N(z,\zeta) \\
&\quad = 2\,\mathrm{Re}\big\{(z-m)\partial_z N(z,\zeta)\big\} \\
&\quad = \mathrm{Re}\left\{\frac{z-m}{\zeta-z} + \frac{(z-m)\overline{\zeta}}{1-z\overline{\zeta}} - \frac{(z-m)(1-m\zeta)}{\zeta+z-m(1+z\zeta)} - \frac{(z-m)(\overline{\zeta}-m)}{1+z\overline{\zeta}-m(\overline{\zeta}+z)}\right\} \\
&\quad = -2,
\end{aligned}$$

if $\zeta \in D$, and if $\zeta \in \partial_r D$

$$\partial_{\nu_z} N(z,\zeta) = \frac{\zeta-m}{\zeta-z} + \frac{\overline{\zeta}-m}{\overline{\zeta-z}} - 3.$$

Remark If one starts with z in the lune L_1 or L_r the process leads to the same formula for the Schwarz, Green, and Neumann kernels. This phenomenon is well known from the Euclidean half planes or from the inside and outside of discs.

Theorem *For* $f \in L_p(\mathbb{D};\mathbb{C})$, $2 < p$, $\gamma \in C(\partial\mathbb{D};\mathbb{C})$, *the Neumann problem*

$$\partial_z\partial_{\overline{z}} w = f \ \text{in } D, \qquad \partial_\nu w = \gamma \ \text{on } \partial D,$$

is solvable if and only if

$$\frac{1}{2\pi}\int_{\partial D} \gamma(\zeta)ds_\zeta = \frac{2}{\pi}\int_D f(\zeta)d\xi d\eta$$

the solution being then

$$w(z) = c + \frac{1}{2\pi}\int_{\partial D}\gamma(\zeta)N(\zeta,z)ds_\zeta - \frac{2}{\pi}\int_D f(\zeta)N(\zeta,z)d\xi d\eta,$$

with

$$c = -\frac{1}{2\pi}\int_{\partial D} w(\zeta)\partial_{\nu_\zeta}N(\zeta,z)ds_\zeta = \frac{1}{\pi}\int_{\partial D} w(\zeta)ds_\zeta.$$

The proof follows from the Neumann representation formula [2]

$$\begin{aligned}
w(z) = &-\frac{1}{2\pi}\int_{\partial D} w(\zeta)\partial_{\nu_\zeta}N(\zeta,z)ds_\zeta + \frac{1}{2\pi}\int_{\partial D}\partial_{\nu_\zeta}w(\zeta)N(\zeta,z)ds_\zeta \\
&-\frac{2}{\pi}\int_D \partial_\zeta\partial_{\overline{\zeta}}w(\zeta)N(\zeta,z)d\xi d\eta.
\end{aligned}$$

Let $w(z)$ be the function defined by the formula in the theorem. On $\partial_1 D$ then

$$\partial_\nu w(z) = \frac{1}{2\pi}\int_{\partial_1 D}\gamma(\zeta)\left[\frac{\zeta}{\zeta-z} + \frac{\overline{\zeta}}{\overline{\zeta-z}} - 1\right]ds_\zeta$$

$$-\frac{1}{\pi}\int_{\partial D}\gamma(\zeta)ds_\zeta+\frac{4}{\pi}\int_D f(\zeta)d\xi d\eta$$

and on $\partial_r D$

$$\partial_\nu w(z)=\frac{1}{2\pi}\int_{\partial_r D}\gamma(\zeta)\left[\frac{\zeta-m}{\zeta-z}+\frac{\overline{\zeta}-m}{\overline{\zeta-z}}-1\right]ds_\zeta$$
$$-\frac{1}{\pi}\int_{\partial D}\gamma(\zeta)ds_\zeta+\frac{4}{\pi}\int_D f(\zeta)d\xi d\eta.$$

From the properties of the Poisson kernels the boundary behavior is seen when observing the solvability condition. For proving

$$\frac{1}{\pi}\int_{\partial D}w(\zeta)ds_\zeta=c$$

the property

$$\int_{\partial D}N(z,\zeta)ds_\zeta=0$$

would be needed.

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On Existence of the Resolvent and Discreteness of the Spectrum of a Class of Differential Operators of Hyperbolic Type

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Abstract The existence and compactness of the resolvent and discreteness of the spectrum of some hyperbolic differential operators are studied in this paper. One of the main results is the criterion of discreteness of the spectrum of a hyperbolic singular differential operator.

Keywords Spectrum · Resolvent · Singular differential operator · Hyperbolic type

Mathematics Subject Classification (2010) Primary 47A10 · Secondary 35L81

1 Problem Statement and the Main Results

Singular differential operators, for example operators defined in an unbounded domain, in general may have not only a discrete but also a continuous spectrum. Therefore in general an arbitrary function cannot be decomposed into a series of eigen functions. For this reason the most important problem in the study of the spectrum in dependence of the behavior of the coefficients in the case of an unbounded domain is the discreteness of the spectrum.

Spectral characteristics of singular elliptic differential operators are well studied and the typical difficulties encountered in connection with bad behaving coefficients clarified. An extensive literature is devoted to their study and we mention [1–3].

Review of the literature shows that such questions as: (1) the existence and compactness of the resolvent, (2) the discreteness of the spectrum of hyperbolic differential operators defined in an unbounded domain are not well studied.

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We consider in the space $L_2(\Omega)$ the differential operator of hyperbolic type

$$A_0 u = u_{xx} - u_{yy} + a(y)u_x + c(y)u$$

with the domain $D(A_0)$ of infinitely differentiable functions satisfying the conditions $u(-\pi; y) = u(\pi; y)$, $u_x(-\pi; y) = u_x(\pi; y)$ and compactly supported with respect to the variable y, where

$$\Omega = \big\{(x, y):\ -\pi < x < \pi,\ -\infty < y < \infty\big\}.$$

Further, we assume that the coefficients $a(y)$, $c(y)$ satisfy the conditions:

(i) $|a(y)| \geq \delta_0 > 0$, $c(y) \geq \delta > 0$ are continuous functions in $R = (-\infty; \infty)$.

It is easy to verify that the operator A_0 admits a closure in the space $L_2(\Omega)$, which is denoted by A.

We note that the operator A corresponds to the problem of propagation of the boundary regime (see [4], p. 106), i.e. the problem without initial conditions. Here the term au_x describes the friction force. The question of the existence of solutions of the problem without initial conditions, in general, depends on the behavior of the coefficients a and c. For example, when $a = 0$, a solution does not always exist.

The main results of this paper are the following theorems.

Theorem 1.1 *Let the condition* (*i*) *be fulfilled. Then the operator* $A + \lambda I$ *is continuously invertible for* $\lambda \geq 0$.

Theorem 1.2 *Let the condition* (*i*) *be fulfilled. Then the resolvent of the operator* A *is compact if and only if for any* $w > 0$

$$\lim_{|y|\to\infty} \int_y^{y+w} c(t)dt = \infty. \qquad (*)$$

The last theorem shows that the condition ($*$) is a necessary and sufficient condition for the discreetness of the spectrum of A.

The question of the existence of the resolvent and discreet spectrum in an unbounded domain with growing and oscillating coefficients was previously studied only in the case of elliptic and pseudodifferential operators [1–3].

Assume that the coefficients of the operator A, in addition to conditions (i), satisfy the condition

(ii)

$$\mu_0 = \sup_{|y-t|\leq 1} \frac{c(y)}{c(t)} < \infty, \qquad \mu = \sup_{|y-t|\leq 1} \frac{a(y)}{a(t)} < \infty.$$

Then, Theorem 1.2 easily implies the following theorem.

Theorem 1.3 *Let the conditions* (i)–(ii) *be fulfilled. Then the resolvent of the operator* A *is compact if and only if* $\lim_{|y|\to\infty} c(y) = \infty$.

2 Auxiliary Lemmas and Inequalities

The following statements below are given without proofs, because computations and arguments that have been used in [5] are similar for their proofs.

Lemma 2.1 *Let the condition* (i) *be fulfilled and* $\lambda \geq 0$. *Then the inequality*

$$\left\|(A+\lambda I)u\right\|_2 \geq c\|u\|_2, \tag{2.1}$$

holds for all $u \in D(A)$, *where* $c = c(\delta, \delta_0)$ *and* $\|\cdot\|_2$ *is the norm in* $L_2(\Omega)$.

Let $\Delta_j = (j-1, j+1)$ $(j \in Z)$, and γ be a constant such that $\gamma a(y) > 0$. Denote by $l_{n,j,\gamma} + \lambda I$ the closure in $L_2(\Delta_j)$ of the differential expression $(l_{n,j,\gamma} + \lambda I)u = -u'' + [-n^2 + in(a(y)+\gamma) + c(y) + \lambda]u$, $(n = 0, \pm 1, \pm 2, \ldots)$ defined on the set $C_0^2(\overline{\Delta}_j)$ of twice continuously differentiable functions u on $\bar{\Delta}_j$ which satisfy the conditions $u(j-1) = u(j+1) = 0$.

Lemma 2.2 *Let the condition* (i) *be fulfilled and* $\lambda \geq 0$. *Then the following inequalities*

(a)

$$\begin{aligned}
&\left\|(l_{n,\gamma,j} + \lambda I)u\right\|_{L_2(\Delta_j)} \\
&\quad \geq c_1\left(\left\|u'\right\|_2 + \left\|\sqrt{c(y)+\lambda}\,u\right\|_{L_2(\Delta_j)} + \left\||n|\sqrt{\left(\left|a(y)\right| + |\gamma|\right)}u\right\|_{L_2(\Delta_j)}\right), \\
&\quad n \neq 0,\ u \in D(l_{n,\gamma,j} + \lambda I);
\end{aligned}$$

(b)

$$\left\|(l_{n,j,\gamma} + \lambda I)^{-1}\right\|_{L_2(\Delta_j)\to L_2(\Delta_j)} \leq \frac{c_0}{(\delta+\lambda)^{1/2}};$$

(c)

$$\left\|\frac{d}{dy}(l_{n,j,\gamma} + \lambda I)^{-1}\right\|_{L_2(\Delta_j)\to L_2(\Delta_j)} \leq \frac{c_2}{(\delta+\lambda)^{1/4}}$$

hold, where $c_0 = c_0(\delta)$, $c_1 = c_1(\delta)$, $c_2 = c_2(\delta)$.

Lemma 2.3 *The operator* $l_{n,j,\gamma} + \lambda I$ *is invertible for* $\lambda \geq 0$ *and the inverse operator* $(l_{n,j,\gamma} + \lambda I)^{-1}$ *is defined in all* $L_2(\Delta_j)$, $j \in Z$, *where* Z *is the set of entire numbers*.

By $l_{n,\gamma} + \lambda I$ $(n = 0, \pm 1, \pm 2, \ldots)$ we denote the closure in $L_2(R)$ $(R = (-\infty, \infty))$ of the differential expression $(l_{n,\gamma} + \lambda I)u = -u'' + (-n^2 + in(a(y) + \gamma) + c(y) + \lambda)u$, defined on the set $C_0^\infty(R)$ of infinitely differentiable functions with compact support.

Lemma 2.4 *Let $\lambda \geq 0$ and condition* (i) *hold. Then for any $u \in D(l_{n,\gamma} + \lambda I)$ the following estimates*

$$\left\|(l_{0,\gamma} + \lambda I)u\right\|_{L_2(R)} \geq \sqrt{\delta + \lambda}\|u\|_{L_2(R)},$$

$$\left\|(l_{n,\gamma} + \lambda I)u\right\|_{L_2(R)} \geq |n|\bigl(\delta_0 + |\gamma|\bigr)\|u\|_{L_2(R)}$$

hold for $n \neq 0$.

Lemma 2.4 is proved by transforming the expression $\langle (l_{n,\gamma} + \lambda I)u, -inu\rangle$, where $u \in C_0^\infty(R)$.

Let $\{\varphi_j(y)\}_{j=-\infty}^{+\infty} \subset C_0^\infty(R)$ be a sequence of functions satisfying the conditions $\varphi_j \geq 0$, $\operatorname{supp}\varphi_j \subseteq \Delta_j (j \in Z)$, $\sum_{j=-\infty}^{+\infty} \varphi_j^2(y) = 1$. Assume

$$cK_{\lambda,\gamma} f = \sum_{j=-\infty}^{+\infty} \varphi_j (l_{n,j,\gamma} + \lambda I)^{-1} \varphi_j f,$$

$$B_{\lambda,\gamma} f = \sum_{j=-\infty}^{+\infty} \varphi_j''(l_{n,j,\gamma} + \lambda I)^{-1}\varphi_j f + 2\sum_{j=-\infty}^{+\infty} \varphi_j' \frac{d}{dy}(l_{n,j,\gamma} + \lambda I)^{-1}\varphi_j f,$$

$$f \in C_0^\infty(R), \quad \lambda \geq 0.$$

Obviously,

$$(l_{n,\gamma} + \lambda I)K_{\lambda,\gamma} f = f - B_{\lambda,\gamma} f. \tag{2.2}$$

Lemma 2.5 *Let the condition* (i) *be fulfilled. Then there exists a number $\lambda_0 > 0$ such that $\|B_{\lambda,\gamma}\|_{L_2(R)\to L_2(R)} < 1$ for all $\lambda \geq \lambda_0$.*

Lemma 2.6 *Let the condition* (i) *be satisfied. Then the operator $l_{n,\gamma} + \lambda I$ is continuously invertible for $\lambda \geq \lambda_0 > 0$, and for the inverse operator $(l_{n,\gamma} + \lambda I)^{-1}$ the following equality*

$$(l_{n,\gamma} + \lambda I)^{-1} = K_{\lambda,\gamma}(I - B_{\lambda,\gamma})^{-1} \tag{2.3}$$

holds.

Lemma 2.6 follows from (2.2) and from Lemmas 2.5 and 2.4.

Lemma 2.7 *Let the condition* (i) *be satisfied and $\rho(y)$ be a continuous function defined on R. Then for $\alpha = 0, 1$ and $\lambda \geq \lambda_0$ the following estimate*

$$\begin{aligned} &\left\|\rho(y)|n|^\alpha (l_{n,\gamma} + \lambda I)^{-1}\right\|^2_{L_2(R)\to L_2(R)} \\ &\quad \leq c_4(\lambda) \sup_{j\in Z} \left\|\rho(y)|n|^\alpha \varphi_j (l_{n,j,\gamma} + \lambda I)^{-1}\right\|^2_{L_2(\Delta_j)\to L_2(\Delta_j)} \end{aligned} \tag{2.4}$$

holds.

The result below follows from Lemma 2.2 and the estimate (2.4).

Lemma 2.8 *Let the condition* (i) *be satisfied and* $\lambda \geq \lambda_0$. *Then*

(a) $\|\sqrt{c(y)+\lambda}(l_{n,\gamma}+\lambda I)^{-1}\|_{L_2(R)\to L_2(R)} < \infty$ $(n = 0, \pm 1, \pm 2, \ldots)$;
(b) $\|in(l_{n,\gamma}+\lambda I)^{-1}\|_{L_2(R)\to L_2(R)} < \infty$ $(n \neq 0)$;
(c) $\|\frac{d}{dy}(l_{n,\gamma}+\lambda I)^{-1}\|_{L_2(R)\to L_2(R)} < \infty$ $(n = 0, \pm 1, \pm 2, \ldots)$.

Consider the equation

$$(l_n + \lambda I)u \equiv -u'' + \big(-n^2 + ina(y) + c(y) + \lambda\big)u = f, \tag{2.5}$$

where $f \in L_2(R)$.

The function $u \in L_2(R)$ is called a solution of (2.5) if there exists a sequence $\{u_n\}_{n=1}^{\infty} \subset C_0^{\infty}(R)$ such that $\|u_n - u\|_{L_2(R)} \to 0$, $\|(l_n + \lambda I)u_n - f\|_{L_2(R)} \to 0$ as $n \to \infty$.

Lemma 2.9 *The operator* $l_n + \lambda I$ $(n = 0, \pm 1, \pm 2, \ldots)$ *is boundedly invertible for* $\lambda \geq \lambda_0$, *and for the inverse operator* $(l_n + \lambda I)^{-1}$ *the equality*

$$(l_n + \lambda I)^{-1} f = (l_{n,\gamma} + \lambda I)^{-1}(I - A_{\lambda,\gamma})^{-1} f, \quad f \in L_2(R) \tag{2.6}$$

holds, where $\|A_{\lambda,\gamma}\|_{L_2(R)\to L_2(R)} < 1$.

Lemma 2.8 and the equality (2.6) imply the following lemma.

Lemma 2.10 *If* $\lambda \geq \lambda_0$, *then the estimates*

(a) $\|\sqrt{c(y)+\lambda}(l_n+\lambda I)^{-1}\|_{L_2(R)\to L_2(R)} < \infty$ $(n = 0, \pm 1, \pm 2, \ldots)$;
(b) $\|in(l_n+\lambda I)^{-1}\|_{L_2(R)\to L_2(R)} < \infty (n \neq 0)$;
(c) $\|\frac{d}{dy}(l_n+\lambda I)^{-1}\|_{L_2(R)\to L_2(R)} < \infty$ $(n = 0, \pm 1, \pm 2, \ldots)$ *hold.*

We will use also the following well-known lemma [6, p. 350].

Lemma 2.11 *Let the operator* $A + \lambda_0 I$ $(\lambda_0 > 0)$ *be boundedly invertible in* $L_2(R)$ *and the estimate* $\|(A + \lambda I)u\|_{L_2(R)} \geq c\|u\|_{L_2(R)}$, $u \in D(A + \lambda I)$ *hold for* $\lambda \in (0, \lambda_0]$. *Then the operator* $A : L_2(R) \to L_2(R)$ *is boundedly invertible also.*

Lemma 2.12 *Let the condition* (i) *be fulfilled and* $\lambda > 0$. *Then the inequality*

$$\big\|(l_n + \lambda I)^{-1}\big\|_{L_2(R)\to L_2(R)} \leq \frac{1}{|n| \cdot \delta_0} \tag{2.7}$$

holds for all n $(n = 0, \pm 1, \pm 2, \ldots)$.

Lemma 2.13 *Let the condition* (i) *be fulfilled and* $\lambda > 0$. *Then the operator* $(l_n + \lambda I)^{-1}$ *is completely continuous for all* n $(n = 0, \pm 1, \pm 2, \ldots)$ *if and only if for any*

$\omega > 0$

$$\lim_{|y|\to\infty} \int_y^{y+\omega} c(t)dt = \infty. \tag{$*$}$$

3 Proofs of Theorems 1.1–1.3

Proof of Theorem 1.1 From Lemma 2.9 we obtain that

$$u_k(x, y) = \sum_{n=-k}^{k} (l_n + \lambda I)^{-1} f_n(y) e^{inx} \tag{3.1}$$

is a solution of the problem

$$(A + \lambda I)u_k(x, y) = f_k(x, y),$$
$$u_k(-\pi, y) = u_k(\pi, y), \qquad u_{kx}(-\pi, y) = u_{kx}(\pi, y),$$

where $f_k(x, y) \xrightarrow{L_2} f(x, y)$, $f_k(x, y) = \sum_{n=-k}^{k} f_n(y)e^{inx}$, $(l_n + \lambda I)^{-1}$ is the inverse operator to the operator $(l_n + \lambda I)$.

By virtue of (2.1) we have

$$\|u_k(x, y)\|_2 \le c\|f_k(x, y)\|_2, \tag{3.2}$$

where $c > 0$ is a constant independent of k.

Since $f_k \xrightarrow{L_2} f$, then from (3.2) we find

$$\|u_k - u_m\|_2 \le c\|f_k - f_m\|_2 \to 0 \quad \text{as } k, m \to \infty.$$

Hence, by virtue of the completeness of the space $L_2(\Omega)$, it follows that there exists a unique function $u \in L_2(\Omega)$ such that

$$u_k \to u \quad \text{as } k \to \infty. \tag{3.3}$$

(3.3) implies that for any $f \in L_2(\Omega)$

$$u(x, y) = (A + \lambda I)^{-1} f(x, y) = \sum_{n=-\infty}^{\infty} (l_n + \lambda I)^{-1} f_n(y) e^{inx} \tag{3.4}$$

is a strong solution of the problem

$$(A + \lambda I)u = f \tag{3.5}$$

$$u(-\pi, y) = u(\pi, y), \qquad u_x(-\pi, y) = u_x(\pi, y) \tag{3.6}$$

Let us recall the definition of a strong solution.

The function $u \in L_2(\Omega)$ is called a strong solution of (3.5)–(3.6), if there exists a sequence $\{u_k\}_{k=1}^{\infty} \subset D(L_0)$ such that $\|u_k - u\|_2 \to 0$ and $\|(A+\lambda I)u_k - f\|_2 \to 0$ as $k \to \infty$.

Now, it is easy to see that (3.4) is the inverse operator to the closed operator $A + \lambda I$. Lemma 2.1 implies that the last statement holds for all $\lambda \geq 0$. Theorem 1.1 is proved. □

Proof of Theorem 1.2 Using Lemma 2.12 it is easy to see that

$$\lim_{|n|\to\infty} \left\| (l_n + \lambda I)^{-1} \right\|_{2\to 2} = 0.$$

Therefore, and using the ε-net, from (3.4) we have that the operator $(A + \lambda I)^{-1}$ is compact if and only if $(l_n + \lambda I)^{-1}$ is continuous. Now, the proof of the theorem follows from Lemma 2.13. □

Proof of Theorem 1.3 Without loss of generality we assume $0 < w \leq 1$, then by the condition (ii) we have

$$\mu_0^{-1} \cdot w \cdot c(y) \leq \int_y^{y+w} c(t)dt \leq \mu_0 \cdot w \cdot c(y).$$

The proof of Theorem 1.3 follows from this inequality and Theorem 1.2. □

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On the Singularities of the Emden–Fowler Type Equations

Radosław Antoni Kycia and Galina Filipuk

Abstract We study the Emden–Fowler type equations and their analytic solutions at the origin. We explain the structure of movable singularities of these solutions and visualize them numerically.

Keywords Lane–Emden equation · Emden–Fowler equation · Movable singularities · Nonlinear ODE

Mathematics Subject Classification (2010) Primary 34A34 · Secondary 34A25

1 Introduction

The Emden–Fowler equation

$$\frac{d^2u(x)}{dx^2} + \frac{\alpha}{x}\frac{du(x)}{dx} + \delta x^n u(x)^p = 0 \tag{1.1}$$

has many application in physics [1, 2, 5, 8]. Throughout this paper we assume that $\alpha > 0$ and $\delta \neq 0$ are real parameters, n is an integer such that $n > -2$ and $p > 1$ is a natural number. Note that when $p = 1$, (1.1) is linear and can be integrated using the Bessel functions. When $n = 0$ and $\delta = 1$ (1.1) is called the Lane–Emden equation and it also has many important applications [2].

Equation (1.1) has two fixed singularities at $x = 0$ and $x = \infty$. In practice it is important to know a solution for the prescribed initial data at $x = 0$ with high precision. As it will be shown below (see also [8] for the Lane–Emden equation), there exists a power series solution at the origin which is convergent in a finite circle centred at the origin in the complex plane because of the existence of singularities on

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the boundary of this circle. These singularities move when we change the initial data and/or the parameters of the equation. Such singularities are called movable singularities (for the reference on singularities of ordinary differential equations (ODEs) see [7, 9]). The generalized isothermal sphere equation (including the isothermal sphere equation for $n = 1$)

$$\frac{d^2u(x)}{dx^2} + \frac{\alpha}{x}\frac{du(x)}{dx} - \delta x^n e^{-u(x)} = 0 \tag{1.2}$$

possesses a similar structure of singularities. This equation can be regarded as a limit of (1.1) as p tends to infinity.

The paper is organized as follows. In the next section we shall obtain an analytic solution around $x = 0$ of (1.1) and discuss the relations and the symmetries of (1.1) and (1.2). Then we shall prove, by generalizing the results from [8] for the Lane–Emden equation, that movable singularities of the analytic solution of (1.1) exist in the finite complex plane. We shall also visualize them using numerical methods.

2 Analytic Solutions of the Generalized Emden–Fowler Equation

It is straightforward to show that (1.1) has a local analytic solution near $x = 0$ of the form

$$u(x) = \sum_{k=0}^{\infty} a_k x^k, \tag{2.1}$$

with

$$a_0 = c, \qquad a_1 = \cdots = a_{n+1} = 0,$$
$$a_{k+n+2} = -\frac{\delta c_k}{(k+n+2)(k+n+1+\alpha)}, \qquad k \geq 0,$$

where the coefficients $\{c_k\}_{k=0}^{\infty}$ are derived from $\{a_k\}_{k=0}^{\infty}$ using the known Cauchy product formula [6]

$$\left(\sum_{l=0}^{\infty} a_l (x - x_0)^l\right)^p = \sum_{l=0}^{\infty} c_l (x - x_0)^l,$$
$$c_0 = a_0^p, \qquad c_m = \frac{1}{m a_0}\sum_{l=1}^{m}(lp - m + l)a_l c_{m-l}, \quad m > 0. \tag{2.2}$$

Here c is an arbitrary parameter corresponding to the initial data $u(0) = c$.

To get the recurrence relation (2.1) for the coefficients in the series we substitute it into (1.1) and use (2.2). The proof of the fact that this formal solution has a nonzero radius of convergence relies on Proposition 1 from [3] for the system

$$\begin{cases} xu' = xv \\ xv' = -\alpha v - x^{n+1}\delta u^p, \end{cases} \tag{2.3}$$

which is equivalent to (1.1). The assumption $\alpha > 0$ is essential in the application of the Proposition 1 of [3].

We remark that we can study analytic solutions at $x = \infty$ of (1.1) and (1.2) using the symmetry

$$x \to 1/x, \qquad n \to -(n+4), \qquad \alpha \to 2-\alpha, \qquad \delta \to \delta. \tag{2.4}$$

One can note that $n = -2$, $\alpha = 1$ is a fixed point of this transformation. Moreover, for $n = -2$ there is no non-trivial series solution. For $n < -2$ by duality (2.4) we get an asymptotic series in powers of $1/x$. In this sense one can imagine that by changing n the singularities $x = 0$ and $x = \infty$ swap as we cross the value $n = -2$.

A similar study of analytic solutions can be performed for (1.2). This generalizes results from [8] (see Eqs. (7) and (8) therein).

Equation (1.1) is invariant under the following change of variables:

$$u(x) = a^{k_2} u_1(x_1), \qquad x_1 = a^{k_1} x, \qquad k_2(p-1) = k_1(n+2), \tag{2.5}$$

where a is arbitrary. In particular, we can scale the initial data from $u(0) = c$ to $u(0) = 1$.

The parameter δ is not essential in (1.1) and (1.2). For instance, by changing the variable $u(x) \to \delta^{-1/(p-1)} u(x)$ in (1.1), we get the same equation with $\delta = 1$.

By using a linear transformation $u(x) \to Au(x) + pB$, where A, B are constants and then using rescaling (2.5) we can bring (1.1) to the form

$$\frac{d^2u(x)}{dx^2} + \frac{\alpha}{x}\frac{du(x)}{dx} - x^n\left(1 - \frac{u(x)}{p}\right)^p = 0, \tag{2.6}$$

which leads to (1.2) in the limit $p \to \infty$ (see [8]). Similarly, by shifting $u \to u - c$ in (1.2) and by using (2.5), we can change the arbitrary initial data $u(0) = c$ to the normalized ones $u(0) = 0$.

In the next section we shall study the radius of convergence of the power series, or, alternatively, study the existence of movable singularities of the solutions.

3 Singularities in the Complex Plane

In this section we shall prove the existence of movable singularities of the solution (2.1) of (1.1). This result can be regarded as a generalization of the results from [8] for the Lane–Emden equation.

Theorem 3.1 *A nonzero analytic solution* (2.1) *of* (1.1) *has* $n+2$ *singularities located symmetrically with respect to the origin on the rays connecting the origin with all* $(n+2)$ *roots of* -1 *in the complex plane.*

Proof Using (2.5) we can assume that $u(0)=1$. Moreover, we can also assume that $\delta=1$. The proof is as follows. First, using the change of variables $z=x^{n+2}$, we are interested in a singularity for $z<0$. Next, by introducing new variables we study a vector field of an autonomous system of equations and show that it gives a singularity for a finite negative $\bar{z}$, which is equivalent to the fact that the singularities in the x variable are symmetric with respect to the origin and are located on the rays connecting the origin with all $(n+2)$ roots of -1 in the complex plane. The strategy of the proof is similar to [8] for the Lane–Emden equation.

Introducing a new variable $z=x^{n+2}$ in (1.1) gives

$$(n+2)^2 z u'' + (n+2)(n+1+\alpha)u' + u^p = 0, \quad ' = d/dz. \tag{3.1}$$

Setting

$$k(z) = \frac{-u^p}{(n+2)u'}, \qquad l(z) = \frac{-(n+2)zu'}{u}, \tag{3.2}$$

(3.1) can be written as follows

$$\frac{dl}{dk} = \frac{l(l+k+1-\alpha)}{k(n+1+\alpha-k-pl)}. \tag{3.3}$$

Using (3.2), the series solution (2.1) around $z=0$ with $u(0)=1$ corresponds to the series expansions for $k(z)$ and $l(z)$ with $k(0)=n+1+\alpha$, $l(0)=0$. The existence of a singularity ($u\to\infty$) for $z<0$ means that $u''(z)>0$, i.e., the function is convex. From the equation

$$u'' = \frac{l(n+1+\alpha-k)u}{(n+2)^2 z^2} \tag{3.4}$$

it occurs when $l<0$ and $k>n+1+\alpha$ ($u>0$ and $u'<0$).

To prove that the singularity is located at finite $z<0$ we employ the fact that the flow of (3.3) keeps the solution which starts initially from the analytic solution (2.1) below the critical line $n+1+\alpha-k-pl=0$. See Fig. 1 for details. It follows that

$$k < n+1+\alpha-pl. \tag{3.5}$$

Since $l<0$, we obtain

$$\frac{k}{-l} < \frac{n+1+\alpha}{-l} + p. \tag{3.6}$$

As $l\to-\infty$ one can find such $z_*<0$ that for $z<z_*$ we have $-l>n+1+\alpha$. For such a z_* the variable k is also finite. Moreover, from $kl=u^{p-1}z$ and $kl^p=$

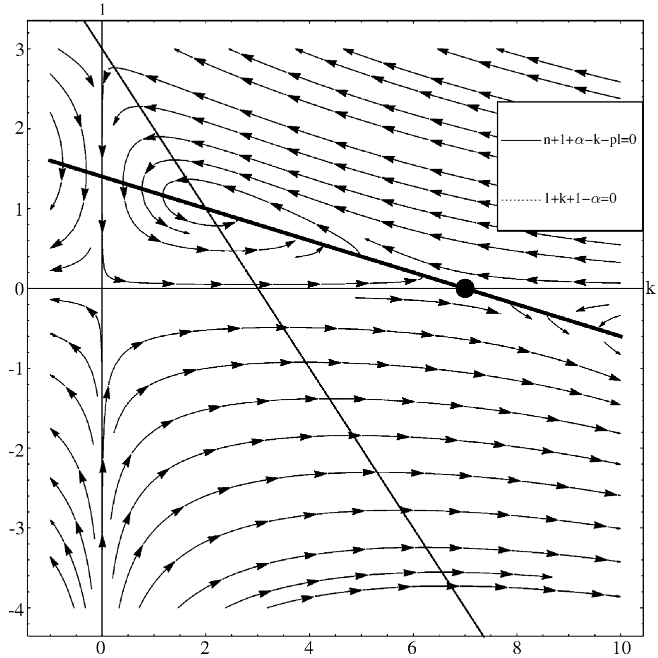

Fig. 1 The vector field (3.3) for $\alpha = 4$, $\delta = 1$, $p = 5$ and $n = 2$. The lines $l + k + 1 - \alpha = 0$, $n + 1 + \alpha - k - pl = 0$ and the axes $l = 0$, $k = 0$ are the critical lines for the vector field (3.3). The *bold point* represents the initial data at $x = 0$ ($z = 0$). The series solutions (2.1) for small $z < 0$ corresponds to the point slightly below this point in the region where the flow keeps the solution below the line $n + 1 + \alpha - k - pl = 0$ and forces it to the limit $k \to \infty$, $l \to -\infty$

$(n+2)^{p-1}(-z)^p(u')^{p-1}$ we see that we have no singularity at this point. Therefore, from (3.2) we get

$$\frac{k}{-l} = \frac{u^{p+1}}{(n+2)^2(-z)(u')^2} < p + 1. \tag{3.7}$$

Integrating from z_* ($u(z_*) = u_*$) to some $\bar{z} < z^*$ at which we assume that $\lim_{z\to\bar{z}} u(z) = \infty$, we obtain a bound on $\bar{z}$ (note that $u' < 0$)

$$\sqrt{-\bar{z}} < \frac{(n+2)\sqrt{p+1}}{p-1} u_*^{(1-p)/2} + \sqrt{-z_*}, \tag{3.8}$$

from which we conclude that the singularities are located at $n + 2$ finite points in the complex plane (i.e., $x = (-|\bar{z}|)^{1/(n+2)}$). □

It is clear that the singular points move along the rays as the initial value $u(0) = c$ changes, which is obvious from the first equation in (2.5) when we set $a = c$.

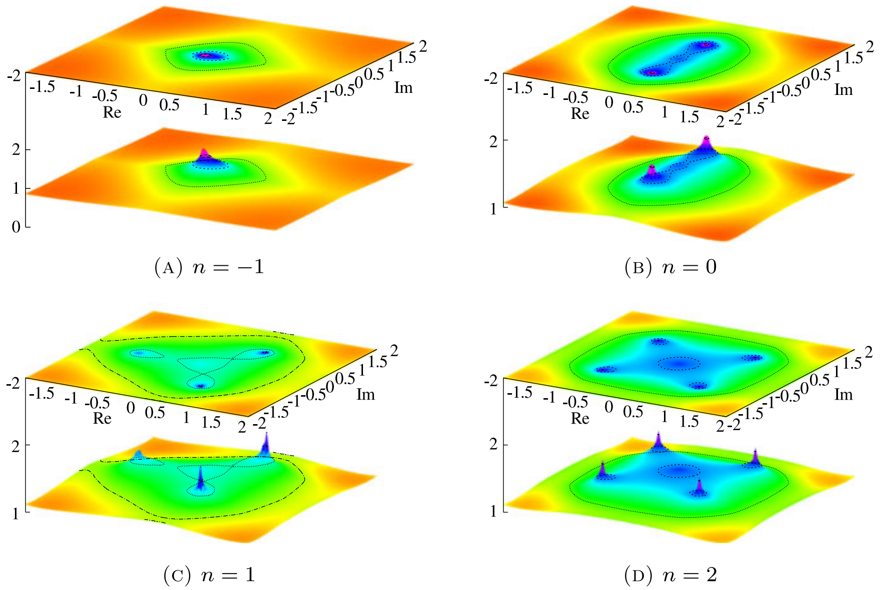

Fig. 2 The plots of $|u(x)|$ of the analytic solution in the complex plane of (1.1) with $\alpha = 3$, $\delta = 1$, $p = 5$ and $c = 1.5$ for the initial data in (2.1)

A similar proof can be given for (1.2) using

$$k(z) = \frac{e^{-u}}{(n+2)u'}, \qquad l(z) = (n+2)zu' \tag{3.9}$$

instead of (3.2). See also [8].

The plots of $|u(x)|$ of the analytic solution (2.1) in the complex plane are presented in Fig. 2. Similar plots can be prepared for the analytic solutions of (1.2). One can see a symmetry of the location of singularities, which is due to the transformation $x \to x^{1/(n+2)}$ mapping (3.1) with only one singularity on the negative axis into (1.1).

The plots were obtained by integrating numerically the analytic initial data from the vicinity of the origin along uniformly distributed (in angle) rays in the complex plane emanating from the origin. The complex version of the classical (4th order) Runge–Kutta method [9] was used. If the modulus of the solution is greater than a prescribed constant value, then it is assumed that the singularity is in the proximity of that point, the integration is stopped and the next ray is chosen until scans along all directions are performed. In this example there was no danger of shadowing singularities on the single ray by the first encountered singularity, however, in more elaborated examples one should prepare the plots for different initial points and check if there is some additional structure of singularities hidden by this shadowing.

4 Discussion

In this paper we showed how to find analytic solutions of the Emden–Fowler type equations. The singularities of these solutions are studied both analytically and numerically.

We can see that when $n \to \infty$ in (1.1) we obtain the so-called natural boundary phenomenon like for the series $\sum_{j=0}^{\infty} x^{2^j}$ when $|x| \to 1^-$ (e.g., see [4]), when the series cannot be extended outside of its circle of convergence because the boundary of this circle consists of an infinite number of singularities.

Equations (1.1) or (1.2) can also be used to construct equations with singularities located at arbitrarily chosen points in the complex plane by choosing an appropriate transformation. To this end one has to construct a mapping of the complex plane which transforms the singular points of these equations into the points of an arbitrary choice. A similar method was proposed for the generation of solutions for equations which can be transformed into (1.1) or (1.2), see [1].

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Differential Equations with Degenerated Variable Operator at the Derivative

B.V. Loginov, Y.B. Rousak, and L.R. Kim-Tyan

Abstract The theory of Jordan chains for multiparameter operator-functions $A(\lambda)$: $E_1 \to E_2$, $\lambda \in \Lambda$, $\dim \Lambda = k$, $\dim E_1 = \dim E_2 = n$ is developed. Here $A_0 = A(0)$ is a degenerated operator, $\dim \operatorname{Ker} A_0 = 1$, $\operatorname{Ker} A_0 = \{\varphi\}$, $\operatorname{Ker} A_0^* = \{\psi\}$ and the operator-function $A(\lambda)$ is supposed to be linear in λ. Applications to degenerate differential equations of the form $[A_0 + R(\cdot, x)]x' = Bx$ are given.

Keywords Jordan chains · Degenerated operator · Degenerated differential equations

Mathematics Subject Classification (2010) 34G20 · 47J15

1 Introduction

Below degenerated differential equations (DE) of the form

$$A(x)x' = G(x) \tag{1.1}$$

are considered. If the opposite is not stipulated, it is supposed $A(x), G(x)$: $E_1 \to E_2$, $\dim E_1 = \dim E_2 = n$, $A(0) = A_0$ is a degenerated operator with

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$\dim \operatorname{Ker} A_0 = \dim \operatorname{Ker} A_0^* = 1$, $\operatorname{Ker} A_0 = N(A_0) = \{\varphi\}$, $\operatorname{Ker} A_0^* = N(A_0^*) = \{\psi\}$. The function $G(x)$ is sufficiently smooth and $G(0) = 0$, $G(x) = Bx - H(x)$, $H(0) = 0$, $H'(0) = 0$. Our aim here is first of all to determine under which conditions the operator $A(x)$ will be nondegenerated in some neighborhood of the point $x = 0$, degenerated in some neighborhood of $x = 0$, or degenerated on some submanifold in a neighborhood of $x = 0$. On this way the theory of Jordan chains for the degenerated operator-functions will be developed (Sects. 2, 3) and applied to DE of the form (4.1), Sect. 4. In the definition of the Jordan chain a slightly more generic case will be studied when the operator-function is linear in the parameter λ.

$$A(\lambda) = A_0 + DA(0)\lambda : E_1 \to E_2, \quad \lambda \in \Lambda, \ \dim \Lambda = k \tag{1.2}$$

depends linearly on the parameter λ acting from Λ into E_1 space.

The case of a polynomial (or analytical) operator-function $A(\lambda)$ will be considered in an extended variant of the article. In the examples provided in Sect. 2 of this article the case $\Lambda = E_1$ will often be considered. Section 4 contains applications to degenerated differential equations.

2 Jordan Chains of Multiparameter Operator-Functions

Let the function $A(\lambda)$ be linear in λ, i.e. $A(\lambda) = A_0 + DA(0)\lambda$, and $DA(0)$ be a mapping from a neighborhood of $0 \in \Lambda$ to the space of square $n \times n$-matrices. The following construction defines a Jordan chain (further JCh) for the operator-function (1.2).

Lemma 2.1 *For the mapping* (1.2) *being non invertible in some neighborhood of* $\lambda = 0$ *the existence of some function* $h(\lambda) : U(0) \to E_1$ *defined in some neighborhood of zero* (*or on a submanifold of* $U(0)$) *is necessary and sufficient, such that* $[A_0 + DA(0)\lambda]h(\lambda) = 0$.

Proof For sufficiently smooth $h(\lambda) = \varphi + Dh(0)\lambda + D^2h(0)\lambda^2 + \cdots + D^s h(0)\lambda^s + \cdots$ where $D^s h(0)$ is an s-linear symmetric operator acting from Λ into E_1 or, that is the same a linear operator acting from $\Lambda \otimes \cdots \otimes \Lambda = \bigotimes_s \Lambda$ into E_1 one has $0 = A_0\varphi + [A_0 Dh(0)\lambda + (DA(0)\lambda)(\varphi)] + \cdots + [A_0 D^s h(0)\lambda^s + (DA(0))\lambda(D^{s-1}h(0)\lambda^{s-1})] + \cdots$ $DA(0)$ in the expression $A_0 Dh(0)\lambda + (DA(0)\lambda)(\varphi)$ can be considered as a bilinear operator of two variables and since the second variable has constant value φ, it presents some known operator acting on λ, $B_1\lambda$, i.e. $A_0 Dh(0)\lambda + (DA(0)\lambda)(\varphi) = [A_0 Dh(0) + B_1]\lambda$. Thus, since $Dh(0) \in L\{\Lambda \to E_1\}$, the operator A_0 generates the operator B_1 acting from the space $L\{\Lambda \to E_1\}$ into the space $L\{\Lambda \to E_2\}$ according to the rule: if $S \in L\{\Lambda \to E_1\}$, then $B_1 S = -A_0 S$.

In order that $S \in \operatorname{Ker} B_1$ it is necessary and sufficient that $\operatorname{Im} S \in \{\varphi\}$. Consequently $\dim \operatorname{Ker} B_1 = k$, and since $\dim \Lambda = k$, then there exist exactly k linearly independent operators S such that $\operatorname{Im} S = \{\varphi\}$. Let the vectors $\xi_1, \ldots, \xi_k$ form a basis in Λ, then a basis in $\operatorname{Ker} B_1$ is composed by the operators $\{\Phi_i \mid \Phi_i \xi_s = \delta_{is}\varphi\}$. The equality $\dim L\{\Lambda \to E_1\} = \dim L\{\Lambda \to E_2\}$ implies that $\dim \operatorname{coKer} B_1 = k$. Since the space $L\{\Lambda \to E_1\}$ is isomorphic to the space $\Lambda^* \otimes E_1$ (designation $L\{\Lambda \to E_1\} \approx \Lambda^* \otimes E_1$), then $L\{\Lambda \to E_2\}^* \approx \Lambda \otimes E_2^*$ and the operators $\Psi_i = \xi_i \otimes \psi$ in the space $L\{\Lambda \to E_2\}^*$ form a basis in the space $\operatorname{coKer} B_1$. Thus for the solvability of the equation $A_0 Dh(0) = -B_1$ the realization of the following equalities $\langle B_1, \Psi_i\rangle = 0$, $i = 1, \dots, n$ is necessary and sufficient.

In this case the operator $Dh(0)$ is defined up to linear combinations of the operators Φ_i.

Suppose by induction that the operator $D^{s-1}h(0)$ is determined and consider the equation

$$A_0 D^s h(0)\lambda^s + \big(DA(0)\lambda\big)\big(D^{s-1}h(0)\lambda^{s-1}\big) = \big[A_0 D^s h(0) + B_s\big]\lambda^s. \tag{2.1}$$

The operator A_0 generates the operator B_s acting from the space $L\{\Lambda \otimes \cdots \otimes \Lambda \to E_1\} = L\{\otimes_s \Lambda \to E_1\}$ into the space $L\{\bigotimes_s \Lambda \to E_2\}$ according to the rule: if $S \in \{\bigotimes_s \Lambda \to E_1\}$, then $B_s S = A_0 S$. Since the space $L\{\bigotimes_s \Lambda \to E_1\}$ is isomorphic to $\bigotimes_s \Lambda^* \otimes E_1$ (represented by the elements of the space), then $\dim \operatorname{Ker} B_s = k^s$ and a basis in $\operatorname{Ker} B_s$ composed by the operators $\Phi_{i_1 \cdots i_s}$, can be constructed in the following manner: let the vectors $\xi_1, \dots, \xi_k$ form the basis in Λ and $\xi_1^*, \dots, \xi_k^*$ be the biorthogonal basis in Λ^*, the vectors $e_1, \dots, e_n$ $(u_1, \dots, u_n)$ form the basis in E_1 (resp. E_2), where for definiteness $e_1 = \varphi$ $(u_1^* = \psi)$, then the vectors $\Phi_{i_1 \cdots i_s}$ are given by the formulae $\Phi_{i_1 \cdots i_s} = \xi_{i_1}^* \otimes \cdots \otimes \xi_{i_s}^* \otimes \varphi$ and $B_s \Phi_{i_1 \cdots i_s} = 0$. In an analogous manner define a basis of $\operatorname{coKer} B_s \subset [L\{\bigotimes_s \Lambda \longrightarrow E_2\}]^* \approx \bigotimes_s \Lambda \otimes E_2^*$ as linearly independent vectors $\Psi_{j_1 \cdots j_s} = \xi_{j_1} \otimes \cdots \otimes \xi_{j_s} \otimes \psi$ forming the space of dimension k^s. In fact, since $S \in L\{\bigotimes_s \Lambda \to E_1\} \approx \bigotimes_s \Lambda^* \otimes E_1$, then S can be represented in the form $S = \sum a_{j_1 \cdots j_s} \bigotimes_{k=1}^s \xi_{j_k}^* \otimes e_j$ and therefore $B_s S = \sum a_{j_1 \cdots j_s} \bigotimes_1^s \xi_{j_k}^* \otimes A_0 e_j$. As far as $\langle B_s S, \Psi_{i_1 \cdots i_s}\rangle = \sum a_{j_1 \cdots j_s} \langle \bigotimes_{k=1}^s \xi_{j_k}^* \otimes A_0 e_j, \bigotimes_{k=1}^s \xi_{i_k} \otimes \psi\rangle = \sum a_{j_1 \cdots j_s} \langle \xi_{j_1}^*, \xi_{j_1}\rangle \cdots \langle A_0 e_j, \psi\rangle = 0$, then $\Psi_{j_1 \cdots j_s}$ form a basis of the space $\operatorname{coKer} B_s$.

Thus for the solvability of (2.1) it is necessary and sufficient that the following conditions hold:

$$\langle B_s, \Psi_{j_1 \cdots j_s}\rangle = 0, \quad j_1, \dots, j_s = \overline{1, n} \tag{2.2}$$

□

Definition 2.2 The elements φ, $Dh(0)$, $D^2h(0), \dots, D^p h(0)$, as far as they are determined, form a Jordan chain of the zero-element φ for the operator-function $A_0 + DA(0)\lambda$.

Lemma 2.3 *For the irreversibility of the operator-function $A_0 + DA(0)\lambda$ everywhere in the neighborhood of the point $\lambda = 0$ the existence of an infinite Jordan chain is necessary and sufficient.*

Sufficiency Let there exists an infinite JCh. For the simplification introduce on every step Schmidt's operator [1]. Then using accepted designations $\widetilde{A}_0 =$

$A_0 + \langle \cdot, e_1^* \rangle u_1$ is the Schmidt's operator for A_0, $\widetilde{A}_0^{-1} = \Gamma$ [1]. Analogously, $\widetilde{B}_1 = B_1 + \sum_{i=1}^k \langle \cdot, \xi_i \otimes e_1^* \rangle \xi_i^* \otimes u_1$. For $L\{\Lambda \to E_1\} \ni S = \sum_m a_{1m} \xi_m^* \otimes e_1$ one has $\widetilde{B}_1 S = B_1 S + \sum_{i=1}^k \langle \sum_m a_{1m} \xi_m^* \otimes e_1, \xi_i \otimes e_1^* \rangle \xi_i^* \otimes u_1 = B_1 S + \sum_{i=1}^k a_{1i} \xi_i^* \otimes u_1$, and since $\langle S\lambda, e_1^* \rangle = \langle \sum_m a_{1,m} \lambda_m e_1, e_1^* \rangle = \sum_m a_{1m} \lambda_m = \sum_m a_{1m} \xi_m^*(\lambda)$ then $\widetilde{B}_1 S = A_0 S + \langle S \cdot, e_1^* \rangle u_1 = \widetilde{A}_0 S$. If $\Gamma_1 = \widetilde{B}_1^{-1}$. From here it follows $\Gamma_1 T = \Gamma T$ for $T \in L\{\Lambda_1 \to E_2\}$. For the proof it is sufficient to set $S = \Gamma T \Rightarrow \widetilde{B}_1 \Gamma T = \widetilde{A}_0 \Gamma T = T \Rightarrow \Gamma_1 \widetilde{B}_1 \Gamma T = \Gamma_1 T \Rightarrow \Gamma T = \Gamma_1 T$.

Analogously, if $B_s : L\{\bigotimes_s \Lambda \to E_1\} \to L\{\bigotimes_s \Lambda \to E_2\}$, then $\widetilde{B}_s S = \widetilde{A}_0 S$ for $S \in L\{\bigotimes_s \Lambda \to E_1\}$. In fact, $\widetilde{B}_s S = B_s S + \sum \langle \cdot, \xi_{j_1} \otimes \ldots \otimes \xi_{j_s} \otimes e_1^* \rangle \xi_{j_1}^* \otimes \cdots \otimes \xi_{j_s}^* \otimes u_1$ and $S = \sum a_{i_1 \cdots i_s, i} \xi_{i_1}^* \otimes \cdots \otimes \xi_{i_s}^* \otimes e_i \Rightarrow \widetilde{B}_s S = A_s S + \sum \langle \sum \langle a_{i_1 \cdots i_s, i} \xi_{i_1}^* \otimes \cdots \otimes \xi_{i_s}^* \otimes e_i, \xi_{i_1} \otimes \cdots \otimes \xi_{i_s} \otimes e_1^* \rangle \xi_{i_1}^* \otimes \cdots \otimes \xi_{i_s}^* \otimes u_1 \rangle = A_0 S + \sum a_{i_1 \cdots i_s} \bigotimes_{k=1}^s \xi_{i_k}^* \otimes u_1 = A_0 S + \langle S \cdot, e_1^* \rangle u_1 = \widetilde{A}_0 S$, since $\langle S\lambda, e_1^* \rangle = \langle \sum a_{i_1 \cdots i_s, i} \lambda_{i_1} \ldots \lambda_{i_s} e_i, e_1^* \rangle = \sum a_{i_1 \cdots i_s, 1} \lambda_{i_1} \ldots \lambda_{i_s} = \sum a_{i_1 \cdots i_s, 1} e_{i_1}^*(\lambda) \otimes \cdots \otimes e_{i_s}^*(\lambda)$. If now $\Gamma_k = (\widetilde{A}_k)^{-1}$, then $\Gamma_k T = \Gamma T$, where $T \in \{\bigotimes_s \Lambda \to E_2\}$. From the last relation the inequality $\|\Gamma_k\| \le \|\Gamma\|$ follows.

In an analogous manner define the operator D_s acting from $L\{\bigotimes_{s-1} \Lambda \to E_1\}$ into $L\{\bigotimes_s \Lambda \to E_2\}$ according to the rule $D_s S\lambda = R(S\lambda, \lambda)$, for $S \in L\{\bigotimes_{s-1} \Lambda \to E_1\}$. Note here, that $D_0 : E_1 \to L\{\Lambda \to E_2\}$. Obviously $\|D_s\| \le \|R\|$ holds. The usage of the introduced notations implies the following formulae for Jordan chain elements $J^{s+1} = (\Gamma_s D_{s-1}) \ldots (\Gamma_1 D_0)\varphi$, $s = 1, 2, \ldots$, $J^1 = \varphi$. The estimate $\|J^{s+1}\| \le (\|R\| \|\Gamma\|)^s \|\varphi\|$ gives the convergence of the series $h(\lambda)$ in some neighborhood of $\lambda = 0$.

Necessity Let in some neighborhood of $\lambda = 0$ the operator-function $A(\lambda)$ be irreversible, i.e. there exists a function $X(\lambda) : D_\varepsilon(0) \to E_1$ such that $A(\lambda)X(\lambda) = 0$, $X(\lambda) \ne 0$, $\|X(\lambda)\| = 1$ or at the usage of accepted notations $A_0 X(\lambda) + R(X(\lambda), \lambda) = 0 \Rightarrow \widetilde{A}_0 X(\lambda) + R(X(\lambda), \lambda) = \langle X(\lambda), e_1^* \rangle u_1 \Rightarrow [I + \Gamma R(\cdot, \lambda)]X(\lambda) = \langle X(\lambda), e_1^* \rangle e_1 \Rightarrow X(\lambda) = \langle X(\lambda), e_1^* \rangle (I + \Gamma R(\cdot, \lambda))^{-1} e_1$, since $\Gamma = (\widetilde{A}_0)^{-1}$ exists, $\Gamma u_1 = e_1$, when λ is sufficiently small. Then $\langle X(\lambda), e_1^* \rangle \ne 0$ at $\|X(\lambda)\| = 1$, and by the application of the functional e_1^* one has $1 = \langle [I + \Gamma R(\cdot, \lambda)]^{-1} e_1, e_1^* \rangle \Rightarrow \langle \Gamma R(e_1, \lambda), e_1^* \rangle - \langle \Gamma R(\Gamma R(e_1, \lambda), \lambda), e_1^* \rangle + \cdots = 0$. Its realization for all sufficiently smooth λ by using the relation $\Gamma^* e_1^* = u_1^*$ gives

$$\begin{aligned} &\langle R(e_1, \lambda), u_1^* \rangle = 0, \qquad \langle R(\Gamma R(e_1, \lambda), \lambda), u_1^* \rangle = 0, \quad \ldots, \\ &\langle R(\ldots(\Gamma R(e_1, \lambda)), \ldots, \lambda), u_1^* \rangle = 0, \quad \ldots \end{aligned} \tag{2.3}$$

From the first equality (2.3) follows that the operator $D_0\varphi$ (here $D_0\varphi = R(e_1, \cdot)$) is orthogonal to the operators $\Psi_i = \xi_i \otimes u_1^*$, $i = 1, \ldots, k$, since according to (2.1) $R(e_1, \cdot) = \sum r_{\sigma\rho} \xi_\sigma^* \otimes u_\rho$, $\rho > 1$. Thus the conditions for the existence of the element J^2 are realized. Analogously the relation $\langle R(\cdots(\Gamma R(e_1, \lambda)), \ldots, \lambda), u_1^* \rangle = 0$ means, that the polylinear function $R(\ldots(\Gamma R(e_1, \lambda)), \ldots, \lambda)$ is equal to $\sum r_{\sigma_1, \ldots, \sigma_s, \rho} \xi_{\sigma_1}^* \otimes \cdots \otimes \xi_{\sigma_s}^* \otimes u_\rho$, $\rho > 1$ and therefore is orthogonal to all $\Psi_{\sigma_1, \ldots, \sigma_s, 1} = \xi_{\sigma_1} \otimes \cdots \otimes \xi_{\sigma_s} \otimes u_1$. Thus the conditions for the existence of any elements J^{s+1} are fulfilled.

Lemma 2.4 *If $\lambda = 0$ is a simple eigenvalue of the operator-function (1.2), i.e. if the Jordan chain consists of only one element φ, then in a small neighborhood of $\lambda = 0$ the operator-function is invertible everywhere with the exception of some hypersurface passing through zero.*

Below the operator-function $A_0^* x + R^*(x, \lambda)$ is considered, where under $R^*(x, \lambda)$ one understands the conjugate to the matrix $R(\cdot, \lambda)$ with regard to the action $R(y, \lambda) = R(\lambda)y$.

Lemma 2.5 *If the operator-function $A(\lambda) = A_0 + R(\cdot, \lambda)$ has zero as a simple eigenvalue, then the operator-function $A^*(\lambda) = A_0^* + R^*(\cdot, \lambda)$ also has zero as a simple eigenvalue. It is non-invertible on the same hypersurface where $A(\lambda)$ is. The zero-element of $A^*(\lambda)$ is determined by the formula $\Psi(y) = (I + \Gamma^* R^*(\cdot, \lambda))^{-1} u_1^*$.*

3 Jordan Chains Along Directions

For every point $0 \neq \lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda$ let $e_\lambda = \frac{\lambda}{\|\lambda\|}$ be the unit vector in the direction of λ. Then the restriction of the operator-function $A(\cdot, \lambda) = A_0 + R(\cdot, \lambda)$ on the straight line $\lambda = \varepsilon e_\lambda$ depends now only on a one-dimensional parameter ε: $A_\lambda(x, \varepsilon) = [A_0 + \varepsilon R(\cdot, e_\lambda)]x$. At the assumption $R(\cdot, e_\lambda) \neq 0$ one can define JChs of the operator-function $A_\lambda(x, \varepsilon)$, which are called JChs of the operator-function $A(x, \lambda)$ along the direction λ. The relevant length of the JCh of $A_\lambda(x, \varepsilon)$ is denoted by $p(\lambda)$.

Lemma 3.1 *Let p be the length of the JCh of the multiparameter operator-function $A(x, \lambda)$. Then for any direction λ $p \leq p(\lambda)$ and for almost all directions λ with the exception of an algebraic set $p = p(\lambda)$.*

Definition 3.2 The directions λ^0 along which $p(\lambda^0) > p$ are called singular, all other ones are nonsingular. A singular direction, along which the operator-function $A_0 + \varepsilon R(\cdot, \lambda^0)$ is non-invertible is called degenerated.

Remark 3.3 Let be $p < \infty$. According to Theorem 30.1 [1] on the set of all nonsingular directions λ^0 the operator-function $A_0 + R(\cdot, \lambda^0)$ is invertible in the ball $0 < |\epsilon| < \rho(\lambda^0)$ for some $\rho(\lambda^0)$.

Remark 3.4 The following example shows that $\rho(\lambda)$ cannot be chosen independently of λ: $A(\lambda)x = \left[\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) + \lambda_1 \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) + \lambda_2 \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\right]\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right)$, $\det A(\lambda) = \lambda_1 - \lambda_1^2 - \lambda_2^2$. $A(\lambda)$ is non-invertible on the curve $(\lambda_1 - \frac{1}{2})^2 + \lambda_2^2 = \frac{1}{4}$. Here any direction $\lambda^0 \neq (0, 1)$, except the vertical ones, is nonsingular and $\rho(\lambda^0)$ is equal to the distance between zero and the intersection point of the direction λ^0 with the curve $\lambda_1 - \lambda_1^2 - \lambda_2^2 = 0$. Obviously, $\rho(\lambda^0) \to 0$ for $\lambda^0 \to (0, 1)$. Along the singular direction $(0, 1)$ $A(\lambda)$ is invertible everywhere except $\lambda = 0$.

Remark 3.5 If along some direction λ^0 $A(\lambda)$ has a maximal JCh, then along this direction it is invertible everywhere except $\lambda = 0$.

Corollary 3.6 *The length of JCh is either equal to infinity, or is not exceeding the space dimension.*

Lemma 3.7 *If for the operator-function $A(\lambda) = A_0 + R(\cdot, \lambda)$ the length of JCh is p, then for any sufficiently small value $\lambda \neq 0$, for which the direction $\lambda^0 = \frac{\lambda}{\|\lambda\|}$ is non-degenerated, the images of the JCh elements in the point λ are linearly independent.*

Remark 3.8 The following example shows that the images of JCh elements in the point λ^0 can be linearly dependent, while the direction λ^0 is degenerated

$$A(\lambda) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.1}$$

This operator-function has JCh consisting of two elements $J^1 = \varphi = (1, 0, 0)^T$ and $J^2(\lambda) = (0, \lambda_1, 0)^T$. This chain is non-prolongated, since $R(J^2(\lambda), \lambda) = (\lambda_1^2, \lambda_1\lambda_2, 0)^T$ and therefore if $\Psi = \xi_1 \otimes \xi_2 \otimes \psi$, then $\langle R(J^2(\lambda), \lambda), \Psi \rangle \neq 0$. However, on the straight line corresponding to the degenerated direction $(0, 1)$ (or $(0, 1, 0)$ if $\dim \Lambda = 3$) the images of JCh elements are linearly dependent since there $J^2(\lambda) = 0$.

Lemma 3.9 *If for $n \leq k$ and for $A_0 + R(\cdot, \lambda)$ the length of JCh is $p > 1$, then always there exists a direction λ^0 along which $A_0 + \varepsilon R(\cdot, \lambda^0)$ is degenerated.*

Remark 3.10 When $n > k$ the degenerated direction λ^0 does not always exist as shows the following example

$$A(\lambda) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

JCh for this $A(\lambda)$ consists of two elements $(1, 0, 0)$ and $(0, \lambda_1, \lambda_2)$. Obviously that the square vector-function $(\lambda_1^2 + \lambda_2^2, 0, 0)$ satisfied one of two conditions $\lambda_1(\lambda_1^2 + \lambda_2^2) \neq 0$ or $\lambda_2(\lambda_1^2 + \lambda_2^2) \neq 0$ and along the directions $(1, 0)$ and $(0, 1)$ $A(\lambda)$ is not degenerated, since $\det\begin{pmatrix} 0 & \lambda_1 & 0 \\ \lambda_1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = -\lambda_1^2 \neq 0$ and $\det\begin{pmatrix} 0 & 0 & \lambda_2 \\ 0 & 1 & 0 \\ \lambda_2 & 0 & 1 \end{pmatrix} = -\lambda_2^2 \neq 0$. For the "intermediate" direction $(1, a)$ one has $\det\begin{pmatrix} 0 & \varepsilon & \varepsilon a \\ \varepsilon & 1 & 0 \\ \varepsilon a & 0 & 1 \end{pmatrix} = -\varepsilon^2 - \varepsilon^2 a^2 \neq 0$. Thus, $A(\lambda)$ has no degenerated directions.

Remark 3.11 However, for more complicated, nonlinear $A(\lambda)$ in λ Lemma 3.9 is not valid, as the following example shows: $A(x, \lambda) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \lambda_2\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} +$

$\lambda_1^2\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$. It has no degenerated direction, since $\det A(\lambda) = \lambda_1^2 + \lambda_2^2 \neq 0$ for $\lambda \neq 0$.

Remark 3.12 The following example shows, that when $A(\lambda)$ has zero as simple eigenvalue, a degenerated direction for it can exist: $A(x) = \begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} - \lambda_1\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$, $R(e_1,\lambda) = \xi_1^* \otimes u_1 \Rightarrow \langle R(e_1,\cdot), \Psi_1\rangle = \langle R(e_1,\cdot), \xi_1 \otimes \psi\rangle = 1$ i.e. $p = 1$, while the direction (0, 1) is degenerated.

4 Degenerated Differential Equations

In this section the results of Sects. 2, 3 are applied to the questions on the existence and uniqueness of the solutions of degenerated DE

$$\left[A_0 + R(\cdot, x)\right]x' = Bx \tag{4.1}$$

Suppose that $A_0 + R(\cdot, x)$ has zero as a simple eigenvalue, i.e. JCh consists of only one zero-element. According to Lemma 2.4 in a neighborhood of $x = 0$ there exists a hypersurface M, on which $A_0 + R(\cdot, x)$ is degenerated, i.e. it has the zero-element $\Phi(x) = [I + \Gamma R(\cdot, x)]^{-1} e_1$ satisfying the condition $\langle\Phi(x), e_1^*\rangle = 1$. The hypersurface M is determined by the equation $\langle R(\cdot, x)(I + \Gamma R(\cdot, x))^{-1} e_1, e_1^*\rangle = 0$ and the tangent space to it in the point $x = 0$ is given by the equation $x_1\langle R(e_1, e_1), \Psi\rangle + \cdots + x_n\langle R(e_1, e_n), \Psi\rangle = 0$.

Outside of M the Cauchy problem for (4.1) has a unique solution, while on M the system (4.1) it can have solution nowhere (except at the point $x = 0$), can have solutions everywhere on M or only on some sub-manifold $M_1 \subset M$, as simple examples show.

Definition 4.1 JCh of $A_0 + R(\cdot, x)$ is breaking along the principal direction e_1, if $\langle R(e_1, e_1), \Psi\rangle \neq 0$.

Definition 4.2 The operator-function $A_0 + R(\cdot, x)$ is non-degenerated along a hypersurface M, if it has no zeroes on the tangent stratification to M.

Lemma 4.3 *If for the operator-function $A(x) = A_0 + R(\cdot, x)$ the JCh is breaking along the principal direction, then $A(x)$ is non-degenerated along the hypersurface M in a neighborhood of the point zero.*

However the following example shows, that if the JCh of $A(x)$ is breaking along the non-principal direction $\langle R(e_1, e_1), \Psi\rangle = 0$, then $A(x)$ can be degenerated along the hypersurface M: $A(x) = \begin{pmatrix}x_2&0\\0&1\end{pmatrix}$. Here M is determined by the equation $x_2 = 0$ and the zero-element e_1 belongs to the tangent space to M in any point of the neighborhood $x = 0$.

Theorem 4.4 *Let zero be a simple eigenvalue of* $A(x)$ *and its JCh be breaking along the principal direction. If in addition* $Bx \in \operatorname{Im} A(x)$ *for any* $x \in M$*, then the unique solution to* (4.1) *belonging to* M *passes through any point of a neighborhood of* $x = 0$ *on* M.

For the case when JCh of $A(x)$ is breaking along a non-principal direction the following analog of Theorem 4.4 is true.

Theorem 4.5 *Let* $A(x)$ *have a simple eigenvalue in zero and also its JCh be breaking along a non-principal direction, but nevertheless* $A(x)$ *be non-degenerate along the hypersurface* M *in some particular neighborhood of zero. If in addition* $Bx \in \operatorname{Im} A(x)$ *for any* $x \in M$*, then the unique solution to* (4.1) *belonging to* M *passes through any point of this neighborhood in* M.

In the following example $\begin{pmatrix} x_2 & x_1 \\ x_1 & 1 \end{pmatrix}\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, M is determined by the equation $x_2 = x_1^2$ and in a particular neighborhood of zero on M the operator-function $A(x) = \begin{pmatrix} x_1^2 & x_1 \\ x_1 & 1 \end{pmatrix}$ has the zero-element $(-1, x_1)$, not belonging to the tangent subspace to M in the point (x_1, x_1^2). This system on the hypersurface M takes the form $\begin{pmatrix} x_1^2 & x_1 \\ x_1 & 1 \end{pmatrix}\begin{pmatrix} x_1' \\ 2x_1' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_1^2 \end{pmatrix}$, and when $x_1 \neq 0$ the obtained equation $(x_1 + 2)x_1' = x_1$ is uniquely solvable.

Theorem 4.6 *Let the operator-function* $A(x)$ *have a simple eigenvalue in zero and its JCh be breaking along a non-principal direction. Let in addition in some neighborhood of* $x = 0$ *the zero-element* $\Phi(x)$ *of* $A(x)$ *belong to the tangent stratification* TM*. If also for any* $x \in M$, $Bx \in \operatorname{Im}(A(x)|_{TM(x)})$*, then in this neighborhood of* $x = 0$ *there exists an* $(n-2)$*-dimensional submanifold* N *of the hypersurface* M *on which* (4.1) *is uniquely solvable.*

The condition $Bx \in \operatorname{Im}(A(x))|_{TM(x)}$ cannot be replaced by the condition by Theorem 4.4 condition: $\begin{pmatrix} x_2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ from Theorem 4.4. The hypersurface M here is $\{x \in \mathbb{R}^3 | x_2 = 0\}$, the system on it has the solution $x_1 = 0$, $x_3 = 0$.

Remark 4.7 There are examples showing that solutions to (4.1) beginning on the hypersurface M can leave M. Here the uniqueness of the solution with a given initial point can be lost. In the assumptions of Theorem 4.4 the conditions are found, which determine solutions beginning on the manifold M and not belonging to it.

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Cauchy Problem for a First Order Ordinary Differential System with Variable Coefficients

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Abstract In this article the general solution of the first order ordinary differential systems is found. The Cauchy problem is solved.

Keywords Ordinary differential system · Variable coefficient · Cauchy problem

Mathematics Subject Classification (2010) 34B05

1 Introduction

Let $-\infty < t_1 < t_2 < \infty$ and $S[t_1, t_2]$ be the set of measurable, essentially bounded functions $f(t)$ in $[t_1, t_2]$ with the norm

$$\|f\|_1 = \text{esssup}_{t\in[t_1,t_2]}\left|f(t)\right| = \lim_{p\to\infty} \|f\|_{L_p[t_1,t_2]}.$$

Here $L_p[t_1, t_2]$, $1 \le p \le \infty$ is the set of functions, the p-th powers of which are integrable on $[t_1, t_2]$ with norm $\|f\|_{L_p[t_1,t_2]} = (\int_{t_1}^{t_2} |f(t)|^p dt)^{\frac{1}{p}}$.

We consider the system

$$u' = f(t)u + g(t)v + h(t), \qquad v' = g(t)u - f(t)v + q(t) \tag{1.1}$$

in $[t_1, t_2]$, where $h(t), q(t) \in L_1[t_1, t_2]$; $f(t), g(t) \in S[t_1, t_2]$.

E. Kamke in his reference book [1] gives a general solution for the system

$$u' = f(t)u - g(t)v, \qquad v' = g(t)u + f(t)v,$$

but he does not consider the system (1.1). This is connected to the fact that the methods used for the latter system are not applicable for (1.1). In the present work we construct the general solution of the system (1.1) in the class

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$$W_\infty^1[t_1,t_2]\cap C[t_1,t_2], \tag{1.2}$$

where $W_\infty^1[t_1,t_2]$ is the class of function $f(t)$ for which $f'(t)\in S[t_1,t_2]$.

Let $t_0\in[t_1,t_2]$. We consider the Cauchy problem:

Problem K *Find the solution of system* (1.1) *from the class* (1.2) *satisfying the Cauchy conditions*

$$u(t_0)=\alpha, \qquad v(t_0)=\beta, \tag{1.3}$$

where α *and* β *are given real numbers.*

2 Construction of the General Solution to (1.1)

For solving the system (1.1) by multiplying the second equation of this system by $i=\sqrt{-1}$ and adding it to the first equation of this system we obtain

$$W'-b(t)\overline{W}=a(t), \tag{2.1}$$

where $b(t)=f(t)+ig(t)$, $a(t)=h(t)+iq(t)$, $W=u+iv$.

Obviously $b(t)\in S[t_1,t_2]$, $a(t)\in L_1[t_1,t_2]$ and $W\in W_\infty^1[t_1,t_2]\cap C[t_1,t_2]$.

By integrating (2.1) we have

$$W(t)=(BW)(t)+A_0(t)+c, \tag{2.2}$$

where

$$(BW)(t)=\int_{t_0}^t b(\tau)\overline{W(\tau)}d\tau, \qquad A_0(t)=\int_{t_0}^t a(\tau)d\tau,$$

c is an arbitrary complex number.

Applying the operator B to both sides of (2.2) we get

$$(BW)(t)=\big(B^2W\big)(t)+A_1(t)+\overline{c}I_1(t), \tag{2.3}$$

where

$$\big(B^2W\big)(t)=\big(B(BW)(t)\big)(t), \qquad I_1(t)=\int_{t_0}^t b(\tau)d\tau,$$

$$A_1(t)=\int_{t_0}^t b(\tau)\overline{A_0(\tau)}d\tau.$$

From (2.3) and (2.2) it follows

$$W(t)=\big(B^2W\big)(t)+A_0(t)+A_1(t)+\overline{c}I_1(t)+c.$$

Continuing this procedure n times we have

$$W(t) = \left(B^{2n}W\right)(t) + \sum_{j=0}^{2n-1} A_j(t) + \overline{c}\cdot\sum_{j=1}^{n} I_{2j-1}(t) + c\left(1 + \sum_{j=1}^{n-1} I_{2j}(t)\right), \tag{2.4}$$

where

$$\left(B^n W\right)(t) = \left(B\left(B^{n-1}W\right)(t)\right)(t), \qquad I_j(t) = \int_{t_0}^{t} b(\tau)\overline{I_{j-1}(\tau)}d\tau,$$

$$A_j(t) = \int_{t_0}^{t} b(\tau)\overline{A_{j-1}(\tau)}d\tau \quad (j = 1, 2, \ldots), \qquad \left(B^1 W\right)(t) = (BW)(t).$$

Taking into consideration the definition of the operators $(B^n W)(t)$ and the functions $I_n(t)$, $A_n(t)$ the following estimates are obtained:

$$\begin{aligned} &\left|\left(B^n W\right)(t)\right| \le |W|_0 \frac{(|b|_1|t-t_0|)^n}{n!}, \qquad \left|I_n(t)\right| \le \frac{(|b|_1|t-t_0|)^n}{n!},\\ &\left|A_n(t)\right| \le |A_0|_0 \frac{(|b|_1|t-t_0|)^n}{n!} \quad (n = 1, 2, \ldots), \end{aligned} \tag{2.5}$$

where

$$|f|_0 = \max_{t\in[t_1,t_2]}\left|f(t)\right|, \qquad |f|_1 = \text{esssup}_{t\in[t_1,t_2]}\left|f(t)\right| = \lim_{p\to\infty}\left\|f(t)\right\|_{L_p[t_1,t_2]}.$$

Passing to the limit with $n \to \infty$ in the representation (2.4) and taking estimates (2.5) into account we receive

$$W(t) = \overline{c}P_1(t) + cP_2(t) + P_3(t), \tag{2.6}$$

where

$$P_1(t) = \sum_{j=1}^{\infty} I_{2j-1}(t), \qquad P_2(t) = 1 + \sum_{j=1}^{\infty} I_{2j}(t), \qquad P_3(t) = \sum_{j=0}^{\infty} A_j(t).$$

Using inequalities (2.5) we obtain

$$\begin{aligned} &\left|P_1(t)\right| \le \sinh\left(|b|_1|t-t_0|\right), \qquad \left|P_2(t)\right| \le \cosh\left(|b|_1|t-t_0|\right),\\ &\left|P_3(t)\right| \le |A_0|_0 \exp\left(|b|_1|t-t_0|\right). \end{aligned} \tag{2.7}$$

From the forms of the functions $P_1(t)$, $P_2(t)$, $P_3(t)$ it follows

$$P_1'(t) = b(t)\overline{P_2(t)}, \qquad P_2'(t) = b(t)\overline{P_1(t)}, \qquad P_3'(t) = b(t)\overline{P_3(t)} + a(t). \tag{2.8}$$

From (2.8) using the equalities $P_1(t_0) = 0$, $P_2(t_0) = 1$ we have the integral equations

$$P_1(t) = \int_{t_0}^{t} b(\tau)\overline{P_2(\tau)}d\tau, \qquad P_2(t) = 1 + \int_{t_0}^{t} b(\tau)\overline{P_1(\tau)}d\tau. \tag{2.9}$$

Integrating by parts n times the integral on the right -hand side of the second (2.9) we obtain

$$P_2(t) - 1 = \overline{P_1(t)} \cdot \sum_{k=1}^{n} I_{2k-1}(t) - P_2(t) \cdot \sum_{k=1}^{n} \overline{I_{2k}(t)} + \int_{t_0}^{t} b(\tau)\overline{I_{2n}(\tau)} \cdot \overline{P_1(\tau)} d\tau.$$

Thus, passing to the limit with $n \to \infty$ and taking into account estimates (2.5) and (2.7) in the last equality we obtain the identity

$$\left|P_2(t)\right|^2 - \left|P_1(t)\right|^2 \equiv 1$$

From the last identity and (2.8) it follows, that the Wronskian of the functions $P_1(t)$ and $P_2(t)$ is equal to $-b(t) \neq 0$. Therefore, the functions $P_1(t)$ and $P_2(t)$ form a linearly independent system in $[t_1, t_2]$. Thus, the general solution of (2.1) is found by the formula (2.6). Highlighting the real and imaginary parts of the equality (2.6) we obtain the solution of the system (1.1):

$$u = c_1 \Re\big(P_1(t) + P_2(t)\big) + c_2 \Im\big(P_1(t) - P_2(t)\big) + \Re P_3(t),$$
$$v = c_1 \Im\big(P_1(t) + P_2(t)\big) - c_2 \Re\big(P_1(t) - P_2(t)\big) + \Im P_3(t),$$

where c_1 and c_2 are arbitrary real numbers.

3 Solving the Cauchy Problem

Now let us solve Problem K. To solve the Cauchy problem we use (2.6). From the form of the functions $P_1(t)$, $P_2(t)$ and $P_3(t)$ it follows

$$P_1(t_0) = P_3(t_0) = 0, \qquad P_2(t_0) = 1.$$

Using these formulas from (2.6) we obtain

$$u(t_0) + iv(t_0) = c_1 + ic_2, \qquad u(t_0) = c_1, \qquad v(t_0) = c_2.$$

Thus, due to (1.3) we get

$$c_1 = \alpha, \qquad c_2 = \beta.$$

The obtained value $c = c_1 + ic_2$ is put in (2.6):

$$W(t) = (\alpha - i\beta)P_1(t) + (\alpha + i\beta)P_2(t) + P_3(t).$$

Highlighting the real and imaginary parts of the last formula we finally obtain the solution of the Cauchy problem:

$$\begin{aligned} u &= \alpha \Re\big(P_1(t) + P_2(t)\big) + \beta \Im\big(P_1(t) - P_2(t)\big) + \Re P_3(t), \\ v &= \alpha \Im\big(P_1(t) + P_2(t)\big) - \beta \Re\big(P_1(t) - P_2(t)\big) + \Im P_3(t). \end{aligned} \tag{3.1}$$

Thus, the following theorem is proved.

Theorem 3.1 *The Cauchy problem has the only solution which is given by the formula* (3.1).

Remark 3.2 The obtained results remain in force and in the case:

$$f(t), g(t), h(t), q(t) \in C[t_1, t_2];$$
$$u(t), v(t) \in C^1[t_1, t_2].$$

4 Cauchy Problem for the Nonlinear First Order Ordinary Differential System

Let $-\infty < t_1 < t_0 < t_2 < \infty$. We consider the nonlinear system

$$u' = f(t)u + g(t)v + h(t, u, v), \qquad v' = g(t)u - f(t)v + q(t, u, v), \tag{4.1}$$

in the interval $[t_1, t_2]$, where $f(t), g(t) \in C[t_1, t_2]$ and the functions $h(t, u, v)$, $q(t, u, v)$ are continuous in the set of variables in the domain $|t - t_0| < \delta$, $|u - \alpha| < \sigma_1$, $|v - \beta| < \sigma_2$. Here $u(t_0) = \alpha$, $v(t_0) = \beta$; δ, α, β, σ_1, σ_2 are real numbers, so that $\delta > 0$, $\sigma_1 > 0$, $\sigma_2 > 0$, $\delta < t_2 - t_1$. The connection between the numbers δ and σ_1, σ_2 will be defined later. Multiplying the second equation of the system (4.1) by $i = \sqrt{-1}$, then adding it to the first equation of (4.1) we get

$$W' - b(t)\overline{W} = a(t, W), \tag{4.2}$$

where $W = u + iv$, $b(t) = f(t) + ig(t)$, $a(t, W) = h(t, u, v) + iq(t, u, v)$.

Obviously $b(t) \in C[t_1, t_2]$ and the function $a(t, W)$ is continuous in the set of variables in the domain $|t - t_0| < \delta$, $|W - \gamma| < \sigma$, where $\gamma = \alpha + i\beta$, $\sigma = \sigma_1 + \sigma_2$. We find the solution to (4.2) from the class $C^1[t_1, t_2]$ satisfying the following condition

$$W(t_0) = \gamma. \tag{4.3}$$

Equality (4.3) follows from (1.3).

Using formula (2.6) we have

$$W(t) = \bar{c}P_1(t) + cP_2(t) + P_3(t, W), \tag{4.4}$$

where

$$P_3(t, W) = \sum_{j=0}^{\infty} A_j(t, W), \qquad A_0(t, W) = \int_{t_0}^{t} a\big(\tau, W(\tau)\big)d\tau,$$

$$A_j(t,W) = \int_{t_0}^{t} b(\tau)\overline{A_{j-1}(\tau,W)}d\tau, \quad (j = \overline{1,\infty}).$$

From (2.7), (2.9) it follows

$$\left|P_1(t)\right| \le |b|_1 \delta \cosh \delta_1, \qquad \left|P_2(t) - 1\right| \le |b|_1 \delta \sinh \delta_1, \tag{4.5}$$

$$\begin{aligned} \left|P_1(t_4) - P_1(t_3)\right| &\le |b|_1 \sinh \delta_1 \cdot (t_4 - t_3), \\ \left|P_2(t_4) - P_2(t_3)\right| &\le |b|_1 \cosh \delta_1 \cdot (t_4 - t_3), \end{aligned} \tag{4.6}$$

where $\delta_1 = |b|_1 \cdot \delta$, $t_1 \le t_3 < t_4 \le t_2$.

Let a_1 be the maximum of the function $|P_3(t,W)|$ in the domain $|t - t_0| < \delta$, $|W - \gamma| < \sigma$. From the form of the function $P_3(t,W)$ it follows

$$\left|P_3(t,W)\right| \le a_1(\exp \delta_1 - 1), \tag{4.7}$$

$$\left|P_3(t_4,W) - P_3(t_3,W)\right| \le a_1\big(1 + |b|_1(\exp \delta_1 - 1)\big)(t_4 - t_3). \tag{4.8}$$

From the form of the functions $P_1(t)$, $P_2(t)$, $P_3(t,W)$ and (2.8) it follows that the right hand-side of the equality (4.4) belongs to the class $C^1[t_1,t_2]$. If we take the derivative of both sides of the equality (4.4) then we obtain (4.2). Therefore, the following theorem holds.

Theorem 4.1 *Any solution of* (4.4) *from the class* $C[t_1,t_2]$ *is a solution to* (4.2) *from the class* $C^1[t_1,t_2]$.

We consider the solutions to (4.4) from the class $C[t_1,t_2]$ satisfying initial condition (4.3). We obtain $c = \gamma$ from (4.4) by taking into account the equalities
$P_1(t_0) = P_3(t_0,W) = 0$, $P_2(t_0) = 1$. Thus, any solution from the class $C[t_1,t_2]$ of the equation

$$W(t) = (DW)(t), \tag{4.9}$$

where

$$(DW)(t) = \overline{\gamma} P_1(t) + \gamma P_2(t) + P_3(t,W)$$

is the solution of the Cauchy problem for (4.2).

Let

$$|\gamma||b|_1 \delta \exp \delta_1 + a_1(\exp \delta_1 - 1) < \sigma, \tag{4.10}$$

where $\delta_1 = \delta|b|_1$, a_1 is the maximum of the function $|P_3(t,W)|$ in the domain

$$|t - t_0| < \delta, \qquad |W - \gamma| < \sigma.$$

Inequality (4.10) always might be obtained for the small value of the number δ. Let us prove the existence of continuous solutions to the system (4.1) in some neighborhood of the point t_0.

Theorem 4.2 *Let $f(t), g(t) \in C[t_1, t_2]$ and the functions $h(t, u, v)$, $q(t, u, v)$ be continuous in the set of variables in the domain $|t - t_0| < \delta$, $|u - \alpha| < \sigma_1$, $|v - \beta| < \sigma_2$. Then on the interval $[t_0 - \delta, t_0 + \delta]$, where the number δ satisfies the condition* (4.10) *there exists at least one solution to the system* (4.1) *from the class $C^1[t_1, t_2]$ satisfying the condition* (1.3).

Proof If there exists a solution to (4.9) from the class $C[t_1, t_2]$, then by highlighting real and imaginary parts of it, we obtain the solution of the system (4.1) from the class $C^1[t_1, t_2]$. Therefore, by Theorem 4.2 it is sufficient to prove the existence of solutions from the class $C[t_0 - \delta, t_0 + \delta]$ of (4.9). Let $\|W\| = \max_{|t-t_0|<\delta} |W(t)|$. We consider the operator D which is defined by the equality

$$(DW)(t) = \overline{\gamma} P_1(t) + \gamma P_2(t) + P_3(t, W)$$

on the sphere $\|W - \gamma\| \le \sigma$ of the space $C[t_0 - \delta, t_0 + \delta]$. Let us show that the operator D is continuous enough on this sphere. Obviously the operator D is continuous on the sphere $\|W - \gamma\| \le \sigma$. For any element $W(t)$ of the sphere $\|W - \gamma\| \le \sigma$ in force of the inequalities (2.7), (4.7) we get

$$\left|(DW)(t)\right| \le |\gamma| \exp \delta_1 + a_1(\exp \delta_1 - 1). \tag{4.11}$$

If t_3 and t_4 are two arbitrary points of the interval $[t_0 - \delta, t_0 + \delta]$, then by inequalities (4.6), (4.8) we have

$$\begin{aligned} &\left|(DW)(t_4) - (DW)(t_3)\right| \\ &\quad \le \big(|\gamma||b|_1 \exp \delta_1 + a_1\big(1 + |b|_1(\exp \delta_1 - 1)\big)\big)(t_4 - t_3). \end{aligned} \tag{4.12}$$

By the Arzela–Ascoli theorem from inequalities (4.11), (4.12) it follows, that the operator D transforms the sphere $\|W - \gamma\| \le \sigma$ into a compact set. We show that the operator D transforms this sphere into itself. Indeed, inequalities (4.5), (4.7) give us

$$\left|(DW)(t) - \gamma\right| \le |b|_1 \delta |\gamma| \exp \delta_1 + a_1(\exp \delta_1 - 1).$$

Finally, inequality (4.10) gives $|(DW)(t) - \gamma| < \sigma$. Therefore, the operator D satisfies all the conditions of Schauder's theorem. Hence, there exists a fixed point of this operator, i.e. such a function $W(t)$, so that

$$W(t) = \bar{\gamma} P_1(t) + \gamma P_2(t) + P_3(t, W).$$

Therefore, by Theorem 4.2 there exists a solution to the Cauchy problem for the system (4.1). The theorem is proved. □

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General Solution of an n-th Order Linear Ordinary Differential Equation with Variable Coefficients

A. Tungatarov and B. Omarbayeva

Abstract In this article the general solution of an n-th order ordinary differential equations is found. The Cauchy problem for this equation is solved.

Keywords General solution · n-th order linear ordinary differential equation · Variable coefficients · Cauchy problem

Mathematics Subject Classification (2010) 34B05

1 Introduction

Let $-\infty < x_1 < x_2 < \infty$. By $S[x_1, x_2]$ we denote the class of measurable essentially bounded functions on $[x_1, x_2]$. The norm of an element from $S[x_1, x_2]$ is defined by the formulas

$$\|f\|_{S[x_1,x_2]} = \sup vrai_{x\in[x_1,x_2]}\left|f(x)\right| = \lim_{p\to\infty} \|f\|_{L_p[x_1,x_2]}.$$

We consider the equation

$$\frac{d^n u}{dx^n} - p(x)u = f(x) \tag{1.1}$$

on $[x_1, x_2]$, where $p(x),\ f(x) \in S[x_1, x_2]$.

The general solution of (1.1) and the solution of the Cauchy problem with the initial problem in the point $x_0 \in [x_1, x_2)$ for this equation will be sought from the class

$$C[x_1, x_2] \cap W^n_\infty[x_1, x_2], \tag{1.2}$$

where $W^n_\infty[x_1, x_2]$ is the class of functions $f(x)$, for which $\frac{d^n f}{dx^n} \in S[x_1, x_2]$.

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The general solution of (1.1) in the particular case $n = 2$ is given in [1, 2] and for $n = 3$ in [3]. Note that in the mathematical literature one cannot find any representation of solutions to (1.1).

If $p(x)$, $f(x) \in C[x_1, x_2]$ the general solution is found in the present article in the class $C^n[x_1, x_2]$.

2 Construction of the General Solution to (1.1)

Let us choose $x_0 \in [x_1, x_2)$. Integrating (1.1) n times gives

$$u(x) = (Bu)(x) + \sum_{k=1}^{n} c_k (x - x_0)^{k-1} + g(x), \tag{2.1}$$

where $c_1, c_2, \ldots, c_n$ are arbitrary real numbers,

$$(Bu)(x) = \int_{x_0}^{x} \int_{x_0}^{y_1} \int_{x_0}^{y_2} \cdots \int_{x_0}^{y_{n-1}} p(t)u(t)dt dy_{n-1} dy_{n-2} \ldots dy_1,$$
$$g(x) = \int_{x_0}^{x} \int_{x_0}^{y_1} \int_{x_0}^{y_2} \cdots \int_{x_0}^{y_{n-1}} f(t)dt dy_{n-1} dy_{n-2} \ldots dy_1.$$

Applying the operator B to (2.1) we get

$$(Bu)(x) = \big(B^2 u\big)(x) + (Bg)(x) + \sum_{k=1}^{n} c_k a_{k,1}(x), \tag{2.2}$$

where

$$\big(B^2 u\big)(x) = \big(B(Bu)(x)\big)(x),$$
$$a_{k,1}(x) = \int_{x_0}^{x} \int_{x_0}^{y_1} \int_{x_0}^{y_2} \cdots \int_{x_0}^{y_{n-1}} (t - x_0)^{k-1} p(t)dt dy_{n-1} dy_{n-2} \ldots dy_1.$$

From (2.1) and (2.2) it follows

$$u(x) = \big(B^2 u\big)(x) + (Bg)(x) + g(x) + \sum_{k=1}^{n} c_k \big((x - x_0)^{k-1} + a_{k,1}(x)\big). \tag{2.3}$$

In the following we use the formulas

$$\big(B^k f\big)(x) = \big(B\big(B^{k-1} f\big)(x)\big)(x),$$

$$a_{k,l}(x) = \big(B a_{k,l-1}(x)\big)(x)$$

$$= \int_{x_0}^{x} \int_{x_0}^{y_1} \int_{x_0}^{y_2} \cdots \int_{x_0}^{y_{n-1}} p(t) a_{k,l-1}(t) dt dy_{n-1} dy_{n-2} \ldots dy_1.$$

Applying the operator B to both sides of (2.3) and using the previous formulas imply

$$(Bu)(x) = \big(B^3 u\big)(x) + \big(B^2 g\big)(x) + (Bg)(x) + \sum_{k=1}^{n} c_k \big(a_{k,1}(x) + a_{k,2}(x)\big) \tag{2.4}$$

From (2.1) and (2.4) it follows

$$u(x) = \big(B^3 u\big)(x) + g(x) + (Bg)(x) + \big(B^2 g\big)(x) + \big(B^3 g\big)(x)$$
$$+ \sum_{k=1}^{n} c_k \big((x - x_0)^{k-1} + a_{k,1}(x) + a_{k,2}(x)\big).$$

Continuing this procedure m times we obtain the following integral equation for the solution of (1.1):

$$u(x) = \big(B^m u\big)(x) + g(x) + \sum_{k=1}^{m-1} \big(B^k g\big)(x)$$
$$+ \sum_{k=1}^{n} c_k \left((x - x_0)^{k-1} + \sum_{l=1}^{m-1} a_{k,l}(x) \right). \tag{2.5}$$

Let us choose $u(x) \in C[x_1, x_2]$. Taking the definition of the iterated operators $(B^k f)(x)$ and the iterated functions $a_{k,l-1}(x)$ into consideration the following estimates are obtained without any difficulties:

$$\big|\big(B^m u\big)(x)\big| \le |u|_1 \frac{(\sqrt[n]{|p|_0} \cdot |x - x_0|)^{nm}}{(mn)!},$$
$$\big|\big(B^k g\big)(x)\big| \le |g|_1 \frac{(\sqrt[n]{|p|_0} \cdot |x - x_0|)^{kn}}{(kn)!}, \tag{2.6}$$
$$\big|a_{k,l}(x)\big| \le |p|_0^l \frac{|x - x_0|^{nl}}{(nl)!},$$

where

$$|f|_0 = \sup vrai_{x \in [x_1, x_2]} \big|f(x)\big|, \qquad |f|_1 = \max_{x \in [x_1, x_2]} \big|f(x)\big|.$$

Passing to the limit $m \to \infty$ in the representation (2.5), by virtue of (2), we conclude

$$u(x) = \sum_{k=1}^{n} c_k \cdot I_k(x) + F(x), \tag{2.7}$$

where

$$I_k(x) = (x - x_0)^{k-1} + \sum_{m=1}^{\infty} a_{k,m}(x), \qquad F(x) = g(x) + \sum_{m=1}^{\infty} \big(B^m g\big)(x).$$

Using the estimates (2) we get

$$\big|I_k(x)\big| \le \sum_{m=0}^{\infty} \frac{(\sqrt[n]{|p|_0} \cdot |x - x_0|)^{nm}}{(mn)!} \quad (k = 1, 2, \dots, n),$$

$$\big|F(x)\big| \le |g|_1 \sum_{m=0}^{\infty} \frac{(\sqrt[n]{|p|_0} \cdot |x - x_0|)^{mn}}{(mn)!}.$$

The following relations for the functions $I_1(x), I_2(x), \dots, I_n(x)$ and $F(x)$ are of importance:

$$\begin{aligned} &\frac{d^n I_k}{dx^n} - p(x) I_k = 0 \quad (k = 1, 2, \dots, n), \\ &\frac{d^n F}{dx^n} - p(x) F(x) = f(x), \end{aligned} \tag{2.8}$$

$$I_k^{l-1}(x_0) = \begin{cases} (l-1)!, & \text{if } k = 1, \\ 0, & \text{if } k \neq 1, \end{cases} \qquad I_k(x_0) = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{if } k \neq 1. \end{cases} \tag{2.9}$$

The formulas (2.8) tell us that the functions $I_1(x), I_2(x), \dots, I_n(x)$ are particular solutions from the class (1.2) of the homogeneous equation

$$\frac{d^n u}{dx^n} - p(x) u = 0$$

and the function $F(x)$ is a particular solution of the inhomogeneous equation (1.1).

From (2.8) we see that the Wronskian $W(x)$ is equal to $2! \cdot 3! \cdot \ldots \cdot (n-1)!$ in $x = x_0$. Therefore the functions $I_1(x), I_2(x), \dots, I_n(x)$ are linearly independent on $[x_1, x_2]$ and the general solution to (1.1) belonging to the class (1.2) is determined by the formula (2.7).

Summarizing we have proved the following theorem.

Theorem 2.1 *The general solution of* (1.1) *from the class* (1.2) *is given by formula* (2.7).

3 Cauchy Problem

Now we are going to solve the Cauchy problem.

Cauchy problem *Find a solution of* (1.1) *from the class* (1.2), *satisfying the Cauchy conditions*

$$\begin{aligned}&\alpha_{11}u(x_0)+\alpha_{12}u^{'}(x_0)+\cdots+\alpha_{1n}u^{n-1}(x_0)=\beta_1,\\&\alpha_{21}u(x_0)+\alpha_{22}u^{'}(x_0)+\cdots+\alpha_{2n}u^{n-1}(x_0)=\beta_2,\\&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\&\alpha_{n1}u(x_0)+\alpha_{n2}u^{'}(x_0)+\cdots+\alpha_{nn}u^{n-1}(x_0)=\beta_n,\end{aligned}\tag{3.1}$$

where α_{kj} $(k, j = 1, 2, \ldots, n)$, β_k $(k = 1, 2, \ldots, n)$ *are given real numbers*,

$$u^k(x_0)=\left.\frac{d^k u}{dx^k}\right|_{x=x_0}\quad (k=1,2,\ldots,n).$$

Solution of the Cauchy problem To solve the Cauchy problem we use the general solution of (1.1) given by formula (2.7). Substituting the function $u(x)$ given by formula (2.7) into the initial problem (3.1), taking (2.9) into consideration we get

$$\begin{cases}\alpha_{11}\cdot c_1+1!\cdot\alpha_{12}\cdot c_2+\cdots+(n-1)!\cdot\alpha_{1n}\cdot c_n=\beta_1,\\ \alpha_{21}\cdot c_1+1!\cdot\alpha_{22}\cdot c_2+\cdots+(n-1)!\cdot\alpha_{2n}\cdot c_n=\beta_2,\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ \alpha_{n1}\cdot c_1+1!\cdot\alpha_{n2}\cdot c_2+\cdots+(n-1)!\cdot\alpha_{nn}\cdot c_n=\beta_n.\end{cases}\tag{3.2}$$

In case $|\Delta| \neq 0$ we get from (3.2)

$$c_k=\frac{|\Delta_k|}{|\Delta|}\quad (k=1,2,\ldots,n),\tag{3.3}$$

where

$$\Delta=\begin{pmatrix}\alpha_{11} & 1!\cdot\alpha_{12} & \ldots & (n-1)!\cdot\alpha_{1n}\\ \alpha_{21} & 1!\cdot\alpha_{22} & \ldots & (n-1)!\cdot\alpha_{2n}\\ \ldots & \ldots & \ldots & \ldots\\ \alpha_{n1} & 1!\cdot\alpha_{n2} & \ldots & (n-1)!\cdot\alpha_{nn}\end{pmatrix},$$

Δ_k is the matrix which is received by replacing the k matrix column of the matrix Δ by the column

$$\beta=\begin{pmatrix}\beta_1\\ \beta_2\\ \vdots\\ \beta_n\end{pmatrix}.$$

If $|\Delta| = 0$ for the solvability of the algebraic system (3.2) the conditions:

$$|\Delta_k|=0\quad (k=1,2,\ldots,n)\tag{3.4}$$

are necessary and sufficient.

Theorem 3.1 (1) *If* $|\Delta| \neq 0$ *then the Cauchy problem has a unique solution given by the formulas* (2.7) *and* (3.3). (2) *If* $|\Delta| = 0$ *then for the solvability of the Cauchy problem it is necessary and sufficient that the equalities* (3.4) *are satisfied. In this case the Cauchy problem has an infinite number of solutions*, *which are given by formula* (2.7), *where the arbitrary constants* $c_1, c_2, \ldots, c_n$ *are connected by the relation* (3.2).

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About a Class of Two Dimensional Volterra Type Integral Equations with Singular Boundary Lines

Lutfya Rajabova

Abstract We examine 2-dimensional integral equations of Volterra type with two singular boundary lines corresponding to $x = a$ and $y = b$. The non-homogeneous integral equation that we can consider involves constants A_1, A_2, B_1, B_2, C_1, C_2, C_3, C_4. Given certain inequalities for A_1, A_2, B_1, B_2, it always has solutions on suitable domains that contain arbitrary functions of one variable. With other hypotheses, the equation has a unique solution in some domain.

Keywords Volterra type integral equation · Singular boundary lines

Mathematics Subject Classification (2010) Primary 45L10 · Secondary 45G05

1 Introduction and Preliminaries

Consider the rectangle $D = \{a < x < a_1,\ b < y < b_1\}$, and the straight lines $\Gamma_1 = \{a < x < a_1,\ y = b\}$, $\Gamma_2 = \{x = a,\ b < y < b_1\}$, where $a < x < a_1$, $b < b < b_1$. In the domain D, we consider the 2-dimensional integral equation

$$\begin{aligned}
u(x,y) &+ A_1 \int_a^x \frac{u(t,y)}{t-a} dt + A_2 \int_a^x \ln\left(\frac{x-a}{t-a}\right) \frac{u(t,y)}{t-a} dt \\
&+ B_1 \int_b^y \frac{u(x,s)}{s-b} ds + B_2 \int_b^y \ln\left(\frac{y-b}{s-b}\right) \frac{u(x,s)}{s-b} ds \\
&+ C_1 \int_a^x \frac{dt}{t-a} \int_b^y \frac{u(t,s)}{s-b} ds \\
&+ C_2 \int_a^x \frac{dt}{t-a} \int_b^y \ln\left(\frac{y-b}{s-b}\right) \frac{u(t,s)}{s-b} ds \\
&+ C_3 \int_a^x \ln\left(\frac{x-a}{t-a}\right) \frac{dt}{t-a} \int_b^y \frac{u(t,s)}{s-b} ds
\end{aligned}$$

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$$+ C_4 \int_a^x \ln\left(\frac{x-a}{t-a}\right)\frac{dt}{t-a}\int_b^y \ln\left(\frac{y-b}{s-b}\right)\frac{ds}{s-b}$$
$$= f(x,y), \tag{1.1}$$

where A_i, B_i, C_j, $i = \overline{1,2}$, $j = \overline{1,4}$, are given constants, $f(x,y) \in C(\overline{D})$.

The solution of (1.1) is sought in the class of functions $u(x,y) \in C(\overline{D})$ which are zero on the singular lines Γ_1 and Γ_2.

The solution of many problems having a significance in applications can be figured out by the help of integral equations in explicit from. For that reason, this article is dedicated to this area.

Early problems concerning 2-dimensional Volterra-type integral equations of the form

$$u(x,y) + \lambda \int_a^x \frac{u(t,y)}{(t-a)^\alpha}dt - \mu \int_y^b \frac{u(x,s)}{(b-s)^\beta}ds$$
$$+ \delta \int_a^x \frac{dt}{(t-a)^\alpha}\int_y^b \frac{u(t,s)}{(b-s)^\beta}ds = f(x,y), \tag{1.2}$$

with two singular and super-singular boundary lines in the domain

$$D_1 = \{a < x < a_0,\ b_0 < y < b\},$$

are investigated in [1–3].

Integral equations of type

$$u(x,y) + \int_a^x \frac{A(t)u(t,y)}{t-a}dt - \int_y^b \frac{B(s)u(x,s)}{b-s}ds$$
$$+ \int_a^x \frac{dt}{t-a}\int_y^b \frac{C(t,s)u(t,s)}{b-s}ds = f(x,y)$$

with singular and super-singular boundary and interior lines, in cases of $\alpha = 1$, $\beta > 1$; $\alpha > 1$, $\beta = 1$; $\alpha < 1$, $\beta > 1$ are investigated in [4, 5, 10].

References [6, 7] are dedicated to the problem of finding continuous solutions of a second-order hyperbolic equation with two singular or super-singular boundary lines Γ_1 and Γ_2, corresponding to the study of integral equations (1.2) in the domain D_1 with $\alpha \geq 1$ and $\beta \geq 1$.

Finally, [8, 9] deals with the integral equation

$$u(x,y) + \int_a^x \frac{K_1(x,y;t)u(t,y)}{(t-a)^\alpha}dt - \int_y^b \frac{K_2(x,y;s)u(x,s)}{(b-s)^\beta}ds+$$
$$+ \int_a^x \frac{dt}{(t-a)^\alpha}\int_y^b \frac{K_3(x,y;t,s)u(t,s)}{(b-s)^\beta}ds = f(x,y)$$

in D_1 in the cases $\alpha = 1$, $\beta = 1$ where

$$\lambda = K_1(a, b; a), \qquad \mu = K_2(a, b; b), \qquad \delta = K_3(a, b; a, b) = -\lambda\mu.$$

For $\alpha > 1$, $\beta > 1$ we set

$$A(t) = K_1(a, b; t), \qquad B(s) = K_2(a, b; s), \qquad C(t, s) = K_3(a, b; t, s)$$

and require $C_1(t, s) = C(t, s) + A(t)B(s)$ not to be identically zero.

In this paper, we find the solution of the 2-dimensional Volterra type linear integral equation with boundary singularities, when $C_1 = A_1B_1$, $C_2 = A_1B_2$, $C_3 = A_2B_1$, $C_4 = A_2B_2$.

In this case we shall prove that, whether parameters in the integral equation are connected with one another in a certain from, dependent on the signs of these parameters and the roots of the characteristic equations the general solution of the homogeneous equation contains a few arbitrary functions of one variable and in some particular cases has a unique solution.

2 A First Theorem

Our first result is the following statement that deals with the case in which $A_1 > 0$, $A_2 < 0$, $B_1 > 0$, $B_2 < 0$.

Theorem 2.1 *Given* (1), *suppose that* $C_1 = A_1B_1$, $C_2 = A_1B_2$, $C_3 = A_2B_1$, $C_4 = A_2B_2$ *and* $A_1 > 0$, $A_2 < 0$, $B_1 > 0$, $B_2 < 0$, $A_1^2 - 4A_2 > 0$, $B_1^2 - 4B_2 > 0$. *Moreover, suppose that* $f(x, y)$ *is a continues function in* $\overline{D}$ *that satisfies* $f(a, b) = 0$ *and has the following asymptotic behavior*

$$f(x, y) = o\big[(x-a)^{\delta_1}\big], \quad \delta_1 > \lambda_1 \text{ as } x \to a,$$
$$f(x, y) = o\big[(y-b)^{\gamma_1}\big], \quad \gamma_1 > \nu_1, \text{ as } y \to b.$$

Then (1) *always admits a solution with* $u(x, y) \in C(\overline{D})$ *and* $u(x, y) \to 0$ *as* $(x, y) \to \Gamma_i$ *for* $i = 1, 2$, *and its general solution contains two arbitrary functions each of one variable.*

We claim that the solution is given by means of the following formula:

$$u(x, y) = (y-b)^{\nu_1}\varphi_1(x) + (x-a)^{\lambda_1} \times \Bigg[\psi_1(y) - \frac{1}{\sqrt{B_1^2 + 4|B_2|}} \int_b^y \Bigg[\nu_2^2\left(\frac{s-b}{y-b}\right)^{|\nu_2|} - \nu_1^2\left(\frac{y-b}{s-b}\right)^{\nu_1}\Bigg] \times \frac{\psi_1(s)}{s-b}\Bigg] ds + f(x, y) - \frac{1}{\sqrt{A_1^2 + 4|A_2|}}$$

$$\times \int_a^x \left[\lambda_2^2\left(\frac{t-a}{x-a}\right)^{|\lambda_2|} - \lambda_1^2\left(\frac{x-a}{t-a}\right)^{\lambda_1}\right]\frac{f(t,y)}{t-a}dt$$
$$- \frac{1}{\sqrt{B_1^2+4|B_2|}}\int_b^y \left[\nu_2^2\left(\frac{s-b}{y-b}\right)^{|\nu_2|} - \nu_1^2\left(\frac{y-b}{s-b}\right)^{\nu_1}\right]$$
$$\times \frac{f(x,s)}{s-b}ds + \frac{1}{\sqrt{A_1^2+4|A_2|}\sqrt{B_1^2+4|B_2|}}$$
$$\times \int_b^y \left[\nu_2^2\left(\frac{s-b}{y-b}\right)^{|\nu_2|} - \nu_1^2\left(\frac{y-b}{s-b}\right)^{\nu_1}\right]\frac{1}{s-b}$$
$$\times \int_a^x \left[\lambda_2^2\left(\frac{t-a}{x-a}\right)^{|\lambda_2|} - \lambda_1^2\left(\frac{x-a}{t-a}\right)^{\lambda_1}\right]\frac{f(t,s)}{t-a}dtds.$$

In this formula $\varphi_1(x)$, $\psi_1(y)$ are arbitrary continuous functions on Γ_1 and Γ_2, satisfying $\varphi_1(x) \to 0$ as $x \to a$ and $\psi_1(y) \to 0$ as $y \to b$ and such that their behavior is governed by the following asymptotic formulas:

$$\varphi_1(x) = o\left[(x-a)^{\varepsilon}\right], \quad as\ x \to 0,$$
$$\psi_1(y) = o\left[(y-b)^{\gamma_2}\right], \quad \gamma_2 > \nu_1\ as\ y \to b,$$
$$\lambda_1 = \frac{\sqrt{A_1^2-4A_2}-A_1}{2}, \qquad \lambda_2 = \frac{-A_1-\sqrt{A_1^2-4A_2}}{2}$$
$$\nu_1 = \frac{\sqrt{B_1^2-4B_2}-B_1}{2}, \qquad \nu_2 = \frac{-B_1-\sqrt{B_1^2-4B_2}}{2}$$

3 Other Theorems

Theorem 3.1 *In* (1.1), *let $C_1 = A_1B_1$, $C_2 = A_1B_2$, $C_3 = A_2B_1$, $C_4 = A_2B_2$ and $A_1 < 0$, $A_2 > 0$, $B_1 < 0$, $B_2 > 0$, $A_1^2 - 4A_2 > 0$, $B_1^2 - 4B_2 > 0$. Moreover, let $f(x,y) \in C(\overline{D})$, $f(a,b) = 0$ with the following asymptotic behavior on the boundaries Γ_1 and Γ_2*:

$$f(x,y) = o\left[(x-a)^{\delta_2}\right], \quad \delta_2 > \lambda_3\ as\ x \to a,$$
$$f(x,y) = o\left[(y-b)^{\gamma_3}\right], \quad \gamma_3 > \nu_3,\ as\ y \to b.$$

Then the non-homogeneous integral equation (1.1) *in the class $C(\overline{D})$, approaching zero at Γ_1 and Γ_2, is always solvable and its general solution contain four arbitrary functions of one variable.*

The solution is in fact given by means of following formula

$$
\begin{aligned}
u(x,y) &= (y-b)^{\nu_3}\varphi_1(x) + (y-b)^{\nu_4}\varphi_2(x) + (x-a)^{\lambda_3} \\
&\quad\times\left[\psi_1(y) - \frac{1}{\sqrt{B_1^2-4B_2}}\int_b^y\left[\nu_4^2\left(\frac{y-b}{s-b}\right)^{\nu_4} - \nu_3^2\left(\frac{y-b}{s-b}\right)^{\nu_3}\right]\frac{\psi_1(s)}{s-b}ds\right] \\
&\quad-(x-a)^{\lambda_4}\left[\psi_2(y) - \frac{1}{\sqrt{B_1^2-4B_2}}\int_b^y\left[\nu_4^2\left(\frac{y-b}{s-b}\right)^{\nu_4} - \nu_3^2\left(\frac{y-b}{s-b}\right)^{\nu_3}\right]\right. \\
&\quad\left.\times\frac{\psi_2(s)}{s-b}ds\right] + f(x,y) - \frac{1}{\sqrt{A_1^2-4A_2}} \\
&\quad\times\int_a^x\left[\lambda_4^2\left(\frac{x-a}{t-a}\right)^{\lambda_4} - \lambda_3^2\left(\frac{x-a}{t-a}\right)^{\lambda_3}\right]\frac{f(t,y)}{t-a}dt \\
&\quad-\frac{1}{\sqrt{B_1^2-4B_2}}\int_b^y\left[\nu_4^2\left(\frac{y-b}{s-b}\right)^{\nu_4} - \nu_3^2\left(\frac{y-b}{s-b}\right)^{\nu_3}\right]\frac{f(x,s)}{s-b}ds \\
&\quad+\frac{1}{\sqrt{A_1^2-4A_2}\sqrt{B_1^2-4B_2}}\int_b^y\left[\nu_4^2\left(\frac{y-b}{s-b}\right)^{\nu_4} - \nu_3^2\left(\frac{y-b}{s-b}\right)^{\nu_3}\right] \\
&\quad\times\frac{1}{s-b}\int_a^x\left[\lambda_4^2\left(\frac{x-a}{t-a}\right)^{\lambda_4} - \lambda_3^2\left(\frac{x-a}{t-a}\right)^{\lambda_3}\right]\frac{f(t,y)}{t-a}dtds.
\end{aligned}
$$

Here $\varphi_i(x),\ \psi_i(y),\ i=1,2,$ *are arbitrary continuous functions on* Γ_1 *and* Γ_2 *respectively. Moreover, as* $x\to a,\ y\to b$ $\varphi_i(y)$ *and* $\psi_i(y)$ *approach zero and their behavior is determined by the following asymptotic formula*:

$$
\begin{aligned}
\varphi_1(x) &= o\big[(x-a)^{\varepsilon}\big], \quad \varepsilon>0,\ \text{as } x\to a, \\
\varphi_2(x) &= o\big[(x-a)^{\varepsilon}\big], \quad \varepsilon>0,\ \text{as } x\to a, \\
\psi_1(y) &= o\big[(y-b)^{\gamma_4}\big], \quad \gamma_4>\nu_3 \text{ as } y\to b, \\
\psi_2(y) &= o\big[(y-b)^{\gamma_5}\big], \quad \gamma_5>\nu_3 \text{ as } y\to b,
\end{aligned}
$$

$$
\lambda_3 = \frac{|A_1|+\sqrt{A_1^2-4A_2}}{2}, \qquad \lambda_4 = \frac{|A_1|-\sqrt{A_1^2-4A_2}}{2},
$$

$$
\nu_3 = \frac{|B_1|+\sqrt{B_1^2-4B_2}}{2}, \qquad \nu_4 = \frac{|B_1|-\sqrt{B_1^2-4B_2}}{2}.
$$

Theorem 3.2 *In* (1.1), *let* $C_1=A_1B_1,\ C_2=A_1B_2,\ C_3=A_2B_1,\ C_4=A_2B_2$ *and* $A_1<0,\ A_2<0,\ B_1<0,\ B_2<0,\ A_1^2>4A_2,\ B_1^2>4B_2$. *Moreover, let* $f(x,y)\in C(\overline{D}),\ f(a,b)=0$ *with the following asymptotic behavior at the boundaries* Γ_1

and Γ_2:

$$f(x,y) = o\left[(x-a)^{\delta_3}\right], \quad \delta_3 > \lambda_5 \text{ as } x \to a,$$
$$f(x,y) = o\left[(y-b)^{\gamma_6}\right], \quad \gamma_6 > \nu_5, \text{ as } y \to b.$$

Then the non-homogeneous integral equation (1.1) *is always solvable in the class of* $C(\overline{D})$*-functions approaching zero at* Γ_1 *and* Γ_2*, and its general solution contains two arbitrary functions of one variable.*

The solution is in fact given by means of the following formula

$$\begin{aligned} u(x,y) &= (y-b)^{\nu_5}\varphi(x) + (x-a)^{\lambda_5} \\ &\times \left[\psi(y) - \frac{1}{\sqrt{B_1^2+4|B_2|}}\int_b^y \left[\nu_6^2\left(\frac{s-b}{y-b}\right)^{|\nu_6|} - \nu_5^2\left(\frac{y-b}{s-b}\right)^{\nu_5}\right]\right. \\ &\times \left.\frac{\psi(s)}{s-b}ds\right] + f(x,y) - \frac{1}{\sqrt{A_1^2+4|A_2|}}\int_a^x \left[\lambda_6^2\left(\frac{t-a}{x-a}\right)^{|\lambda_6|}\right. \\ &- \left.\lambda_5^2\left(\frac{x-a}{t-a}\right)^{\lambda_5}\right]\frac{f(t,y)}{t-a}dt - \frac{1}{\sqrt{B_1^2+4|B_2|}} \\ &\times \int_b^y \left[\nu_6^2\left(\frac{s-b}{y-b}\right)^{|\nu_6|} - \nu_5^2\left(\frac{y-b}{s-b}\right)^{\nu_5}\right] \\ &\times \frac{f(x,s)}{s-b}ds + \frac{1}{\sqrt{A_1^2+4|A_2|}\sqrt{B_1^2+4|B_2|}}\int_b^y \left[\nu_6^2\left(\frac{s-b}{y-b}\right)^{|\nu_6|}\right. \\ &- \left.\nu_5^2\left(\frac{y-b}{s-b}\right)^{\nu_5}\right]\frac{1}{s-b}\int_a^x \left[\lambda_6^2\left(\frac{t-a}{x-a}\right)^{|\lambda_6|}\right. \\ &- \left.\lambda_5^2\left(\frac{x-a}{t-a}\right)^{\lambda_5}\right]\frac{f(t,s)}{t-a}dtds. \end{aligned}$$

Here $\varphi(x)$, $\psi(y)$ *are arbitrary continuous functions on* Γ_1 *and* Γ_2 *respectively. Moreover, at* $x \to a$, $y \to b$ $\varphi(x)$ *and* $\psi(y)$ *approach zero and their behavior is determined by the following asymptotic formulas*:

$$\varphi(x) = o\left[(x-a)^{\varepsilon}\right], \quad \varepsilon > 0, \text{ as } x \to a,$$
$$\psi(y) = o\left[(y-b)^{\gamma_7}\right], \quad \gamma_7 > \nu_5 \text{ as } y \to b,$$
$$\lambda_5 = \frac{|A_1| + \sqrt{A_1^2 - 4A_2}}{2}, \qquad \lambda_6 = \frac{|A_1| - \sqrt{A_1^2 - 4A_2}}{2}$$

$$\nu_5 = \frac{|B_1| + \sqrt{B_1^2 - 4B_2}}{2}, \qquad \nu_6 = \frac{|B_1| - \sqrt{B_1^2 - 4B_2}}{2}.$$

Theorem 3.3 *In (1.1), let $C_1 = A_1B_1$, $C_2 = A_1B_2$, $C_3 = A_2B_1$, $C_4 = A_2B_2$ and $A_1 > 0$, $A_2 > 0$, $B_1 > 0$, $B_2 > 0$, $A_1^2 - 4A_2 > 0$, $B_1^2 - 4B_2 > 0$. Moreover, let $f(x,y) \in C(\overline{D})$, $f(a,b) = 0$ with the following asymptotic behavior at the boundaries Γ_1 and Γ_2:*

$$f(x,y) = o\big[(x-a)^{\delta_8}\big], \quad \delta_8 > \lambda_8 \text{ as } x \to a,$$
$$f(x,y) = o\big[(y-b)^{\gamma_8}\big], \quad \gamma_8 > \nu_8, \text{ as } y \to b.$$

Then the non-homogeneous integral equation (1.1) has a unique solution in the class $C(\overline{D})$, approaching zero at Γ_1 and Γ_2, and given by means of the following formula

$$\begin{aligned}
u(x,y) &= f(x,y) - \frac{1}{\sqrt{A_1^2 - 4A_2}} \int_a^x \left[\lambda_8^2\left(\frac{t-a}{x-a}\right)^{\lambda_8} - \lambda_7^2\left(\frac{t-a}{x-a}\right)^{\lambda_7}\right]\frac{f(t,y)}{t-a}dt \\
&\quad - \frac{1}{\sqrt{B_1^2 - 4B_2}} \int_b^y \left[\gamma_8^2\left(\frac{s-b}{y-b}\right)^{\gamma_8} - \gamma_7^2\left(\frac{s-b}{y-b}\right)^{\nu_7}\right]\frac{f(x,s)}{s-b}ds \\
&\quad + \frac{1}{\sqrt{A_1^2 - 4A_2}\sqrt{B_1^2 - 4B_2}} \int_b^y \left[\gamma_8^2\left(\frac{s-b}{y-b}\right)^{\gamma_8} - \gamma_7^2\left(\frac{s-b}{y-b}\right)^{\nu_7}\right]\frac{1}{s-b} \\
&\quad \times \int_a^x \left[\lambda_8^2\left(\frac{t-a}{x-a}\right)^{\lambda_8} - \lambda_7^2\left(\frac{t-a}{x-a}\right)^{\lambda_7}\right]\frac{f(t,s)}{t-a}dtds.
\end{aligned}$$

Here

$$\lambda_7 = \frac{-A_1 + \sqrt{A_1^2 - 4A_2}}{2}, \qquad \lambda_8 = \frac{-A_1 - \sqrt{A_1^2 - 4A_2}}{2}$$
$$\nu_7 = \frac{-B_1 + \sqrt{B_1^2 - 4B_2}}{2}, \qquad \nu_8 = \frac{-B_1 - \sqrt{B_1^2 - 4B_2}}{2}.$$

Theorem 3.4 *In (1.1), let $C_1 = A_1B_1$, $C_2 = A_1B_2$, $C_3 = A_2B_1$, $C_4 = A_2B_2$ and $A_1 < 0$, $B_1 < 0$, $A_1^2 = 4A_2$, $B_1^2 = 4B_2$. Moreover, let $f(x,y) \in C(\overline{D})$, $f(a,b) = 0$ with the following asymptotic behavior at the boundaries Γ_1 and Γ_2:*

$$f(x,y) = o\big[(x-a)^{\delta_8}\big], \quad \delta_8 > \frac{|A_1|}{2} \text{ as } x \to a,$$
$$f(x,y) = o\big[(y-b)^{\gamma_9}\big], \quad \gamma_9 > \frac{|B_1|}{2}, \text{ as } y \to b.$$

Then the non-homogeneous integral equation (1.1) *is always solvable in the class of* $C(\overline{D})$*-functions approaching zero at* Γ_1 *and* Γ_2, *and its general solution contains two arbitrary functions of one variable.*

The solution is in fact given by means the following formula:

$$u(x,y)=(x-a)^{\frac{|A_1|}{2}}\big[\psi_1(y)+\ln(x-a)\psi_2(y)\big]+\frac{|A_1|}{2}\int_a^x\left(\frac{x-a}{t-a}\right)^{\frac{|A_1|}{2}}$$
$$\times\left[2+\frac{|A_1|}{2}\ln\left(\frac{x-a}{t-a}\right)\right]\frac{f(t,y)}{t-a}dt+\frac{|B_1|}{2}\int_b^y\left(\frac{y-b}{s-b}\right)^{\frac{|B_1|}{2}}$$
$$\times\left[2+\frac{|B_1|}{2}\ln\left(\frac{y-b}{s-b}\right)\right]\frac{f(x,s)}{s-b}ds+\frac{|A_1B_1|}{4}\int_b^y\left(\frac{y-b}{s-b}\right)^{\frac{|B_1|}{2}}$$
$$\times\left[2+\frac{|B_1|}{2}\ln\left(\frac{y-b}{s-b}\right)\right]\frac{1}{s-b}\int_a^x\left(\frac{x-a}{t-a}\right)^{\frac{|A_1|}{2}}$$
$$\times\left[2+\frac{|A_1|}{2}\ln\left(\frac{x-a}{t-a}\right)\right]\frac{f(t,s)}{t-a}dtds.$$

Here $\varphi_j(y),\ j=1,2$ *are arbitrary continuous functions on* Γ_2. *Moreover, as* $y\to b$ $\varphi_j(y)$ *approaches zero and its asymptotic behavior is determined by the following asymptotic formula*:

$$\varphi_j(y)=o\big[(y-b)^{\varepsilon}\big],\quad \varepsilon>0,\ \text{as } y\to b,\ j=1,2.$$

Theorem 3.5 *In* (1.1), *let* $C_1=A_1B_1,\ C_2=A_1B_2,\ C_3=A_2B_1,\ C_4=A_2B_2$ *and* $A_1>0,\ B_1>0,\ A_1^2=4A_2,\ B_1^2=4B_2$. *Moreover, let* $f(x,y)\in C(\overline{D}),\ f(a,b)=0$ *with the following asymptotic behavior at the boundaries* Γ_1 *and* Γ_2:

$$f(x,y)=o\big[(x-a)^{\varepsilon}\big],\quad \text{as } x\to a,$$
$$f(x,y)=o\big[(y-b)^{\varepsilon}\big],\quad \text{as } y\to b.$$

Then the non-homogeneous integral equation (1.1) *has a unique solution in the class* $C(\overline{D})$ *approaching zero at* Γ_1 *and* Γ_2, *and given by means of the following formula*:

$$u(x,y)=f(x,y)-\frac{A_1}{2}\int_a^x\left(\frac{t-a}{x-a}\right)^{\frac{A_1}{2}}\left[2-\frac{A_1}{2}\ln\left(\frac{x-a}{t-a}\right)\right]\frac{f(t,y)}{t-a}dt$$
$$-\frac{B_1}{2}\int_b^y\left(\frac{s-b}{y-b}\right)^{\frac{B_1}{2}}\left[2-\frac{B_1}{2}\ln\left(\frac{y-b}{s-b}\right)\right]\frac{f(x,s)}{s-b}ds$$
$$+\frac{A_1B_1}{4}\int_b^y\left(\frac{s-b}{y-b}\right)^{\frac{B_1}{2}}$$

$$\times \left[2 - \frac{B_1}{2}\ln\left(\frac{y-b}{s-b}\right)\right]\frac{1}{s-b}\int_a^x \left(\frac{t-a}{x-a}\right)^{\frac{A_1}{2}}$$
$$\times \left[2 - \frac{A_1}{2}\ln\left(\frac{x-a}{t-a}\right)\right]\frac{f(t,s)}{t-a}dtds.$$

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Optimal Control Problem on Optimization of Resources Productivity

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Abstract The paper is devoted to the optimal control problem which is based on the model of optimization of resources productivity. Model analysis is implemented within the framework of Pontryagin maximum principle for the problems with infinite time horizon. Qualitative analysis of the Hamiltonian system allows to formulate necessary and sufficient conditions of existence of a steady state in terms of the model parameters. Under these conditions and an assumption on the saddle character of the steady state, we construct a nonlinear regulator which allows to approximate optimal trajectories by the solutions of the stabilized Hamiltonian system at a vicinity of the steady state. Finally, comparative analysis of results of numerical simulations is carried out.

Keywords Optimal control · Stabilization · Numerical algorythm · Feedback control

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1 Introduction

The paper is devoted to investigation of the model on optimization of resources productivity by means of balanced investment in economy's dematerialization. The

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model is based on the economic growth theory [3, 5, 7, 12, 13] and inherits elements of economic growth models introduced in [1, 2, 6, 9, 14, 16].

It is assumed that the investment in resources productivity stimulates its relative growth. Output is represented as an exponential production function of the Cobb–Douglas type, it depends on main production factor which is material use. Balance relation takes into account three components: investment in raising resources productivity, expenditures on natural resources and consumption. Quality of the control process is valued by means of the integrated consumption index discounted on the infinite time interval. The connection between expenditures on natural resources and their rests is expressed with help of the price formation mechanism which supposes inverse proportionality between stocks of materials and prices on them. Increasing prices negatively influence the consumption index which should be maximized in the model as the basic element of the utility function.

The problem is to find the optimal proportion of investment in the dynamic process with maximization of the utility function given as the integrated consumption index over trajectories of the economic system. The model is examined within the framework of the Pontryagin maximum principle [11] with special characteristics of infinite horizon [4]. Specific features of the corresponding Hamiltonian system are examined within the qualitative theory of differential equations [8]. Particularly, necessary and sufficient conditions of existence of a steady state are formulated. Owing to the numerical calculations for the data by Chinese economy it is shown that the steady state has a saddle character. Due to this fact the nonlinear regulator [14] can be constructed, which allows to stabilize the Hamiltonian system at a steady state neighborhood. Integration of the original Hamiltonian system is performed using stabilized solutions in the backward time. Finally, the comparison of the stabilized solutions and optimal trajectories at a steady state neighborhood is carried out.

2 Model Description

The main phase variables of the model are presented by the resource use $m = m(t)$, the cumulative resource consumption, which is introduced as the integrated material use $M(t) = \int_0^t m(s)ds \leq M_0$, where parameter M_0 is the initial stock of the natural resources, and the current production $y = y(t)$. Resources productivity $z = z(t)$ is the ratio of the production $y(t)$ to the used materials $m(t)$

$$z(t) = \frac{y(t)}{m(t)}. \tag{2.1}$$

The exponential production function of the Cobb–Douglas type is chosen for the first version of the model

$$y(t) = ae^{bt}m^{\alpha}(t), \quad a > 0,\ b \geq 0,\ 0 \leq \alpha < 1. \tag{2.2}$$

Here the parameter a is a scale factor; the growth rate b indicates the growth process of production $y(t)$ due to development of basic production factors such as capital, labor, technology, etc.; the symbol α denotes the elasticity coefficient of natural resources.

2.1 Price Formation Mechanism

The price formation mechanism provides the growth of prices $p(t)$ on nature resource in the case of their exhaustion. It is assumed that prices are growing according to the inversely proportional rule of resource exhaustion

$$p = p(t) = p_0 \left(1 - \frac{M(t)}{M_0}\right)^{-1}. \tag{2.3}$$

Here the symbol p_0 denotes the initial price on natural resources. Formula (2.3) envisages that price $p(t)$ can grow rapidly to infinity according to the hyperbolic law when the total material use $M(t)$ reaches its limitation M_0.

2.2 Balance Equation

In the balance equation it is taken into account that production $y(t)$ in period t is shared between consumption $c(t)$, from the one hand, and the growing cost of natural resources $p(t)m(t)$ plus investment $u(t)y(t)$ in improving the resource productivity, from the other hand,

$$y(t) = c(t) + p(t)m(t) + u(t)y(t). \tag{2.4}$$

Let us assume that there exists lower and upper bounds $\underline{u}$, $\overline{u}$ for investment intensity $u(t)$, i.e. $0 < \underline{u} \leq u(t) \leq \overline{u} < 1$. Deducing the consumption intensity $c(t)/y(t)$ from (2.4) through the resource intensity $m(t)/y(t)$ we obtain the following relation

$$\frac{c(t)}{y(t)} = 1 - p(t)\frac{m(t)}{y(t)} - u(t). \tag{2.5}$$

Using representation of the consumption $c(t)$ (2.5), we introduce an integrated logarithmic consumption index discounted with the discount rate ρ, $\rho > 0$, on the infinite time interval

$$J = \int_0^{+\infty} e^{-\rho t} \ln c(t) dt = \int_0^{+\infty} e^{-\rho t} \ln \big(y(t) - p(t)m(t) - u(t)y(t)\big) dt.$$

2.3 Model Dynamics

Let assume that the relative raise in the resource productivity $z(t)$ is proportional to the portion of the assigned investment $u(t)$ which can be interpreted as investment in "green" technology (see [7, 16])

$$\frac{\dot{z}(t)}{z(t)} = \beta u(t), \quad \beta \geq 0. \tag{2.6}$$

Here the parameter β describes the effectiveness of investment $u(t)$ in raising the resources productivity. Taking into account the definition (2.1) of the resources productivity $z(t)$, its dynamic (2.6) and the production function (2.2) for the output $y(t)$ one can obtain the differential equation for the used materials $m(t)$

$$\dot{m}(t) = \frac{m(t)}{1-\alpha}\big(b - \beta u(t)\big). \tag{2.7}$$

Equation (2.7) shows that the rate of the resource consumption is influenced by the production growth rate b and can be reduced only by investment $u(t)$ in raising the resource productivity.

Let us introduce phase variables

$$\begin{aligned} x_1(t) &= \frac{m(t)}{M_0 - M(t)}, \qquad x_2(t) = \frac{p(t)m(t)}{y(t)}, \\ x_1(0) &= x_1^0 = \frac{m_0}{M_0}, \qquad x_2(0) = x_2^0 = \frac{p_0 m_0^{1-\alpha}}{a}. \end{aligned} \tag{2.8}$$

where the first variable $x_1(t)$ is the share of material use $m(t)$ in the current stock $(M_0 - M(t))$ of natural resources, and second one is the ratio of expenditures $p(t)m(t)$ on natural resources to the production output $y(t)$.

3 Optimal Control Problem

The problem is to maximize the utility function

$$J(\cdot) = \int_0^{+\infty} e^{-\rho t}\big(\ln x_1(t) - \ln x_2(t) + \ln\big(1 - x_2(t) - u(t)\big)\big)dt \tag{3.1}$$

over the control process $(x_1(t), x_2(t), u(t))$ of the dynamic system

$$\dot{x}_1(t) = x_1(t)\left(x_1(t) + \frac{b - \beta u(t)}{1-\alpha}\right), \qquad \dot{x}_2(t) = x_2(t)\big(x_1(t) - \beta u(t)\big) \tag{3.2}$$

satisfying initial conditions $x_1(0) = x_1^0$, $x_2(0) = x_2^0$ (2.8) and subject to constraints for the control variable

$$0 < \underline{u} \leq u(t) \leq \overline{u} < 1.$$

Remark 3.1 It is supposed that the following inequality is valid $\beta \underline{u} \geq b + (1 - \alpha)\frac{m_0}{M_0} > 0$.

3.1 Hamiltonian Function

Analysis of the optimal control problem is based on generalized Pontryagin maximum principle [4]

The stationary Hamiltonian function [16] for the optimal control problem (3.1)–(3.2) has the following structure

$$\widehat{H}(x, \psi, u) = \ln x_1 - \ln x_2 + \ln(1 - x_2 - u) + \psi_1 x_1 \left(x_1 + \frac{b - \beta u}{1 - \alpha} \right) + \psi_2 x_2 (x_1 - \beta u)$$

where $\psi = (\psi_1, \psi_2)$ and $x = (x_1, x_2)$ are vectors of adjoint and phase variables respectively.

Maximizing the Hamiltonian function $\widehat{H}(x, \psi, u)$ with respect to the control variable u, one can obtain the following structure of the control $\widehat{u}$

$$\widehat{u} = \begin{cases} u_1 = \underline{u}, & D_1 = \{(x, \psi) : 1 - x_2 - v(x, \psi) \leq \underline{u}\}, \\ u_2 = 1 - x_2 - v(x, \psi), & D_2 = \{(x, \psi) : \underline{u} \leq 1 - x_2 - v(x, \psi) \leq \overline{u}\}, \\ u_3 = \overline{u}, & D_3 = \{(x, \psi) : 1 - x_2 - v(x, \psi) \geq \overline{u}\}, \end{cases} \tag{3.3}$$

where symbol $v(x, \psi)$ is defined as follows $v(x, \psi) = -\frac{1-\alpha}{\beta} \frac{1}{\psi_1 x_1 + (1-\alpha)\psi_2 x_2}$. Thus, one can single out three domains D_i $(i = 1, 2, 3)$ of definition of the maximized Hamiltonian function.

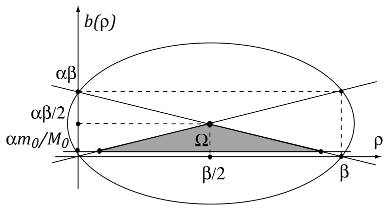
Hamiltonian systems in each domain can be found by formulas

$$\dot{x}_i(t) = \left. \frac{\partial H_i(x, \psi)}{\partial \psi_i} \right|_{\substack{x=x(t), \\ \psi=\psi(t)}}, \qquad \dot{\psi}_i(t) = \rho \psi_i(t) - \left. \frac{\partial H_i(x, \psi)}{\partial x_i} \right|_{\substack{x=x(t), \\ \psi=\psi(t)}}, \quad i = 1, 2.$$

Let us introduce new adjoint variables $z_i = x_i \psi_i$, $i = 1, 2$. In these variables $x(t) = (x_1(t), x_2(t))$, $z(t) = (z_1(t), z_2(t))$ Hamiltonian systems have the following structures

In domains D_1, D_3	In the domain D_2
$\dot{x}_1(t) = x_1(t)(x_1(t) + \frac{b - \beta u_i}{1-\alpha})$	$\dot{x}_1(t) = x_1(t)(x_1(t) + \frac{\beta(x_2(t) + v(z(t)) - 1) - b}{1-\alpha})$
$\dot{x}_2(t) = x_2(t)(x_1(t) - \beta u_i)$	$\dot{x}_2(t) = x_2(t)(x_1(t) - \beta(x_2(t) + v(z(t)) - 1))$
$\dot{z}_1(t) = \rho z_1(t) - x_1(t)(z_1(t) + z_2(t)) - 1$	$\dot{z}_1(t) = \rho z_1(t) - x_1(t)(z_1(t) + z_2(t)) - 1$
$\dot{z}_2(t) = \rho z_2(t) + \frac{1-u_i}{1-u_i-x_2(t)}$	$\dot{z}_2(t) = \rho z_2(t) + \frac{x_2(t)}{v(z(t))} + 1$

Fig. 1 Area Ω of existence of the steady state



3.2 Qualitative Analysis of the Hamiltonian System

First of all let us formulate necessary and sufficient conditions of existence of a steady state in the domain D_2 of the non-constant control u_2.

Proposition 3.2 *The Hamiltonian system in the domain D_2 of the transient control regime u_2 has a steady state $(x^*, z^*) = (x_1^*, x_2^*, z_1^*, z_2^*)$ if and only if model parameters are located in the area Ω depicted in Fig.* 1.

If the steady state exists then its coordinates can be found analytically by formulas

$$x_1^* = \frac{b}{\alpha}, \qquad x_2^* = \frac{(\alpha\rho - b)((1-\alpha)\rho + \alpha\beta - b)}{\alpha\beta\rho},$$

$$z_1^* = (1-\alpha)\left(1 - \frac{\alpha\rho}{\alpha\beta - b}\right) z_2^*, \qquad z_2^* = -\frac{\alpha\beta - b}{\alpha\rho(1-\alpha)(\beta-\rho) + b(\alpha\beta - b)}.$$

Numerical experiments which have been carried out for the statistical data by Chinese economy [10] for the period from 1980 to 2010, show that the steady state has the saddle character, i.e. Jacobi matrix, evaluated at the steady state, has four different real eigenvalues, two of which are negative numbers and two others are positive ones $\lambda_1 < \lambda_2 < 0 < \lambda_3 < \lambda_4$. It allows to construct a nonlinear regulator (stabilizer) which stabilizes Hamiltonian system at a neighborhood of the steady state.

4 Nonlinear Stabilizer

Let us formulate the theorem which summarizes conditions for constructing nonlinear regulator and its properties.

Theorem 4.1 (Nonlinear stabilizer) *Let the Hamiltonian system at the domain D_2 of the transient control regime u_2 has a steady state (x^*, z^*) satisfying following conditions*

A1. *Jacobi matrix, evaluated at the steady state, has four different real eigenvalues, two of which are negative numbers and two others are positive ones* $\lambda_1 < \lambda_2 < 0 < \lambda_3 < \lambda_4$;

A2. *Eigenvectors* $h_i = (h_{i1}, h_{i2}, h_{i3}, h_{i4})$ *corresponding to negative eigenvalues* λ_i $(i = 1, 2)$ *meet inequality*: $h_{11}h_{22} \neq h_{12}h_{21}$.

Then there exists the nonlinear stabilizer with the following properties:

P1. *the stabilized Hamiltonian system generated by the nonlinear regulator is closed with respect to the phase variables* x_1, x_2;
P2. *the steady state of the stabilized system has coordinates* $x^* = (x_1^*, x_2^*)$ *which coincide with the phase coordinates* x^* *of the steady state* (x^*, z^*) *of the original Hamiltonian system*;
P3. *the stabilized system has two negative eigenvalues which are equal to* λ_1, λ_2;
P4. *the eigenvectors of the linearized system generated by the stabilizer can be evaluated by formulas*

$$\overline{h}_1 = \begin{pmatrix} h_{11} \\ h_{12} \end{pmatrix}, \qquad \overline{h}_2 = \begin{pmatrix} h_{21} \\ h_{22} \end{pmatrix}.$$

4.1 Algorithm for Constructing Nonlinear Stabilizer

The algorithm for constructing nonlinear stabilizer can be applied if the assumptions A1, A2 indicated at Theorem 4.1 are fulfilled.

1. The first step consists in constructing a plane by two eigenvectors h_1, h_2 corresponding to negative eigenvalues λ_1, λ_2.

$$\begin{aligned} \nu_{11}x_1 + \nu_{12}x_2 + \nu_{13}z_1 + \nu_{14}z_2 &= C_1, \\ \nu_{21}x_1 + \nu_{22}x_2 + \nu_{23}z_1 + \nu_{24}z_2 &= C_2. \end{aligned} \tag{4.1}$$

   Let us call this flat surface as eigen-plane. Since the steady state is located in the eigen-plane constants C1 and C2 can be uniquely found by substituting coordinates $(x_1^*, x_2^*, z_1^*, z_2^*)$ of the steady state in equalities (4.1).
2. Using (4.1) of the eigen-plane one can derive adjoint variables through the phase ones $z_1 = \widehat{z}_1(x)$, $z_2 = \widehat{z}_2(x)$.

$$\begin{aligned} \widehat{z}_1(x) &= z_1^* + \gamma_{11}\big(x_1 - x_1^*\big) + \gamma_{12}\big(x_2 - x_2^*\big), \\ \widehat{z}_2(x) &= z_2^* + \gamma_{21}\big(x_1 - x_1^*\big) + \gamma_{22}\big(x_2 - x_2^*\big), \end{aligned} \tag{4.2}$$

   where coefficients γ_{ij} are found by formulas due to the condition A2 of Theorem 4.1

$$\gamma_{11} = -\frac{\begin{vmatrix} h_{12} & h_{13} \\ h_{22} & h_{23} \end{vmatrix}}{\begin{vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{vmatrix}}, \qquad \gamma_{12} = \frac{\begin{vmatrix} h_{11} & h_{13} \\ h_{21} & h_{23} \end{vmatrix}}{\begin{vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{vmatrix}},$$

$$\gamma_{21} = -\frac{\begin{vmatrix} h_{12} & h_{14} \\ h_{22} & h_{24} \end{vmatrix}}{\begin{vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{vmatrix}}, \qquad \gamma_{22} = \frac{\begin{vmatrix} h_{11} & h_{14} \\ h_{21} & h_{24} \end{vmatrix}}{\begin{vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{vmatrix}}.$$

3. On this step the representation of adjoint variables (4.2) is substituted in the relation for the control u_2 (3.3) corresponding to the domain D_2.

$$\widehat{u}(x) = 1 - x_2 + \frac{1-\alpha}{\beta}\frac{1}{\widehat{z}_1(x) + (1-\alpha)\widehat{z}_2(x)} = 1 - x_2 - v\big(\widehat{z}(x)\big). \tag{4.3}$$

4. Stabilized Hamiltonian system in the domain D_2 of the transient control regime u_2 can be obtained by replacing adjoint variables z_1, z_2 by their expressions $\widehat{z}_1(x), \widehat{z}_2(x)$ in the first two equations of the original Hamiltonian system.

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)\left(x_1(t) + \frac{\beta}{1-\alpha}\big(x_2(t) + v\big(\widehat{z}\big(x(t)\big)\big) - 1\big) + \frac{b}{1-\alpha}\right), \\ \dot{x}_2(t) &= x_2(t)\big(x_1(t) + \beta\big(x_2(t) + v\big(\widehat{z}\big(x(t)\big)\big) - 1\big)\big). \end{aligned} \tag{4.4}$$

4.2 Construction of the Optimal Trajectories

Relying on the qualitative theory of differential equations [8], it is known that the solution of ordinary differential equations converges to the saddle steady state by the tangent plane built by two eigenvectors corresponding to negative eigenvalues. Due to this fact, construction of the optimal trajectories is carried out by the following way.

1. Algorithm for constructing optimal solutions is based on stabilized trajectories $(\widehat{x}_1(t), \widehat{x}_2(t))$ obtained as a solution of the stabilized Hamiltonian system at the steady state ε-vicinity [15].
2. Original Hamiltonian system is solved in backward time starting from points located both at the steady state ε-vicinity (O^*_ε) and in a δ-neighborhood ($\widehat{O}_\delta$) of stabilized trajectories $(\widehat{x}_1(t), \widehat{x}_2(t))$.
3. Algorithm is stopped when the constructed trajectory reaches original initial position (x^0_1, x^0_2).
4. If the integrated trajectory does not achieve the original initial point (x^0_1, x^0_2), then we exhaustively search for another initial position by looking through the points from the intersection of neighborhoods O^*_ε and $\widehat{O}_\delta$.

4.3 Numerical Experiments

The calculations have been carried out on the basis of the data on Chinese economy from 1980 to 2010, reduced to the values of 1980. The calibration of the parameters of the model has yielded the following values of the parameters $p_0 = 100$, $M_0 = 1.8121 \cdot 10^6$, $\beta = 1.5226$, $a = 64.3348$, $\alpha = 0.4091$, $b = 0.0686$, $\rho = 0.18$, $m_0 = 0.08565$, $\underline{u} = 0.0451$, $\overline{u} = 0.6360$. Steady state of the Hamiltonian system has coordinates $x^*_1 = 0.1676$, $x^*_2 = 0.0296$, $z^*_1 = 2.9449$, $z^*_2 = -5.7473$. Jacobi

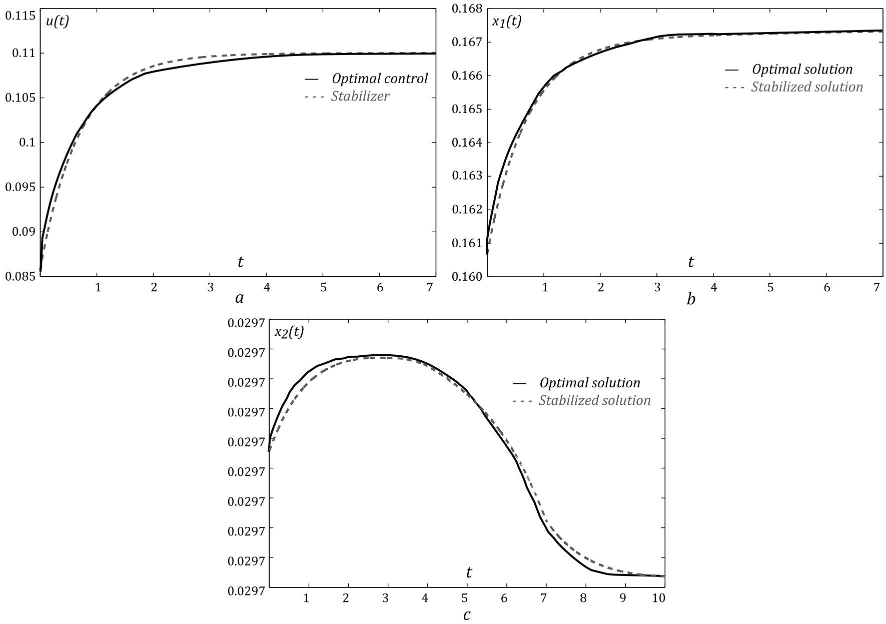


Fig. 2 Comparison of the optimal and stabilized solutions: (a) for the control variable $u(t)$, (b), (c) for the phase variables $(x_1(t), x_2(t))$

matrix evaluated at the steady state has the following eigenvalues $\lambda_1 = -1.457$, $\lambda_2 = -0.005$, $\lambda_3 = 0.185$, $\lambda_4 = 1.637$. Eigen-plane equations (4.1) look as follows

$$\widehat{z}_1(x) = 2.945 - 2.17\big(x_1 - x_1^*\big) + 2.5\big(x_2 - x_2^*\big),$$
$$\widehat{z}_2(x) = -5.747 + 0.44\big(x_1 - x_1^*\big) - 6.29\big(x_2 - x_2^*\big).$$

Figure 2(a) depicts optimal programming control $u(t)$ in comparison with the stabilizer $\widehat{u}(\widehat{x}(t))$ (4.3) at the ε-neighborhood of the steady state, where radius $\varepsilon = 10^{-3}$. Graphs of the optimal solutions $(x_1(t), x_2(t))$ and corresponding trajectories $(\widehat{x}_1(t), \widehat{x}_2(t))$ of the stabilized Hamiltonian system (4.4), constructed at the same ε-neighborhood of the steady state, are shown at Fig. 2(b)–(c).

One can see that the stabilized system solution and original one are very close to each other in the steady state neighborhood, moreover they have the similar behavior.

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Part III
Spaces of Differentiable Functions of Several Real Variables and Applications

Organizers: Victor Burenkov, Stefan Samko

The Amalgam Spaces $W(L^{p(x)}, \ell^{\{p_n\}})$ and Boundedness of Hardy–Littlewood Maximal Operators

A. Turan Gürkanlı

Abstract Let $L^{q(x)}(\mathbb{R})$ be variable exponent Lebesgue space and $\ell^{\{q_n\}}$ be discrete analog of this space. In this work we define the amalgam spaces $W(L^{p(x)}, L^{q(x)})$ and $W(L^{p(x)}, \ell^{\{q_n\}})$, and discuss some basic properties of these spaces. Since the global components $L^{q(x)}(\mathbb{R})$ and $\ell^{\{q_n\}}$ are not translation invariant, these spaces are not a Wiener amalgam space. But we show that there are similar properties of these spaces to the Wiener amalgam spaces. We also show that there is a variable exponent $q(x)$ such that the sequence space $\ell^{\{q_n\}}$ is the discrete space of $L^{q(x)}(\mathbb{R})$. By using this result we prove that $W(L^{p(x)}, \ell^{\{p_n\}}) = L^{p(x)}(\mathbb{R})$. We also study the frame expansion in $L^{p(x)}(\mathbb{R})$. At the end of this work we prove that the Hardy–Littlewood maximal operator from $W(L^{s(x)}, \ell^{\{t_n\}})$ into $W(L^{u(x)}, \ell^{\{v_n\}})$ is bounded under some assumptions.

Keywords Amalgam space · Variable exponent Lebesgue space · Hardy–Littlewood maximal operator

Mathematics Subject Classification (2010) 42B25 · 42B35

1 Introduction

The first appearance of amalgam can be traced to N. Wiener [20]. But the first systematic study of these spaces was undertaken by F. Holland [15]. A generalization of Wiener's definition was given by H.G. Feichtinger in [9]. Later a number of authors worked on these spaces, for example [11, 12, 16]. Also J.J. Fournier and J.S. Steward gave a good historical background of amalgam space cite18. A.T. Gürkanlı and I. Aydın was defined the weighted variable exponent amalgam space $W(L^{p(x)}, \ell^p_w)$ by using variable exponent Lebesgue space $L^{p(x)}$ as local component and investigated some properties of these spaces, [2, 13].

This paper is in final form and no version of it will be submitted for publication elsewhere.

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Also the function spaces with variable exponents have been studied in recent years by a significant number of authors. But the modern development started by O. Kovacik and J. Rakosnik [17]. The boundedness of the Maximal operator was open problem in $L^{p(x)}$ for long time. It was proved by L. Diening over bounded domains in [5] and by D.C. Ulribe, A. Fiorenza and C.J. Neugebauer over open domains in [3, 4].

In the present work we define the variable exponent amalgam spaces $W(L^{p(x)}, L^{q(x)})$ and $W(L^{p(x)}, \ell^{\{q_n\}})$, and discuss some basic properties of these spaces, where $\ell^{\{q_n\}}$ is discrete analog of $L^{q(x)}(\mathbb{R})$. Since the global components $L^{q(x)}(\mathbb{R})$ and $\ell^{\{q_n\}}$ are not translation invariant, these amalgam spaces are not a Wiener amalgam space. We also show that there is a variable exponent $q(x)$ such that the sequence space $\ell^{\{q_n\}}$ is the discrete space of $L^{q(x)}(\mathbb{R})$. By using this result we prove that $W(L^{p(x)}, \ell^{\{p_n\}}) = L^{p(x)}(\mathbb{R})$. Later we study on the frame expansion in $L^{p(x)}(\mathbb{R})$ and boundedness of the Hardy–Littlewood maximal operator from $W(L^{s(x)}, \ell^{\{t_n\}})$ into $W(L^{u(x)}, \ell^{\{v_n\}})$.

2 Notations

Let $\Omega \subset \mathbb{R}$ be an open subset. For a measurable function $p : \Omega \to [1, \infty)$ (called the variable exponent on Ω) put

$$p_* = \operatorname*{ess\,inf}_{x\in\Omega} p(x), \qquad p^* = \operatorname*{ess\,sup}_{x\in\Omega} p(x).$$

Define $P(\Omega)$ to be the set of measurable functions $p : \Omega \to [1, \infty)$ such that $1 < p_* \le p(x) \le p^* < \infty$. Throughout this paper we will assume that $p^* < \infty$. The generalized Lebesgue spaces (or Lebesgue space with variable exponent) $L^{p(x)}(\Omega)$ is defined to be the space of all measurable functions (equivalent classes) f on Ω for which

$$\varrho_p(f) = \int_\Omega \left|f(x)\right|^{p(x)} dx < \infty.$$

Equipped with the Luxemburg norm

$$\|f\|_{L^{p(x)}} = \inf\left\{\lambda > 0 : \varrho_p\left(\frac{f}{\lambda}\right) \le 1\right\},$$

$L^{p(x)}(\Omega)$ is a Banach space. If $p(x) = p$ is a constant function, then the norm $\|.\|_{L^{p(x)}}$ coincides with the usual Lebesgue norm $\|.\|_{L^p}$, [6, 17]. It is also known that if $p^* < \infty$ then $L^{p(x)}(\Omega)$ is solid, i.e if any measurable function g, for which there exists $f \in L^{p(x)}(\Omega)$ such that $|g(x)| \le |f(x)|$ locally almost everywhere, belongs to $L^{p(x)}(\Omega)$, with $\|g\|_{L^{p(x)}} \le \|f\|_{L^{p(x)}}$, [1].

Let $\mathcal{M}$ be the set of all mappings $a : \mathbb{Z} \to \mathbb{R}$, $a = \{a_n\}$. Denote by

$$\varepsilon = \{p \in \mathcal{M} : p \ge 1 \text{ for all } n \in \mathbb{Z}\},$$

$$p^* = \sup\{p_n : n \in \mathbb{Z}\}$$

and

$$\mathcal{B} = \left\{p \in \varepsilon : p^* < \infty\right\}.$$

Let $p \in \varepsilon$. Define the space $\ell^{\{p_n\}}$ by

$$\ell^{\{p_n\}} = \left\{a : \|a\|_{\{p_n\}} < \infty\right\},$$

where

$$\|a\|_{\{p_n\}} = \inf\left\{\lambda > 0 : \sum_{n\in\mathbb{Z}} \left|\frac{a_n}{\lambda}\right|^{p_n} \leq 1\right\}. \tag{2.1}$$

If the mapping $r \in \varepsilon$ is a constant, then we use the usual symbol ℓ^r and,

$$\|a\|_r = \left(\sum_{n\in\mathbb{Z}} |a_n|^r\right)^{\frac{1}{r}}.$$

If $p \in \mathcal{B}$, then by Lemma 2.5 in [18]

$$\ell^{\{p_n\}} = \left\{a : \sum_{n\in\mathbb{Z}} |a_n|^{p_n} < \infty\right\}.$$

Let $p, q \in \varepsilon$ and let T be a linear function from $\ell^{\{p_n\}}$ into $\ell^{\{q_n\}}$. We say that T is bounded if

$$\|T\| = \sup\left\{\|Ta\|_{\{p_n\}} : \|a\|_{\{p_n\}} \leq 1\right\} < \infty.$$

Let $\varepsilon \in \mathcal{M}$. We say that $\varepsilon \in \mathcal{P}$ if there exists a real number $C > 0$ such that

$$\sum_{n\in P(\varepsilon)} \varepsilon_n C^{\frac{1}{\varepsilon_n}} < \infty,$$

where

$$P(\varepsilon) = \{n \in \mathbb{Z} : \varepsilon_n > 0\},$$

[7, 18].

A sequence space X_d is called a Banach coordinate space (or shortly BK space) if it is a Banach space and the coordinate functions $P_n : X_d \to \mathbb{C}$ are continuous on X_d, i.e., the relations $x_n = (\alpha_j^n)$, $x = (\alpha_j) \in X_d$, $\lim_{n\to\infty} x_n = x$ imply $\lim_{n\to\infty} \alpha_j^n = \alpha_j$ $(j = 1, 2, \ldots)$.

Given any neighbourhood U of $0 \in \mathbb{R}$, a family $X = (x_i)_{i\in I} \subset \mathbb{R}$ is called U-dense if the family $(x_i + U)_{i\in I}$ covers $\mathbb{R}$. That is $\bigcup_{i\in I}(x_i + U) = \mathbb{R}$. The family X is called seperated if the sets $(x_i + U)_{i\in I}$ are pairwise disjoint. The family X is called relatively separated proved that it is a finite union of separated sets. We will call a family $X = (x_i)_{i\in I} \subset \mathbb{R}$ is well-spread in $\mathbb{R}$ if it is both U-dense and relatively separated.

A family of functions $\psi = \{\psi_i\}_{i\in I}$ on $\mathbb{R}$ is a bounded uniform partition of unity (BUPU or U-BUPU) if the following properties hold:

(a) $\sum_{i\in I} \psi_i \equiv 1$;
(b) $\sup \|\psi_i\|_{L^\infty} < \infty$;
(c) There exists a compact set $U \subset \mathbb{R}$ with nonempty interior and $X = (y_i)_{i\in I} \subset \mathbb{R}$ such that $\operatorname{supp}\psi_i \subset y_i + U$ for all $i \in I$;
(d) For each compact $K \subset \mathbb{R}$,

$$\sup_{x\in\mathbb{R}} \sharp\{i \in I : x \in K + y_i\} = \sup_{i\in I} \sharp\{j \in I : K + y_i \cap K + y_j \neq \phi\} < \infty$$

Let $1 \le r, s \le \infty$. Fix a compact $Q \subset \mathbb{R}$ with nonempty interior. Then the Wiener amalgam space $W(L^r, L^s)(\mathbb{R})$ with local component $L^r(\mathbb{R})$ and global component $L^s(\mathbb{R})$ is defined as the space of all measurable functions (equivalent classess) $f : \mathbb{R} \to \mathbb{C}$ such that $f\chi_K \in L^r(\mathbb{R})$ for each compact subset $K \subset \mathbb{R}$, for which the norm

$$\|f\|_{W(L^r,L^s)} = \|F_f\|_s = \big\| \|f\chi_{Q+x}\|_r \big\|_s$$

is finite, where χ_K is the characteristic function of K and

$$F_f(x) = \|f\chi_{Q+x}\|_r \in L^s(\mathbb{R}).$$

It is known that if $r_1 \ge r_2$ and $s_1 \le s_2$ then $W(L^{r_1}, L^{s_1})(\mathbb{R}) \subset W(L^{r_2}, L^{s_2})(\mathbb{R})$. If $r = s$ then $W(L^r, L^r)(\mathbb{R}) = L^r(\mathbb{R})$, [9, 10, 14].

For $f \in L^1_{loc}(\Omega)$, we define the (centered) Hardy–Littlewood maximal function Mf of f by

$$Mf(x) = \sup_{r>0} \frac{1}{|\tilde{B}(x,r)|} \int_{\tilde{B}(x,r)} |f(y)| dy, \quad \tilde{B}(x,r) = B(x,r) \cap \Omega, \tag{2.2}$$

where the supremum is taken over all balls $\tilde{B}(x,r)$ and $|.|$ denotes the volume of $\tilde{B}(x,r)$.

We will often need to assume that $p(x)$ satisfies the following two log-Hölder continuity conditions:

$$|p(x) - p(y)| \le \frac{C}{-\ln|x-y|}, \quad x, y \in \Omega,\ |x-y| \le \frac{1}{2} \tag{2.3}$$

and

$$|p(x) - p(y)| \le \frac{C}{\ln(e+|x|)}, \quad x, y \in \Omega,\ |y| > |x|. \tag{2.4}$$

Condition (2.4) is the natural analogue of (2.3) at infinity. It implies that there is some number p_∞ such that $p(x) \to p_\infty$ as $|x| \to \infty$, and this limit holds uniformly in all directions.

It is known that if Ω is open and bounded and (2.3) holds, then the Hardy–Littlewood maximal operator is bounded on $L^{p(x)}(\Omega)$, [5]. It is also known that if Ω is open and both (2.3) and (2.4) hold, then again the Hardy–Littlewood maximal operator is bounded on $L^{p(x)}(\Omega)$, [3, 4]. Although the Hardy–Littlewood maximal operator is bounded on $L^{p(x)}$ under some conditions, it is not bounded on many of the Wiener amalgam spaces.

3 The Amalgam Spaces $W(L^{p(x)}, \ell^{\{p_n\}})$ and $W(L^{p(x)}, L^{q(x)})$

In this section we will define the spaces $W(L^{p(x)}, \ell^{\{p_n\}})$ and $W(L^{p(x)}, L^{q(x)})$ like the Wiener amalgam space in [2, 9, 10, 14] and investigate some properties. Later we will compare these spaces with the Wiener amalgam spaces.

Suppose that $(y_i)_{i\in\mathbb{Z}}$ and U such that $(U+i)_{i\in\mathbb{Z}}$ is a partition of $\mathbb{R}$. Then $(\psi_i) = (\chi_{i+U})_{i\in\mathbb{Z}}$ is a BUPU.

Definition 3.1 Let $p(x)$ be a variable exponent on $\mathbb{R}$ and $(\psi_i)_{i\in\mathbb{Z}}$ is a BUPU as above. Then the *amalgam space* $W(L^{p(x)}, \ell^{\{q_n\}})$ consists of all (classes of) measurable functions f on $\mathbb{R}$ such that $f\chi_K \in L^{p(x)}(\mathbb{R})$ for each compact subset $K \subset \mathbb{R}$, and $\|f\|_W = \|f\|_{W(L^{p(x)},\ell^{\{q_n\}})} < \infty$, where

$$\|f\|_W = \|f\|_{W(L^{p(x)},\ell^{\{q_n\}})} = \left\| \left\{ \|f\psi_n\|_{L^{p(x)}} \right\}_{n\in\mathbb{Z}} \right\|_{\{q_n\}}.$$
$$= \left\| \left\{ \|f\chi_{U+n}\|_{L^{p(x)}} \right\}_{n\in\mathbb{Z}} \right\|_{\{q_n\}}. \quad (3.1)$$

Remark 3.1 By the definition of Wiener amalgam space $W(L^p, L^q)$, the global component L^q (or in the discrete case $W(L^p, \ell^q)$ the global component ℓ^q) is translation invariant. Hence as a consequence of Theorem 1 in [9], the definition of Wiener amalgam space $W(L^p, L^q)$ is not depend on the particular choice of compact subset $Q \subset \mathbb{R}$. But in the amalgam spaces $W(L^{p(x)}, \ell^{\{p_n\}})$, the global components $\ell^{\{p_n\}}$ is not translation invariant. We notice that the definition of the space $W(L^{p(x)}, \ell^{\{p_n\}})$ are depend on the choice of the subset $U \subset \mathbb{R}$. Throughout this work we will choice that $U = (0, 1]$.

Now we will discuss whether some known properties of Wiener amalgam space true or not true for the space $W(L^{p(x)}, \ell^{\{p_n\}})$.

Lemma 3.1 *If $p \in \mathcal{B}$, then $\ell^{\{p_n\}}$ is a BK space under the norm*

$$\|x\|_{\{p_n\}} = \inf\left\{ \lambda > 0 : \sum_{n\in\mathbb{Z}} \left| \frac{x_n}{\lambda} \right|^{p_n} \le 1 \right\}, \quad x = (x_n)_{n\in\mathbb{Z}}.$$

Proof It is known that $\ell^{\{p_n\}}$ is a Banach function space by Lemma 2.5 in [8]. Hence it is a Banach space. Now let $x \in \ell^{\{p_n\}}$ and $x = (x_n)_{n\in\mathbb{Z}}$. We have the inequality

$$\left|P_n(x)\right| = |x_n| \le \inf\left\{\lambda > 0 : |x_n| \le \lambda\right\}$$
$$\le \inf\left\{\lambda > 0 : \sum_{n\in\mathbb{Z}} \left|\frac{x_n}{\lambda}\right|^{p_n} \le 1\right\} = \|x\|_{\{p_n\}}. \tag{3.2}$$

That means the coordinate functions P_n are continuous for all $n \in \mathbb{Z}$. Hence $\ell^{\{p_n\}}$ is a BK space. □

Proposition 3.1 *Let $1 \le p(x) \le s(x) < \infty$ and $q = \{q_n\}$, $t = \{t_n\}$. If $q, t \in \mathcal{B}$ and $t - q \in \mathcal{P}$, then*

$$W\left(L^{s(x)}, \ell^{\{t_n\}}\right) \hookrightarrow W\left(L^{p(x)}, \ell^{\{q_n\}}\right),$$

equivalently

$$\|f\|_{W(L^{p(x)},\ell^{\{q_n\}})} \le C\left(\mu(U)+1\right)\|f\|_{W(L^{p(x)},\ell^{\{t_n\}})} \quad \left(f \in W\left(L^{s(x)}, \ell^{\{t_n\}}\right)\right)$$

for some constant $C > 0$ independent of f.

Proof Let $f \in W(L^{s(x)}, \ell^{\{t_n\}})$ be given. Then $\{\|\chi_{U+n} f\|_{L^{s(x)}}\}_{n\in\mathbb{Z}} \in \ell^{\{t_n\}}$. Since $p(x) \le s(x)$, by Theorem 2.8 in [17] we have $L^{s(x)}(U+n) \hookrightarrow L^{p(x)}(U+n)$ and

$$\|\chi_{U+n} f\|_{L^{p(x)}} \le \left(\mu(U+n)+1\right)\|\chi_{U+n} f\|_{L^{s(x)}}$$
$$= \left(\mu(U)+1\right)\|\chi_{U+n} f\|_{L^{s(x)}}, \tag{3.3}$$

for all $n \in \mathbb{Z}$, where μ is the Lebesgue measure of U. Hence by (3.3) and the solidness of $\ell^{\{t_n\}}$ one has $\{\|\chi_{U+n} f\|_{L^{p(x)}}\}_{n\in\mathbb{Z}} \in \ell^{\{t_n\}}$. On the other hand since $t - q \in \wp$, by Theorem 4.1 in [18] we have the continuous embedding $\ell^{\{t_n\}} \hookrightarrow \ell^{\{q_n\}}$ and there exists $C > 0$ such that

$$\|.\|_{\{q_n\}} \le C\|.\|_{\{t_n\}} \tag{3.4}$$

This implies $\{\|\chi_{U+n} f\|_{L^{q(x)}}\}_{n\in\mathbb{Z}} \in \ell^{\{q_n\}}$. Thus $f \in W(L^{p(x)}, \ell^{\{q_n\}})$. Also since $\ell^{\{t_n\}} \hookrightarrow \ell^{\{q_n\}}$, by using (3.4) we obtain

$$\begin{aligned}
&\|f\|_{W(L^{p(x)},\ell^{\{q_n\}})} \\
&= \left\|\left\{\|f\chi_{U+n}\|_{L^{p(x)}}\right\}_{n\in\mathbb{Z}}\right\|_{\{q_n\}} \le C\left\|\left\{\|f\chi_{U+n}\|_{L^{p(x)}}\right\}_{n\in\mathbb{Z}}\right\|_{\{t_n\}} \\
&= C\left[\inf\left\{\lambda > 0 : \sum_{n\in\mathbb{Z}} \left|\frac{\|f\chi_{U+n}\|_{L^{p(x)}}}{\lambda}\right|^{t_n} \le 1\right\}\right] \\
&\le C\left[\inf\left\{\lambda > 0 : \sum_{n\in\mathbb{Z}} \left|\frac{(\mu(U)+1)\|f\chi_{U+n}\|_{L^{s(x)}}}{\lambda}\right|^{t_n} \le 1\right\}\right] \\
&= C\left[\inf\left\{\lambda > 0 : \sum_{n\in\mathbb{Z}} \left|\frac{\|f\chi_{U+n}\|_{L^{s(x)}}}{\frac{\lambda}{(\mu(U)+1)}}\right|^{t_n} \le 1\right\}\right]
\end{aligned}$$

$$= C\big(\mu(U)+1\big)\left[\inf\left\{\frac{\lambda}{(\mu(U)+1)} > 0 : \sum_{n\in\mathbb{Z}}\left|\frac{\|f\chi_{U+n}\|_{L^{s(x)}}}{\frac{\lambda}{(\mu()+1)}}\right|^{t_n} \le 1\right\}\right]$$

$$= C\big(\mu(U)+1\big)\left\|\left\{\|f\chi_{U+n}\|_{L^{s(x)}}\right\}_{n\in\mathbb{Z}}\right\|_{\{t_n\}}$$

$$= C\big(\mu(U)+1\big)\|f\|_{W(L^{p(x)},\ell^{\{t_n\}})}$$

Then $W(L^{s(x)}, \ell^{\{t_n\}}) \hookrightarrow W(L^{p(x)}, \ell^{\{q_n\}})$ and

$$\|f\|_{W(L^{p(x)},\ell^{\{q_n\}})} \le C\big(\mu(U)+1\big)\|f\|_{W(L^{p(x)},\ell^{\{t_n\}})}. \qquad \square$$

Corollary 3.1 *Let* $1 \le p(x) < \infty$, $q = \{q_n\}$, $t = \{t_n\}$ *and* $q, t \in \mathcal{B}$. *If* $|t - q| \in \mathcal{P}$, *then*

$$W\big(L^{p(x)}, \ell^{\{t_n\}}\big) = W\big(L^{p(x)}, \ell^{\{q_n\}}\big).$$

Proof Since $|t - p| \in \mathcal{P}$, then by Theorem 4.3 in [14], $\ell^{\{q_n\}} = \ell^{\{t_n\}}$. The proof is completed by Theorem 3.2. $\square$

Theorem 3.1 *If* $p \in \mathcal{B}$, *then* $W(L^{p(x)}, \ell^{\{q_n\}})$ *is a Banach space under the norm*

$$\|f\|_W = \|f\|_{W(L^{p(x)},\ell^{\{q_n\}})} = \left\|\left\{\|f\chi_{U+n}\|_{L^{p(x)}}\right\}_{n\in\mathbb{Z}}\right\|_{\{q_n\}}.$$

Proof Since $q^* \ge q_n$ for all $n \in \mathbb{N}$, by Proposition 3.1 we have the inclusion $W(L^{p(x)}, \ell^{\{q_n\}}) \subset W(L^{p(x)}, \ell^{q^*})$ and there exists $C > 0$ such that

$$\|.\|_{W(L^{p(x)},\ell^{q^*})} \le C\|.\|_{W(L^{p(x)},\ell^{\{q_n\}})}. \tag{3.5}$$

Let $(f_i)_{i\in\mathbb{N}}$ be a Cauchy sequence of functions in $W(L^{p(x)}, \ell^{\{q_n\}})$. Then by (3.5), $(f_i)_{i\in\mathbb{N}}$ is also a Cauchy sequence in $W(L^{p(x)}, \ell^{q^*})$. Since $W(L^{p(x)}, \ell^{q^*})$ is complete, $(f_i)_{i\in\mathbb{N}}$ converges to a function $f_0 \in W(L^{(x)}, \ell^{q^*})$. Thus

$$\lim_{i\to\infty}\|f_i - f_0\|_{W(L^{p(x)},\ell^{q^*})} = \lim_{i\to\infty}\left\|\left\{\left\|(f_i - f_0)\chi_{U+n}\right\|_{L^{p(x)}}\right\}_{n\in\mathbb{Z}}\right\|_{\ell^{q^*}} = 0$$

for all $n \in \mathbb{Z}$. Then by Lemma 3.1, for every fixed $n \in \mathbb{Z}$,

$$\lim_{i\to\infty}\left\|(f_i - f_0)\chi_{U+n}\right\|_{L^{p(x)}} = 0.$$

Now we will show that $(f_i)_{i\in\mathbb{N}}$ converges to f_0 in $W(L^{p(x)}, \ell^{\{q_n\}})$. For given any $\varepsilon > 0$, choice any $\lambda > \varepsilon$. Since for fixed $i \in \mathbb{N}$ one has

$$\lim_{j\to\infty}\inf\left|\frac{\|(f_i - f_0)\chi_{U+n}\|_{L^{p(x)}}}{\lambda}\right|^{q_n} = \lim_{j\to\infty}\left|\frac{\|(f_i - f_j)\chi_{U+n}\|_{L^{p(x)}}}{\lambda}\right|^{q_n}$$

$$= \left|\frac{\|(f_i - f_0)\chi_{U+n}\|_{L^{p(x)}}}{\lambda}\right|^{q_n} \tag{3.6}$$

then by the Fatou's Lemma for series and (3.6),

$$\liminf_{j\to\infty}\sum_{n\in\mathbb{Z}}\left|\frac{\|(f_i-f_j)\chi_{U+n}\|_{L^{p(x)}}}{\lambda}\right|^{q_n} \geq \sum_{n\in\mathbb{Z}}\liminf_{j\to\infty}\left|\frac{\|(f_i-f_0)\chi_{U+n}\|_{L^{p(x)}}}{\lambda}\right|^{q_n}$$
$$= \sum_{n\in\mathbb{Z}}\left|\frac{\|(f_i-f_0)\chi_{U+n}\|_{L^{p(x)}}}{\lambda}\right|^{q_n} \tag{3.7}$$

Also since $(f_i)_{i\in\mathbb{N}}$ is a Cauchy sequence in $W(L^{p(x)},\ell^{\{q_n\}})$, there exists $k_0\in\mathbb{N}$ such that for all $i,j\geq k_0$

$$\|f_i-f_j.\|_{W(L^{p(x)},\ell^{\{q_n\}})} = \left\|\left\{\|f\chi_{U+n}\|_{L^{p(x)}}\right\}_{n\in\mathbb{Z}}\right\|_{\{q_n\}}$$
$$= \inf\left\{\lambda>0:\sum_{n\in\mathbb{Z}}\left|\frac{\|(f_i-f_j)\chi_{U+n}\|_{L^{p(x)}}}{\lambda}\right|^{q_n}\leq 1\right\}<\varepsilon.$$

Thus if for fixed $i\geq k_0$ and for all $j\geq k_0$

$$\sum_{n\in\mathbb{Z}}\left|\frac{\|(f_i-f_j)\chi_{U+n}\|_{L^{p(x)}}}{\lambda}\right|^{q_n}\leq 1,$$

then

$$\liminf_{j\to\infty}\sum_{n\in\mathbb{Z}}\left|\frac{\|(f_i-f_j)\chi_{U+n}\|_{L^{p(x)}}}{\lambda}\right|^{q_n}\leq 1.$$

Hence by (3.7), for all $j\geq k_0$

$$\sum_{n\in\mathbb{Z}}\left|\frac{\|(f_i-f_0)\chi_{U+n}\|_{L^{p(x)}}}{\lambda}\right|^{q_n}\leq 1.$$

Since these results are true for every $\lambda>\varepsilon$ and for all $j\geq k_0$, we have

$$\|f_i-f_0\|_{W(L^{p(x)},\ell^{\{q_n\}})} = \left\|\left\{\left\|(f_i-f_0)\chi_{U+n}\right\|_{L^{p(x)}}\right\}_{n\in\mathbb{Z}}\right\|_{\{q_n\}}$$
$$= \inf\left\{\lambda>0:\sum_{n\in\mathbb{Z}}\left|\frac{\|(f_i-f_0)\chi_{U+n}\|_{L^{p(x)}}}{\lambda}\right|^{q_n}\leq 1\right\}$$
$$\leq \inf\left\{\lambda>0:\sum_{n\in\mathbb{Z}}\left|\frac{\|(f_i-f_j)\chi_{U+n}\|_{L^{p(x)}}}{\lambda}\right|^{q_n}\leq 1\right\}$$
$$= \|f_i-f_j\|_{W(L^{p(x)},\ell^{\{q_n\}})}\leq\varepsilon$$

for all $i\geq k_0$. Hence $(f_i)_{i\in\mathbb{N}}$ tends to f_0 in $W(L^{p(x)},\ell^{\{q_n\}})$. To show that $f_0\in W(L^{p(x)},\ell^{\{q_n\}})$, take any $\varepsilon>0$. Since $(f_i)_{i\in\mathbb{N}}$ tends to f_0 there exists $i_0\in\mathbb{N}$ such that

$$\|f_i-f_0\|_{W(L^{p(x)},\ell^{\{q_n\}})}<\varepsilon.$$

On the other hand since $(f_i)_{i\in\mathbb{N}}$ is bounded, there exists a number $M > 0$ such that $\|f_i\|_{W(L^{p(x)},\ell^{\{q_n\}})} < M$. Hence we have

$$\begin{aligned}\|f_0\|_{W(L^{p(x)},\ell^{\{q_n\}})} &= \|f_0 + f_i - f_i\|_{W(L^{p(x)},\ell^{\{q_n\}})} \\ &= \|f_i - f_0\|_{W(L^{p(x)},\ell^{\{q_n\}})} + \|f_i\|_{W(L^{p(x)},\ell^{\{q_n\}})} < M + \varepsilon.\end{aligned}$$

This completes the proof. □

Definition 3.2 Let $k \in \mathbb{Z}$. Define a shift operator S_k from $\mathcal{M}$ into itself by

$$(S_k a)_n = a_{n-k}, \quad n \in \mathbb{Z},\ a \in \mathcal{M}.$$

Set

$$D = \sup\bigl\{\|S_k\|_{p_n \to p_n} : k \in \mathbb{Z}\bigr\}.$$

Proposition 3.2 *Let $1 \le p(x) < \infty$, $q = \{q_n\}$ and $q \in \mathcal{B}$. If $D < \infty$, then there exists $r \in [1,\infty)$ such that*

$$W\bigl(L^{p(x)}, \ell^{\{q_n\}}\bigr) = W\bigl(L^{p(x)}, \ell^r\bigr) = W\bigl(L^{p(x)}, L^r\bigr).$$

Proof Assume that $D < \infty$. Then by Lemma 5.10 in [18] there exists $r \in [1,\infty)$ such that the norms in $\ell^{\{q_n\}}$ and ℓ^r are equivalent. Thus $\ell^{\{q_n\}} = \ell^r$. Hence

$$W\bigl(L^{p(x)}, \ell^{\{q_n\}}\bigr) = W\bigl(L^{p(x)}, \ell^r\bigr). \tag{3.8}$$

Also by Proposition 3 in [13]

$$W\bigl(L^{p(x)}, \ell^r\bigr) = W\bigl(L^{p(x)}, L^r\bigr). \tag{3.9}$$

Finally by (3.8) and (3.9) we obtain

$$W\bigl(L^{p(x)}, \ell^{\{q_n\}}\bigr) = W\bigl(L^{p(x)}, \ell^r\bigr) = W\bigl(L^{p(x)}, L^r\bigr).$$ □

As a easy consequence of Proposition 3.2 and Corollary 3.1 we obtain the following corollary.

Corollary 3.2 *Let $1 \le p(x) < \infty$, $q = \{q_n\}$, $t = \{t_n\}$ and $q, t \in \mathcal{B}$. If $|t - q| \in \mathcal{P}$ and $D < \infty$, then there exists $r \in [1,\infty)$ such that the norms in $W(L^{p(x)}, \ell^{\{q_n\}})$, $W(L^{p(x)}, \ell^{\{t_n\}})$ and $W(L^{p(x)}, \ell^r)$ are equivalent.*

Proposition 3.3 *Let $1 \le p(x), s(x) < \infty$, $q = \{q_n\}$, $t = \{t_n\}$ and $q, t \in \mathcal{B}$. Also assume that $|t - q| \in \mathcal{P}$ and $D < \infty$. Then there exists $r \in [1,\infty)$ such that*

$$W\bigl(L^{p(x)}, \ell^{\{q_n\}}\bigr) = W\bigl(L^{s(x)}, \ell^{\{t_n\}}\bigr) = W\bigl(L^{p(x)}, \ell^r\bigr)$$

if and only if $p(x) = s(x)$.

Proof Assume that $p(x) = s(x)$. Since $|t - q| \in \mathcal{P}$, then by Corollary 3.1,

$$W\big(L^{p(x)}, \ell^{\{q_n\}}\big) = W\big(L^{s(x)}, \ell^{\{t_n\}}\big). \tag{3.10}$$

Also since $D < \infty$, according to Corollary 3.2, there exists $r \in [1, \infty)$ such that

$$W\big(L^{p(x)}, \ell^{\{q_n\}}\big) = W\big(L^{p(x)}, \ell^{r}\big). \tag{3.11}$$

Hence by (3.10) and (3.11) we obtain

$$W\big(L^{p(x)}, \ell^{\{q_n\}}\big) = W\big(L^{s(x)}, \ell^{\{t_n\}}\big) = W\big(L^{p(x)}, \ell^{r}\big).$$

Conversely assume that

$$W\big(L^{p(x)}, \ell^{\{q_n\}}\big) = W\big(L^{s(x)}, \ell^{\{t_n\}}\big). \tag{3.12}$$

Since $|t - q| \in \mathcal{P}$, then $\ell^{\{q_n\}} = \ell^{\{t_n\}}$. Also since $D < \infty$, by Lemma 5.10 in [18] there exists $r \in [1, \infty)$ such that

$$\ell^{\{q_n\}} = \ell^{\{t_n\}} = \ell^{r}.$$

Hence by (3.12)

$$W\big(L^{p(x)}, \ell^{\{q_n\}}\big) = W\big(L^{p(x)}, \ell^{r}\big)$$

and

$$W\big(L^{s(x)}, \ell^{\{t_n\}}\big) = W\big(L^{s(x)}, \ell^{r}\big).$$

Then

$$W\big(L^{p(x)}, \ell^{r}\big) = W\big(L^{s(x)}, \ell^{r}\big).$$

Finally by Corollary 1 in [13] one has $p(x) = s(x)$. □

Now we define the general case of amalgam space by taking $L^{q(x)}(\mathbb{R})$ instead of $\ell^{\{p_n\}}$ and compare it with the Wiener amalgam space.

Definition 3.3 Let $p(x)$ and $q(x)$ be variable exponents on $\mathbb{R}$. Fix a compact subset $U \subset \mathbb{R}$ with nonempty interior. Then the amalgam space $W(L^{p(x)}(\mathbb{R}), L^{q(x)}(\mathbb{R}))$, (shortly $W(L^{p(x)}, L^{q(x)})$) consists of all functions $f \in L^{p(x)}_{loc}(\mathbb{R})$ such that $\|f\|_W = \|f\|_{W(L^{p(x)}, L^{q(x)})} < \infty$, where the norm of f is

$$\|f\|_W = \|f\|_{W(L^{p(x)}, L^{q(x)})} = \big\| \|f\chi_{U+z}\|_{L^{p(x)}} \big\|_{L^{q(z)}}, \quad z \in \mathbb{Z}.$$

Remark 3.2 Since the global component $L^{q(x)}$ of the amalgam space $W(L^{p(x)}, L^{q(x)})$ is not translation invariant, we can not say that the definition is independent of the choice of U. Hence the space $W(L^{p(x)}, L^{q(x)})$ is not a Wiener amalgam space.

Definition 3.4 Let $X = (x_i)_{i \in I} \subset \mathbb{R}$ be well-spread family in $\mathbb{R}$. For any variable exponent Lebesgue space $L^{p(x)}$, we define the associate discrete space $(L^{p(x)})_d$ as

$$\left(L^{p(x)}\right)_d = \left\{ \Lambda : \Lambda = (\lambda_i)_{i \in I} \text{ with } \sum_{i \in I} |\lambda_i| \chi_{x_i+U} \in L^{p(x)} \right\}$$

with the norm

$$\|\Lambda\|_{(L^{p(x)})_d} = \left\| \sum_{i \in I} |\lambda_i| \chi_{x_i+U} \right\|_{L^{p(x)}}.$$

Since $L^{p(x)}$ is not translation invariant, $(L^{p(x)})_d$ is depend on the choice of U.

Theorem 3.2 *Let $q : \mathbb{Z} \to \mathbb{R}$, $q = \{p_n\} \in \mathcal{B}$ and $U = (0, 1]$. Define a function $p : \mathbb{R} \to [1, \infty)$ as $p(x) = p_n$ if $x \in U + n$ for $n \in \mathbb{Z}$. Then $(L^{p(x)})_d = \ell^{\{p_n\}}$.*

Proof If we get $x_i = i$, $i \in \mathbb{Z}$, then $(x_i)_{i \in I} \subset \mathbb{R}$ is a well spread family in $\mathbb{R}$. Define the associate discrete space $(L^{p(x)})_d$ as in Definition 3.4. Let $\lambda = (\lambda_i)_{i \in \mathbb{Z}} \in (L^{p(x)})_d$. By the definition of $(L^{p(x)})_d$, $\sum_{i \in \mathbb{Z}} |\lambda_i| \chi_{U+i} \in L^{p(x)}$. On the other hand we have the equality

$$\begin{aligned}
\int_{\mathbb{R}} \left| \sum_{i \in \mathbb{Z}} |\lambda_i| \chi_{U+i}(x) \right|^{p(x)} dx &= \int_{\bigcup_{n \in \mathbb{Z}}(U+n)} \left| \sum_{i \in \mathbb{Z}} |\lambda_i| \chi_{U+i}(x) \right|^{p(x)} dx \\
&= \sum_{n \in \mathbb{Z}} \int_{U+n} \left| \sum_{i \in \mathbb{Z}} |\lambda_i| \chi_{U+i}(x) \right|^{p(x)} dx \\
&= \sum_{n \in \mathbb{Z}} \int_{U+n} \left| \sum_{i \in \mathbb{Z}} |\lambda_i| \chi_{U+i}(x) \right|^{p_n} dx \\
&= \sum_{i \in \mathbb{Z}} |\lambda_i|^{p_i} \qquad (3.13)
\end{aligned}$$

Since the left side of the equality (3.13) is finite, then right side is also finite. Then we have the inclusion

$$\left(L^{p(x)}\right)_d \subset \ell^{\{p_n\}} \qquad (3.14)$$

Conversely let $\lambda = (\lambda_i)_{i \in \mathbb{Z}} \in \ell^{\{p_n\}}$. Then

$$\sum_{i \in \mathbb{Z}} |\lambda_i|^{p_i} < \infty.$$

Thus again by (3.13),

$$\int_{\mathbb{R}} \left| \sum_{i \in \mathbb{Z}} |\lambda_i| \chi_{U+i}(x) \right|^{p(x)} dx = \sum_{i \in \mathbb{Z}} |\lambda_i|^{p_i} < \infty.$$

This implies

$$\ell^{\{p_n\}} \subset \left(L^{p(x)}\right)_d. \tag{3.15}$$

If we combine (3.14) and (3.15) obtain $(L^{p(x)})_d = \ell^{\{p_n\}}$. □

Theorem 3.3 *Let $q = \{q_n\}$, $q \in \mathcal{B}$. Define any function $p : \mathbb{R} \to [1, \infty)$ as $p(x) = q_n$ if $x \in U + n$ for $n \in \mathbb{Z}$. Then $W(L^{p(x)}, \ell^{\{q_n\}}) = L^{p(x)}$.*

Proof By Definition 3.1, $f \in W(L^{p(x)}, \ell^{\{q_n\}})$ if and only if

$$\left\{\|f\chi_{U+n}\|_{L^{p(x)}}\right\}_{n\in\mathbb{Z}} = \left\{\|f\chi_{U+n}\|_{L^{q_n}}\right\}_{n\in\mathbb{Z}} \in \ell^{\{q_n\}}.$$

Hence $f \in W(L^{p(x)}, \ell^{\{q_n\}})$ if and only if

$$\begin{aligned}\sum_{n\in\mathbb{Z}} \|f\chi_{U+n}\|_{L^{q_n}}^{q_n} &= \sum_{n\in\mathbb{Z}}\left[\left\{\int_{\mathbb{R}} \left|f\chi_{U+n}(x)\right|^{q_n} dx\right\}^{\frac{1}{q_n}}\right]^{q_n} \\ &= \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} \left|f\chi_{U+n}(x)\right|^{q_n} dx < \infty \end{aligned} \tag{3.16}$$

On the other hand for any $f \in L^{p(x)}$ we have

$$\begin{aligned}\rho_p(f) &= \int_{\mathbb{R}} \left|f(x)\right|^{p(x)} dx = \sum_{n\in\mathbb{Z}} \int_{U+n} \left|f(x)\right|^{p(x)} dx \\ &= \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} \left|f\chi_{U+n}(x)\right|^{p(x)} dx = \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} \left|f\chi_{U+n}(x)\right|^{q_n} dx. \end{aligned} \tag{3.17}$$

Since $f \in L^{p(x)}$ if and only if $\rho_p(f) < \infty$ and the right sides of (3.16) and (3.17) are equal, then $f \in W(L^{p(x)}, \ell^{\{q_n\}})$ if and only if $f \in L^{p(x)}$. This implies $W(L^{p(x)}, \ell^{\{q_n\}}) = L^{p(x)}$. □

4 Frame Expansions in $L^{p(x)}(\mathbb{R})$

Definition 4.1 Let $\frac{1}{p(x)} + \frac{1}{s(x)} = 1$. A countable family $(g_i)_{n\in\mathbb{Z}} \subset L^{s(x)}(\mathbb{R})$ is called a $p(x)$-frame for $L^{p(x)}(\mathbb{R})$ if $(g_i(f))_{n\in\mathbb{Z}} \in \ell^{\{q_n\}}$ and there exist constants $A, B > 0$ such that

$$A\|f\|_{L^{p(x)}} \leq \left\|\left(g_i(f)\right)_{n\in\mathbb{Z}}\right\|_{\ell^{\{p_n\}}} \leq B\|f\|_{L^{p(x)}}$$

for all $f \in L^{p(x)}(\mathbb{R})$.

Theorem 4.1 *Let $q : \mathbb{Z} \to \mathbb{R}$, $q = \{p_n\} \in \mathcal{B}$ and $U = (0, 1]$. Define a function $p : \mathbb{R} \to [1, \infty)$ as $p(x) = p_n$ if $x \in I_n = U + n$ for $n \in \mathbb{Z}$. Assume that $\frac{1}{p(x)} + \frac{1}{s(x)} = 1$.*

1. Then $(\chi_{I_n})_{n\in\mathbb{Z}} \subset L^{s(x)}(\mathbb{R})$ is a $p(x)$-frame for $L^{p(x)}(\mathbb{R})$.
2. Every $f \in L^{p(x)}(\mathbb{R})$ has the expansions

$$f = \sum_{n\in\mathbb{Z}} \langle f, \chi_{I_n}\rangle \chi_{I_n}.$$

Proof 1. By Lemma 3.1, $\ell^{\{p_n\}}$ is a BK-space. It is easy to see that $(\chi_{I_n})_{n\in\mathbb{Z}} \subset L^{p(x)}(\mathbb{R})$ and $(\chi_{I_n})_{n\in\mathbb{Z}} \subset L^{s(x)}(\mathbb{R})$, where $L^{s(x)}(\mathbb{R})$ is the dual of $L^{p(x)}(\mathbb{R})$. By Proposition 3.2,

$$W\big(L^{p(x)}, \ell^{\{p_n\}}\big) \hookrightarrow W\big(L^1, \ell^{\{p_n\}}\big) \tag{4.1}$$

and then by (4.1) and Theorem 3.3, there exists $B > 0$ such that

$$\|f\|_{W(L^1,\ell^{\{p_n\}})} \le B\|f\|_{W(L^{p(x)},\ell^{\{p_n\}})} = B\|f\|_{L^{p(x)}},\ f \in L^{p(x)}(\mathbb{R}). \tag{4.2}$$

On the other hand

$$\begin{aligned}\|f\|_{W(L^1,\ell^{\{p_n\}})} &= \|F_f\|_{\ell^{\{p_n\}}} = \big\|\big(\|f\chi_{U+n}\|_{L^1}\big)_{n\in\mathbb{Z}}\big\|_{\ell^{\{p_n\}}}\\ &= \left\|\left(\int_{\mathbb{R}} \big|f\chi_{U+n}(x)\big|dx\right)_{n\in\mathbb{Z}}\right\|_{\ell^{\{p_n\}}}\\ &= \left\|\left(\int_{I_n} \big|f(x)\big|dx\right)_{n\in\mathbb{Z}}\right\|_{\ell^{\{p_n\}}}.\end{aligned} \tag{4.3}$$

Define the analysis operator $C : L^{p(x)}(\mathbb{R}) \to \ell^{\{p_n\}}$ by

$$f \to \left(\int_{I_n} f(x)dx\right)_{n\in\mathbb{Z}} = \big(\langle f, \chi_{I_n}\rangle\big)_{n\in\mathbb{Z}}.$$

From (4.2) and (4.3) we have

$$\begin{aligned}\big\|\big(\langle f, \chi_{I_n}\rangle\big)_{n\in\mathbb{Z}}\big\|_{\ell^{\{p_n\}}} &= \left\|\int_{I_n} C(f)\right\|_{\ell^{\{p_n\}}} = \left\|\left(\int_{I_n} f(x)dx\right)_{n\in\mathbb{Z}}\right\|_{\ell^{\{p_n\}}}\\ &\le \left\|\left(\int_{I_n} \big|f(x)\big|dx\right)_{n\in\mathbb{Z}}\right\|_{\ell^{\{p_n\}}}\\ &= \|f\|_{W(L^1,\ell^{\{p_n\}})} \le B\|f\|_{L^{p(x)}}.\end{aligned} \tag{4.4}$$

This implies that $(\langle f, \chi_{I_n}\rangle)_{n\in\mathbb{Z}} \in (L^{p(x)}(\mathbb{R}))_d = \ell^{\{p_n\}}$. That means the analysis operator C is well defined. Also by (4.4), it is bounded. It is easy to see that C is injective. To show that C is surjective take any $(a_n)_{n\in\mathbb{Z}} \in \ell^{\{p_n\}}$. If we say

$$\sum_{n\in\mathbb{Z}} a_n\chi_{I_n} = f,$$

by the definition of $(L^{p(x)}(\mathbb{R}))_d$ we have $f \in L^{p(x)}(\mathbb{R})$. Also we write $C(f) = (\langle \sum_{n\in\mathbb{Z}} a_n \chi_{I_n}, \chi_{I_m}\rangle)_{n\in\mathbb{Z}} = (a_n)_{n\in\mathbb{Z}}$. Then C is surjective. Finally since C is bijective and bounded, then C^{-1} exits and bounded by Theorem 4.2.H in [19]. Since C^{-1} is bounded, there exists $M > 0$ such that

$$\begin{aligned} \|f\|_{L^{p(x)}} &= \left\|C^{-1}(Cf)\right\|_{L^{p(x)}} \leq M\left\|C(f)\right\|_{\ell\{p_n\}} \\ &= M\left\|\left(\langle f, \chi_{I_n}\rangle\right)_{n\in\mathbb{Z}}\right\|_{\ell\{p_n\}}, \quad f \in L^{p(x)}. \end{aligned} \tag{4.5}$$

Combining (4.4) and (4.5) we have

$$\frac{1}{M}\|f\|_{L^{p(x)}} \leq \left\|\left(\langle f, \chi_{I_n}\rangle\right)_{n\in\mathbb{Z}}\right\|_{\ell\{p_n\}} \leq \|f\|_{L^{p(x)}}.$$

Hence $(\langle f, \chi_{I_n}\rangle)_{n\in\mathbb{Z}}$ is a $p(x)$-frame for $L^{p(x)}(\mathbb{R})$.

2. It is easy to show that $(e_n)_{n\in\mathbb{Z}}$ is a Schauder basis of $\ell^{\{p_n\}}$, where $e_n = (0, 0, \ldots, 1^{n\text{-th}}, \ldots)$. Let $f_i = C^{-1}(e_i)$ for each $i \in \mathbb{Z}$. Since C^{-1} is bounded and linear, then for every $f \in L^{p(x)}(\mathbb{R})$ we have

$$\begin{aligned} f &= C^{-1}(Cf) = C^{-1}\left(\left(\langle f, \chi_{I_n}\rangle\right)_{n\in\mathbb{Z}}\right) \\ &= C^{-1}\left(\sum_{n\in\mathbb{Z}}\langle f, \chi_{I_n}\rangle e_n\right) \\ &= \sum_{n\in\mathbb{Z}}\langle f, \chi_{I_n}\rangle C^{-1}(e_n) = \sum_{n\in\mathbb{Z}}\langle f, \chi_{I_n}\rangle f_n. \end{aligned} \tag{4.6}$$

By the definition of f_n we write

$$e_n = C(f_n) = \left(\ldots, \int_{I_{n-1}} f_n(x)dx, \int_{I_n} f_n(x)dx, \int_{I_{n+1}} f_n(x)dx \cdots\right), \quad \forall n \in \mathbb{N}. \tag{4.7}$$

This implies if $m \neq n$, $\int_{I_n} f_m(x)dx = 0$ and if $m = n$, $\int_{I_n} f_m(x)dx = 1$. On the other hand

$$C(\chi_{I_m}) = \left(\int_{I_n} \chi_{I_m}(x)dx\right)_{n\in\mathbb{Z}} = e_m. \tag{4.8}$$

Since C is bijective, by (4.7) and (4.8) we have $f_m = \chi_{I_m}$. Finally from (4.6) we obtain

$$f = \sum_{n\in\mathbb{Z}}\langle f, \chi_{I_n}\rangle f_n = \sum_{n\in\mathbb{Z}}\langle f, \chi_{I_n}\rangle \chi_{I_n}.$$

This completes the proof. □

5 Boundedness of the Hardy–Littlewood Maximal Operator

Theorem 5.1 *Let $q = \{q_n\}$ and $q \in \mathcal{B}$. Define any function $p : \mathbb{R} \to [1, \infty)$ by $p(x) = q_n$ if $x \in U + n$ for $n \in \mathbb{Z}$. Also let $t = \{t_n\}$, $v = \{v_n\}$ and $t, v \in \mathcal{B}$. If $1 \leq u(x) \leq p(x) < s(x) < \infty$, $t - q \in \mathcal{P}$, $q - v \in \mathcal{P}$ and the Hardy–Littlewood maximal operator*

$$M : L^{p(x)}(\mathbb{R}) \to L^{p(x)}(\mathbb{R})$$

is bounded, then the Hardy–Littlewood maximal operator

$$M : W\big(L^{s(x)}, \ell^{\{t_n\}}\big) \to W\big(L^{u(x)}, \ell^{\{v_n\}}\big)$$

is bounded.

Proof By Theorem 3.3, we write

$$W\big(L^{p(x)}, \ell^{\{q_n\}}\big) = L^{p(x)}.$$

Also by Proposition 3.2 we have

$$W\big(L^{s(x)}, \ell^{\{t_n\}}\big) \hookrightarrow W\big(L^{p(x)}, \ell^{\{q_n\}}\big) = L^{p(x)} \hookrightarrow W\big(L^{u(x)}, \ell^{\{v_n\}}\big). \tag{5.1}$$

If we denote the unit maps from $W(L^{s(x)}, \ell^{\{t_n\}})$ into $L^{p(x)}(\mathbb{R})$ and from $L^{p(x)}(\mathbb{R})$ into $W(L^{u(x)}, \ell^{\{v_n\}})$ by I_1 and I_2 respectively, then there exists $C_1 > 0$ and $C_2 > 0$ such that

$$\big\| I_1(f) \big\|_{L^{p(x)}} \leq C_1 \| f \|_{W(L^{s(x)}, \ell^{\{t_n\}})} \tag{5.2}$$

and

$$\big\| I_2(f) \big\|_{W(L^{u(x)}, \ell^{\{v_n\}})} \leq C_2 \| f \|_{L^{p(x)}}. \tag{5.3}$$

Since $M : L^{p(x)}(\mathbb{R}) \to L^{p(x)}(\mathbb{R})$ is bounded there exists $C_3 > 0$ such that

$$\| Mf \|_{L^{p(x)}} \leq C_3 \| f \|_{L^{p(x)}}. \tag{5.4}$$

Then combining (5.2), (5.3) and (5.4) we obtain

$$\begin{aligned} \| Mf \|_{W(L^{u(x)}, \ell^{\{v_n\}})} &= \big\| I_2(Mf) \big\|_{W(L^{u(x)}, \ell^{\{v_n\}})} \leq C_2 \| Mf \|_{L^{p(x)}} \\ &\leq C_2 C_3 \| f \|_{L^{p(x)}} = C_2 C_3 \big\| I_1(f) \big\|_{L^{p(x)}} \\ &\leq C_1 C_2 C_3 \| f \|_{W(L^{s(x)}, \ell^{\{t_n\}})} = K \| f \|_{W(L^{s(x)}, \ell^{\{t_n\}})}, \end{aligned}$$

where $K = C_1 C_2 C_3$. This completes the proof. □

Corollary 5.1 *Let $q = \{q_n\}$ and $q \in \mathcal{B}$. Define any function $1 \leq p(x) < \infty$ by $p(x) = q_n$ if $x \in U + n$ for $n \in \mathbb{Z}$. Also let $t = \{t_n\}$, $v = \{v_n\}$ and $t, v \in \mathcal{B}$. If*

$1 \leq u(x) \leq p(x) < s(x) < \infty$, $t - q \in \mathcal{P}$, $q - v \in \mathcal{P}$ and $p(x)$ satisfies the Log-Hölder continuity conditions (2.3) and (2.4) then the Hardy–Littlewood maximal operator

$$M : W\left(L^{s(x)}, \ell^{\{t_n\}}\right) \to W\left(L^{u(x)}, \ell^{\{v_n\}}\right)$$

is bounded.

Proof Since $p(x)$ satisfies Log-Hölder continuity conditions (2.3) and (2.4), then the Hardy–Littlewood maximal operator

$$M : L^{p(x)} \to L^{p(x)}$$

is bounded (see [4] and [3]). Hence by Theorem 5.1, the Hardy–Littlewood maximal operator

$$M : W\left(L^{s(x)}, \ell^{\{t_n\}}\right) \to W\left(L^{u(x)}, \ell^{\{v_n\}}\right)$$

is bounded. □

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Spaces of Generalised Smoothness in Summability Problems for Φ-Means of Spectral Decomposition

Tsegaye G. Ayele and Mikhail L. Goldman

Abstract We establish conditions for localization of generalised Riesz means of spectral decomposition by system of fundamental functions of Laplace operator in terms of belongingness of the decomposing function to the spaces of generalised smoothness.

Keywords Spectral decomposition · Summation method · Principle of localization · Spases of generalised smoothness

Mathematics Subject Classification (2010) 46E35 · 54C35 · 35B65

1 Introduction

In spectral theory of differential operators one is often confronted with problems, solutions of which are to be sought in terms of spaces of generalised smoothness. The problem on conditions for localization of spectral decompositions for different differential operators (in particular, for multiple Fourier series and Fourier integrals) is among them. Such problems were investigated by several mathematicians. However, we shall restrict ourselves to works of Ilin and Alimov, particularly to the results in [4]. Throughout the paper, we use the following notations:

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For n-dimensional Euclidean space $\mathbb{R}^n$ and for $1 \leq p \leq \infty$, $L_p(\mathbb{R}^n)$ denotes the Lebesgue space with the norm

$$\|f\|_{L_p(\mathbb{R}^n)} = \begin{cases} \left[\int_{\mathbb{R}^n} \left|f(x)\right|^p\right]^{1/p}, & \text{if } 1 \leq p < \infty, \\ \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \left|f(x)\right|, & \text{if } p = \infty \end{cases}$$

whereas $A \hookrightarrow B$ denotes the topological inclusion of space A in space B and observe that $A = B$ if and only if $A \hookrightarrow B \cap B \hookrightarrow A$. The notation $a \asymp b$ means that $c \leq \frac{a}{b} \leq d$ with $0 < c \leq d$ depending on non significant parameters. For $u > 0$ and function φ in $\mathbb{R}^1$, the notations $\varphi(u)$ al. $\downarrow$ and $\varphi(u)$ al. $\uparrow$ are used to mean that $\varphi(u)$ is almost decreasing and $\varphi(u)$ is almost increasing respectively.

In this paper, we establish conditions for localisation of Φ-means of spectral decomposition by fundamental functions for Laplace operator in arbitrary multidimensional domain. The result which is formulated in terms of Nikol'skii type spaces of generalised smoothness generalises results in [4] and extends our publication [1].

2 Statement of the Problem

Let $G \subset \mathbb{R}^n$ be an arbitrary domain, $(-\widehat{\Delta})$ arbitrary self-adjoint nonnegative extension of Laplace operator in G, and u an ordered spectral representation of $L_2(G)$ with respect to $(-\widehat{\Delta})$, $d\rho(t)$ the corresponding spectral measure and $\{u_i(x,t)\}_{i=1}^m$ fundamental function system with multiplicity $m \leq \infty$. Moreover, for any fixed $t \geq 0$, the fundamental functions $u_i(x,t)$ belong to the class $C^\infty(G)$ and satisfy the differential equation:

$$\Delta u_i(x,t) + t^2 u_i(x,t) = 0, \quad x \in G. \tag{2.1}$$

For a function $f \in L_2(G)$, define the Fourier transform

$$\widehat{f}(t) := \left\{\widehat{f}(t)\right\}_{i=1}^m \quad \text{and} \quad \widehat{f_i}(t) := \int_G f(x) u_i(x,t) dx \tag{2.2}$$

the spectral decomposition by the system $u(x,t) = \{u_i(x,t)\}_{i=1}^m$

$$S_\mu(f,x) = \int_0^\mu \widehat{f}(t) u(x,t) d\rho(t), \quad \mu > 0 \tag{2.3}$$

and Riesz means of spectral decomposition by

$$\sigma_\mu^s(f,x) = \int_0^\mu \widehat{f}(t) u(x,t) \left(1 - \frac{t^2}{\mu^2}\right)^s d\rho(t). \tag{2.4}$$

Now introduce Φ-means of spectral decomposition

$$\sigma_\mu^\Phi(f,x) = \int_0^\mu \widehat{f}(t)u(x,t)\Phi\left(1-\frac{t^2}{\mu^2}\right)d\rho(t) \tag{2.5}$$

where $\Phi(u)$ is sth Reimann–Liouville integral.

$$\Phi(u) = \frac{1}{\Gamma(s)}\int_0^u (u-v)^{s-1}\varphi(v)dv; \quad u \in (0,1], \tag{2.6}$$

with $s > 0$ and a function φ defined on $\mathbb{R}$ possessing properties: $\varphi(u) \equiv 0$ for $u < 0$, $\varphi(u) > 0$, $\varphi(u)$ almost decreasing on $(0,1]$ and $\varphi(u) \asymp \varphi(v)$ when $u \asymp v$ for $u, v \in (0,1]$. Moreover, for $s > 0$, set $s_0 = s$ if $s < 1$, $s_0 = 1$ if $s \geq 1$ and require that

1.

$$\varphi_{s_0}(u) = \int_0^u v^{s_0-1}\varphi(v)dv < \infty, \quad u \in (0,1]$$

2.

$$\varphi \in C^2(0,1), \quad \left|\varphi'(u)\right| \leq c\varphi(u)u^{-1}, \qquad \left|\varphi''(u)\right| \leq c\varphi(u)u^{-2}$$

3.

$$\int_0^1 (1-v)^{s-1}\varphi(v)dv = \Gamma(s).$$

We observe that $\Phi(1) = 1$ and if we set $\varphi(u) = \Gamma(s+1)$ for $u \in (0,1]$, then $\Phi(u) = u^s$ and Φ-means reduce to Riesz means of spectral decomposition, $\sigma_\mu^s(f,x)$.

The problem consists in obtaining conditions for localisation of Φ-means of spectral decomposition. For Riesz means a similar problem has been solved completely by Ilin and Alimov [4] in the framework of spaces of exponential order of smoothness. The result, substantially, was due to the possibility of giving a two-sided estimate of exponential type for Lebesgue function of Riesz means in spaces with exponential order of smoothness.

For Φ-means of spectral decomposition such an estimate does not posses exponential character. Thus, for multiple Fourier integral with assumption of convexity on $\varphi(u)$, the following estimate for Lebesgue constant was established by Goldman in [2].

$$\int_{R_1\leq|x|\leq R_2} \left|D_\mu^\Phi\right|dx \asymp \frac{1}{\omega_0(1/\mu)}, \quad \omega_0(u) = \frac{u^{(n-1)/2-s}}{\varphi(u)} \tag{2.7}$$

Here $0 < R_1 < R_2 < \infty$ are fixed and for $x, \xi \in \mathbb{R}^n$, $D_\mu^\Phi(x) = F^{-1}[\Phi(1-|\xi|^2/\mu^2)](x)$ is the Kernel of Φ-means.

This estimate hints that the results on conditions for localization of Φ-means of spectral decomposition to be formulated in terms of spaces with generalised smoothness.

3 Formulation of the Result

Let $\Omega \subset \mathbb{R}^n$ be a domain and $\Omega \subset\subset G$, that is, $\overline{\Omega}$ is compact and $\overline{\Omega} \subset G$. Let $\omega(0) = 0$, $\omega(u)$ is increasing and $\omega(u)u^{-k}$ is almost decreasing for $k \in \mathbb{N}$.

Definition 3.1 The Nikol'skii type space with generalised smoothness $H_p^{\omega(.)}(\Omega)$ is defined as

$$H_p^{\omega(.)}(\Omega) = \left\{ f \in L_p(\Omega) : \|f\|_{H_p^{\omega(.)}(\Omega)} < \infty \right\}, \tag{3.1}$$

where

$$\|f\|_{H_p^{\omega(.)}(\Omega)} = \|f\|_{L_p(\Omega)} + \sup_{0<u\leq 1} \left[\frac{\omega_{p,\Omega}^k(f;u)}{\omega(u)} \right] \tag{3.2}$$

and

$$\omega_{p,\Omega}^k(f;u) = \sup_{|h|\leq u} \|\Delta_{h,\Omega}^k f\|_{L_{p(\Omega)}}, \quad u > 0 \tag{3.3}$$

is the modulus of continuity of order k for function $f \in L_p(\Omega)$ with

$$\Delta_{h,\Omega}^k f(x) = \begin{cases} \Delta_h^k f(x) = \sum_{m=0}^{k} (-1)^{k-m} C_k^m f(x+mh), & [x, x+kh] \subset \Omega, \\ 0, & [x, x+kh] \subset \Omega^c. \end{cases}$$

Theorem 3.2 (Main Theorem) *Let $\mathring{H}_p^{\omega(.)}(\Omega)$ be the closure in $H_p^{\omega(.)}(\Omega)$ of $C_0^\infty(\Omega)$ and $s > 0$, $0 \leq \alpha, \beta$ are such, that*

$$\frac{n-2}{2} - s < \alpha \leq \beta < \min\left\{ \alpha + \frac{3}{2}, \frac{n}{2} + 1 \right\} \tag{3.4}$$

and function $\omega(u)$ satisfies the requirement

$$\omega(u)u^{-\alpha} \text{ al. } \uparrow, \qquad \omega(u)u^{-\beta} \text{ al. } \downarrow \quad \textit{on } (0, 1], \tag{3.5}$$

$$\omega(u) \leq c\omega_0(u), \quad u \in (0, 1], \qquad \omega_0(u) = \frac{u^{((n-1)/2)+s_0-s}}{\varphi_{s_0}(u)}. \tag{3.6}$$

Let $D \subset \Omega \subset\subset G$ and $f \in \mathring{H}_p^{\omega(.)}(\Omega)$ be a function satisfying the condition $f(x) \equiv 0$ for all x in D. Then for each compact $K \subset D$ uniformly in x holds the relation:

$$\lim_{\mu\to\infty} \sigma_\mu^\Phi(f, x) = 0.$$

To prove the main theorem we require the following results which we state without proof.

Lemma 3.3 *Let condition (3.6) be satisfied for $\alpha \le \beta < \alpha + 3/2$. Then for any domain $\Omega \subset\subset G$ and function $f \in C_0^\infty(\Omega)$*

$$\sup_{\mu \ge 1} \left\{ \int_\mu^{3\mu} \left|\hat{f}(t)\right|^2 d\rho(t) \right\}^{1/2} \le c(\Omega) \|f\|_{H_2^\omega} \omega\left(\mu^{-1}\right) \tag{3.7}$$

The proof of the lemma is similar to that of Lemma 3.1 in [4], giving the estimate for H_2^r.

Proposition 3.4 *Let $\delta = \frac{\nu+1}{2} - s$*

1. *For $\mu \ge \frac{3}{R}$, $0 < t < 1/R$, $\delta \le 0$ holds true the estimate*

$$\left|\Delta_R(\mu, t)\right| \le c_1 (\mu R)^\delta \varphi_1\left(\frac{1}{\mu R}\right) \tag{3.8}$$

2. *For $\mu \ge \frac{1}{R}$, $t > 1/R$ holds true the estimates*

$$\left|\Delta_R(\mu, t)\right| \le c_2 R^{-s} \mu^{\delta+1} \varphi_1\left(\frac{1}{\mu R}\right) t^{-(\nu+1/2)} |\mu - t|^{-1}, \quad |\mu - t| > \frac{1}{R} \tag{3.9}$$

$$\left|\Delta_R(\mu, t)\right| \le c_3 R^{s_0 - s} \mu^{\delta+s_0} \varphi_{s_0}\left(\frac{1}{\mu R}\right) t^{-(\nu+1/2)}, \quad |\mu - t| \le \frac{1}{R} \tag{3.10}$$

where c_1, c_2, c_3 do not depend on t, μ, R.

Proposition 3.5 *Let $s > 0$, α, β and $\omega(u)$ be as stated in the theorem. $\Omega \subset\subset G$, $x_0 \in \Omega$, $0 < R < \rho(x_0, \partial\Omega)$, $R \le 1$, $f \in C_0^\infty(\Omega)$, $f(x) \equiv 0$ for $|x - x_0| \le R$. Then for all $\mu \ge 3/R$, holds true the following inequality,*

$$\left|\sigma_\mu^\Phi(f, x_0)\right| \le c\|f\|_{H_2^\omega} \left\{ (\mu R)^{s_0 - s - \frac{1}{2}} \varphi_{s_0}\left(\frac{1}{\mu R}\right) \omega\left(\frac{1}{\mu}\right) \mu^{\frac{n}{2}} + (\mu R)^\delta \varphi_1\left(\frac{1}{\mu R}\right) \omega(R) V_\beta(R) \right\}. \tag{3.11}$$

Here c does not depend on μ, R, f, x_0

$$V_\beta(R) = \begin{cases} R^{-\frac{n}{2}}, & \beta < \frac{n}{2} \\ R^{-\frac{n}{2}} \log_2\left(\frac{1}{R}\right) & \beta = \frac{n}{2} \\ R^{-\beta} & \beta > \frac{n}{2} \end{cases}$$

and

$$\delta \le \min\left\{0, (n+1)/2 - s - \beta\right\}.$$

Proof of Theorem 3.2 Let $R = \rho(x, \partial D)$. Then for any function $f \in C_0^\infty(\Omega)$, $f \equiv 0$ in D and any point $x_0 \in K$ holds estimate (3.11) in which $R = R(K) > 0$ is fixed. Then

$$\left|\sigma_\mu^\Phi(f, x_0)\right| \le c(K)\|f\|_{H_2^{\omega(.)}(\Omega)}\left[\frac{\omega(\mu^{-1})}{\omega_0(\mu^{-1})} + \mu^\delta\varphi_1\big(\mu^{-1}\big)\right] \tag{3.12}$$

Since $\delta \le 0$ and $\mu^\delta\varphi_1(\mu^{-1}) \to 0$ as $\mu \to \infty$. Finally applying (3.6), we arrive at

$$\left|\sigma_\mu^\Phi(f, x_0)\right| \le c(K)\|f\|_{H_2^{\omega(.)}(\Omega)} \tag{3.13}$$

for all $x_0 \in K$ and for all $\mu \ge 1$. Using denseness of $C_0^\infty(\Omega)$ in $\mathring{H}_p^{\omega(.)}(\Omega)$ and uniform convergence of spectral decomposition and hence Φ-means for $f \in C_0^\infty(\Omega)$, by standard scheme we obtain that $\sigma_\mu^\Phi(f, x) \to f(x)$ as $\mu \to \infty$ uniformly in $x \in K$ for any function $f \in \mathring{H}_p^{\omega(.)}(\Omega)$, $f \equiv 0$ in D. □

Remark 3.6 The theorem gives the conditions for localization of Φ-means of spectral decomposition. In typical situations, when

$$\varphi_{s_0}(u) \asymp u^{s_0}\varphi(u), \qquad \omega_0(u) \asymp \frac{u^{\frac{n-1}{2}-s}}{\varphi(u)}, \quad u \in (0, 1], \tag{3.14}$$

there appears a function $\omega_0(u)$ of the form (2.7). In particular, for Riesz means of spectral decomposition, that is, for $\varphi(u) = \Gamma(s+1)$ we obtain the requirement:

$$\omega(u) \le cu^{\frac{n-1}{2}-s} \tag{3.15}$$

which for $s < \frac{n-1}{2}$ reduces to the sharp condition of localization in terms of exponential order of smoothness for $f \in \mathring{H}_2^\alpha(\Omega)$ with $\alpha = \frac{n-1}{2} - s$, established by Ilin and Alimov in [4]. Moreover, Goldman in [2] proved that this condition is sharp in terms of spaces of generalised smoothness.

Namely, if

$$\overline{\lim_{u\to+0}}\big[\omega(u)u^{s-\frac{n-1}{2}}\big] = \infty \quad \text{and} \quad 0 \le \frac{n-1}{2}, \tag{3.16}$$

then for all $x_0 \in \Omega$, $1 \le p \le \infty$, there exists a function $f_0 \in \mathring{H}_p^{\omega(.)}(\Omega)$, which is equal to zero in some neighbourhood of x_0, such that the Riesz means of spectral decomposition $\sigma_\mu^s(f_0, x_0)$ is unbounded as $\mu \to \infty$. To obtain these results, the generalised kernels of fractional order and corresponding integral operators in spaces of generalised smoothness investigated by Goldman in [3] were implemented.

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Viewing the Steklov Eigenvalues of the Laplace Operator as Critical Neumann Eigenvalues

Pier Domenico Lamberti and Luigi Provenzano

Abstract We consider the Steklov eigenvalues of the Laplace operator as limiting Neumann eigenvalues in a problem of boundary mass concentration. We discuss the asymptotic behavior of the Neumann eigenvalues in a ball and we deduce that the Steklov eigenvalues minimize the Neumann eigenvalues. Moreover, we study the dependence of the eigenvalues of the Steklov problem upon perturbation of the mass density and show that the Steklov eigenvalues violates a maximum principle in spectral optimization problems.

Keywords Steklov boundary conditions · Eigenvalues · Optimization

Mathematics Subject Classification (2010) Primary 35J25 · Secondary 35B25 · 35P15

1 Introduction

Let Ω be a bounded domain (i.e. a bounded connected open set) of class C^2 in $\mathbb{R}^N$, $N \geq 2$. We consider the Steklov eigenvalue problem for the Laplace operator

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = \lambda \rho u, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

in the unknowns λ (the eigenvalue) and u (the eigenfunction). Here ρ denotes a positive function on $\partial\Omega$ bounded away from zero and infinity and ν the unit outer normal to $\partial\Omega$.

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Keeping in mind important problems in linear elasticity (see e.g. Courant and Hilbert [4]), we shall think of the weight ρ as a mass density. In fact, for $N = 2$ problem (1.1) arises for example in the study of the vibration modes of a free elastic membrane the total mass of which is concentrated at the boundary. Note that the total mass is given by $\int_{\partial\Omega} \rho d\sigma$. This mass concentration phenomenon can be described as follows.

For any $\epsilon > 0$ sufficiently small, we consider the ϵ-neighborhood of the boundary $\Omega_\epsilon = \{x \in \Omega : d(x, \partial\Omega) < \epsilon\}$ and for a fixed $M > 0$ we define a function ρ_ϵ in the whole of Ω as follows

$$\rho_\epsilon(x) = \begin{cases} \epsilon, & \text{if } x \in \Omega \setminus \overline{\Omega}_\epsilon, \\ \dfrac{M - \epsilon|\Omega \setminus \overline{\Omega}_\epsilon|}{|\Omega_\epsilon|}, & \text{if } x \in \Omega_\epsilon. \end{cases} \tag{1.2}$$

Note that for any $x \in \Omega$ we have $\rho_\epsilon(x) \to 0$ as $\epsilon \to 0$, and $\int_\Omega \rho_\epsilon dx = M$ for all $\epsilon > 0$. Then we consider the following eigenvalue problem for the Laplace operator with Neumann boundary conditions

$$\begin{cases} -\Delta u = \lambda \rho_\epsilon u, & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

We recall that for $N = 2$ problem (1.3) provides the vibration modes of a free elastic membrane with mass density ρ_ϵ and total mass M. It is not difficult to prove that the eigenvalues and eigenfunctions of problem (1.3) converge as ϵ goes to zero to the eigenvalues and eigenfunctions of problem (1.1) with $\rho = \frac{M}{|\partial\Omega|}$. Thus the Steklov problem can be considered as a limiting Neumann problem. We refer to [1, Arrieta, Jiménez-Casas, Rodríguez-Bernal] for a general approach to this type of problems.

The aim of this paper is to highlight a few properties of the Steklov problem which, compared to the Neumann problem, reveals a critical nature.

First, we study the asymptotic behavior of the eigenvalues of problem (1.3) as $\epsilon \to 0$, when Ω is a ball. We prove that such eigenvalues are differentiable with respect to $\epsilon \geq 0$ and establish formulas for the first order derivatives at $\epsilon = 0$, see Theorem 2.2. It turns our that such derivatives are positive, hence the Steklov eigenvalues minimize the Neumann eigenvalues of problem (1.3) for ϵ sufficiently small, see Remark 2.3.

Second, we consider the problem of optimal mass distributions for problem (1.1) under the condition that the total mass is fixed. This problem has been largely investigated in the case of Dirichlet boundary conditions, see e.g. Henrot [5] for references. As for Steklov boundary conditions, we quote the classical paper by Bandle and Hersch [3].

By following the approach developed in [6], we prove that simple eigenvalues and the symmetric functions of the multiple eigenvalues of (1.1) depend real analytically on ρ and we characterize the corresponding critical mass densities under mass constraint. See Theorem 3.1 and Corollary 3.2. Again, the Steklov problem exhibits a critical behavior and violates the maximum principle discussed in [10] for general elliptic operators of arbitrary order subject to homogeneous boundary conditions of Dirichlet, Neumann and intermediate type for which critical mass densities do not exist. Indeed, it turns out that if Ω is a ball then the constant function is a critical mass density for the Steklov problem (1.1), see Corollary 3.3, Remark 3.4 and Theorem 3.5.

2 Asymptotic Behavior of Neumann Eigenvalues

Given a bounded domain Ω in $\mathbb{R}^N$ of class C^2 and $M > 0$ we denote by λ_j, $j \in \mathbb{N}$, the eigenvalues of problem (1.1) corresponding to the constant surface density $\rho = \frac{M}{|\partial\Omega|}$. Similarly, for $\epsilon > 0$ sufficiently small, we denote by $\lambda_j(\epsilon)$, $j \in \mathbb{N}$, the eigenvalues of problem (1.3). Note that in this paper we always assume that $N \geq 2$. Moreover, by $\mathbb{N}$ we denote the set of natural numbers including zero, hence $\lambda_0(\epsilon) = \lambda_0 = 0$ for all $\epsilon > 0$.

As is well-known, by the Min–Max Principle we get the following variational characterization of the two sequences of eigenvalues:

$$\lambda_j(\epsilon) = \inf_{\substack{E \subset H^1(\Omega) \\ \dim E = j+1}} \sup_{0 \neq u \in E} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega u^2 \rho_\epsilon \, dx}, \qquad j \in \mathbb{N},$$

$$\lambda_j = \inf_{\substack{E \subset H^1(\Omega) \\ \dim E = j+1}} \sup_{\substack{u \in E \\ \mathrm{Tr}\, u \neq 0}} \frac{\int_\Omega |\nabla u|^2 dx}{\int_{\partial\Omega} (\mathrm{Tr}\, u)^2 \frac{M}{|\partial\Omega|} \, d\sigma}, \qquad j \in \mathbb{N}.$$

Here $H^1(\Omega)$ denotes the standard Sobolev space of real-valued functions in $L^2(\Omega)$ with weak derivatives up to first order in $L^2(\Omega)$ and $\mathrm{Tr}\, u$ denotes the trace in $\partial\Omega$ of a function $u \in H^1(\Omega)$. We note that, for each fixed $u \in H^1(\Omega)$ we have

$$\lim_{\epsilon \to 0} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega u^2 \rho_\epsilon \, dx} = \frac{\int_\Omega |\nabla u|^2 dx}{\int_{\partial\Omega} (\mathrm{Tr}\, u)^2 \frac{M}{|\partial\Omega|} \, d\sigma}. \tag{2.1}$$

By looking at (2.1) one could expect the spectral convergence of the Neumann problems under consideration to the Steklov problem. In fact the following statement holds.

Theorem 2.1 *If Ω is bounded domain in $\mathbb{R}^N$ of class C^2 then $\lim_{\epsilon \to 0} \lambda_j(\epsilon) = \lambda_j$ for all $j \in \mathbb{N}$.*

This theorem can be proved directly by using the notion of compact convergence for the resolvent operators but can also be obtained as a consequence of the more general results proved in [1, Arrieta, Jiménez-Casas, Rodríguez-Bernal].

By Theorem 2.1, it follows that the function $\lambda_j(\cdot)$ can be extended with continuity at $\epsilon = 0$ by setting $\lambda_j(0) = \lambda_j$ for all $j \in \mathbb{N}$. This will be understood in the sequel. If Ω is a ball then we are able to establish the asymptotic behavior of $\lambda_j(\epsilon)$ as $\epsilon \to 0$. Indeed, we can prove that $\lambda_j(\epsilon)$ is differentiable with respect to ϵ and compute the derivative $\lambda_j'(0)$ at $\epsilon = 0$.

Theorem 2.2 *If Ω is the unit ball in $\mathbb{R}^N$ then $\lambda_j(\epsilon)$ is differentiable for any $\epsilon \geq 0$ and*

$$\lambda_j'(0) = \frac{2M\lambda_j^2(0)}{3N|\Omega|} + \frac{2\lambda_j^2(0)|\Omega|}{2M\lambda_j(0) + N^2|\Omega|}.$$

The proof of this theorem relies on the use of Bessel functions which allow to recast the Neumann eigenvalue problem in the form of an equation $F(\lambda, \epsilon) = 0$ in the unknowns λ, ϵ. Then, after some preparatory work, it is possible to apply the Implicit Function Theorem and conclude. We note that, despite the idea of the proof is rather simple and used also in other contexts (see e.g. [9]), this method requires standard but lengthy computations, suitable Taylor's expansions and estimates on the corresponding remainders, as well as recursive formulas for the cross-products of Bessel functions and their derivatives. We refer to [12] for details.

Remark 2.3 By Theorem 2.2 it follows that for $\epsilon > 0$ sufficiently small the functions $\epsilon \mapsto \lambda_j(\epsilon)$ are strictly increasing. In particular, it follows that for all $\epsilon > 0$ sufficiently small, we have that $\lambda_j(0) < \lambda_j(\epsilon)$.

It is interesting to compare our result with the monotonicity result by Ni and Wang [11] who have proved that if Ω is the unit disk in the plane then the first positive eigenvalue of the Neumann Laplacian in Ω_ϵ, i.e. the first positive eigenvalue of the problem

$$\begin{cases} -\Delta u = \lambda u, & \text{in } \Omega_\epsilon, \\ \dfrac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega_\epsilon, \end{cases} \tag{2.2}$$

is a strictly increasing function of $\epsilon > 0$.

3 Existence of Critical Mass Densities for the Steklov Problem

Given a bounded domain Ω in $\mathbb{R}^N$ of class C^2, we denote by $\mathcal{R}$ the subset of $L^\infty(\partial\Omega)$ of those functions $\rho \in L^\infty(\partial\Omega)$ such that $\operatorname{ess\,inf}_{\partial\Omega} \rho > 0$. For any $\rho \in \mathcal{R}$, we denote by $\lambda_j[\rho]$, $j \in \mathbb{N}$, the eigenvalues of problem (1.1). By classical results in perturbation theory, one can prove that $\lambda_j[\rho]$ depends real-analytically on ρ as long as ρ is such that $\lambda_j[\rho]$ is a simple eigenvalue. This is no longer true if the multiplicity of $\lambda_j[\rho]$ varies. As it was pointed out in [6, 7], in the case of multiple eigenvalues, analyticity can be proved for the symmetric functions of the eigenvalues. Namely, given a finite set of indexes $F \subset \mathbb{N}$, one can consider the symmetric functions of the eigenvalues with indexes in F

$$\Lambda_{F,h}[\rho] = \sum_{\substack{j_1,\dots,j_h \in F \\ j_1 < \cdots < j_h}} \lambda_{j_1}[\rho] \cdots \lambda_{j_h}[\rho], \quad h = 1, \dots, |F|$$

and prove that such functions are real-analytic on

$$\mathcal{R}[F] \equiv \left\{ \rho \in \mathcal{R} : \lambda_j[\rho] \neq \lambda_l[\rho],\ \forall j \in F,\ l \in \mathbb{N} \setminus F \right\}. \tag{3.1}$$

In fact, we can prove the following theorem where in order to establish formulas for the Frechét differentials, we find it convenient to set

$$\Theta[F] \equiv \left\{ \rho \in \mathcal{R}[F] : \lambda_{j_1}[\rho] = \lambda_{j_2}[\rho],\ \forall j_1, j_2 \in F \right\}.$$

Theorem 3.1 *Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^2 and F a finite subset of $\mathbb{N}$. Then $\mathcal{R}[F]$ is an open set in $L^\infty(\partial\Omega)$ and the functions $\Lambda_{F,h}$ are real-analytic in $\mathcal{R}[F]$. Moreover, if $F = \cup_{k=1}^n F_k$ and $\rho \in \cap_{k=1}^n \Theta[F_k]$ is such that for each $k = 1, \dots, n$ the eigenvalues $\lambda_j[\rho]$ assume the common value $\lambda_{F_k}[\rho]$ for all $j \in F_k$, then the differentials of the functions $\Lambda_{F,h}$ at the point ρ are given by the formula*

$$d\Lambda_{F,h}[\rho][\dot\rho] = -\sum_{k=1}^{n} c_k \sum_{l \in F_k} \int_{\partial\Omega} (\mathrm{Tr}\, u_l)^2 \dot\rho \, d\sigma, \tag{3.2}$$

for all $\dot\rho \in L^\infty(\partial\Omega)$, where

$$c_k = \sum_{\substack{0 \le h_1 \le |F_1| \\ \dots\dots \\ 0 \le h_n \le |F_n| \\ h_1 + \cdots + h_n = h}} \binom{|F_k| - 1}{h_k - 1} \lambda_{F_k}^{h_k}[\rho] \prod_{\substack{j=1 \\ j \neq k}}^{n} \binom{|F_j|}{h_j} \lambda_{F_j}^{h_j}[\rho],$$

and for each $k = 1, \dots, n$, $\{u_l\}_{l \in F_k}$ is a basis of the eigenspace of $\lambda_{F_k}[\rho]$ normalized by the condition $\int_{\partial\Omega} \mathrm{Tr} u_i \,\mathrm{Tr}\, u_j \rho d\sigma = \delta_{ij}$ for all $i, j \in F_k$.

The proof of this theorem follows the lines of the corresponding result proved in [10] for general elliptic operators subject to homogeneous boundary conditions of Dirichlet, Neumann and intermediate type. In the same spirit of [10], we can use formula (3.2) in order to investigate the existence of critical mass densities for the eigenvalues of the Steklov problem subject to mass constraint. We note that a typical optimization problem in the analysis of composite materials consists in finding mass densities ρ, with given total mass, which minimize a cost functional $F[\rho]$ associated with the solutions of suitable partial differential equations depending on ρ. Namely, in the case of Steklov boundary conditions one can consider the following problems

$$\min_{\int_{\partial\Omega} \rho d\sigma = \text{const.}} F[\rho] \quad \text{or} \quad \max_{\int_{\partial\Omega} \rho d\sigma = \text{const.}} F[\rho].$$

More in general, setting $M[\rho]=\int_{\partial\Omega}\rho d\sigma$ one can consider the problem of finding critical mass densities ρ under mass constraint, i.e. mass densities ρ which satisfy the condition $\mathrm{Ker}\, dM[\rho]\subset \mathrm{Ker} dF[\rho]$. As in [10] we can give a characterization of critical mass densities which immediately follows by formula (3.2) combined with the Lagrange Multipliers Theorem.

Corollary 3.2 *Let all assumptions of Theorem 3.1 hold. Then, $\rho\in\mathcal{R}$ is a critical mass density for $\Lambda_{F,h}$ for some $h=1,\dots,|F|$, subject to mass constraint if and only if there exists $c\geq 0$ such that*

$$\sum_{k=1}^{n} c_k \sum_{l\in F_k} (\mathrm{Tr}\, u_l)^2 = c, \quad \text{a.e. on } \partial\Omega. \tag{3.3}$$

The analysis carried out in [10] has pointed out that for a large class of non-negative elliptic operators subject to homogeneous boundary conditions of intermediate type (including the case of Dirichlet boundary conditions), there are no critical mass densities for simple eigenvalues and the symmetric functions of multiple eigenvalues. For example, in the case of Dirichlet or Neumann boundary conditions, (3.3) has to be replaced by

$$\sum_{k=1}^{n} c_k \sum_{l\in F_k} u_l^2 = c, \quad \text{a.e. in } \Omega, \tag{3.4}$$

which is clearly not satisfied in the Dirichlet case. As for Neumann boundary conditions the same non existence result can be easily proved for simple eigenvalues in which case only a summand appears in (3.4). The situation is not completely clear for multiple eigenvalues. Under suitable regularity assumptions on the eigenfunctions u_1 and u_2 associated with the same Neumann eigenvalue λ one can prove that the condition $u_1^2+u_2^2=c$ in Ω implies that $\lambda=0$, but the proof in the case of multiplicities higher than two seems not straightforward. However, well-known explicit formulas for the eigenfunctions of the Neumann Laplacian in the ball clearly show that condition (3.4) is not satisfied, hence *no critical mass densities exist for the Neumann Laplacian in the ball*. In the case of Steklov boundary conditions the situation is much different. Indeed, if Ω is a ball then a critical mass density exists.

Corollary 3.3 *Let Ω be the unit ball in $\mathbb{R}^N$, $M>0$ and $F\subset\mathbb{N}$ be a finite set such that the constant mass density $\rho=M/|\partial\Omega|$ belongs to $\mathcal{R}[F]$. Then $\rho=M/|\partial\Omega|$ is critical for $\Lambda_{F,h}$ for all $h=1,\dots,|F|$ under the constraint $M[\rho]=M$.*

The proof can be carried out as in [8]. Namely, assume that λ is an eigenvalue of problem (1.1) with multiplicity m and consider a basis $u_1,\dots,u_m$ of the corresponding eigenspace. Assume that this basis is orthonormal in $L^2(\partial\Omega)$ with respect to the scalar product defined by $\int_{\partial\Omega}\mathrm{Tr} u\,\mathrm{Tr}\, v\rho d\sigma$. Then for any isometry R in $\mathbb{R}^N$ also $u_1\circ R,\dots,u_m\circ R$ is an orthonormal basis of the same eigenspace, hence $\sum_{i=1}^m u_i^2=\sum_{i=1}^m u_i^2\circ R$. It follows that $\sum_{i=1}^m u_i^2$ is constant on $\partial\Omega$.

Remark 3.4 It is interesting to compare Corollary 3.3 with a classical result proved by Bandle and Hersch [2] in the case of a class of symmetric planar domains. For the convenience of the reader we formulate such result assuming directly that Ω is the unit disk in $\mathbb{R}^2$ centered at zero. For any $n \in \mathbb{N}$ we set

$$\mathcal{R}_n = \left\{\rho \in \mathcal{R} :\ \rho\left(e^{2\pi i/n} z\right) = \rho(z),\ \forall\, z \in \partial\Omega\right\},$$

where the use of the complex variable z is clearly understood. Then we have the following result

Theorem 3.5 (Bandle and Hersch) *Let Ω be the unit disk in $\mathbb{R}^2$ centered at zero, $M > 0$, $n \in \mathbb{N}$. Then*

$$\lambda_j[\rho] \leq \lambda_j\left[\frac{M}{2\pi}\right]$$

for all $j = 0, \dots, n$ and $\rho \in \mathcal{R}_n$ such that $M[\rho] = M$. Equality holds only if $\rho = M/2\pi$.

Thus in the case of a ball in $\mathbb{R}^2$ the constant mass density is in fact a maximizer among all mass densities satisfying the symmetry condition above. We refer to Bandle [2] for further discussions.

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Generalized Fractional Integrals on Central Morrey Spaces and Generalized σ-Lipschitz Spaces

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Abstract For the generalized fractional integrals $\tilde{I}_{\alpha,d}$, which were defined in Function Spaces X, to appear, when $n/\alpha \le p < \infty$, we will consider their boundedness from the central Morrey spaces $B^{p,\lambda}(\mathbb{R}^n)$ to the generalized σ-Lipschitz spaces $\mathrm{Lip}_{\beta,\sigma}^{(d)}(\mathbb{R}^n)$.

Keywords Central Morrey space · λ-Central mean oscillation space · σ-Lipschitz space · Generalized λ-central mean oscillation space · Generalized σ-Lipschitz space · Fractional integral · Generalized fractional integral

Mathematics Subject Classification (2010) Primary 42B35 · Secondary 26A33 · 46E30 · 46E35

1 Introduction

In 1964, the spaces $B^p(\mathbb{R}^n)$, $1 \le p < \infty$, were introduced by A. Beurling [3], together with their preduals $A^p(\mathbb{R}^n)$, so-called the Beurling algebras. Further, in 1989, the central mean oscillation spaces $\mathrm{CMO}^p(\mathbb{R}^n)$, $1 \le p < \infty$, which contain the spaces $B^p(\mathbb{R}^n)$ modulo constants, were introduced by Y. Chen and K. Lau [6] and J. García-Cuerva [9].

Later, in 2000, the non-homogeneous central Morrey spaces $B^{p,\lambda}(\mathbb{R}^n)$ and the λ-central mean oscillation spaces $\mathrm{CMO}^{p,\lambda}(\mathbb{R}^n)$, $1 \le p < \infty$ and $\lambda \in \mathbb{R}$, were introduced by J. Alvarez, M. Guzmán-Partida and J. Lakey [1]. The spaces $B^{p,\lambda}(\mathbb{R}^n)$ contain the spaces $B_{p,q}(\mathbb{R}^n)$, $1 \le p < \infty$ and $0 < q \le 1$, which were defined by J. García-Cuerva and M. J. Herrero [10] in 1994, as special cases. Here we note that the spaces $B^{p,\lambda}(\mathbb{R}^n)$ are the non-homogeneous Herz spaces $K_{p,\infty}^{-n/p-\lambda}(\mathbb{R}^n)$ (cf. [7] and [13]). Moreover, in [18] (cf. [14, 15]), we defined the weak λ-central mean oscillation spaces $W\mathrm{CMO}^{p,\lambda}(\mathbb{R}^n)$.

Dedicated to Professor Kichi-Suke Saito in celebration of his 65th birthday.

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Afterward, in 2011, we ([20]; cf. [16]) introduced the B_σ-function spaces, $0 \le \sigma < \infty$, in order to unify $B^{p,\lambda}(\mathbb{R}^n)$, $\mathrm{CMO}^{p,\lambda}(\mathbb{R}^n)$ and the usual Morrey–Campanato spaces.

Recently, in [17, 19] (cf. [22]), we introduced the generalized λ-central mean oscillation spaces $\Lambda^{(d)}_{p,\lambda}(\mathbb{R}^n)$, the generalized weak λ-central mean oscillation spaces $W\Lambda^{(d)}_{p,\lambda}(\mathbb{R}^n)$, $1 \le p < \infty$, $d \in \mathbb{N} \cup \{0\}$ and $\lambda \in \mathbb{R}$, in order to consider the further boundedness of fractional integrals I_α for $B^{p,\lambda}(\mathbb{R}^n)$. The spaces $\Lambda^{(d)}_{p,\lambda}(\mathbb{R}^n)$ also contain the spaces $\Lambda_{p,q}(\mathbb{R}^n)$, $1 \le p < \infty$ and $0 < q \le 1$, which were defined in [10], as special cases.

On the other hand, for the fractional integrals I_α and the modified fractional integrals $\tilde{I}_\alpha$, $0 < \alpha < n$, which are defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy$$

and

$$\tilde{I}_\alpha f(x) = \int_{\mathbb{R}^n} f(y) \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1-\chi_{Q_1}(y)}{|y|^{n-\alpha}} \right) dy \tag{1.1}$$

(for the notation χ_{Q_1}, see Sect. 2) respectively, the following boundedness results are well-known: For $0 < \alpha < n$, $1 \le p < \infty$ and $1/q = 1/p - \alpha/n$,

(i) $I_\alpha : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$, $1 < p < n/\alpha$;
(ii) $I_\alpha : L^1(\mathbb{R}^n) \to WL^q(\mathbb{R}^n)$, $p = 1$;
(iii) $\tilde{I}_\alpha : L^p(\mathbb{R}^n) \to \mathrm{Lip}_\beta(\mathbb{R}^n)$, $0 < \beta = \alpha - n/p < 1$;
(iv) $\tilde{I}_\alpha : L^{n/\alpha}(\mathbb{R}^n) \to \mathrm{BMO}(\mathbb{R}^n)$, $p = n/\alpha$.

Here (i) is due to G. H. Hardy and J. E. Littlewood [11, 12] for the 1-dimensional case and S. L. Sobolev [28] for the n-dimensional case, (ii) belongs to A. Zygmund [29], and (iii) and (iv) were proved by J. Peetre [24].

After that, for the fractional integrals I_α on $B^{p,\lambda}(\mathbb{R}^n)$, the following boundedness result was obtained by Z. W. Fu, Y. Lin and S. Z. Lu [8]: For $0 < \alpha < n$, $1 < p < n/\alpha$, $-n/p + \alpha \le \lambda + \alpha = \mu < 0$ and $1/q = 1/p - \alpha/n$,

(i') $I_\alpha : B^{p,\lambda}(\mathbb{R}^n) \to B^{q,\mu}(\mathbb{R}^n)$.

Furthermore, in [16, 20] (cf. [15]), from several B_σ-Morrey–Campanato estimates for I_α and $\tilde{I}_\alpha$, the following boundedness results are obtained as the corollaries: For $0 < \alpha < n$, $-n + \alpha \le \lambda + \alpha = \mu < 0$ and $1/q = 1 - \alpha/n$,

(ii') $I_\alpha : B^{1,\lambda}(\mathbb{R}^n) \to WB^{q,\mu}(\mathbb{R}^n)$.

Also for $0 < \alpha < n$, $1 \le p < \infty$, $-n/p + \alpha \le \lambda + \alpha = \mu < 1$, $1 \le q \le pn/(n - p\alpha)$ and $\sigma = \lambda + n/p$,

(i'') $\tilde{I}_\alpha : B^{p,\lambda}(\mathbb{R}^n) \to \mathrm{CMO}^{q,\mu}(\mathbb{R}^n)$, $1 < p < n/\alpha$;
(ii'') $\tilde{I}_\alpha : B^{1,\lambda}(\mathbb{R}^n) \to W\mathrm{CMO}^{q,\mu}(\mathbb{R}^n)$, $p = 1$;

(iii') $\tilde{I}_\alpha : B^{p,\lambda}(\mathbb{R}^n) \to \mathrm{Lip}_{\beta,\sigma}(\mathbb{R}^n)$, $0 < \beta = \alpha - n/p < 1$;
(iv') $\tilde{I}_\alpha : B^{n/\alpha,\lambda}(\mathbb{R}^n) \to \mathrm{BMO}_\sigma(\mathbb{R}^n)$, $p = n/\alpha$.

As for the spaces $\mathrm{Lip}_{\beta,\sigma}(\mathbb{R}^n)$ and $\mathrm{BMO}_\sigma(\mathbb{R}^n)$, refer to Remark 2.10 below.

Recently, as we stated above, for the whole of λ such that $-n/p \le \lambda < \infty$, in order to investigate the boundedness of I_α for $B^{p,\lambda}(\mathbb{R}^n)$, we introduced the "new" modification of I_α, i.e., the generalized fractional integrals $\tilde{I}_{\alpha,d}$, $0 < \alpha < n$ and $d \in \mathbb{N} \cup \{0\}$, and showed the following boundedness results: For $0 < \alpha < n$, $1 \le p < n/\alpha$, $d \in \mathbb{N}_0$, $-n/p + \alpha \le \lambda + \alpha = \mu < d + 1$ and $q = pn/(n - p\alpha)$, i.e., $1/q = 1/p - \alpha/n$,

(i''') $\tilde{I}_{\alpha,d} : B^{p,\lambda}(\mathbb{R}^n) \to \Lambda^{(d)}_{q,\mu}(\mathbb{R}^n)$, $1 < p < n/\alpha$;
(ii''') $\tilde{I}_{\alpha,d} : B^{1,\lambda}(\mathbb{R}^n) \to W\Lambda^{(d)}_{q,\mu}(\mathbb{R}^n)$, $p = 1$.

The above (i''') and (ii''') are the results for the whole of λ such that $-n/p \le \lambda < \infty$ and further the condition $1 \le p < n/\alpha$. In this paper, therefore, under the condition $n/\alpha \le p < \infty$, we investigate the boundedness of $\tilde{I}_{\alpha,d}$ for $B^{p,\lambda}(\mathbb{R}^n)$. In order to do so, we shall use the generalized σ-Lipschitz spaces $\mathrm{Lip}^{(d)}_{\beta,\sigma}(\mathbb{R}^n)$, the special cases of which are $\mathrm{Lip}_{\beta,\sigma}(\mathbb{R}^n)$ and $\mathrm{BMO}_\sigma(\mathbb{R}^n)$ (see Definition 2.8 below).

We note that the same results in this paper still hold for the homogeneous versions of the function spaces.

2 Generalized σ-Lipschitz Spaces

First we explain the notations used in the present paper. We use the symbol $A \lesssim B$ to denote that there exists a constant $C > 0$ such that $A \le CB$. If $A \lesssim B$ and $B \lesssim A$, we then write $A \sim B$. For $r > 0$, by Q_r, we mean the following: $Q_r = \{y \in \mathbb{R}^n : |y| < r\}$ or $Q_r = \{y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n : \max_{1 \le i \le n} |y_i| < r\}$. And for $x \in \mathbb{R}^n$, we set $Q(x, r) = x + Q_r = \{x + y : y \in Q_r\}$. For a measurable set $G \subset \mathbb{R}^n$, we denote the Lebesgue measure of G by $|G|$ and the characteristic function of G by χ_G. Further, for a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and a measurable set $G \subset \mathbb{R}^n$ with $|G| > 0$, let

$$f_G = \fint_G f(y)\,dy = \frac{1}{|G|} \int_G f(y)\,dy$$

and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Next we state the definition of the non-homogeneous central Morrey space $B^{p,\lambda}(\mathbb{R}^n)$ (see [1] and [8]).

Definition 2.1 For $1 \le p < \infty$ and $-n/p \le \lambda < \infty$,

$$B^{p,\lambda}\big(\mathbb{R}^n\big) = \big\{f \in L^p_{loc}\big(\mathbb{R}^n\big) : \|f\|_{B^{p,\lambda}} < \infty\big\},$$

where

$$\|f\|_{B^{p,\lambda}} = \sup_{r \ge 1} \frac{1}{r^\lambda} \left(\fint_{Q_r} \big|f(y)\big|^p\,dy \right)^{1/p}.$$

Remark 2.2 The spaces $B^{p,\lambda}(\mathbb{R}^n)$ are the particular cases of local general Morrey-type spaces $LM_{p\theta,w(\cdot)}(\mathbb{R}^n)$ with $\theta = \infty$ and $w(r) = r^{-n/p-\lambda}$. In the last decade there were many results on the boundedness of various operators of real analysis from one local Morrey-type space $LM_{p_1\theta_1,w_1(\cdot)}(\mathbb{R}^n)$ to another one $LM_{p_2\theta_2,w_2(\cdot)}(\mathbb{R}^n)$. A survey of these results is given in [4, 5].

Now we define the generalized σ-Lipschitz spaces $\mathrm{Lip}^{(d)}_{\beta,\sigma}(\mathbb{R}^n)$ (see [17] and [22]).

Definition 2.3 (Definition 8.1 of [22]; cf. [17]) Let $U = \mathbb{R}^n$ or $U = Q_r$ with $r > 0$. For $d \in \mathbb{N}_0$ and $0 \le \beta \le 1$, the continuous function f will be said to belong to the generalized Lipschitz space on U, i.e., $\mathrm{Lip}^{(d)}_{\beta}(U)$ if and only if

$$\|f\|_{\mathrm{Lip}^{(d)}_{\beta}(U)} = \sup_{x,x+h\in U, h\neq 0} \frac{1}{|h|^\beta}\left|\Delta_h^{d+1} f(x)\right| < \infty,$$

where Δ_h^k is a difference operator, which is defined inductively by

$$\Delta_h^0 f = f, \qquad \Delta_h^1 f = \Delta_h f = f(\cdot + h) - f(\cdot),$$
$$\Delta_h^k f = \Delta_h^{k-1} f(\cdot + h) - \Delta_h^{k-1} f(\cdot), \quad k = 2, 3, \ldots.$$

In particular,

$$\mathrm{BMO}^{(d)}(U) = \mathrm{Lip}^{(d)}_0(U),$$

which we call the generalized BMO space on U.

Remark 2.4 For $0 < \beta < 1$, $d \in \mathbb{N}_0$ and $\beta = 1$, $d \in \mathbb{N}$ the spaces $\mathrm{Lip}^{(d)}_{\beta}(U)$ concide with Nikol'skiĭ spaces $h^\beta_\infty(U)$. For the well developed theory of Nikol'skiĭ spaces $h^\beta_p(U)$ and their generalizations see books [2] and [23].

Remark 2.5 (Theorem 8.3 of [22]) Let $U = \mathbb{R}^n$ or $U = Q_r$ with $r > 0$. For $1 \le p < \infty$, $d \in \mathbb{N}_0$, $0 \le \beta \le 1$ and $f \in L^p_{loc}(\mathbb{R}^n)$, we have

$$\|f\|_{\mathrm{Lip}^{(d)}_{\beta}(U)} \sim \|f\|_{\mathcal{L}^{(d)}_{p,\beta}(U)},$$

where $\mathcal{L}^{(d)}_{p,\beta}(U)$ is the generalized Campanato space on U defined in the following.

Definition 2.6 Let $U = \mathbb{R}^n$ or $U = Q_r$ with $r > 0$. For $1 \le p < \infty$, $d \in \mathbb{N}_0$ and $-n/p \le \lambda < d+1$, the function $f \in L^p_{loc}(U)$ will be said to belong to the generalized Campanato space on U, i.e., $\mathcal{L}^{(d)}_{p,\lambda}(U)$ if and only if for every $Q(x,s) \subset U$, there is a polynomial $P^d_{Q(x,s)} f$ of degree at most d such that

$$\|f\|_{\mathcal{L}^{(d)}_{p,\lambda}(U)} = \sup_{Q(x,s)\subset U} \frac{1}{s^\lambda}\left(⨍_{Q(x,s)} \left|f(y) - P^d_{Q(x,s)} f(y)\right|^p dy\right)^{1/p} < \infty.$$

Remark 2.7 (Remark 6.2 of [22]) Let $U = \mathbb{R}^n$ or $U = Q_r$ with $r > 0$. For $1 \le p < \infty$, $d \in \mathbb{N}_0$, $-n/p \le \lambda < d+1$ and $f \in L^p_{loc}(U)$, we have

$$\|f\|_{\mathcal{L}^{(d)}_{p,\lambda}(U)} \sim \sup_{Q(x,s)\subset U} \inf_{P\in\mathcal{P}^d(U)} \frac{1}{s^\lambda}\left(⨍_{Q(x,s)} \left|f(y) - P(y)\right|^p dy\right)^{1/p}$$

where $\mathcal{P}^d(U)$ is the set of all polynomials of degree at most d.

Definition 2.8 (cf. Definition 11 of [17]) For $d \in \mathbb{N}_0$, $0 \le \beta \le 1$ and $0 \le \sigma < \infty$, the continuous function f will be said to belong to the generalized σ-Lipschitz (σ-Lip) space, i.e., $\mathrm{Lip}^{(d)}_{\beta,\sigma}(\mathbb{R}^n)$ if and only if

$$\|f\|_{\mathrm{Lip}^{(d)}_{\beta,\sigma}} = \sup_{r\ge 1} \frac{1}{r^\sigma}\|f\|_{\mathrm{Lip}^{(d)}_\beta(Q_r)} < \infty.$$

In particular,

$$\mathrm{BMO}^{(d)}_\sigma\big(\mathbb{R}^n\big) = \mathrm{Lip}^{(d)}_{0,\sigma}\big(\mathbb{R}^n\big),$$

which we call the generalized σ-BMO space.

Remark 2.9 Very close spaces to $\mathrm{Lip}^{(d)}_{\beta,\sigma}(\mathbb{R}^n)$ are studied in much detail and generality in papers by T. Runst, W. Sickel and other mathematicians. See book [25] and survey paper [26, 27].

Identifying functions which differ by a polynomial of degree at most d, a.e., we see that $\mathrm{Lip}^{(d)}_\beta(\mathbb{R}^n)$ and $\mathrm{Lip}^{(d)}_{\beta,\sigma}(\mathbb{R}^n)$ are the Banach spaces (see [17] and [22]).

Remark 2.10 ([18]; cf. [15]) We note that particularly

$$\mathrm{Lip}^{(0)}_\beta\big(\mathbb{R}^n\big) = \mathrm{Lip}_\beta\big(\mathbb{R}^n\big), \qquad \mathrm{Lip}^{(0)}_{\beta,\sigma}\big(\mathbb{R}^n\big) = \mathrm{Lip}_{\beta,\sigma}\big(\mathbb{R}^n\big)$$

and

$$\mathrm{BMO}^{(0)}\big(\mathbb{R}^n\big) = \mathrm{BMO}\big(\mathbb{R}^n\big), \qquad \mathrm{BMO}^{(0)}_\sigma\big(\mathbb{R}^n\big) = \mathrm{BMO}_\sigma\big(\mathbb{R}^n\big).$$

Here, for $0 \le \beta \le 1$ and $0 \le \sigma < \infty$, $\mathrm{Lip}_{\beta,\sigma}(\mathbb{R}^n)$ and $\mathrm{BMO}_\sigma(\mathbb{R}^n) = \mathrm{Lip}_{0,\sigma}(\mathbb{R}^n)$, so-called the σ-Lipschitz (σ-Lip) space and the σ-BMO space respectively, are defined by

$$\|f\|_{\mathrm{Lip}_{\beta,\sigma}} = \sup_{r\ge 1} \frac{1}{r^\sigma}\|f\|_{\mathrm{Lip}_\beta(Q_r)},$$

where

$$\|f\|_{\mathrm{Lip}_\beta(Q_r)} = \sup_{Q(x,s)\subset Q_r} \frac{1}{s^\beta}\left(⨍_{Q(x,s)} \left|f(y) - f_{Q(x,s)}\right| dy\right).$$

3 Generalized Fractional Integrals

In [19], in order to consider the boundedness of fractional integrals I_α on $B^{p,\lambda}(\mathbb{R}^n)$ under the conditions $0 < \alpha < n$, $1 \le p < \infty$, $-n/p \le \lambda < \infty$ and $\lambda + \alpha \ge 1$, we introduced the following definition of generalized fractional integrals $\tilde{I}_{\alpha,d}$.

Definition 3.1 (Definition 3.1 of [19]) For $0 < \alpha < n$ and $d \in \mathbb{N}_0$, we define the generalized fractional integral (of order α), i.e., $\tilde{I}_{\alpha,d}$, as follows: For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$,

$$\tilde{I}_{\alpha,d} f(x) = \int_{\mathbb{R}^n} f(y)\left\{K_\alpha(x-y) - \left(\sum_{\{l:|l|\le d\}} \frac{x^l}{l!}\big(D^l K_\alpha\big)(-y)\right)\big(1-\chi_{Q_1}(y)\big)\right\} dy,$$

where

$$K_\alpha(x) = \frac{1}{|x|^{n-\alpha}}$$

and for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $l = (l_1, l_2, \ldots, l_n) \in \mathbb{N}_0^n$, $|l| = l_1 + l_2 + \cdots + l_n$, $x^l = x_1^{l_1} x_2^{l_2} \cdots x_n^{l_n}$ and D^l is the partial derivative of order l, i.e.,

$$D^l = (\partial/\partial x_1)^{l_1}(\partial/\partial x_2)^{l_2} \cdots (\partial/\partial x_n)^{l_n}.$$

We note that particularly

$$\tilde{I}_{\alpha,0} = \tilde{I}_\alpha$$

(see (1.1) above). For further details of $\tilde{I}_{\alpha,d}$, refer to [19].

Then for a generalized fractional integral $\tilde{I}_{\alpha,d}$, we can show the following estimates on $B^{p,\lambda}(\mathbb{R}^n)$.

Theorem 3.2 *Let $0 < \alpha < n$, $n/\alpha \le p < \infty$, $d \in \mathbb{N}_0$, $-n/p + \alpha + d \le \lambda + \alpha < d + 1$, $\beta = \alpha - n/p$ and $\sigma = \lambda + n/p$. Then $\tilde{I}_{\alpha,d}$ is bounded from $B^{p,\lambda}(\mathbb{R}^n)$ to $\mathrm{Lip}^{(d)}_{\beta,\sigma}(\mathbb{R}^n)$, that is, there exists a constant $C > 0$ such that*

(iii”) *when $n/\alpha < p < \infty$,*

$$\|\tilde{I}_{\alpha,d} f\|_{\mathrm{Lip}^{(d)}_{\beta,\sigma}} \le C\|f\|_{B^{p,\lambda}}, \quad f \in B^{p,\lambda}\big(\mathbb{R}^n\big);$$

(iv”) *when $p = n/\alpha$, i.e., $\beta = 0$,*

$$\|\tilde{I}_{\alpha,d} f\|_{\mathrm{BMO}^{(d)}_\sigma} \le C\|f\|_{B^{p,\lambda}}, \quad f \in B^{p,\lambda}\big(\mathbb{R}^n\big).$$

In the above theorem, if $d = 0$, then we obtain the following estimates for $\tilde{I}_\alpha$ (cf. Sect. 1).

Corollary 3.3 (cf. Theorem 2.3 of [20]) *Let* $0 < \alpha < n$, $n/\alpha \le p < \infty$, $-n/p + \alpha \le \lambda + \alpha < 1$, $\beta = \alpha - n/p$ *and* $\sigma = \lambda + n/p$. *Then* $\tilde{I}_\alpha$ *is bounded from* $B^{p,\lambda}(\mathbb{R}^n)$ *to* $\mathrm{Lip}_{\beta,\sigma}(\mathbb{R}^n)$, *that is, there exists a constant* $C > 0$ *such that*

(iii') *when* $n/\alpha < p < \infty$,

$$\|\tilde{I}_\alpha f\|_{\mathrm{Lip}_{\beta,\sigma}} \le C\|f\|_{B^{p,\lambda}}, \quad f \in B^{p,\lambda}(\mathbb{R}^n);$$

(iv') *when* $p = n/\alpha$, *i.e.*, $\beta = 0$,

$$\|\tilde{I}_\alpha f\|_{\mathrm{BMO}_\sigma} \le C\|f\|_{B^{p,\lambda}}, \quad f \in B^{p,\lambda}(\mathbb{R}^n).$$

4 Proof of Theorem 3.2

First of all, we state the following well-definedness of $\tilde{I}_{\alpha,d}$ for $B^{p,\lambda}(\mathbb{R}^n)$, which was shown in [19].

Lemma 4.1 (Lemma 4.1 of [19]) *Let* $0 < \alpha < n$, $1 \le p < \infty$, $d \in \mathbb{N}_0$ *and* $-n/p + \alpha \le \lambda + \alpha < d + 1$. *Then for* $f \in B^{p,\lambda}(\mathbb{R}^n)$, $\tilde{I}_{\alpha,d} f$ *is well-defined.*

Also, in order to prove Theorem 3.2, it is necessary to use the following two lemmas.

Lemma 4.2 (cf. Lemma 7.3 of [21]) *Let* $x \in \mathbb{R}^n$, $0 < \alpha < n$ *and* $d \in \mathbb{N}_0$. *If* $y \in \mathbb{R}^n \setminus Q_{2|x|}$, *then*

$$\left|\Delta_h^{d+1}\left(K_\alpha(x-y) - \sum_{\{l:|l|\le d\}} \frac{x^l}{l!}\left(D^l K_\alpha\right)(-y)\right)\right| \le C\frac{|h|^{d+1}}{|y|^{n-\alpha+d+1}}.$$

Proof By Taylor's theorem, there exists $\xi \in Q_{|x|} \setminus \{0\}$ such that

$$K_\alpha(x-y) - \sum_{\{l:|l|\le d\}} \frac{x^l}{l!}\left(D^l K_\alpha\right)(-y) = \sum_{\{l:|l|=d+1\}} \frac{x^l}{l!}\left(D^l K_\alpha\right)(\xi - y).$$

Thus

$$\left|\Delta_h^{d+1}\left(K_\alpha(x-y) - \sum_{\{l:|l|\le d\}} \frac{x^l}{l!}\left(D^l K_\alpha\right)(-y)\right)\right|$$
$$= \left|\sum_{\{l:|l|\le d\}} h^l \left(D^l K_\alpha\right)(\xi - y)\right| \lesssim \frac{|h|^{d+1}}{|y|^{n-\alpha+d+1}}.$$

□

Lemma 4.3 (Lemma 4.3 of [19]) *Let* $1 \le p < \infty$ *and* $\lambda \in \mathbb{R}$. *If* $\beta + \lambda < 0$, *then there exists a constant* $C > 0$ *such that*

$$\int_{\mathbb{R}^n \setminus Q_r} \frac{|f(y)|}{|y|^{n-\beta}}\, dy \le C r^{\beta+\lambda} \|f\|_{B^{p,\lambda}} \quad \textit{for all } f \in B^{p,\lambda}(\mathbb{R}^n) \textit{ and } r \ge 1.$$

Proof of Theorem 3.2 Let $f \in B^{p,\lambda}(\mathbb{R}^n)$, $r \ge 1$ and $x \in Q_r$. Since $\tilde{I}_{\alpha,d} f$ is well-defined by Lemma 4.1, we prove only that

$$\|\tilde{I}_{\alpha,d} f\|_{\mathrm{Lip}^{(d)}_{\beta,\sigma}} \le C \|f\|_{B^{p,\lambda}}.$$

Now, we decompose $\tilde{I}_{\alpha,d} f(x)$ as follows:

$$\begin{aligned}
&\tilde{I}_{\alpha,d} f(x) \\
&\quad = \tilde{I}_{\alpha,d}(f\chi_{Q_{2r}})(x) + \tilde{I}_{\alpha,d}\big(f(1-\chi_{Q_{2r}})\big)(x) \\
&\quad = I_\alpha(f\chi_{Q_{2r}})(x) - \sum_{\{l:|l|\le d\}} \frac{x^l}{l!} \int_{Q_{2r}\setminus Q_1} f(y)\big(D^l K_\alpha\big)(-y)\, dy \\
&\qquad + \int_{\mathbb{R}^n \setminus Q_{2r}} f(y) \left(K_\alpha(x-y) - \sum_{\{l:|l|\le d\}} \frac{x^l}{l!} \big(D^l K_\alpha\big)(-y) \right) dy. \qquad (4.1)
\end{aligned}$$

Here note that the second term is a polynomial of degree at most d (see the proof of Lemma 4.1 of [19]). Consequently, in (4.1), letting

$$R_r^d f(x) = -\sum_{\{l:|l|\le d\}} \frac{x^l}{l!} \int_{Q_{2r}\setminus Q_1} f(y)\big(D^l K_\alpha\big)(-y)\, dy$$

and

$$J_{\alpha,d,r} f(x) = \int_{\mathbb{R}^n \setminus Q_{2r}} f(y) \left(K_\alpha(x-y) - \sum_{\{l:|l|\le d\}} \frac{x^l}{l!} \big(D^l K_\alpha\big)(-y) \right) dy,$$

$R_r^d f \in \mathcal{P}^d(Q_r)$ and it follows from Remark 2.7 that

$$\begin{aligned}
\|\tilde{I}_{\alpha,d} f\|_{\mathrm{Lip}^{(d)}_\beta(Q_r)} &\sim \|\tilde{I}_{\alpha,d} f\|_{\mathcal{L}^{(d)}_{1,\beta}(Q_r)} \\
&\sim \sup_{Q(x,s)\subset Q_r} \inf_{P\in\mathcal{P}^d(Q_r)} \frac{1}{s^\beta} \left(⨍_{Q(x,s)} \big|\tilde{I}_{\alpha,d} f(y) - P(y)\big|\, dy \right) \\
&\le \big\| I_\alpha(f\chi_{Q_{2r}}) \big\|_{\mathrm{Lip}_\beta(Q_r)} + \|J_{\alpha,d,r} f\|_{\mathcal{L}^{(d)}_{1,\beta}(Q_r)} \\
&\equiv I_1 + I_2.
\end{aligned}$$

First we estimate I_1. When $n/\alpha < p < \infty$, if we use the $(L^p, \mathrm{Lip}_\beta)$ boundedness of I_α, then we obtain

$$I_1 \le \left\| I_\alpha(f\chi_{Q_{2r}})\right\|_{\mathrm{Lip}_\beta} \lesssim \|f\chi_{Q_{2r}}\|_{L^p} \lesssim r^{\lambda+n/p}\|f\|_{B^{p,\lambda}} \\ = r^\sigma \|f\|_{B^{p,\lambda}}.$$

Similarly, when $p = n/\alpha$, by applying the $(L^{n/\alpha}, \mathrm{BMO})$ boundedness of I_α, we have

$$I_1 \le \left\| I_\alpha(f\chi_{Q_{2r}})\right\|_{\mathrm{BMO}} \lesssim r^\sigma \|f\|_{B^{p,\lambda}}.$$

Next we estimate I_2. By Remark 2.5 it follows that

$$I_2 \sim \|J_{\alpha,d,r}f\|_{\mathrm{Lip}_\beta^{(d)}(Q_r)} = \sup_{x,x+h\in Q_r, h\neq 0} \frac{1}{|h|^\beta}\left|\Delta^{d+1}J_{\alpha,d,r}f(x)\right|.$$

In order to estimate $\Delta^{d+1}J_{\alpha,d,r}f(x)$, if we use Lemmas 4.2, 4.3 and the assumption $\lambda+\alpha < d+1$, then we get for $x \in Q_r$ and $y \in \mathbb{R}^n \setminus Q_{2r}$,

$$\left|\Delta^{d+1}J_{\alpha,d,r}f(x)\right| \\ = \left|\int_{\mathbb{R}^n\setminus Q_{2r}} f(y)\left\{\Delta_h^{d+1}\left(K_\alpha(x-y) - \sum_{\{l:|l|\le d\}} \frac{x^l}{l!}\left(D^l K_\alpha\right)(-y)\right)\right\} dy\right| \\ \lesssim |h|^{d+1}\int_{\mathbb{R}^n\setminus Q_{2r}} \frac{|f(y)|}{|y|^{n-\alpha+d+1}} \lesssim |h|^{d+1} r^{\alpha-d-1+\lambda}\|f\|_{B^{p,\lambda}}.$$

Therefore

$$I_2 \lesssim \sup_{x,x+h\in Q_r, h\neq 0} \frac{1}{|h|^\beta}\cdot |h|^{d+1} r^{\alpha-d-1+\lambda}\|f\|_{B^{p,\lambda}} \\ \le r^{\lambda+\alpha-\beta}\|f\|_{B^{p,\lambda}} = r^\sigma\|f\|_{B^{p,\lambda}}.$$

Thus we have

$$\|\tilde{I}_{\alpha,d}f\|_{\mathrm{Lip}_{\beta,\sigma}^{(d)}} = \sup_{r\ge 1}\frac{1}{r^\sigma}\|\tilde{I}_{\alpha,d}f\|_{\mathrm{Lip}_\beta^{(d)}(Q_r)} \lesssim \|f\|_{B^{p,\lambda}}.$$

This concludes the proof. □

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Part IV
Qualitative Properties of Evolution Models

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Well-Posedness for a Generalized Boussinesq Equation

Alessia Ascanelli and Chiara Boiti

Abstract We consider a generalization of the Boussinesq equation obtained by adding a term of the form $a(t,x,u)\partial_x^3 u$. We prove local in time well-posedness of the Cauchy problem in Sobolev spaces under a suitable decay condition on the real part of the coefficient $a(t,x,u)$, as $x \to \infty$.

Keywords Boussinesq equation · Non-linear evolution equations · Well-posedness in Sobolev spaces

Mathematics Subject Classification (2010) Primary 35G25 · Secondary 35Q35

1 A Well-Posedness Result

In this paper we consider the equation

$$\partial_t^2 u + (u\partial_x u)_x + a(t,x,u)\partial_x^3 u + \partial_x^4 u = 0,$$

with $a \in C([0,T]; C^\infty(\mathbb{R} \times \mathbb{C}))$. This is the classical Boussinesq equation, widely used in hydrodynamics and physics, if $a \equiv 0$.

It is well known that the Cauchy problem for the classical Boussinesq equation is locally in time well-posed in Sobolev spaces. In this paper we show that the same result holds also in the case $a \neq 0$, under suitable decay conditions on the coefficient $a(t,x,u)$ (see condition (1.2)).

The Boussinesq equation describes the propagation of long waves with small amplitude in shallow water of constant depth. The introduction of a term of the third order with a coefficient that depends on x might be useful to take into account irregular seabeds (e.g. presence of dunes); the condition that this coefficient vanishes

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for $|x| \to \infty$ implies the assumption that for large $|x|$ (e.g. closer to the beach) we recover the ideal conditions of Boussinesq (shallow water and constant depth).

To state our result, let us write the equation in the form $Pu(t,x) = 0$, where

$$P(t,x,u,D_t,D_x) := D_t^2 - D_x^4 + ia(t,x,u)D_x^3 + uD_x^2 - i(\partial_x u)D_x \tag{1.1}$$

for $D = -i\partial$. This is a semi-linear operator of 2-evolution order $m = 2$ with real characteristics (in the sense of Petrowsky) $\tau = \pm\xi^2$. We know, from the pioneering paper [8], that to obtain well-posedness in Sobolev spaces of the Cauchy problem for a 2-evolution equation with real characteristics, some decay conditions at infinity on the lower order terms are necessary. These kind of conditions already appeared in [5] for the vibrating beam equation (which has the same principal part as the Boussinesq equation); a more general case of second order Schrödinger type equations with decay conditions at infinity was considered in [6].

We are going to prove the following result:

Theorem 1.1 *Consider P in (1.1) for $a \in C([0,T]; C^\infty(\mathbb{R}\times\mathbb{C}))$ and $a(t,\cdot,u) \in \mathcal{B}^\infty(\mathbb{R})$, satisfying*

$$\left|\mathrm{Re}\, a(t,x,w)\right| \le \frac{C}{\langle x\rangle^{1+\varepsilon}}h(w) \quad \forall (t,x,w) \in [0,T]\times\mathbb{R}\times\mathbb{C}, \tag{1.2}$$

for some constants $C, \varepsilon > 0$ and a function $h : \mathbb{C} \to \mathbb{R}^+$ bounded on compact sets, where $\langle x\rangle := \sqrt{1+x^2}$.

Then the Cauchy problem

$$\begin{cases} P\big(t,x,u(t,x),D_t,D_x\big)u(t,x) = f(t,x) \\ u(0,x) = u_0(x) \\ u_t(0,x) = u_1(x) \end{cases} \tag{1.3}$$

is locally in time well-posed in H^∞. More precisely, for every $s > 3/2$ and for all $f \in C([0,T]; H^s(\mathbb{R}))$, $u_0 \in H^{s+2}(\mathbb{R})$ and $u_1 \in H^s(\mathbb{R})$, there exists $0 < T^* \le T$ and a unique solution $u \in C([0,T^*]; H^{s+2}(\mathbb{R})) \cap C^1([0,T^*]; H^s(\mathbb{R}))$ of (1.3) satisfying, for some $\sigma > 0$ depending on s,

$$\begin{aligned} &\left\|u(t,\cdot)\right\|_{s+2}^2 + \left\|D_t u(t,\cdot)\right\|_s^2 \\ &\quad \le e^{\sigma t}\left(\|u_0\|_{s+2}^2 + \|u_1\|_s^2 + \int_0^t \left\|f(\tau,\cdot)\right\|_s^2 d\tau\right) \end{aligned} \tag{1.4}$$

for all $t \in [0,T^*]$.

To prove Theorem 1.1 we linearize equation (1.1) as done in [4], reduce it to a first order 2-evolution system following [2] and finally apply the results of [1, 3], in the framework of pseudo-differential operators with symbol $p(x,\xi)$ in the classical class S^m defined by

$$\left|\partial_\xi^\alpha D_x^\beta p(x,\xi)\right| \le C_{\alpha,\beta,h}\langle\xi\rangle_h^{m-\alpha} \quad \forall\alpha,\beta\in\mathbb{N},\ h\ge 1,$$

for some $C_{\alpha,\beta,h} > 0$, where $\langle\xi\rangle_h := \sqrt{h^2+\xi^2}$.

In the next section we shall use the notation $\mathrm{op}(p(x,\xi))$ to denote a pseudo-differential operator with symbol $p(x,\xi)$.

2 Proof of Theorem 1.1

The first step is to linearize the operator P defined in (1.1). To this aim we fix $u \in C([0,T]; H^{s+2}(\mathbb{R})) \cap C^1([0,T]; H^s(\mathbb{R}))$ and look for a solution $v \in C([0,T]; H^{s+2}(\mathbb{R})) \cap C^1([0,T]; H^s(\mathbb{R}))$ of the linear Cauchy problem

$$\begin{cases} P\big(t,x,u(t,x),D_t,D_x\big)v(t,x) = f(t,x) \\ v(0,x) = u_0(x) \\ v_t(0,x) = u_1(x). \end{cases} \tag{2.1}$$

In order to study well-posedness of (2.1) in Sobolev spaces, we follow [2] and look for a factorization of the principal part of P (in the sense of Petrowsky) by means of pseudo-differential operators: defining

$$\mu(t,x,u,D_x) := D_x^2 - \frac{i}{2}a(t,x,u)D_x \tag{2.2}$$

we have that

$$P(t,x,u,D_t,D_x) = (D_t-\mu)(D_t+\mu) + b_2(t,x,u)D_x^2 + b_1(t,x,u)D_x$$

with

$$\begin{aligned} b_2(t,x,u) &:= u - iD_x\big(a(t,x,u)\big) - \frac{1}{4}a^2(t,x,u) \\ b_1(t,x,u) &:= -i(\partial_x u) + \frac{i}{2}D_t\big(a(t,x,u)\big) - \frac{i}{2}D_x^2\big(a(t,x,u)\big) \\ &\qquad - \frac{1}{4}a(t,x,u)D_x\big(a(t,x,u)\big). \end{aligned}$$

Now, following again [2] we perform a reduction to a first order system: we set $V = (v_1, v_2)$ with

$$\begin{cases} v_1 = \langle D_x\rangle_h^2 v \\ v_2 = (D_t+\mu)v \end{cases} \tag{2.3}$$

and obtain that v is a solution of the equation $Pv = f$ if and only if V is a solution of the system

$$(D_t - A + R)V = F \tag{2.4}$$

where D_t is the matrix $D_t \cdot I$, $A = \begin{pmatrix} -\mu & \langle D_x\rangle_h^2 \\ 0 & \mu \end{pmatrix}$, $F = {}^t(0, f)$ and $R(t, x, u, D_x)$ is a matrix of pseudo-differential operators of order 0.

We now diagonalize A (modulo terms of order 0) by means of the following matrix of pseudo-differential operators of order 0:

$$K(t, x, u, D_x) := \begin{pmatrix} 1 & \operatorname{op}(\frac{\langle\xi\rangle_h^2}{2\mu(t,x,\xi)}) \\ 0 & 1 \end{pmatrix}.$$

The operator K is invertible and the inverse operator

$$K^{-1}(t, x, u, D_x) = \begin{pmatrix} 1 & -\operatorname{op}(\frac{\langle\xi\rangle_h^2}{2\mu(t,x,\xi)}) \\ 0 & 1 \end{pmatrix}$$

coincides with the inverse matrix of K. We have that

$$K^{-1}AK = \tilde{A} + \tilde{R} \tag{2.5}$$

with $\tilde{A} = \begin{pmatrix} -\mu & 0 \\ 0 & \mu \end{pmatrix}$ and $\tilde{R} = \begin{pmatrix} 0 & \tilde{r}_{12} \\ 0 & 0 \end{pmatrix}$ of order 0 since $\tilde{r}_{12}$ has symbol

$$\begin{aligned} \sigma(\tilde{r}_{12}) &= \sigma\left(-\mu \operatorname{op}\left(\frac{\langle\xi\rangle_h^2}{2\mu}\right) + \langle D_x\rangle_h^2 - \operatorname{op}\left(\frac{\langle\xi\rangle_h^2}{2\mu}\right)\mu\right) \\ &= -\mu\frac{\langle\xi\rangle_h^2}{2\mu} + \langle\xi\rangle_h^2 - \frac{\langle\xi\rangle_h^2}{2\mu}\mu - \sum_{\alpha\geq 1}\frac{1}{\alpha!}\left[\partial_\xi^\alpha \mu \cdot D_x^\alpha \frac{\langle\xi\rangle_h^2}{2\mu} + \partial_\xi^\alpha \frac{\langle\xi\rangle_h^2}{2\mu} D_x^\alpha \mu\right] \end{aligned}$$

with principal part $\partial_\xi \mu \cdot D_x \frac{\langle\xi\rangle_h^2}{2\mu} + \partial_\xi \frac{\langle\xi\rangle_h^2}{2\mu} D_x \mu \in S^0$ by (2.2).

Applying then K^{-1} to (2.4) we obtain from (2.5) that V is a solution of (2.4) if and only if $W = K^{-1}V$ is a solution of

$$(D_t - \tilde{A} + R_0)W = \tilde{F}, \tag{2.6}$$

where $\tilde{F} = K^{-1}F$ and $R_0 = -\tilde{R} + K^{-1}RK$ is a matrix of pseudo-differential operators of order zero.

To study well-posedness of system (2.6), we follow [3, formula (2.4) and Remark 3.1] (see also [1, Remark 4.1]) and define

$$\lambda(x, \xi) := M\omega\left(\frac{\xi}{h}\right)\int_0^x \langle y\rangle^{-1-\varepsilon}\psi\left(\frac{\langle y\rangle}{\langle\xi\rangle_h}\right)dy$$

for $M > 0$ to be chosen in the following, $\omega \in C^\infty(\mathbb{R})$ and $\psi \in C_0^\infty(\mathbb{R})$ with $0 \leq \omega, \psi \leq 1$ such that

$$\omega(y) = \begin{cases} 0 & |y| \leq 1 \\ \operatorname{sgn} y & |y| \geq 2, \end{cases} \qquad \psi(y) = \begin{cases} 1 & |y| \leq 1/2 \\ 0 & |y| \geq 1. \end{cases}$$

Then, by Lemma 2.3 of [3] the operator $e^{\lambda(x,D_x)}$ with symbol $e^{\lambda(x,\xi)} \in S^0$ is invertible, for h large enough, and

$$\left(e^{\lambda}\right)^{-1} = e^{-\lambda}(I + R_{-1})$$

for a pseudo-differential operator R_{-1} with principal symbol $r_{-1}(x,\xi) = \partial_\xi \lambda \cdot D_x\lambda \in S^{-1}$.

We define then the matrix

$$E^{\Lambda(x,D_x)} := \begin{pmatrix} e^{\lambda(x,D_x)} & 0 \\ 0 & e^{-\lambda(x,D_x)} \end{pmatrix}.$$

This is an invertible matrix and its inverse matrix is a the diagonal matrix with entries $(E^\Lambda)^{-1}_{11} = (e^\lambda)^{-1}$ and $(E^\Lambda)^{-1}_{22} = (e^{-\lambda})^{-1}$.

Defining then

$$L_\Lambda := \left(E^\Lambda\right)^{-1}(D_t - \tilde{A} + R_0)E^\Lambda = D_t - iA_\Lambda + R_\Lambda$$

with

$$A_\Lambda := -i\left(E^\Lambda\right)^{-1}\tilde{A}E^\Lambda, \qquad R_\Lambda := \left(E^\Lambda\right)^{-1}R_0E^\Lambda$$

we have that W is solution of (2.6) if and only if $Z := (E^\Lambda)^{-1}W$ is solution of $L_\Lambda Z = F_\Lambda$ with $F_\Lambda := (E^\Lambda)^{-1}\tilde{F}$.

We are thus reduced to study well-posedness of the Cauchy problem

$$\begin{cases} L_\Lambda Z(t,x) = F_\Lambda(t,x), & (t,x) \in [0,T] \times \mathbb{R} \\ Z(0,x) = Z_0(x), & x \in \mathbb{R}, \end{cases} \tag{2.7}$$

with $Z_0(x) = (E^\Lambda)^{-1}W(0,x) = (E^\Lambda)^{-1}K^{-1}V(0,x)$.

This system satisfies all assumptions of Theorem 1.1 of [1], so that we have the following energy estimate with no loss of derivatives (see also [1, Remark 1.3]):

$$|||Z(t,\cdot)|||_s^2 \le C_s(u)\left(|||Z_0|||_s^2 + \int_0^t |||F_\Lambda(\tau,\cdot)|||_s^2 d\tau\right),$$

where $|||Z|||_s^2 := \|z_1\|_s^2 + \|z_2\|_s^2$ for $Z = (z_1, z_2)$.

However, as in [4], we need to determine the constant $C_s(u)$ in order to apply a fixed point theorem.

To this aim we set $\langle\langle Z, Z'\rangle\rangle_s = \langle z_1, z_1'\rangle_s + \langle z_2, z_2'\rangle_s$ for $Z = (z_1, z_2)$ and $Z' = (z_1', z_2')$, and we estimate:

$$\begin{aligned}\frac{d}{dt}\|\!|Z|\!\|_0^2 &= 2\,\mathrm{Re}\big\langle\langle\partial_t Z, Z\rangle\big\rangle_0 = 2\,\mathrm{Re}\big\langle\langle iL_\Lambda Z, Z\rangle\big\rangle_0\\ &\quad - 2\,\mathrm{Re}\big\langle\langle A_\Lambda Z, Z\rangle\big\rangle_0 - 2\,\mathrm{Re}\big\langle\langle iR_\Lambda Z, Z\rangle\big\rangle_0\\ &\le C\big(\|\!|L_\Lambda Z|\!\|_0^2 + \|\!|Z|\!\|_0^2\big) - 2\,\mathrm{Re}\big\langle\langle A_\Lambda Z, Z\rangle\big\rangle_0 \end{aligned} \tag{2.8}$$

for some $C > 0$ since R_Λ has order zero.

Now,

$$\begin{aligned}\mathrm{Re}\big\langle\langle A_\Lambda Z, Z\rangle\big\rangle_0 &= \mathrm{Re}\big\langle\big(e^{\lambda}\big)^{-1} i\mu e^{\lambda} z_1, z_1\big\rangle_0 - \mathrm{Re}\big\langle\big(e^{-\lambda}\big)^{-1} i\mu e^{-\lambda} z_2, z_2\big\rangle_0\\ &=: \mathrm{Re}\big\langle A_\Lambda^1 z_1, z_1\big\rangle_0 + \mathrm{Re}\big\langle A_\Lambda^2 z_2, z_2\big\rangle_0, \end{aligned} \tag{2.9}$$

and, for $j = 1, 2$ and $|\xi| \ge 2h$:

$$\begin{aligned}\mathrm{Re}\,\sigma\big(A_\Lambda^j\big) &= 2\xi\,\partial_x\lambda \pm \frac{1}{2}\mathrm{Re}\,a(t,x,u)\xi\\ &= 2\xi\frac{M}{\langle x\rangle^{1+\varepsilon}}\omega\left(\frac{\xi}{h}\right)\psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h}\right) \pm \frac{1}{2}\mathrm{Re}\,a(t,x,u)\xi\\ &\ge \frac{\langle\xi\rangle_h}{\langle x\rangle^{1+\varepsilon}}\left(\frac{4}{\sqrt{5}}M - \frac{1}{2}Ch(u)\right)\psi - \frac{C}{2}h(u)\frac{\langle\xi\rangle_h}{\langle x\rangle^{1+\varepsilon}}(1-\psi)\\ &\ge \frac{\langle\xi\rangle_h}{\langle x\rangle^{1+\varepsilon}}\left(\frac{4}{\sqrt{5}}M - \frac{1}{2}Ch(u)\right)\psi - Ch(u) \end{aligned} \tag{2.10}$$

since $\langle\xi\rangle_h \le 2\langle x\rangle$ on $\mathrm{supp}(1-\psi)$.

We now define

$$\big\|\big\|u(t,\cdot)\big\|\big\|_s := \big\|u(t,\cdot)\big\|_{s+2} + \big\|D_t u(t,\cdot)\big\|_s,$$

$$\mathcal{B}_r := \left\{u \in C\big([0,T]; H^{s+2}\big) \cap C^1\big([0,T]; H^s\big) : \sup_{t\in[0,T]} \big\|\big\|u(t,\cdot)\big\|\big\|_s \le r\right\}$$

and

$$c_r := \sup_{\substack{(t,x)\in[0,T]\times\mathbb{R}\\ u\in\mathcal{B}_r}} h(u).$$

We choose $M > \sqrt{5}Cc_r/8$ and obtain, from (2.10), that

$$\mathrm{Re}\,\sigma\big(A_\Lambda^j\big) \ge -Cc_r, \quad j = 1, 2$$

and hence, by the sharp-Gårding inequality:

$$\mathrm{Re}\langle A_{\Lambda}^{j} z_j, z_j \rangle_0 \geq -c(1+c_r)\|z_j\|_0^2, \quad j = 1, 2$$

for some $c > 0$.

Substituting in (2.9) and (2.8) we have the estimate:

$$\frac{d}{dt}\|\!|Z|\!\|_0^2 \leq (4c + C)(1 + c_r)\|\!|Z|\!\|_0^2 + C\|\!|L_{\Lambda} Z|\!\|_0^2;$$

this leads, by standard arguments, to the following energy estimate for the solution of (2.7):

$$\|\!|Z(t,\cdot)|\!\|_s^2 \leq e^{c_1(1+c_r)t}\left(\|\!|Z_0|\!\|_s^2 + \int_0^t \|\!|F_{\Lambda}(\tau,\cdot)|\!\|_s^2 d\tau\right)$$

for some $c_1 > 0$ depending on $s \in \mathbb{R}$.

Since E^{Λ} and K have order zero, then for $V = KE^{\Lambda}Z$ and $F = KE^{\Lambda}F_{\Lambda}$ we have:

$$\|\!|V(t,\cdot)|\!\|_s^2 \leq e^{c_2(1+c_r)t}\left(\|\!|V(0)|\!\|_s^2 + \int_0^t \|\!|F(\tau,\cdot)|\!\|_s^2 d\tau\right) \tag{2.11}$$

for some $c_2 > 0$.

Now, $\|\!|F(\tau,\cdot)|\!\|_s^2 = \|\!|f(\tau,\cdot)|\!\|_s^2$ and moreover, by (2.3),

$$\begin{aligned}\|\!|V|\!\|_s^2 = \|v_1\|_s^2 + \|v_2\|_s^2 &= \left\|\langle D_x\rangle_h^2 v\right\|_s^2 + \left\|(D_t + \mu)v\right\|_s^2 \\ &\leq c_3\left(\|v\|_{s+2}^2 + \|D_t v\|_s^2\right)\end{aligned}$$

and, vice versa,

$$\begin{aligned}\|v\|_{s+2} + \|D_t v\|_s &= \left\|\langle D_x\rangle_h^{-2} v_1\right\|_{s+2} + \left\|v_2 - \mu\langle D_x\rangle_h^{-2} v_1\right\|_s \\ &\leq c_4\left(\|v_1\|_s + \|v_2\|_s\right)\end{aligned}$$

for some $c_3, c_4 > 0$.

Therefore (2.11) is equivalent to

$$\begin{aligned}\left\|v(t,\cdot)\right\|_{s+2}^2 + \left\|D_t v(t,\cdot)\right\|_s^2 \leq e^{c_5(1+c_r)t}\Big(&\left\|v(0,\cdot)\right\|_{s+2}^2 + \left\|v_t(0,\cdot)\right\|_s^2 \\ &+ \int_0^t \left\|f(\tau,\cdot)\right\|_s^2 d\tau\Big)\end{aligned}$$

and hence

$$\left\|v(t,\cdot)\right\|_{s+2}+\left\|D_t v(t,\cdot)\right\|_s \le e^{C'(1+c_r)t}\Big(\|u_0\|_{s+2}+\|u_1\|_s + \sqrt{t}\sup_{t\in[0,T]}\left\|f(t,\cdot)\right\|_s\Big) \tag{2.12}$$

for some c_5, $C'>0$.

We have thus defined a map

$$\begin{aligned} S:\ \mathcal{B}_r &\longrightarrow C\big([0,T];H^{s+2}\big)\cap C^1\big([0,T];H^s\big)\\ u &\longmapsto v \end{aligned}$$

which associates to each $u\in\mathcal{B}_r$ the unique solution $v\in C([0,T];H^{s+2})\cap C^1([0,T];H^s)$ of the Cauchy problem (2.1) satisfying (2.12).

If we now choose

$$r>2e\max\Big\{\|u_0\|_{s+2}+\|u_1\|_s,\ \sup_{t\in[0,T]}\left\|f(t,\cdot)\right\|_s\Big\},$$

then, from (2.12),

$$\left\|\left\|v(t,\cdot)\right\|\right\|_s \le \frac{1}{2}re^{C'(1+c_r)t-1}(1+\sqrt{t})<r$$

if $t\in[0,T_0]$ for T_0 sufficiently small.

Defining then

$$\mathcal{B}_r^0:=\Big\{v\in C\big([0,T_0];H^{s+2}\big)\cap C^1\big([0,T_0];H^s\big):\ \sup_{t\in[0,T_0]}\left\|\left\|v(t,\cdot)\right\|\right\|_s\le r\Big\}$$

we have that S maps $\mathcal{B}_r^0$ into itself.

We prove now that S is a contraction. To this aim we fix $u,\tilde{u}\in\mathcal{B}_r^0$ and set $v:=S(u)$, $\tilde{v}:=S(\tilde{u})$, $w:=v-\tilde{v}$.

From

$$\begin{aligned} &D_t^2v-D_x^4v+ia(t,x,u)D_x^3v+uD_x^2v-i(\partial_x u)D_xv=f\\ &D_t^2\tilde{v}-D_x^4\tilde{v}+ia(t,x,\tilde{u})D_x^3\tilde{v}+\tilde{u}D_x^2\tilde{v}-i(\partial_x\tilde{u})D_x\tilde{v}=f \end{aligned}$$

we have that

$$\begin{aligned} &D_t^2w-D_x^4w+ia(t,x,u)D_x^3w+uD_x^2w-i(\partial_x u)D_xw\\ &\quad+i\big(a(t,x,u)-a(t,x,\tilde{u})\big)D_x^3\tilde{v}+(u-\tilde{u})D_x^2\tilde{v}-i(\partial_x u-\partial_x\tilde{u})D_x\tilde{v}=0, \end{aligned}$$

i.e.

$$\begin{cases} P(t,x,u,D_t,D_x)w=\tilde{f}(t,x,u,\tilde{u},\tilde{v})\\ w(0,x)=0\\ w_t(0,x)=0 \end{cases} \tag{2.13}$$

with

$$\tilde{f}(t, x, u, \tilde{u}, \tilde{v}) := -i\big(a(t, x, u) - a(t, x, \tilde{u})\big)D_x^3\tilde{v} \\ - (u - \tilde{u})D_x^2\tilde{v} + i(\partial_x u - \partial_x\tilde{u})D_x\tilde{v}.$$

Since $u, \tilde{u}, \tilde{v} \in C([0, T]; H^{s+2}) \cap C^1([0, T]; H^s)$ then $\tilde{f} \in C([0, T]; H^{s-1})$ and, applying (2.12) to (2.13) with $s - 1$ instead of s:

$$\|w(t, \cdot)\|_{s+1} + \|D_t w(t, \cdot)\|_{s-1} \leq e^{C'(1+c_r)t}\sqrt{t} \sup_{t\in[0,T_0]} \|\tilde{f}(t, \cdot)\|_{s-1}$$

for all $t \in [0, T_0]$. Now, for $s > 3/2$:

$$\begin{aligned}
\|\tilde{f}(t, \cdot)\|_{s-1} &\leq \big\|\big(a(t, x, u) - a(t, x, \tilde{u})\big)D_x^3\tilde{v}\big\|_{s-1} + \|(u - \tilde{u})D_x^2\tilde{v}\|_{s-1} \\
&\quad + \|(\partial_x u - \partial_x\tilde{u})D_x\tilde{v}\|_{s-1} \\
&\leq \|a(t, x, u) - a(t, x, \tilde{u})\|_{s-1} \cdot \|D_x^3\tilde{v}\|_{s-1} \\
&\quad + \|u - \tilde{u}\|_{s-1} \cdot \|D_x^2\tilde{v}\|_{s-1} + \|\partial_x u - \partial_x\tilde{u}\|_{s-1} \cdot \|D_x\tilde{v}\|_{s-1} \\
&\leq C_{s,r}\|u - \tilde{u}\|_s
\end{aligned}$$

for some $C_{s,r} > 0$ depending on s and r.

Therefore

$$\|w(t, \cdot)\|_{s+1} + \|D_t w(t, \cdot)\|_{s-1} \leq C_{s,r}e^{C'(1+c_r)t}\sqrt{t} \sup_{t\in[0,T_0]} \|u - \tilde{u}\|_s$$

for all $t \in [0, T_0]$, i.e.

$$\big\|\|S(u) - S(\tilde{u})\|\big\|_{s-1} \leq C_{s,r}e^{C'(1+c_r)t}\sqrt{t} \sup_{t\in[0,T_0]} \big\|\|u - \tilde{u}\|\big\|_{s-1}.$$

We now choose $0 < T^* < T_0$ sufficiently small so that

$$C_{s,r}e^{C'(1+c_r)T^*}\sqrt{T^*} < 1$$

and hence

$$\sup_{t\in[0,T^*]} \big\|\|S(u) - S(\tilde{u})\|\big\|_{s-1} \leq L \sup_{t\in[0,T_0]} \big\|\|u - \tilde{u}\|\big\|_{s-1} \tag{2.14}$$

for $0 < L < 1$. Setting

$$\mathcal{B}_r^* := \Big\{v \in C\big([0, T^*]; H^{s+2}\big) \cap C^1\big([0, T^*]; H^s\big) : \sup_{t\in[0,T^*]} \big\|\|v(t, \cdot)\|\big\|_s \leq r\Big\},$$

we have that $S : \mathcal{B}_r^* \longrightarrow \mathcal{B}_r^*$ is a contraction with the norm $\|\|\cdot\|\|_{s-1}$.

Defining then recursively $u_{n+1} = S(u_n)$ we can prove, by the same fixed point arguments used in [4], that $u_n \to u$ and $S(u_n) \to S(u)$ in $C([0, T^*]; H^{s+1}) \cap C^1([0, T^*]; H^{s-1})$. Moreover, $u \in C([0, T^*]; H^{s+2}) \cap C^1([0, T^*]; H^s)$ and so the fixed point $u = S(u)$ is a solution of the Cauchy problem (1.3).

Finally, uniqueness follows from (2.14) and the proof is complete. □

Remark 2.1 Assumption (1.2) with $\varepsilon > 0$ is needed to obtain the energy estimate (2.11) without loss of derivatives. This implies that the map S is a contraction (see (2.14)) and hence allows us to apply a fixed point argument.

In the case $\varepsilon = 0$ the fixed point method fails because of the loss of derivatives. However, this limit case could be probably treated by a different approach, based on the Nash–Moser Theorem in the tame space H^∞ (cf. [7]).

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Energy Solutions for Nonlinear Klein–Gordon Equations in de Sitter Spacetime

Makoto Nakamura

Abstract The Cauchy problem for nonlinear Klein–Gordon equations is considered in de Sitter spacetime. The nonlinear terms are power type or exponential type. The local and global solutions are shown in the energy class.

Keywords Klein–Gordon equation · De Sitter spacetime

Mathematics Subject Classification (2010) Primary 35L70 · Secondary 35L15

1 Introduction

We report some fundamental results of local and global energy solutions for the Cauchy problem of semi-linear Klein–Gordon equations in de Sitter spacetime. Let $n \geq 1$, $M > 0$, $H > 0$, $c > 0$, and let us consider the Cauchy problem given by

$$\begin{cases} \left(\partial_t^2 - c^2 e^{-2Ht}\Delta + M^2\right)u(t,x) + c^2 e^{nHt/2} f\left(e^{-nHt/2}u(t,x)\right) = 0 \\ \quad \text{for } (t,x) \in [0,T) \times \mathbb{R}^n \\ u(0,\cdot) = u_0(\cdot) \in H^1\left(\mathbb{R}^n\right), \qquad \partial_t u(0,\cdot) = u_1(\cdot) \in L^2\left(\mathbb{R}^n\right), \end{cases} \tag{1.1}$$

where u_0, u_1, f are real-valued functions, $\Delta := \sum_{j=1}^n \partial^2/\partial x_j^2$, $H^1(\mathbb{R}^n)$ denotes the Sobolev space and $L^2(\mathbb{R}^n)$ denotes the Lebesgue space.

D'Ancona and Giuseppe have shown in [5] and [6] global classical solutions for $(\partial_t^2 - a(t)\Delta)u + |u|^{p-1}u = 0$ with some additional conditions on $a(t) \geq 0$ and p when $n = 1, 2, 3$. Yagdjian has shown in [19] small global solutions for the first equation in (1.1) when the nonlinear term f is of power type and the norm of initial data $\|u_0\|_{H^s(\mathbb{R}^n)} + \|u_1\|_{H^s(\mathbb{R}^n)}$ is sufficiently small for some $s > n/2 \geq 1$ (see also [20] for the system of the equations). Baskin has shown in [3] small global solution for $(\Box_g + \lambda)u + f(u) = 0$ when $f(u)$ is a type of $|u|^{p-1}u$, $p = 1 + 4/(n-1)$,

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$\lambda > n^2/4$, $(u_0, u_1) \in H^1 \oplus L^2$, where g gives the asymptotic de Sitter spacetime (see also [2] for the cases $p = 5$ with $n = 3$, $p = 3$ with $n = 4$). Blow-up phenomena are considered in [18]. See also the references in the summary [21] by Yagdjian. The aim of this paper is to give some results for the well-posedness of the Cauchy problem (1.1) with power type nonlinear terms in the energy space, and we also consider exponential type nonlinear terms in two spatial dimensions for the limiting case in terms of Sobolev embeddings.

To denote power type nonlinear terms of order p, we define the following set $N(p)$. We note that the nonlinear terms $f(u) = \lambda|u|^{p-1}u$ and $f(u) = \lambda|u|^p$ for $\lambda \in \mathbb{R}$ satisfy $f \in N(p)$. We remark that in [19–21], the nonlinear terms must be Lipschitz continuous in the Sobolev space $H^s(\mathbb{R}^n)$ of order $s > n/2$, which requires additional regularity for initial data (see Condition $\mathcal{L}$ in the papers).

Definition 1.1 Let $p \geq 1$. We denote by $N(p)$ the set of functions f from $\mathbb{R}$ to $\mathbb{R}$ which satisfies $f(0) = 0$ and

$$\left| f(u) - f(v) \right| \leq C \max_{w=u,v} |w|^{p-1} |u - v| \tag{1.2}$$

for any u and $v \in \mathbb{R}$, where $C > 0$ is a constant independent of u and v.

For $T > 0$, we define a function space $X(T) := \{u : \|u\|_{X(T)} < \infty\}$, where

$$\begin{aligned}\|u\|_{X(T)} := \max\Big\{ & M\|u\|_{L^\infty((0,T),L^2(\mathbb{R}^n))},\ \|\partial_t u\|_{L^\infty((0,T),L^2(\mathbb{R}^n))}, \\ & c\left\|e^{-Ht}\nabla u\right\|_{L^\infty((0,T),L^2(\mathbb{R}^n))}, \\ & c\sqrt{H}\left\|e^{-Ht}\nabla u\right\|_{L^2((0,T)\times\mathbb{R}^n))}\Big\}.\end{aligned} \tag{1.3}$$

We start from the Cauchy problem for power type nonlinear terms.

Theorem 1.2 *Let p satisfy*

$$1 \leq p \begin{cases} < \infty & \text{if } n = 1, 2 \\ \leq 1 + \dfrac{2}{n-2} & \text{if } n \geq 3. \end{cases} \tag{1.4}$$

Let $f \in N(p)$. Then we have the following results.

(1) *For any u_0 and u_1, there exists $T = T(\|u_0\|_{H^1(\mathbb{R}^n)} + \|u_1\|_{L^2(\mathbb{R}^n)}) > 0$ such that (1.1) has a unique solution u in $C([0, T), H^1(\mathbb{R}^n)) \cap C^1([0, T), L^2(\mathbb{R}^n))$. Here, u satisfies $u \in X(T)$, and for any fixed p_0 with $1 \leq p_0 < 1 + 4/n$, there exists a constant $C > 0$ dependent on p_0 but independent of u_0 and u_1 such that T can be estimated from below as*

$$T \geq C\left\{\|u_0\|_{H^1(\mathbb{R}^n)} + \|u_1\|_{L^2(\mathbb{R}^n)}\right\}^{-(p-1)/\{1-n(p_0-1)/4\}}.$$

(2) *Let* $n \le 4$. *If* $\|u_0\|_{H^1(\mathbb{R}^n)} + \|u_1\|_{L^2(\mathbb{R}^n)}$ *is sufficiently small, and* $1 + 4/n \le p$, *then* (1.1) *has a unique solution* u *in* $C([0,\infty), H^1(\mathbb{R}^n)) \cap C^1([0,\infty), L^2(\mathbb{R}^n))$. *And* u *satisfies* $u \in X(\infty)$.

Theorem 1.2 shows that any growth order p of polynomial type is subcritical for $n = 1, 2$. When $n = 1$, we are able to see that any growth order of C^1 type is also subcritical by the embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$. Namely, for any real-valued function $g \in C^1(\mathbb{R})$, the result (1) of Theorem 1.2 is valid even if we replace $f(u)$ with $f(u)g(u)$. This result shows the existence of time local solutions, while the existence of time global solutions for small data, namely (2) of Theorem 1.2 could not be shown for $f(u)g(u)$ since the energy estimates are too weak to treat the nonlinear terms when $n = 1$. When $n = 2$, the embedding $H^1(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$ does not hold and it is critical embedding. In this case, the next theorem shows that the exponential growth order is also subcritical. We use the following Gagliardo–Nirenberg interpolation inequality with asymptotic.

Lemma 1.3 *Let* $n = 1, 2$. *There exist* $\beta > 0$ *and* $q_0 \ge 2$ *such that*

$$\|u\|_{L^q(\mathbb{R}^n)} \le \beta q^{1/2} \|\nabla u\|_{L^2(\mathbb{R}^n)}^{n(1/2-1/q)} \|u\|_{L^2(\mathbb{R}^n)}^{1-n(1/2-1/q)}$$

for any q *with* $q_0 \le q < \infty$ *and nonconstant* u. *Here,* β *can be taken for any number with* $\beta > (8\pi e)^{-1/2}$ *when* $n = 2$.

Indeed, this lemma easily follows from the Moser–Trudinger inequality

$$\|u\|_{L^q(\mathbb{R}^n)} \le \beta q^{1/2} \|u\|_{\dot{H}^{n/2}(\mathbb{R}^n)}^{1-2/q} \|u\|_{L^2(\mathbb{R}^n)}^{2/q}$$

(see [10, Corollary 1.6], [11, Theorem 1.1] and the references therein) and the interpolation inequality

$$\|u\|_{\dot{H}^{n/2}(\mathbb{R}^n)} \le \|u\|_{\dot{H}^1(\mathbb{R}^n)}^{n/2} \|u\|_{L^2(\mathbb{R}^n)}^{1-n/2}.$$

The exponential nonlinear terms have been considered for Schrödinger equations in [4, 14], wave equations in [7, 15], Klein–Gordon equations in [9, 16], heat equations in [8], complex Ginzburg–Landau equations and dissipative wave equations in [13], damped Klein–Gordon equations in [1]. We show the corresponding result for Klein–Gordon equations in de Sitter spacetime. We note that the result (1) in the next theorem is weaker than the aforementioned case for the time local solutions when $n = 1$, however, we are able to consider time global solutions for small data in (2).

Theorem 1.4 *Let* $n = 1, 2$. *Let* $\lambda \in \mathbb{R}$, $\alpha > 0$, $0 < \nu \le 2$, $j_0 \ge 0$. *Let* $f(u) = \lambda u (e^{\alpha |u|^\nu} - \sum_{0 \le j < j_0} \frac{\alpha^j}{j!} |u|^{\nu j})$ *for* $j_0 \ge 1$, *and* $f(u) = \lambda u e^{\alpha |u|^\nu}$ *for* $j_0 = 0$. *Put* $D := \|u_0\|_{H^1(\mathbb{R}^n)} + \|u_1\|_{L^2(\mathbb{R}^n)}$. *Then we have the following results.*

(1) *Let* $\nu < 2$. *For any* u_0 *and* u_1, *there exists* $T > 0$ *such that* (1.1) *has a unique time local solution* u *in* $C([0,T), H^1(\mathbb{R}^2)) \cap C^1([0,T), L^2(\mathbb{R}^2))$. *Here,* u *satisfies* $u \in X(T)$, *and for any fixed* p_0 *with* $j_0 \geq (p_0-1)/\nu$ *and* $1 \leq p_0 < 1+4/n$, *there exists a constant* C_0 *independent of* D *such that* T *can be estimated from below as*

$$T \geq \left(2C_0 \sum_{j \geq j_0} p(j)a(j)(2C_0 D)^{p(j)-1} \right)^{-1/\{1-n(p_0-1)/4\}},$$

where $p(j) := \nu j + 1$, $a(j) := \alpha^j \beta^{p(j)} (2p(j))^{p(j)/2}/j!$, *and* β *is any real number by which Lemma* 1.3 *holds*.

(2) *If* $4/n\nu \leq j_0$ *and* D *is sufficiently small, then* (1.1) *has a unique time global solution* u *in* $C([0,\infty), H^1(\mathbb{R}^2)) \cap C^1([0,\infty), L^2(\mathbb{R}^2))$. *And* u *satisfies* $u \in X(\infty)$.

Remark 1.5 The assumption $\nu \leq 2$ seems to be optimal in the energy class $H^1(\mathbb{R}^2)$ in view of the Trudinger inequality (see [17]). A typical example for (2) is the case $\nu = 2$, $f(u) = \lambda u(e^{\alpha|u|^2} - 1 - \alpha|u|^2)$ when $n = 1$, $f(u) = \lambda u(e^{\alpha|u|^2} - 1)$ when $n = 2$, where the lower order of $f(u)$ are 5 when $n = 1$ and 3 when $n = 2$ which are both critical in L^2 theory.

We have considered the existence of solutions so far. Our solutions have the continuous dependence on the initial data, and they have asymptotic profiles to free solutions as follows.

Theorem 1.6 *Let* u *be the solution obtained in the above theorems for initial data* u_0 *and* u_1, *and let* $0 < T \leq \infty$ *be the existence time of* u *there.*

(1) *Let* $v_0 \in H^1(\mathbb{R}^n)$ *and* $v_1 \in L^2(\mathbb{R}^n)$, *and let* v *be the solution obtained in the above theorems for initial data* v_0 *and* v_1. *If* v_0 *converges to* u_0 *in* $H^1(\mathbb{R}^n)$, *and* v_1 *converges to* u_1 *in* $L^2(\mathbb{R}^n)$, *then* $\|u - v\|_{X(T)}$ *tends to zero.*

(2) *If* u *is the time global solution given by* (2) *of Theorems* 1.2 *and* 1.4, *then there exist* $v_0 \in L^2(\mathbb{R}^n)$ *and* $v_1 \in H^{-1}(\mathbb{R}^n)$ *such that*

$$\lim_{t\to\infty} \left\{ e^{-Ht} \left\| u(t) - v(t) \right\|_{L^2(\mathbb{R}^n)} + \left\| \partial_t u(t) - \partial_t v(t) \right\|_{H^{-1}(\mathbb{R}^n)} \right\} = 0,$$

where v *is the free solution of* $(\partial_t^2 - c^2 e^{-2Ht}\Delta + M^2)v = 0$, $v(0,\cdot) = v_0(\cdot)$, $\partial_t v(0,\cdot) = v_1(\cdot)$.

Finally, we consider global solutions for large data when the nonlinear term f in (1.1) has an energy conservative potential function.

Theorem 1.7 *Let* $\lambda \geq 0$. *Let* $u_0 \in H^1(\mathbb{R}^n)$ *and* $u_1 \in L^2(\mathbb{R}^n)$. *Let* $f(u)$ *be given by the following* (1) *or* (2).

(1) *We put* $f(u) = \lambda|u|^{p-1}u$, *where* p *satisfies*

$$1 \leq p \begin{cases} < \infty & \text{if } n = 1, 2 \\ \leq 1 + \dfrac{2}{n-2} & \text{if } n \geq 3. \end{cases}$$

(2) *Let* $n = 2$, $0 < \alpha < \infty$, $0 < \nu \leq 2$, $0 \leq j_0 < \infty$. *Let* $f(u) = \lambda u(e^{\alpha|u|^\nu} - \sum_{0\leq j<j_0} \frac{\alpha^j}{j!}|u|^{\nu j})$ *for* $j_0 \geq 1$, *and* $f(u) = \lambda u e^{\alpha|u|^\nu}$ *for* $j_0 = 0$. *When* $\nu = 2$, *we assume*

$$\frac{1}{2}\int_{\mathbb{R}^2} c^2\left|\nabla u_0(x)\right|^2 + M^2 u_0^2(x) + \left|u_1(x)\right|^2 + c^2\lambda \sum_{j\geq j_0}^{\infty} \frac{\alpha^j}{j!2(j+1)}|u_0|^{2(j+1)}dx \leq \frac{c^2\pi}{\alpha}. \tag{1.5}$$

Then (1.1) *has a unique global solution* u *in* $C([0,\infty), H^1(\mathbb{R}^n)) \cap C^1([0,\infty), L^2(\mathbb{R}^n))$. *And* u *satisfies* $u \in X(\infty)$.

We refer to [12] for the proofs of the above results.

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A Benefit from the L^∞ Smallness of Initial Data for the Semilinear Wave Equation with Structural Damping

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Abstract In this note, we prove the global existence of small data solutions for a semilinear wave equation with *structural* damping,

$$u_{tt} - \Delta u + \mu(-\Delta)^{\frac{1}{2}} u_t = |u|^p,$$

for any $n \geq 2$ and $p > 1 + 2/(n-1)$. The damping term allows us to derive linear $L^{q_1} - L^{q_2}$ estimates, for $1 \leq q_1 \leq q_2 \leq \infty$, without loss of regularity, in any space dimension. These estimates provide the basic tool to state our result, in which we assume initial data to be small in $(L^1 \cap H^1 \cap L^\infty) \times (L^1 \cap L^n)$.

Keywords Semilinear equations · Global existence · Structural damping · Critical exponent

Mathematics Subject Classification (2010) 35L71

We study the global existence of small data solutions to

$$\begin{cases} u_{tt} - \Delta u + \mu(-\Delta)^{\frac{1}{2}} u_t = f(u), \\ u(0,x) = u_0(x), \\ u_t(0,x) = u_1(x), \end{cases} \tag{1}$$

where $\mu > 0$, and $f(u) = |u|^p$, or more in general,

$$f(0) = 0, \qquad \big|f(u) - f(v)\big| \lesssim |u - v|\big(|u|^{p-1} + |v|^{p-1}\big), \tag{2}$$

for some $p > 1$. The linear part of the model in (1), i.e.

$$u_{tt} - \Delta u + \mu(-\Delta)^{\frac{1}{2}} u_t = 0, \tag{3}$$

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is frequently used in the determination of lifespan for primary or rechargeable batteries (see [8]). Equation (3) is a special case of a wave equation with *structural* damping:

$$u_{tt} - \Delta u + \mu(-\Delta)^\sigma u_t = 0, \quad \sigma > 0. \tag{4}$$

Different kind of estimates in Sobolev spaces for (4) with $\sigma \in (0, 1)$ have been recently studied in [1, 11, 17]. In the limit case $\sigma = 1$, the damping is also called *visco-elastic* and it has been studied in [13, 19] and, in abstract setting, in [10, 12]. Smoothing effects for (4) are studied in [9], including the case $\sigma \geq 1$. Some $L^2 - L^2$ estimates have been derived in the case with time-dependent damping $b(t)(-\Delta)^\sigma u_t$ in [14, 18].

The semilinear problem corresponding to (4) has been studied by the author and M. Reissig in [6], for $\sigma \in (0, 1]$. In particular, if $\sigma = 1/2$, by assuming small initial data in the energy space and in L^1, we proved global existence of energy solutions for any $p > 3$ in space dimension $n = 2$ and for any $p \in (2, 3]$ in space dimension $n = 3$. We also proved the optimality of the *critical exponent* $1 + 2/(n - 1)$. More precisely, for $\mu = 2$, $u_0 \equiv 0$ and $u_1 \geq 0$, nontrivial, global solutions to (1) cannot exist if $1 < p \leq 1 + 2/(n - 1)$ (see later, Theorem 2).

In this note, we show how the smallness of initial data in

$$\mathcal{A} := \left(L^1 \cap H^1 \cap L^\infty\right) \times \left(L^1 \cap L^n\right)$$

may be used to prove global existence of small data solutions for any $p > 1 + 2/(n - 1)$, in any space dimension $n \geq 2$. The case $n = 2$ has been completely solved in [6]. Indeed, in this case, the L^∞ smallness assumption of u_0 brings no benefit on the range of admissible exponents p.

Our approach takes advantage of the special structure of linear $L^{q_1} - L^{q_2}$ estimates, $1 \leq q_1 \leq q_2 \leq \infty$, which can be obtained for (3). In particular, no loss of regularity appears in high space dimension, a difficulty which occurs, for instance, for the classical damped wave equation [16].

For any $u \in \mathcal{C}([0, T], H^1) \cap \mathcal{C}^1([0, T], L^2)$, we define its *energy* as

$$\mathcal{E}[u](t) := \left\|\nabla u(t, \cdot)\right\|_{L^2} + \left\|u_t(t, \cdot)\right\|_{L^2}.$$

Theorem 1 *Let $n \geq 2$ and $p > 1 + 2/(n - 1)$. Then there exists $\varepsilon > 0$ such that for any initial data $(u_0, u_1) \in \mathcal{A}$ with*

$$\left\|(u_0, u_1)\right\|_{\mathcal{A}} \leq \varepsilon,$$

there exists a $\mathcal{C}([0, \infty), L^1 \cap H^1 \cap L^\infty) \cap \mathcal{C}^1([0, \infty), L^2)$ solution to (1). The solution satisfies the estimates

$$\left\|u(t, \cdot)\right\|_{L^q} \leq C(1 + t)^{1-n(1-1/q)} \left\|(u_0, u_1)\right\|_{\mathcal{A}}, \quad \text{for any } q \in [1, \infty], \tag{5}$$

$$\mathcal{E}[u](t) \leq C(1 + t)^{-\frac{n}{2}} \left\|(u_0, u_1)\right\|_{\mathcal{A}}. \tag{6}$$

Estimate (5) follows by interpolation, once we prove that

$$\|u(t,\cdot)\|_{L^1} \le C(1+t)\|(u_0,u_1)\|_{\mathcal{A}},$$
$$\|u(t,\cdot)\|_{L^\infty} \le C(1+t)^{-(n-1)}\|(u_0,u_1)\|_{\mathcal{A}}.$$

From the linear estimates in [17] for the solution to

$$\begin{cases} v_{tt} - \Delta v + \mu(-\Delta)^{\frac{1}{2}} v_t = 0, \\ v(0,x) = v_0(x), \\ v_t(0,x) = v_1(x), \end{cases} \tag{7}$$

we may derive

$$\|v(t,\cdot)\|_{L^q} \le t^{-n(\frac{1}{q_0}-\frac{1}{q})}\|v_0\|_{L^{q_0}} + t^{1-n(\frac{1}{q_1}-\frac{1}{q})}\|v_1\|_{L^{q_1}}, \tag{8}$$

for any $q_0, q_1 \ge 1$ and $q \ge \max\{q_0, q_1\}$. In particular, if $q = 1$, then (8) gives

$$\|v(t,\cdot)\|_{L^1} \le \|v_0\|_{L^1} + t\|v_1\|_{L^1}. \tag{9}$$

Let $q = \infty$. Taking $q_0 = \infty$, $q_1 = n$ for any $t \ge 0$, from (8) it follows that

$$\|v(t,\cdot)\|_{L^\infty} \le \|v_0\|_{L^\infty} + \|v_1\|_{L^n}. \tag{10}$$

Combining (10) with (8) for $q_0 = q_1 = 1$ and $t \ge 1$, we get

$$\|v(t,\cdot)\|_{L^\infty} \le (1+t)^{-(n-1)}\big(\|v_0\|_{L^1\cap L^\infty} + \|v_1\|_{L^1\cap L^n}\big). \tag{11}$$

The solution to (7) also satisfies the energy estimates (see [6]):

$$\mathcal{E}[v](t) \le (1+t)^{-\frac{n}{2}}\big(\|v_0\|_{L^1\cap H^1} + \|v_1\|_{L^1\cap L^2}\big), \tag{12}$$
$$\mathcal{E}[v](t) \le \|v_0\|_{H^1} + \|v_1\|_{L^2}. \tag{13}$$

We are now ready to prove our statement.

Proof of Theorem 1 For any $T > 0$, we introduce the space

$$X(T) := \mathcal{C}\big([0,T], L^1 \cap H^1 \cap L^\infty\big) \cap \mathcal{C}^1\big([0,T], L^2\big), \tag{14}$$

with norm given by

$$\|w\|_{X(T)} := \max_{t\in[0,T]}\big\{(1+t)^{-1}\|w(t,\cdot)\|_{L^1} + (1+t)^{n-1}\|w(t,\cdot)\|_{L^\infty} + (1+t)^{\frac{n}{2}}\mathcal{E}[w](t)\big\}. \tag{15}$$

If $w \in X(T)$ then we derive that

$$\mathcal{E}[w](t) \le (1+t)^{-\frac{n}{2}}\|w\|_{X(T)}, \tag{16}$$

and, by interpolation, that

$$\left\|w(t,\cdot)\right\|_{L^q} \leq (1+t)^{1-n(1-\frac{1}{q})}\|w\|_{X(T)}, \quad \text{for any } q \in [1,\infty]. \tag{17}$$

We consider the operator N defined by

$$Nw := u^{\text{lin}}(t,x) + Gw(t,x), \quad \text{with } Gw := \int_0^t E(t-\tau,x) *_{(x)} f(w)d\tau, \tag{18}$$

where $u^{\text{lin}}(t,x)$ is the solution to (7) with $(v_0,v_1) = (u_0,u_1)$, and $E(t,x)$ is the fundamental solution to (7) for $v_0 \equiv 0$ and $v_1 = \delta$. A function $w \in X(T)$ is a solution to (1) for any $t \in [0,T]$ if, and only if, $w = Nw$ in $X(T)$. If we prove that

$$\|Nw\|_{X(T)} \leq C_1\left\|(u_0,u_1)\right\|_{\mathcal{A}} + C_2\|w\|_{X(T)}^p, \tag{19}$$

$$\|Nw - N\tilde{w}\|_{X(T)} \leq C\|w-\tilde{w}\|_{X(T)}\big(\|w\|_{X(T)}^{p-1} + \|\tilde{w}\|_{X(T)}^{p-1}\big), \tag{20}$$

where C_1, C_2 and C do not depend on T, by standard arguments (see, for instance, [5]), we may derive the existence of a unique fixed point of N in $X(T)$, and then the existence of small data global solutions to (1), satisfying

$$\|u\|_{X(T)} \leq \tilde{C}\left\|(u_0,u_1)\right\|_{\mathcal{A}}, \quad \text{uniformly for any } T > 0. \tag{21}$$

By (21), estimates (5)–(6) for the solution follow.

Recalling the definition of $\mathcal{A}$ and setting $u_0 = v_0$, $u_1 = v_1$, in (9), (11) and (12), we immediately derive that

$$\left\|u^{\text{lin}}\right\|_{X(T)} \leq C_1\left\|(u_0,u_1)\right\|_{\mathcal{A}},$$

and (19) follows once we prove that

$$\|Gw\|_{X(T)} \leq C\|w\|_{X(T)}^p. \tag{22}$$

On the other hand, it is clear that we may rewrite (20) as

$$\|Gw - G\tilde{w}\|_{X(T)} \leq C\|w-\tilde{w}\|_{X(T)}\big(\|w\|_{X(T)}^{p-1} + \|\tilde{w}\|_{X(T)}^{p-1}\big). \tag{23}$$

We first prove (22). Recalling the definition of the operator G in (18), thanks to (9) and (17), we obtain

$$\begin{aligned}
\left\|Gw(t,\cdot)\right\|_{L^1} &\leq C\int_0^t (t-\tau)\left\|f(w)(\tau,\cdot)\right\|_{L^1} d\tau \\
&\leq C_1\int_0^t (t-\tau)\left\|w(\tau,\cdot)\right\|_{L^p}^p d\tau \\
&\leq C_1\|w\|_{X(T)}^p \int_0^t (t-\tau)(1+\tau)^{-(n-1)(p-1)+1} d\tau.
\end{aligned}$$

We notice that $(n-1)(p-1) > 2$ if, and only if, $p > 1 + 2/(n-1)$. Therefore we derive (see, for instance, [15]):

$$\left\|Gw(t,\cdot)\right\|_{L^1} \leq C(1+t)\|w\|_{X(T)}^p.$$

In order to deal with the L^∞ norm of $Gw(t,\cdot)$ we need to modify our approach. We split the interval $[0,t]$ into $[0,t/2]$ and $[t/2,t]$. In the first interval, we use (11), whereas in the second one we use (10):

$$\begin{aligned}\left\|Gw(t,\cdot)\right\|_{L^\infty} \leq C &\int_0^{t/2} (1+t-\tau)^{-(n-1)} \left\|f(w)(\tau,\cdot)\right\|_{L^1\cap L^n} d\tau \\ &+ C\int_{t/2}^t \left\|f(w)(\tau,\cdot)\right\|_{L^n} d\tau.\end{aligned}$$

Thanks to (17), and using

$$1+t-\tau \geq (1+t)/2 \quad \text{if } \tau \in [0,t/2], \qquad 1+\tau \geq (1+t)/2 \quad \text{if } \tau \in [t/2,t],$$

it follows that

$$\begin{aligned}\left\|Gw(t,\cdot)\right\|_{L^\infty} &\leq C\|w\|_{X(T)}^p (1+t)^{-(n-1)} \int_0^{t/2} (1+\tau)^{-(n-1)(p-1)+1} d\tau \\ &\quad + C\|w\|_{X(T)}^p (1+t)^{-(n-1)-(n-1)(p-1)+1} \int_{t/2}^t 1 d\tau \\ &\leq C_1 (1+t)^{-(n-1)} \|w\|_{X(T)}^p,\end{aligned}$$

where we used $(n-1)(p-1) > 2$ to prove the last inequality. Thanks to (12) and (13), we proceed similarly to estimate

$$\begin{aligned}\mathcal{E}[w](t) &\leq C\int_0^{t/2} (1+t-\tau)^{-\frac{n}{2}} \left\|f(w)(\tau,\cdot)\right\|_{L^1\cap L^2} d\tau \\ &\quad + C\int_{t/2}^t \left\|f(w)(\tau,\cdot)\right\|_{L^2} d\tau \\ &\leq C_1\|w\|_{X(T)}^p (1+t)^{-\frac{n}{2}} \int_0^{t/2} (1+\tau)^{-(n-1)(p-1)+1} d\tau \\ &\quad + C\|w\|_{X(T)}^p (1+t)^{-\frac{n}{2}-(n-1)(p-1)+1} \int_{t/2}^t 1 d\tau \\ &\leq C_2 (1+t)^{-\frac{n}{2}} \|w\|_{X(T)}^p.\end{aligned}$$

Therefore we proved (22). To prove (23), it is sufficient to use (2) and Hölder inequality, to estimate

$$\left\|f(w)-f(\tilde{w})\right\|_{L^q} \leq \|w-\tilde{w}\|_{L^{qp}} \big(\|w\|_{L^{qp}}^{p-1} + \|\tilde{w}\|_{L^{qp}}^{p-1}\big),$$

where needed, then we proceed as in the proof of (22). This concludes the proof of the theorem. □

To motivate the sharpness of the *critical exponent* in Theorem 1, we present the following.

Theorem 2 *We consider the Cauchy problem*

$$\begin{cases} u_{tt} - \Delta u + 2(-\Delta)^{\frac{1}{2}} u_t = u^p, & t \geq 0, \ x \in \mathbb{R}^n, \\ u(0,x) = 0, \\ u_t(0,x) = u_1(x). \end{cases} \tag{24}$$

Let us assume that the data $u_1 \in L^1_{\text{loc}}$ is non-negative. If $1 < p \leq 1 + 2/(n-1)$, then there exists no global nontrivial L^p_{loc} solution to (24).

Theorem 2 has been first proved in Theorem 9 in [6], with stronger assumptions on the data and solution spaces. Our proof is also simpler.

An essential role in the proof of Theorem 2 is played by the fact that any local or global solution to (24) with non-negative initial data u_1, is non-negative. This property is used to extend the test function method (see, for instance, [4, 7]) to the operator in (24), which contains the nonlocal term $(-\Delta)^{\frac{1}{2}} u_t$. The requirement of non-negativity of the solution do not appear in the case of a classical damping [20].

Proof of Theorem 2 We assume by contradiction that $u \in L^p_{\text{loc}}$ is a global solution to (24). Therefore, for any test function $\psi \in \mathcal{C}_c^\infty([0,\infty) \times \mathbb{R}^n)$ it holds

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}^n} u\big(\psi_{tt} - \Delta\psi - 2(-\Delta)^{\frac{1}{2}}\psi_t\big) dx dt \\ &\quad = \int_0^\infty \int_{\mathbb{R}^n} u^p \psi \, dx dt + \int_{\mathbb{R}^n} u_1(x)\psi(0,x) dx. \end{aligned} \tag{25}$$

Let $\phi \in \mathcal{C}_c^\infty([0,\infty))$ be a nontrivial, nonincreasing function, compactly supported in $[0,1]$, and let $\ell > p'$. First, we assume that $1 < p < 1 + 2/(n-1)$. For any $R > 1$, we set $\psi(t,x) = \phi(t/R)^\ell \phi(|x|/R)^\ell$ in (25), for some $\ell > p' = p/(p-1)$. Recalling that ϕ, $-\phi'$, and u_1 are nonnegative and that (see [2])

$$(-\Delta)^\theta \phi\big(|x|/R\big)^\ell \leq \ell \phi\big(|x|/R\big)^{\ell-1} (-\Delta)^\theta \phi\big(|x|/R\big)$$

for any $\theta \in (0,1]$ and $\ell > 1$, we may derive

$$I_R := \int_0^\infty \int_{\mathbb{R}^n} u^p \psi \, dx dt \leq R^{-2} \ell \int_0^\infty \int_{\mathbb{R}^n} u \psi^{\frac{\ell-1}{\ell}} h\big(t/R, |x|/R\big) dx dt,$$

$$h\big(t,|x|\big) = \big(\phi''(t) + (\ell-1)\big(\phi'\big)^2(t)/\phi(t)\big)\phi\big(|x|\big)$$

$$- \phi(t)\Delta\phi\big(|x|\big) - 2\ell\phi'(t)(-\Delta)^{\frac{1}{2}}\phi\big(|x|\big)$$

We notice that $\psi^\epsilon(t,x)h(t/R,|x|/R)$ is a bounded function with compact support in $[0,R]\times B_R$, for any $\epsilon>0$, since h is bounded.

Setting $\epsilon=(\ell-1)/\ell-\ell/p$, by Hölder's inequality, we obtain:

$$I_R \lesssim R^{-2} I_R^{\frac{1}{p}} \left(\int_0^R \int_{B_R} \big(\psi^\epsilon \big|h\big(t/R,|x|/R\big)\big|\big)^{p'} dxdt\right)^{\frac{1}{p'}} \lesssim R^{-2+\frac{n+1}{p'}} I_R^{\frac{1}{p}}.$$

This gives $I_R \lesssim R^{-2p'+n+1}$, which vanishes as $R\to\infty$. By Beppo–Levi convergence theorem, it follows that $u\equiv 0$.

Now let $n\geq 2$ and $p=1+2/(n-1)$, i.e. $2p'=n+1$. The previous approach only gives a uniform bound for I_R, that is, $u^p\psi\in L^1$. We now fix ϕ such that it also satisfies $\phi(\rho)=1$ for any $\rho\in[0,1/2]$, and we define $\psi(t,x)=\phi(t)^\ell\phi(|x|/(\delta R))^\ell$, for some $\delta>0$. Following the reasoning above, we derive, in particular,

$$\begin{aligned}
-\int_0^\infty \int_{\mathbb{R}^n} u\Delta\psi\,dxdt &= -\int_0^\infty \int_{|x|>\delta R/2} u\Delta\psi\,dxdt \\
&\lesssim J_R^{\frac{1}{p}} \equiv \left(\int_0^\infty \int_{|x|>\delta R/2} u^p\psi\,dxdt\right)^{\frac{1}{p}} \to 0, \quad \text{as } R\to\infty,
\end{aligned}$$

thanks to $u^p\psi\in L^1$, for any fixed $\delta>0$.

For the other two terms, we proceed as before, and taking into account of the presence of $\delta>0$ in the definition of ψ, we obtain

$$I_R \lesssim J_R^{\frac{1}{p}} + \big(\delta^{\frac{n}{p'}} + \delta^{\frac{n}{p'}-1}\big) I_R^{\frac{1}{p}}.$$

Being δ arbitrarily small and $n=2p'-1>p'$, we get again $u\equiv 0$. □

Some ideas contained in this note have been employed in a forthcoming paper of the author to study the influence of a nonlinear memory on (3), extending the results in [3] for the classical damped wave equation.

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A Regularity Criterion for the Schrödinger Map

Jishan Fan and Tohru Ozawa

Abstract We prove a regularity criterion

$$\nabla u \in L^2\big(0, T; BMO(\mathbb{R}^n)\big)$$

with $2 \le n \le 5$ for the Schrödinger map. Here BMO is the space of functions with bounded mean oscillations.

Keywords Landau–Lifshitz · Schrödinger map · Regularity criterion

Mathematics Subject Classification (2010) Primary 35Q05 · Secondary 35Q35

1 Introduction

In this paper, we consider the regularity criterion of the Schrödinger map:

$$u_t = u \times \Delta u, \tag{1.1}$$

$$u(\cdot, 0) = u_0(\cdot) \quad \text{in } \mathbb{R}^n, \tag{1.2}$$

where $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{S}^2$ is a three-dimensional vector field, representing the magnetization and $\times$ denotes the cross product in $\mathbb{R}^3$.

By the standard stereographic projection $\mathbb{S}^2 \to \mathbb{C} \cup \{\infty\}$, (1.1) can be rewritten as the derivative Schrödinger equation

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$$iw_t + \Delta w + 4\frac{(\nabla w)^2}{1+|w|^2}\overline{w} = 0,$$

where $w : \mathbb{R}^n \times \mathbb{R} \to \mathbb{C}$.

The problem (1.1) and (1.2) has been studied by many authors [1–8, 10–17, 19–25, 27–33]. In [32], Sulem, Sulem, and Bardos proved the well-posedness of local smooth solutions for any $n \geq 2$. Guo and Han [10] show a regularity criterion

$$\nabla u \in L^2\big(0,T; L^\infty\big(\mathbb{R}^n\big)\big) \tag{1.3}$$

with $n = 2, 3$ and $0 < T < \infty$.

The aim of this paper is to refine (1.3) further. We will prove

Theorem 1.1 *Let* $2 \leq n \leq 5$ *and let* $u_0 : \mathbb{R}^n \to \mathbb{S}^2$ *satisfy* $\nabla u_0 \in H^3(\mathbb{R}^n)$. *Let* u *be a local smooth solution of the problem* (1.1) *and* (1.2). *If* u *satisfies*

$$\nabla u \in L^2\big(0,T; BMO\big(\mathbb{R}^n\big)\big) \tag{1.4}$$

with $0 < T < \infty$, *then the solution* u *can be extended beyond* $T > 0$.

Remark 1.2 It is an open problem to prove (1.4) for $n \geq 6$. In (2.10) and (2.11) we need $n \leq 5$.

2 Proof of Theorem 1.1

We only need to prove a priori estimates.

First, testing (1.1) by $-\Delta u$ and using $(a \times b) \cdot b = 0$, we have the well-known energy equality

$$\frac{d}{dt}\int |\nabla u|^2 dx = 0. \tag{2.1}$$

Here it should be noted that (2.1) is an equality instead of an inequality because we deal with the local strong solutions.

To prove further estimates, we derive a new equation from (1.1). Applying ∂_t to (1.1) and using the formula $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$, and $|u| = 1$, we find that

$$\begin{aligned}
u_{tt} &= u_t \times \Delta u + u \times \Delta u_t \\
&= (u \times \Delta u) \times \Delta u + u \times \Delta(u \times \Delta u) \\
&= -\Delta u \times (u \times \Delta u) + u \times \big(u \times \Delta^2 u\big) + 2\sum_i u \times (\partial_i u \times \partial_i \Delta u) \\
&= -u|\Delta u|^2 + (u \cdot \Delta u)\Delta u + \big(u \cdot \Delta^2 u\big)u - \Delta^2 u \\
&\quad + 2\sum_i (u \cdot \partial_i \Delta u)\partial_i u.
\end{aligned} \tag{2.2}$$

Now using $|u| = 1$ and the formulae

$$u \cdot \Delta u = -|\nabla u|^2, \tag{2.3}$$

$$\begin{aligned} 0 &= \frac{1}{2}\partial_i \Delta |u|^2 = \partial_i \left(u \Delta u + |\nabla u|^2\right) \\ &= u \partial_i \Delta u + \partial_i u \Delta u + 2 \sum_j \partial_j u \partial_i \partial_j u, \end{aligned} \tag{2.4}$$

$$\begin{aligned} 0 &= \frac{1}{2}\Delta^2 |u|^2 = \sum_i \Delta(u_i \Delta u_i + \nabla u_i \nabla u_i) \\ &= \sum_i \Big[u_i \Delta^2 u_i + |\Delta u_i|^2 + 4 \nabla u_i \nabla \Delta u_i + 2 \sum_j (\partial_j \nabla u_i)^2\Big], \end{aligned}$$

we obtain

$$u \Delta^2 u = -|\Delta u|^2 - 4 \sum_{i,j} \partial_j u_i \partial_j \Delta u_i - 2 \sum_{i,j} (\partial_j \partial_i u)^2. \tag{2.5}$$

Putting (2.3), (2.4) and (2.5) into (2.2), we arrive at:

$$\begin{aligned} u_{tt} + \Delta^2 u &= -2u|\Delta u|^2 - 4u \sum_{i,j} \partial_j u_i \partial_j \Delta u_i - 2u \sum_{i,j} (\partial_j \partial_i u)^2 \\ &\quad - |\nabla u|^2 \Delta u - 2 \sum_i (\partial_i u \cdot \Delta u) \partial_i u - 4 \sum_{i,j} (\partial_j u \cdot \partial_i \partial_j u) \partial_i u. \end{aligned} \tag{2.6}$$

The new (2.6) is similar to the biharmonic wave map introduced by the authors [9].

In the following calculations, we will use the following bilinear product estimates due to Kato and Ponce [18]:

$$\left\| \Lambda^s (fg) \right\|_{L^p} \le C\left(\left\| \Lambda^s f \right\|_{L^{p_1}} \|g\|_{L^{q_1}} + \|f\|_{L^{p_2}} \left\| \Lambda^s g \right\|_{L^{q_2}} \right) \tag{2.7}$$

with $s > 0$, $\Lambda := (-\Delta)^{1/2}$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$.

We will also use the following formula

$$0 = \Delta^2 (u u_t) = u \Delta^2 u_t + u_t \Delta^2 u + C_1 D u_t D^3 u + C_2 D^2 u_t D^2 u + C_3 D^3 u_t D u, \tag{2.8}$$

where C_1, C_2, C_3 are suitable constant tensors.

Testing (2.6) by $\Delta^2 u_t$ and using (2.8), we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt} \int \left(|\Delta u_t|^2 + |\Delta^2 u|^2\right) dx \\ &= -\int \Big[2u|\Delta u|^2 + 2u \sum_{i,j} (\partial_i \partial_j u)^2 \end{aligned}$$

$$+ |\nabla u|^2 \Delta u + 2\sum_i (\partial_i u \cdot \Delta u)\partial_i u + 4\sum_{i,j} (\partial_j u \cdot \partial_i \partial_j u)\partial_i u \Big] \Delta^2 u_t dx$$

$$+ 4\sum_{i,j} \int \partial_j u_i \partial_j \Delta u_i \big(u_t \Delta^2 u + C_1 D u_t D^3 u + C_2 D^2 u_t D^2 u + C_3 D^3 u_t D u\big) dx$$

$$= \int \Big(2|\Delta u|^2 + 2\sum_{i,j} |\partial_i \partial_j u|^2\Big)\big(u_t \Delta^2 u + C_2 D^2 u_t D^2 u\big) dx$$

$$- \int 2C_1 u_t D\Big[D^3 u\Big(|\Delta u|^2 + \sum_{i,j} |\partial_i \partial_j u|^2\Big)\Big] dx$$

$$- \int 2C_3 D^2 u_t D\Big[Du\Big(|\Delta u|^2 + \sum_{i,j} |\partial_i \partial_j u|^2\Big)\Big] dx$$

$$- \int \Delta\Big[|\nabla u|^2 \Delta u + 2\sum_i (\partial_i u \cdot \Delta u)\partial_i u + 4\sum_{i,j} (\partial_j u \cdot \partial_i \partial_j u)\partial_i u\Big] \Delta u_t dx$$

$$+ 4\sum_{i,j} \int \partial_j u_i \partial_j \Delta u_i \big(u_t \Delta^2 u + C_2 D^2 u_t D^2 u\big) dx$$

$$- 4C_1 \sum_{i,j} \int u_t D\big(\partial_j u_i \partial_j \Delta u_i D^3 u\big) dx$$

$$- 4C_3 \sum_{i,j} \int D^2 u_t \cdot D(\partial_j u_i \partial_j \Delta u_i D u) dx$$

$$=: I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7. \tag{2.9}$$

We will use (1.1) to bounded u_t by Δu, we bound I_i $(i = 1, \dots, 7)$ as follows. We estimate I_1 as

$$\begin{aligned} |I_1| &\le C\|\Delta u\|_{L^6}^3 \big(\|\Delta^2 u\|_{L^2} + \|\Delta u_t\|_{L^2}\big) \\ &\le C\|\nabla u\|_{L^\infty}^2 \big(\|\Delta^2 u\|_{L^2}^2 + \|\Delta u_t\|_{L^2}^2\big), \end{aligned}$$

where we have used the Gagliardo–Nirenberg inequality

$$\|\Delta u\|_{L^6} \le C\|\nabla u\|_{L^\infty}^{\frac{2}{3}} \|\Delta^2 u\|_{L^2}^{\frac{1}{3}}. \tag{2.10}$$

We estimate I_2 as

$$\begin{aligned} |I_2| &\le C\|\Delta u\|_{L^6}^3 \|\Delta^2 u\|_{L^2} + C\|\Delta u\|_{L^6}^2 \|D^3 u\|_{L^3}^2 \\ &\le C\|\Delta u\|_{L^6}^3 \|\Delta^2 u\|_{L^2} \\ &\le C\|\nabla u\|_{L^\infty}^2 \|\Delta^2 u\|_{L^2}^2, \end{aligned}$$

where we have used (2.10) and the Gagliardo–Nirenberg inequality

$$\|D^3u\|_{L^3}^2 \le C\|\Delta u\|_{L^6}\|\Delta^2 u\|_{L^2} \le C\|\nabla u\|_{L^\infty}^{\frac{2}{3}}\|\Delta^2 u\|_{L^2}^{\frac{4}{3}}. \tag{2.11}$$

We use (2.10) and (2.11) to bound

$$\begin{aligned}
|I_3| &\le C\|\Delta u_t\|_{L^2}\big(\|\Delta u\|_{L^6}^3 + C\|\nabla u\|_{L^\infty}\|\Delta u\|_{L^6}\|D^3u\|_{L^3}\big)\\
&\le C\|\Delta u_t\|_{L^2}\|\nabla u\|_{L^\infty}^2\|\Delta^2 u\|_{L^2}\\
&\le C\|\nabla u\|_{L^\infty}^2\big(\|\Delta^2 u\|_{L^2}^2 + \|\Delta u_t\|_{L^2}^2\big),\\
|I_5| &\le C\|\nabla u\|_{L^\infty}\|\Delta u\|_{L^6}\|D^3u\|_{L^3}\big(\|\Delta^2 u\|_{L^2} + \|\Delta u_t\|_{L^2}\big)\\
&\le C\|\nabla u\|_{L^\infty}^2\big(\|\Delta^2 u\|_{L^2}^2 + \|\Delta u_t\|_{L^2}^2\big),\\
|I_6| &\le C\|\Delta u\|_{L^6}^2\|D^3u\|_{L^3}^2 + C\|\nabla u\|_{L^\infty}\|\Delta u\|_{L^6}\|D^3u\|_{L^3}\|\Delta^2 u\|_{L^2}\\
&\le C\|\nabla u\|_{L^\infty}^2\|\Delta^2 u\|_{L^2}^2,\\
|I_7| &\le C\|\Delta u_t\|_{L^2}\big(\|\nabla u\|_{L^\infty}^2\|\Delta^2 u\|_{L^2} + \|\nabla u\|_{L^\infty}\|\Delta u\|_{L^6}\|D^3u\|_{L^3}\big)\\
&\le C\|\nabla u\|_{L^\infty}^2\big(\|\Delta^2 u\|_{L^2}^2 + \|\Delta u_t\|_{L^2}^2\big).
\end{aligned}$$

Finally, we use (2.7), (2.10) and (2.11) to bound I_4 as

$$\begin{aligned}
|I_4| &\le C\big(\|\nabla u\|_{L^\infty}^2\|\Delta^2 u\|_{L^2} + \|\Delta u\|_{L^6}\|\nabla u\|_{L^\infty}\|D^3u\|_{L^3}\big)\|\Delta u_t\|_{L^2}\\
&\le C\|\nabla u\|_{L^\infty}^2\|\Delta^2 u\|_{L^2}\|\Delta u_t\|_{L^2}\\
&\le C\|\nabla u\|_{L^\infty}^2\big(\|\Delta^2 u\|_{L^2}^2 + \|\Delta u_t\|_{L^2}^2\big).
\end{aligned}$$

Inserting the above estimates into (2.9) and using (2.1) and the logarithmic Sobolev inequality [26]:

$$\|\nabla u\|_{L^\infty}^2 \le C\big(1 + \|\nabla u\|_{BMO}^2 \log\big(e + \|\nabla u\|_{H^3}\big)\big),$$

we arrive at

$$\|\nabla u\|_{L^\infty(0,T;H^3)} + \|u_t\|_{L^\infty(0,T;H^2)} \le C.$$

This completes the proof.

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Microlocal Analysis for Hyperbolic Equations in Einstein-de Sitter Spacetime

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Abstract We consider the waves propagating in the Einstein-de Sitter spacetime, which obey the covariant d'Alembert's equation. We construct the parametrixes in the terms of Fourier integral operators and discuss the propagation and reflection of the singularities phenomena.

Keywords Einstein-de Sitter model · Parametrix · Propagation of singularities

Mathematics Subject Classification (2010) Primary 35Q75 · 35A21 · 58J47 · Secondary 83C05 · 35S30

1 Introduction

The current note is concerned with the wave propagating in the universe modeled by the cosmological models with expansion.

The homogeneous and isotropic cosmological models possess highest symmetry, which makes them more amenable to rigorous study. Among them, Friedmann–Lemaître–Robertson–Walker (FLRW) models are mentioned. The metric of the EdeS universe is a member of the FLRW metrics $ds^2 = -dt^2 + a^2(t)[\frac{dr^2}{1-Kr^2} + r^2 d\Omega^2]$, where $K = -1, 0,$ or $+1$, for a hyperbolic, flat or spherical spatial geometry, respectively. The time dependence of the function $a(t)$ is determined by the Einstein field equations for gravity $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$. For pressureless matter distributions and vanishing spatial curvature in the EdeS universe the solution to the Einstein field equations is $a(t) = a_0 t^{2/3}$ [4]. The universe expands, and its expansion decelerates since $\ddot{a} < 0$. Even though the EdeS spacetime is conformally flat, its causal structure is quite different from asymptotically flat geometries. In particular, and unlike Minkowski or Schwarzschild, the past particle horizons exist. The EdeS spacetime is a good approximation to the large scale structure of the universe during a matter dominated phase, when the averaged (over space and time)

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energy density evolves adiabatically and pressures are vanishingly small, as, e.g., immediately after inflation [4]. This justifies why such a metric is adopted to model the collapse of overdensity perturbations in the early matter dominated phase that followed inflation.

The solution $a(t) = a_0 t^{2/3}$ is singular at $t = 0$: at that moment the scale factor is equal to zero, and the energy density is infinite. This is an example of the cosmological singularity, the moment of Big Bang. The EdeS model of the universe is the simplest non-empty expanding model. It was first proposed jointly by Einstein and de Sitter (the EdeS model) [1]. In the EdeS spacetime the covariant wave equation with the source term f is

$$\psi_{tt} - t^{-4/3}\Delta\psi + 2t^{-1}\psi_t = f, \quad x \in \mathbb{R}^3. \tag{1.1}$$

In this note we give the parametrix for the initial value problem for this equation with $x \in \mathbb{R}^n$ and the wave front sets $WF(\psi)$ of the solution.

Equation (1.1) is strictly hyperbolic in the domain with $t > 0$. It is known that the wave front set $WF(\psi) \setminus WF(f)$ is contained in the characteristic set $p^{-1}(0)$, where $p = p(t, x; \tau, \xi)$ is the principal symbol, $p(t, x; \tau, \xi) = \tau^2 - t^{-4/3}|\xi|^2$, $(t, x) \in (0, \infty) \times \mathbb{R}^n$, $(\tau, \xi) \in \mathbb{R}^{1+n}$. The set $WF(\psi) \setminus WF(f)$ is invariant under the flow defined by the Hamiltonian field H_p of the principal symbol p. On the hypersurface $t = 0$ its coefficients have singularities that make the study of the initial value problem and the reflection of the singularities difficult.

In [2] have been used the approach suggested in [6], which reduces the problem for (1.1) to the Cauchy problem for the free wave equation in Minkowski spacetime. More precisely, in [2] have been utilized the solution $v = v(x, t; b)$ to the Cauchy problem

$$v_{tt} - \Delta v = 0, \qquad v(x, 0) = \varphi(x, b), \qquad v_t(x, 0) = 0, \quad t > 0,\ x \in \mathbb{R}^n, \tag{1.2}$$

with the parameter $b \in \mathbb{R}$. Here Δ is the Laplace operator on the flat metric, $\Delta := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$. We denote that solution by $v_\varphi = v_\varphi(x, t; b)$. In the case of function φ independent of parameter, we simply write $v_\varphi = v_\varphi(x, t)$.

The straightforward application of the formulas obtained in [6] to the Cauchy problem for (1.1) decidedly does not work, but it reveals a surprising link to the EdeS spacetime. Indeed, we note that the "principal part" of (1.1) belongs to the family of the Tricomi-type equations $u_{tt} - t^l \Delta u = 0$, where $l \in \mathbb{N}$. According to [6] the solution to the Cauchy problem $u_{tt} - t^l \Delta u = f(x, t)$, $u(x, 0) = 0$, $u_t(x, 0) = 0$, with the smooth functions f, φ_0, and φ_1, can be represented as follows:

$$u(x, t) = 2c_k \int_0^t db \int_0^{\phi(t)-\phi(b)} dr\, E(r, t; 0, b) v_f(x, r; b), \quad x \in \mathbb{R}^n,\ t > 0,$$

with the kernel $E(r, t; 0, b)$ containing $F(\gamma, \gamma; 1; \frac{(\phi(t)-\phi(b))^2 - r^2}{(\phi(t)+\phi(b))^2 - r^2})$, where $\gamma := \frac{\ell}{2(\ell+2)}$, $\phi(t) := \frac{t^{1+\ell/2}}{1+\ell/2}$, and $F(\gamma, \gamma; 1; \zeta)$ is the hypergeometric function. Suppose now that we are looking for the simplest possible kernel $E(r, t; 0, b)$ of the last integral. In the hierarchy of the hypergeometric functions the simplest one, that is different from the constant, is $F(-1,-1;1;\zeta)=1+\zeta$. The parameter l leading to $F(-1,-1;1;\zeta)$ is exactly the exponent $l=-4/3$ of the wave equation (and of the metric tensor) in the EdeS spacetime.

Theorem 1.1 [2] shows how the "lower order term" of (1.1) affects the initial value problem. More precisely, the initial conditions can be prescribed as follows:

$$\begin{cases}\psi_{tt}-t^{-4/3}\Delta\psi+2t^{-1}\psi_t=f(x,t), \quad t>0,\ x\in\mathbb{R}^n,\\ \lim\limits_{t\to 0} t\psi(x,t)=\varphi_0(x), \quad x\in\mathbb{R}^n,\\ \lim\limits_{t\to 0}\big(t\psi_t(x,t)+\psi(x,t)+3t^{-1/3}\Delta\varphi_0(x)\big)=\varphi_1(x), \quad x\in\mathbb{R}^n,\end{cases} \tag{1.3}$$

For the solutions of the problem(1.3) Theorem 1.1 [2] gives representation

$$\begin{aligned}\psi(x,t)=&\frac{3}{2}t^2\int_0^1 db\int_0^{1-b^{1/3}} ds b v_f\big(x,3t^{1/3}s;tb\big)\big(1+b^{2/3}-s^2\big)\\&+t^{-1}v_{\varphi_0}\big(x,3t^{1/3}\big)-3t^{-2/3}(\partial_t v_{\varphi_0})\big(x,3t^{1/3}\big)\\&+\frac{3}{2}\int_0^1 v_{\varphi_1}\big(x,3t^{1/3}s\big)\big(1-s^2\big)ds.\end{aligned} \tag{1.4}$$

The initial conditions of (1.3) are the so-called *weighted initial conditions*. Theorem 1.1 [2] has been used to obtain in [2] some important properties of the solutions of the wave equation in E&deS spacetime. In particular, for the initial value problem (1.3) the so-called L^p-L^q estimates are derived in [2].

In this present paper we supplement the results of [2] with the microlocal analysis, which, in particular, describes the wave front sets of solutions. Denote by $\widehat{\varphi_0}(\xi)$ the Fourier transform of $\varphi_0(x)$ and by $\widehat{f}(t,\xi)$ the partial Fourier transform of $f(t,x)$. The main result of the present paper is the following theorem. Denote $\phi(t,\xi):=3t^{\frac{1}{3}}|\xi|$ and set $n\geq 3$.

Theorem 1.1 *The solution of the problem* (1.3) *is given by the Fourier integral operators as follows*:

$$\begin{aligned}\psi(t,x)=&\sum_{+,-}\frac{\pm 1}{(2\pi)^n}\int_{\mathbb{R}^n}e^{i(x\cdot\xi\mp\phi(t,\xi))}\frac{1}{2t}\big(i\phi(t,\xi)\pm 1\big)\widehat{\varphi_0}(\xi)d\xi\\&+\sum_{+,-}\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}e^{i(x\cdot\xi\mp\phi(t,\xi))}\frac{i}{18t|\xi|^3}\big(i\phi(t,\xi)\pm 1\big)\widehat{\varphi_1}(\xi)d\xi\\&+\sum_{+,-}\frac{\pm 1}{(2\pi)^n}\int_0^t\int_{\mathbb{R}^n}e^{ix\cdot\xi}e^{\mp i(\phi(\tau,\xi)-\phi(t,\xi))}\frac{i\tau}{18t|\xi|^3}\\&\times\big(\mp i\big[\phi(\tau,\xi)-\phi(t,\xi)\big]-\phi(\tau,\xi)\phi(t,\xi)-1\big)\widehat{f}(\tau,\xi)d\xi d\tau.\end{aligned} \tag{1.5}$$

Thus, the wave front set of the solution, $WF(\psi)$, of the waves without source, $f=0$, in the E & dS spacetime is the union of the bicharacteristics, that is curves in the Hamilton foliation of zero set of the principal symbol, $p^{-1}(0)$, emanating from the wave front sets of the initial data: $WF(\psi)=\Gamma^{+}_{\varphi_0}(t)\cup\Gamma^{-}_{\varphi_0}(t)\cup\Gamma^{+}_{\varphi_1}(t)\cup\Gamma^{-}_{\varphi_1}(t)$, where with $\mu=+,-$, is denoted $\Gamma^{\mu}_{\varphi}(t):=\{(x,\xi)\in\mathbb{R}^n\times(\mathbb{R}^n_\xi\setminus\{0\});\ \exists(x_0,\xi_0)\in WF(\varphi),\ x=x_0+\mu 3t^{\frac{1}{3}}\xi_0/|\xi_0|,\ \xi=\xi_0\}$. To discuss a reflection of singularities we note that the limits $\lim_{t\to 0^+}\psi$ and $\lim_{t\to 0^+}\psi_t$ for the solutions of (1.1), in general, do not exist and we change the formulation of the known reflection of the singularities theorems (see, e.g., [3, Sect. 9 Ch. 2]). Let $\psi\in C^2((0,T);\mathcal{E}'(\mathbb{R}^n))$ be a solution of the equation of (1.1) without source, $f=0$. Then according to Lemma 2.2 $\lim_{t\to 0^+}(t\psi)\in\mathcal{E}'(\mathbb{R}^n)$ and $\lim_{t\to 0^+}(t\psi_t+\psi+3t^{-1/3}\Delta\lim_{t\to 0^+}(t\psi))\in\mathcal{E}'(\mathbb{R}^n)$ exist. The next theorem shows that the solutions obey the reflection of the singularities phenomena. For $(x_0,\xi_0)\in\mathbb{R}^{2n}$, $\xi_0\neq 0$, denote

$$\Gamma^{\mu}(x_0,\xi_0):=\Big\{(t,x;\tau,\xi)\in\mathbb{R}^{2n+2};\ x=x_0+\mu 3t^{\frac{1}{3}}\xi_0/|\xi_0|,\\ \xi=\xi_0,\ \tau=\mu t^{-2/3}|\xi|,\ t\in[0,\infty)\Big\},$$

$\mu=+,-$, the bicharacteristics emanating from the point (x_0,ξ_0) and lying in the half-space $t\geq 0$. It is sufficient to consider the case of point $x_0=0$.

Theorem 1.2 *Let* $\psi\in C^2((0,T);\mathcal{E}'(\mathbb{R}^n))$ *be a solution of* (1.1) *with* $f\equiv 0$. *If* $WF(\lim_{t\to 0^+}(t\psi))\cap\Gamma^{+}(0,\xi_0)=\emptyset$ *and* $(0,\xi_0)\notin WF(\lim_{t\to 0^+}(t\psi))$, *or* $(0,\xi_0)\notin WF(\lim_{t\to 0^+}(t\psi_t+\psi+3t^{-1/3}\Delta\lim_{t\to 0^+}(t\psi)))$ *then* $WF(\psi)\cap\Gamma^{-}(0,\xi_0)=\emptyset$ *and for every* $t>0$ *the point* $(-3t^{\frac{1}{3}}\xi_0/|\xi_0|,\xi_0)$ *does not belong to* $WF(\psi|_t)$. *Similar statement is true if we permute* $\Gamma^{+}(0,\xi_0)$ *and* $\Gamma^{-}(0,\xi_0)$.

The EdeS model recently became a focus of interest for an increasing number of authors. We believe that the microlocal analysis of the solution operators obtained in the present paper fills the gap in the existing literature on the wave equation in the EdeS spacetime. In this note we present only the outline of the proof; the complete proof and further results will be published in the forthcoming paper.

2 The Microlocal Representation Formulas

If we denote $\mathcal{L}:=\partial_t^2-t^{-\frac{4}{3}}\Delta+2t^{-1}\partial_t$, $\mathcal{S}:=\partial_t^2-t^{-\frac{4}{3}}\Delta$, then we can easily check for $t\neq 0$ the following operator identity $t^{-1}\circ\mathcal{S}\circ t=\mathcal{L}$. The last equation suggests a change of unknown function ψ with u such that $\psi=t^{-1}u$. Then the problem for u is as follows:

$$\begin{cases} u_{tt}-t^{-4/3}\Delta u=g(x,t), \quad t>0,\ x\in\mathbb{R}^n,\\ \lim\limits_{t\to 0}u(x,t)=\varphi_0(x), \qquad \lim\limits_{t\to 0}\big(u_t(x,t)+3t^{-\frac{1}{3}}\Delta\varphi_0(x)\big)=\varphi_1(x), \quad x\in\mathbb{R}^n, \end{cases} \tag{2.1}$$

where $g(x,t) = tf(x,t)$. It is enough to consider the solution of the last problem. We consider two cases: $(\boldsymbol{\varphi_0}, \boldsymbol{\varphi_1})$ with $f = 0$; $(\boldsymbol{f})$ with $\varphi_0 = \varphi_1 = 0$.

Consider the ordinary differential equation

$$y'' + t^{\frac{4}{3}}|\xi|^2 y = 0, \tag{2.2}$$

with a parameter $\xi \in \mathbb{R}^n$. Set $\tau = 3t^{1/3}|\xi|$. The next lemma is an extension to negative γ of the corresponding result from [5].

Lemma 2.1 *The functions*

$$V_1(t,|\xi|) = e^{-3it^{1/3}|\xi|}\big(3it^{1/3}|\xi| + 1\big), \tag{2.3}$$

$$V_2(t,|\xi|) = \sum_{+,-} \frac{i}{18|\xi|^3} e^{\mp 3it^{\frac{1}{3}}|\xi|}\big(3it^{\frac{1}{3}}|\xi| \pm 1\big), \tag{2.4}$$

on $\mathbb{R}_t \times \mathbb{R}^n_\xi$, *form the fundamental system for* (2.2) *such that*

$$V_1(0,|\xi|) = 1, \qquad \lim_{t\to 0^+}\big(V_1'(t,|\xi|) - 3t^{-1/3}|\xi|^2 e^{-3it^{1/3}|\xi|}\big) = 0, \tag{2.5}$$

$$V_2(0,|\xi|) = 0, \qquad \lim_{t\to 0^+} V_2'(t,|\xi|) = 1. \tag{2.6}$$

Case of $(\boldsymbol{\varphi_0}, \boldsymbol{\varphi_1})$ The second relation of (2.5) generates the second initial condition from (2.2) [2]. In fact, any solution of the equation

$$u_{tt} - t^{-4/3}\Delta u = 0, \quad t > 0,\ x \in \mathbb{R}^n, \tag{2.7}$$

according to the next lemma has certain asymptotic behavior at $t = 0$, which allows us to set up a proper weighted initial value problem.

Lemma 2.2 *Let* $u \in C^2((0,T);\mathcal{E}'(\mathbb{R}^n))$ *be a solution of the equation of* (2.7). *Then the* $\lim_{t\to 0} u \in \mathcal{E}'(\mathbb{R}^n)$ *exists. Moreover, if we denote* $u_0 := \lim_{t\to 0} u$, *then* $\lim_{t\to 0}(u_t(x,t) + 3t^{-1/3}\Delta u_0(x)) \in \mathcal{E}'(\mathbb{R}^n)$ *exists as well.*

Thus, the following initial value problem

$$\begin{cases} u_{tt} - t^{-4/3}\Delta u = 0, \quad t > 0,\ x \in \mathbb{R}^n, \\ \lim\limits_{t\to 0^+} u(x,t) = \varphi_0(x), \qquad \lim\limits_{t\to 0^+}\big(u_t(x,t) + 3t^{-\frac{1}{3}}\Delta\varphi_0(x)\big) = \varphi_1(x). \end{cases} \tag{2.8}$$

has been justified by Lemma 2.2. After partial Fourier transform we obtain

$$\begin{cases} y_{tt}(t,|\xi|) + t^{-4/3}|\xi|^2 y(t,|\xi|) = 0, \quad t > 0,\ \xi \in \mathbb{R}^n, \\ \lim\limits_{t\to 0^+} y(t,|\xi|) = \widehat{\varphi}_0(\xi), \qquad \lim\limits_{t\to 0^+}\big(y_t(t,|\xi|) - 3t^{-\frac{1}{3}}|\xi|^2\widehat{\varphi}_0(\xi)\big) = \widehat{\varphi}_1(\xi). \end{cases}$$

The solution $y(t, |\xi|)$ of the last problem can be represented as

$$y(t,|\xi|) = p_{01}\big(|\xi|\big)V_1(t,|\xi|)\widehat{\varphi}_0(\xi) + p_{02}\big(|\xi|\big)V_2\big(t,|\xi|\big)\widehat{\varphi}_0(\xi) \\ + p_{11}\big(|\xi|\big)V_1\big(t,|\xi|\big)\widehat{\varphi}_1(\xi) + p_{12}\big(|\xi|\big)V_2\big(t,|\xi|\big)\widehat{\varphi}_1(\xi). \tag{2.9}$$

This allows us to prove the following microlocal representation theorem. The operator $\mathcal{G} : \mathcal{E}'(\mathbb{R}^n) \times \mathcal{E}'(\mathbb{R}^n) \longrightarrow C^2((0, T]; C^\infty(\mathbb{R}^n))$, is said to be a *parametrix* of the problem (2.8), if $\mathcal{G}\{\varphi_0, \varphi_1\} \in C^2((0, T]; C^\infty(\mathbb{R}^n))$ and

$$\begin{cases} \mathcal{S}\mathcal{G}\{\varphi_0,\varphi_1\} \in C\big((0,T];C^\infty\big(\mathbb{R}^n\big)\big), \ \lim\limits_{t\to 0^+}\big(\mathcal{G}\{\varphi_0,\varphi_1\}-\varphi_0\big) \in C^\infty\big(\mathbb{R}^n\big), \\ \lim\limits_{t\to 0^+}\big(\big(\mathcal{G}\{\varphi_0,\varphi_1\}\big)_t + 3t^{-1/3}\Delta\varphi_0 - \varphi_1\big) \in C^\infty\big(\mathbb{R}^n\big), \end{cases}$$

for every $\varphi_0, \varphi_1 \in \mathcal{E}'(\mathbb{R}^n)$. Let $\chi \in C^\infty(\mathbb{R}^n)$ be a cut-off function such that $\chi(\xi) = 0$ if $|\xi| \le 1$ and $\chi(\xi) = 1$ if $|\xi| \ge 2$.

Theorem 2.3 *The parametrix $\mathcal{G}$ of the problem* (2.8) *is given by the Fourier integral operators as follows*:

$$\mathcal{G}\{\varphi_0,\varphi_1\}(t,x) = \sum_{+,-}\frac{\pm 1}{(2\pi)^n}\int_{\mathbb{R}^n}\frac{1}{2}e^{i(x\cdot\xi \mp 3t^{\frac{1}{3}}|\xi|)}\big(3it^{\frac{1}{3}}|\xi| \pm 1\big)\chi(\xi)\widehat{\varphi}_0(\xi)d\xi \\ + \sum_{+,-}\frac{\pm 1}{(2\pi)^n}\int_{\mathbb{R}^n}e^{i(x\cdot\xi \mp 3t^{\frac{1}{3}}|\xi|)}\frac{i}{18|\xi|^3}\big(3it^{\frac{1}{3}}|\xi| \pm 1\big)\chi(\xi)\widehat{\varphi}_1(\xi)d\xi.$$

Proof From (2.9) and the first initial condition we derive $\lim_{t\to 0^+} y(t, |\xi|) = p_{01}(|\xi|)\widehat{\varphi}_0(\xi) + p_{11}(|\xi|)\widehat{\varphi}_1(\xi)$ and obtain for the coefficients $p_{01}(|\xi|) = 1$, $p_{11}(|\xi|) = 0$. Consequently,

$$y\big(t,|\xi|\big) = V_1\big(t,|\xi|\big)\widehat{\varphi}_0(\xi) + p_{02}\big(|\xi|\big)V_2\big(t,|\xi|\big)\widehat{\varphi}_0(\xi) + p_{12}\big(|\xi|\big)V_2\big(t,|\xi|\big)\widehat{\varphi}_1(\xi).$$

It follows

$$\lim_{t\to 0^+}\big(y_t\big(t,|\xi|\big) - 3t^{-\frac{1}{3}}|\xi|^2\widehat{\varphi}_0(\xi)\big) \\ = \lim_{t\to 0^+}\big(\big[3t^{-\frac{1}{3}}|\xi|^2e^{-3it^{\frac{1}{3}}|\xi|} + p_{02}\big(|\xi|\big) - 3t^{-\frac{1}{3}}|\xi|^2\big]\widehat{\varphi}_0(\xi) + p_{12}\big(|\xi|\big)\widehat{\varphi}_1(\xi)\big).$$

We set $p_{02}(|\xi|) = 9i|\xi|^3$, $p_{12}(|\xi|) = 1$ and obtain

$$\lim_{t\to 0^+}\big(\big[3t^{-1/3}|\xi|^2e^{-3it^{1/3}|\xi|} + 9i|\xi|^3 - 3t^{-1/3}|\xi|^2\big]\widehat{\varphi}_0(\xi)\big) = 0.$$

Thus,

$$y\big(t,|\xi|\big) = \sum_{+,-}\frac{\pm 1}{2}e^{\mp 3it^{\frac{1}{3}}|\xi|}\big(3it^{\frac{1}{3}}|\xi| \pm 1\big)\widehat{\varphi}_0(\xi)$$

$$+\sum_{+,-}\frac{i}{18|\xi|^3}e^{\mp 3it^{\frac{1}{3}}|\xi|}\big(3it^{\frac{1}{3}}|\xi|\pm 1\big)\widehat{\varphi}_1(\xi).$$

To complete the proof of the theorem we have to prove that the difference between the exact solution u_{ext} of the problem (2.8) and the function $u := \mathcal{G}\{\varphi_0,\varphi_1\}$ of (2.10) is the smooth function. □

Corollary 2.4 *The wave front set of the solution, $WF(u)$, is the union*

$$WF\big(u(t)\big)=\Gamma^{+}_{\varphi_0}(t)\cup\Gamma^{-}_{\varphi_0}(t)\cup\Gamma^{+}_{\varphi_1}(t)\cup\Gamma^{-}_{\varphi_1}(t),$$

where with $\mu=+,-$ is denoted $\Gamma^{\mu}_{\varphi}(t):=\{(x,\xi)\in\mathbb{R}^n\times(\mathbb{R}^n_\xi\setminus\{0\});\ \exists(x_0,\xi_0)\in WF(\varphi),\ x=x_0+\mu 3t^{\frac{1}{3}}\xi_0/|\xi_0|,\ \xi=\xi_0\}$.

The corollary below shows that the problem obeys the reflection of the singularities phenomena similar to the operators with the regular coefficients. For $(x_0,\xi_0)\in\mathbb{R}^{2n}$, $\xi_0\neq 0$, denote $\Gamma^{\mu}(x_0,\xi_0):=\{(t,x;\tau,\xi)\in\mathbb{R}^{2n+2};\ x=x_0+\mu 3t^{\frac{1}{3}}\xi_0/|\xi_0|,\ \xi=\xi_0,\ \tau=\mu t^{-2/3}|\xi|,\ t\in[0,\infty)\}$, $\mu=+,-$, the bicharacteristics emanating from the point (x_0,ξ_0) and lying in the half-space $t\geq 0$. It is sufficient to consider the case of point $x_0=0$ only, since the equation is invariant with respect to translations in x.

Corollary 2.5 *Let $u\in C^2((0,T);\mathcal{E}'(\mathbb{R}^n))$ be a solution of (2.7). If $WF(u)\cap\Gamma^{+}(0,\xi_0)=\emptyset$ and $(0,\xi_0)\notin WF(\lim_{t\to 0^+}(u_t+3t^{-1/3}\Delta\lim_{t\to 0^+}u))$ or $(0,\xi_0)\notin WF(\lim_{t\to 0^+}u)$, then $WF(u)\cap\Gamma^{-}(0,\xi_0)=\emptyset$ and for every $t>0$ the point $(-3t^{\frac{1}{3}}\xi_0/|\xi_0|,\xi_0)$ does not belong to $WF(u|_t)$. Similar statement is true if we permute $\Gamma^{+}(0,\xi_0)$ and $\Gamma^{-}(0,\xi_0)$.*

Case of (*f*) We look for the solution of the Cauchy problem for the equation with the source term,

$$\begin{cases} u_{tt}-t^{-4/3}\Delta u=g(x,t), & t>0,\ x\in\mathbb{R}^n,\\ \lim\limits_{t\to 0^+}u(x,t)=0, \qquad \lim\limits_{t\to 0^+}u_t(x,t)=0, & x\in\mathbb{R}^n. \end{cases} \tag{2.10}$$

The operator $\mathcal{G}_s: C((0,T];\mathcal{E}'(\mathbb{R}^n))\longrightarrow C^2((0,T];\mathcal{E}'(\mathbb{R}^n))$, is said to be a *parametrix* of the problem (2.10), if $\mathcal{G}_s g\in C^2((0,T];\mathcal{E}'(\mathbb{R}^n))$ and

$$\begin{cases} \mathcal{S}\mathcal{G}_s g-g\in C\big((0,T];C^\infty\big(\mathbb{R}^n\big)\big),\\ \lim\limits_{t\to 0^+}\mathcal{G}_s g\in C^\infty\big(\mathbb{R}^n\big), \qquad \lim\limits_{t\to 0^+}(\mathcal{G}_s g)_t\in C^\infty\big(\mathbb{R}^n\big), \end{cases}$$

for every distribution-valued function $g\in C([0,T];\mathcal{E}'(\mathbb{R}^n))$.

Theorem 2.6 *Let* $\phi(t,\xi) = 3t^{\frac{1}{3}}|\xi|$, *then the parametrix* $\mathcal{G}_s$ *of the problem* (2.10) *is given by*

$$\mathcal{G}_s g(t,x) = \sum_{+,-} \frac{\pm 1}{(2\pi)^n} \int_0^t \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{\mp i(\phi(\tau,\xi)-\phi(t,\xi))} \frac{i}{18|\xi|^3}$$
$$\times \big(\mp i\big[\phi(\tau,\xi) - \phi(t,\xi)\big] - \phi(\tau,\xi)\phi(t,\xi) - 1\big)\chi\big(|\xi|\big)\widehat{g}(\tau,\xi)d\xi d\tau.$$

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Nonlinear Evolution Equations with Strong Dissipation and Proliferation

Akisato Kubo and Hiroki Hoshino

Abstract We investigate the global existence in time and asymptotic profile of the solution of some nonlinear evolution equations with strong dissipation and proliferation arising in mathematical biology. We apply our result to mathematical models of tumour angiogenesis and invasion with proliferation of tumour cells.

Keywords Nonlinear evolution equation · Mathematical biology

Mathematics Subject Classification (2010) Primary 35L53 · Secondary 35K15

1 Introduction

In this paper we consider the initial Neumann-boundary value problem of nonlinear evolution equations arising from chemotaxis models with logistic term;

$$(NE)\begin{cases} u_{tt} = D\nabla^2 u_t + \nabla\cdot\left(\chi\left(u_t, e^{-u}\right)e^{-u}\nabla u\right) + \mu(1-u_t)u_t \\ \quad \text{in } \boldsymbol{\Omega}\times(0,T) & (1.1)\\ \left.\dfrac{\partial}{\partial\nu}u\right|_{\partial\boldsymbol{\Omega}} = 0 \quad \text{on } \partial\boldsymbol{\Omega}\times(0,T) & (1.2)\\ u(x,0)=u_0(x), \qquad u_t(x,0)=u_1(x) \quad \text{in } \boldsymbol{\Omega} & (1.3)\end{cases}$$

$$\frac{\partial}{\partial t} = \partial_t, \qquad \frac{\partial}{\partial x_i} = \partial_{x_i}, \quad i=1,\dots,n,$$

$$\nabla u = \mathrm{grad}_x u = (\partial_{x_1}u,\dots,\partial_{x_n}u) \tag{1.4}$$

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$$\nabla^2 u = \nabla \cdot \nabla u = \Delta u = \partial_{x_1}^2 u + \cdots + \partial_{x_n}^2 u$$

where the coefficient of the non-linear term $\chi(\cdot,\cdot)$ will be specified later in (A), D is a positive constant, Ω is a bounded domain in R^n and $\partial\Omega$ is a smooth boundary of Ω and ν is the outer unit normal vector. We frequently denote different positive constants on the same set of arguments, we use the same letter C.

Our purpose is to establish global in time existence of solutions to (*NE*). However we first consider a special case $(NE)'$ where we put $\mu = 0$ in (*NE*),

$$(NE)' \quad \begin{cases} u_{tt} = D\nabla^2 u_t + \nabla \cdot \big(\chi(u_t, e^{-u})e^{-u}\nabla u\big), \\ \dfrac{\partial}{\partial \nu} u \Big|_{\partial \boldsymbol{\Omega}} = 0, \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x). \end{cases}$$

We deal with it in Sect. 2 and after that, applying the result about $(NE)'$, we will get the desired result for (*NE*) in Sect. 3.

In order to discuss the existence of the solution and its asymptotic behaviour of $(NE)'$ we seek the solution in the form of $u(x,t) = a + bt + v(x,t)$ for positive parameters a and b. Then $(NE)'$ is rewritten by

$$(RP)' \quad \begin{cases} Q[v] = v_{tt} - D\nabla^2 v_t - \nabla \cdot \big(\chi_{a,b}(v)e^{-a-bt-v}\nabla v\big) = 0, \\ \dfrac{\partial}{\partial \nu} v \Big|_{\partial \boldsymbol{\Omega}} = 0, \\ v(x,0) = v_0(x), \qquad v_t(x,0) = v_1(x), \end{cases} \tag{1.5}$$

where we donote $\chi(v_t + b, e^{-a-bt-v})$ by $\chi_{a,b}(v)$. Levine and Sleeman [9] obtained explicit solutions of the form: $u = \gamma t + v$ $(\gamma > 0)$ of a simplified equation of (1.5) for $n = 1$ (cf. [11]). In this line the papers [5–8] showed the existence of the solution in the same form as above of a special case of $(NE)'$ for any spatial dimensions, which arises from mathematical biology and biomedicine (see [1, 10]).

Let $B_{r+} = B_r \cap (R \times R_+)$ where B_r is a ball of radius r at 0 in R^2. We assume that for a constant $r > 0$ and $(s_1, s_2) \in B_{r+}$ there exists a positive constant c_r such that for any integer $m \geq [n/2] + 3$

$$(A): \quad \chi(s_1 + b, s_2) \in C^m(R \times R_+), \quad m \geq [n/2] + 3,$$
$$\chi(s_1 + b, s_2) \leq c_r(b+1).$$

Remark 1.1 In the previous paper [4] additionally we need to assume that $\chi(s_1 + b, s_2)$ is positive to obtain the global existence theorem in time of $(NE)'$. In this paper without the additional condition we can get the same result as [4]. For example, this assumption covers the case where $\chi(s_1, s_2) = s_1^{p_1} s_2^{p_2}$, p_1 is a positive integer and p_2 is a positive constant. Even if $p_1 < 0$, it is admissible by taking b sufficiently large.

Now let us introduce function spaces used in this paper. First, $H^l(\Omega)$ denotes the usual Sobolev space $W^{l,2}(\Omega)$ of order l on Ω. For functions $h(x,t)$ and $k(x,t)$ defined in $\Omega \times [0,\infty)$, we put

$$(h,k)(t) = \int_\Omega h(x,t)k(x,t)dx, \qquad \|h\|_l^2(t) = \sum_{|\beta|\le l} \left|\partial_x^\beta h(\cdot,t)\right|^2_{L^2(\boldsymbol{\Omega})}(t)$$

and sometimes we write $\|h\|_0(t)$ by $\|h\|(t)$ for simplicity where β is a multi-index for $\beta = (\beta_1,\ldots,\beta_n)$.

The eigenvalues of $-\Delta$ with the homogeneous Neumann boundary conditions are denoted by $\{\lambda_i \mid i = 0,1,2,\ldots\}$, which are arranged as $0 = \lambda_0 < \lambda_1 \le \cdots \to +\infty$ and $\varphi_i = \varphi_i(x)$ indicates the L^2 normalized eigenfunction corresponding to λ_i. Then we put for $h(x), k(x) \in H^l(\boldsymbol{\Omega})$, if $l = 2j$, for a non-negative integer l,

$$(h,k)_l = (h,k) + \left(\Delta^j h, \Delta^j k\right), \quad |h|_l^2 = (h,h)_l$$

and if $l = 2j+1$

$$(h,k)_l = (h,k) + \left(\nabla\cdot\Delta^j h, \nabla\cdot\Delta^j k\right), \quad |h|_l^2 = (h,h)_l.$$

We set $W^l(\Omega)$ as a closure of $\{\varphi_1,\varphi_2,\ldots,\varphi_n,\ldots\}$ in the functional space $H^l(\Omega)$. Taking $\lambda_1 > 0$ into account, it is noticed that we have $\int_\Omega h(x) = 0$ for $h(x) \in W^l(\Omega)$, which enables us to use Poincare's inequality. Since the trace operator γ is continuous from $H^{1/2}(\Omega)$ to $L^2(\partial\Omega)$, it holds that $\gamma(\nabla u) = 0$ for $u \in W^m(\Omega)$ for $m \ge [n/2]+2$. We know the equivalence of norms $|\cdot|_l$, $\|\cdot\|_{l,}$ which will be used frequently.

2 Existence and Asymptotic Behaviour of Solutions

We prepare the following lemmas to derive energy estimates of $(RP)'$. Let $v \in \bigcap_{i=0}^2 C^i([0,\infty); W^{m+1-i}(\Omega))$ and let $(v_t, e^{-l-v}) \in B_{r+}$ for $l = a+bt$. Lemmas 2.1 and 2.2 are obtained in [5].

Lemma 2.1 *If $u = u(x,t)$ satisfies the above regularity conditions, then it holds that for $M \ge [n/2]+1$*

$$\|u\|_M^2(t) \le 4t\|u_t\|^2_{L^2((0,\infty);H^M(\boldsymbol{\Omega}))} + 2\|u\|_M^2(0)$$

for any $t \in (0,\infty)$.

Lemma 2.2 *If $u = u(x,t)$ satisfies the above regularity conditions with $M \ge [n/2]+1$, then it holds that*

(i) $\|e^{-u}\|_M(t) \le C_1 \exp(C_1\sqrt{t})$
(ii) *for any constant* $0 < b' < b$

$$e^{-a-bt-u} < \left\|e^{-a-bt-u}\right\|_M(t) \le C_2 e^{-b't}$$

where $C_2 \to 0$ *as* $a \to \infty$.

Lemma 2.3 *Assume that* $u = u(x,t)$ *satisfies the above inequality conditions. For* $0 < b' < b$, *it holds that for* $i = 1, 2, \dots, n$

$$\left\|e^{-b't}u_{x_i}\right\|^2(t) + \int_0^t e^{-2b'\tau}\|u_{x_i}\|^2(\tau)d\tau \le C\left(\int_0^t e^{-2b'\tau}\|u_{x_i\tau}\|^2(\tau)d\tau + \|u_{x_i}\|^2(0)\right).$$

The sketch of proof In the same way as Lemma 2.1 for any $\varepsilon > 0$, we have

$$\left\|e^{-b't}u_{x_i}\right\|^2(t) + 2b'\int_0^t e^{-2b'\tau}\|u_{x_i}\|^2(\tau)d\tau \le C\left(\frac{1}{\varepsilon}\int_0^t e^{-2b'\tau}\|u_{x_i\tau}\|^2(\tau) + \varepsilon\int_0^t e^{-2b'\tau}\|u_{x_i}\|^2(\tau)d\tau + \|u_{x_i}\|^2(0)\right).$$

Taking ε sufficiently small, we have the desired result. □

Lemma 2.4 (Basic estimate of $(RP)'$) *We have a basic energy estimate of* $(RP)'$ *under the same assumption for* $v = v(x,t)$ *as above and sufficiently large* a.

$$\|v_t\|^2(t) + \int_0^t D\|\nabla v_t\|^2 d\tau \le CE[v](0),$$

where $E[v] = \|v_t\|^2 + \|\nabla v\|^2$.

The sketch of proof We consider $(Q[v], v_t) = 0$ in order to obtain a basic estimate of $(RP)'$. Then we have

$$2\big(\partial_t^2 v - D\Delta v_t - \nabla\cdot\big(\chi_{a,b}(v)e^{-a-bt-v}\nabla v, v_t\big)\big)$$

by the integration by parts

$$= \frac{\partial}{\partial t}\|v_t\|^2 + 2D\|\nabla v_t\|^2 + 2\big(\chi_{a,b}(v)e^{-a-bt-v}\nabla v, \nabla v_t\big) = 0. \tag{2.1}$$

Since we have for any $\varepsilon > 0$ by using Dionne [3] for the estimate of nonlinear terms and Lemma 2.2,

$$\int_0^t \big(\chi_{a,b}(v)e^{-a-bt-v}\nabla v, \nabla v_t\big)d\tau$$
$$\leq C\left(\varepsilon^{-1}\int_0^t \big(e^{-2a-2b't}\nabla v, \nabla v\big)dt + \varepsilon\int_0^t \|\nabla v_\tau\|^2 d\tau\right),$$

we get by integrating the equality (2.1) over $(0, t)$ and using the above estimate

$$\|v_t\|^2(t) + \int_0^t 2D\|\nabla v_t\|^2(\tau)d\tau$$
$$\leq CE[v](0) + C\left(\varepsilon^{-1}\int_0^t \big(e^{-2a-2b't}\nabla v, \nabla v\big)d\tau + \varepsilon\int_0^t \|\nabla v_\tau\|^2 d\tau\right). \quad (2.2)$$

Since the last term of the right hand side of (2.2) is negligible for sufficiently small ε, we derive by using Lemma 2.3 for the second term of the right hand side of (2.2)

$$\|v_t\|^2(t) + \int_0^t 2D\|\nabla v_t\|^2(\tau)d\tau \leq CE[v](0) + Ce^{-a}\int_0^t \|\nabla v_\tau\|^2 d\tau. \quad (2.3)$$

Taking a sufficiently large the last term of (2.3) can be negligible. Hence we have a basic energy estimate of $(RP)'$. □

Lemma 2.5 (Higher order estimate for $(RP)'$) *Under the same assumption for $v = v(x,t)$ as above we have the result of higher order energy estimate $(RP)'$ for sufficiently large a:*

$$\sum_{j=1}^{M+1}\left(\left\|\nabla^{j-1}v_t\right\|^2(t) + \int_0^t D\left\|\nabla^j v_t\right\|^2(\tau)d\tau\right) \leq CE_M[v](0) \quad (2.4)$$

where we denote for any non-negative integer $k \leq M \leq m$, $E_k[v](t) = E[\nabla^k v]$.

The sketch of proof Suppose that the estimate (2.4) holds for $M = k-1 \geq 0$. Considering $\nabla^k v$ instead of v in (2.1), in the same way as in Lemma 2.4 we obtain (2.4) for $M = k$. In fact, we used the following estimate for $l = a + bt$ and a parameter $\kappa > 0$

$$\big(\nabla^{k+1}\big(\chi_{a,b}(v)e^{-l-v}\nabla v\big) - \chi_{a,b}(v)e^{-l-v}\nabla^{k+1}v, \nabla^k v_t\big)$$

by using the above assumption

$$\leq C\left(\kappa^{-1}\sum_{j=1}^{k+1}\big(e^{-a-b't}\nabla^j v_t, \nabla^j v_t\big) + \kappa|v_t|_k^2 + E_{k-1}[v](0)\right) \quad (2.5)$$

where the first and second terms in (2.5) are negligible for sufficiently large a and small $\kappa > 0$ respectively. Hence we obtain (2.4). □

Now we state our result for $(NE)'$.

Theorem 2.6 *Assume that* (A) *holds and* $(v_0(x), v_1(x)) \in W^{m+1}(\boldsymbol{\Omega}) \times W^m(\boldsymbol{\Omega})$ *for* $v_0(x) = u_0(x) - a$ *and* $v_1(x) = u_1(x) - b$. *For sufficiently large* a *and* r, *there is a solution* $u(x,t) = a + bt + v(x,t) \in \bigcap_{i=0}^{1} C^i([0,\infty); W^{m-i}(\boldsymbol{\Omega}))$ *to* $(NE)'$ *such that for* $\overline{u}_1 = |\boldsymbol{\Omega}|^{-1}\int_{\boldsymbol{\Omega}} u_1(x)dx$

$$\lim_{t\to\infty} \left\| u_t(x,t) - \overline{u}_1 \right\|_{m-1} = 0.$$

The sketch of proof The proof is given by the same way as used in [5]. Actually we consider the following iteration scheme and derive the energy estimate of it by using Lemma 2.5,

$$(i+1) \quad \begin{cases} Q_i[v_{i+1}] = \partial_t^2 v_{i+1} - \partial_t \boldsymbol{\Delta} v_{i+1} \\ \quad - \nabla \cdot \left(e^{-a-bt} \chi(u_{it}, e^{-u_i}) e^{-v_i} \nabla v_{i+1} \right) = 0, \\ \dfrac{\partial}{\partial \nu} v_{i+1} \Big|_{\partial \Omega} = 0, \\ v_{i+1}(x,0) = v_0(x), \qquad v_{i+1t}(x,0) = v_1(x), \end{cases}$$

where $v_i = \sum_{j=1}^{\infty} f_{ij}(t)\varphi_j(x)$, $v_0(x) = \sum_{j=1}^{\infty} h_j \varphi_j(x)$, $v_1(x) = \sum_{j=1}^{\infty} h'_j \varphi_j(x)$. The energy estimate (2.4) guarantees the uniform estimate of each $(i+1)$ for $i = 1, 2, \ldots$. We determine $f_{ij}(t)$ by the solution of the following system of ordinary equations with initial data. For $j = 1, 2, \ldots$

$$\begin{cases} \left(Q_i[v_{i+1}], \varphi_j \right) = 0, \\ f_{i+1j}(0) = h_{i+1}, \qquad f_{i+1jt}(0) = h'_{i+1}. \end{cases}$$

It is not difficult to assure the local existence in time of $f_{ij}(t)$ by the theory of ordinary differential equations. Therefore, deriving the energy estimates, the global existence in time of the solution $\{u_i\}$ satisfying the regularity assumption in Sect. 2 and justification of the limiting process are assured by the standard method. The energy estimate enables us to get the solution by considering $Q_i[v_{i+1}] - Q_{i-1}[v_i]$ and standard argument of convergence for $v_{i+1} - v_i = w_i$. □

Remark 2.7 We can apply Theorem 2.6 to mathematical models of tumour angiogenesis by Anderson and Chaplain [1] and Othmer and Stevens [10] (see [4–8]).

3 The Case of $\mu \neq 0$

Now let's consider the case of $\mu \neq 0$ in (1.1). Putting $u(x,t) = L_a(t) + v(x,t)$ we have in (1.1) for $\mu \neq 0$

$$v_{tt} = D\nabla^2 v_t + \nabla \cdot \big(\chi\big(l(t) + v_t, e^{-L_a(t)-v}\big) e^{-L_a(t)-v} \cdot \nabla v\big) \\ + \mu v_t(t)\big(1 - 2l(t) - v_t\big)$$

where $L_a(t) = \int_0^t l(\tau)d\tau + a$, a is a positive parameter, $l(t)$ satisfies $1 - 2l(0) < 0$ and the logistic equation: $l_t(t) = \mu l(t)(1 - l(t))$, $l(0) = l_0 > 1/2$, then (*NE*) is rewritten by

$$(RP) \quad \begin{cases} P[v] = v_{tt} - D\nabla^2 v_t - \nabla \cdot \big(\chi\big(l(t) + v_t, e^{-L_a(t)-v}\big) e^{-L_a(t)-v} \cdot \nabla v\big) \\ \qquad - \mu v_t(t)\big(1 - 2l(t) - v_t\big) = 0, \\ \dfrac{\partial}{\partial \nu} v \Big|_{\partial \boldsymbol{\Omega}} = 0, \\ v(x, 0) = v_0(x), \qquad v_t(x, 0) = v_1(x). \end{cases}$$

In the same way as used in Sect. 2, we obtain the following result of (*NE*).

Theorem 3.1 *Assume that* (*A*) *holds and* $(v_0(x), v_1(x)) \in W^{m+1}(\boldsymbol{\Omega}) \times W^m(\boldsymbol{\Omega})$ *for* $v_0(x) = u_0(x) - a$, $v_1(x) = u_1(x) - l_0$ *and* $l_0 > 1/2$. *For sufficiently large* a *and* r, *there is a solution* $u(x, t) = L_a(t) + v(x, t) \in \bigcap_{i=0}^1 C^i([0, \infty); W^{m-i}(\boldsymbol{\Omega}))$ *to* (*NE*) *such that*

$$\lim_{t \to \infty} \big\| u_t(x, t) - l(t) \big\|_{m-1} = 0.$$

Remark 3.2 In [5–8] our solution is in the form of $u(x, t) = bt + v(x, t)$ for sufficiently large $b > 0$, but in the previous paper [4] and Sect. 2 in this paper we can get the solution in more general form $u(x, t) = a + bt + v(x, t)$ for any $b > 0$, which enables us to deal with (1.1) with proliferation term.

4 Application to a Mathematical Model

The following is a mathematical model of tumour invasion by the Chaplain–Lolas [2].

$$\begin{cases} \dfrac{\partial n}{\partial t} = d_n \dfrac{\partial^2 n}{\partial x^2} - \gamma \dfrac{\partial}{\partial x}\left(n \dfrac{\partial f}{\partial x}\right) + \mu_1 n(1 - n - f) & (4.1) \\ \dfrac{\partial f}{\partial t} = -\eta m f + \mu_2 f(1 - n - f) & (4.2) \\ \dfrac{\partial m}{\partial t} = d_m \dfrac{\partial^2 m}{\partial x^2} + \alpha n - \beta m & (4.3) \end{cases}$$

where $n := n(x, t)$ is the density of tumour cells, $m := m(x, t)$ is degradation enzymes concentration and $f := f(x, t)$ is the extra cellular matrix density and d_n, γ,

μ_1, η, μ_2, d_m, α and β are positive constants. In the below we consider only the case of $\mu_2 = 0$ for our convenience. It is seen in (4.2) that $f(x,t)$ is written by

$$f(x,t) = f_0(x) \cdot \exp\left(-\eta \int_0^t m ds\right). \tag{4.4}$$

Substituting $f(x,t)$ by the right hand side of (4.4) and putting $\Psi = \int_0^t n ds$ and $\Phi = \int_0^t m ds$, from (4.1) and (4.3) it follows that

$$\begin{aligned}\Psi_{tt} &= d_n \partial_x^2 \Psi_t + \gamma\eta\partial_x\left(f_0(x)\Psi_t\Phi_x e^{-\eta\Phi}\right) - \gamma\partial_x\left(f_{0x}(x)\Psi_t e^{-\eta\Phi}\right) \\ &\quad + \mu_1\Psi_t\left(1 - \Psi_t - f_0(x)e^{-\eta\Phi}\right)\end{aligned} \tag{4.5}$$

and

$$\Phi_{tt} = d_m \partial_x^2 \Phi_t + \alpha\Psi_t - \beta\Phi_t, \tag{4.6}$$

which are essentially regarded as the same type of equation as (1.1). From (2.4) the energy estimates of (4.5) and (4.6) follow and combining these estimates we obtain the desired estimate with respect to Ψ and Φ. Hence applying the same argument as used for Theorem 3.1 to the above mathematical model, we have existence and asymptotic behaviour of the solutions to our mathematical model.

Remark 4.1 The full proofs of our results obtained in this paper will be published somewhere soon later.

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A Note on Real Powers of Time Differentiation

Rainer Picard

Abstract A Hilbert space framework for fractional calculus is presented. The utility of the approach is exemplified by applications to abstract ordinary fractional differential equations with or without delay.

Keywords Fractional derivatives · Fractional integrals · Fractional calculus · Evolutionary equations · Causality

Mathematics Subject Classification (2010) Primary 34K37 · Secondary 34A08 · 26A33

1 Introduction

The idea of considering fractional derivatives goes back to the beginnings of calculus and was first raised and discussed in a correspondence between L'Hôpital and Leibniz in 1695. In 1823, Abel was the first to note the connection to a class of integral operators, an observation which dominated the ongoing development of this field, see [9] for a detailed history. This observation also allowed to consider arbitrary real orders of differentiation, so that speaking of fractional, i.e. rational, orders of differentiation actually turns out to be a misnomer. Nevertheless, labels such as "fractional derivatives" and "fractional calculus" are retained in recognition of the historical sources of this flourishing research field. Although there is a vast literature on the topic of fractional derivatives (see for example the monographs [3, 9] and the references therein), this article is largely self-contained, so that there is no much need to refer to classical statements on fractional derivatives to understand the results in this work.

To the best of the author's knowledge it has not been widely noted that ∂_0 can be established as a normal operator, see [5], which then comes with its own standard function calculus via the spectral theorem for such operators, see e.g. [1]. Indeed,

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it turns out that a unitary variant of the Fourier–Laplace transformation yields a spectral representation for $\mathfrak{Im}\,\partial_0 = \frac{1}{2i}\overline{(\partial_0 - \partial_0^*)}$ and $\mathfrak{Re}\,\partial_0$ is just multiplication by a number. This simplifies matters considerably and since we are staying in a Hilbert space setting there is indeed no need to utilize more intricate results of fractional calculus in other spaces. This approach was already successfully used for a large class of evolutionary partial differential equation problems. Here we want to focus on fractional ordinary differential equations, which allows for far more general right-hand sides. The approach relies on an observation discovered in [2].

2 Functional Analytic Framework

2.1 *Fractional Calculus and Operator-Valued Functions of Time Differentiation*

As indicated in the introduction, we start by establishing time differentiation ∂_0 as a normal operator. We consider the weighted H-valued L^2-type space $H_{\rho,0}(\mathbb{R}, H)$, generated by completion of $\mathring{C}_\infty(\mathbb{R}, H)$, the space of smooth H-valued functions with compact support, with respect to the inner product $\langle \cdot \mid \cdot \rangle_{\rho,0}$ given by the weighted integral

$$(\varphi, \psi) \mapsto \int_{\mathbb{R}} \big\langle \varphi(t) \bigm| \psi(t) \big\rangle_H \exp(-2\rho t)\,dt,$$

where $\rho \in \mathbb{R}$ is a parameter. The associated norm will be denoted by $|\cdot|_{\rho,0}$. The multiplication operator

$$\begin{aligned} \mathring{C}_\infty(\mathbb{R}, H) \subseteq H_{\rho,0}(\mathbb{R}, H) &\to \mathring{C}_\infty(\mathbb{R}, H) \subseteq H_{0,0}(\mathbb{R}, H) = L^2(\mathbb{R}, H), \\ \varphi &\mapsto \big(t \mapsto \exp(-\rho t)\varphi(t)\big), \end{aligned}$$

clearly has a unitary continuous extension, which we shall denote briefly by $\exp(-\rho m_0)$. Its inverse (adjoint) is given by

$$\begin{aligned} H_{0,0}(\mathbb{R}, H) &\to H_{\rho,0}(\mathbb{R}, H), \\ \varphi &\mapsto \big(t \mapsto \exp(\rho t)\varphi(t)\big). \end{aligned}$$

By taking the closure of the operator

$$\begin{aligned} \mathring{C}_\infty(\mathbb{R}, H) \subseteq H_{\rho,0}(\mathbb{R}, H) &\to H_{\rho,0}(\mathbb{R}, H) \\ \phi &\mapsto \phi', \end{aligned}$$

the time-derivative ∂_0 can be established as a normal operator on $H_{\rho,0}(\mathbb{R}, H)$ with real part ρ and so with $\frac{1}{i}(\partial_0 - \rho)$ as imaginary part. The domain of ∂_0 can be characterized by functions belonging to $H_{\rho,0}(\mathbb{R}, H)$, whose weak derivatives also lie in $H_{\rho,0}(\mathbb{R}, H)$. For $\rho \in \mathbb{R} \setminus \{0\}$ we have the continuous invertibility of ∂_0. The inverse of the normal operator ∂_0 is continuous and can be described by

$$\left(\partial_0^{-1} f\right)(t) = \begin{cases} \int_{-\infty}^{t} f(s)ds & \text{if } \rho > 0, \\ -\int_{t}^{\infty} f(s)ds & \text{if } \rho < 0 \end{cases}$$

for every $f \in H_{\rho,0}(\mathbb{R}, H)$ and almost every $t \in \mathbb{R}$ as a Bochner integral. Henceforth we shall focus on the case $\rho \in]0, \infty[$, which is associated with (forward) causality. The Fourier–Laplace transform $\mathcal{L}_\rho := \mathcal{F} \exp(-\rho m_0) : H_{\rho,0}(\mathbb{R}, H) \to L^2(\mathbb{R}, H)$, given as a composition of the (temporal) Fourier transform $\mathcal{F}$ and the unitary weight operator $\exp(-\rho m_0)$, is a spectral representation associated with ∂_0. It is

$$\partial_0 = \mathcal{L}_\rho^*(\mathrm{i}m_0 + \rho)\mathcal{L}_\rho,$$

where m_0 denotes the multiplication-by-argument operator given as the closure of

$$\begin{aligned} \mathring{C}_\infty(\mathbb{R}, H) \subseteq L^2(\mathbb{R}, H) &\to L^2(\mathbb{R}, H) \\ \varphi &\mapsto m_0\varphi \end{aligned}$$

with

$$(m_0\varphi)(\lambda) := \lambda\varphi(\lambda) \quad \text{in } H$$

for every $\lambda \in \mathbb{R}$. This observation allows us to consistently define an operator function calculus associated with ∂_0 in a standard way, [1]. Clearly, we can even extend this calculus to operator-valued functions by letting

$$M\left(\partial_0^{-1}\right) := \mathcal{L}_\rho^* M\left(\frac{1}{im_0 + \rho}\right)\mathcal{L}_\rho.$$

Here the linear operator $M(\frac{1}{im_0+\rho}) : L^2(\mathbb{R}, H) \to L^2(\mathbb{R}, H)$ is determined uniquely via

$$\left(M\left(\frac{1}{im_0 + \rho}\right)\varphi\right)(\lambda) := M\left(\frac{1}{i\lambda + \rho}\right)\varphi(\lambda) \quad \text{in } H$$

for every $\lambda \in \mathbb{R}$, $\varphi \in \mathring{C}_\infty(\mathbb{R}, H)$ by an operator-valued function M. For a material law the operator-valued function M needs to be bounded and an analytic function $z \mapsto M(z)$ in an open ball $B_\mathbb{C}(r, r)$ with some positive radius r centered at r. This is not an artificial assumption, rather a necessary constraint enforced by the requirement of causality, see [7] or [10, Theorem 9.1].

In terms of the associated operator-valued function calculus we also know what

$$\partial_0^{-\alpha}, \quad \alpha \in [0, 1[,$$

means.[1] With this we may define for $\gamma \in \mathbb{R}$

$$\partial_0^\gamma := \partial_0^{\lceil \gamma \rceil} \partial_0^{\gamma - \lceil \gamma \rceil} \tag{2.1}$$

as a natural generalization of differentiation to arbitrary real orders. Here $\lceil \alpha \rceil$ denotes the smallest integer greater or equal to α. The fact that $(\partial_0^\gamma)_{\gamma \in \mathbb{R}}$ is a family of commuting operators appears to be useful in applications, see e.g. [4] and the quoted literature. In contrast, the variant[2]

$${}_a D_t^\gamma := \partial_0^\gamma \chi_{]a,\infty[}(m_0) = \partial_0^{\lceil \gamma \rceil} \left(\partial_0^{\gamma - \lceil \gamma \rceil} \chi_{]a,\infty[}(m_0) \right)$$

with an appropriate choice of domain, known as Riemann–Liouville fractional derivative, $a \in \mathbb{R}$ a parameter, [3, 9], lacks this property. This is also true for the frequently used alternative fractional derivative, the Caputo fractional derivative, [3, 9]:

$${}_a^C D_t^\gamma := \partial_0^{\gamma - \lceil \gamma \rceil} \chi_{]a,\infty[}(m_0) \partial_0^{\lceil \gamma \rceil}$$

with a suitable domain. With these fractional derivatives being mere variants of limited usefulness in our context, we shall utilize only the above spectral definition (2.1) for our fractional differentiation/integration.

Let us inspect more closely some properties of ∂_0^α for $\alpha \in]0, 1[$. We recall our first lemma from [6, 8].

Lemma 2.1 *For $\alpha \in [0, 1]$ we have*

$$\mathfrak{Re}\, \partial_0^\alpha \geq \rho^\alpha \tag{2.2}$$

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[1] It should be noted that $\partial_0^{-\alpha}$ is largely independent of the particular choice of $\rho \in]0, \infty[$. Indeed, since

$$\mathcal{L}_\rho \chi_{]0,\infty[}(m_0) m_0^{\alpha-1} = \frac{1}{\sqrt{2\pi}} \frac{\Gamma(\alpha)}{(im + \rho)^\alpha}$$

we have for $\varphi \in \mathring{C}_\infty(\mathbb{R}, H)$

$$\frac{1}{\Gamma(\alpha)} \chi_{]0,\infty[}(m_0) m_0^{\alpha-1} * \varphi = \partial_0^{-\alpha} \varphi$$

and

$$\left(\frac{1}{\Gamma(\alpha)} \chi_{]0,\infty[}(m_0) m_0^{\alpha-1} * \varphi \right)(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^t \frac{1}{(t-s)^{1-\alpha}} \varphi(s) ds.$$

From this convolution integral representation we can also read off that $\partial_0^{-\alpha}$ is causal.

[2] In the limit $a \to -\infty$ the spectral fractional derivative is formally recovered:

$$\partial_0^\gamma = {}_{-\infty} D_t{}^\gamma = {}_{-\infty}^{\;\;C} D_t{}^\gamma .$$

There is, however, a domain issue here. Whereas ∂_0^γ is a well-defined closed operator, the operators ${}_{-\infty} D_t{}^\gamma$, ${}_{-\infty}^{\;\;C} D_t{}^\gamma$ are usually considered in terms of their integral representation leading to slightly different constraints and different choices of underlying spaces.

and for $\alpha \in [0, \infty[$

$$\left\| \partial_0^{-\alpha} \right\|_{H_{\rho,0}(\mathbb{R},H)} \le \rho^{-\alpha}. \tag{2.3}$$

2.2 Sobolev-Chains Associated with ∂_0

For $\rho > 0$ the operator $\partial_0 : D(\partial_0) \subseteq H_{\rho,0}(\mathbb{R}, H) \to H_{\rho,0}(\mathbb{R}, H)$ is a densely defined, normal operator. Moreover, since $\mathfrak{Re}\,\partial_0 = \rho$, we have $0 \in \rho(\partial_0)$. Thus, we can define the Sobolev-chain associated with the time-derivative, which is the family of Hilbert spaces $(H_{\rho,\alpha}(\mathbb{R}, H))_{\alpha\in\mathbb{R}}$, where $H_{\rho,\alpha}(\mathbb{R}, H)$ is the completion of $D(\partial_0^\alpha)$ with respect to the norm $|\cdot|_{\rho,\alpha,0}$ induced by the inner product $\langle u \mid v\rangle_{\rho,\alpha,0} = \langle \partial_0^\alpha u \mid \partial_0^\alpha v\rangle_{\rho,0}$, $\alpha \in \mathbb{R}$. Note that for $\alpha \in [0, \infty[$ the domain $D(\partial_0^\alpha)$ is already complete $D(\partial_0^\alpha) = H_{\rho,\alpha}(\mathbb{R}, H)$, so that the completion process is superfluous. For $\alpha \in]-\infty, 0[$, however, we obtain in this fashion larger spaces extending $D(\partial_0^\alpha)$ (extrapolation spaces). For example the Dirac-distribution δ which satisfies[3] $\partial_0^{-1}\delta = \chi_{[0,\infty[} \in H_{\rho,0}(\mathbb{R}, H)$ lies in $H_{\rho,\alpha}(\mathbb{R}, H)$ for $\alpha \in]-\infty, -1/2[$.

The following observations are completely analogous to the integer index case and will therefore be recorded without proof. It is for $\alpha \le \beta$

$$H_{\rho,\beta}(\mathbb{R}, H) \hookrightarrow H_{\rho,\alpha}(\mathbb{R}, H)$$

in the sense of continuous and dense embedding. Indeed, we have in this case with Lemma 2.1

$$|u|_{\rho,\alpha,0} \le \frac{1}{\rho^{\beta-\alpha}} |u|_{\rho,\beta,0} \tag{2.4}$$

for all $u \in H_{\rho,\beta}(\mathbb{R}, H)$. If we consider ∂_0^γ as a mapping from $H_{\rho,\alpha+\gamma}(\mathbb{R}, H)$ to $H_{\rho,\alpha}(\mathbb{R}, H)$, i.e. we consider

$$H_{\rho,\alpha+\gamma}(\mathbb{R}, H) \to H_{\rho,\alpha}(\mathbb{R}, H),$$
$$u \mapsto \partial_0^\gamma u,$$

then this mapping is an isometry with dense range extending canonically to a unitary mapping for which we shall retain the notation ∂_0^γ. With these observations we have established the basis for a—somewhat surprisingly powerful—solution theory for ordinary differential equations (with or without delay).

3 Abstract Fractional Differential Equations

$$\partial_0^\alpha u = f(u) \tag{3.1}$$

[3]Here χ_M denotes the characteristic function or indicator function of the set M.

Without loss of generality we shall assume that $\alpha \in]0, 1]$. If $\alpha > 1$ we utilize a variant of the re-formulation of higher order differential equations and consider instead

$$\partial_0^{\alpha/N}\begin{pmatrix} u \\ \partial_0^{\alpha/N} u \\ \vdots \\ \vdots \\ \partial_0^{(N-1)\alpha/N} u \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & 0 \end{pmatrix}\begin{pmatrix} u \\ \partial_0^{\alpha/N} u \\ \vdots \\ \vdots \\ \partial_0^{(N-1)\alpha/N} u \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ \vdots \\ f(u) \end{pmatrix},$$

where $N := \lfloor \alpha \rfloor$. Note that at this point of our general discussion we do not exclude that $f(u)$ is actually of the form $F(u, \partial_0^{\alpha/N} u, \ldots, \partial_0^{(N-1)\alpha/N} u)$. For equations of the form (3.1) with $\alpha \in]0, 1]$ we have the following Picard–Lindelöf type result.

Theorem 3.1 *Let $\alpha, \gamma \in \mathbb{R}$ and let*

$$f : H_{\rho,\gamma}(\mathbb{R}, H) \to H_{\rho,-\alpha+\gamma}(\mathbb{R}, H)$$

be Lipschitz continuous with best Lipschitz constant[4] $|f|_{\rho,\mathrm{Lip}} < 1$ *for all sufficiently large* $\rho \in]0, \infty[$. *Then* (3.1) *has a unique solution* $u \in H_{\rho,\gamma}(\mathbb{R}, H)$.

Proof The result follows the familiar line of reasoning, observing that combining f with the unitary mapping $\partial_0^{-\alpha} : H_{\rho,-\alpha+\gamma}(\mathbb{R}, H) \to H_{\rho,\gamma}(\mathbb{R}, H)$ we get a contraction

$$\partial_0^{-\alpha} f : H_{\rho,\gamma}(\mathbb{R}, H) \to H_{\rho,\gamma}(\mathbb{R}, H).$$

Indeed,

$$\left|\partial_0^{-\alpha} f(u) - \partial_0^{-\alpha} f(v)\right|_{\rho,\gamma,0} = \left|f(u) - f(v)\right|_{\rho,-\alpha+\gamma,0} \le |f|_{\rho,\mathrm{Lip}}|u - v|_{\rho,\gamma,0}$$

for all $u, v \in H_{\rho,\gamma}(\mathbb{R}, H)$. As long as $\rho \in]0, \infty[$ is sufficiently large we have the desired contraction property due to the assumption that $\liminf_{\rho\to\infty} |f|_{\rho,\mathrm{Lip}} < 1$. The uniquely determined fixed point $u \in H_{\rho,\gamma}(\mathbb{R}, H)$ of the contraction $\partial_0^{-\alpha} f$ satisfies

$$u = \partial_0^{-\alpha} f(u) \tag{3.2}$$

and so, by applying the (inverse) ∂_0^{α} as a unitary mapping from $H_{\rho,\gamma}(\mathbb{R}, H)$ onto $H_{\rho,-\alpha+\gamma}(\mathbb{R}, H)$ we regain

$$\partial_0^{\alpha} u = f(u),$$

[4] That is

$$|f|_{\rho,\mathrm{Lip}} := \inf\bigl\{L \in]0, \infty[\bigm| \left|f(u) - f(v)\right|_{\rho,-\alpha+\gamma,0} \le L|u - v|_{\rho,\gamma,0} \text{ for all } u, v \in H_{\rho,\gamma}(\mathbb{R}, H)\bigr\}.$$

which is now an equality holding in $H_{\rho,-\alpha+\gamma}(\mathbb{R}, H)$. Since (3.1) and (3.2) are equivalent, uniqueness of this solution is also clear. □

3.1 Two Simple Application Cases

For illustration purposes, we want to apply the abstract result of Theorem 3.1 to two particular cases.

So, let first $(\beta_s)_{s=0,\ldots,m} \in [0,1]^{m+1}$ be a strictly monotone decreasing family of numbers with $\alpha_0 > \beta_0$ and consider an equation of the form

$$\partial_0^{\alpha_0} u = \sum_{s=0}^{m} \partial_0^{\beta_s} F_s \circ u,$$

where $F_s : \mathbb{C}^N \to \mathbb{C}^N$ are Lipschitz continuous functions and $u(t) \in H := L^2(\Omega, \mathbb{C}^N)$ for almost every $t \in \mathbb{R}$. As a result $u \mapsto F_s \circ u$ generates a Lipschitz continuous mapping (of Nemicky type) in $H_{\rho,0}(\mathbb{R}, H)$, $\rho \in]0,\infty[$, with a uniform Lipschitz constant L_s. This in turn yields that

$$f(u) := \sum_{s=0}^{m} \partial_0^{\beta_s} F_s(u)$$

induces a Lipschitz continuous mapping $f : H_{\rho,0}(\mathbb{R}, H) \to H_{\rho,-\beta_0}(\mathbb{R}, H)$, $\epsilon, \rho \in]0,\infty[$. Indeed, we have

$$\begin{aligned}
&\left|f(u) - f(v)\right|_{\rho,-\alpha_0} \\
&\quad \le \rho^{\beta_0-\alpha_0}\left|f(u) - f(v)\right|_{\rho,-\beta_0} \\
&\quad \le \rho^{\beta_0-\alpha_0} \sum_{s=0}^{m} \rho^{\beta_s-\beta_0}\left|F_s(u) - F_s(v)\right|_{\rho,0}, \\
&\quad \le \rho^{\beta_0-\alpha_0} \sqrt{(m+1)\sum_{s=0}^{m} \rho^{2(\beta_s-\beta_0)} L_s^2} |u-v|_{\rho,0}.
\end{aligned}$$

We have a case of Theorem 3.1 with $\gamma = 0$.

As our second example we consider a class of neutral delay differential equations of the form

$$\partial_0^{\alpha_0} u = F\big(\big(\partial_0^{\alpha_k} u\big)_{k=1,\ldots,n}, \big(\tau_{-h_k} \partial_0^{\alpha_k} u\big)_{k=0,\ldots,n}\big),$$

where now $(\alpha_k)_{k=0,\ldots,n} \in [0,1]^{n+1}$ is a strictly monotone decreasing family of numbers and $(h_k)_{k=0,\ldots,n} \in]0,\infty[^{n+1}$ and $F : \mathbb{C}^{N\times n} \times \mathbb{C}^{N\times(n+1)} \to \mathbb{C}^N$ is Lipschitz continuous with Lipschitz constant $L \in]0, \infty[$ in the sense that for the canonical matrix norms

$$\left|F(U_0, U_1) - F(V_0, V_1)\right| \leq L\sqrt{|U_0 - V_0|^2 + |U_1 - V_1|^2}$$

for all $U_0, V_0 \in \mathbb{C}^{N\times n}$ and $U_1, V_1 \in \mathbb{C}^{N\times(n+1)}$. Observing that

$$\begin{aligned}&\left|\left(\partial_0^{\alpha_k}(u-v)\right)_{k=1,\ldots,n}\right|_{\rho,0}^2 + \left|\left(\tau_{-h_k}\partial_0^{\alpha_k}(u-v)\right)_{k=0,\ldots,n}\right|_{\rho,0}^2 \\ &\quad\leq \left(\rho^{2(\alpha_1-\alpha_0)}n + (n+1)\max\left\{\exp(-2h_k\rho)\;\middle|\; k=0,\ldots,n\right\}\right)|u-v|_{\rho,\alpha_0}^2\end{aligned}$$

for all $u, v \in H_{\rho,\alpha_0}(\mathbb{R}, H)$ and $\rho \in]0, \infty[$ sufficiently large, we obtain that

$$f(u) := F\left(\left(\partial_0^{\alpha_k}u\right)_{k=1,\ldots,n}, \left(\tau_{-h_k}\partial_0^{\alpha_k}u\right)_{k=0,\ldots,n}\right)$$

yields a Lipschitz continuous $f : H_{\rho,\alpha_0}(\mathbb{R}, H) \to H_{\rho,0}(\mathbb{R}, H)$. Indeed, with $U_0 := (\partial_0^{\alpha_k}u)_{k=1,\ldots,n}$, $U_1 := (\tau_{-h_k}\partial_0^{\alpha_k}u)_{k=0,\ldots,n}$, $V_0 := (\partial_0^{\alpha_k}v)_{k=1,\ldots,n}$ and $V_1 := (\tau_{-h_k}\partial_0^{\alpha_k}v)_{k=0,\ldots,n}$ we obtain

$$\begin{aligned}&\left|f(u) - f(v)\right|_{\rho,0} \\ &\quad= \left|F(U_0, U_1) - F(V_0, V_1)\right|_{\rho,0} \\ &\quad\leq L\sqrt{\left(\rho^{2(\alpha_1-\alpha_0)}n + (n+1)\max\left\{\exp(-2h_k\rho)\;\middle|\; k=0,\ldots,n\right\}\right)}|u-v|_{\rho,\alpha_0}.\end{aligned}$$

So, also in this situation we have an instance of Theorem 3.1 choosing $\gamma = \alpha_0$. More general examples can be tailored in analogy of examples treated in [2].

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A Stationary Approach to the Scattering on Noncompact Star Graphs Containing Finite Rays

Kiyoshi Mochizuki and Igor Trooshin

Abstract In this paper we consider Schrödinger operators on noncompact star-shaped graphs including some finite rays. We show that our spectral representation formula provides the time dependent formulation of the scattering theory. The scattering operator S is constructed in the configuration space, and then is related to the scattering matrix $S(\lambda)$ in the momentum space. Corresponding inverse scattering problem is investigated.

Keywords Star graph · Schrödinger operator · Scattering

Mathematics Subject Classification (2010) 34L25 · 81Q35

1 Introduction

In this paper we consider Schrödinger operators on non-compact star-shaped graphs including some finite rays. The origin of each ray is identified with the single vertex of the graph. The potential is real valued and satisfies suitable decay conditions on each infinite ray. To guarantee the selfadjointness of the operator we require the Dirichlet condition at each end point of the finite ray and the natural Kirchhoff conditions at the origin.

There are many works which generalize the classical results [1–3] on the half line or the whole line. Among them in [5] is treated the star graph which consists

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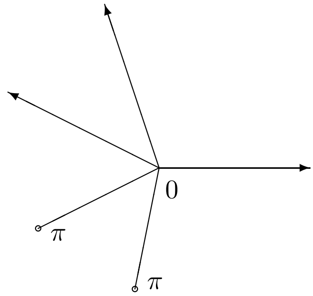

Fig. 1 Star graph

of a finite number of half lines. These results are generalized in [10] to the star-shaped graph which contains some finite rays. Note that we need precise low energy estimates of resolvent kernel in this case.

In this paper we present that our spectral representation formula provides the time dependent formulation of the scattering theory. The scattering operator S is constructed in the configuration space, and then is related to the scattering matrix $S(\lambda)$ in the momentum space. Corresponding inverse scattering problem is investigated.

$S(\lambda)$ is originally defined based on the generalized eigenfunctions. So, it is independent of the time dependent theory. Note that the main concern of [1, 4–7, 11] are the stationary theory for inverse scattering problems, and there are no descriptions of the time dependent theory there. Importance of the time dependent treatment in the scattering theory is claimed by Faddeev [2, 3]. Faddeev's approach is simply and clearly described in [12].

2 Schrödinger Operators on a Star-Shaped Graph

Let $\Gamma = \gamma_1 \times \cdots \times \gamma_{p_1} \times \gamma_{p_1+1} \cdots \times \gamma_p$ be a star-shaped graph which consists of p_1 semi-infinite rays $\gamma_j = \mathbf{R}_+ = \{x_j \in (0, \infty)\}$ $(j = 1, \dots, p_1)$ and $p - p_1$ finite rays $\gamma_j = \{x_j \in (0, \pi)\}$ $(j = p_1 + 1, \dots, p)$, with the origin of each ray identified with the single vertex of the graph (Fig. 1).

For each function $u(x)$ on $x \in \Gamma$, its restriction to the ray e_j is denoted by $u_j(x_j) = u(x)|_{e_j}$. We identify $u(x) = (u_j(x_j))_{j=1}^p$, and consider on Γ the Schrödinger operator

$$Lu = -\frac{d^2u}{dx^2} + q(x)u = \left(-\frac{d^2u_j}{dx_j^2} + q_j(x_j)u_j\right)_{j=1}^p, \quad x_j \in \gamma_j, \tag{2.1}$$

defined for functions u satisfying the natural Kirchhoff boundary conditions on the vertex:

$$u_1(0) = u_2(0) = \cdots = u_p(0), \tag{2.2}$$

$$u_1'(0) + u_2'(0) + \cdots + u_p'(0) = 0, \tag{2.3}$$

where $u'_j = du_j/dx_j$. These represent the continuity and conservation of flux at the origin. As for functions $u_j(x_j)$, $j \geq p_1 + 1$, we further require the Dirichlet boundary conditions on the end point $x_j = \pi$:

$$u_{p_1+1}(\pi) = u_{p_1+2}(\pi) = \cdots = u_p(\pi) = 0. \tag{2.4}$$

Each potential $q_j(x)$ $(j = 1, \ldots, p)$ is assumed to be real, continuous on γ_j (hereafter we simply write x for each x_j since there are no possibility of confusion). Moreover, we require for the first p_1 components

$$\int_{\gamma_j} (1 + x)\left|q_j(x)\right| dx < \infty \quad (j = 1, \ldots, p_1). \tag{2.5}$$

Let $\mathcal{H} = L^2(\Gamma)$ be the Hilbert space with norm

$$\|f\|_{\widetilde{\Gamma}} = \left(\sum_{j=1}^{p} \|f_j\|^2_{\gamma_j}\right)^{1/2}, \qquad \|f_j\|^2_{\gamma_j} = \int_{\gamma_j} \left|f_j(x)\right|^2 dx.$$

Under the above conditions on $q_j(x)$ the operator

$$Lu = \left(-u''_j + q_j(x)u_j\right)_{j=1}^{p}$$

restricted to the domain

$$\mathcal{D}(L) = \left\{ u \in \prod_{j=1}^{p} H^2_{\mathrm{loc}}(\overline{\gamma_j}) \text{ satisfying (2.2)–(2.4)}, \ \left(-u''_j + q_j(x)u\right)_{j=1}^{p} \in \mathcal{H} \right\}$$

forms a lower semi-bounded selfadjoint operator in $\mathcal{H}$. Moreover, we see that the essential spectrum of L fills the nonnegative half line $[0, \infty)$.

3 Expression of the Resolvent Kernel

For more study of spectral and scattering problems of L, we shall prepare precise expressions of the resolvent kernel near the spectrum of L. For this aim we consider on Γ the generalized eigenvalue problem

$$-u''_j + \left\{q_j(x) - \lambda^2\right\}u_j = 0, \quad x \in \gamma_j, \ j = 1, \ldots, p, \tag{3.1}$$

with the Kirchhoff conditions (2.2), (2.3).

Here λ is a complex parameter in $\overline{\mathbf{C}_+} = \mathbf{C}_+ \cup \mathbf{R}$.

We put $\mathbf{P} = \{1, \ldots, p\}$, $\mathbf{P}_1 = \{1, \ldots, p_1\}$ and $\mathbf{P}_2 = \{p_1 + 1, \ldots, p\}$.

Let $\omega_j(x, \lambda)$, $j \in \mathbf{P}$, be the regular solution $\omega(x_j, \lambda)$ on γ_j, and a function $e_j(x, \lambda)$, $j \in \mathbf{P}_1$ be the Jost solution $e(x_j, \lambda)$ on γ_j. Let $e_j(x, \lambda)$, $j \in \mathbf{P}_2$, be the regular solution of (3.1) satisfying the initial condition $e_j(\pi,\lambda)=0$, $e'_j(\pi,\lambda)=1$ at $x=\pi$. Each Jost solution $e_j(x,\lambda)$ $(j\in\mathbf{P}_1)$ of (3.1) is represented in the form

$$e_j(x,\lambda)=e^{i\lambda x}+\int_x^\infty K_j(x,y)e^{i\lambda y}dy \tag{3.2}$$

where the kernel $K_j(x,y)$ is a real function of $0\le x\le y<\infty$ satisfying the following conditions:

$$K_j(x,x)=\frac{1}{2}\int_x^\infty q_j(y)dy,\quad x>0. \tag{3.3}$$

The function $e_j(0,\lambda)$ $(j\in\mathbf{P}_2)$ has a countable number of simple zeros, a finite number of which is on $i\mathbf{R}_+$, and the remainders are on the real line $\mathbf{R}$.

Let K_e and K_G be the sets defined respectively by

$$K_e=\left\{\lambda\in\overline{\mathbf{C}_+};\ \prod_{j=1}^p e_j(0,\lambda)=0\right\},\qquad K_G=\left\{\lambda\in\overline{\mathbf{C}_+};\ G(\lambda)=0\right\},$$

where $G(\lambda)=\sum_{j=1}^p\frac{e'_j(0,\lambda)}{e_j(0,\lambda)}$.

For each $\lambda\in\overline{\mathbf{C}_+}\backslash\{K_e\cup K_G\}$ the problem (3.1), (2.2), (2.3) has linearly independent p solutions $\varphi_k(x,\lambda)=(\varphi_{kj}(x,\lambda))_{j=1}^p$ $(k=1,\dots,p)$:

$$\begin{aligned}\varphi_{kk}(x,\lambda)&=\frac{2i\lambda}{e_k(0,\lambda)}\left\{\omega_k(x,\lambda)-\frac{e_k(x,\lambda)}{e_k(0,\lambda)G(\lambda)}\right\},\\ \varphi_{kj}(x,\lambda)&=\frac{2i\lambda e_j(x,\lambda)}{e_k(0,\lambda)e_j(0,\lambda)G(\lambda)}\quad(j\ne k)\end{aligned} \tag{3.4}$$

Theorem 3.1 *For each $\lambda\in\mathbf{C}_+\backslash\{K_G\cup K_e\}$, the square λ^2 is in the resolvent set of L, and the resolvent $R(\lambda^2)=(L-\lambda^2)^{-1}$ forms an integral operator with kernel $R(x,y;\lambda)=(R_{kj}(x,y;\lambda))_{k,j=1}^p$ defined by*

$$\begin{aligned}R_{kk}(x,y;\lambda)&=\begin{cases}\dfrac{e_k(x,\lambda)\varphi_{kk}(y,\lambda)}{-2i\lambda}, & 0\le y\le x\\ \dfrac{\varphi_{kk}(x,\lambda)e_k(y,\lambda)}{-2i\lambda}, & 0\le x\le y,\end{cases}\\ R_{kj}(x,y;\lambda)&=\frac{e_k(x,\lambda)\varphi_{kj}(y,\lambda)}{-2i\lambda}\quad(j\ne k).\end{aligned} \tag{3.5}$$

To classify the singular points of the resolvent, we divide $K_e=K_e(\mathrm{I})\cup K_e(\mathrm{II})\cup K_e(\mathrm{III})$, where

$$K_e(\mathrm{I})=\left\{\lambda;\ e_j(0,\lambda)=0\text{ for at least two } j\text{'s, among which }\sharp\{j\in\mathbf{P}_2\}\le 1\right\},$$

$$K_e(\mathrm{II}) = \left\{\lambda;\ e_j(0,\lambda) = 0 \text{ for at least two } j\text{'s, among which } \sharp\{j \in \mathbf{P}_2\} \geq 2\right\},$$

$$K_e(\mathrm{III}) = \left\{\lambda;\ \exists_1 j \in \mathbf{P} \text{ verifying } e_j(0,\lambda) = 0\right\}.$$

For $\mu \in K_e$ we put $\mathbf{P}^0(\mu) = \{j \in \mathbf{P}; e_j(0,\mu) = 0\}$.

Lemma 3.2

(i) $[K_G \cup K_e] \cap \mathbf{C}_+$ *is a finite set on upper half-imaginary axis* $i\mathbf{R}_+$.
(ii) $[K_G \cup K_e(\mathrm{I})] \cap \mathbf{R} \subset \{0\}$.
(iii) $K_e(\mathrm{II}) \cap \mathbf{R}$ *is at most a countable set which is symmetric with respect to the origin and has no accumulation points.*
(iv) $K_e(\mathrm{III})$ *has removable singularities of* $R(x,y,\lambda)$.

In the following we put

$$\mathcal{K}_1 = \left[K_G \cup K_e(\mathrm{I}) \cup K_e(\mathrm{II})\right] \cap i\mathbf{R}_+, \qquad \mathcal{K}_2 = K_e(\mathrm{II}) \cap \overline{\mathbf{R}_+},$$
$$\mathcal{K} = \mathcal{K}_1 \cup \mathcal{K}_2.$$

Theorem 3.3

(i) *The eigenvalues of L forms the set* $\{\mu^2 :\ \mu \in \mathcal{K}\}$.
(ii) *The negative eigenvalues* μ^2, $\mu \in \mathcal{K}_1$, *are finite, and the dimension of the corresponding eigenspaces are* 1 *if* $\mu \in K_G$ *and are* $\sharp\mathbf{P}^0(\mu) - 1$ *if* $\mu \in K_e(\mathrm{I}) \cup K_e(\mathrm{II})$.
(iii) *Each* μ^2, $\mu \in \mathcal{K}_2$, *are eigenvalues contained in the essential spectrum of* L. *The dimension of the corresponding eigenvalues are* $\sharp\mathbf{P}^0(\mu) - 1$ *if* $\mu \neq 0$ *and are* $\sharp\mathbf{P}^0(\mu) \cap \mathbf{P}_2 - 1$ *if* $\mu = 0$. *The support of each eigenfunctions does not exceed the finite rays* $\Gamma_2 = \gamma_{p_1+1} \times \cdots \times \gamma_p$.
(iv) *If* $0 \in K_e(\mathrm{I}) \cup K_G$, *then* L *has a resonance at this point*: *Namely, the equation* $-u'' + q(x)u = 0$ *has a non-trivial bounded solution satisfying the Kirchhoff conditions* (2.2), (2.3), *and the Dirichlet condition* (2.4) *also. Its multiplicity is* $\sharp\mathbf{P}^0(0) - 1$ *if* $0 \in K_e(\mathrm{I})$, *and is* 1 *if* $0 \in K_G$.

Later on we single out the following negative eigenvalues $-\lambda_n^2$; $n = 1, \ldots, N$: $\{\mu = i\lambda_n;\ n = 1, \ldots, N\} = [K_G \cup K_e(\mathrm{I}) \cup K_e(\mathrm{II}_1)]\backslash\{0\}$, where $K_e(\mathrm{II}_1) = \{\mu \in K_e(\mathrm{II});\ \mathbf{P}^0(\mu) \cap \mathbf{P}_1 \neq \phi\}$. Such eigenvalues possess eigenfunctions with non-compact support.

4 Spectral Representations of *L*

We put

$$\begin{aligned} \Phi_1(x,\lambda) &= (\varphi_{kj}(x,\lambda))_{k=1,\ldots,p_1}^{j=1,\ldots,p_1}, \\ \Phi_2(x,\lambda) &= \left(\varphi_{kj}(x,\lambda)\right)_{k=1,\ldots,p_1}^{j=p_1+1,\ldots,p}, \end{aligned} \tag{4.1}$$

$$\Psi(x,\lambda) = \left(\Phi_1(x,\lambda) \;\middle|\; \Phi_2(x,\lambda)\right) = \begin{pmatrix} \varphi_1(x,\lambda) \\ \cdots \\ \varphi_{p_1}(x,\lambda) \end{pmatrix}. \tag{4.2}$$

Lemma 4.1 *$\mathcal{K}_2$ is of removable singularities of $\Psi(x,\lambda)$ $(\lambda \in \mathbf{R})$, and it is continuously extended to $\mathbf{R}\backslash\{0\}$.*

Remark 4.2 The continuity at $\lambda = 0$ of $\Psi(x,\lambda)$ is not guaranteed when $0 \in K_G$, but we can show its integrability near $\lambda = 0$.

The following is a key proposition to obtain the spectral representations.

Proposition 4.3 *For $\lambda \in \mathbf{R}\backslash[\mathcal{K}_2 \cup \{0\}]$ we have*

$$R(x,y;\lambda) - R(x,y;-\lambda) = \frac{1}{-2i\lambda}\,{}^t\overline{\Psi(x,\lambda)}\Psi(y,\lambda). \tag{4.3}$$

Thus, $2\lambda\{R(x,y,\lambda) - R(x,y;-\lambda)\}$ is continuous in $\lambda \in \mathbf{R}\backslash\{0\}$, and is integrable near $\lambda = 0$.

Note that each component of the above identity is given by

$$R_{kj}(x,y,\lambda) - R_{kj}(x,y;-\lambda) = \frac{1}{-2i\lambda}\sum_{l=1}^{p_1}\overline{\varphi_{lk}(x,\lambda)}\varphi_{lj}(y,\lambda). \tag{4.4}$$

Applying the Stone formula with Proposition 4.3, we obtain

Theorem 4.4

(i) *For each $f(x) = {}^t(f_1(x_1),\ldots,f_p(x_p)) \in \mathcal{H}$, the spectral representation of L is given by*

$$f(x) = \sum_{\mu\in\mathcal{K}}\left[P(\mu)f\right](x) + \frac{1}{\sqrt{2\pi}}\int_0^\infty {}^t\overline{\Psi(x,\pm\lambda)}[\mathcal{F}_\pm f](\lambda)d\lambda, \tag{4.5}$$

where $P(\mu)$ is the orthogonal projection onto the eigenspace corresponding to the eigenvalue μ^2, and

$$\begin{aligned}[\mathcal{F}_\pm f](\lambda) &= \frac{1}{\sqrt{2\pi}}\int_\Gamma \Psi(y,\pm\lambda)f(y)dy \\ &= \frac{1}{\sqrt{2\pi}}\sum_{j=1}^{p}{}^t\Big(\int_{\gamma_j}\varphi_{1j}(y,\pm\lambda)f_j(y)dy,\ldots, \\ &\qquad \int_{\gamma_j}\varphi_{p_1 j}(y,\pm\lambda)f_j(y)dy\Big).\end{aligned} \tag{4.6}$$

(ii) *Put* $P = \sum_{\mu \in \mathcal{K}} P(\mu)$. *Then*

$$\mathcal{F}_\pm^* \mathcal{F}_\pm = I - P, \qquad \mathcal{F}_\pm \mathcal{F}_\pm^* = I_\lambda. \tag{4.7}$$

Here $I = I_x$ *is the identity in configuration space* $\mathcal{H}$, *and* I_λ *is the identity in the momentum space* $\hat{\mathcal{H}}_1 = \{h(\lambda) \in [L^2_\lambda(\mathbf{R}_+)]^{p_1}\}$.

Put $\Gamma = \Gamma_1 \times \Gamma_2$, $\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2$, $E(x,\lambda) = \begin{pmatrix} E_1(x,\lambda) & 0 \\ 0 & E_2(x,\lambda) \end{pmatrix}$, where

$$\Gamma_1 = \gamma_1 \times \cdots \times \gamma_{p_1}, \qquad \Gamma_2 = \gamma_{p_1+1} \times \cdots \times \gamma_p,$$

$$\mathcal{H}_1 = L^2(\Gamma_1), \qquad \mathcal{H}_2 = L^2(\Gamma_2),$$

$$E_1(x,\lambda) = \operatorname{diag}\big(e_j(x,\lambda)\big)_{j=1}^{p_1}, \qquad E_2(x,\lambda) = \operatorname{diag}\big(e_j(x,\lambda)\big)_{j=p_1+1}^{p}.$$

Proposition 4.5

(i) *For* $\lambda \in \mathbf{R}\backslash\{0\}$, $\Phi_1(x,\lambda)$ *is rewritten as*

$$\Phi_1(x,\lambda) = E_1(x,-\lambda) - S(\lambda) E_1(x,\lambda). \tag{4.8}$$

Here $S(\lambda) = (s_{kj}(\lambda))_{k,j=1}^{p_1}$ *is given by*

$$\begin{aligned} s_{kk}(\lambda) &= \frac{e_k(0,-\lambda)}{e_k(0,\lambda)} - \frac{2i\lambda}{e_k^2(0,\lambda)G(\lambda)}, \\ s_{kj}(\lambda) &= \frac{-2i\lambda}{e_k(0,\lambda)e_j(0,\lambda)G(\lambda)} \quad (j \neq k). \end{aligned} \tag{4.9}$$

(ii) $S(\lambda)$ *is a unitary matrix on* $\mathbf{C}^{p_1}$, *and*

$$S(\lambda)^* = \overline{S(\lambda)} = S(-\lambda), \tag{4.10}$$

$$S(-\lambda)\Psi(x,\lambda) = -\Psi(x,-\lambda). \tag{4.11}$$

(iii) *Let* $0 \notin K_G$. *Then* $S(\lambda)$ *is continuous on the whole* $\mathbf{R}$ *and*

$$s_{kj}(0) = -\iota_k \delta_{kj} - \begin{cases} 0, \quad k \text{ or } j \notin \mathbf{P}^0(0), \text{ or } \mathbf{P}^0(0) \cap \mathbf{P}_2 \neq \phi, \\ \dfrac{2i}{\dot{e}_k(0,0)\dot{e}_j(0,0)} \left(\displaystyle\sum_{j \in \mathbf{P}^0(0)} \frac{e_j'(0,0)}{\dot{e}_j(0,0)} \right)^{-1}, \\ \quad k, j \in \mathbf{P}^0(0) \text{ and } \mathbf{P}^0(0) \cap \mathbf{P}_2 = \phi. \end{cases}$$

where $\iota_k = 1$ *if* $e_k(0,0) \neq 0$, *and* $= -1$ *if* $e_k(0,0) = 0$; $\dot{e}_j(0,0) = \frac{\partial e_j(0,\lambda)}{\partial \lambda}|_{\lambda=0}$.

(iv) *Let* $0 \in K_G$. *Then* $S(\lambda)$ *is also continuous on the whole line if each* $q_j(x)$ *is required to satisfy the stronger condition* $\int_{\gamma_j}(1+x)^2|q_j(x)|dx < \infty$, *and in this case we have*

$$s_{kj}(0) = \delta_{kj} - \frac{2i}{e_k(0,0)e_j(0,0)\dot{G}(0)}, \qquad \dot{G}(0) = \left.\frac{\partial G(\lambda)}{\partial \lambda}\right|_{\lambda=0}.$$

5 Møller's Scattering Operator and Its Expression

Let L_0 be the operator L with $q(x) \equiv 0$. Corresponding to L_0, we staff with superscript 0.

We put $P = \sum_{\mu \in \mathcal{K}} P(\mu)$, $P^0 = \sum_{n=1}^{\infty} P^0(n)$. Then

$$\mathcal{F}_{\pm} : (I - P)\mathcal{H} \to \hat{\mathcal{H}}_1, \qquad \mathcal{F}^0 : \big(I - P^0\big)\mathcal{H} \to \hat{\mathcal{H}}_1$$

are both unitary operators. With these operators, we define the stationary wave and scattering operators as follows:

$$U_+ = \mathcal{F}_+^* \mathcal{F}^0, \qquad U_- = -\mathcal{F}_-^* S^0(\lambda)\mathcal{F}^0, \tag{5.1}$$

$$\mathcal{S} = U_+^* U_- = -\mathcal{F}^{0*}\mathcal{F}_+\mathcal{F}_-^* S^0(\lambda)\mathcal{F}^0, \tag{5.2}$$

where $S^0(\lambda) = (s_{kj}^0(\lambda))_{k,j=1}^{p_1}$.

Now consider the Schrödinger evolution operators e^{-itL} and e^{-itL_0}. Choose $f \in (I-P)\mathcal{H}$, $f^0 \in (I - P^0)\mathcal{H}$. Then by use of the spectral representation formulae, we have

$$e^{-itL} f = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} {}^t\overline{\Psi(x, \pm\lambda)} e^{-i\lambda^2 t} [\mathcal{F}_{\pm} f](\lambda) d\lambda, \tag{5.3}$$

$$e^{-itL_0} f^0 = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} {}^t\overline{\Psi^0(x, \lambda)} e^{-i\lambda^2 t} \big[\mathcal{F}^0 f^0\big](\lambda) d\lambda. \tag{5.4}$$

By use of these expressions, we can prove the following theorem

Theorem 5.1

(i) *The Møller wave operator exists and coincides with* $U_{\pm}$:

$$s - \lim_{t \to \pm\infty} e^{itL} e^{-itL_0}\big(I - P^0\big) = U_{\pm}. \tag{5.5}$$

Thus, the Møller scattering operator is given by $\mathcal{S} = U_+^* U_-$.

(ii) *The matrix function* $S(\lambda)$ *represents the scattering operator in the momentum space*:

$$\big[\mathcal{F}_0 \mathcal{S} \mathcal{F}_0^* \hat{f}\big](\lambda) = S(\lambda) S_0(\lambda) \hat{f}(\lambda) \quad \textit{for } \hat{f}(\lambda) \in \hat{\mathcal{H}}_1.$$

6 Inverse Scattering Problem

The triplet of quantities $\{S(\lambda), -\lambda_n^2, M_1(i\lambda_n) :\ n = 1, 2, \ldots, N\}$ with $M_1(i\lambda_n) = -iE_1^{-1}(x, i\lambda_n)\mathrm{Res}_{\lambda=i\lambda_n}\Phi_1(x, \lambda)$ defines the scattering data for the operator L. We investigate the following inverse scattering problem : Given scattering data, recover potential $q(x)$, $x \in \gamma_j$, $j \in \mathbf{P}_1$.

For each $j \in \mathbf{P}_1$, let $K_j(x, y) = K_j(x_j, y_j)$ be the kernel of the transformation operator on γ_j (see (3.2)–(3.3)).

Theorem 6.1 *Each kernel $K_j(x, y)$ satisfies the following integral equation*:

$$F_j(x + y) + K_j(x, y) + \int_x^{\infty} K_j(x, z)F_j(z + y)dz \quad (0 \le x \le y < \infty) \tag{6.1}$$

with

$$F_j(x) = \sum_{n=1}^{N} e^{-\lambda_n x} m_{jj}(i\lambda_n x) + \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x}\big\{s_{0,jj}(\lambda) - s_{jj}(\lambda)\big\}d\lambda. \tag{6.2}$$

Here $S_0(\lambda) = (s_{0,kj}(\lambda))_{k,j=1}^{p_1}$ is the scattering matrix corresponding to the operator with $q(x) \equiv 0$, and the last integral is understood as the Fourier transform of the L^2-function.

Let $F_j(x)$ be constructed by (6.2) in terms of the diagonal entry

$$\big\{s_{jj}(\lambda), -\lambda_n^2, m_{jj}(i\lambda_n)\big\} \tag{6.3}$$

of the scattering data. Then the Marchenko equation (6.1) has a unique solution $K_j(x, y)$ for every $x \ge 0$, and the potential $q_j(x)$ is uniquely recovered on γ_j by the formula

$$-2\frac{d}{dx}K_j(x, x) = q_j(x), \quad x \in \gamma_j,\ j \in \mathbf{P}_1. \tag{6.4}$$

Thus, the knowledge of the scattering data allows us to recover the potential on the infinite rays γ_j, $j \in \mathbf{P}_1$.

Remark 6.2 Similar inverse scattering problem is investigated in the case of graph which consists of a loop $\kappa = \{z \mid 0 < z < 2\pi\}$ and N half lines $\gamma_i = \{x_i \mid 0 < x_i < \infty\}$, $i = 1, \ldots, N$, joined at the points $\{x_i = 0\} = \{z = \alpha_i\}$, where $0 = \alpha_1 < \alpha_2 < \cdots < \alpha_N < 2\pi$. (See [8, 9] for the case $N = 1$. The case $N \ge 2$ will be discussed in forthcoming paper.)

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Integral Transform Approach to the Cauchy Problem for the Evolution Equations

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Abstract In this note we describe some integral transform that allows to write solutions of the Cauchy problem for one partial differential equation via solution of another one. It was suggested by author in J. Differ. Equ. 206:227–252, 2004 in the case when the last equation is a wave equation, and then used in the series of articles (see, e.g., Yagdjian in J. Differ. Equ. 206:227–252, 2004, Yagdjian and Galstian in J. Math. Anal. Appl. 346(2):501–520, 2008, Yagdjian and Galstian in Commun. Math. Phys. 285:293–344, 2009, Yagdjian in Rend. Ist. Mat. Univ. Trieste 42:221–243, 2010, Yagdjian in J. Math. Anal. Appl. 396(1):323–344, 2012, Yagdjian in Commun. Partial Differ. Equ. 37(3):447–478, 2012, Yagdjian in Semilinear Hyperbolic Equations in Curved Spacetimepp, pp. 391–415, 2014 and Yagdjian in J. Math. Phys. 54(9):091503, 2013) to investigate several well-known equations such as Tricomi-type equation, the Klein–Gordon equation in the de Sitter and Einstein–de Sitter spacetimes. The generalization given in this note allows us to consider also evolution equations with x-dependent coefficients.

Keywords Klein–Gordon Equation · Representation of Solutions · Tricomi Equation · De Sitter Spacetime · Fundamental Solution

Mathematics Subject Classification (2010) Primary 47G10 · 35C15 · Secondary 35Q75

1 Introduction

A novel approach to study second order hyperbolic equations with variable coefficients was suggested in [1]. It was used in a series of papers [1–8] to investigate in a unified way several equations such as the linear and semilinear Tricomi and Tricomi-type equations, Gellerstedt equation, the wave equation in Einstein–de Sitter (EdeS) spacetime, the wave and the Klein–Gordon equations in the de Sitter

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and anti-de Sitter spacetimes. The listed equations play an important role in the gas dynamics, elementary particle physics, quantum field theory in curved spaces, and cosmology. For all above mentioned equations, we have obtained among other things, fundamental solutions, representation formulas for the initial-value problem, $L^p - L^q$-estimates, local and global solutions for the semilinear equations, blow up phenomena, sign-changing phenomena, Huygens' principle, self-similar solutions and number of other results.

Consider the solution $v = v(x, t; b)$ to the Cauchy problem

$$\begin{aligned} &v_{tt} - \Delta v = 0, \quad (t, x) \in \mathbb{R}^{1+n}, \\ &v(x, 0; b) = \varphi(x, b), \qquad v_t(x, 0) = 0, \quad x \in \mathbb{R}^n, \end{aligned} \tag{1.1}$$

with the parameter $b \in B \subseteq \mathbb{R}$. Denote that solution by $v_\varphi = v_\varphi(x, t; b)$; if φ is independent of the second variable b, then we write $v_\varphi(x, t)$. There are well-known explicit representation formulas for the solution of the problem (1.1).

The starting point of the approach suggested in [1] is the Duhamel's principle, which is revised in order to prepare the ground for generalization. Our *first observation* is that we obtain the following representation

$$u(x, t) = \int_{t_0}^{t} db \int_0^{t-b} w_f(x, r; b) dr, \tag{1.2}$$

of the solution of the Cauchy problem $u_{tt} - \Delta u = f(x, t)$ in $\mathbb{R}^{n+1}$, and $u(x, t_0) = 0$, $u_t(x, t_0) = 0$ in $\mathbb{R}^n$, where the function $w_f = w_f(x; t; b)$ is the solution of the problem (1.2).

The *second observation* is that in (1.2) the upper limit $t - b$ of the inner integral is generated by the propagation phenomena with the speed which equals to one. In fact, that is a distance function.

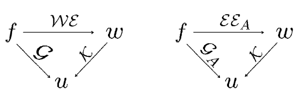
Our *third observation* is that the solution operator $\mathcal{G} : f \longmapsto u$ can be regarded as a composition of two operators. The first one

$$\mathcal{WE} : f \longmapsto w$$

is a Fourier Integral Operator (FIO), which is a solution operator of the Cauchy problem for wave equation in the Minkowski spacetime. The second operator

$$\mathcal{K} : w \longmapsto u$$

is the integral operator given by (1.2). We regard the variable b in (1.2) as a "subsidiary time". Thus, $\mathcal{G} = \mathcal{K} \circ \mathcal{WE}$ and we arrive at the first diagram:



Based on the first diagram, we generated in [4] a class of operators for which we obtained explicit representation formulas for the solutions. That means also that we have obtained representations for the *fundamental solutions of the partial differential operator*. In fact, this diagram brings into a single hierarchy several different partial differential operators. Indeed, if we take into account the propagation cone by introducing the distance function $\phi(t)$, and if we provide the integral operator (1.2) with the kernel $K(t; r, b)$ as follows:

$$\mathcal{K}[w](x,t) = 2\int_{t_0}^{t} db \int_0^{|\phi(t)-\phi(b)|} K(t; r, b) w(x, r; b) dr, \quad x \in \mathbb{R}^n, \ t > t_0, \quad (1.3)$$

then we actually generated new representations for the solutions of different well-known equations with x-independent coefficients. (See, e.g., [4].)

In the present note we extend the class of the equations by allowing the first mapping be a resolving operator for the Cauchy problem for a wider class of equations, which, of course, includes a wave equation itself, but which is not restricted to it. More precisely, consider the second diagram where $w = w_{A,\varphi}(x, t; b)$ is a solution to the Cauchy problem

$$\begin{aligned} &v_{tt} - A(x, \partial_x)v = 0, \qquad v(x, 0; b) = \varphi(x, b), \\ &v_t(x, 0) = 0, \quad t \in \mathbb{R}, x \in \mathbb{R}^n, \end{aligned} \quad (1.4)$$

with the parameter $b \in B \subseteq \mathbb{R}$. If we have a resolving operator of the problem (1.4), by applying (1.3) we can generate new representations for the solutions of different equations. Thus, $\mathcal{G}_A = \mathcal{K} \circ \mathcal{E}\mathcal{E}_A$. The new class contains operators with x-depending coefficients, not necessarily hyperbolic, which allows to obtain more results, especially interesting in the applications.

Example (Tricomi-type equations.) This operator is generated by the kernel $K(t; r, b) = 2E(0, t; r, b)$, where the function $E(x, t; y, b)$ [1] is defined by

$$\begin{aligned} E(x,t;y,b) := c_k\big(\big(\phi(t)+\phi(b)\big)^2 - (x-y)^2\big)^{-\gamma} \\ \times F\left(\gamma, \gamma; 1; \frac{(\phi(t)-\phi(b))^2 - (x-y)^2}{(\phi(t)+\phi(b))^2 - (x-y)^2}\right), \end{aligned} \quad (1.5)$$

with $\gamma := k/(2k+2)$, $c_k = (k+1)^{-k/(k+1)} 2^{-1/(k+1)}$, $k \neq -1$, $k \in \mathbb{R}$, and the distance function is $\phi(t) = t^{k+1}/(k+1)$, while $F(a, b; c; \zeta)$ is the Gauss's hypergeometric function. For the simplicity, in (1.5) we use the notation $x^2 = x \cdot x = |x|^2$ if $x \in \mathbb{R}^n$. We can prove that for the smooth function $f = f(x,t)$, the function

$$u(x,t) = 2\int_0^t db \int_0^{\phi(t)-\phi(b)} E(0, t; r, b) w_{A,f}(x, r; b) dr, \quad t > 0,$$

solves the Tricomi-type equation ($\ell = 2k \in \mathbb{N}$)

$$u_{tt} - t^{\ell} A(x, \partial_x)u = f(x,t) \quad \text{in } \mathbb{R}^{n+1}_+ := \big\{(x,t) \,\big|\, x \in \mathbb{R}^n, \ t > 0\big\},$$

and takes vanishing initial values

$$u(x,0)=0, \qquad u_t(x,0)=0 \quad \text{in } \mathbb{R}^n. \tag{1.6}$$

Example (The wave equation in the Friedmann–Lemaître–Roberson–Walker models: de Sitter spacetime.) In this example $K(t;r,b)=2E(0,t;r,b)$, where the function $E(x,t;y,b)$ [3] is defined by

$$E(x,t;y,b) := \left((e^{-b}+e^{-t})^2-(x-y)^2\right)^{-\frac{1}{2}} \times F\left(\frac{1}{2},\frac{1}{2};1;\frac{(e^{-t}-e^{-b})^2-(x-y)^2}{(e^{-t}+e^{-b})^2-(x-y)^2}\right), \tag{1.7}$$

and $\phi(t):=1-e^{-t}$. We can prove that, defined by the integral transform (1.3) with the kernel (1.7) the function

$$u(x,t)=2\int_0^t db\int_0^{e^{-b}-e^{-t}} \left((e^{-b}+e^{-t})^2-r^2\right)^{-\frac{1}{2}} \times F\left(\frac{1}{2},\frac{1}{2};1;\frac{(e^{-t}-e^{-b})^2-r^2}{(e^{-t}+e^{-b})^2-r^2}\right) w_{A,f}(x,r;b)dr$$

solves the wave equation in the Friedmann–Lemaître–Roberson–Walker (FLRW) space arising in the de Sitter model of the universe,

$$u_{tt}-e^{-2t}A(x,\partial_x)u=f(x,t)$$

in $\mathbb{R}^{n+1}_+$, and takes vanishing initial data (1.6).

Example (The wave equation in the FLRW-models: anti-de Sitter spacetime.) The third example we obtain if we set $K(t;r,b)=2E(0,t;r,b)$, where the function $E(x,t;r,b)$ is defined by (see [2])

$$E(x,t;r,b) := \left((e^{b}+e^{t})^2-(x-r)^2\right)^{-\frac{1}{2}} \times F\left(\frac{1}{2},\frac{1}{2};1;\frac{(e^{t}-e^{b})^2-(x-r)^2}{(e^{t}+e^{b})^2-(x-r)^2}\right), \tag{1.8}$$

while the distance function is $\phi(t):=e^t-1$. In that case the function $u=u(x,t)$ produced by the integral transform (1.3) with $t_0=0$ and the kernel (1.8), solves the wave equation in the FLRW space arising in the anti-de Sitter model of the universe,

$$u_{tt}-e^{2t}A(x,\partial_x)u=f(x,t) \quad \text{in } \mathbb{R}^{n+1}_+.$$

Moreover, it takes vanishing initial values (1.6).

Example (The wave equation in the Einstein–de Sitter spacetime.) If we allow negative $k \in \mathbb{R}$ in (1.5), then we obtain another way to get new operators of the above described hierarchy. In fact, in the hierarchy of the hypergeometric functions $F(a,b;c;\zeta)$ the simplest non-constant function is $F(-1,-1;1;\zeta) = 1+\zeta$. The exponent l leading to $F(-1,-1;1;\zeta)$ is exactly the exponent $l = -4/3$ of the *wave equation* (and of the *metric tensor*) *in the Einstein–de Sitter spacetime*. In that case the kernel is $K(t;r,b) = \frac{1}{18}(9t^{2/3} + 9b^{2/3} - r^2)$. Consequently, the function

$$u(x,t) = \int_0^t db \int_0^{3t^{1/3}-3b^{1/3}} \frac{1}{18}\big((3t^{1/3})^2 + (3b^{1/3})^2 - r^2\big) w_{A,f}(x,r;b)dr,$$

$x \in \mathbb{R}^n$, $t > 0$, solves the equation

$$u_{tt} - t^{-4/3}A(x,\partial_x)u = f \quad \text{in } \mathbb{R}^{n+1}_+,$$

and takes vanishing initial data (1.6) provided that $w_{A,f} = \mathcal{E}\mathcal{E}_A(f)$.

2 The Klein–Gordon Equation in the de Sitter Spacetime

We introduce the following notations. First, we define a *chronological future* $D_+(x_0,t_0)$ and a *chronological past* $D_-(x_0,t_0)$ of the point (x_0,t_0), $x_0 \in \mathbb{R}^n$, $t_0 \in \mathbb{R}$, as follows: $D_\pm(x_0,t_0) := \{(x,t) \in \mathbb{R}^{n+1};\, |x-x_0| \le \pm(e^{-t_0} - e^{-t})\}$. Then, we define for $(x_0,t_0) \in \mathbb{R}^n \times \mathbb{R}$ the function

$$\begin{aligned} E(x,t;x_0,t_0;M) &:= 4^{-M} e^{M(t_0+t)}\big((e^{-t}+e^{-t_0})^2 - (x-x_0)^2\big)^{-\frac{1}{2}+M} \\ &\quad \times F\Big(\frac{1}{2}-M, \frac{1}{2}-M; 1; \frac{(e^{-t_0}-e^{-t})^2 - (x-x_0)^2}{(e^{-t_0}+e^{-t})^2 - (x-x_0)^2}\Big), \end{aligned}$$

in $D_+(x_0,t_0) \cup D_-(x_0,t_0)$, where $F(a,b;c;\zeta)$ is the hypergeometric function. The kernels $K_0(z,t)$, $K_1(z,t)$, $K_0(z,t;M)$, and $K_0(z,t;M)$ are defined by

$$\begin{aligned} &K_0(z,t;M) \\ &:= 4^{-M} e^{tM}\big((1+e^{-t})^2 - z^2\big)^M \frac{1}{[(1-e^{-t})^2 - z^2]\sqrt{(1+e^{-t})^2 - z^2}} \\ &\quad \times \Big[\big(e^{-t} - 1 + M(e^{-2t} - 1 - z^2)\big) F\Big(\frac{1}{2}-M, \frac{1}{2}-M; 1; \frac{(1-e^{-t})^2 - z^2}{(1+e^{-t})^2 - z^2}\Big) \\ &\quad + \big(1 - e^{-2t} + z^2\big)\Big(\frac{1}{2}+M\Big) F\Big(-\frac{1}{2}-M, \frac{1}{2}-M; 1; \frac{(1-e^{-t})^2 - z^2}{(1+e^{-t})^2 - z^2}\Big)\Big], \end{aligned}$$

$$K_1(z,t;M) := 4^{-M} e^{Mt}\big((1+e^{-t})^2 - z^2\big)^{-\frac{1}{2}+M}$$

$$\times F\left(\frac{1}{2}-M, \frac{1}{2}-M; 1; \frac{(1-e^{-t})^2-z^2}{(1+e^{-t})^2-z^2}\right).$$

In the following theorem, the equation with mass $M \in \mathbb{C}$, in the case of second order elliptic operator A, can be derived from the covariant Klein–Gordon equation by simple change of the unknown function (see, e.g., [3]). In order to avoid more specific description of the order of the operator $A(x, \partial_x)$ and regularity of its coefficients, the smoothness of all functions in the theorem is set frightfully redundant. In general, it can be relaxed to the necessary smoothness. Let δ_{ik} be the Kronecker symbol.

Theorem 2.1 *Assume that* $f \in C^\infty(\mathbb{R}^{n+1}_+)$, $\varphi_0, \varphi_1 \in C_0^\infty(\mathbb{R}^n)$, *and that the function* $v_{A,f}(x,t;b) \in C^\infty(\mathbb{R}^{n+2}_{++})$ *is a solution to the Cauchy problem*

$$v_{tt} - A(x, \partial_x)v = 0, \qquad v(x,0;b) = f(x,b), \qquad v_t(x,0;b) = 0, \quad t > 0,$$

while $v_{A,\varphi_k}(x,t) \in C^\infty(\mathbb{R}^{n+1}_+)$, *is a solutions of the problem*

$$v_{tt} - A(x, \partial_x)v = 0, \qquad v(x,0) = \delta_{0k}\varphi_0(x), \qquad v_t(x,0) = \delta_{1k}\varphi_1(x),$$

$k = 0, 1$. *Then the function* $u = u(x,t)$ *given by the integral transform*

$$\begin{aligned} u(x,t) &= 2\int_0^t db \int_0^{e^{-b}-e^{-t}} E(0,t;r,b;M) v_{A,f}(x,r;b)dr \\ &\quad + e^{\frac{t}{2}} v_{A,\varphi_0}\big(x, 1-e^{-t}\big) + 2\int_0^{1-e^{-t}} K_0(s,t;M) v_{A,\varphi_0}(x,s)ds \\ &\quad + 2\int_0^{1-e^{-t}} K_1(s,t;M) v_{A,\varphi_1}(x,s)ds, \quad x \in \mathbb{R}^n,\ t > 0, \end{aligned} \tag{2.1}$$

solves the Cauchy problem

$$\begin{aligned} &u_{tt} - e^{-2t}A(x,\partial_x)u - M^2 u = f, \\ &u(x,0) = \varphi_0(x), \qquad u_t(x,0) = \varphi_1(x). \end{aligned} \tag{2.2}$$

Outline of the proof The proof Theorem 2.1 is straightforward; we just substitute the function $u = u(t,x)$ of (2.1) into (2.2) and then check the initial conditions. It is straightforward, but not short; the proof contains very long calculations. The full proof of the results presented in the present note will be published soon later. □

Some Applications We mention here the $L^p - L^q$ estimates, Strichartz estimates, Huygens' principle, global and local existence theorem for semilinear and quasilinear equations. The applications of the above presented method to the second-order hyperbolic equations with x-independent coefficients one can find in [1–8]. Below we give examples of the equations with x-dependent coefficients those are amenable to the integral transform method.

Example The metric g in the Friedmann–Lemaître–Roberson–Walker spacetime $g_{00} = g^{00} = -1$, $g_{0j} = g^{0j} = 0$, $g_{ij} = a^2(t)\delta_{ij}(x)$, $|g| = a^{2n}(t)|\det\delta(x)|$, $g^{ij} = a^{-2}(t)\delta^{ij}(x)$, $i, j = 1, 2, \ldots, n$, where $\delta^{ij}(x)\delta_{jk}(x) = \delta_{ik}$, $a(t) = e^t$. The linear covariant Klein–Gordon equation in the coordinates is

$$\psi_{tt} - \frac{e^{-2t}}{\sqrt{|\det\delta(x)|}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i}\left(\sqrt{|\det\delta(x)|}\delta^{ij}(x)\frac{\partial}{\partial x^j}\psi\right)$$
$$+ n\psi_t + m^2\psi = f.$$

This example includes also equations in the FLRW metric with hyperbolic or spherical spatial geometry.

Example The degenerating at $t = 0$ equation

$$\psi_{tt} + t^\ell \sum_{i,j=1}^{n} a^{ij}(x)\frac{\partial^2}{\partial x^i \partial x^j}\psi + t^\ell \sum_{i=1}^{n} b^i(x)\frac{\partial}{\partial x^i}\psi = f$$

that is elliptic in the domain with $t^\ell > 0$. Here $\sum_{i,j=1}^n a^{ij}(x)\xi_i\xi_j \geq 0$.

Example The Euler–Bernoulli beam equation with the variable coefficients

$$\psi_{tt} + a^2(t) \sum_{i,j=1}^{n} a^{ij}(x)\partial_{x_i}^2\partial_{x_j}^2\psi = f,$$

where $a(t)$ is one of the following functions t^ℓ, e^t, or e^{-t}. Here $\sum_{i,j=1}^n a^{ij}(x)\xi_i\xi_j \geq 0$.

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Part V
Nonlinear PDE and Control Theory

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On Some Solutions of Certain Versions of "Sigma" Model and Some Skyrme-Like Models

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Abstract Some results concerning certain versions of "sigma" model and some Skyrme-like models, are presented.

Keywords "Sigma" model · $O(3)$ model · Baby Skyrme model · Skyrme–Faddeev model · Bogomolny decomposition

Mathematics Subject Classification (2010) Primary 35Q51 · 37K10 · Secondary 37K40

1 Motivation

The nonlinear "sigma" (σ) $O(3)$ model was born by an inspiration in the late 50's of the 20th Century, by the development of high-energy physics, [9]. In [47] stable soliton solutions for this model have been found. Some new class of $O(3)$ models was studied in [49]. Its static (time independent) version describes Heisenberg ferromagnet (static Heisenberg model), [50].

Baby Skyrme model is another very interesting model. It is an analogical model on plane, to Skyrme model, which lagrangian includes, apart from the term of nonlinear $O(3)$ sigma model also the quartic term—so-called Skyrme term, it is necessary in order to overcome Derrick–Hobart theorem and the potential. Skyrme model possesses solitonic solutions, useful for describing phenomena in world of baryons, it provides good description of low-energy physics of strong interactions,

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[26]. The target space of Skyrme model is $SU(2)$, [30–32], and the target space of baby Skyrme model is S^2. In these both models: Skyrme and baby Skyrme, static field configurations can be classified topologically by their winding numbers. The presence of the potential in the case of static field configurations with finite energy, in baby Skyrme model, is necessary. However, the form of this potential—not restricted, its different forms were investigated in [13, 24, 25, 29] and [1, 2, 22]. An application of this model is the description of the quantum Hall efect, [3, 39, 48]. So-called spinning baby Skyrmions in the restricted baby Skyrme model, were investigated in [19]. In [2, 18], Bogomolny equations have been derived for some special cases of the potential in restricted (without $O(3)$ term) baby Skyrme model (some more general results have been obtained in [40] and [45]). Some other interesting Skyrme-like model is so called Skyrme–Faddeev model, [14, 20] (and other bibliographical entries in [15]). In [15] some exact vortex solutions of CP^N version of this model have been found. In [16] exact vortex solutions for extended $CP4^N$ Skyrme–Faddeev model, have been found and investigated. In this paper we present some results concerning the models: extended CP^N Skyrme–Faddeev, nonlinear "sigma" (σ) $O(3)$ and restricted baby Skyrme one.

2 Decomposition (Non-Bogomolny) Method and Some Solutions

The main idea of this method was published in [43, 44]. Namely, we assume a possibility of a decomposition of an investigated NPDE for several smaller "pieces", which are characterized by **a homogenity of the derivatives** of the unknown function u (if it is possible). Next, we equal them to zero, and try to solve. The exact meaning of the property "homogenity of the derivatives" was explained in [43, 44]. Here we say only that if given NPDE possesses this property, then we can change the problem of solving of NPDE in the problem of solving of some system of nonlinear algebraic equations. In this version of this method, the absence of free term (as a constant term or a function $g(u)$—not multiplied by these derivatives) is required. We use an ansatz:

$$u(x^\mu) \equiv \omega(x^\mu) = \beta_1 + f\left(a_\mu x^\mu + \beta_2, b_\mu x^\mu + \beta_3, c_\mu x^\mu + \beta_4, d_\mu x^\mu + \beta_5\right), \quad (2.1)$$

where: $f \in \mathcal{C}^2$ and we work in Minkowski space-time with the signature: $(-, +, +, +)$. The fourth argument of f: $d_\mu x^\mu + \beta_5$, has been absent in [43, 44].

If we seek localized solutions, then we must assume asymptotic boundary conditions: $\lim_{x\to\pm\infty} f = \text{const}$ and analogical condition for energy density. We wish the function was *arbitrary*. Such solutions are **functionally invariant solutions**. Some methods for obtaining of such solutions and some special classes of them, were published in: [6, 7, 10–12, 17, 27, 28, 33, 34, 41]. The decomposition method, described in [43, 44], is just some method for finding such solutions (they were called there as "classes of solutions"). Also enough general functionally invariant solutions of some nonlinear equations, have been found in [43, 44].

After applying decomposition method to the given equation(s), we have to solve the corresponding algebraic system. The solutions of it, are the values of the coefficients $a_\mu, b_\mu, \ldots$. The solutions of this system determine the shape of dependance of the function f on x^μ, $\mu = 0, 1, 2, 3$.

In [15] and [16], extended Skyrme–Faddeev model, and its CP^N version (in (3 + 1)-dimensions), correspondingly, have been investigated and so-called vortex solutions, of the form:

$$\omega(x, y, z, t) = f(x + i\varepsilon_1 y, t + \varepsilon_2 z), \quad f \in \mathcal{C}, \ \varepsilon_k = \pm 1 \tag{2.2}$$

have been found and studied. These solutions have been found by solving the equations: $\partial_\mu \partial^\mu \omega = 0$, $\partial_\mu \omega \partial^\mu \omega = 0$, $\omega \in \mathbb{C}$, (of course, ω is twice differentiable function in complex sense). Owing to it, these solutions are simultaneously the solutions of: “sigma” $O(3)$ model (CP^1 model) and extended CP^N Skyrme–Faddeev model. Namely, one can see, after looking at the idea of the decomposition method, that it is possible owing to the fact that the field equations of these models are homogeneous with respect to the derivatives of the unknown function u and this is the condition for applying of decomposition method. After applying it, three sets of the functionally invariant solutions of the models: the models: nonlinear “sigma” $O(3)$ (CP^1) model and extended Skyrme–Faddeev one, in (3 + 1)-dimensions, have been found. These solutions are established by the values of the coefficients $a_\mu, b_\mu, c_\mu, d_\mu, \mu = 0, \ldots, 3$, together with the ansatz (2.1). We present here three sets of the values of the coefficients:

1. $a_0 = a_2$, $a_3 = -ia_1$, $b_0 = b_2$, $b_1 = ib_3$, $c_0 = c_2$, $c_1 = ic_3$ and a_1, a_2, b_2, b_3, c_2, c_3—arbitrary (these values of the coefficients have been published in [43, 44] (the first paper) and the coefficients d_j, $j = 0, \ldots, 3$, have been absent there),
2. $a_0 = -a_1$, $a_3 = -ia_2$, $b_0 = i\frac{b_2 d_1}{d_3}$, $b_1 = -i\frac{b_2 d_1}{d_3}$, $b_3 = -ib_2$, $c_0 = -c_1$, $c_2 = ic_3$, $d_0 = -d_1$, $d_2 = id_3$ and a_1, a_2, b_2, c_1, c_3, d_1—arbitrary and $d_3 \neq 0$
3. $a_0 = \frac{a_1 d_1 + a_2 d_2}{\sqrt{d_1^2 + d_2^2}}$, $a_3 = i\frac{a_1 d_2 - a_2 d_1}{\sqrt{d_1^2 + d_2^2}}$, $b_0 = \frac{b_1 \sqrt{d_1^2 + d_2^2}}{d_1}$, $b_2 = \frac{b_1 d_2}{d_1}$, $b_3 = 0$, $c_0 = \frac{c_1 \sqrt{d_1^2 + d_2^2}}{d_1}$, $c_2 = \frac{c_1 d_2}{d_1}$, $c_3 = 0$, $d_0 = \sqrt{d_1^2 + d_2^2}$, $d_3 = 0$ and a_1, a_2, b_1, c_1, d_2—arbitrary and $d_1 \neq 0$, $d_1 \neq \pm id_2$.

If we look now at the first set, we can notice that after putting $a_1 = 0$, $a_2 = 1$, $b_2 = 0$, $b_3 = 1$, $c_2 = 0$, $c_3 = 0$, we get a special case of the solution, established by the ansatz (2.1) and the first set of the values of the coefficients (of course, $\beta_k = 0$, $k = 1, \ldots, 5$):

$$\omega(x, y, z, t) = f(t + y, ix + z) \tag{2.3}$$

Hence, the solutions (2.2), with $\varepsilon_n = 1$, $n = 1, 2$, obtained in [16], possess very similar form to the solutions, which are special case of some solutions found in [43, 44] (the first paper) and presented here in point 1 (for $a_1 = 0$, $a_2 = 1$, $b_2 = 0$, $b_3 = 1$, $c_2 = 0$, $c_3 = 0$, $\beta_k = 0$, $k = 1, \ldots, 5$).

Besides, if we look at the ansatz (2.1) and at these three sets of the values of the coefficients, we notice that we can talk about superposition of the solutions for the models: nonlinear "sigma" $O(3)$ (CP^1) model and extended Skyrme–Faddeev one, in $(3+1)$-dimensions. We can see that there are two kinds of these superpositions:

1. I kind—i.e. each component solution is completely independent on the others
2. II kind—in this case, by using a coefficient(s) of one of the superposed solutions, we can change the values of independent variables in the others superposed solutions, especially, by putting a zero coefficient in one of superposed solutions, we can make the other solution as time independent.

3 Bogomolny Equations

3.1 Classical Approach

This is well-known fact that Euler–Lagrange equations of many models in physics are nonlinear partial differential equations of second order. However, in [8] Bogomolny derived the equations, called as Bogomolny equations (sometimes also, as Bogomol'nyi equations), although historically, they were derived earlier in [4], for another model—$SU(2)$ Yang–Mills theory. Similar problem was considered by Hosoya in [21], Hosoya's paper was cited in [5]. Bogomolny made it for scalar field theory so-called ϕ^4 model, with spontaneous symmetry breaking. The energy functional of this model is $E=\int_{-\infty}^{\infty}(\frac{1}{2}(\frac{d\phi}{dx})^2+\frac{\lambda}{2}(\phi^2-\gamma^2)^2)dx$, $\phi(x)\in\mathbb{R}$, $\lim_{x\to\pm\infty}\phi(x)=\pm\gamma$ and Euler–Lagrange equations for this model are $\frac{d^2\phi}{dx^2}=2\lambda\phi(\phi^2-\gamma^2)$. Bogomolny showed that by proper writing down the energy functional of this model, one can avoid solving of Euler–Lagrange equations, namely: $E=\int_{-\infty}^{\infty}(\frac{1}{2}(\frac{d\phi}{dx}+\sqrt{\lambda}(\phi^2-\gamma^2))^2-\sqrt{\lambda}\frac{d\phi}{dx}(\phi^2-\gamma^2))dx$. Next, we integrate the second (non-quadratic) term in this functional and we require reaching the minimum by the functional. Hence, the following equation must be satisfied: $\frac{d\phi}{dx}=\sqrt{\lambda}(\gamma^2-\phi^2)$ (Bogomolny equation) and the following inequality (Bogomolny bound) is satisfied $E\geq\frac{2\sqrt{\lambda}}{3}\gamma^2|Q|$, where $Q=\phi(\infty)-\phi(-\infty)$ is a topological charge. The very well-known solution of this Bogomolny equation is so called "kink" $\phi(x)=\gamma\tanh(\gamma\sqrt{\lambda}(x-x_0))$, obtained in [8].

3.2 Variational Approach—the Concept of Strong Necessary Conditions

From the extremum principle, applied to the functional $\Phi[u]=\int_{E^2}F(u,u_{,x},u_{,t})dxdt$, follow the Euler–Lagrange equations

$$F_{,u}-\frac{d}{dx}F_{,u_{,x}}-\frac{d}{dt}F_{,u_{,t}}=0, \tag{3.1}$$

Instead of (3.1) we consider strong necessary conditions, we call them here shortly as SNC (this method was first time described in [35] and it was developed later in [36, 37])

$$F_{,u} = 0, \qquad F_{,u_{,t}} = 0, \qquad F_{,u_{,x}} = 0, \tag{3.2}$$

where $F_{,u} \equiv \frac{\partial F}{\partial u}$, etc. Obviously, all solutions of the system of (3.2) satisfy the Euler–Lagrange equation (3.1), but these solutions, if they exist, are very often trivial. We may avoid it by making gauge transformation of the functional $\Phi[u]$:

$$\Phi \to \Phi + Inv, \tag{3.3}$$

where *Inv* is such functional that its local variation with respect to $u(x,t)$ vanishes: $\delta Inv \equiv 0 \Longrightarrow$ E.-L. equations are **invariant** with respect to the gauge transformation (3.3). **Non-invariance** of the SNC (3.2), with respect to the gauge transformation (3.3) $\Longrightarrow$ some non-trivial solutions are possible. Now, we apply the SNC (3.2) to the gauged functional: $\tilde{\Phi} = \Phi + Inv$. We obtain so called *dual equations*. Some early version of the variational approach for deriving Bogomolny equations was applied in [23]. The application of SNC for deriving Bogomolny equations (decomposition) was later developed in [38, 42, 46].

3.2.1 An Example: Bogomolny Decomposition for Ungauged Restricted Baby Skyrme Model in (2 + 0)-Dimensions

We consider the energy functional for restricted baby Skyrme model in (2 + 0) dimensions (the static σ term is absent), of the following form, [2, 45]: $H = \frac{1}{2}\int d^2x\mathcal{H} = \frac{1}{2}\int d^2x(\frac{\beta}{4}(\epsilon_{ij}\partial_i\vec{S} \times \partial_j\vec{S})^2 + \gamma^2 V(\vec{S}))$, where $|\vec{S}|^2 = 1$ and we assume **nothing** about the form of the potential V (of course, $V \in \mathcal{C}$). After making stereographic projection: $\vec{S} = [\frac{\omega+\omega^*}{1+\omega\omega^*}, \frac{-i(\omega-\omega^*)}{1+\omega\omega^*}, \frac{1-\omega\omega^*}{1+\omega\omega^*}]$, where $\omega = \omega(x,y) \in \mathbb{C}$ and $x, y \in \mathbb{R}$, the density of the energy functional H has the form, [45]: $\mathcal{H} = -4\beta\frac{(\omega_{,x}\omega^*_{,y} - \omega_{,y}\omega^*_{,x})^2}{(1+\omega\omega^*)^4} + V(\omega,\omega^*)$. Now, we make gauge transformation on the invariants, [45]: $\sum_{k=1}^{3} I_k$, where I_k are the densities of the invariants: $I_1 = G_1 \cdot (\omega_{,x}\omega^*_{,y} - \omega_{,y}\omega^*_{,x})$ is the density of topological invariant, $I_2 = D_x G_2$, $I_3 = D_y G_3$, $D_x \equiv \frac{d}{dx}$, $D_y \equiv \frac{d}{dy}$, $\omega = \omega(x,y)$, $\omega^* = \omega^*(x,y) \in \mathcal{C}^2$ and $G_k = G_k(\omega,\omega^*) \in \mathcal{C}^2$ $(k = 1, 2, 3)$, are some functions, which are to be determinated.

If we apply the concept of strong necessary conditions to $\tilde{\mathcal{H}}$, we get six dual equations [45]. One can see their form in [45]. We say here only that all these equations include the derivatives of ω and ω^* with respect to x and y, and additionally, two of them include the derivatives of the potential V, with respect to ω and ω^* (these last equations, we call here shortly as "A" equations and other dual equations as "B" equations). In order to derive Bogomolny equations, making these equations self-consistent is necessary. So, there is the necessity of the reduction of the number of independent equations by an appropriate choice of the functions G_k $(k = 1, 2, 3)$. Usually, such ansatzes exist only for some special $V(\omega,\omega^*)$, i.e. in most cases of $V(\omega, \omega^*)$, for many nonlinear field models, the reduction of the system of corresponding dual equations, to Bogomolny equations, is impossible. The appropriate reduction of this system of dual equations, has been done in [45], for $G_2(\omega, \omega^*) = \text{const}$, $G_3(\omega, \omega^*) = \text{const}$. Hence, after inserting these relations into the "B" equations, we get, [45]:

$$\omega_{,x}\omega^*_{,y} - \omega_{,y}\omega^*_{,x} = \frac{1}{8\beta} G_1\big(\omega, \omega^*\big)\big(1 + \omega\omega^*\big)^4. \tag{3.4}$$

Hence, all solutions of (3.4) satisfy the "B" equations. Now, we make the "A" equations, consistent with the other dual equation: after eliminating $\omega_{,x}$, $\omega_{,y}$, $\omega^*_{,x}$, $\omega^*_{,y}$, from "A" equations and integrating obtained equations, we get the condition for the potential:

$$V\big(\omega, \omega^*\big) = -\frac{1}{16\beta} G_1^2\big(\omega, \omega^*\big)\big(1 + \omega\omega^*\big)^4 + C, \quad C = \text{const}. \tag{3.5}$$

This result (without integrating constant C) was obtained in [45], but by using Hamilton–Jacobi equation. In this current paper we have not applied Hamilton–Jacobi equation. Then of course, $G_1 = \frac{4\sqrt{\beta}}{(1+\omega\omega^*)^2}\sqrt{C - V(\omega, \omega^*)}$. For some simplicity, we put here $C = 0$. We insert this relation into (3.4) and we obtain Bogomolny decomposition for the given potential $V(w, w^*)$, [45]: $\omega_{,x}\omega^*_{,y} - \omega_{,y}\omega^*_{,x} = \frac{i}{2\sqrt{\beta}}\sqrt{V(\omega, \omega^*)}(1 + \omega\omega^*)^2$. Then, this last equation is Bogomolny decomposition (Bogomolny equation) for restricted baby Skyrme model in $(2 + 0)$ dimensions, for *arbitrary* potential. Because of the limit of number of pages, we present here only the exact form of the solution of this Bogomolny equation, (here $u = \Re(\omega)$, $v = \Im(\omega)$ and $F_1(x)$, $F_2(x)$—are arbitrary functions), which we have found for $V(u, v) = \frac{1}{(1+u^2+v^2)^4}$:

$$u(x, y) = F_1(x), \qquad v(x, y) = -\frac{1}{4\sqrt{\beta}} \frac{y}{\frac{d}{dx} F_1(x)} + F_2(x) \tag{3.6}$$

4 Summary

We have found some exact functionally invariant solutions for the models: "sigma" $O(3)$ and extended CP^N Skyrme–Faddeev one, in $(3 + 1)$-dimensions, by applying decomposition (non-Bogomolny) method. One of special cases of the solutions found in [43, 44] (the first paper) and presented also here, are very similar to the vortex solutions, found in (2.2). Hence, one may say that one of the solutions, obtained in this paper, as the special case of the solutions found in [43, 44] (the first paper), are also vortex solutions. Besides, the general form of these solutions satisfies some superposition principles. We can say about two kinds of these superpositions—I kind, when each component solution is quite independent on the others and II kind—in this case, by using a coefficient(s) of one of the superposed solutions, we have an impact on the values of independent variables in the others superposed solutions, especially, by putting a zero coefficient in one of superposed solutions, we can make the other solution as time independent.

Besides, we have derived for ungauged restricted baby Skyrme model in $(2+0)$-dimensions, the condition for the potential, necessary for exitence of Bogomolny decomposition (Bogomolny equations) for this model, without applying Hamilton–Jacobi equation. We have found also some exact solution of Bogomolny decomposition (Bogomolny equations). Further investigation of obtained results, is in progress.

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Sharp Sobolev–Strichartz Estimates for the Free Schrödinger Propagator

Neal Bez, Chris Jeavons, and Nikolaos Pattakos

Abstract We consider gaussian extremisability of sharp linear Sobolev–Strichartz estimates and closely related sharp bilinear Ozawa–Tsutsumi estimates for the free Schrödinger equation.

Keywords Sobolev–Strichartz estimates · Extremisers · Schrödinger equation

Mathematics Subject Classification (2010) Primary 35B45 · Secondary 35Q40

1 Introduction

For $d \geq 1$ and $s \in [0, \frac{d}{2})$, it is well-known that the solution $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ of the free Schrödinger equation

$$i\partial_t u + \Delta u = 0, \qquad u(0) = u_0 \in \dot{H}^s\left(\mathbb{R}^d\right) \tag{1.1}$$

satisfies the global space-time estimate

$$\|u\|_{L^{p(d,s)}} \leq \mathbf{S}(d, s)\|u_0\|_{\dot{H}^s} \tag{1.2}$$

for some finite constant $\mathbf{S}(d, s)$, which we assume to be the optimal (i.e. smallest) such constant. Here, $p(d, s) = \frac{2(d+2)}{d-2s}$ and, as usual, $\dot{H}^s(\mathbb{R}^d)$ denotes the homogeneous Sobolev space with norm $\|f\|_{\dot{H}^s} = \|(-\Delta)^{s/2} f\|_{L^2}$. This article will be

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concerned with optimal constants and extremisers, and we note immediately that $\mathbf{S}(d,s)$ and the shape of corresponding extremisers are only known in the rather special cases $(d,s) \in \{(1,0),(2,0)\}$ (see Foschi [5] and also Hundertmark–Zharnitsky [6]). In such cases, the isotropic gaussian initial data $u_0(x) = \exp(-|x|^2)$ is an extremiser.

Very closely related are the sharp bilinear estimates

$$\left\|(-\Delta)^{\frac{2-d}{4}}(u\overline{v})\right\|_{L^2} \leq \mathbf{OT}(d)\|u_0\|_{L^2}\|v_0\|_{L^2} \tag{1.3}$$

due to Ozawa–Tsutsumi [7], where $d \geq 2$ and

$$\mathbf{OT}(d) = \Gamma\left(\tfrac{d}{2}\right)^{-\frac{1}{2}} 2^{-\frac{d}{2}} \pi^{\frac{2-d}{4}}.$$

In (1.3), u and v are solutions of (1.1) with square-integrable initial data u_0 and v_0. The constant $\mathbf{OT}(d)$ is optimal and (u_0, v_0) is an extremising pair when $u_0(x) = v_0(x) = \exp(-|x|^2)$.

The sharp estimate (1.3) was motivated by the case of one spatial dimension, in which case (1.3) is an identity

$$\left\|(-\Delta)^{\frac{1}{4}}(u\overline{v})\right\|_{L^2} = \mathbf{OT}(1)\|u_0\|_{L^2}\|v_0\|_{L^2}. \tag{1.4}$$

This basic tool was established and used in [7] to prove local-wellposedness for certain nonlinear Schrödinger equations with nonlinearities including $\partial(|u|^2)u$ and initial data in $H^{1/2}$. Thus, (1.4) gives control on the null gauge form $\partial(u\overline{v})$ for the one-dimensional Schrödinger equation, and (1.3) gives estimates for the null gauge form in higher dimensions.

We remark that taking $d = 2$ and $u_0 = v_0$ in (1.3) immediately yields the optimal constant $\mathbf{S}(2,0)$ and its gaussian extremisability (this was not explicitly observed in [7]). The approach in [7] is different to the approaches in [5] and [6], so this provides an alternative derivation of this optimal constant.

As far as we know, for the Sobolev–Strichartz estimate (1.1), no conjecture has been made on the shape of extremising initial data in the case where s is *strictly positive*. Extremising initial data certainly exist for all admissible $d \geq 1$ and $s \in [0, \frac{d}{2})$ (see, for example, [8]). In this direction, our first observation is the following.

Theorem 1.1 *Only if $s = 0$ are gaussians u_0 such that*

$$\widehat{u_0}(\xi) = \exp\bigl(a|\xi|^2 + ib\cdot\xi + c\bigr)$$

for some $a, c \in \mathbb{C}$, $b \in \mathbb{R}^d$, $\mathrm{Re}(a) < 0$, *critical points for the functional*

$$u_0 \mapsto \frac{\|e^{it\Delta}u_0\|_{L^{p(d,s)}}}{\|u_0\|_{\dot{H}^s}}. \tag{1.5}$$

Theorem 1.1 of course implies that gaussians are not amongst the class of extremisers for (1.2) for any admissible s which are *strictly positive*; i.e. $s \in (0, \frac{d}{2})$.

We find this outcome particularly interesting when measured against the analogous sharp estimates for the wave propagator $e^{it\sqrt{-\Delta}}$. Here, it is known that if $d \geq 2$ and $s \in [0, \frac{d-1}{2})$ then

$$\|u\|_{L^{p(d-1,s)}} \leq \mathbf{W}(d, s)\|u_0\|_{\dot{H}^{s+\frac{1}{2}}} \tag{1.6}$$

for all solutions of the (pseudo) wave equation

$$i\partial_t u + \sqrt{-\Delta}u = 0, \qquad u(0) = u_0 \in \dot{H}^{s+\frac{1}{2}}(\mathbb{R}^d).$$

Again, we take $\mathbf{W}(d, s)$ to be the optimal constant, which is finite for the given range of parameters (d, s). It is known that initial data u_0 for which

$$\widehat{u}_0(\xi) = |\xi|^{-1} \exp(-|\xi|) \tag{1.7}$$

are extremisers for (1.6) when $(d, s) \in \{(2, 0), (3, 0), (5, \frac{1}{2})\}$ (uniquely, up to certain transformations). The cases $(2, 0)$ and $(3, 0)$ were established by Foschi [5] and the case $(5, \frac{1}{2})$ was established in [2]. Thus, u_0 satisfying (1.7) is an extremiser for $\mathbf{W}(d, s)$ *for two distinct values of* s. Theorem 1.1 shows that this phenomena does not occur for $\mathbf{S}(d, s)$ and gaussian u_0.

The condition $s = 0$ is, in fact, necessary and sufficient for gaussians to be critical points for the functional in (1.5). The sufficiency part was demonstrated by Hundertmark–Zharnitsky [6]. See also the work of Christ–Quilodrán [3] where a closely related result to Theorem 1.1 was established in the context of adjoint Fourier restriction estimates for the paraboloid; in fact, we prove Theorem 1.1 by making small modifications to their argument.

For the cases $(d, s) \in \{(1, 0), (2, 0)\}$, it is known that the isotropic gaussian $u_0(x) = \exp(-|x|^2)$ is, up to certain transformations, the *only* extremising initial data for (1.1); see [5, 6]. Furthermore, it is conjectured ([5, 6]) that gaussians are the only extremisers for $\mathbf{S}(d, 0)$ *for all* $d \geq 1$. Providing a full characterisation of the set of extremisers often requires delicate arguments, and applications of sharp estimates frequently demand that such a characterisation is established. For example, recent work of Duyckaerts–Merle–Roudenko [4] considered extremisers for the global Strichartz norm for solutions of the mass-critical nonlinear Schrödinger equation

$$i\partial_t u + \Delta u + \gamma|u|^{\frac{4}{d}} u = 0, \qquad u(0) = u_0 \in L^2(\mathbb{R}^d),$$

where $\gamma = 1$ in the focusing case and $\gamma = -1$ in the defocusing case. In particular, it was shown in [4] that for $\delta > 0$ sufficiently small,

$$\mathfrak{I}(\delta) := \sup_{\|u_0\|_2=\delta} \|u\|_{L^{p(d,0)}}$$

is attained for some initial data $u_0(\delta) \in L^2(\mathbb{R}^d)$ with $\|u_0(\delta)\|_2 = \delta$. When $d = 1, 2$ they prove significantly more; it is shown that, as $\delta \to 0$,

$$\mathfrak{I}(\delta) = \mathbf{S}(d, 0)\delta + \gamma\Lambda(d)\delta^{1+\frac{4}{d}} + O(\delta^{1+\frac{8}{d}}),$$

where $\Lambda(d)$ is some positive constant, and that any extremising initial data $u_0(\delta)$ is, for δ sufficiently small and up to certain transformations, close to δG_0, where G_0 is the isotropic centred gaussian which has been L^2-normalised. For this additional information concerning $\mathfrak{I}(\delta)$ when $d = 1, 2$, it was vital to know a full characterisation of the extremisers for $\mathbf{S}(d, 0)$.

Our next result establishes a full characterisation of extremisers for the bilinear Sobolev–Strichartz estimate (1.3) of Ozawa–Tsutsumi; this question was left open in [7] and the following theorem says that extremisers for (1.3) must be isotropic centred gaussians, up to certain transformations.

Theorem 1.2 *Let $d \geq 2$. We have equality in the estimate (1.3) if and only if there exist $a, c, d \in \mathbb{C}$, $b \in \mathbb{C}^d$, with $\mathrm{Re}(a) < 0$, so that*

$$u_0(x) = \exp\bigl(a|x|^2 + b \cdot x + c\bigr), \qquad v_0(x) = \exp\bigl(a|x|^2 + b \cdot x + d\bigr). \tag{1.8}$$

2 Further Remarks and Proofs

For $d \geq 1$, $q \in (1, \frac{2(d+1)}{d})$ and $p = \frac{(d+2)q'}{d}$, it was shown in [3] that gaussians are critical points for the $L^q(\mathbb{P}^d, \mathrm{d}\sigma) \to L^p(\mathbb{R}^{d+1})$ adjoint Fourier restriction estimates associated to the paraboloid

$$\mathbb{P}^d = \bigl\{\bigl(\xi, |\xi|^2\bigr) : \xi \in \mathbb{R}^d\bigr\} \subset \mathbb{R}^{d+1}$$

if and only if $q = 2$. Here, $\mathrm{d}\sigma$ is the measure supported on $\mathbb{P}^d$ given by $\int_{\mathbb{P}^d} F \,\mathrm{d}\sigma := \int_{\mathbb{R}^d} F(\xi, |\xi|^2)\,\mathrm{d}\xi$ and $\mathrm{d}\xi$ is Lebesgue measure on $\mathbb{R}^d$. A mixed-norm generalisation of this is also established in [3] and we remark that Theorem 1.1 may also be extended by measuring the solution in appropriate $L^r_t L^p_x(\mathbb{R}^{d+1})$ norms.

To prove Theorem 1.1 we make minor modifications to the argument in [3] associated with replacing $L^q(\mathbb{P}^d, \mathrm{d}\sigma)$ by $\dot{H}^s(\mathbb{R}^d)$.

Proof of Theorem 1.1 We fix $d \geq 1$, $s \in (0, \frac{d}{2})$ and let $p = p(d, s)$. If Ψ is the functional

$$\Psi(u_0) = \frac{\|e^{it\Delta}u_0\|^p_{L^p}}{\|u_0\|^p_{\dot{H}^s}},$$

defined for nonzero $u_0 \in \dot{H}^s(\mathbb{R}^d)$, then u_0 is a critical point if

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\bigl(\Psi(u_0 + \varepsilon v_0) - \Psi(u_0)\bigr) = 0$$

for any $v_0 \in \dot{H}^s(\mathbb{R}^d)$, where ε is a complex parameter. For brevity, we write $u(t, \cdot) = e^{it\Delta}u_0$ and $v(t, \cdot) = e^{it\Delta}v_0$.

Using Lemma 2.3 of [3], which gives an expansion of $\|F+\varepsilon G\|_{L^p}^p$ as $\varepsilon \to 0$, $\varepsilon \in \mathbb{C}$, we obtain some constant $\gamma > 1$ such that

$$\|u+\varepsilon v\|_{L^p}^p = \|u\|_{L^p}^p + p\int_{\mathbb{R}^{d+1}} \big|u(t,x)\big|^p \operatorname{Re}\Big(\varepsilon \frac{v(t,x)}{u(t,x)}\Big)\, dx\, dt + O\big(|\varepsilon|^\gamma\big)$$

and

$$\|u_0+\varepsilon v_0\|_{\dot H^s}^p = \|u_0\|_{\dot H^s}^p + (2\pi)^d p \|u_0\|_{\dot H^s}^{p-2} \operatorname{Re}\Big(\varepsilon \int_{\mathbb{R}^d} \widehat{u_0}(\xi)\overline{\widehat{v_0}(\xi)}|\xi|^{2s}\, d\xi\Big) + O\big(|\varepsilon|^\gamma\big)$$

as $\varepsilon \to 0$. It then follows that u_0 is a critical point if and only if there exists $\lambda > 0$ such that

$$\int_{\mathbb{R}^{d+1}} \big|u(t,x)\big|^{p-2} u(t,x) \exp\big(-i\big(x\cdot\xi - t|\xi|^2\big)\big)\, dx\, dt = \lambda |\xi|^{2s}\widehat{u_0}(\xi) \tag{2.1}$$

for almost all $\xi \in \mathbb{R}^d$. For Theorem 1.1, it suffices to show that u_0 is not a critical point, where $u_0(x) = \exp(-\frac14|x|^2)$. This reduction follows because u_0 such that $\widehat{u_0}(\xi) = \exp(a|\xi|^2 + ib\cdot\xi + c)$, with $a, c \in \mathbb{C}$ and $b \in \mathbb{R}^d$, can be generated from a centred isotropic gaussian under the action of the group generated by:

(1) *space-time translations*: $u(t,x) \to u(t+t_0, x+x_0)$ with $(t_0,x_0) \in \mathbb{R}^{d+1}$;
(2) *parabolic dilations*: $u(t,x) \to u(\mu^2 t, \mu x)$ with $\mu > 0$;
(3) *change of scale*: $u(t,x) \to \mu u(t,x)$ with $\mu > 0$;
(4) *phase shift*: $u(t,x) \to e^{i\theta} u(t,x)$ with $\theta \in \mathbb{R}$.

The Euler–Lagrange equation (2.1) is invariant under each of the above actions.

For $u_0(x) = \exp(-\frac14|x|^2)$ we have $\widehat{u_0}(\xi) = C_d \exp(-|\xi|^2)$ and

$$u(t,x) = C_d \frac{1}{(1+it)^{\frac d2}} \exp\Big(-\frac{|x|^2}{4(1+it)}\Big)$$

for some positive constants C_d (which may differ). Thus, (2.1) is equivalent to

$$\mathrm{I}(a) = C_{d,s} a^s \tag{2.2}$$

for all $a \in [0,\infty)$, where $C_{d,s}$ is some positive constant,

$$\mathrm{I}(a) := \int_{\mathbb{R}} \frac{H(t)}{(1+it)^{\frac d4(p-2)}} \exp\Big(a\frac{(p-2)(1+it)}{p-1-it}\Big)\, dt$$

and

$$H(t) := (1-it)^{-\frac d4(p-4)}(p-1-it)^{-\frac d2}.$$

A power series expansion of the exponential term leads to

$$\mathrm{I}(a)=\sum_{j=0}^{\infty}\frac{(p-2)^j\mathrm{I}_j}{j!}a^j, \tag{2.3}$$

where

$$\mathrm{I}_j:=\int_{\mathbb{R}}(1+it)^{j-\frac{d(s+1)}{d-2s}}H_j(t)\,\mathrm{d}t$$

and

$$H_j(t)=(1-it)^{-\frac{d}{4}(p-4)}(p-1-it)^{-\frac{d}{2}-j}.$$

Since H_j is holomorphic in the upper half-plane and

$$\left|(1+it)^{j-\frac{d(s+1)}{d-2s}}H_j(t)\right|\le C|t|^{-\frac{2d(s+1)}{d-2s}},$$

with $\frac{2d(s+1)}{d-2s}>1$, it follows (using Lemma 4.1 of [3]) that for $j>\frac{d(s+1)}{d-2s}-1$ we have

$$\mathrm{I}_j=-2\sin(\gamma_j\pi)\int_0^{\infty}r^{\gamma_j}H_j(i+ir)\,\mathrm{d}r, \tag{2.4}$$

where $\gamma_j:=j-\frac{d(s+1)}{d-2s}$. Since $H_j(i+ir)>0$ for all $r\ge 0$, it is clear that $\mathrm{I}_j=0$ if and only if $\gamma_j\in\mathbb{Z}$.

In the case where $s\in(0,\frac{d}{2})\cap\mathbb{N}$, using (2.2), (2.3) and a power series uniqueness argument, it follows that $\mathrm{I}_j=0$ for all $j\neq s$ and $\mathrm{I}_j\neq 0$ for $j=s$. If, additionally, $\frac{d(s+1)}{d-2s}\notin\mathbb{N}$, then (2.4) implies $\mathrm{I}_j\neq 0$ for any $j>\max\{\frac{d(s+1)}{d-2s}-1,s\}$, which gives a contradiction. If, instead, $\frac{d(s+1)}{d-2s}\in\mathbb{N}$, then for $j_\star=\frac{d(s+1)}{d-2s}-1$ we may use Cauchy's residue theorem to obtain

$$I_{j_\star}=\int_{\mathbb{R}}(1+it)^{-1}H_{j_\star}(t)\,\mathrm{d}t=2\pi H_{j_\star}(i)\neq 0.$$

Since $s>0$ we have $j_\star\neq s$ and so this is also a contradiction.

In the remaining case where $s\in(0,\frac{d}{2})$ and $s\notin\mathbb{N}$, one can see that (2.2) cannot hold for all $a\in[0,\infty)$ since (2.3) implies that $\mathrm{I}(a)$ is k times (right) differentiable at $a=0$ for each $k\in\mathbb{N}$, whereas $a\mapsto a^s$ is not. □

Regarding Theorem 1.2, we begin with the observation that the proof of (1.3) in [7], involving several well-chosen changes of variables, leads to the representation

$$\left\|(-\Delta)^{\frac{2-d}{4}}(u\overline{v})\right\|_{L^2}^2=C_d\int_{\mathfrak{M}}\left|\int_{\mathbb{R}^d}\widehat{u}_0\big((r-p)\omega-\eta\big)\widehat{\overline{v}}_0(\eta)\,\mathrm{d}\Sigma_{\omega,r}(\eta)\right|^2\mathrm{d}\sigma(\omega)\,\mathrm{d}r\,\mathrm{d}p,$$

where $\mathfrak{M}=\mathbb{S}^{d-1}\times\mathbb{R}^2$, $\mathrm{d}\Sigma_{\omega,r}(\eta)=\delta(r-\omega\cdot\eta)\,\mathrm{d}\eta$, δ is the Dirac measure on $\mathbb{R}$ supported at the origin, and $\mathrm{d}\sigma$ is the induced Lebesgue measure on $\mathbb{S}^{d-1}$. The constant

C_d is explicitly computable (and whose value depends on the chosen convention for the Fourier transform).

An application of Cauchy–Schwarz with respect to the measure $\mathrm{d}\Sigma_{r,\omega}$ for each fixed $(\omega, r, p) \in \mathfrak{M}$ yields (1.3). Using the standard fact that equality holds in the Cauchy–Schwarz inequality precisely when the constituent functions are linearly dependent, we see that if (u_0, v_0) is an extremising pair, then there exists a scalar function Λ such that

$$\widehat{u}_0\big((r-p)\omega - \eta\big) = \Lambda(\omega, r, p)\overline{\widehat{v}_0(\eta)} \tag{2.5}$$

for almost all $\eta \in \mathbb{R}^d$ (with respect to $\mathrm{d}\Sigma_{\omega,r}$) in the support of the measure $\mathrm{d}\Sigma_{\omega,r}$ and almost all $(\omega, r, p) \in \mathfrak{M}$ (with respect to the induced Lebesgue measure). A complete justification that (u_0, v_0) satisfies the geometric functional equation in (2.5) if and only if (u_0, v_0) have the gaussian form in (1.8) requires a multiple-stage argument.

The strategy behind the characterisation is to first argue that u_0 and v_0 must be equal (up to non-zero constants), and then establish that $\widehat{u}_0$ must have a certain amount of regularity. In fact, a delicate geometric argument shows that $\widehat{u}_0$ must be at least continuous. Once equipped with this information, and furthermore, that $\widehat{u}_0$ never vanishes, it is possible to solve (2.5) by decomposing $\widehat{u}_0 = fg$ into a product of logarithmically even and odd functions, where

$$f(\eta) = \big(\widehat{u}_0(\eta)\widehat{u}_0(-\eta)\big)^{\frac{1}{2}} \quad \text{and} \quad g(\eta) = \left(\frac{\widehat{u}_0(\eta)}{\widehat{u}_0(-\eta)}\right)^{\frac{1}{2}}.$$

The functional equation inherited by f and g, from $\widehat{u}_0$, is the classical orthogonal Cauchy functional equation

$$h(\eta_1 + \eta_2) = h(\eta_1)h(\eta_2)$$

whenever η_1 and η_2 are orthogonal vectors in $\mathbb{R}^d$. If f and g are normalised so that $f(0) = g(0) = 1$, this forces $f(\eta) = \exp(a|\eta|^2)$ and $g(\eta) = \exp(b \cdot \eta)$ for some $a \in \mathbb{C}$ and $b \in \mathbb{C}^d$, and hence $\widehat{u}_0$ has the desired form (1.8).

Full details of this argument can be found in [1] as part of a substantial analysis of sharp bilinear estimates of Ozawa–Tsutsumi type.

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Nonlinear PDE as Immersions

Zhanat Zhunussova

Abstract Investigating of the nonlinear PDE including their geometric nature is one of the topical problems. With geometric point of view the nonlinear PDE are considered as immersions. We consider some aspects of the simplest soliton immersions in multidimensional space in Fokas–Gelfand's sense (Ceyhan et al. in J. Math. Phys. 41:2551–2270, 2000). In $(1+1)$-dimensional case nonlinear PDE are given in compatibility condition some system of linear equations (Lakshmanan and Myrzakulov in J. Math. Phys. 39:3765–3771, 1998). In this case there is a surface with immersion function. We find the second quadratic form in Fokas–Gelfand's sense associated to one soliton solution of nonlinear Schrödinger equation.

Keywords Immersion · Soliton · Surface · Evolution equation

1 Introduction

Over the last twenty years in the field of mathematical physics a large number of researches is devoted to the study of nonlinear equations. Some nonlinear wave equations can occur in problems of the different physical nature [1, 2]. For example, such equations are the well-known Korteweg de Vries equation, the nonlinear Schrödinger equation, sin-Gordon equation.

Soliton theory is a powerful apparatus for studying nonlinear equations including their geometrical nature. With a geometrical point of view soliton systems are considered as immersion of infinite-dimensional spaces. In other words, the hierarchy of soliton equations considered as a system of defining immersion of a manifold V^n in space V^m, where $n < m$. Connection between theory of solitons and theory of surfaces is set by introducing evolution equations that associated with algebra. The relation $(1+1)$-dimensional soliton equations with the theory of surfaces are given by the Gauss–Codazzi–Mainardi equation. In this case, the soliton equations are considered as some integrable reductions of the Gauss–Codazzi–Mainardi equation.

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In this work, we would like to review the simplest aspects of soliton investments in multi-dimensional space, in Fokas–Gelfand sense [1].

In $(1+1)$-dimensional case nonlinear partial differential equations are given as a condition of zero curvature $U_t - V_x + [U, V] = 0$, where $[U, V] = UV - VU$, matrix U is given, and the matrix V is expressed in terms of elements of the matrix U. Also the nonlinear partial differential equation is the compatibility condition the system of linear equations $\phi_x = U\phi$, $\phi_t = V\phi$. In this case there is a surface with immersion function $P(x,t)$ defined by the formulas $\frac{\partial P}{\partial x} = \phi^{-1}X\phi$, $\frac{\partial P}{\partial t} = \phi^{-1}Y\phi$. Surface defined by $P(x,t)$ identified a surface in three-dimensional space defined by the coordinates [1] $x_j = P_j(x,t)$, $j = 1, 2, 3$. Frame on the surface is given by a triple [1] $\frac{\partial P}{\partial x} = \phi^{-1}X\phi$, $\frac{\partial P}{\partial t} = \phi^{-1}Y\phi$, $N = \phi^{-1}J\phi$, where $J = \frac{[X,Y]}{|[X,Y]|}$, $|X| = \sqrt{\langle X, X\rangle}$. Here, by definition, $\langle X, Y\rangle = -\frac{1}{2}\,\mathrm{tr}(XY)$, where X, Y are some matrixes. And the first and second quadratic forms of the surface are given by

$$I = \langle X, X\rangle dx^2 + 2\langle X, Y\rangle dxdt + \langle Y, Y\rangle dt^2, \tag{1.1}$$

$$II = \left\langle \frac{\partial X}{\partial x} + [X, U], J\right\rangle dx^2 + 2\left\langle \frac{\partial X}{\partial t} + [X, V], J\right\rangle dxdt + \left\langle \frac{\partial Y}{\partial t} + [Y, V], J\right\rangle dt^2. \tag{1.2}$$

As shown in [1] immersion function P can be defined as $P = \gamma_0\phi^{-1}\phi_\lambda + \phi^{-1}M_1\phi = \sum_{j=1}^{3} P_j f_j$, where M_1 is a matrix function, which depends on λ, x, t. Here $f_j = -\frac{i}{2}\sigma_j$ is basis of the corresponding algebra, σ_j are Pauli matrices and $[f_i, f_j] = f_k$. In this case, X, Y can be written as $X = \gamma_0 U_\lambda + M_{1x} + [M_1, U]$, $Y = \gamma_0 V_\lambda + M_{1t} + [M_1, V]$.

2 Soliton Immersions in $(1+1)$-Dimension

Let the matrixes X, Y, J have the form

$$X = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad Y = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \qquad J = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}. \tag{2.1}$$

In this case, the elements of the matrix J are expressed through the elements of the matrix X and Y in accordance with the formulas

$$c_{11} = \frac{a_{12}b_{21} - b_{12}a_{21}}{|[X, Y]|}, \qquad c_{21} = \frac{a_{21}(b_{11} - b_{22}) + b_{21}(a_{22} - a_{11})}{|[X, Y]|}, \tag{2.2}$$

$$c_{12} = \frac{b_{12}(a_{11} - a_{22}) + a_{12}(b_{22} - b_{11})}{|[X, Y]|}, \qquad c_{22} = \frac{a_{21}b_{12} - b_{21}a_{12}}{|[X, Y]|}. \tag{2.3}$$

Then the first fundamental form (1.1) of two-dimensional surface becomes $I = Edx^2 + 2Fdxdt + Gdt^2$, where

$$E = -\frac{1}{2}\left(a_{11}^2 + 2a_{12}a_{21} + a_{22}^2\right),$$
$$F = -\frac{1}{2}(a_{11}b_{11} + a_{12}b_{21} + a_{21}b_{12} + a_{22}b_{22}), \tag{2.4}$$

$$G = -\frac{1}{2}\left(b_{11}^2 + 2b_{12}b_{21} + b_{22}^2\right). \tag{2.5}$$

As an example of a soliton equation that yields such immersion we consider the nonlinear Schrödinger equation $i\psi_t + \psi_{xx} + 2\beta|\psi|^2\psi = 0$, where $\beta = +1$, ψ is complex function. In this case, the matrix U, V have the form [3]

$$U = \frac{\lambda\sigma_3}{2i} + U_0, \quad U_0 = i\begin{pmatrix} 0 & \bar{q} \\ q & 0 \end{pmatrix},$$
$$V = \frac{i\lambda^2}{2}\sigma_3 + i|q|^2\sigma_3 - i\lambda\begin{pmatrix} 0 & \bar{q} \\ q & 0 \end{pmatrix} + \begin{pmatrix} 0 & \bar{q}_x \\ -q_x & 0 \end{pmatrix}. \tag{2.6}$$

The theorem is held.

Theorem 2.1 *Second quadratic form in the sense of Fokas–Gelfand corresponding to soliton solution q of nonlinear Schrödinger equation has the form*

$$II = Ldx^2 + 2Mdxdt + Ndt^2, \tag{2.7}$$

where

$$\begin{aligned} L = -\frac{1}{2}\big\{ & a_{11x}c_{11} + a_{12x}c_{21} + a_{21x}c_{12} + a_{22x}c_{22} \\ & - \lambda i(a_{21}c_{12} - a_{12}c_{21}) \\ & + iq(a_{12}c_{11} + a_{22}c_{12} - a_{11}c_{12} - a_{12}c_{22}) \\ & + i\bar{q}(a_{21}c_{22} + a_{11}c_{21} - a_{22}c_{21} - a_{21}c_{11})\big\}, \end{aligned} \tag{2.8}$$

$$\begin{aligned} M = -\frac{1}{2}\big\{ & a_{11t}c_{11} + a_{12t}c_{21} + a_{21t}c_{12} + a_{22t}c_{22} \\ & + i\left(\lambda^2 + 2|q|^2\right)(a_{21}c_{12} - a_{12}c_{21}) \\ & + (q_x + \lambda iq)(a_{11}c_{12} + a_{12}c_{22} - a_{12}c_{11} - a_{22}c_{12}) \\ & + (\bar{q}_x - \lambda i\bar{q})(a_{11}c_{21} + a_{21}c_{22} - a_{21}c_{11} - a_{22}c_{21})\big\}, \end{aligned} \tag{2.9}$$

$$\begin{aligned} N = -\frac{1}{2}\big\{ & b_{11t}c_{11} + b_{12t}c_{21} + b_{21t}c_{12} + b_{22t}c_{22} \\ & + i\left(\lambda^2 + 2|q|^2\right)(b_{21}c_{12} - b_{12}c_{21}) \end{aligned}$$

$$+ (q_x + \lambda i q)(b_{11}c_{12} + b_{12}c_{22} - b_{12}c_{11} - b_{22}c_{12})$$
$$+ (\bar{q}_x - \lambda i \bar{q})(b_{11}c_{21} + b_{21}c_{22} - b_{21}c_{11} - b_{22}c_{21})\big\}, \qquad (2.10)$$

Proof By direct substitution of the matrix (2.1), (2.6) to (1.2) we obtain (2.7), (2.8)–(2.10). Theorem is proved. □

3 One-Soliton Solution of the Nonlinear Schrödinger Equation Corresponding to the Surface

We consider the partial case of immersion at $\gamma_0 = 1$, $M_1 = 0$. For this case we have

$$X = U_\lambda = \frac{1}{2i}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad Y = V_\lambda = -i\begin{pmatrix} -\lambda & \bar{q} \\ q & \lambda \end{pmatrix},$$
$$J = \begin{pmatrix} 0 & -\frac{\bar{q}}{\sqrt{q\bar{q}}} \\ \frac{q}{\sqrt{q\bar{q}}} & 0 \end{pmatrix}, \qquad (3.1)$$

and $P = \phi^{-1}\phi_\lambda$. To calculate the explicit expressions for the functions of immersion P we consider the one-soliton solution of the nonlinear Schrödinger equation, which has the form

$$q(x,t) = Q\frac{\exp\{i(\varphi_0 + \frac{vx}{2} + \frac{(u^2-v^2)t}{4} - \frac{\pi}{2})\}}{\mathrm{ch}\{\frac{u}{2}(x - vt - x_0)\}}, \qquad (3.2)$$

where we put $\lambda = \frac{u+iv}{2}$, Q is constant.

Theorem 3.1 (Main Theorem) *One-soliton solution of the nonlinear Schrödinger equation corresponds to the surface in the sense of Fokas–Gelfand with the corresponding coefficients of the first and second quadratic form*

$$E = \frac{Q^2(u^2+v^2)}{(\lambda - \bar{\lambda}_1)^4\,\mathrm{ch}^2\{\frac{u}{2}(x - vt - x_0)\}},$$
$$F = -\frac{v(u^2+v^2)Q^2}{2(\lambda - \bar{\lambda}_1)^4\,\mathrm{ch}^2\{\frac{u}{2}(x - vt - x_0)\}}, \qquad (3.3)$$

$$G = \frac{(u^2+v^2)^2 Q^2}{4(\lambda - \bar{\lambda}_1)^4\,\mathrm{ch}^2\{\frac{u}{2}(x - vt - x_0)\}},$$
$$L = -\frac{u(u^2+v^2)}{4(\lambda - \bar{\lambda}_1)^2\,\mathrm{ch}^2\{\frac{u}{2}(x - vt - x_0)\}}, \qquad (3.4)$$

$$M = \frac{uv(u^2+v^2)}{8(\lambda-\bar{\lambda}_1)^2 \operatorname{ch}^2\{\frac{u}{2}(x-vt-x_0)\}},$$
$$N = -\frac{u(u^2+v^2)}{16(\lambda-\bar{\lambda}_1)^2 \operatorname{ch}^2\{\frac{u}{2}(x-vt-x_0)\}}, \tag{3.5}$$

where λ_1 *is constant.*

Proof The solution of the linear system we find in the form

$$\psi = \phi e^{-(\frac{\lambda\sigma_3}{2i}x + \frac{i\lambda^2}{2}\sigma_3 t)}. \tag{3.6}$$

Taking into account (3.6), apply (2.6) we have

$$\psi_x = \left(\frac{\lambda\sigma_3}{2i} + U_0\right)\psi - \psi\frac{\lambda\sigma_3}{2i} = \frac{\lambda\sigma_3}{2i}\psi - \psi\frac{\lambda\sigma_3}{2i} + U_0\psi$$
$$= \left[\frac{\lambda\sigma_3}{2i}, \psi\right] + U_0\psi. \tag{3.7}$$

We take

$$\psi = I - \frac{\tilde{A}}{\lambda - \lambda_1^*}, \quad \text{where } \tilde{A} = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix}, \ I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \lambda_1^*\text{-const.} \tag{3.8}$$

We substitute (3.8) to (3.7)

$$\psi_x = U_0 - \frac{U_0\tilde{A}}{\lambda-\lambda_1^*} - \frac{1}{2i}[\sigma_3, \tilde{A}] - \frac{\lambda_1^*}{2i(\lambda-\lambda_1^*)}[\sigma_3, \tilde{A}]. \tag{3.9}$$

On the other side of (3.8) follows

$$\psi_x = -\frac{\tilde{A}_x}{\lambda-\lambda_1^*}. \tag{3.10}$$

From (3.9) and (3.10) we have

$$-\frac{\tilde{A}_x}{\lambda-\lambda_1^*} = U_0 - \frac{U_0\tilde{A}}{\lambda-\lambda_1^*} - \frac{1}{2i}[\sigma_3, \tilde{A}] - \frac{\lambda_1^*}{2i(\lambda-\lambda_1^*)}[\sigma_3, \tilde{A}]. \tag{3.11}$$

Thus

$$\tilde{A}_x = U_0\tilde{A} + \frac{\lambda_1^*}{2i}[\sigma_3, \tilde{A}], \quad U_0 = \frac{1}{2i}[\sigma_3, A]. \tag{3.12}$$

Note that

$$[\sigma_3, \tilde{A}] = \sigma_3\tilde{A} - \tilde{A}\sigma_3 = 2\begin{pmatrix} 0 & \tilde{b} \\ -\tilde{c} & 0 \end{pmatrix}. \tag{3.13}$$

Then substituting (3.13) into (3.28), we have

$$U_0 = \frac{1}{i}\begin{pmatrix} 0 & \tilde{b} \\ -\tilde{c} & 0 \end{pmatrix}. \tag{3.14}$$

Substituting (3.13) to (3.12), we have

$$\begin{pmatrix} \tilde{a}_x & \tilde{b}_x \\ \tilde{c}_x & \tilde{d}_x \end{pmatrix} = \frac{1}{i}\begin{pmatrix} \tilde{b}\tilde{c} & \tilde{b}\tilde{d} \\ -\tilde{c}\tilde{a} & -\tilde{c}\tilde{b} \end{pmatrix} + \frac{\lambda_1^*}{i}\begin{pmatrix} 0 & \tilde{b} \\ -\tilde{c} & 0 \end{pmatrix}. \tag{3.15}$$

From (2.6) and (3.14) we have

$$i\begin{pmatrix} 0 & \bar{q} \\ q & 0 \end{pmatrix} = \frac{1}{i}\begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix} \quad \Rightarrow \quad \begin{cases} i\bar{q} = \frac{1}{i}b \\ iq = -\frac{1}{i}c \end{cases} \quad \Rightarrow \quad \begin{cases} b = -\bar{q} \\ c = q. \end{cases} \tag{3.16}$$

Thus we have found a matrix $\tilde{A}$ implicitly, with components (3.15). From (3.15), (3.16) follows $\tilde{a} = -\frac{i\tilde{c}_x}{c} - \lambda_1^* \Rightarrow \tilde{a} = -\frac{iq_x}{q} - \lambda_1^*$. Using (3.2) we obtain

$$\tilde{a} = \frac{iu}{2}\,\mathrm{th}\left\{\frac{u}{2}(x - vt - x_0)\right\} + \frac{v}{2} - \lambda_1^*. \tag{3.17}$$

From (3.15) follows $\tilde{a}_x = \frac{1}{i}\tilde{b}\tilde{c} \Rightarrow \tilde{a}_x = \frac{1}{i}(-\bar{q})q, \Rightarrow \tilde{a} = -\frac{1}{i}\int \bar{q}q\,dx$. Using (3.2) we obtain

$$\tilde{a} = -\frac{2|Q|^2}{iu}\,\mathrm{th}\left\{\frac{u}{2}(x - vt - x_0)\right\} - \frac{2|Q|^2 c_1}{iu}. \tag{3.18}$$

From (3.17), (3.18) follows

$$\begin{cases} -\frac{2|Q|^2}{iu} = \frac{iu}{2}, \quad \Rightarrow \quad \frac{2|Q|^2}{iu} = -\frac{iu}{2}, \quad \Rightarrow \quad 4|Q|^2 = u^2, \quad \Rightarrow \quad |Q|^2 = \frac{u^2}{4}, \\ \left(\frac{v}{2} - \lambda_1^*\right) = -c_1\frac{2|Q|^2}{iu}, \quad \Rightarrow \quad c_1 = -\frac{iu}{2|Q|^2}\left(\frac{v}{2} - \lambda_1^*\right). \end{cases} \tag{3.19}$$

From (3.15), (3.16) follows

$$\tilde{d} = \frac{i\tilde{b}_x}{\tilde{b}} - \lambda_1^* \quad \Rightarrow \quad \tilde{d} = \frac{i(-\bar{q})_x}{(-\bar{q})} - \lambda_1^* \quad \Rightarrow \quad \tilde{d} = \frac{i\bar{q}_x}{\bar{q}} - \lambda_1^*. \tag{3.20}$$

Using (3.2) we have

$$\tilde{d} = -\frac{iu}{2}\,\mathrm{th}\left\{\frac{u}{2}(x - vt - x_0)\right\} + \left(\frac{v}{2} - \lambda_1^*\right). \tag{3.21}$$

From (3.15), (3.16) follows

$$\tilde{d}_x = -\frac{1}{i}\tilde{c}\tilde{b} \quad \Rightarrow \quad \tilde{d} = \frac{1}{i}\int q\bar{q}\,dx \quad \Rightarrow \quad \tilde{d} = -\tilde{a}. \tag{3.22}$$

We denote $c = c_1 \frac{2|Q|^2}{iu}$. From (3.21), (3.22) follows

$$\begin{cases} \left(\frac{v}{2} - \lambda_1^*\right) = c_1 \frac{2|Q|^2}{iu}, & \Rightarrow \quad c_1 = \frac{iu}{2|Q|^2}\left(\frac{v}{2} - \lambda_1^*\right), \\ \frac{2|Q|^2}{iu} = -\frac{iu}{2}, & \Rightarrow \quad 4|Q|^2 = u^2, \quad \Rightarrow \quad |Q|^2 = \frac{u^2}{4} \end{cases} \tag{3.23}$$

Taking into account c, (3.23), we obtain (3.18) in the form

$$\tilde{a} = -\frac{u}{2i} \operatorname{th}\left\{\frac{u}{2}(x - vt - x_0)\right\} - c. \tag{3.24}$$

Thus, the matrix $\tilde{A}$ for one-soliton solution (3.2) of the nonlinear Schrödinger equation takes the form

$$\tilde{A} = \begin{pmatrix} -\frac{u}{2i} \operatorname{th}\{\frac{u}{2}(x - vt - x_0)\} - c & -Q \frac{\exp\{-i(\varphi_0 + \frac{vx}{2} + \frac{(u^2 - v^2)}{4} t - \frac{\pi}{2})}{\operatorname{ch}\{\frac{u}{2}(x - vt - x_0)\}} \\ Q \frac{\exp\{i(\varphi_0 + \frac{vx}{2} + \frac{(u^2 - v^2)}{4} t - \frac{\pi}{2})}{\operatorname{ch}\{\frac{u}{2}(x - vt - x_0)\}} & \frac{u}{2i} \operatorname{th}\{\frac{u}{2}(x - vt - x_0)\} + c \end{pmatrix}. \tag{3.25}$$

Now we take $\phi = I - \frac{A}{(\lambda - \bar{\lambda}_1)^2}$, where λ_1 is constants, then from (3.1) we have

$$P = \phi^{-1}\phi_\lambda = \left(I + \frac{\tilde{A}}{\lambda - \lambda_1}\right)\frac{\tilde{A}}{(\lambda - \bar{\lambda}_1)^2} \tag{3.26}$$

On the other hand, we obtain

$$P = \sum_{j=1}^{3} P_j f_j = -\frac{i}{2}\sum_{j=1}^{3} P_j \sigma_j = \begin{pmatrix} -\frac{i}{2} P_3 & -\frac{i}{2} P_1 - \frac{1}{2} P_2 \\ -\frac{i}{2} P_1 + \frac{1}{2} P_2 & \frac{i}{2} P_3 \end{pmatrix}. \tag{3.27}$$

From (3.26), (3.27) by (3.22) we have $P_3 = \frac{2i\tilde{a}}{(\lambda - \bar{\lambda}_1)^2}$. Now with the help of (3.24) we find P_3 explicitly for solution of the nonlinear Schrödinger equation

$$P_3 = -\frac{4|Q|^2 c_1}{u(\lambda - \bar{\lambda}_1)^2} - \frac{u \operatorname{th}\{\frac{u}{2}(x - vt - x_0)\}}{(\lambda - \bar{\lambda}_1)^2}. \tag{3.28}$$

From (3.26), (3.27) we have $P_2 = \frac{\tilde{c} - \tilde{b}}{(\lambda - \bar{\lambda}_1)^2}$. Thus $P_1 = \frac{i(\tilde{c} + \tilde{b})}{(\lambda - \bar{\lambda}_1)^2}$, $P_2 = \frac{(\tilde{c} - \tilde{b})}{(\lambda - \bar{\lambda}_1)^2}$, $P_3 = \frac{2i\tilde{a}}{(\lambda - \bar{\lambda}_1)^2}$. From (3.26), (3.2) using the known formulas

$$\begin{aligned} \operatorname{sh}\zeta &= \frac{e^{\zeta} - e^{-\zeta}}{2}; \qquad \operatorname{ch}\zeta = \frac{e^{\zeta} + e^{-\zeta}}{2}; \\ \cos\zeta &= \frac{e^{i\zeta} + e^{-i\zeta}}{2}; \qquad \sin\zeta = \frac{e^{i\zeta} - e^{-i\zeta}}{2i}, \end{aligned} \tag{3.29}$$

where $\zeta = (\varphi_0 + \frac{vx}{2} + \frac{(u^2-v^2)}{4}t - \frac{\pi}{2})$ we obtain the explicit values of P_1, P_2, P_3 matrix P

$$P_1 = -\frac{2Q\sin(\varphi_0 + \frac{vx}{2} + \frac{(u^2-v^2)}{4}t - \frac{\pi}{2})}{(\lambda - \bar{\lambda}_1)^2 \operatorname{ch}\{\frac{u}{2}(x - vt - x_0)\}},$$
$$P_2 = \frac{2Q\cos(\varphi_0 + \frac{vx}{2} + \frac{(u^2-v^2)}{4}t - \frac{\pi}{2})}{(\lambda - \bar{\lambda}_1)^2 \operatorname{ch}\{\frac{u}{2}(x - vt - x_0)\}}, \tag{3.30}$$

$$P_3 = -\frac{4|Q|^2 c_1}{u(\lambda - \bar{\lambda}_1)^2} - \frac{u \operatorname{th}\{\frac{u}{2}(x - vt - x_0)\}}{(\lambda - \bar{\lambda}_1)^2}. \tag{3.31}$$

Now we can calculate the coefficients on the first quadratic form i.e.

$$E = P_{1x}^2 + P_{2x}^2 + P_{3x}^2. \tag{3.32}$$

For this, we compute P_{1x}, P_{2x}, P_{3x}. Now the first derivatives are raised separately to the 2nd power and substitute into (3.32), then

$$E = \frac{Q^2(u^2 + v^2)}{(\lambda - \bar{\lambda}_1)^4 \operatorname{ch}^2\{\frac{u}{2}(x - vt - x_0)\}}.$$

Similarly, according to the formulae $F = P_{1x}P_{1t} + P_{2x}P_{2t} + P_{3x}P_{3t}$, $G = P_{1t}^2 + P_{2t}^2 + P_{3t}^2$ we obtain the values

$$F = -\frac{v(u^2 + v^2)Q^2}{2(\lambda - \bar{\lambda}_1)^4 \operatorname{ch}^2\{\frac{u}{2}(x - vt - x_0)\}},$$
$$G = \frac{Q^2(u^2 + v^2)^2}{4(\lambda - \bar{\lambda}_1)^4 \operatorname{ch}^2\{\frac{u}{2}(x - vt - x_0)\}}. \tag{3.33}$$

Now, using (3.30), (3.31) we calculate coefficients of the second form L, M, N. For this, we have to calculate

$$\mathbf{n} = \frac{\mathbf{r}_x \times \mathbf{r}_t}{\sqrt{\Lambda}}, \quad \sqrt{\Lambda} = \sqrt{EG - F^2}. \tag{3.34}$$

Directly substituting the values of (3.30)–(3.31) to (3.34) we calculate the components vector $\mathbf{n}$. Here we present the calculation

$$n_1 = -\frac{u^2(u^2 + v^2)Q\sin(\varphi_0 + \frac{vx}{2} + \frac{(u^2-v^2)}{4}t - \frac{\pi}{2})}{4\sqrt{\Lambda}(\lambda - \bar{\lambda}_1)^4 \operatorname{ch}^3\{\frac{u}{2}(x - vt - x_0)\}}. \tag{3.35}$$

$$n_2 = \frac{u^2(u^2 + v^2)Q\cos(\varphi_0 + \frac{vx}{2} + \frac{(u^2-v^2)}{4}t - \frac{\pi}{2})}{4\sqrt{\Lambda}(\lambda - \bar{\lambda}_1)^4 \operatorname{ch}^3\{\frac{u}{2}(x - vt - x_0)\}}, \tag{3.36}$$

$$n_3 = -\frac{Q^2 u(u^2+v^2)\,\mathrm{sh}\{\frac{u}{2}(x-vt-x_0)\}}{2\sqrt{\Lambda}(\lambda-\bar{\lambda}_1)^4\,\mathrm{ch}^3\{\frac{u}{2}(x-vt-x_0)\}}. \tag{3.37}$$

We calculate with the help of (3.33)

$$\sqrt{\Lambda} = \left(EG-F^2\right)^{\frac{1}{2}} = \left\{\frac{Q^4(u^2+v^2)^2u^2}{4(\lambda-\bar{\lambda}_1)^8\,\mathrm{ch}^4\{\frac{u}{2}(x-vt-x_0)\}}\right\}^{\frac{1}{2}}. \tag{3.38}$$

Now we find P_{1xx}, P_{2xx}, P_{3xx}. Then we can find L. By the similar way we calculate M, N. Now, using (3.38), (3.34) Gaussian and mean curvature K and H can be calculated

$$K = \frac{1}{4u^2}\left(1-v^2\right)(\lambda-\bar{\lambda}_1)^4, \qquad H = \frac{1}{2u^3}\left(v^2-u^2-1\right)(\lambda-\bar{\lambda}_1)^2. \tag{3.39}$$

Now, from (2.4), (2.5) using (3.1) for the case γ_0, $M_1=0$ we have coefficients of the first fundamental form corresponding to (3.2) as $E=\frac{1}{4}$, $F=-\frac{\lambda}{2}$, $G=\lambda^2+\bar{q}q$. Respectively, from (2.8)–(2.10) using (3.1), we have coefficients of the second quadratic form. Now we can calculate $\Lambda = EG-F^2 = \frac{1}{4}\bar{q}q$. Theorem is proved. □

4 Conclusion

Thus, we have examined the soliton immersion in $(1+1)$-dimension and obtained the corresponding formulae. As an example of such immersion we consider $(1+1)$-dimensional nonlinear Schrodinger equation. It is found integrable surface corresponding to the one-soliton solution of the nonlinear Schrödinger equation given by the first and second quadratic forms with coefficients (3.3)–(3.5). We have calculated the Gaussian and mean curvature of found surface. We see, that the geometric equation describing the n-curvilinear coordinate systems in flat Euclidean and pseudo-Euclidean space allow some integrable reductions. In addition, we have assumed that immersion 3- and 4-dimensional manifolds arbitrarily embedded in R^{μ} admit integrable cases.

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Blow-Up for Nonlinear Inequalities with Singularities on Unbounded Sets

Evgeny Galakhov and Olga Salieva

Abstract Many physical phenomena are described by nonlinear equations and inequalities with singular coefficients, for which blow-up situation occurs. In this paper we establish sufficient conditions of blow-up situation for some classes of nonlinear differential inequalities with singularities on unbounded sets.

Keywords Nonlinear differential inequality · Blow-up · Singularity

Mathematics Subject Classification (2010) Primary 35J60 · Secondary 35K55

1 Introduction

A blow-up situation is growth of a solution of a differential equation or inequality towards infinity in the neighborhood of finite values of the argument. The theory of blow-up of solutions to nonlinear differential equations is used for prediction of many disastrous events in physics and technology, such as crash of buildings [1] and phase transitions in the Ginzburg–Landau–Allen–Cahn model [2].

Most known results in the blow-up theory have to do with differential equations of the second order. A method for investigating blow-up for a wider class of problems by using asymptotic a priori estimates was developed by S. Pohozaev and E. Mitidieri [1, 3].

In this paper we obtain sufficient conditions for a blow-up situation to occur for several classes of equations and inequalities that have singular coefficients on unbounded sets, such as lines, planes, smooth curves and surfaces in $\mathbb{R}^n$.

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2 Main Results

Let $S \subset \mathbb{R}^n$ be a closed unbounded set.

Let $\varepsilon > 0$, $x \in \mathbb{R}^n$. Denote $\rho(x) = \mathrm{dist}(x, S)$ and $S^\varepsilon = \{x \in \mathbb{R}^n : \rho(x, S) < \varepsilon\}$. Suppose that there exists a family of functions $\xi_R \in C_0^{2k}(\mathbb{R}^n \setminus S; [0,1])$ such that

$$\xi_R(x) = \begin{cases} 0 & \left(x \in S^{\frac{1}{2R}} \cup \left(\mathbb{R}^n \setminus S^{2R}\right)\right), \\ 1 & \left(x \in S^R \setminus S^{\frac{1}{R}}\right) \end{cases} \tag{2.1}$$

and there exists a constant $c > 0$ such that

$$\left|D^\alpha \xi_R(x)\right| \le c\rho^{-|\alpha|} \quad \left(x \in \mathbb{R}^n\right). \tag{2.2}$$

Example As the set S we can consider a hyperplane $S = \Pi_n = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_n = 0\}$. In that case we can choose $\xi_R(x) = \tilde{\xi}_R(x_n)$, where

$$\tilde{\xi}_{\frac{1}{R}}(x_n) = \begin{cases} 0 & \left(|x_n| \le \dfrac{1}{2R} \text{ or } |x_n| \ge 2R\right), \\ 1 & \left(\dfrac{1}{R} \le |x_n| \le R\right). \end{cases}$$

For definiteness, we formulate our results in special cases, namely for inequalities

$$(-\Delta)^k u \ge u^q \rho^{-\alpha} |x|^\beta \quad \left(x \in \mathbb{R}^n \setminus S\right), \tag{2.3}$$

$$(-\Delta)^k u \ge |Du|^q \rho^{-\alpha} |x|^\beta \quad \left(x \in \mathbb{R}^n \setminus S\right) \tag{2.4}$$

with some $k \in \mathbb{N}$, $q, \beta > 0$.

Our main results for problems (2.3)–(2.4) can be formulated as follows.

Theorem 2.1 *Let $q > 1$ and $n + \frac{|\alpha - 2kq| - \beta}{q} \le 0$.*

Then problem (2.3) has no nonnegative nontrivial solutions $u \in L^q_{\mathrm{loc}}(\mathbb{R}^n \setminus S)$.

Theorem 2.2 *Let $q > 1$ and $n + \frac{|\alpha - (2k-1)q| - \beta}{q} \le 0$.*

Then problem (2.4) has no nontrivial solutions $u \in W^{1,q}_{\mathrm{loc}}(\mathbb{R}^n \setminus S)$.

A nonexistence result also takes place for an evolution inequality

$$u_t - (-\Delta)^k u \ge u^q \rho^{-\alpha} |x|^\beta \quad \left(x \in \mathbb{R}^n \setminus S; t \in \mathbb{R}_+\right) \tag{2.5}$$

with an initial condition

$$u(x, 0) = u_0(x) \ge 0 \quad \left(x \in \mathbb{R}^n \setminus S\right). \tag{2.6}$$

We assume that the initial function $u_0 \in L^1_{\mathrm{loc}}(\mathbb{R}^n \setminus S)$.

A typical result for problem (2.5), (2.6) can be formulated as follows.

Theorem 2.3 *Let $q > 1$, $\alpha > 2kq$ and $n + 2k + \frac{\alpha - 2kq - \beta}{q} \le 0$.*

Then problem (2.5), (2.6) has no nonnegative nontrivial solutions $u \in L^q_{\mathrm{loc}}((\mathbb{R}^n \setminus S) \times [0, T])$ for any $T > 0$.

Remark 2.4 Nonexistence of solutions to (2.5), (2.6) is called *instantaneous blow-up.*

3 Proof of Theorem 2.1

We introduce a family of test functions $\varphi = \varphi_R \in C_0^{2k}(\mathbb{R}^n \setminus S; [0, 1])$ of the form

$$\varphi_R(x) = \xi_R^\kappa(x)\psi_R^\kappa(x)$$

with $\kappa > 2kq'$, $\xi_R \in C_0^{2k}(\mathbb{R}^n \setminus S; [0, 1])$ that satisfy (2.1) and (2.2), and $\psi_R \in C_0^{2k}(\mathbb{R}^n; [0, 1])$ such that

$$\psi_R(x) = \begin{cases} 1 & \big(|x| \le R\big), \\ 0 & \big(|x| \ge 2R\big), \end{cases} \tag{3.1}$$

such that for some constant $c > 0$ one has

$$\big|D^\alpha \psi_R(x)\big| \le cR^{-|\alpha|} \quad \big(x \in \mathbb{R}^n\big) \tag{3.2}$$

for all multi-indices α with $0 \le |\alpha| \le 2k$.

The structure of these test functions allows to get rid of singularities both on S and at infinity.

Now suppose that a solution u of (2.3) does exist. Multiplying both sides of (2.3) by φ_R and integrating by parts $2k$ times, we get

$$\begin{aligned}
&\int_{\operatorname{supp}\varphi_R} u^q \rho^{-\alpha} |x|^\beta \varphi_R \, dx \\
&\quad \le \int_{\operatorname{supp}\varphi_R} u(-\Delta)^k \varphi_R \, dx \le \int_{\operatorname{supp}\varphi_R} u\big|\Delta^k \varphi_R\big| \, dx \\
&\quad \le \left(\int_{\operatorname{supp}\varphi_R} u^q \rho^{-\alpha} |x|^\beta \varphi_R \, dx\right)^{\frac{1}{q}} \cdot \left(\int_{\operatorname{supp}\varphi_R} \rho^{\frac{\alpha q'}{q}} |x|^{-\frac{\beta q'}{q}} \big|\Delta^k \varphi_R\big|^{q'} \varphi_R^{1-q'} \, dx\right)^{\frac{1}{q'}},
\end{aligned}$$

whence

$$\int_{\operatorname{supp}\varphi_R} u^q \rho^{-\alpha} |x|^\beta \varphi_R \, dx \le \int_{\operatorname{supp}\varphi_R} \rho^{\frac{\alpha q'}{q}} |x|^{-\frac{\beta q'}{q}} \big|\Delta^k \varphi_R\big|^{q'} \varphi_R^{1-q'} \, dx.$$

Denote $A_R = (S^R \setminus S^{\frac{1}{R}}) \cap B_R(0)$. Due to the choice of φ_R, we can restrict the domains of integration on both sides of the inequality:

$$\int_{A_R} u^q \rho^{-\alpha} |x|^\beta \varphi_R \, dx \le \int_{A_{2R}} \rho^{\frac{\alpha q'}{q}} |x|^{-\frac{\beta q'}{q}} \left|\Delta^k \varphi_R\right|^{q'} \varphi_R^{1-q'} \, dx.$$

Note that $\varphi_R \equiv 1$ in the whole domain of integration on the left-hand side. Using conditions (2.2) and (3.2), we obtain

$$\int_{A_R} u^q \rho^{-\alpha} |x|^\beta \, dx \le c R^{n + \frac{|\alpha - 2kq| - \beta}{q}},$$

which leads to a contradiction as $R \to \infty$, if the exponent on the right-hand side is negative. The case of a zero exponent is considered in a standard way (see [1]).

Remark 3.1 This result can be extended to a wider class of stationary higher order differential operators with constant or variable coefficients, including systems of the form

$$\begin{cases} (-\Delta)^k u \ge v^q \rho^{-\alpha} |x|^\beta & \big(x \in \mathbb{R}^n \setminus S\big), \\ (-\Delta)^l v \ge u^p \rho^{-\gamma} |x|^\delta & \big(x \in \mathbb{R}^n \setminus S\big) \end{cases}$$

with appropriate parameters $k, l \in \mathbb{N}$, $p, q > 1$, $\alpha, \beta, \gamma, \delta \in \mathbb{R}$.

4 Proof of Theorem 2.2

To prove Theorem 2.2, we take $\varphi = \varphi_R \in C_0^{2k-1}(\mathbb{R}^n \setminus S; [0, 1])$ of the same form as in the previous section, with $\kappa > (2k-1)q'$, $\xi_R \in C_0^{2k-1}(\mathbb{R}^n \setminus S; [0, 1])$, which satisfy (2.1) and (2.2), and $\psi_R \in C_0^{2k-1}(\mathbb{R}^n; [0, 1])$, which satisfy (3.1) and (3.2). It suffices that estimates (2.2) and (3.2) hold for $0 \le |\alpha| \le 2k - 1$.

Assume that a nontrivial (not constant a.e.) solution u of (2.4) does exist. Multiplying both sides of (2.4) by φ_R and integrating by parts $2k - 1$ times, we get

$$\begin{aligned} &\int_{\operatorname{supp} \varphi_R} |Du|^q \rho^{-\alpha} |x|^\beta \varphi_R \, dx \\ &\quad \le \int_{\operatorname{supp} \varphi_R} \big(Du, D\big((-\Delta)^{k-1} \varphi_R\big)\big) \, dx \\ &\quad \le \int_{\operatorname{supp} \varphi_R} |Du| \cdot \big|D\big(\Delta^{k-1} \varphi_R\big)\big| \, dx \le \left(\int_{\operatorname{supp} \varphi_R} |Du|^q \rho^{-\alpha} |x|^\beta \varphi_R \, dx \right)^{\frac{1}{q}} \\ &\qquad \times \left(\int_{\operatorname{supp} \varphi_R} \rho^{\frac{\alpha q'}{q}} |x|^{-\frac{\beta q'}{q}} \big|D\big(\Delta^{k-1} \varphi_R\big)\big|^{q'} \varphi_R^{1-q'} \, dx \right)^{\frac{1}{q'}}, \end{aligned}$$

whence

$$\int_{\operatorname{supp}\varphi_R} |Du|^q \rho^{-\alpha}|x|^\beta \varphi_R\, dx \le \int_{\operatorname{supp}\varphi_R} \rho^{\frac{\alpha q'}{q}} |x|^{-\frac{\beta q'}{q}} \left|D\left(\Delta^{k-1}\varphi_R\right)\right|^{q'} \varphi_R^{1-q'}\, dx.$$

Due to the choice of φ_R, we can restrict the domains of integration on both sides of the inequality:

$$\int_{A_R} |Du|^q \rho^{-\alpha}|x|^\beta \varphi_R\, dx \le \int_{A_{2R}} \rho^{\frac{\alpha q'}{q}} |x|^{-\frac{\beta q'}{q}} \left|D\left(\Delta^{k-1}\varphi_R\right)\right|^{q'} \varphi_R^{1-q'}\, dx.$$

Note that $\varphi_R \equiv 1$ in the whole domain of integration on the left-hand side. Using conditions (2.2) and (3.2), we obtain

$$\int_{A_R} |Du|^q \rho^{-\alpha}|x|^\beta\, dx \le c R^{n+\frac{|\alpha-(2k-1)q|-\beta}{q}},$$

which leads to a contradiction as $R \to \infty$, if the exponent on the right-hand side is negative. The case of a zero exponent is considered in a standard way.

Remark 4.1 This result can be extended to a wider class of stationary higher order differential operators with constant or variable coefficients, including systems of the form

$$\begin{cases} (-\Delta)^k u \ge |Dv|^q \rho^{-\alpha}|x|^\beta & \left(x \in \mathbb{R}^n \setminus S\right), \\ (-\Delta)^l v \ge |Du|^p \rho^{-\gamma}|x|^\delta & \left(x \in \mathbb{R}^n \setminus S\right) \end{cases}$$

with appropriate parameters $k, l \in \mathbb{N}$, $p, q > 1$, $\alpha, \beta, \gamma, \delta \in \mathbb{R}$.

5 Proof of Theorem 2.3

For the Cauchy problem (2.5), (2.6) we introduce two families of test functions, namely $\varphi_R(x)$ in space variables and $T_\tau(t)$ in time. Here $\varphi_R(x)$ is defined as in the previous sections, and $T_\tau \in C^1([0,\tau];[0,1])$ with $\tau > 0$ are such that

$$T_\tau(t) = \begin{cases} 1 & (0 \le t \le \tau/2), \\ 0 & (3\tau/4 \le t \le \tau) \end{cases}$$

and, moreover,

$$\int_{\tau/2}^{3\tau/4} \frac{|T_\tau'|^{q'}}{|T_\tau|^{q'-1}}\, dt \le c\tau^{1-q'} \tag{5.1}$$

with some constant $c > 0$.

Multiplying both sides of (2.5) by $\varphi_R(x)T_\tau(t)$ and integrating by parts, we get

$$\int_{\mathbb{R}_+} T_\tau dt \int_{\mathbb{R}^n} u^q \rho^{-\alpha}|x|^\beta \varphi_R dx + \int_{\mathbb{R}^n} u_0 \varphi_R dx$$

$$\leq \int_{\mathbb{R}_+} T_\tau dt \int_{\mathbb{R}^n} u(-\Delta)^k \varphi_R dx - \int_{\mathbb{R}_+} T_\tau' dt \int_{\mathbb{R}^n} u\varphi_R dx$$
$$\leq \int_{\mathbb{R}_+} T_\tau dt \int_{\mathbb{R}^n} u\left|\Delta^k \varphi_R\right| dx + \int_{\mathbb{R}_+} \left|T_\tau'\right| dt \int_{\mathbb{R}^n} u\varphi_R dx.$$

Applying the parametric Young inequality to both terms on the right, we get

$$\frac{1}{2}\int_{\mathbb{R}_+} T_\tau dt \int_{\mathbb{R}^n} u^q \rho^{-\alpha}|x|^\beta \varphi_R dx + \int_{\mathbb{R}^n} u_0 \varphi_R dx$$
$$\leq c_1 \int_{\mathbb{R}_+} T_\tau dt \int_{\mathbb{R}^n} \left|\Delta^k \varphi_R\right|^{q'} \rho^{-\frac{\alpha q'}{q}} |x|^{\frac{\beta q'}{q}} \varphi_R^{1-q'} dx$$
$$+ c_2 \int_{\mathbb{R}_+} \left|T_\tau'\right|^{q'} T_\tau^{1-q'} dt \int_{\mathbb{R}^n} \rho^{-\frac{\alpha q'}{q}} |x|^{\frac{\beta q'}{q}} \varphi_R^{1-q'} dx$$

with some constants $c_1, c_2 > 0$.

Due to the choice of $\varphi_R(x)$ and $T_\tau(t)$, we can restrict domains of integration on both sides of the inequality:

$$\frac{1}{2}\int_0^{2\tau} T_\tau dt \int_{A_R} u^q \rho^{-\alpha}|x|^\beta \varphi_R dx + \int_{A_R} u_0 \varphi_R dx$$
$$\leq c_1 \int_0^{2\tau} T_\tau dt \int_{A_{2R}} \left|\Delta^k \varphi_R\right|^{q'} \rho^{-\frac{\alpha q'}{q}} |x|^{\frac{\beta q'}{q}} \varphi_R^{1-q'} dx$$
$$+ c_2 \int_\tau^{2\tau} \left|T_\tau'\right|^{q'} T_\tau^{1-q'} dt \int_{A_{2R}} \rho^{-\frac{\alpha q'}{q}} |x|^{\frac{\beta q'}{q}} \varphi_R^{1-q'} dx.$$

Note that the second term on the left-hand side of the inequality is nonnegative and $\varphi_R(x) \equiv 1$ in the whole domain of integration. Using conditions (2.2) and (5.1), we obtain

$$\int_0^{2\tau} T_\tau dt \int_{A_R} u^q \rho^{-\alpha}|x|^\beta dx \leq c R^{n+\frac{\beta-\alpha}{q-1}} \tau \left(\tau^{-\frac{q}{q-1}} + R^{\frac{2kq}{q-1}}\right). \tag{5.2}$$

It is easily seen that the right-hand side of (5.2) attains its minimum at

$$\tau = c R^{-2k}. \tag{5.3}$$

Substituting (5.3) into (5.2) and taking $R \to \infty$, we reach a contradiction.

6 Conclusion

We have proven the main results formulated in Sect. 2 on blow-up conditions for stationary and evolutional semilinear differential inequalities of higher order with respect to the space variables, as well as for quasilinear elliptic differential inequalities of the second order.

Similarly we can obtain blow-up conditions for other classes of differential inequalities, such as:

- Semilinear and quasilinear elliptic systems.
- Elliptic inequalities with gradient terms.
- More general evolutional differential inequalities, etc.

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Well-Posedness and Stability of a Mindlin–Timoshenko Plate Model with Damping and Sources

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Abstract This note gives a concise summary of results concerning the well-posedness and long-time behavior of (Reissner)–Mindlin–Timoshenko plate equations as presented in Pei et al. (Local and global well-posedness for semilinear Reissner–Mindlin–Timoshenko plate equations, 2013 and Global well-posedness and stability of semilinear Mindlin–Timoshenko system, 2013). The main feature of the considered model is the interplay between nonlinear viscous interior damping and nonlinear source terms. The results include Hadamard local well-posedness, global existence, blow-up theorems, as well as estimates on the uniform energy decay rates.

Keywords Plate · Mindlin–Timoshenko · Reissner–Mindlin · Source · Damping · Local existence · Global existence · Blow-up · Potential well · Stability · Stability · Energy decay

Mathematics Subject Classification (2010) Primary 35L71 · Secondary 74K20 · 35A01 · 35B44 · 35B35

1 Introduction

The classical Euler–Bernoulli beam theory and its Kirchhoff–Love plate counterpart are of limited accuracy when describing high-frequency vibrations or when the deflections are relatively large with respect to the thickness of the plate. A linear refinement on the beam model was developed around 1920 by Timoshenko (see e.g.

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[11]) and accounts for shear deformations. About thirty years later Mindlin [5] derived a generalization for plate models (an analogous though somewhat different theory had also been proposed earlier by Reissner [10] in 1945). A large body of research literature has been devoted to Reissner–Mindlin–Timoshenko (RMT) plates (e.g. see [1, 2, 6] and the many references therein); however, the study of interaction between nonlinear sources and damping, rather common to the research on wave problems, has not been addressed so thoroughly in the context of the RMT plates. These questions were recently investigated in [8, 9] and the purpose of this note is to provide a concise summary of the new results.

1.1 The Model

Let the open bounded domain $\Omega \subset \mathbb{R}^2$ of class C^2 represent the mid-surface of a plate in the state of an equilibrium. The deformation is quantified by vector $u = (w, \psi, \phi)$ dependent on the coordinate $\mathbf{x} = (x, y) \in \Omega$ and time $t \geq 0$. The component $w = w(t, \mathbf{x})$ represents the out-of-plane displacement, whereas $\psi = \psi(t, \mathbf{x})$ and $\phi = \phi(t, \mathbf{x})$ quantify shear deformations and are proportional to the angles of the linearized filaments. At the principal level, the model is comprised of a scalar wave equation for w and a 2D system of elasticity for (ψ, ϕ) (see e.g. [4, pp. 25–26]):

$$\begin{cases} \rho h w_{tt} - D\Delta w - K(\psi_x + \phi_y) + g_1(w_t) = f_1(w, \psi, \phi), \\ \quad \text{in } Q_T := \Omega \times (0, T), \\ \rho h \psi_{tt} - D\left(\psi_{xx} + \dfrac{1-\mu}{2}\psi_{yy} + \dfrac{1+\mu}{2}\phi_{xy}\right) + K(\psi + w_x) + g_2(\psi_t) \\ \quad = f_2(w, \psi, \phi), \quad \text{in } Q_T, \\ \rho h \phi_{tt} - D\left(\dfrac{1-\mu}{2}\phi_{xx} + \phi_{yy} + \dfrac{1+\mu}{2}\psi_{xy}\right) + K(\phi + w_y) + g_3(\phi_t) \\ \quad = f_3(w, \psi, \phi), \quad \text{in } Q_T, \\ w = \psi = \phi = 0 \quad \text{on } \Gamma \times (0, T), \quad \text{where } 0 < \mu < 1/2. \end{cases} \tag{1.1}$$

The initial data $(u(0), u_t(0))$ comes from the associated *finite energy space* described later. The positive constants ρ, h, D, and K depend on the material of the plate. We will set these parameters to 1 throughout the paper as their magnitude does not affect the well-posedness analysis. The parameter $0 < \mu < 1/2$ stands for the Poisson's ratio and its value will not be normalized since it plays a central role in the positivity of the associated stress operator. The scalar feedback maps f_1, f_2, f_3 represent the source terms, while feedbacks g_1, g_2, g_3 could be regarded as viscous damping or stabilizing interior controls.

1.2 Function Spaces and Solutions

For scalar-valued functions on Ω we will use the following norms and inner products:

$$\|w\|_s := \|w\|_{L^s(\Omega)}, \qquad (v,w)_\Omega := (v,w)_{L^2(\Omega)}.$$

Similarly, for $u = (w,\psi,\phi)$ and $\tilde{u} = (\tilde{w},\tilde{\psi},\tilde{\phi})$ we will employ the notation:

$$(u,\tilde{u})_\Omega := (w,\tilde{w})_\Omega + (\psi,\tilde{\psi})_\Omega + (\phi,\tilde{\phi})_\Omega,$$

$$\|u\|_s := \left(\|w\|_s^s + \|\psi\|_s^s + \|\phi\|_s^s\right)^{\frac{1}{s}}.$$

Let

$$V := \left(H_0^1(\Omega)\right)^3 \quad \text{and} \quad \mathbb{H} := V \times \left(L^2(\Omega)\right)^3.$$

As a consequence of Korn's inequality the space V is a Hilbert space equipped with an with an equivalent inner product (e.g. see [9])

$$\begin{aligned}(u,\tilde{u})_V = \int_\Omega \Big(&(1-\mu)(\psi_x\tilde{\psi}_x + \phi_y\tilde{\phi}_y) + \mu(\psi_x+\phi_y)(\tilde{\psi}_x+\tilde{\phi}_y) \\ &+ \frac{1-\mu}{2}(\psi_y+\phi_x)(\tilde{\psi}_y+\tilde{\phi}_x)\Big) d\mathbf{x} + (w_x+\psi, \tilde{w}_x+\tilde{\psi})_\Omega \\ &+ (w_y+\phi, \tilde{w}_y+\tilde{\phi})_\Omega. \end{aligned} \tag{1.2}$$

We equip the *finite energy space* $\mathbb{H}$ with the corresponding graph norm and inner product. In addition, define:

$$\mathscr{G}(u_t) = \big(g_1(w_t), g_2(\psi_t), g_3(\phi_t)\big), \qquad \mathscr{F}(u) = \big(f_1(u), f_2(u), f_3(u)\big). \tag{1.3}$$

We will also make use of the following concept:

Definition 1.1 (Linearly bounded) A function $\gamma : \mathbb{R} \to \mathbb{R}$ will be said to be *linearly bounded near the origin* if there exist $c_1, c_2 > 0$ such that

$$c_1|s| \le \big|\gamma(s)\big| \le c_2|s| \quad \text{for all } |s| < 1. \tag{1.4}$$

Now we are ready to formulate the notion of a weak solution to our problem:

Definition 1.2 (Weak solution) A function $u \in H^1(0,T;V)$ is said to be a weak solution of (1.1) on $[0,T]$ if:

- $(u,u_t) \in C([0,T];\mathbb{H})$ and

$$u_t \in L^{m+1}(Q_T) \times L^{r+1}(Q_T) \times L^{q+1}(Q_T);$$

- in addition u satisfies

$$\big(u_t(t),\theta(t)\big)_\Omega-\big(u_t(0),\theta(0)\big)_\Omega+\int_0^t\Big(-\big(u_t(\tau),\theta_t(\tau)\big)_\Omega+\big(u(\tau),\theta(\tau)\big)_V\Big)d\tau$$
$$+\int_0^t\big(\mathscr{G}(u_t),\theta\big)_\Omega d\tau=\int_0^t\big(\mathscr{F}(u),\theta\big)_\Omega d\tau,\tag{1.5}$$

for all $t\in[0,T]$ and all test functions θ in the space

$$\Theta:=\big\{\theta\in C\big([0,T];V\big),\ \theta_t\in L^1\big(0,T;\big[L^2(\Omega)\big]^3\big)\big\}.$$

2 Assumptions and Preliminaries

2.1 Properties of the Nonlinear Terms

Assumption 2.1 (Well-posedness assumptions)

- ***Damping:*** $g_i:\mathbb{R}\to\mathbb{R}$, $i=1,2,3$, are continuous, monotone increasing functions with $g_i(0)=0$. In addition, there exist positive constants α and β such that for all $|s|\ge 1$,

$$\alpha|s|^{p_i+1}\le g_i(s)s\le\beta|s|^{p_i+1},\tag{2.1}$$

  with $p_i\ge 1$ where $p_1=m$, $p_2=r$, $p_3=q$.
- ***Sources:*** $f_j\in C^1(\mathbb{R}^3)$, $j=1,2,3$, and there are constants $C>0$, $p\ge 1$ such that for all $(w,\psi,\phi)\in\mathbb{R}^3$

$$\big|\nabla f_j(w,\psi,\phi)\big|\le C\big(|w|^{p-1}+|\psi|^{p-1}+|\phi|^{p-1}+1\big).$$

Assumption 2.2 (Additional conditions for blow-up results) Suppose there exists a nonnegative function $F(w,\psi,\phi)\in C^2(\mathbb{R}^3)$ such that for some $\alpha_0>0$, $c_1>2$ and all $u=(w,\psi,\phi)\in\mathbb{R}^3$,

$$f_1(u)=\partial_w F(u),\qquad f_2(u)=\partial_\psi F(u),\qquad f_3(u)=\partial_\phi F(u),$$
$$F(u)\ge\alpha_0\big(|w|^{p+1}+|\psi|^{p+1}+|\phi|^{p+1}\big),\tag{2.2}$$
$$u\cdot\nabla F(u)\ge c_1F(u).$$

Remark 2.3 There is a large class of functions satisfying Assumption 2.2. For instance, those of the form

$$F(w,\psi,\phi)=a|w+\psi|^{p+1}+b|w\psi|^{\frac{p+1}{2}}+c|\phi|^{p+1},\tag{2.3}$$

where a, b, c are positive constants and $p\ge 3$.

2.2 Potential Well

Here, we introduce the *potential energy functional* J and highlight its connections with system (1.1) and the Mountain Pass Theorem. The potential well theory of Payne and Sattinger [7] will then be formulated for the problem in question.

Assumption 2.4 (Potential well solutions) There exists a nonnegative function $F \in C^2(\mathbb{R}^3)$ that satisfies (2.2) and furthermore is homogeneous of order $p+1$, i.e.,

$$F(\lambda u) = \lambda^{p+1} F(u), \quad \text{for all } \lambda > 0,\ u \in \mathbb{R}^3.$$

Remark 2.5 The function in (2.3) would be an example of such an F.

Define the **potential energy** functional $J : V \to \mathbb{R}$ as

$$J(u) := \frac{1}{2}\|u\|_V^2 - \int_\Omega F(u)d\mathbf{x}. \tag{2.4}$$

The *total energy* of the system (1.1) will be defined as follows

$$\mathscr{E}(t) := \frac{1}{2}\big(\|u\|_V^2 + \|u_t\|_2^2\big) - \int_\Omega F\big(u(t)\big)d\mathbf{x}, \tag{2.5}$$

and so, $\mathscr{E}(t) = \frac{1}{2}\|u_t(t)\|_2^2 + J(u(t))$. As [8] demonstrates, the Fréchet derivative of J at $u \in V$ is given by: $D_u J(\theta) = (u,\theta)_V - \int_\Omega \mathscr{F}(u)\cdot\theta d\mathbf{x}$, for all $\theta \in V$, (where $\mathscr{F}$ is given in (1.3)), which implies that the critical points of the functional J are the weak solutions to the elliptic problem:

$$\begin{cases} -\Delta w - (\psi_x + \phi_y) = f_1(w,\psi,\phi), & \text{in } Q_T, \\ -\left(\psi_{xx} + \frac{1-\mu}{2}\psi_{yy}\right) - \frac{1+\mu}{2}\phi_{xy} + (\psi + w_x) = f_2(w,\psi,\phi), & \text{in } Q_T, \\ -\left(\frac{1-\mu}{2}\phi_{xx} + \phi_{yy}\right) - \frac{1+\mu}{2}\psi_{xy} + (\phi + w_y) = f_3(w,\psi,\phi), & \text{in } Q_T. \end{cases} \tag{2.6}$$

Associated with the functional J is the *Nehari manifold*

$$\begin{aligned} \mathscr{N} &:= \big\{u \in V\setminus\{0\} : D_u J(u) = 0\big\} \\ &= \left\{u \in V\setminus\{0\} : \|u\|_V^2 = (p+1)\int_\Omega F(u)d\mathbf{x}\right\}. \end{aligned}$$

It follows from [3, Lemma 4.2.1] that J fulfills the hypothesis of the Mountain Pass Theorem, moreover, the *mountain pass level* $d := \inf_{u\in\mathscr{N}} J(u)$ satisfies:

Lemma 2.6 *In addition to Assumptions 2.1 and 2.4, further assume that $p > 1$ then*

$$d := \inf_{u\in\mathscr{N}} J(u) = \inf_{u\in V\setminus\{0\}} \sup_{\lambda\geq 0} J(\lambda u) > 0.$$

We also introduce the following sets:

$$\begin{aligned}
\mathscr{W} &:= \{u \in V : J(u) < d\};\\
\mathscr{W}_1 &:= \left\{u \in \mathscr{W} : \|u\|_V^2 > (p+1)\int_\Omega F(u)d\mathbf{x}\right\} \cup \{0\}; \qquad (2.7)\\
\mathscr{W}_2 &:= \left\{u \in \mathscr{W} : \|u\|_V^2 < (p+1)\int_\Omega F(u)d\mathbf{x}\right\}.
\end{aligned}$$

Clearly, $\mathscr{W}_1 \cap \mathscr{W}_2 = \emptyset$, and $\mathscr{W}_1 \cup \mathscr{W}_2 = \mathscr{W}$. We refer to $\mathscr{W}$ as the **potential well** and d as the *depth* of the well. The set $\mathscr{W}_1$ can be formally regarded as the "good" part of the well, as it will be shown that every weak solution starting therein exists globally provided initial energy is under the level d. On the other hand, if the initial data are taken from $\mathscr{W}_2$ and the source exponents dominate those of the damping, then solutions with nonnegative initial energy $\mathscr{E}(0)$ may blow up in finite time.

Due to some technicalities in the proofs, the following approximation of the "good" part of the potential well is useful. Let $\mathcal{G}(s) := \frac{1}{2}s^2 - MRs^{p+1}$, where the constant $M > 0$ only depends on F from Assumption 2.4. If $p > 1$, a straightforward calculation shows that $\mathcal{G}$ attains its absolute maximum on $[0, \infty)$ at the unique critical point: $s_0 = ((p+1)MR)^{1/(1-p)}$. For sufficiently small $\delta > 0$, we can define a closed subset of $\mathscr{W}_1$, given by

$$\tilde{\mathscr{W}}_1^{\delta} := \{u \in V : \|u\|_V \le s_0 - \delta,\, J(u) \le \mathcal{G}(s_0 - \delta)\}. \qquad (2.8)$$

3 Theorems

Theorem 3.1 (Local well-posedness [9]) *Suppose Assumption 2.1 holds, then for any initial condition* $(u_0, u_1) \in \mathbb{H}$, *there exists a* ***unique local*** *weak solution* u *to* (1.1) *defined on* $[0, T]$, *for some* $T > 0$ *dependent on the initial quadratic energy* $E(0)$ *which is defined via* $E(t) := \frac{1}{2}(\|u\|_V^2 + \|u_t\|_2^2) = \frac{1}{2}\|(u, u_t)\|_{\mathbb{H}}^2$. *The solution* u *satisfies the following energy identity for all* $t \in [0, T]$:

$$E(t) + \int_0^t \big(\mathscr{G}(u_t), u_t\big)_\Omega d\tau = E(0) + \int_0^t \big(\mathscr{F}(u), u_t\big)_\Omega d\tau. \qquad (3.1)$$

Furthermore, weak solutions depend continuously in $C([0, T]; \mathbb{H})$ *on the initial data in* $\mathbb{H}$.

Theorem 3.2 (Global solution [9]) *Assume the validity of Assumption 2.1. If, in addition,* $p \le \min\{m, r, q\}$, *then the said solution* u *in Theorem 3.1 is* ***global*** *in the sense that* T *can be chosen arbitrarily large.*

Theorem 3.3 (Blow-up in finite time [9]) *Suppose Assumptions 2.1 and 2.2 hold. If* $p > \max\{m, r, q\}$ *and* $\mathscr{E}(0) < 0$, *then any weak solution* u *to* (1.1) *furnished by Theorem 3.1 blows up in finite time in the sense that* $\limsup_{t \to T^-} E(t) = \infty$ *for some* $T < \infty$.

Theorem 3.4 (Global potential well Solutions [8]) *In addition to Assumption 2.1 and Assumption 2.4, further assume $u(0) \in \mathscr{W}_1$ and $\mathscr{E}(0) < d$. If $p > 1$, then the unique weak solution u provided by Theorem 3.1 is a global solution. Furthermore, if $\rho = \frac{p+1}{p-1}$, then for all $t \geq 0$ we have*:

(i) $J(u(t)) \leq \mathscr{E}(t) \leq \mathscr{E}(0)$
(ii) $u(t) \in \mathscr{W}_1$
(iii) $E(t) < d \cdot \rho$
(iv) $\rho^{-1} E(t) \leq \mathscr{E}(t) \leq E(t)$

Theorem 3.5 (Potential well blow-up [8]) *Suppose Assumptions 2.1, 2.2, and 2.4 hold. Further assume $p > \max\{m, r, q\}$, $0 \leq \mathscr{E}(0) < d$, and $u(0) \in \mathscr{W}_2$, then any weak solution u blows up in finite time.*

Theorem 3.6 (Uniform stabilization rates [8]) *In addition to Assumptions 2.1, 2.4 suppose that $p > 1, u_0 \in L^{m+1}(\Omega) \times L^{r+1}(\Omega) \times L^{q+1}(\Omega), u_0 \in \tilde{\mathscr{W}}_1^{\delta}$, as defined in (2.8), and $\mathscr{E}(0) < \mathcal{G}(s_0 - \delta)$ for some $\delta > 0$. Let $\varphi_j : [0, \infty) \mapsto [0, \infty)$ be continuous strictly increasing concave functions vanishing at the origin such that*

$$\varphi_j\big(g_j(s)s\big) \geq \big|g_j(s)\big|^2 + s^2 \quad \text{for } |s| < 1, \ j = 1, 2, 3.$$

Define

$$\Phi(s) := \varphi_1(s) + \varphi_2(s) + \varphi_3(s) + s, \qquad s \geq 0. \tag{3.2}$$

Then, for any $T > 0$ there exists a concave increasing map $H = (I + \tilde{C}\Phi)^{-1}$, where $\tilde{C} = \tilde{C}(T, \mathscr{E}(0))$ (instead of dependence on $\mathscr{E}(0)$ one may use dependence on $d \cdot \rho$) such that $\rho^{-1} E(t) \leq \mathscr{E}(t) \leq S(T^{-1}t - 1)$ for all $t \geq T$, where scalar function S satisfies the nonlinear monotone ODE

$$S'(t) + H\big(S(t)\big) = 0, \qquad S(0) = \mathscr{E}(0). \tag{3.3}$$

Here are some explicit examples of these energy decay rates:

Corollary 3.7 (Exponential decay [8]) *Under the hypotheses of Theorem 3.6, if g_1, g_2, and g_3 are linearly bounded near the origin, then $H(s) = \omega s$ for some $\omega = \omega(T, \mathscr{E}(0)) > 0$ and the total energy $\mathscr{E}(t)$ and quadratic energy $E(t)$ decay exponentially*:

$$\rho^{-1} E(t) \leq \mathscr{E}(t) \leq C\mathscr{E}(0)e^{-(\omega/T)t}, \quad \text{for some } C > 0 \text{ and all } t \geq 0. \tag{3.4}$$

Here $\omega = [1 + C_2(1 + \mathscr{E}(0)^{p-1})]^{-1}$ with $C, C_2 > 0$ independent of $\mathscr{E}(0)$.

Corollary 3.8 (Algebraic decay [8]) *Under the hypotheses of Theorem 3.6, if at least one of the feedback mapping g_i, $i = 1, 2, 3$ is not linearly bounded near the* *origin and instead for* $|s| < 1$ *is bounded above and below by functions of the form* $c_i|s|^{\gamma_i}$ *some* $c_i > 0$, $\gamma_i > 0$, *then there exists* $t_0 > 0$ *such that*

$$E(t) \leq \left(\mathscr{E}(0)^{-1/b} + Ct\right)^{-b}, \quad \textit{for some } C, b > 0 \textit{ and all } t \geq t_0, \tag{3.5}$$

where b is independent of $\mathscr{E}(0)$, *while* $C = (\frac{C_3}{1+\mathscr{E}(0)^{p-1}})^{\frac{b+1}{b}}$

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On Deterministic and Stochastic Linear Quadratic Control Problems

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Abstract The numerical treatment of linear quadratic regulator (LQR), linear quadratic Gaussian (LQG) design and stochastic control problems of certain type require solving Riccati equations. In the finite time horizon case, the Riccati differential equation (RDE) arises. We show that within a Galerkin projection framework the solutions of the finite-dimensional RDEs converge in the strong operator topology to the solutions of the infinite-dimensional RDEs. A discussion about LQG design in the context of receding horizon control for nonlinear problems as well as a brief discussion about stochastic control is also addressed. Numerical experiments validate the proposed convergence result.

Keywords LQR · LQG · Stochastic control · RDEs · SRDEs

Mathematics Subject Classification (2010) 34H05 · 95B40 · 49J55 · 15A24

1 Introduction

We consider optimal control problems for parabolic diffusion-convection and diffusion-reaction systems which can be linearized. The variational formulation leads to an abstract Cauchy problem for a linear evolution equation of the form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{x}(0) = \mathbf{x}_0 \in \mathcal{H}, \; t \in [0, T] \tag{1.1}$$

for linear operators $\mathbf{A} : \mathrm{dom}(\mathbf{A}) \subset \mathcal{H} \to \mathcal{H}$, $\mathbf{B} : \mathcal{U} \to \mathcal{H}$, $\mathbf{C} : \mathcal{H} \to \mathcal{Y}$, where the state space $\mathcal{H}$, the observation space $\mathcal{Y}$, and the control space $\mathcal{U}$ are assumed to be

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separable Hilbert spaces. Additionally, $\mathcal{U}$ is assumed to be finite-dimensional, i.e., there is only a finite number of independent control inputs. Here, $\mathbf{C}$ maps the states of the system into its outputs and $\mathbf{y} = \mathbf{Cx}$ for $\mathbf{x} \in \mathcal{H}$. Moreover we consider the cost functional to be given in a quadratic form

$$\mathbf{J}(\mathbf{x}, \mathbf{u}) = \int_0^T \left\{ \langle \mathbf{x}, \mathbf{Qx} \rangle_{\mathcal{H}} + \langle \mathbf{u}, \mathbf{Ru} \rangle_{\mathcal{U}} \right\} dt + \langle \mathbf{x}_T, \mathbf{Gx}_T \rangle_{\mathcal{H}}, \tag{1.2}$$

where $\mathbf{Q}$, $\mathbf{G}$ are self-adjoint operators on the state space $\mathcal{H}$, $\mathbf{R}$ is a self-adjoint positive definitive operator on the control space $\mathcal{U}$. We denote $\mathbf{x}_T = \mathbf{x}(T)$ for fixed $0 \leq T < \infty$. Usually, only a few measurements of the state are available as the outputs of the system. We assumed that operators $\mathbf{Q} := \mathbf{C}^*\mathbf{C}$ and $\mathbf{G}$ are in general positive semidefinite. If $\mathbf{A}$ is the infinitesimal generator of a strongly continuous semigroup $\mathbf{T}(t)$, $\mathbf{B}$, $\mathbf{C}$ are linear bounded operators and for every initial value there exists an admissible control $\mathbf{u} \in L^2([0, T]; \mathcal{U})$ hold, then the solution of the abstract LQR problem can be obtained, analogously to the finite-dimensional case, as a *feedback control*

$$\mathbf{u}(t) = -\mathbf{R}^{-1}\mathbf{B}^*\Pi(t)\mathbf{x}(t), \tag{1.3}$$

where $\Pi(t)$ represents the unique nonnegative solution of the operator Riccati differential equation

$$\dot{\Pi}(t) = -\big(\mathbf{Q} + \mathbf{A}^*\Pi(t) + \Pi(t)\mathbf{A} - \Pi(t)\mathbf{BR}^{-1}\mathbf{B}^*\Pi(t)\big) \tag{1.4}$$

with the terminal condition $\Pi(T) = \mathbf{G}$. Some of the required conditions, particularly the restrictive assumption that $\mathbf{B}$ is bounded, can be weakened [8, 11, 12].

In this paper we present an approximation framework for the computation of the Riccati operators (1.4), Sect. 2. Moreover, in Sect. 3 we consider nonlinear optimal control problems in which LQG design is applied. There, linear problems have to be solved in subintervals of the time horizon. These are the so called receding horizon (RHC) and model predictive control (MPC) approaches. A brief discussion about stochastic control is also addressed. A numerical example for a nonlinear control problem is shown in Sect. 4.

2 Convergence Result

Numerical schemes for Riccati equations in infinite-dimensional spaces as well as convergence rates for some types of control problems have been proposed in recent years [3, 8, 10, 12, 13]. An approximation scheme in terms of differential Riccati equations follows from the abstract theory developed by Gibson [8], and from the ideas for the infinite-time horizon case presented in [3]. These results as well as novel numerical methods for large-scale RDEs are proposed in [5]. We summarize these results here and discuss the suboptimality behavior of the solution.

We consider a control system in $\mathcal{H}$ given by (1.1) and the cost functional (1.2). As in [7, 12], we assume that (1.1) has a unique solution. Moreover, $\mathbf{Q}, \mathbf{G} \in \mathcal{L}(\mathcal{H})$, $\mathbf{R} \in \mathcal{L}(\mathcal{U})$ are self-adjoint with $\mathbf{R} > 0$, $\mathbf{G} \geq 0$. A function $\mathbf{u} \in L^2([0,T];\mathcal{U})$ is an admissible control for the initial state $\mathbf{x}_0 \in \mathcal{H}$ if $J(\mathbf{x}_0, \mathbf{u})$ in (1.2) is finite.

Note that any solution of (1.4) is self-adjoint, and that $\Pi(.)$ is nonnegative operator as long as $\mathbf{G}$ is nonnegative. In order to solve numerically the operator Riccati differential equation for practical problems, we have to find suitable finite-dimension approximations of its solution. Therefore, let $\mathcal{H}^N$, $N = 1, 2, \ldots$, be a sequence of finite-dimensional linear subspaces of $\mathcal{H}$ and $P^N : \mathcal{H} \to \mathcal{H}^N$ be the canonical orthogonal projections. Assume that $T^N(t)$ is a sequence of strongly continuous semigroups on $\mathcal{H}^N$ with infinitesimal generator $A^N \in \mathcal{L}(\mathcal{H}^N)$. Given operators $B^N \in \mathcal{L}(\mathcal{U}, \mathcal{H}^N)$, $G^N, Q^N \in \mathcal{L}(\mathcal{H}^N)$, $G^N \geq 0$, we consider the family of linear-quadratic regulator problems on $\mathcal{H}^N$, denoted by $(\mathcal{R}^N)$, associated to the cost functional:

$$J_N\big(x_0^N, \mathbf{u}\big) := \int_0^T \big\{\big\langle x^N, Q^N x^N\big\rangle_{\mathcal{H}^N} + \langle \mathbf{u}, \mathbf{R}\mathbf{u}\rangle_{\mathcal{U}}\big\}\, dt + \big\langle x_T^N, G^N x_T^N\big\rangle_{\mathcal{H}^N}$$

and the state equation:

$$\dot{x}^N(t) = A^N x^N(t) + B^N \mathbf{u}(t), \quad x^N(0) = x_0^N := P^N \mathbf{x}_0,\ t \in]0, T].$$

$(\mathcal{R}^N)$ is a linear regulator problem in the finite-dimensional state space $\mathcal{H}^N$. If $Q^N \geq 0$, $\mathbf{R} > 0$, then the optimal control for $(\mathcal{R}^N)$ is given in a feedback form by

$$u(t)^N = -\mathbf{R}^{-1} B^{N*} \Pi^N(t) x^N(t),$$

where $x^N(t)$ is the corresponding solution of the state equation with $\mathbf{u}(t) = u(t)^N$ and $\Pi^N(t) \in \mathcal{L}(\mathcal{H}^N)$ is the unique nonnegative self-adjoint solution of the Riccati differential equation, see [1],

$$\begin{aligned} &\dot{\Pi}^N(t) = -\big(Q^N + A^{N*}\Pi^N(t) + \Pi^N(t)A^N - \Pi^N(t)B^N\mathbf{R}^{-1}B^{N*}\Pi^N(t)\big), \\ &\Pi^N(T) = G^N. \end{aligned} \tag{2.1}$$

Similar to [3, (H2)], we assume for $N \to \infty$:

(H)

(i) For all $\varphi \in \mathcal{H}$ it holds that $T^N(t)P^N\varphi \to \mathbf{T}(t)\varphi$ uniformly on any bounded subinterval of $[0, T]$.

(ii) For all $\varphi \in \mathcal{H}$ it holds that $T^N(t)^* P^N \phi \to \mathbf{T}(t)^*\varphi$ uniformly on any bounded subinterval of $[0, T]$.

(iii) For all $v \in \mathcal{U}$ it holds $B^N v \to \mathbf{B}v$ and for all $\varphi \in \mathcal{H}$ it holds that $B^{N*}P^N\varphi \to \mathbf{B}^*\varphi$.

(iv) For all $\varphi \in \mathcal{H}$ it holds that $Q^N P^N \varphi \to \mathbf{Q}\varphi$.

(v) For all $\varphi \in \mathcal{H}$ it holds that $G^N P^N \varphi \to \mathbf{G}\varphi$.

Assumption (ii) implies that $P^N\varphi \to \varphi$ for all $\varphi \in \mathcal{H}$, in this sense the subspaces $\mathcal{H}^N$ approximate $\mathcal{H}$.

Theorem 2.1 *Let* (H) *hold, then for* $N \to \infty$ *the sequences* $u^N \to \mathbf{u}$, *uniformly on* $[0, T]$, $x^N \to \mathbf{x}$ *uniformly on* $[0, T]$, *and for* $\varphi \in \mathcal{H}$,

$$\Pi^N(t)P^N\varphi \to \Pi(t)\varphi \quad \text{uniformly in } t \in [0, T]. \tag{2.2}$$

Here u^N, $\mathbf{u}$, x^N, $\mathbf{x}$ *denote optimal controls and trajectories of the problems* ($\mathcal{R}^N$) *and the infinite dimensional problem, respectively.*

The proof follows from of the result proposed in [8, Theorem 5.1, p. 560]. This theorem can be extended for approximating schemes where $\mathcal{H}^N \not\subseteq \mathcal{H}$, e.g., if boundary elements are applied. Moreover, the result can be extended to the non-autonomous case, for a detail explanation see [5]. This is particularly useful for solving nonlinear problems in model predictive control and receding horizon context. There the LQG approach is applied to a linearization around a reference trajectory. This requires the solution of RDEs, in which the coefficient matrices are time dependent. We will briefly review the later in next section.

Note that the solution of the RDE is suboptimal in terms of the optimal cost which is of interest in applications. The optimal cost for the infinite and finite dimensional control problems, can be found, respectively, as

$$\bar{\mathbf{J}}(\mathbf{x}, \mathbf{u}) = \mathbf{x}_0{}^* \Pi(0)\mathbf{x}_0, \qquad \bar{J}_N(x, \mathbf{u}) = x_0^{N*} \Pi^N(0) x_0^N. \tag{2.3}$$

3 LQG Design and Stochastic Control Problems

One of the most important classes of stochastic control problems is a class of LQG problems. This approach represents an extension of the LQR which allows Gaussian noise, see for instance [15]. We consider a nonlinear stochastic control system

$$\dot{x}(t) = f\big(x(t)\big) + Bu(t) + Fv(t), \quad x(0) = x_0 + \eta_0, \tag{3.1}$$

where $v(t)$ is an unknown Gaussian disturbance process and η_0 denotes the noise in the initial condition. The observation process $y(t) = Cx(t) + w(t)$ provides partial observations of the state $x(t)$, where $w(t)$ is a measurement noise process which is assumed to be Gaussian. If we linearize f around a reference trajectory $x^*(t)$ we obtain the time-varying system

$$\dot{z}(t) = A(t)z(t) + B\tilde{u}(t) + Fv(t), \quad z(0) = \eta_0,$$

where $z(t) = x(t) - x^*(t)$, $A(t) = A(x^*(t)) = f'(x^*(t))$ and $\tilde{u}(t) = u(t) - u^*(t)$ and u^* is the associated control to x^*.

Let $Q \in \mathbb{R}^{n \times n}$ denote a positive definite matrix and consider the tracking problem for the pair (x^*, u^*), with the functional cost

$$J(z_0, \tilde{u}) = \int_0^T \big\{z(t)^T C^T QCz(t) + \tilde{u}(t)^T R\tilde{u}(t)\big\}\, dt + z(T)^T Gz(T)$$

and the state and the output equations

$$\dot{z}(t) = A(t)z(t) + B\tilde{u}(t) + Fv(t), \quad z(0) = \eta_0;$$
$$y(t) = Cx(t) + w(t), \quad t \in [0, T].$$

For the feedback law we use an estimated state of the process which is based on the measured output $\tilde{y}$, i.e., $\tilde{u}(t) = -K(t)\hat{z}(t)$, where $\hat{z}(t)$ denotes the estimated state of the system. Applying a Kalman filter, [6], the estimated state $\hat{z}(t)$ is given by

$$\hat{z}(t) = A(t)\hat{z}(t) + B\tilde{u}(t) + L(t)\big(y(t) - C\hat{x}(t)\big).$$

The feedback law can be represented as $u(t) = u_*(t) + K(t)^T(\hat{x}(t) - x^*(t))$, where $K(t)$ is the feedback matrix defined as $K(t) = -X_*(t)BR^{-1}$ and $X_*(t)$ is the unique nonnegative self-adjoint solution of the RDE

$$\dot{X}(t) = -\big(C^T QC + A(t)^T X(t) + X(t)A(t) - X(t)BR^{-1}B^T X(t)\big). \tag{3.2}$$

Moreover, the filter gain matrix $L(t)$ is given by $L(t) = \Sigma_*(t)C^T W^{-1}$ and $\Sigma_*(t)$ is the symmetric solution of the filter RDE

$$\dot{\Sigma}(t) = F^T VF + A(t)\Sigma(t) + \Sigma(t)A(t)^T - \Sigma(t)C^T W^{-1}C\Sigma(t). \tag{3.3}$$

For a detail explanation we refer the reader to [15] and references therein.

The solution on $[0, T]$ is obtained by concatenation of the solutions on $[T_i, T_{i+1}]$ for $i = 0, 1, \ldots$. The optimal control for the problem on $[T_i, T_{i+1}]$ is computed via LQG applying (3.2) and (3.3). A similar to Theorem 2.1 is proved in [9].

We consider now an abstract stochastic Cauchy problem in a separable Hilbert space $\mathcal{H}$ given by an Itô stochastic differential equation

$$d\mathbf{x}(t) = \big[\mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) + \mathbf{b}\big]dt + \big[\mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t) + \mathbf{d}\big]dW(t),$$
$$\mathbf{x}(0) = \mathbf{x}_0 \in \mathcal{H},$$

for $t \in [0, T]$, where $\mathbf{A}$ is a linear closed operator in $\mathcal{H}$, $\mathbf{d} \in \mathcal{L}([0, T], \mathcal{H})$, operators $\mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{b}$ are linear and bounded and $\{W(t), t \in [0, T]\}$ is an one dimensional standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ over $t \in [0, T]$, with $W(0) = 0$ a.e. and $\{\mathcal{F}_t, t \in [0, T]\}$ is the filtration of the process $\{W(t), t \in [0, T]\}$. The aim of the stochastic linear quadratic problem (SLQ) is to minimize the cost functional represented in terms of the mathematical expectation with respect to the measure $\mathbb{P}$

$$\mathbf{J}(\mathbf{u}) = E\left[\int_0^T \big\{\langle \mathbf{x}, Q\mathbf{x}\rangle_{\mathcal{H}} + \langle \mathbf{u}, \mathbf{R}\mathbf{u}\rangle_{\mathcal{U}}\big\}\,dt + \langle \mathbf{x}_T, \mathbf{G}\mathbf{x}_T\rangle_{\mathcal{H}}\right],$$

over a set of square integrable controls $\mathbf{u} \in L^2(\Omega, \mathcal{U})$ which are adapted in the filtration $\mathcal{F}_t$. The operators $\mathbf{Q}, \mathbf{G}$ are self adjoint on $\mathcal{H}$. Following [2, 4, 17] one can state and prove a theorem similar to Theorem 2.1 for stochastic case. Denote

$\mathbf{M} = \mathbf{R} + \mathbf{D}^*\mathbf{P}\mathbf{D}$ and $\mathbf{F} = \mathbf{P}\mathbf{B}^* + \mathbf{C}^*\mathbf{P}\mathbf{D}$. Let $\mathbf{A}$ be the generator of an exponentially bounded semigroup in $\mathcal{H}$ and $\overline{dom\mathbf{A}} = \mathcal{H}$. If we assume that $\mathbf{R} > 0$, $\mathbf{D} > 0$, $\mathbf{Q} > 0$ and $\mathbf{M} > 0$ and $\mathbf{M}$ has inverse a.e., then there exists a feedback control $\mathbf{u}_*(t) = -\mathbf{M}^{-1}(\mathbf{B}^*\phi + \mathbf{D}^*\mathbf{P}\mathbf{d})(\mathbf{x}(t) + 1)$ where $\mathbf{P}$ represents the unique nonnegative solution to the stochastic Riccati differential equation

$$\dot{\mathbf{P}} = -\big(\mathbf{Q} + \mathbf{A}^*\mathbf{P} + \mathbf{P}\mathbf{A}^* + \mathbf{C}^*\mathbf{P}^*\mathbf{C} - \mathbf{F}\mathbf{M}^{-1}\mathbf{F}^*\big),$$
$$\mathbf{P}(T) = \mathbf{G} \text{ a.e. } t \in [0, T]$$

and ϕ is the solution of the backward SDE

$$\dot{\phi} + \big(\mathbf{A}^* - \mathbf{F}\mathbf{M}^{-1}\mathbf{B}^*\big)\phi + \big(\mathbf{C}^* - \mathbf{F}\mathbf{M}^{-1}\mathbf{D}^*\big)\mathbf{P}\mathbf{d} + \mathbf{P}\mathbf{b} = 0,$$
$$\phi(T) = 0 \text{ a.e. } t \in [0, T].$$

If $\mathbf{D} = 0$ then SRDE reduces to the deterministic one. In order to find numerically the solution of the SRDE one needs also to consider the sequence of finite dimensional subspaces of a Hilbert space of random variables $L^2(\Omega, \mathcal{H})$, i.e., the finite dimensional Wiener chaos spaces $\mathcal{H}_k$, $k \geq 0$, [14]. Note that all the coefficients appearing are not random. Assumption $\mathbf{M} > 0$ ensures the existence of a solution of finite-dimensional SRDE [2, 17]. Due to the presence of the control in the diffusion term, some of SLQ problems could be well-posed even when $\mathbf{R}$ is negative definite for almost all $t \in [0, T]$, which makes SLQ problem more difficult to solve than deterministic ones. A convergence result for the SLQ problem, similar to Theorem 2.1, as well as other types of stochastic control problems will be reported somewhere else.

4 Numerical Results

We consider the Burgers equation which is used as a model for description of basic phenomena of flow problems

$$\begin{aligned} x_t(t,\xi) &= \nu x_{\xi\xi}(t,\xi) - x(t,\xi)x_\xi(t,\xi) + B(\xi)u(t) + F(\xi)v(t), \\ x(t,0) &= x(t,1) = 0, \quad t > 0, \\ x(0,\xi) &= x_0(\xi) + \eta_0(\xi), \quad \xi \in]0,1[\end{aligned} \tag{4.1}$$

where t is the variable in time, ξ the variable in space, and ν is a viscosity parameter, and the observation process $y(t,\xi) = Cx(t,\xi) + w(t,\xi)$. The aim is to control the state to 0. The initial condition is given in terms of *sine* function and parameters are defined as in [16]. The state is shown in Fig. 1 as well as a plot of the convergence of the cost functional (2.3) over the mesh size. This let us visualize the convergence result, Theorem 2.1.

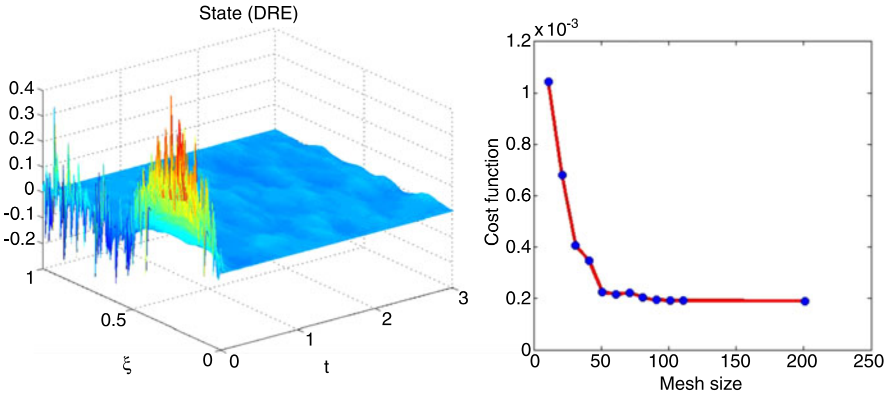

Fig. 1 Burgers equation with noise in the initial condition (**a**) state for refined mesh and (**b**) functional cost for different mesh sizes

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Part VI
Topological and Geometrical Methods

Organizers: Anatoly Prykarpatskyi, Yurii Zelinskyi, Kamal Soltanov, Alexander Schmitt

Ergodic Theory, Boole Type Transformations, Dynamical Systems Theory

Anatolij K. Prykarpatski

Abstract The arithmetic properties of generalized one-dimensional ergodic Boole type transformations are studied in the framework of the operator-theoretic approach. Some invariant measure statements and ergodicity conjectures concerning generalized multi-dimensional Boole-type transformations are formulated.

Keywords Generalized Boole type transformations · Arithmetic properties · Ergodic dynamical systems · Invariant measures · Frobenius–Perron operator

Mathematics Subject Classification (2010) Primary 34A30 · 34B05 · 34B15 · Secondary 35D35 · 35J60 · 35Q82

1 Introduction

With its origins, going back several centuries, discrete analysis becomes now an increasingly central methodology for many mathematical problems related to discrete dynamical systems and algorithms, widely applied in modern science. Our theme, being related with studying ergodic aspects and the related arithmetic properties of discrete Boole type dynamical systems [3, 7], is of deep interest in many branches of modern science and technology [6, 19], especially in discrete mathematics, numerical analysis, statistics and probability theory as well as in electrical and electronic engineering. But the important viewpoint is that this topic belongs to a much more general realm of mathematics, namely, to calculus, differential equations and differential geometry, because of the remarkable analogy of the subject especially to these branches of mathematics. Nonetheless, although the topic is discrete, our approach to treating ergodicity and the related arithmetic properties of the generalized Boole type discrete dynamical systems will be completely analytical.

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The generalized Boole transformation looks as

$$\mathbb{R} \ni x \to \varphi(x) := \alpha x + a - \sum_{j=1}^{N} \frac{\beta_j}{x - b_j} \in \mathbb{R}, \tag{1.1}$$

where a and $b_j \in \mathbb{R}$, $j = \overline{1, N}$, are some real and $\alpha, \beta_j \in \mathbb{R}_+$, $j = \overline{1, N}$, and was analyzed in [3, 15]. It generalizes that classical [7] Boole transformation

$$\mathbb{R} \ni x \to \phi(x) := x - 1/x \in \mathbb{R}, \tag{1.2}$$

which appeared to be ergodic [5] with respect to the invariant standard infinite Lebesgue measure on $\mathbb{R}$. This, in particular, means that the following Boole's [7] equalities

$$\int_{\mathbb{R}} f(x - 1/x)dx = \int_{\mathbb{R}} f(x)dx, \tag{1.3}$$

and

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n-1} f(\phi^n x)}{\sum_{k=0}^{n-1} g(\phi^n x)} = \frac{\int_{\mathbb{R}} f(x)dx}{\int_{\mathbb{R}} g(x)dx} \tag{1.4}$$

hold for any $f \in L_1(\mathbb{R}; \mathbb{R})$ and $g \in L_1(\mathbb{R}; \mathbb{R}_+)$. In the case $\alpha = 1$, $a = 0$, a similar ergodicity result was proved in [1–3] making use of a specially devised inner function method. The related spectral aspects of the mapping (1.1) were in part studied also in [3]. In spite of these results the case $\alpha \neq 1$ still persists to be challenging as the only relating result [4] concerns the following special case of (1.1):

$$\mathbb{R} \ni x \to \varphi(x) := \alpha x + a - \frac{\beta}{x - b} \in \mathbb{R} \tag{1.5}$$

for $0 < \alpha < 1$, and arbitrary $a, b \in \mathbb{R}$ and $\beta \in \mathbb{R}_+$. The ergodicity of the Boole type mapping (1.5) can be easily enough stated. Really, concerning a general nonsingular mapping $\varphi : \mathbb{R} \to \mathbb{R}$, the problem of constructing the measure preserving ergodic measures was analyzed [4, 12, 14] by means of studying the spectral properties of the adjoint Frobenius–Perron operator $\hat{T}_\varphi \rho : L_2(\mathbb{R}; \mathbb{R}) \to L_2(\mathbb{R}; \mathbb{R})$, where, by definition,

$$\hat{T}_\varphi \rho(x) := \sum_{y \in \{\varphi^{-1}(x)\}} \rho(y) J_\varphi^{-1}(y) \tag{1.6}$$

for any $\rho \in L_2(\mathbb{R}; \mathbb{R}_+)$ and $J_\varphi^{-1}(y) := |\frac{d\varphi(y)}{dy}|$, $y \in \mathbb{R}$. Then, if $\hat{T}_\varphi \rho = \rho$, $\rho \in L_2(\mathbb{R}; \mathbb{R}_+)$, then the expression $d\mu(x) := \rho(x)dx$, $x \in \mathbb{R}$, will be invariant, in general infinite, measure with respect to the mapping $\varphi : \mathbb{R} \to \mathbb{R}$. Another way to finding a general algorithm for finding such an invariant measure was devised in [15–17], making use of the generating measure function method. (1.5) at $\alpha = 1/2$ and $b = 2a \in \mathbb{R}$ appears to be measure preserving and ergodic. Namely, the following propositions [15] hold.

Proposition 1.1 *The Boole type transformation* (1.5) *at* $\alpha = 1/2$ *and* $b = 2a \in \mathbb{R}$ *is measure preserving and ergodic with respect to the measure*

$$d\mu(x) := \frac{|\gamma| dx}{\pi[(x-2a)^2 + \gamma^2]}, \tag{1.7}$$

where $x \in \mathbb{R}$ *and* $\gamma^2 = 2\beta \in \mathbb{R}_+$.

Proof (*Sketch*) A proof follows easily from the fact that the function

$$\rho(x) := \frac{\gamma}{\pi[(x-2a)^2 + \gamma^2]} \tag{1.8}$$

satisfies for all $x \in \mathbb{R}$ the determining condition (1.6):

$$\hat{T}_{\varphi}\rho(x) := \sum_{I} \rho(y_{\pm}) \left| y'_{\pm}(x) \right|, \tag{1.9}$$

where, by definition, $\varphi(y_{\pm}(x)) := x$ for any $x \in \mathbb{R}$. The relationship (1.9) is, evidently, equivalent to the next infinitesimal invariance condition

$$\sum_{\pm} d\mu\big(y_{\pm}(x), y_{\pm}(x) + dy\big) = d\mu(x) := \mu(x, x + dx) \tag{1.10}$$

for any infinitesimal subset $[x, x + dx) \subset \mathbb{R}$. □

Proposition 1.2 *The measure* (1.7) *is ergodic with respect to the Boole type transformation* (1.5) *at* $\alpha = 1/2$ *and* $b = 2a \in \mathbb{R}$ *as it is equivalent to the canonical ergodic mapping* $\mathbb{R}/\mathbb{Z} \ni s :\rightarrow \psi(s) := 2s \pmod{\mathbb{Z}} \in \mathbb{R}/\mathbb{Z}$ *with respect to the standard Lebesgue measure on* $\mathbb{R}/\mathbb{Z}$.

Proof (*Sketch*) Put, by definition, $\mathbb{R}/\mathbb{Z} \ni s :\rightarrow \xi(s) = y \in \mathbb{R}$, where

$$\xi(s) := \gamma \cot \pi s + 2a, \tag{1.11}$$

Then transformation (1.5) at $\alpha = 1/2$, $b = 2a \in \mathbb{R}$ and $\gamma^2 := 2\beta \in \mathbb{R}_+$, owing to the mapping (1.11), yields

$$\begin{aligned}
\varphi(y) = \varphi\big(\xi(s)\big) &= \frac{\gamma}{2} \cot \pi s + 2a - \frac{\gamma}{2} \tan \pi s \\
&= \frac{\gamma(\cos^2 \pi s - \sin^2 \pi s)}{2 \sin \pi s \cos \pi s} + 2a = \gamma \frac{\cos 2\pi s}{\sin 2\pi s} + 2a \\
&= \gamma \cot 2\pi s + 2a := \xi(2s)
\end{aligned} \tag{1.12}$$

for any $s \in \mathbb{R}/\mathbb{Z}$. The result (1.12) means that the transformation (1.5) is conjugated [3, 12] to the transformation

$$\mathbb{R}/\mathbb{Z} \ni s :\rightarrow \psi(s) = 2s \bmod 1 \in \mathbb{R}/\mathbb{Z}, \tag{1.13}$$

that is the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{R}/\mathbb{Z} & \stackrel{\psi}{\to} & \mathbb{R}/\mathbb{Z} \\ \xi\downarrow & & \downarrow\xi \\ \mathbb{R} & \stackrel{\varphi}{\to} & \mathbb{R} \end{array} \tag{1.14}$$

that is $\xi \cdot \psi = \varphi \cdot \xi$, where $\xi : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ is the conjugation mapping defined by (1.11). It is easy now to check that the measure (1.7) under the conjugation (1.11) transforms into the standard normalized Lebesgue measure on $\mathbb{R}/\mathbb{Z}$:

$$\begin{aligned} d\mu(x)|_{x=\gamma\cot\pi s+2a} &= \frac{ds\gamma^2|d(\cot\pi s)/ds|}{(\gamma^2\cot^2\pi s+\gamma^2)} \\ &= \frac{\sin^2\pi s\cdot\sin^{-2}\pi s ds}{\cos^2\pi s+\sin^2\pi s} = ds, \end{aligned} \tag{1.15}$$

where $s \in \mathbb{R}/\mathbb{Z}$. The infinitesimal measure ds on $\mathbb{R}/\mathbb{Z}$ as well as the infinitesimal measure (1.7) on $\mathbb{R}$ are normalized, being thus probabilistic. Now it is enough to make use of the fact that the measure ds on $\mathbb{R}/\mathbb{Z}$ on the interval $[0, 1) \simeq \mathbb{R}/\mathbb{Z}$ is ergodic [4, 12] with respect to the mapping $\psi : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$. □

It is important to mention that in the framework of the theory of inner functions in [4] there was stated that there exists an invariant measure $d\mu(x)$, $x \in \mathbb{R}$, on the axis $\mathbb{R}$, such that the generalized Boole type transformation (1.1) for any $N > 1$, $\alpha = 1$ and $a = 0$ is ergodic. If $\alpha = 1$ and $a \neq 0$, the transformation (1.1) appears to be not ergodic, being totally dissipative, that is the wandering set $\mathcal{D}(\varphi) := \bigcup \mathcal{W}(\varphi) = \mathbb{R}$, where $\mathcal{W}(\varphi) \subset \mathbb{R}$ are such subsets that all sets $\varphi^{-n}(\mathcal{W})$, $n \in \mathbb{Z}_+$, are disjoint. Similar the above statement can be also [1, 4] formulated for the mostly generalized Boole type transformation

$$\mathbb{R} \ni x \to \varphi(x) := \alpha x + a + \int_{\mathbb{R}} \frac{d\nu(s)}{s-x} \in \mathbb{R}, \tag{1.16}$$

where $a \in \mathbb{R}$, $\alpha \in \mathbb{R}_+$ and a measure ν on $\mathbb{R}$ has the compact support $\operatorname{supp}\nu \subset \mathbb{R}$, being such that the following natural conditions

$$\int_{\mathbb{R}} \frac{d\nu(s)}{1+s^2} = a, \quad \int_{\mathbb{R}} d\nu(s) < \infty, \tag{1.17}$$

hold.

Below we will analyze the related arithmetic aspects of the generalized ergodic Boole type transformation (1.1), making use of the approaches recently initiated in [8, 11, 13, 20–22] concerning the old general Baudet arithmetic progression conjecture.

2 The Generalized Boole Type Ergodic Transformations and Their Arithmetic Properties

Consider the generalized Boole type transformation (1.1) and its right orbit $Or(\varphi; x_0) := \{\varphi^j(x_0) \in \mathbb{R} : x_0 \in \mathbb{R}, j \in \mathbb{Z}_+\}$ for an arbitrary $x_0 \in \mathbb{R}\backslash\{\alpha \in \mathbb{R} : \varphi(\alpha) = \alpha\}$. Owing to the ergodicity of the mapping (1.1), one easily obtains that the closure $\overline{Or(\varphi; x_0)} = \bar{\mathbb{R}}$. Thus, for any point $\alpha \in \mathbb{R}$ one can find a convergent subsequence $\{\varphi^{n_j}(x_0) \in \mathbb{R} : n_j := n_j(\alpha) \in \mathbb{Z}_+, j \in \mathbb{Z}_+\} \subset Or(\varphi; x_0)$, such that

$$\lim_{j\to\infty} \varphi^{n_j}(x_0) = \alpha. \tag{2.1}$$

The corresponding integer subsequence

$$A(\alpha) := \big\{n_1(\alpha), n_2(\alpha), \dots, n_j(\alpha), \dots\big\} \subset \mathbb{Z}_+, \tag{2.2}$$

owing to the condition (2.1), *a priori* possesses, owing to the Weil theorem [9, 12, 14] the following upper density property:

$$\bar{d}\big(A(\alpha)\big) := \overline{\lim_{m\to\infty}} \sharp\big(A(\alpha) \cap \{0, 1, 2, \dots, m\}\big)/(m+1) > 0. \tag{2.3}$$

Consider now the left shift mapping

$$\theta : l_\infty(\mathbb{Z}_+; \mathbb{R}) \ni (c_0, c_1, \dots, c_n, \dots) \to (c_1, \dots, c_n, \dots) \in l_\infty(\mathbb{Z}_+; \mathbb{R}) \tag{2.4}$$

and put, by definition, the set

$$\mathcal{A}(\alpha) := \overline{\big\{\theta^n 1_{A(\alpha)} \in l_\infty(\mathbb{Z}_+; \mathbb{R}) : n \in \mathbb{Z}_+\big\}}, \tag{2.5}$$

the closure with respect to the weak σ^*-topology of $l_\infty(\mathbb{Z}_+; \mathbb{R})$. The constructed set (2.5) is, by definition, θ-invariant and σ^*-weakly compact in $l_\infty(\mathbb{Z}_+; \mathbb{R})$. Its subset

$$\mathcal{A}_0(\alpha) := \big\{(c_0, c_1, \dots, c_n, \dots) \in \mathcal{A}(\alpha) : c_0 := 1\big\}, \tag{2.6}$$

as well as its preimages $\theta^{-j}(\mathcal{A}_0(\alpha))$, $j \in \mathbb{N}$, are open-closed subsets of $\mathcal{A}(\alpha)$. It is easy to observe the following characteristic [8, 10] property of the set $\mathcal{A}_0(\alpha)$:

$$n \in A(\alpha) \quad \text{iff} \quad \theta^n 1_{A(\alpha)} \in \mathcal{A}_0(\alpha). \tag{2.7}$$

Following the classical Furstenberg scheme [10] one can construct an invariant probabilistic measure ν on the compact set $\mathcal{A}(\alpha)$. Namely, owing to the condition (2.3) one can chose an infinite subsequence $\{m_j \in \mathbb{Z}_+ : j \in \mathbb{Z}\}$, such that there exists the limit

$$\lim_{j\to\infty} \frac{1}{m_j + 1} \sum_{k=0}^{m_j} \delta_k\big(A(\alpha)\big) = \bar{d}\big(A(\alpha)\big). \tag{2.8}$$

Now making use of the property (2.8) one can define an infinite sequence of probability measures $\{\nu_j : j \in \mathbb{Z}_+\}$ on $\mathcal{A}(\alpha)$

$$\nu_j(\mathcal{B}) := \frac{1}{m_j+1}\sum_{k=0}^{m_j}\delta_{\{\theta^k 1_{A(\alpha)}\}}(\mathcal{B}) \tag{2.9}$$

for any Borel subsets of $\mathcal{A}(\alpha)$. In particular, one has

$$\nu_j\big(\mathcal{A}_0(\alpha)\big) := \frac{1}{m_j+1}\sum_{k=0}^{m_j}\delta_{\{\theta^k 1_{A(\alpha)}\}}(\mathcal{B}) \overset{j\to\infty}{\to} \bar{d}\big(A(\alpha)\big). \tag{2.10}$$

Based on the Banach–Alaoglu theorem and on the metrisability of the set $\mathcal{A}(\alpha) \subset l_\infty(\mathbb{Z}_+;\mathbb{R})$ with respect to the weak σ^*-topology one obtains that there exists a convergent subsequence of measures (2.10) to some probability measure ν on $\mathcal{A}(\alpha)$, which can be re-denoted as $\{\nu_j : j \in \mathbb{Z}_+\}$. It is important that the obtained above measure ν on $\mathcal{A}(\alpha)$ is θ-invariant:

$$\begin{aligned}
&\nu(\mathcal{B}) - \nu\big(\theta^{-1}\mathcal{B}\big)\\
&\quad= \lim_{j\to\infty}\frac{1}{m_j+1}\left(\sum_{k=0}^{m_j}\delta_{\{\theta^k 1_{A(\alpha)}\}}(\mathcal{B}) - \delta_{\{\theta^{k+1} 1_{A(\alpha)}\}}(\mathcal{B})\right)\\
&\quad= \lim_{j\to\infty}\frac{1}{m_j+1}\left(\sum_{k=0}^{m_j}\delta_{\{1_{A(\alpha)}\}}(\mathcal{B}) - \delta_{\{\theta^{m_j} 1_{A(\alpha)}\}}(\mathcal{B})\right) = 0
\end{aligned} \tag{2.11}$$

for any Borel subset of $\mathcal{B} \subset \mathcal{A}(\alpha)$, as $m_j \to \infty$ if $j \to \infty$. Thus, the constructed measure-theoretic dynamical system $(\mathcal{A}(\alpha), \theta; \nu)$ is characterized by the condition $\nu(\mathcal{A}_0(\alpha)) = \bar{d}(A(\alpha)) > 0$. Moreover, taking into account that the ergodic measures are extreme points [9, 12, 14] of the set of invariant measures on $\mathcal{A}(\alpha)$, one can choose this limiting invariant measure ν on the set $\mathcal{A} = \mathcal{A}(\alpha)$ to be ergodic.

Define now for the mapping (1.1) a linear operator $T_\varphi : L_2^{(\nu)}(\mathcal{A};\mathbb{R}) \to L_2^{(\nu)}(\mathcal{A};\mathbb{R})$, which satisfies for any $f \in L_2^{(\nu)}(\mathcal{A};\mathbb{R})$ the shift property $T_\theta f(c) := f(\theta(c))$, $c \in \mathcal{A}$. Based on the existence of the invariant and ergodic measure ν on $\mathcal{A}(\alpha)$, one can state the following characteristic proposition.

Proposition 2.1 *If the set $A(\alpha) \subset \mathbb{Z}_+$ possesses a positive upper density $\bar{d}(A(\alpha)) > 0$, then for any strongly positive function $f \in L_2^{(\nu)}(\mathcal{A};\mathbb{R}_+) \cap L_\infty^{(\nu)}(\mathcal{A};\mathbb{R}_+)$ and for arbitrary ergodic measure ν on the set $\mathcal{A} = \mathcal{A}(\alpha)$ there holds the following strong inequality*:

$$\overline{\lim}_{N\to\infty}\frac{1}{N+1}\sum_{n=0}^{N}\int_{\mathcal{A}}\left(\prod_{j=0}^{N} T_\theta^{n_j} f\right) d\nu > 0. \tag{2.12}$$

And conversely, if for a chosen subsequence $A = \{n_1, n_2, \dots, n_j, \dots\} \subset \mathbb{Z}_+$ *and any strongly positive function* $f \in L_2^{(\nu)}(\mathcal{A}; \mathbb{R}_+) \cap L_\infty^{(\nu)}(\mathcal{A}; \mathbb{R}_+)$ *on a compact set* $\mathcal{A}$ *there holds the strong inequality* (2.12), *then the upper density* $\bar{d}(A) > 0$ *and there exists such a point* $x_0 \in \mathbb{R}$ *and a real value* $\alpha \in \mathbb{R}$ *that there exists the limit* $\lim_{j\to\infty} \varphi^{n_j}(x_0) = \alpha$. *Moreover, the corresponding sets* $\mathcal{A}(\alpha)$ *and* $\mathcal{A}$ *coincide.*

Proof (*Sketch*) We can easily observe, taking into account (2.7) that the following one-to-one mapping holds between the sets $A(\alpha)$ and the subset $\{\theta^{n_1} 1_{A(\alpha)}, \theta^{n_2} 1_{A(\alpha)}, \dots, \theta^{n_j} 1_{A(\alpha)}, \dots\} \subset \mathcal{A}_0(\alpha)$. Thus, since each set $\theta^{-j}(\mathcal{A}_0(\alpha)) \subset \mathcal{A}(\alpha)$ is open and the points $\theta^n 1_{A(\alpha)}$, $n \in \mathbb{Z}_+$, are dense in $\mathcal{A}(\alpha)$, one finds that the set $A(\alpha)$ is in one-to-one correspondence to the condition that $\bigcap_{j\in\mathbb{Z}_+} \theta^{-n_j}(\mathcal{A}_0(\alpha)) \neq \emptyset$. The latter easily reduces to the relationship $\prod_{j=0}^{N} T_\theta^{n_j} 1_{\mathcal{A}_0(\alpha)} \neq 0$ for any $N \in \mathbb{Z}_+$, which allows to formulate a sufficient integral condition [10] in the form (2.12), thus proving the first part of the proposition. Having followed back by the reasonings above, one can state that for every ergodic measure ν on a chosen weakly σ^*-compact set $\mathcal{A} \subset l_\infty(\mathbb{Z}_+; \mathbb{R})$ one can find a subset of integers $A := \{n_j \in \mathbb{Z}_+ : j \in \mathbb{Z}_+\}$ with a nonzero upper density $\bar{d}(A) > 0$, for which there exists the standard representation

$$\mathcal{A} := \overline{\left\{\theta^n 1_A \in l_\infty(\mathbb{Z}_+; \mathbb{R}) : n \in \mathbb{Z}_+\right\}}. \tag{2.13}$$

Moreover, for the open subset

$$\mathcal{A}_0 := \left\{(c_0, c_1, \dots, c_n, \dots) \in \mathcal{A} : c_0 := 1\right\} \tag{2.14}$$

there holds the equality $\bar{d}(A) = \nu(\mathcal{A}_0)$. Now making use of the condition $\prod_{j=0}^{N} T_\theta^{n_j} 1_{\mathcal{A}_0} \neq 0$, $N \in \mathbb{Z}_+$, one can find such a point $x_0 \in \mathbb{R}$ that the corresponding iterations $\{\varphi^{n_j}(x_0) \in \mathbb{R} : n_j \in A, j \in \mathbb{Z}_+\}$ are convergent to some real value $\alpha := \lim_{j\to\infty} \varphi^{n_j}(x_0)$ and, simultaneously, the whole orbit $\{\varphi^n(x_0) \in \mathbb{R} : n \in \mathbb{Z}_+\}$ is dense on the axis $\mathbb{R}$. The letter proves the second part of the proposition. □

3 Conclusion

Recently in [18] there was proposed a set of multi-dimensional Boole type transformations $\varphi_{\sigma|\eta} : \mathbb{R}^n \to \mathbb{R}^n$, where

$$\begin{aligned}\varphi_{\sigma|\eta}&(x_1, x_2, \dots, x_n) \\ &:= (x_{\eta(1)} - 1/x_{\sigma(1)}, x_{\eta(2)} - 1/x_{\sigma(2)}, \dots, x_{\eta(n)} - 1/x_{\sigma(n)})\end{aligned} \tag{3.1}$$

for any $n \in \mathbb{N}$ and arbitrary permutations σ and $\eta \in S_n$. For the case $n = 2$ one obtains the following two-dimensional Boole type mappings:

$$\varphi_{1|1}(x, y) := (x - 1/x, y - 1/y), \tag{3.2}$$

$$\varphi_{2|2}(x, y) := (y - 1/y, x - 1/x), \tag{3.3}$$

and

$$\varphi_{1|2}(x, y) := (x - 1/y, y - 1/x), \tag{3.4}$$

$$\varphi_{2|1}(x, y) := (y - 1/x, x - 1/y) \tag{3.5}$$

for all $(x, y) \in \mathbb{R}^2 \backslash \{0, 0\}$. It is easy to observe that the infinitesimal measure $d\mu(x, y) := dxdy$ on the plane $\mathbb{R}^2$ is, by the Fubini theorem, invariant subject to the mapping (3.2) as the tensor product of two one-dimensional measures dx and dy, every one of which is invariant with respect to the corresponding true Boole transformation. The latter entails right away that the generalized Boole type transformation (3.2) is also ergodic. In the case of the generalized two-dimensional transformation (3.4) the infinitesimal invariance property of the measure $d\mu(x, y) := dxdy$ holds owing to the following lemma, stated in [18].

Lemma 3.1 *The mapping* (3.4) *subject to the measure* $d\mu(x, y)$ *on* $\mathbb{R}^2$ *satisfies the following infinitesimal invariance property*:

$$\begin{aligned} &\mu\big(\psi_{1|2}^{-1}\big([x, x + dx] \times [y, y + dy]\big)\big) \\ &\quad = dxdy = \mu\big([x, x + dx] \times [y, y + dy]\big) \end{aligned} \tag{3.6}$$

for all $(x, y) \in \mathbb{R}^2 \backslash \{0, 0\}$.

It is easy to check that a similar statement and the infinitesimal invariance property like (3.6) hold also in the case of the two-dimensional Boole type transformations (3.3), (3.4) and (3.5). As the problem of ergodicity of the mappings (3.3)–(3.5) is of great interest, we formulate the following conjecture, generalizing that from [18].

The constructed above mappings (3.2)–(3.5) are ergodic with respect to the invariant infinitesimal measure $d\mu(x, y) := dxdy$ on $\mathbb{R}^2$. Moreover, for any $n \in \mathbb{N}$ the infinitesimal measure $d\mu(x_1, x_2, \ldots, x_n) := \prod_{j=1}^{n} dx_j$ is invariant and ergodic with respect to generalized multi-dimensional Boole type transformations (3.1) for arbitrary chosen permutations σ and $\eta \in S_n$.

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Fixed Points Theorems for Multivalued Mappings

Yuri Zelinskii

Abstract In this article we discuss the solvability of some class of multivalued inclusions in Euclidean spaces based on a generalization of the "conditions of an acute angle". As corollary we receive fixed-point theorems for multivalued mappings (continuous and non continuous).

Keywords Multivalued mappings · Multivalued inclusions · Fixed-point · Conditions of acute angle

Mathematics Subject Classification (2010) Primary 54H25 · 54C60 · Secondary 47H10

1 Introduction

We consider theorems on the existence of solutions of multivalued inclusions in Euclidean spaces, including fixed point theorems for multivalued mappings, based on some generalization of the "conditions of an acute angle" [1]. This approach appeared in K.N. Soltanov's papers [2–4], where he proposed to use the above method for study of the fixed points for discontinuous mappings. Specificity of a discontinuity of mappings is a reason why to use traditional technique of mapping degree is impossible (cf., e.g., [5]).

Let E^n be n-dimensional Euclidean (real or complex) space, $\langle \cdot, \cdot \rangle$ be a scalar product in E^n, $\text{conv}(A)$ be a convex hull of a set A, $\overline{A}$ be a closure of A. If $\overline{A}$ is a closed domain of the Euclidean space E^n then we will say that $\overline{A}$ is a domain.

For any z in the complex plane by $\text{Re}(z)$ we mean its real part, if z belongs to the field of real numbers then $\text{Re}(z) = z$.

We will consider multivalued (continuous and non continuous) mappings of subsets of Euclidean space. If $F_1 : X \to Y$ and $F_2 : X \to Y$ be two multivalued mappings we will say that F_2 is a *restriction* of F_1 to the set $K \subset X$ and denote

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$F_2 = F_1|_K$ if and only if $F_1(x) \supset F_2(x)$ for every $x \in K$ ($F_2(x)$ can be empty for some x).

We say that a mapping F satisfied *conditions of (strong) acute angle* if and only if $\mathrm{Re}(\langle x, y\rangle)(>) \geq 0$ for every pair (x, y), $x \in X$, $y \in F(x)$. We denote

$$G = Id - F \quad \text{if and only if} \quad G(x) = \{x - y : x \in F(x)\}.$$

2 Main Results

By using the geometrical form of the Hahn–Banach theorems we prove the next results (cf., e.g., [6]).

Theorem 2.1 *Let D be a domain in Euclidean space E^n containing the origin 0. Let $K \subset \overline{D}$ be a subset of the closure of this domain and K has the following property (α): any ray, emanating from the origin, contains at least one point belonging to K. Suppose that the restriction $F|_K$ of multivalued mapping $F : \overline{D} \to E^n$ satisfies the "acute angle condition" and $\mathrm{conv}(F(K))$ is a compact set. If $\mathrm{conv}(F(K)) \subset F(\overline{D})$ then $0 \in F(\overline{D})$.*

It follows the following corollaries from Theorem 2.1.

Corollary 2.2 *Let $K \subset \overline{D}$ be a subset of the domain $\overline{D}$ and K has the property (α). Suppose that the restriction $F|_K$ of the multivalued mapping $F : \overline{D} \to E^n$ has a restriction $F_1 \neq \emptyset$ and $\mathrm{conv}(F_1(K))$ is a compact set. Let $\mathrm{conv}(F_1(K)) \subset F(\overline{D})$. If $0 \notin F(\overline{D})$ then there exists a pair of points (x, y), $x \in K$, $y \in F(x)$, such that $\mathrm{Re}(\langle x, y\rangle) < 0$.*

Corollary 2.3 *Let $K \subset \overline{D}$ be a subset of the domain $\overline{D}$ and K has the property (α). Suppose that the restriction $F|_K$ of the multivalued mapping $F : \overline{D} \to E^n$ has a restriction F_1 which satisfies the "condition of acute angle" and $\mathrm{conv}(F_1(K))$ is a compact. If $F(\overline{D}) \supset \mathrm{conv}(F_1(K))$ then $0 \in F(\overline{D})$.*

Corollary 2.4 *Let $K \subset \overline{D}$ be a subset of the domain $\overline{D}$ and K has the property (α). Suppose that the restriction $G|_K$ of the multivalued mapping $G = Id - F : \overline{D} \to E^n$ has a restriction G_1 satisfied the "condition of acute angle" and $\mathrm{conv}(G_1(K))$ is a compact. If $G(\overline{D}) \supset \mathrm{conv}(G_1(K))$ then the mapping F has a fixed point $x \in F(x)$.*

Theorem 2.5 *Let D be a domain in Euclidean space E^n containing the origin 0. Let $K \subset \overline{D}$ be a subset of $\overline{D}$ and K has the property (α). Suppose that the restriction $F|_K$ of the multivalued mapping $F : \overline{D} \to E^n$ satisfying the "strict acute angle condition". If $\mathrm{conv}(F(K)) \subset F(\overline{D})$ then $0 \in F(\overline{D})$.*

Corollary 2.6 *Let $K \subset \overline{D}$ be a subset of the domain $\overline{D}$ and K has the property (α). Suppose that the restriction $F|_K$ of multivalued mapping $F : \overline{D} \to E^n$ has the* *restriction* $F_1 \neq \emptyset$ *and* $\text{conv}(F_1(K)) \subset F(\overline{D})$. *If* $0 \notin F(\overline{D})$ *then there exists a pair of points* (x, y), $x \in K$, $y \in F(x)$ *such, that* $\text{Re}(\langle x, y\rangle) \leq 0$.

Let Y^* be dual (conjugated) space to the space Y. We say that a mapping F satisfies the *condition of (strong) coacute angle* if and only if for every point $y^* \in Y^*$, $y^* \neq 0$, there exists a point $x \in X$ such, that $\text{Re}(\langle y, y^*\rangle) \geq 0$ ($\text{Re}(\langle y, y^*\rangle) > 0$) for every point $y \in F(x)$.

Theorem 2.7 *Let* $K \subset \overline{D}$ *be a subset of the domain* $\overline{D} \subset E^n$. *Suppose that the restriction* $F|_K$ *of multivalued mapping* $F : \overline{D} \to E^n$ *has a restriction* F_1 *satisfying the "condition of coacute angle" and* $\text{conv}(F_1(K))$ *is a compact. If* $F(\overline{D}) \supset \text{conv}(F_1(K))$ *then* $0 \in F(\overline{D})$.

Proof We suppose that $0 \notin F(\overline{D})$ then $0 \notin \text{conv}(F_1(K))$. By the geometrical form of the Hahn–Banach theorem, obtain that there exists a hyperplane L which separates the origin 0 and $F_1(K)$. We chose the ray l, emanating from the origin, and perpendicular to the hyperplane L, which directed in the opposite side to $\text{conv}(F_1(K))$. For the Euclidean space a duality mapping $\mathfrak{I} : Y \to Y^*$ is a bijection. Fix any point $y^* \in l$. From the one hand, $y^* \notin \text{conv}(F_1(K))$, but from the another one, according to the condition of the coacute angle, there exists a point $x \in K$, which image $F_1(x)$ must be in the same hyperspace as the point y^* with respect to the hyperplane L. This contradiction completed the proof. □

Remark 1 If in previous results $K \subset D$ (the subset K lies in the interior of the domain D), that all stated results remain true if we replace the considered mapping to a mapping of an open domain.

Remark 2 For the validity of the previous results it is enough an existence in the space invariant for considered mappings F subspace E^n (i.e. $F(T) \subset T$) for the restriction $F|_T$ on which conditions of the corresponding statements are fulfilled.

Theorem 2.8 *Let D be a domain in Euclidean space* $X = E^n$ *and let* $K \subset \overline{D}$ *be a subset of the closure* $\overline{D}$ *of* D. *Suppose that there exists a restriction* F_1 *of multivalued mapping* $F : \overline{D} \to E^n = Y$ *to the subset* K *which satisfies the condition of a strong coacute angle. If* $F(\overline{D}) \supset \text{conv}\, F_1(K)$ *then* $0 \in F(\overline{D})$.

Proof We suppose that $0 \notin F(\overline{D})$ then $0 \notin \text{conv}(F_1(K))$. The interior $\text{Int}(\text{conv}(F_1(K)))$ is an open convex set which does not contain the origin 0. If $\text{Int}(\text{conv}(F_1(K))) = \emptyset$ then the dimension of the set $\text{conv}\, F_1(K)$ is not greater than $n - 1$. Thus, this set is a subset of some hyperplane. If $\text{Int}(\text{conv}(F_1(K))) \neq \emptyset$ then there exists a hyperplane L, which passes through the origin and does not intersect the set $\text{Int}(\text{conv}(F_1(K)))$. In both cases, the set $\text{conv}(F_1(K))$ is a subset of the one closed half-space, on which the plane L divides the whole space E^n. Then the proof of the theorem completed by similar statements as in the proof of the previous theorem. □

It is possible to intensify the obtained above results if to require performing conditions of the type of the acute angle only for some directions chosen in a specific way.

Consider *conditions of a (strong) acute ε-angle:*

$$\mathrm{Re}\big(\langle x, y\rangle\big)(>) \geq \varepsilon\|x\|\|y\| \quad \text{for every pair } (x, y),\ x \in X,\ y \in F(x).$$

Conditions of a (strong) coacute ε-angle means, that for every point $y^* \in S^n$ exists a pair (x, y) of points $x \in E^n$ and $y \in F(x)$ such, that $\mathrm{Re}(\langle y, y^*\rangle) \geq (>)\varepsilon\|y\|$.

Remark 3 Analogs of Theorems 2.1, 2.5, 2.8 are true, if in formulations of similar statements to Theorems 2.1, 2.5 to require only, for any ray, emanating from the origin, an existence a ray which crosses the set K such, that an angle between these two rays did not exceed then $\varepsilon/2$. In Theorem 2.8 it is sufficiently to require only the validity of the inequality $\mathrm{Re}(\langle y, y^*\rangle) \geq 0$ for some ε-net on a sphere.

Theorem 2.9 *Let D be a domain in Euclidean space $X = E^n$ and let $K \subset \overline{D}$ be a subset of the closure of D. Suppose that there exists the restriction F_1 of multivalued mapping $F : \overline{D} \to E^n = Y$ on the subset K which satisfied conditions of a coacute ε-angle for some $\delta/2$-net Σ on the sphere $S^* = \{y^* \in Y^* : \|y^*\| = 1\}$ in Y^*, $\delta > \varepsilon/2$. If $F(\overline{D}) \supset \mathrm{conv}(F_1(K))$ then $0 \in F(\overline{D})$.*

Proof Let $y_1^* \in S^* \subset Y^*$ be arbitrary points. By conditions of the theorem there exist points $y^* \in \Sigma \subset S^* \subset Y^*$, $\|y^* - y_1^*\| < \varepsilon/2$ and $x \in X$ such, that the following relations $\mathrm{Re}(\langle y, y^*\rangle) = \cos(\angle y0y^*)\|y\| > \sin\delta\|y\| > \varepsilon\|y\|/2$ are true for all points $y \in F_1(x)$. Then, $\mathrm{Re}(\langle \frac{y}{\|y\|}, y_1^*\rangle) = \mathrm{Re}(\langle \frac{y}{\|y\|}, y^*\rangle) + \mathrm{Re}(\langle \frac{y}{\|y\|}, y_1^* - y^*\rangle) > \varepsilon/2 - \|y_1^* - y^*\| > 0$.

Now the proof of the theorem is completed by virtue of Theorem 2.8. □

Example Let $f : B^2 \to B^2$, $\partial B^2 = S^1 = \{z = e^{i\varphi}, 0 \leq \varphi < 2\pi\}$, be a non continuous mapping of the unit ball, which expressed by the formula

$$f\big(e^{i\varphi}\big) = \begin{cases} e^{i(\varphi-\pi/2)}, & 0 \leq \varphi \leq \pi/2, \\ e^{i(\varphi+\pi/2)}, & \pi/2 < \varphi \leq \pi, \\ e^{i\varphi}, & \pi < \varphi < 2\pi \end{cases}$$

Let the mapping f on the interior of B be an arbitrary homeomorphism of $\mathrm{Int}(B^2)$ on the open half ball

$$B_-^2 = \big\{z \in \mathrm{Int}\, B^2, \mathrm{Im}\, z < 0\big\}$$

Obviously, that image $f(B^2)$ coincides with the convex hull of the set $f(S^1)$ but $0 \notin f(B^2)$.

This example shows an importance of the restrictions on mappings made in quoted theorems.

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Parametric Continuity of Choquet and Sugeno Integrals

Ion Chiţescu

Abstract For a probabilistic space $(T, \mathcal{T}, \mu)$, a fixed measurable set A and a fixed positive measurable function f, the continuity with respect to the real parameter λ of the Choquet or Sugeno integral $\int_A dm(\lambda, \mu)$ is proved. Here $m(\lambda, \mu)$ are all possible λ-Sugeno measures generated by μ. Asymptotical properties are studied too.

Keywords Classical and generalized probability · Sugeno measure · Choquet integral · Sugeno integral

Mathematics Subject Classification (2010) Primary 28A25 · 28E10 · Secondary 26E50

1 Introduction

Classical measure theory deals with additive measures. In the last period, generalized measure theory, dealing with (possibly) non-additive measures, became more and more important and a huge number of research papers and monographs pertaining to it appeared. The explanation of this phenomen is double, in our opinion: on one hand appears the scientific tentation of a new theory, on the other hand appears the objective fact that many aspects of concrete life or of science are better modelated by this new theory. A natural companion of the theory of non-additive measures is the theory of the integrals built with these measures—the non-linear integrals, especially the Choquet and Sugeno integrals.

From historical and scientifical point of view, the most important contribution to the theory of non-additive measures and non-linear integrals is the work of the distinguished Japanese scholar M. Sugeno who, in his Ph.D. Thesis [5] introduced the concept of λ-additive measures or fuzzy measures (we call these last measures *Sugeno measures*), together with his fuzzy integral (we call this integral *Sugeno*

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integral). For the sake of completeness, let us recall that a λ-additive measure (λ real, subject to some restrictions) is a measure μ having the property that

$$\mu(A \cup B) = \mu(A) + \mu(B) + \lambda\mu(A)\mu(B),$$

for any disjoint sets A, B. The Sugeno measures are special normalized λ-additive measures.

Subsequent developments are due to Z. Wang who showed (see [6]) that any Sugeno measure m has the form $m(\lambda, \mu) = h_\lambda \circ \mu$, where μ is a classical probability and h_λ is the canonical T-function (see the preliminary Part) and to D. Schmeidler who formally introduced the Choquet integral using [1] (see [4]).

In the present paper we show that for any fixed A, f, μ, the function $\lambda \mapsto \int_A f dm(\lambda, \mu)$ is continuous (here the integral is either Choquet or Sugeno, see the Preliminary Part). Besides, because $\lambda \in (-1, \infty)$, we study the marginal measures $m(-1, \mu)$, $m(\infty, \mu)$ and the asymptotic behaviour of the integral (with respect to the marginal measures).

The main theoretical tool used throughout the paper is the monograph [7]. For classical measure theory, see [2] and [3].

Note Because of the restricted volume of these Proceedings, proofs presented in the present paper are short. Extended proofs will appear elsewhere.

2 Preliminary Part

Throughout the paper $\mathbb{R}$ = the real numbers, $\mathbb{R}_+$ = the positive real numbers, $\overline{\mathbb{R}}_+ = \mathbb{R} \cup \{\infty\}$, $\mathbb{N}$ = the natural numbers.

We shall consider a fixed probabilistic space $(T.\mathcal{T}, \mu)$, where: T is a nonempty set, $\mathcal{T}$ is a σ-algebra of subsets of T and $\mu : \mathcal{T} \to [0, \infty)$ is a (σ-additive) probability. For any $\mathcal{T}$-measurable function $f : T \to \overline{\mathbb{R}}_+$ and any $\alpha \in \mathbb{R}$, we define

$$F_\alpha(f) = \left\{t \in T \;\middle|\; f(t) \geq \alpha\right\} \in \mathcal{T}.$$

A generalized probability is a function $m : \mathcal{T} \to [0, 1]$ such that $m(\phi) = 0$, $m(T) = 1$, m is increasing (i.e. $m(A) \leq m(B)$ whenever A, B are in $\mathcal{T}$, $A \subset B$). For any $\lambda \in (-1, \infty)$, the λ-*Sugeno measure generated by* μ is the generalized probability $m(\lambda, \mu) = h_\lambda \circ \mu$, where $h_\lambda : [0, 1] \to [0, 1]$ is *the canonical T-function* given via

$$h_\lambda(t) = \begin{cases} t, & \text{if } \lambda = 0 \\ \dfrac{(\lambda+1)^t - 1}{\lambda}, & \text{if } \lambda \neq 0 \end{cases}$$

(one knows that $h_\lambda(0) = 0$, $h_\lambda(1) = 1$, h_λ is strictly increasing and continuous).

Supplementarily, it follows that $m(\lambda, \mu)$ is *continuous* (i.e. $m(\lambda, \mu)(\lim_n A_n) = \lim_n m(\lambda, \mu)(A_n)$, whenever $(A_n)_n \subset \mathcal{T}$ is a monotone sequence).

Let $f : T \to \overline{\mathbb{R}}_+$ be $\mathcal{T}$-measurable, $A \in \mathcal{T}$ and $m : \mathcal{T} \to [0, 1]$ a generalized probability. We define (write F_α instead of $F_\alpha(f)$)

(a) *The Choquet integral of f over A with respect to m* (generalization of standard integral), namely

$$(C)\int_A f dm = \int m(F_\alpha \cap A) dL(\alpha) \in \overline{\mathbb{R}}_+$$

where L is the Lebesgue measure on $[0, \infty)$ (the function $\alpha \mapsto m(F_\alpha \cap A)$ is decreasing!).

(b) *The Sugeno integral of f over A with respect to m*, namely

$$(S)\int_A f dm = \sup_{\alpha \in \mathbb{R}_+} \alpha \wedge m(F_\alpha \cap A) \in [0, 1].$$

In case $A = T$ we write $(C)\int f dm$ and $(S)\int f dm$. If $(C)\int f dm < \infty$ we say that f is *Choquet integrable*.

3 Results

Lemma 3.1 (Elementary fact) *Let $0 \le m \le 1$. Define $\varphi : (-1, \infty) \to \mathbb{R}_+$, via*

$$\varphi(\lambda) = \begin{cases} m, & \text{if } \lambda = 0 \\ \dfrac{(\lambda + 1)^m - 1}{\lambda}, & \text{if } \lambda \neq 0. \end{cases}$$

Then φ is continuously differentiable and decreasing. In case $m = 0$ or $m = 1$, φ is constant. In case $0 < m < 1$, $\varphi'(\lambda) < 0$ and $0 < \varphi(\lambda) < 1$ for any $\lambda \in [-1, \infty)$.

In the sequel we shall always write F_α instead of $F_\alpha(f)$. We shall also use the fact that we always have $(C)\int_A f dm = (C)\int f\varphi_A dm$ and $(S)\int_A f dm = (S)\int f\varphi_A dm$, where φ_A is the characteristic (indicator) function of A.

Proposition 3.2 *Assume $f : T \to \mathbb{R}_+$ is μ-integrable. Then*:

1. *The function f is Choquet integrable with respect to $m(\lambda, \mu)$ for any $\lambda \in (-1, \infty)$.*
2. *For any $A \in \mathcal{T}$, the function $V : (-1, \infty) \to \mathbb{R}_+$, given via*

$$V(\lambda) = (C)\int_A f dm(\lambda, \mu)$$

is continuous.

Proof 1. Working in the non trivial cases $\lambda \neq 0$ and $\mu(F_\alpha) > 0$ for any $\alpha \geq 0$, we apply Cauchy's integral criterion to the monotone function $\alpha \mapsto (\lambda + 1)^{\mu(F_\alpha)} - 1$ and use the hypothesis.

2. One can work for $A = T$. Fixing $\lambda_0 \in (-1, \infty)$, we must show that $\lim_{\lambda \to \lambda_0} (C) \int f dm(\lambda, \mu) = (C) \int f dm(\lambda_0, \mu)$. The idea is to prove that the function $W : (-1, \infty) \to \mathcal{L}^1(L)$, given via $W(\lambda) = h(\lambda)$ is continuous and to use the continuous linear functional $H : (-1, \infty) \to \mathcal{L}^1(L)$, $H(\varphi) = \int \varphi dL$. Here $\mathcal{L}^1$ is seminormed as usual with the seminorm $\|f\|_1 = \int |f| dL$ and $h(\lambda)(\alpha) = m(\lambda, \mu)(F_\alpha)$ for any $\alpha \in \mathbb{R}_+$.

The proof is done separately for $\lambda_0 \neq 0$ and $\lambda_0 = 0$.

Case $\boldsymbol{\lambda_0 \neq 0}$ For arbitrary $\varepsilon > 0$ we get $\delta > 0$ such that if $|\lambda - \lambda_0| < \delta$ one has $\|h(\lambda) - h(\lambda_0)\|_1 < \varepsilon$. This can be done imposing some preliminary conditions on δ and using the Mac Laurin expansion of the functions $x \mapsto (\lambda + 1)^x = \exp(x \ln(\lambda + 1))$ and $x \mapsto (\lambda_0 + 1)^x = \exp(x \ln(\lambda_0 + 1))$, for $x = \mu(F_\alpha)$ followed by term by term integration.

Case $\boldsymbol{\lambda_0 = 0}$ For any $\alpha \in \mathbb{R}_+$ and $\lambda \in (-1, 1)$, one expands $\lambda \mapsto (\lambda + 1)^{\mu(F_\alpha)}$ as a binomial series and, after some other computations (especially term by term integration) we get an evaluation (for $0 \neq \lambda \in (-1, 1)$) of the form

$$\left| m(\lambda, \mu)(F_\alpha) - \mu(F_\alpha) \right| \leq \mu(F_\alpha) \sum_{p=2}^{\infty} |\lambda|^{p-1} = \mu(F_\alpha) \frac{|\lambda|}{1 - |\lambda|}.$$

This leads to

$$\left\| h(\lambda) - h(0) \right\|_1 \leq \frac{|\lambda|}{1 - |\lambda|} \int f d\mu \xrightarrow[\lambda \to 0]{} 0. \qquad \square$$

Proposition 3.3 *For any $A \in \mathcal{T}$, the function $V(-1, \infty) \to \mathbb{R}_+$ given via*

$$V(\lambda) = (S) \int_A f dm(\lambda, \mu)$$

is continuous.

Proof Again we can work for $A = T$. Let us fix $\lambda_0 \in (-1, \infty)$ and take a strictly monotone sequence of numbers $(\lambda_n)_n$ with $\lambda_0 \neq \lambda_n \underset{n}{\to} \lambda_0$. According to the Transformation Theorem (Theorem 9.13 from [7]) we have the equalities:

$$(S) \int f dm(\lambda_n, \mu) = (S) \int h_n dL$$

$$(S) \int f dm(\lambda_0, \mu) = (S) \int h dL,$$

where $h_n : \mathbb{R}_+ \to \mathbb{R}_+$, $h_n(\alpha) = m(\lambda_n, \mu)(F_\alpha)$ and $h : \mathbb{R}_+ \to \mathbb{R}_+$, $h(\alpha) = m(\lambda_0, \mu)(F_\alpha)$. We must prove that

$$\lim_n (S) \int h_n dL = (S) \int h dL. \tag{3.1}$$

In case $\lambda_n > \lambda_0$ ($(\lambda_n)_n$ strictly decreasing), we use Lemma 3.1 to deduce that $(h_n)_n$ is increasing with pointwise limit h. Equality (3.1) follows from Theorem 9.5 from [7].

In case $\lambda_n < \lambda_0$ ($(\lambda_n)_n$ strictly increasing), Lemma 3.1 says that $(h_n)_n$ is decreasing with pointwise limit h. We have

$$C = \int h dL \le 1 < \infty.$$

Define, for any $n \in \mathbb{N}$:

$$A_n = \left\{ \alpha \in \mathbb{R}_+ \;\middle|\; h(\alpha) > C \right\}.$$

According to the same Theorem 9.5 from [7], a sufficient condition for the validity of (3.1) consists in the existence of $n_0 \in \mathbb{N}$ such that $L(A_{n_0}) < \infty$. We shall prove that this in the case.

Actually, we shall prove a stronger assertion, namely that $L(A_n) < \infty$ for any n. Indeed, assuming the existence of some n such that $L(A_n) = \infty$, we shall arrive at a contradiction, as follows. Firstly, A_n is an interval with extremity 0, hence $A_n = \mathbb{R}_+$, which implies the fact that $m(\lambda_n, \mu)(F_\alpha) > C$ for any $\alpha \in \mathbb{R}_+$. Using a strictly increasing $(\alpha_p)_p$ such that $\alpha_p \underset{p}{\to} \infty$ and the fact that μ is continuous, we infer that all $m(\lambda_n, \mu)$ are continuous and get, for any n:

$$\lim_p m(\lambda_n, \mu)(F_{\alpha_p}) = m(\lambda_n, \mu)\left(\bigcap_p F_{\alpha_p} \right) \ge C,$$

contradiction, because $\bigcap_p F_{\alpha_p} = \emptyset$ (f is finite). □

The remainder of our paper will be concerned with the asymptotic behaviour. The pointwise limit measures of $m(\lambda, \mu)$ when $\lambda \to -1$ or $\lambda \to \infty$ exist:

Proposition 3.4 *The marginal generalized probabilities* $m(-1, \mu) : \mathcal{T} \to [0, 1]$ *and* $m(\infty, \mu) : \mathcal{T} \to [0, 1]$ *exist. They are defined pointwise, for any* $A \in \mathcal{T}$, *as follows*:

$$m(-1, \mu)(A) = \lim_{\lambda \to -1} m(\lambda, \mu)(A) = \begin{cases} 0, & \textit{if } \mu(A) = 0 \\ 1, & \textit{if } \mu(A) > 0. \end{cases}$$

$$m(\infty, \mu)(A) = \lim_{\lambda \to -1} m(\lambda, \mu)(A) = \begin{cases} 0, & \textit{if } \mu(A) < 1 \\ 1, & \textit{if } \mu(A) = 1. \end{cases}$$

Supplementarily: $m(-1,\mu)$ *is countably subadditive and* $m(\infty,\mu)$ *is countably superadditive.*

The "continuity" of the functions $\lambda \mapsto (C)\int_A f dm(\lambda,\mu)$, $\lambda \mapsto (S)\int_A f dm(\lambda,\mu)$ can be "extended" at the points -1 and ∞, as the following results shows:

Proposition 3.5 *Let* $A \in \mathcal{T}$.

1. *Assume* $f : T \to \overline{\mathbb{R}}_+$ *is* μ*-integrable. Then one has*

$$\lim_{\lambda\to -1} (C)\int_A dm(\lambda,\mu) = (C)\int_A f dm(-1,\mu)$$

$$\lim_{\lambda\to \infty} (C)\int_A f d(\lambda,\mu) = (C)\int_A f dm(\infty,\mu).$$

2. *One has*

$$\lim_{\lambda\to -1} (S)\int_A f dm(\lambda,\mu) = (S)\int_A f dm(-1,\mu)$$

$$\lim_{\lambda\to \infty} (S)\int_A f dm(\lambda,\mu) = (S)\int_A f dm(\infty,\mu).$$

Proof 1. We use Beppo Levi's theorem or Lebesgue's dominated convergence theorem, taking monotone sequences $\lambda_n \underset{n}{\to} \lambda_0$.

2. We use the fact that for two non empty sets and for a bounded from below function $g : A \times B \to \mathbb{R}$, one has

$$\sup_{a\in A}\sup_{b\in B} g(a,b) = \sup_{b\in B}\sup_{a\in A} g(a,b) = \sup_{(a,b)\in A\times B} g(a,b)$$

$$\sup_{a\in A}\inf_{b\in B} g(a,b) = \inf_{b\in B}\sup_{a\in A} g(a,b).$$

□

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Nonlinear Operators, Fixed-Point Theorems, Nonlinear Equations

Kamal N. Soltanov

Abstract In this article we discuss the solvability of some class of fully nonlinear equations, and equations with p-Laplacian in more general conditions by using a new approach given by Soltanov in Nonlinear Anal. 72:164–175, 2010 for studying the nonlinear continuous operator. Moreover we reduce certain general results for the continuous operators acting on Banach spaces, and investigate their image. Here we also consider the existence of a fixed-point of the continuous operators under various conditions.

Keywords Nonlinear continuous operator · Closed image · Fixed-point · Solvability theorem · Nonlinear BVP

Mathematics Subject Classification (2010) Primary 46T20 · 47H10 · 47J05 · Secondary 35D35 · 35J60 · 35Q82

1 Introduction

In the present paper we consider the boundary-value problem for the fully nonlinear equation of the second order

$$F(x, u, Du, \Delta u) = h(x), \quad x \in \Omega, \tag{1.1}$$

and also for the nonlinear equations with p-Laplacian that depend upon the parameters λ and ρ

$$-\nabla\left(|\nabla u|^{p-2}\nabla u\right) + G(x, u, Du, \lambda, \rho) = h(x), \quad x \in \Omega, \tag{1.2}$$

on the smooth bounded domain $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$, where $F(x, \xi, \eta, \zeta)$ and $G(x, \xi, \eta, \lambda, \rho)$ are Caratheodory functions. We deal with the properties of the nonlinear operators generated by the posed problems and study the solvability of these

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problems by using the general results of such type as in [1]. It should be noted that equations of such type arise in the diffusion processes, reaction-diffusion processes etc., in the steady-state case (see, for example [2–13] and their references). Furthermore we discuss of some nonlinear continuous mappings acting on Banach spaces and an equation (inclusion) with mappings of such type. The problems of such type were studied earlier under various conditions in the semilinear ([2, 3, 7–11, 13] etc.) and in the fully nonlinear cases ([4, 12, 14] etc.). In the mentioned articles, the known general results having such conditions that cannot be applicable to the problems considered here, were used, whereas in this article we want to investigate these problems under more general conditions. Indeed, under here assumed conditions not is possible to use a methods that require of the compactness of the operators generated by the studied problems, since these operators here only are continuous. Therefore we need to use a general result that will be applicable to the considered problems here and consequently, the conditions of the general result differ from the conditions of the known results. The results of [1] and the general results adduced here allow us to study the imposed problem in more general conditions.

So for our goal we lead to a fixed-point theorem for nonlinear continuous mappings acting on Banach spaces and also a solvability theorem for nonlinear equations involving continuous operators. These general results allow us to study various nonlinear mappings and also nonlinear problems under more general conditions. That is demonstrated under study of the boundary value problems (BVP) for the nonlinear differential equations of reduced here. Moreover we obtain the existence of the fixed point for the operator associated with imposed problem. We would like also to note that Theorem 2.1 of present article is a result of the type of Lax–Milgram theorem.

2 Some General Results on Solvability

Let X, Y be reflexive Banach spaces and X^*, Y^* be their dual spaces, moreover let Y be reflexive with strictly convex norm together with Y^* (this condition is not complementary condition; see, for example, [15]) and $f : D(f) \subseteq X \longrightarrow Y$ be an operator.

So, we will conduct here the special case of the main result of [1]. Consider the following conditions. Let the closed ball $B^X_{r_0}(0)$ of X is contained in $D(f)$, i.e. $B^X_{r_0}(0) \subseteq D(f) \subset X$ and on $B^X_{r_0}(0)$ are fulfilled the conditions:

(i) $f : B^X_{r_0}(0) \subseteq D(f) \subseteq X \longrightarrow Y$ be a continuous operator that bounded on $B^X_{r_0}(0)$, i.e.

$$\left\| f(x) \right\|_Y \leq \mu\big(\|x\|_X\big), \quad \forall x \in B^X_{r_0}(0);$$

(ii) there is a mapping $g : D(g) \subseteq X \longrightarrow Y^*$ such that $D(f) \subseteq D(g)$, and for any $S^X_r(0) \subset B^X_{r_0}(0)$, $0 < r \leq r_0$, $\mathrm{cl}\, g(S^X_r(0)) = \overline{g(S^X_r(0))} \equiv S^{Y^*}_r(0)$, $S^X_r(0) \subseteq g^{-1}(S^{Y^*}_r(0))$

$$\big\langle f(x), g(x)\big\rangle \geq \nu\big(\|x\|_X\big)\|x\|_X, \quad \text{a.e.} x \in B^X_{r_0}(0) \ \&\ \nu(r_0) \geq \delta_0 > 0 \tag{2.1}$$

holds,[1] where $\mu : R^1_+ \longrightarrow R^1_+$ and $\nu : R^1_+ \longrightarrow R^1$ are continuous functions ($\mu, \nu \in C^0$), moreover ν is the nondecreasing function for $\tau \geq \tau_0$, $r_0 \geq \tau_0 \geq 0$; $\tau_0, \delta_0 > 0$ are constants;

(iii) almost each $\widetilde{x} \in int B^X_{r_0}(0)$ possesses a neighborhood $V_\varepsilon(\widetilde{x})$, $\varepsilon \geq \varepsilon_0 > 0$ such that the following inequality

$$\left\| f(x_2) - f(x_1) \right\|_Y \geq \Phi\left(\|x_2 - x_1\|_X, \widetilde{x}, \varepsilon\right), \tag{2.2}$$

holds for any $\forall x_1, x_2 \in V_\varepsilon(\widetilde{x}) \cap B^X_{r_0}(0)$, where $\Phi(\tau, \widetilde{x}, \varepsilon) \geq 0$ is a continuous function at τ and $\Phi(\tau, \widetilde{x}, \varepsilon) = 0 \Leftrightarrow \tau = 0$ (in particular, $\widetilde{x} = 0$, $\varepsilon = \varepsilon_0 = r_0$ and $V_\varepsilon(\widetilde{x}) = V_{r_0}(0) \equiv B^X_{r_0}(0)$, consequently $\Phi(\tau, \widetilde{x}, \varepsilon) \equiv \Phi(\tau, 0, r_0)$ on $B^X_{r_0}(0)$),

Theorem 2.1 (Main Theorem) *Let X, Y be Banach spaces such as above and $f : D(f) \subseteq X \longrightarrow Y$ be an operator. Assume that on the closed ball $B^X_{r_0}(0) \subseteq D(f) \subset X$ the conditions (i) and (ii) are fulfilled then the image $f(B^X_{r_0}(0))$ of the ball $B^X_{r_0}(0)$ contains an everywhere dense subset of M that has the form*

$$M \equiv \left\{ y \in Y \mid \left\langle y, g(x) \right\rangle \leq \left\langle f(x), g(x) \right\rangle, \ \forall x \in S^X_{r_0}(0) \right\}.$$

Furthermore if in addition the image $f(B^X_{r_0}(0))$ of the ball $B^X_{r_0}(0)$ is closed or the condition (iii) is fulfilled then the image $f(B^X_{r_0}(0))$ is a bodily subset of Y, moreover $f(B^X_{r_0}(0))$ contains the above bodily subset M.

The proof of this theorem is obtained from general result that was proven in [14] (see also, [1]). (Theorem 2.1 is the generalization of theorem of such type from [11].)

Remark 2.2 **1.** It is easy to see that the condition $B^X_{r_0}(0) \subseteq D(f)$ is not essential, because if $D(f)$ comprises a bounded closed subset $U(x_0) \subseteq X$ of some element $x_0 \in D(f)$ such that $U(x_0)$ is topologically equivalent to $B^X_1(0)$ and $U(x_0) \subseteq D(f) \cap D(g)$, then we can formulate the conditions and statement of this theorem analogously, i.e. for this we determine the operator $\widetilde{f}(x) = f(x) - f(x_0)$ and assume that

$$\left\| \widetilde{f}(x) \right\|_{X^*} \leq \mu\left(\|x - x_0\|_X\right),$$

holds for any $x \in U(x_0)$ and

$$\left\langle \widetilde{f}(x), g(x - x_0) \right\rangle \geq \nu\left(\|x - x_0\|_X\right) \|x - x_0\|_X \tag{2.3}$$

holds for almost all $x \in U(x_0)$. Moreover $\nu(\|x - x_0\|_X) \geq \delta_0 > 0$ for any $x \in \partial U(x_0)$, where $g : D(g) \subseteq X \longrightarrow Y^*$ such that $D(f) \subseteq D(g)$ and g satisfies a claim

[1] In particular, the mapping g can be a linear bounded operator as $g \equiv L : X \longrightarrow Y^*$ that satisfy the conditions of (ii).

analogously of the condition (ii) respect to $U(x_0)$. In this case, we define subset $\widetilde{M}_{x_0}$ in the form

$$\widetilde{M}_{x_0} \equiv \left\{ y \in Y \;\middle|\; 0 \le \left\langle f(x) - y, g(x - x_0) \right\rangle, \ \forall x \in \partial U(x_0) \right\} \tag{2.4}$$

2. In the formulation of Theorem 2.1 we use the equality $\|g(x)\|_{Y^*} \equiv \|x\|_X$ that can be determined by the known way, i.e. $g'(x) \equiv \frac{\|x\|_X}{\|g(x)\|_{Y^*}} g(x)$ for any $x \in D(g) \subseteq X$.

Condition (iii) of Theorem 2.1 can be generalized, for example, as in the following proposition.

Corollary 2.3 *Let all conditions of Theorem 2.1 be fulfilled except inequality (2.2), and instead of that, let the following inequality*

$$\left\| f(x_2) - f(x_1) \right\|_Y \ge \Phi\left(\|x_2 - x_1\|_X, \widetilde{x}, \varepsilon\right) + \psi\left(\|x_1 - x_2\|_Z, \widetilde{x}, \varepsilon\right), \tag{2.5}$$

be fulfilled for any $x_1, x_2 \in V_\varepsilon(\widetilde{x}) \cap B^X_{r_0}(0)$, where $\Phi(\tau, \widetilde{x}, \varepsilon)$ is a function such as in the condition (iii), Z is a Banach space and the inclusion $X \subset Z$ is compact, and $\psi(\cdot, \widetilde{x}, \varepsilon) : R^1_+ \longrightarrow R^1$ is a continuous function at τ and $\psi(0, \widetilde{x}, \varepsilon) = 0$. Then the statement of Theorem 2.1 is correct.

Note It should be noted that this result is a generalization of the known Lax–Milgram theorem [16] to the nonlinear case in the class of Banach spaces when all conditions of this theorem are fulfilled on whole space. Indeed, we can formulate the Lax–Milgram theorem for the linear operator T acting on the real Hilbert space X in the form:

(a) There exists a positive constant γ such that

$$\left|(Tx, y)\right| \le \gamma \|x\|_X \cdot \|y\|_X, \quad \forall (x, y) \in X \times X;$$

that is equivalent to boundedness of operator $T : X \longrightarrow X$.

(b) there exists a positive constant δ such that

$$(Tx, x) \ge \delta \|x\|^2_X;$$

that is equivalent to the coerciveness of $T : X \longrightarrow X$, i.e. T satisfies the condition (ii) for the special case ($g \equiv id$).

Then equation $Tx = y$ is solvable for any $y \in X$.

From the condition b, it follows that

$$\left\| T(x_1 - x_2) \right\|_X \ge \delta \|x_1 - x_2\|_X;$$

i.e. inequality (2.2) holds for any $x_1, x_2 \in X$.

Theorem 2.4 (Fixed-Point Theorem) *Let X be a reflexive separable Banach space and $f_1 : D(f_1) \subseteq X \longrightarrow X$ be a bounded continuous operator. Moreover, let on a* *closed ball* $B^X_{r_0}(x_0) \subseteq D(f_1)$, *where* $x_0 \in D(f_1)$, *operator* $f \equiv Id - f_1$ *satisfy the following conditions*

$$\begin{aligned} \left\| f_1(x) - f_1(x_0) \right\|_X \le \mu\big(\|x - x_0\|_X\big), \quad \forall x \in B^X_{r_0}(x_0), \\ \big\langle f(x) - f(x_0), g(x - x_0)\big\rangle \ge \nu\big(\|x - x_0\|_X\big)\|x - x_0\|_X, \quad \forall x \in B^X_{r_0}(x_0), \end{aligned} \tag{2.6}$$

and almost each $\widetilde{x} \in intB^X_{r_0}(x_0)$ *possesses a neighborhood* $V_\varepsilon(\widetilde{x})$, $\varepsilon \ge \varepsilon_0 > 0$ *such that the following inequality*

$$\left\| f(x_2) - f(x_1) \right\|_X \ge \varphi\big(\|x_2 - x_1\|_X, \widetilde{x}, \varepsilon\big),$$

holds for any $x_1, x_2 \in V_\varepsilon(\widetilde{x}) \cap B^X_{r_0}(x_0)$, *where* $g : D(g) \subseteq X \longrightarrow X^*$ *such that* $B^X_{r_0}(0) \subseteq D(g)$ *and* g *satisfies condition* (*ii*), μ *and* ν *are such functions as in Theorem* 2.1, *function* $\varphi(\tau, \widetilde{x}, \varepsilon)$ *has such a form as the right hand side of inequality* (2.3)[2] *Then the operator* f_1 *possesses a fixed-point on the ball* $B^X_{r_0}(x_0)$.

Definition 2.5 We call that an operator $f : D(f) \subseteq X \longrightarrow Y$ possesses the P-property iff any precompact subset M of Y from $\operatorname{Im} f$ has a (general) subsequence $M_0 \subset M$ such that there exists a precompact subset G of X that satisfies the inclusions $f^{-1}(M_0) \subseteq G$ and $f(G \cap D(f)) \supseteq M_0$.

Remark 2.6 We can take the following condition instead of condition (iii) of Theorem 2.1: f possesses the P-property on the ball $B^X_{r_0}(0)$. It should be noted that an operator $f : D(f) \subseteq X \longrightarrow Y$ possesses of the P-property if f^{-1} is a lower or upper semi-continuous mapping (cf. A-property [8]).

In the above results for the completeness of the image ($\operatorname{Im} f$) of the imposed operator f, the condition (iii) and P-property (and also the generalizations of the conditions (iii)) are used. But there are some other types of the complementary conditions on f under which $\operatorname{Im} f$ will be a closed subset. These types of conditions are described in [1, 12, 14]. Therefore we do not conduct them here again.

3 Fully Nonlinear Equations of Second Order

Now, we study some nonlinear BVP with using the general results. Let $\Omega \subset R^n$ ($n \ge 1$) be an open bounded domain with sufficiently smooth boundary $\partial\Omega$. Consider the following problem

$$f(u) \equiv -\Delta u + F(x, u, Du, \Delta u) = 0, \quad x \in \Omega,\ u|_{\partial\Omega} = 0, \tag{3.1}$$

[2] In particular, $g \equiv J : X \longrightarrow X^*$, i.e. g be a duality mapping.

where $F(x,\xi,\eta,\zeta)$ is a Caratheodory function on $\Omega \times R^2 \times R^n$ as $F: \Omega \times R^2 \times R^n \longrightarrow R^1$, $D \equiv (D_1, D_2, \dots, D_n)$, $\Delta \equiv \sum_{i=1}^n D_i^2$ (is Laplacian), $D_i \equiv \frac{\partial}{\partial x_i}$.

Let the following conditions be fulfilled

(i) there are Caratheodory functions $F_0(x,\xi)$, $F_1(x,\xi,\eta)$, $F_2(x,\xi,\eta,\zeta)$: F_0, F_1, $F_2 : \Omega \times R^{n+2} \longrightarrow R^1$ such that $F(x,\xi,\eta,\zeta) = F_0(x,\xi) + F_1(x,\xi,\eta) + F_2(x,\xi,\eta,\zeta)$ for any $(x,\xi,\eta,\zeta) \in \Omega \times R^1 \times R^n \times R^1$, moreover

(a) there exist a Caratheodory function $a_1(x,\xi)$ and numbers $m_0, \nu \geq 0$, $\mu, M > 0$, $\mu + 2 > 2\nu$ such that

$$F_0(x,\xi) \equiv M|\xi|^\mu \xi + a_1(x,\xi),$$
$$\left|a_1(x,\xi)\right| \leq m_0|\xi|^\nu + \psi(x), \quad \psi \in L_p(\Omega),\ p > 2 \tag{3.2}$$

hold for a.e. $x \in \Omega$ and any $\xi \in R^1$, and

(b) there exist a number $2 \geq \rho \geq 0$ and a nonnegative Caratheodory function $m_1(x,\xi,\eta) \geq 0$ such that

$$\left|F_1(x,\xi,\eta)\right| \leq m_1(x,\xi,\eta)|\eta|^\rho + k(x), \tag{3.3}$$

$\forall(x,\xi,\eta) \in \Omega \times R^1 \times R^n$ holds, where $m_1(x,\xi,\eta) \leq M_1(x)$, and $2(\widehat{C}(\mu,\rho,n) \times \|M_1\|_\infty^2)^2 \leq M$, $k \in L_{p_1}(\Omega)$, $p_1 > 2$, $\widehat{C}(\mu,\rho,n)$ is the coefficient of Gagliardo–Nirenberg–Sobolev (G-N-S) inequality (see, [16]).

(c) there exist a Caratheodory function, $c(x,\xi,\eta,\zeta)$, $\widetilde{F}_2(x,\xi,\eta)$ and continuous function $k(\zeta)$ such that the following inequalities

$$\left|F_2(x,\xi,\eta,\zeta) - F_2(x,\xi,\eta,\zeta_1)\right| \leq c\big(x,\xi,\eta,\zeta(\zeta,\zeta_1)\big)|\zeta - \zeta_1|, \tag{3.4}$$

$$\left|F_2(x,\xi,\eta,\zeta) - F_2(x,\xi_1,\eta_1,\zeta)\right| \leq k(\zeta)\left|\widetilde{F}_2(x,\xi,\eta) - \widetilde{F}_2(x,\xi_1,\eta_1)\right| \tag{3.5}$$

hold for a.e. $x \in \Omega$ and any $(\xi,\eta), (\xi_1,\eta_1) \in R^{n+1}$, $\forall\zeta,\zeta_1 \in R^1$, and $F_2(x,\xi,\eta,0) = 0$, moreover there exists a function $\psi_1 \in L^\infty(\Omega)$ such that $\psi_1(x) \geq 0$, $c(x,\xi,\eta,\zeta) \leq \psi_1(x)$ hold for a.e. $x \in \Omega$ and any $(\xi,\eta,\zeta) \in R^N$, here $\|\psi_1\|_{L^\infty(\Omega)} \equiv \|\psi_1\|_\infty \leq 4^{-1}$.

Assume the following denotations: $\|u\|_{L_p(\Omega)} \equiv \|u\|_p$ for any $p \in [1,\infty]$, and $\|u\|_{W^{l,p}(\Omega)} \equiv \|u\|_{l,p}$, for $u \in W^{l,p}(\Omega)$, $l \geq 1$. So, we consider the operator $f : W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega) \longrightarrow L^2(\Omega)$, which is generated by the imposed problem.

Theorem 3.1 *Let conditions (i), (a), (b), (c) be fulfilled and parameters ρ_0 and ρ satisfy the following relations*

$$1 < \rho_0 \leq \frac{4}{n-4} \quad \text{if}\, n \geq 5 \qquad \& \qquad 1 < \rho_0 < \infty \quad \text{if}\, n = 2,3,4,$$
$$\rho \geq \frac{(\rho_0+2)n}{2(n+\rho_0)} \qquad \& \qquad \rho \leq 1 + \min\left\{\frac{\rho_0+2}{n+\rho_0};\ \frac{2}{n-2};\ \frac{\rho_0}{\rho_0+2}\right\}.$$

Then problem (3.1) is solvable in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$.

Proof For the proof, it is enough to show that operator $f : W^{2,2}(\Omega) \cap \mathring{W}^{1,2}(\Omega) \longrightarrow L_2(\Omega)$ fulfills all conditions of Theorem 2.1. For this, we will estimate the following dual form

$$\big\langle f(u), g(u)\big\rangle = \big\langle -\Delta u + F(x, u, Du, \Delta u), -\Delta u\big\rangle$$

where the operator g is determined in the form $g \equiv -\Delta : W^{2,2}(\Omega) \cap \mathring{W}^{1,2}(\Omega) \longrightarrow L_2(\Omega)$, then we have

$$\big\langle -\Delta u + F(x, u, Du, \Delta u), -\Delta u\big\rangle = \|\Delta u\|_2^2 + \big\langle F(x, u, Du, \Delta u), -\Delta u\big\rangle,$$

consequently we need to estimate the second term of this equation, i.e. the dual form: $\langle F(x, u, Du, \Delta u), -\Delta u\rangle$.

Thus using conditions (i), (a), (b), (c) we obtain

$$\begin{aligned}
&\big\langle F(x, u, Du, \Delta u), -\Delta u\big\rangle \\
&\quad = -\big\langle F_0(x, u), \Delta u\big\rangle - \big\langle F_1(x, u, Du), \Delta u\big\rangle \\
&\qquad - \big\langle F_2(x, u, Du, \Delta u), \Delta u\big\rangle = -\big\langle M|u|^{\mu} u + a_1(x, u), \Delta u\big\rangle \\
&\qquad - \big\langle F_1(x, u, Du), \Delta u\big\rangle - \big\langle F_2(x, u, Du, \Delta u), \Delta u\big\rangle \\
&\quad \ge \big\langle M(\mu + 1)|u|^{\mu} \nabla u, \nabla u\big\rangle \\
&\qquad - \big\| m_0|u|^{\nu} + \psi\big\|_2 \|\Delta u\|_2 - \big\| m_1(x, u, Du)|\nabla u|^{\rho} + k(x)\big\|_2 \|\Delta u\|_2 \\
&\qquad - \big\| c(x, u, Du, \Delta u)|\Delta u|\big\|_2 \|\Delta u\|_2 \ge M(\mu + 1)\big\| |u|^{\frac{\mu}{2}} \nabla u\big\|_2^2 \\
&\qquad - \varepsilon_1 \big\| |u|^{\frac{\mu+2}{2}}\big\|_2 - \|M_1\|_\infty^2 \big\| |\nabla u|^{\rho}\big\|_2^2 - \big(\varepsilon + 4^{-1} + \|\psi_1\|_\infty\big)\|\Delta u\|_2^2 \\
&\qquad - C\big(\varepsilon, \varepsilon_1, m_0, \|\psi\|_p, \|k\|_p\big), \quad \varepsilon, \varepsilon_1 \in (0, 1).
\end{aligned} \tag{3.6}$$

Hence we need to estimate the term $\||\nabla u|^{\rho}\|_2^2$ with using $\||u|^{\frac{\mu}{2}} u\|_{1,2}^2$ and $\|u\|_{2,2}^2$ for which we will use the known inequality (G-N-S). Using that we get

$$\begin{aligned}
&\big\langle F(x, u, Du, \Delta u), -\Delta u\big\rangle \\
&\quad \ge \big\langle M(\varepsilon_1)(\mu + 1)|u|^{\mu} \nabla u, \nabla u\big\rangle - \big(\varepsilon + 2^{-1}\big)\|\Delta u\|_2^2 \\
&\qquad - \widehat{C}(\mu, \rho, n)\|M_1\|_\infty^2 \|\Delta u\|_2^{\theta\rho} \|u\|_{\frac{(\mu+2)n}{n-2}}^{(1-\theta)\rho} - C\big(\varepsilon, \varepsilon_1, m_0, \|\psi\|_p, \|k\|_p\big),
\end{aligned}$$

where $\theta = [2\rho(n + \mu) - (\mu + 2)n] \cdot [4\rho(\mu + 1) - \rho\mu n]^{-1}$ for the considered case, and here $\widehat{C}(\mu, \rho, n)$ is the coefficient of the inequality G-N-S.

From here we obtain

$$\begin{aligned}
&\big\langle F(x, u, Du, \Delta u), -\Delta u\big\rangle \\
&\quad \ge \big\langle M_{\varepsilon_1}(\mu + 1)|u|^{\mu} \nabla u, \nabla u\big\rangle - \left(\varepsilon + \frac{3}{4}\right)\|\Delta u\|_2^2
\end{aligned}$$

$$- \left(\widehat{C}(\mu, \rho, n) \|M_1\|_\infty^2\right)^2 \|u\|_{\frac{(\mu+2)n}{n-2}}^{\mu+2} - C\left(\varepsilon, \varepsilon_1, m_0, \|\psi\|_p, \|k\|_p\right), \tag{3.7}$$

where $(\widehat{C}(\mu, \rho, n)\|M_1\|_\infty^2)^2 < M$ by the condition (b).

Now taking account (3.6) in (3.7) we get

$$\left\langle f(u), g(u)\right\rangle = \left\langle -\Delta u + F(x, u, Du, \Delta u), -\Delta u\right\rangle \geq \left(\frac{1}{4} - \varepsilon\right) \|\Delta u\|_2^2$$
$$+ \widetilde{M}(\mu+1) \left\| |u|^{\frac{\mu}{2}} \nabla u\right\|_2^2 - C\left(\varepsilon, \varepsilon_1, m_0, \|\psi\|_p, \|k\|_p\right). \tag{3.8}$$

So, condition (i) is fulfilled as the operator $f : W^{2,2}(\Omega) \cap \mathring{W}^{1,2}(\Omega) \longrightarrow L_2(\Omega)$ is bounded that can be seen easily from its expression and the conditions of this theorem.

Thus it follows that problem (3.1) is densely solvable in $L_2(\Omega)$. Consequently, it is remained to show that image $f(W^{2,2}(\Omega) \cap \mathring{W}^{1,2}(\Omega))$ of mapping f is closed in the space $L_2(\Omega)$.

Let $h_0 \in L_2(\Omega)$ then there is a sequence $\{h_m\} \subset \operatorname{Im} f \subseteq L_2(\Omega)$ that converges to the given h_0 in $L_2(\Omega)$, as $\operatorname{cl}\operatorname{Im} f \equiv L_2(\Omega)$. For any h_m we have the subset $f^{-1}(h_m)$ and as $\{h_m\}$ is a bounded subset in $L_2(\Omega)$ then there is a bounded subset G of $W^{2,2}(\Omega) \cap \mathring{W}^{1,2}(\Omega)$ such that $G \cap f^{-1}(h_m) \neq \varnothing$. Then we can choose a sequence $\{u_m\} \subset G$ such that $f(u_m) = h_m$ which belongs to the bounded subset G. From here using the reflexivity of the space $W^{2,2}(\Omega) \cap \mathring{W}^{1,2}(\Omega)$ we can select a subsequence $\{u_{m_k}\} \subseteq \{u_m\}$ that is a weakly convergent sequence in $W^{2,2}(\Omega) \cap \mathring{W}^{1,2}(\Omega)$, i.e. there is an element u_0 such that $u_{m_k} \rightharpoonup u_0$ in $W^{2,2}(\Omega) \cap \mathring{W}^{1,2}(\Omega)$ (may be after the choice of a subsequence of $\{u_{m_k}\}$), and consequently $u_{m_k} \longrightarrow u_0$ in $W^{1,p}(\Omega)$, $1 \leq p < 2^*$.

Thus we get $F_0(x, u_{m_k}) \longrightarrow F_0(x, u_0)$, $F_1(x, u_{m_k}, Du_{m_k}) \longrightarrow F_1(x, u_0, Du_0)$ in $L_2(\Omega)$ by the conditions of Theorem 3.1 that $F_i : W^{2,2}(\Omega) \cap \mathring{W}^{1,2}(\Omega) \longrightarrow L_2(\Omega)$, $i = 0, 1$, are continuous operators.

On the other hand for any $\varepsilon > 0$ there exist $m_k, m_l \geq m_k(\varepsilon) \geq 1$ such that the inequality

$$\begin{aligned}
\varepsilon &> \|h_{m_k} - h_{m_l}\|_2 \equiv \left\| f(u_{m_k}) - f(u_{m_l})\right\|_2 \\
&\geq \frac{3}{4}\left\|\Delta(u_{m_k} - u_{m_l})\right\|_2 + \widehat{M}\|u_{m_k} - u_{m_l}\|_{2(\mu+1)}^{\mu+1} \\
&\quad - \left\| F_0(x, u_{m_k}) - F_0(x, u_{m_l})\right\|_2 - \left\| F_1(x, u_{m_k}, Du_{m_k}) - F_1(x, u_{m_l}, Du_{m_l})\right\|_2 \\
&\quad - k\left(\|\Delta u_0\|_2\right)\left\| \widetilde{F}_2(x, u_{m_k}, Du_{m_k}) - \widetilde{F}_2(x, u_{m_l}, Du_{m_l})\right\|_2
\end{aligned} \tag{3.9}$$

holds, where $u_{m_k} \longrightarrow u_0$ in $W^{1,p}(\Omega)$, $1 \leq p < 2^*$ and $u_{m_k} \rightharpoonup u_0$ in $W^{2,2}(\Omega)$. Hence we obtain that the last terms of (3.9) converge to zero under $m_k \nearrow \infty$, then we get

$$\left\|\Delta(u_{m_k} - u_{m_l})\right\|_2 \searrow 0 \quad \text{if } m_k, m_l \nearrow \infty.$$

Consequently $\Delta u_{m_k} \longrightarrow \Delta u_0$ in $L_2(\Omega)$, $F_2(x, u_{m_k}, Du_{m_k}, \Delta u_{m_k}) \longrightarrow F_2(x, u_0, Du_0, \Delta u_0)$ in $L_2(\Omega)$ and from the equality

$$\begin{aligned}
&\left\langle -\Delta u_{m_k} + F(x, u_{m_k}, Du_{m_k}, \Delta u_{m_k}), v\right\rangle \\
&\quad = \langle -\Delta u_{m_k}, v\rangle + \left\langle F_0(x, u_{m_k}), v\right\rangle \\
&\qquad - \left\langle F_1(x, u_{m_k}, Du_{m_k}), v\right\rangle - \left\langle F_2(x, u_{m_k}, Du_{m_k}, \Delta u_{m_k}), v\right\rangle \\
&\quad = \langle h_{m_k}, v\rangle, \quad \forall v \in L_2(\Omega)\ \&\ \forall k \ge 1
\end{aligned}$$

we obtain that

$$\left\langle -\Delta u_0 + F(x, u_0, Du_0, \Delta u_0), v\right\rangle = \langle h_0, v\rangle, \quad \forall v \in L_2(\Omega).$$

Hence it follows $h_0 \in \operatorname{Im} f$, i.e. $\operatorname{Im} f \equiv f(W^{2,2}(\Omega) \cap \mathring{W}^{1,2}(\Omega)) \equiv L_2(\Omega)$. □

Remark 3.2 The result of this theorem shows that we can consider the following problem

$$-\Delta u + M|u|^{\mu} u = -a_1(x, u) - F_1(x, u, Du) - F_2(x, u, Du, \Delta u)$$

let the operators $G_0 : W^{2,2}(\Omega) \cap \mathring{W}^{1,2}(\Omega) \longrightarrow L_2(\Omega)$ and $G_1 : W^{2,2}(\Omega) \cap \mathring{W}^{1,2}(\Omega) \longrightarrow L_2(\Omega)$ be defined by the expressions

$$\begin{aligned}
&G_0(u) \equiv -\Delta u + M|u|^{\mu} u, \\
&G_1(u) \equiv -a_1(x, u) - F_1(x, u, Du) - F_2(x, u, Du, \Delta u)
\end{aligned}$$

respectively, then as known $G_0^{-1} : L_2(\Omega) \longrightarrow W^{2,2}(\Omega) \cap \mathring{W}^{1,2}(\Omega)$ exists and is a bounded continuous operator. Now we determine the operator $G(u) \equiv (G_0^{-1} \circ G_1)(u)$ that acts from $W^{2,2}(\Omega) \cap \mathring{W}^{1,2}(\Omega)$ to $W^{2,2}(\Omega) \cap \mathring{W}^{1,2}(\Omega)$ and is bounded continuous operator under the conditions of Theorem 3.1. Hence we obtain, that the operator G possesses a fixed point by virtue of Theorem 2.4, which allows us to impose the problem $G_0(u) = \lambda G_1(u)$ on the existence of the eigenvalue of operator G_0 respective of operator G_1. But in this case we can only conclude that λ will be dependent of u.

Remark 3.3 From the proof of this theorem it follows that the result of such type is true in the following case: let in problem (3.1) operator $F(x, u, Du, \Delta u)$ be independent of Δu, i.e. it has the representation $F(x, \xi, \eta, \zeta) \equiv F(x, \xi, \eta) \equiv F_0(x, \xi) + F_1(x, \xi, \eta)$ for $(x, \xi, \eta) \in \Omega \times \Re^{n+1}$.

4 Nonlinear Equation with p-Laplacian

On the open bounded domain $\Omega \subset R^n$ with sufficiently smooth boundary $\partial\Omega$ consider the following problem

$$f(u) \equiv -\nabla\left(|\nabla u|^{p-2}\nabla u\right) + G(x, u, Du, \lambda, \rho) = h(x), \quad x \in \Omega \subset R^n, \tag{4.1}$$

$$u|_{\partial\Omega} = 0, \quad n \geq 1, \quad \Omega \in Lip, \quad h \in W^{-1,q}(\Omega). \tag{4.2}$$

Assume that

$$G(x, \xi, \eta, \lambda) = \rho G_0(x, \xi) + \lambda G_1(x, \xi, \eta), \tag{4.3}$$

holds for a.e. $x \in \Omega$ and any $(\xi, \eta) \in R^{n+1}$, where $G_1(x, \xi, \eta)$ and $G_0(x, \xi)$ are some Caratheodory functions, $\lambda \in R$, $\mu \geq 0$ are some parameters.

4.1 Dense Solvability

Let the following conditions

$$G_0(x, \xi) \cdot \xi \geq a_0(x)|\xi|^{p_0} - a_1(x), \quad a_0(x) \geq A_0 > 0; \tag{4.4}$$

$$\left|G_0(x, \xi)\right| \leq \widetilde{a}_0(x)|\xi|^{p_0-1} + \widetilde{a}_1(x), \quad \widetilde{a}_0(x), \widetilde{a}_1(x) \geq 0,$$

$$\left|G_1(x, \xi, \eta)\right| \leq b_0(x)|\eta|^{p_1} + b_1(x)|\xi|^{p_2} + b_2(x), \tag{4.5}$$

$$b_j(x) \geq 0, \ j = 0, 1, 2,$$

hold for a.e. $x \in \Omega$ and any $(\xi, \eta) \in R^{n+1}$ where $p_0, p_1, p_2 \geq 0$, $p > 1$ are some numbers, $a_k(x)$, $\widetilde{a}_k(x)$ and $b_j(x)$ are some functions, $k = 0, 1$ and $j = 0, 1, 2$.

Here we study the solvability of problem (4.1)–(4.2) in the generalized sense, i.e. a function $u \in W_0^{1,p}(\Omega)$ is called a solution of the problem (4.1), (4.2) if u satisfies the equation

$$\left\langle f(u), v\right\rangle = \langle h, v\rangle, \quad v \in W_0^{1,p}(\Omega),$$

for any $v \in W_0^{1,p}(\Omega)$.

Theorem 4.1 *Let conditions (4.3)–(4.5) be fulfilled and* $p_0 - 1 \geq p_2 \geq 0$, $p > 1$, $p > p_1 \geq 0$. *Moreover, let* $a_k(x)$, $\widetilde{a}_k(x)$, $b_j(x)$ *be such functions that* $a_0, \widetilde{a}_0 \in L^\infty(\Omega)$, $a_1, \widetilde{a}_1 \in L^q(\Omega)$ *and* $b_0, b_1 \in L^\infty(\Omega)$, $b_2 \in L^q(\Omega)$. *If* p_1, p_0 *satisfy the inequalities* $p_1 \leq p - \frac{p}{p_0}$, $p_0 \leq p^* \equiv \frac{pn}{n-p}$, *then there exist a subset* $\mathcal{M} \subseteq W^{-1,q}(\Omega)$ *and some numbers* $\mu_0 > 0$, $C_0 > 0$ *such that* $\overline{\mathcal{M}}^{W^{-1,q}} \equiv W^{-1,q}(\Omega)$, $q = \frac{p}{p-1} \equiv p'$, *and* $\rho \geq \mu_0 > 0$, $\lambda : |\lambda| \leq C_0$, *problem (4.1)–(4.2) is solvable in* $W_0^{1,p}(\Omega)$ *for* *any* $h \in \mathcal{M}$; *moreover if* $\widetilde{p} = p_0$ *or* $p_2 + 1 = p_0$ *then* $C_0 \equiv C_0(A_0, b_0, b_1, \rho)$ *is sufficiently small number.*

Proof Let $u \in W_0^{1,p}(\Omega) \cap L^{p_0}(\Omega)$ and consider the dual form

$$\begin{aligned}\big\langle f(u), u\big\rangle &\equiv \|\nabla u\|_p^p + \big\langle G(x, u, Du, \lambda), u\big\rangle \\ &= \|\nabla u\|_p^p + \big\langle \rho G_0(x, u), u\big\rangle + \lambda\big\langle G_1(x, u, Du), u\big\rangle,\end{aligned}$$

then by using conditions (4.4) and (4.5), we get

$$\begin{aligned}\big\langle f(u), u\big\rangle \geq{}& \|\nabla u\|_p^p + \rho\big\langle a_0(x)|u|^{p_0-2}u, u\big\rangle - \rho\|a_1\|_1 \\ &- |\lambda|\big\langle b_0(x)|\nabla u|^{p_1}, |u|\big\rangle - |\lambda|\big\langle b_1(x)|u|^{p_2}, |u|\big\rangle - |\lambda|\big\langle b_2(x), |u|\big\rangle,\end{aligned}$$

or

$$\begin{aligned}\big\langle f(u), u\big\rangle \geq{}& \|\nabla u\|_p^p + \rho A_0\|u\|_{p_0}^{p_0} - \rho\|a_1\|_1 \\ &- |\lambda|\int_\Omega b_0(x)|\nabla u|^{p_1}|u|dx - |\lambda|\int_\Omega b_1(x)|u|^{p_2+1}dx \\ &- |\lambda|\int_\Omega b_2(x)|u|dx. \end{aligned} \tag{4.6}$$

Since $b_j \in L^\infty(\Omega)$, $j = 0, 1$, it is enough to estimate first integral of the right side of inequality (4.6). For this, we have

$$|\lambda|\int_\Omega b_0(x)|\nabla u|^{p_1}|u|dx \leq |\lambda|\|b_0\|_\infty\big[\varepsilon\big\||\nabla u|\big\|_p^p + c(\varepsilon)\|u\|_{\widetilde{p}}^{\widetilde{p}}\big].$$

By using the last inequality in (4.6) and taking into account the condition on p_1, we obtain

$$\begin{aligned}\big\langle f(u), u\big\rangle \geq{}& \big(1 - \varepsilon|\lambda|\|b_0\|_\infty\big)\big\||\nabla u|\big\|_p^p + (\rho A_0 - \varepsilon_1)\|u\|_{p_0}^{p_0} \\ &- c(\varepsilon)|\lambda|\|b_0\|_\infty\|u\|_{\widetilde{p}}^{\widetilde{p}} - |\lambda|\int_\Omega b_1(x)|u|^{p_2+1}dx \\ &- C_{\varepsilon_1}\big(|\lambda|, \rho, \|a_1\|_q, \|b_2\|_q\big).\end{aligned}$$

since $p_1 \leq p(1 - p_0^{-1})$ by the conditions $\widetilde{p} \leq p_0$. Hence either of these cases take place: $\widetilde{p} < p_0$ and $p_2 + 1 < p_0$ or one of equations $\widetilde{p} = p_0$ or $p_2 + 1 = p_0$ holds, if $\widetilde{p} < p_0$ and $p_2 + 1 < p_0$ then we can estimate it by second term from right side of the previous inequality with using Young inequality, and if $\widetilde{p} = p_0$ or $p_2 + 1 = p_0$ then it is enough to choose number $|\lambda|$ sufficiently small. Thus we obtain the following inequality:

$$\langle f(u),u\rangle \geq \bigl(1-\varepsilon|\lambda|\|b_0\|_\infty\bigr)\bigl\||\nabla u|\bigr\|_p^p + (\rho A_0 - \varepsilon_1 - \varepsilon_2)\|u\|_{p_0}^{p_0}$$
$$- C_{\varepsilon_1}\bigl(|\lambda|,\rho,\varepsilon_2,\|a_1\|,\|b_1\|,\|b_2\|\bigr).$$

Consequently, inequality (2.1) of Theorem 2.1 is fulfilled, $|\lambda|$ must be sufficiently small, i.e. the statement of Theorem 4.1 is true since the operator

$$f : W_0^{1,p}(\Omega)\cap L^{p_0}(\Omega) \longrightarrow W^{-1,q}(\Omega)+L^{q_0}(\Omega),\ q_0=\frac{p_0}{p_0-1},$$

is bounded by virtue of the obtained estimations here. □

4.2 Everywhere Solvability

Now we reduce the conditions under which imposed problem (4.1)–(4.2) is everywhere solvable. Let conditions (4.3)–(4.5) be fulfilled and consider the following conditions

$$\begin{aligned} \bigl|G_0(x,\xi)-G_0(x,\xi_1)\bigr| &\leq c_0(x,\widetilde{\xi})|\xi-\xi_1|,\\ \bigl|G_1(x,\xi,\eta)-G_1(x,\xi_1,\eta_1)\bigr| &\leq c_1(x,\widetilde{\xi},\widetilde{\eta})|\eta-\eta_1|, \end{aligned} \tag{4.7}$$

hold for a.e. $x\in\Omega$, and any (ξ,η), $(\xi_1,\eta_1)\in\Re\times\Re^n$, where $c_0(x,\xi)$, $c_1(x,\xi,\eta)$ are some Caratheodory functions such that $\widetilde{\xi}=\widetilde{\xi}(\xi,\xi_1)$, $\widetilde{\eta}=\widetilde{\eta}(\eta,\eta_1)$ are continuous functions, and moreover $c_1(x,v,\nabla v)$, $c_0(x,v)$ are bounded operators such that if $v(x)$ belongs to a bounded subset D of $W_0^{1,p}(\Omega)\cap L^{p_0}(\Omega)$, i.e. if $\|v\|_{W_0^{1,p}(\Omega)\cap L^{p_0}(\Omega)}\leq K_0$, then

$$\bigl\|c_1(x,v,\nabla v)\bigr\|_{L^\infty(\Omega)}\leq K_1,$$
$$\bigl\|c_0(x,v)\bigr\|_{L^\infty(\Omega)}\leq K_2,$$

for some numbers $K_0,K_1,K_2>0$, i.e.

$$c_j(x,\cdot,\cdot) : W_0^{1,p}(\Omega)\cap L_{p_0}(\Omega)\longrightarrow L^\infty(\Omega),\quad j=0,1, \tag{4.8}$$

are bounded operators.

Then

$$\bigl|\bigl\langle G_1(x,u,\nabla u)-G_1(x,v,\nabla v),u-v\bigr\rangle\bigr|$$
$$\leq \bigl\|c_1\bigl(x,\widetilde{u}(u,v),\nabla\widetilde{u}(\nabla u,\nabla v)\bigr)\bigr\|_\infty\|\nabla u-\nabla v\|_p\|u-v\|_q,$$

holds for any $u,v\in W_0^{1,p}(\Omega)\cap L^{p_0}(\Omega)$. Hence we get

$$\begin{aligned}
&\langle f(u) - f(v), u - v\rangle \\
&\quad \equiv \langle |\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v, \nabla(u - v)\rangle \\
&\qquad + \rho\langle G_0(x,u) - G_0(x,v), u - v\rangle + \lambda\langle G_1(x,u,Du) - G_1(x,v,Dv), u - v\rangle \\
&\quad \geq \widehat{C}_0\|\nabla(u - v)\|_p^p + \rho\langle a(x)(|u|^{p_0-2}u - |v|^{p_0-2}v), u - v\rangle,
\end{aligned}$$

and we obtain the following inequality by using conditions (4.4), (4.7) and (4.8)

$$\begin{aligned}
&\langle f(u) - f(v), u - v\rangle \\
&\quad \geq \widehat{C}\|\nabla(u - v)\|_p^p + \widehat{A}_0\|u - v\|_{p_0}^{p_0} \\
&\qquad - \|c_1(x,u,\nabla u, v, \nabla v)\|_\infty \|\nabla u - \nabla v\|_p \|u - v\|_q.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
&\|f(u) - f(v)\|_{W_q^{-1}(\Omega)} \cdot \|\nabla(u - v)\|_p \\
&\quad \geq \widehat{C}\|\nabla(u - v)\|_p^p + \widehat{A}_0\|u - v\|_{p_0}^{p_0} \\
&\qquad - \|c_1(x,u,\nabla u, v, \nabla v)\|_\infty \left[\varepsilon\|\nabla(u - v)\|_p^p + c(\varepsilon)\|u - v\|_q^q\right],
\end{aligned}$$

or

$$\begin{aligned}
&\|f(u) - f(v)\|_{W_q^{-1}(\Omega)} \cdot \|\nabla(u - v)\|_p \\
&\quad \geq \widehat{C}_1\|\nabla(u - v)\|_p^{p-1} + \widehat{A}_0\|u - v\|_{p_0}^{p_0} - c(\varepsilon)\|u - v\|_q^q.
\end{aligned}$$

Thus we get that the conditions of Corollary 2.3 are fulfilled, i.e. if we continue this proof as in Sect. 3 then we obtain that the following result is true.

Theorem 4.2 *Let conditions* (4.3)–(4.5), (4.7) *and* (4.8) *be fulfilled and the numbers* λ, ρ *satisfy the conditions of Theorem* 4.1, *then problem* (4.1)–(4.2) *is solvable in* $W_0^{1,p}(\Omega) \cap L^{p_0}(\Omega)$ *for any* $h \in W^{-1,q}(\Omega)$.

Remark 4.3 It should be noted that a remark similar to Remark 3.2 takes place for the problem investigated here. Moreover, we can obtain the same conclusion for the considered general case. Therefore, for the investigation of the spectrum of the nonlinear operators by using the fixed-point theorem mentioned above, we need to consider some particular cases of these problems.

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Some Remarks About Chow, Hilbert and K-stability of Ruled Threefolds

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Abstract Given a rank 2 holomorphic vector bundle E over a projective surface, we explain some relationships between the Gieseker stability of E and the Chow, Hilbert and K-stability of the polarized ruled manifold $\mathbb{P}E$ with respect to polarizations that make fibres sufficiently small.

Keywords Ruled manifold · Projective bundle · K-stability · Chow stability · Hilbert stability · Mumford stability · Gieseker stability · G.I.T

Mathematics Subject Classification (2010) Primary 14L24 · Secondary 14J60

In this paper, we pursue our study of the stability in the sense of Geometric Invariant Theory (G.I.T in short) of ruled manifolds given as projectivisation of rank 2 vector bundles over projective surfaces. The purpose of this note is to observe that the notion of Gieseker stability for the underlying vector bundle plays a key role in the Chow, Hilbert and K-stability of the associated ruled threefold when the first Chern class of the base is proportional to the considered polarization. This has to be compared with the simpler case of ruled manifolds over a curve where the notion of Mumford stability is central, and we refer to [1, 3] on this topic. We want to point that checking stability algebraically is a difficult problem. Our proofs rely mainly on two ingredients. One is coming from geometric analysis with the connection between existence of canonical Kähler metrics (namely Kähler metrics with constant scalar curvature) and stability notions. The other one is a brute force computation of the G.I.T weights for certain test configurations associated to the deformation to the normal cone of the projectivisation of a subbundle. These ingredients have appeared in [11] but in this paper we are carrying the computations to a greater extent and draw some simple and natural consequences from them. In particular we provide

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some new examples of asymptotically Hilbert or Chow semistable polarizations that are not asymptotically Hilbert or Chow stable.

1 About Chow Stability, Hilbert Stability and K-Stability

In this section, we recall briefly some well known facts about Chow stability and K-stability of a polarized scheme. We refer to [6, 7, 16, 21] for details and examples.

Consider (X, L) a polarized subscheme of complex dimension n and $X \subset \mathbb{P}H^0(X, L^k)^* = \mathbb{P}V$ the closed immersion associated to the complete linear system $|L^k|$. Let $Z_X = \{P \in Gr(V, n-1) : P \cap X \neq \emptyset\}$ which is a divisor of degree $d = \deg L$ in the Grassmannian $\mathcal{G} = Gr(V, n-1)$. Thus there exists $s_{X,V} \in H^0(\mathcal{G}, \mathcal{O}_{\mathcal{G}}(d))$, such that one has $Z_X = \{s_{X,V} = 0\}$ and this induces a Chow point

$$\mathrm{Chow}(X) = [s_{X,V}] \in \mathbb{P}H^0\big(\mathcal{G}, \mathcal{O}_{\mathcal{G}}(d)\big)$$

on which one can consider the action of $SL(V)$. The polarized scheme (X, L^k) is said to be Chow stable (resp. Chow semistable) if the Chow point $\mathrm{Chow}(X)$ is G.I.T stable (resp. G.I.T semistable).

We say that it is asymptotically Chow stable (resp. asymptotically Chow semistable) if (X, L^k) is Chow stable (resp. Chow semistable) for $k \gg 1$.

Let us discuss now Hilbert stability. For $X \subset \mathbb{P}V$ a closed subscheme such that the restriction map

$$\rho : H^0\big(\mathbb{P}V, \mathcal{O}(m)\big) \to H^0\big(X, \mathcal{O}(m)\big)$$

is surjective, one sets

$$W_m = \bigwedge^{h^0(X,\mathcal{O}(m))} H^0\big(\mathbb{P}V, \mathcal{O}(m)\big)^\vee.$$

Thus, from the map ρ and taking the wedge product, one can consider the m-Hilbert point

$$[X]_m = \left[\bigwedge^{h^0(X,\mathcal{O}(m))} H^0\big(\mathbb{P}V, \mathcal{O}(m)\big) \to \bigwedge^{h^0(X,\mathcal{O}(m))} H^0\big(X, \mathcal{O}(m)\big)\right] \in \mathbb{P}(W_m).$$

Now, the polarized scheme (X, L) is said to be Hilbert stable (resp. Hilbert semistable) if the induced m-Hilbert points $[X]_m$ defined by the closed immersion associated to the complete linear system $|L^m|$ are all G.I.T semistable (resp. G.I.T stable) for $m \gg 1$.

The polarized scheme (X, L) is said to be asymptotically Hilbert stable (resp. asymptotically Hilbert semistable) if (X, L^k) is Hilbert stable (resp. Hilbert semistable) for $k \gg 1$.

We recall now the notion of test configuration [5, 6].

Definition 1.1 A test configuration for a polarized scheme (X, L) is a polarized scheme $(\mathcal{X}, \mathcal{L})$ with:

- a $\mathbb{C}^\times$ action and a proper flat morphism $\pi : \mathcal{X} \to \mathbb{C}$ which is $\mathbb{C}^\times$ equivariant for the usual action on $\mathbb{C}$,
- a $\mathbb{C}^\times$ equivariant line bundle $\mathcal{L} \to \mathcal{X}$ which is ample over all fibers of π such that for $z \neq 0$, (X, L^s) is isomorphic to $(\mathcal{X}_z, \mathcal{L}_{\mathcal{X}_z})$ for some positive integer s, called the exponent.

A product test configuration is a test configuration with $\mathcal{X} \simeq X \times \mathbb{C}$. A test configuration is trivial in codimension 2 if it is $\mathbb{C}^\times$-equivariantly isomorphic to a product test configuration $X \times \mathbb{C}$, with trivial $\mathbb{C}^\times$-action, away from a closed subscheme of codimension at least 2.

From [18], we know that there is a correspondence between the data of a test configuration $(\mathcal{X}, \mathcal{L})$ of exponent s and the data of a 1-parameter subgroup of $GL(H^0(X, L^s))$. Thus using the Hilbert–Mumford criterion, it is sufficient to consider the weights of the $\mathbb{C}^\times$ action to check the stability of (X, L). More precisely, let us call $w(Ks)$ the total weight of the induced action on $\pi_*\mathcal{L}^K_{|0} = H^0_{\mathcal{X}_0}(\mathcal{L}^K)$ for $K \gg 0$, for a test configuration associated to (X, L^{Ks}). Remark that $w(Ks)$ is a polynomial of degree $n+1$ in the $k = Ks$ variable. Let us denote $P(k) = \dim H^0(X, L^k)$ which is equal to the Hilbert polynomial $\chi(X, L^k)$ for k large. The normalized weight after taking the $sP(s)$-th power of the $\mathbb{C}^\times$ action on $\pi_*\mathcal{L}^K_{|0}$ is

$$\tilde{w}(s,k) = w(k)sP(s) - w(s)kP(k) \tag{1.1}$$

which is a polynomial of degree $n+1$ in the k variable. It is the Hilbert weight of (X, L^s) and thus (X, L) is asymptotically Hilbert stable (resp. asymptotically Hilbert semistable) if and only if $\tilde{w}(s,k) > 0$ (resp. $\tilde{w}(s,k) \geq 0$) for all $k \gg 1$ ($k > k_0(s)$ large enough), $s \gg 1$.

One can decompose $\tilde{w}(s,k)$ as

$$\tilde{w}(s,k) = \sum_{i=0}^{n+1} e_i k^i \tag{1.2}$$

where $e_i = \sum_{j=0}^{n+1} e_{i,j}s^j$ are polynomials of degree $n+1$ in the s variable with $e_{n+1,n+1} = 0$ due to the normalisation.

We refer to [16, Lemma 2.11] and [18, Theorem 3.9] for a proof of the next result.

Lemma 1.1 *The coefficient $e_{n+1}(s)s^{n+1}(n+1)!$ is the Chow weight of $X \subset \mathbb{P}H^0(X, L^s)$. In particular, (X, L) is asymptotically Chow stable (resp. asymptotically Chow semistable) if and only if $e_{n+1}(s) > 0$ (resp. ≥ 0) for all $s \gg 1$. (X, L) is asymptotically Chow polystable if it is asymptotically Chow semistable and any not strictly stable test configuration is a product test configuration.*

The following definition is a refinement of Donaldson's definition of K-stability [6] and is due to Stoppa [20].

Definition 1.2 The polarized variety (X, L) is K-stable (resp. K-semistable) if for any test configuration which is non trivial in codimension 2, the leading coefficient $e_{n+1,n}$ of $e_{n+1}(s)$ is positive (resp. ≥ 0). It is said to be K-polystable if it is K-semistable and any not strictly stable test configuration is a product test configuration.

Let us finish this section by recalling certain well-known relationships between the various notions of stability that we shall use later (see [13, 21]):

$$\begin{aligned} &\text{Asymptotic Chow stability} \\ \Leftrightarrow\quad &\text{Asymptotic Hilbert stability} \\ \Rightarrow\quad &\text{Asymptotic Hilbert semistability} \\ \Rightarrow\quad &\text{Asymptotic Chow semistability} \\ \Rightarrow\quad &\text{K-semistability.} \end{aligned}$$

2 Rank 2 Vector Bundles over Surfaces and the Stability of Their Projectivisation

Let us fix B a projective surface polarized by L and $\pi : E \to B$ an indecomposable holomorphic vector bundle on B. We shall compute in our setting the Donaldson–Futaki invariant $F_1(\mathcal{T})$ induced by the degeneration $\mathcal{T}$ to the normal cone of $\mathbb{P}(F)$ where F is a subbundle of E and with respect to the polarization $\mathcal{L}_{r,m}$. Let us give now some explanations on this computation (we refer to [11, 17] for details of the test configuration we construct).

We consider the family of bundles $\mathcal{E} \to B \times \mathbb{C} \to \mathbb{C}$ with general fibre E and central fibre $F \oplus G$ over $0 \in \mathbb{C}$ where G is the quotient bundle. Then $\mathcal{E}$ admits a $\mathbb{C}^*$ action that covers the usual action on the base $\mathbb{C}$, and whose restriction to $F \oplus G$ scales the fibres of F with weight 1 and acts trivially on G. Setting $\mathcal{X} = \mathbb{P}(\mathcal{E}) \to \mathbb{C}$ and

$$\mathcal{L}_{r,m} = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(r) \otimes \pi^* L^m$$

with (r, m) such that $\mathcal{L}_{r,m}$ is ample, we obtain a flat family of polarized varieties with $\mathbb{C}^*$ action whose general fibre is the polarized ruled manifold $(\mathbb{P}E, \mathcal{L}_{r,m})$. It is a non trivial test configuration that we shall denote by $\mathcal{T}$.

Conventions If $\pi : E \to B$ is a vector bundle then $\pi : \mathbb{P}(E) \to B$ shall denote the space of complex *hyperplanes* in the fibres of E. Thus $\pi_* \mathcal{O}_{\mathbb{P}(E)}(r) = S^r E$ for $r \geq 0$.

Notation 1 *For L a line bundle (not necessarily ample) and $\mathcal{F}$ a coherent subsheaf over B, one can define the slope of $\mathcal{F}$ by the normalised degree of $\mathcal{F}$, i.e.*

$$\mu_L(\mathcal{F}) = \frac{\deg_L(\mathcal{F})}{rk(\mathcal{F})} = \frac{c_1(L)c_1(\mathcal{F})}{rk(\mathcal{F})},$$

and the normalised Hilbert polynomial by

$$\mathcal{P}_{\mathcal{F}}(k) = \frac{\chi(\mathcal{F} \otimes L^k)}{rk(\mathcal{F})}.$$

We recall some well known definitions about stability of bundles.

Definition 2.1 Let L be an ample line bundle on the projective manifold B. A vector bundle E is said to be L-Mumford–Takemoto stable if for any proper coherent subsheaf $\mathcal{F}$ of E one has the slope inequality $\mu_L(\mathcal{F}) < \mu_L(E)$.

We say that E is Gieseker stable (resp Gieseker semistable) with respect to L if for all proper coherent subsheaves $F \subset E$ one has the following inequality for the normalized Hilbert polynomials

$$\mathcal{P}_F(k) < \mathcal{P}_E(k) \quad \text{for } k \gg 0 \text{ (resp. } \leq),$$

and strictly Gieseker semistability E is Gieseker semistable but not Gieseker stable. A Gieseker semistable bundle is said to be Gieseker polystable if it is a direct sum of Gieseker stable bundles with respect to the same polarization.

These stability notions are related; using that $\mu_L(F)$ is the leading order term in k of $\mathcal{P}_F(k)$ one sees immediately that

$$\begin{array}{ccccccc} \text{Mumford} & & \text{Gieseker} & & \text{Gieseker} & & \text{Mumford} \\ \text{stable} & \Rightarrow & \text{stable} & \Rightarrow & \text{semistable} & \Rightarrow & \text{semistable.} \end{array}$$

For simplicity we will work in the sequel of the paper with rank 2 vector bundles over surfaces.

Notation 2 *Let us assume that the vector bundle E has rank $rk(E) = 2$ and B is a surface. We set*

$$\delta_L = \mu_L(E) - \mu_L(F)$$
$$\Delta = \frac{\mathrm{ch}_2(E)}{2} - \mathrm{ch}_2(F) + \frac{1}{2}\delta_{K_B^*}$$

so that one can write $\mathcal{P}_E(k) - \mathcal{P}_F(k) = k\delta_L + \Delta$.

In the following proposition, we express the Donaldson–Futaki invariant for the polarization $\mathcal{L}_{r,m}$ associated to the test configuration we have just described.

Proposition 2.1 *The Donaldson–Futaki invariant of the test configuration $\mathcal{T}$ for a rank 2 vector bundle E over a polarized surface (B, L) induced by the deformation to the normal cone of $\mathbb{P}F$ where F is a subbundle of E is given by*

$$F_1(\mathcal{T}) = \frac{r^6}{36}(\delta_{K_B^*})^2 - \frac{r^4}{72}\Gamma_1\delta_{K_B^*} + \frac{r^3}{24}\Gamma_2(m\delta_L + r\Delta),$$

with

$$\begin{aligned}\Gamma_1 &= r^2\big(c_1(E)^2 - 4c_1(F)^2\big) + 3c_1\big(F^r \otimes L^m\big)^2 + 4r^2\Delta + 12rm\delta_L \\ &\quad - 3rc_1(B)c_1\big(F^r \otimes L^m\big), \\ \Gamma_2 &= \big(rc_1(E) + 2mc_1(L)\big)^2 - 2rc_1\big(F^r \otimes L^m\big)c_1(B).\end{aligned}$$

Proof The proposition is a consequence of [11, Proposition 19–Corollary 21] where it is proved by a direct computation that

$$e_{4,3}(\mathcal{T}) = F_1(\mathcal{T}) = C_1r^3m^3 + C_2r^4m^2 + C_3r^5m + C_4r^6 \tag{2.1}$$

where

$$\begin{aligned}C_1(E,F) &= \frac{c_1(L)^2}{6}\big(\mu_L(E) - \mu_L(F)\big), \\ C_2(E,F) &= \frac{c_1(L)^2}{48}\big(c_1(E) - 2c_1(F)\big)c_1(B) \\ &\quad + \frac{c_1(L)^2}{12}\big(\mathrm{ch}_2(E) - 2\,\mathrm{ch}_2(F)\big) \\ &\quad + \frac{1}{12}\big(2c_1(E)c_1(L) - c_1(B)c_1(L)\big)\big(\mu_L(E) - \mu_L(F)\big), \\ C_3(E,F) &= -\frac{1}{12}\deg_L(E)c_1(F)^2 + \frac{1}{12}\deg_L(E)\,\mathrm{ch}_2(E) \\ &\quad + \frac{1}{48}\deg_L(E)c_1(E)^2 - \frac{1}{24}\deg_L(F)c_1(E)^2 \\ &\quad + \frac{1}{24}c_1(L)c_1(B)\cdot c_1(F)^2 - \frac{1}{24}c_1(L)c_1(B)\cdot \mathrm{ch}_2(E) \\ &\quad + \frac{1}{24}\deg_L(F)c_1(E)c_1(B) - \frac{1}{24}\deg_L(E)c_1(B)c_1(F), \\ C_4(E,F) &= \frac{1}{288}c_1(E)^2\cdot c_1(B)c_1(E) - \frac{1}{144}c_1(E)^2\cdot c_1(B)c_1(F) \\ &\quad + \frac{1}{48}c_1(F)^2\cdot c_1(E)c_1(B) \\ &\quad - \frac{1}{72}\big(c_1(B)c_1(F) + c_1(E)c_1(B)\big)\,\mathrm{ch}_2(E)\end{aligned}$$

$$+\frac{1}{48}c_1(E)^2\big(\mathrm{ch}_2(E)-c_1(F)^2\big).$$

By a simple algebraic manipulation one obtains from (2.1) the expected result. □

Proposition 2.2 *In the same setting as in Proposition 2.1 and with Notations 2, the Chow weight associated to the test configuration $\mathcal{T}$ is given by*

$$\begin{aligned}\mathrm{Chow}_s(\mathcal{T})=e_4(s)=&\frac{sr^4(rs-1)(rs+1)}{36}\delta_{K_B^*}{}^2\\&-\frac{sr^2(rs+1)}{72}A_1\delta_{K_B^*}\\&+\frac{sr^2(rs+1)}{24}A_2(m\delta_L+r\Delta)\end{aligned}$$

with

$$A_1=sr\Gamma_1-A_1'$$
$$A_2=s\Gamma_2-4r\,\mathrm{Todd}_2(B),$$

where we set

$$A_1'=\Gamma_1+3c_1\big(F^r\otimes L^m\big)^2+3rc_1(B)c_1\big(F^r\otimes L^m\big)+6\,\mathrm{Todd}_2(B).$$

Moreover,

$$\mathrm{Chow}_s=s^3F_1(\mathcal{T})+s^2F_2(\mathcal{T})+sF_3(\mathcal{T})$$

with higher Futaki invariants $F_2(\mathcal{T})$, $F_3(\mathcal{T})$ given by

$$F_2(\mathcal{T})=\left(\frac{1}{r}F_1(\mathcal{T})+rF_3(\mathcal{T})\right),$$
$$F_3(\mathcal{T})=-\frac{1}{36}r^4\delta_{K_B^*}{}^2+\frac{1}{72}r^2A_1'\delta_{K_B^*}-\frac{1}{6}r^3\,\mathrm{Todd}_2(B)(m\delta_L+r\Delta),$$

with $\mathrm{Todd}_2(B)$ *the second Todd class of B.*

Proof Writing the weight of the action as $w(s)=\sum_{l=0}^{n+1}b_ls^{n+1-l}$ and

$$P(s)=\dim H^0\big(\mathbb{P}E,\mathcal{L}_{r,m}^s\big)=\sum_{l=0}^{n}a_ls^{n-l}$$

with $n=3$ and s large enough (see Sect. 1), we get

$$e_4(s)=\sum_{l=1}^{3}(b_0a_l-a_0b_l)s^{4-l}-a_0b_4.$$

In the case we are considering, we have

$$
\begin{aligned}
a_0 &= \frac{1}{2}rm^2c_1(L)^2 + \frac{1}{2}mr^2\deg_L(E) + \frac{1}{6}r^3\,\mathrm{ch}_2(E) + \frac{1}{12}r^3c_1(E)^2,\\
a_1 &= \frac{r^2}{4}c_1(E)c_1(B) + \frac{m^2}{2}c_1(L)^2 + \frac{rm}{2}\big(c_1(L)c_1(B) + \deg_L(E)\big) + \frac{r^2}{2}\,\mathrm{ch}_2(E),\\
a_2 &= -\frac{r}{12}c_1(E)^2 + r\,\mathrm{Todd}_2(B) + \frac{r}{4}c_1(E)c_1(B) + \frac{m}{2}c_1(L)c_1(B) + \frac{r}{3}\,\mathrm{ch}_2(E),\\
a_3 &= \mathrm{Todd}_2(B),
\end{aligned}
$$

and

$$
\begin{aligned}
b_0 &= \frac{r^4}{24}c_1(E)^2 + \frac{r^4}{12}c_1(F)^2 + \frac{m^2r^2}{4}c_1(L)^2 + \frac{mr^3}{6}\big(\deg_L(E) + \deg_L(F)\big),\\
b_1 &= \frac{r^3}{4}c_1(F)^2 + \frac{r^3}{12}c_1(F)c_1(B) + \frac{r^3}{12}c_1(E)c_1(B) + \frac{rm^2}{4}c_1(L)^2\\
&\quad + \frac{mr^2}{4}\big(c_1(L)c_1(B) + 2\deg_L(F)\big),\\
b_2 &= \frac{r^2}{2}\,\mathrm{Todd}_2(B) + \frac{r^2}{6}c_1(F)^2 - \frac{r^2}{24}c_1(E)^2 + \frac{r^2}{4}c_1(F)c_1(B)\\
&\quad + \frac{rm}{3}\deg_L(F) - \frac{rm}{6}\deg_L(E) + \frac{rm}{4}c_1(L)c_1(B),\\
b_3 &= \frac{r}{2}\,\mathrm{Todd}_2(B) + \frac{r}{6}c_1(F)c_1(B) - \frac{r}{12}c_1(E)c_1(B),\\
b_4 &= 0.
\end{aligned}
$$

We refer to [11, Proposition 20] and [3] for the details of computing the terms a_l, b_l where most of them have been explicitly identified using Hirzebruch–Riemann–Roch theorem. □

We dress now some easy consequences of the two previous results. We get the following theorem which strengthens [11, Proposition 21].

Theorem 2.1 *Consider E an irreducible rank 2 holomorphic vector bundle on a polarized surface (B, L) with $c_1(B)$ proportional to $c_1(L)$.*

1. *Assume that E is strictly Gieseker semistable and F is a subbundle of E with $\mathcal{P}_F = \mathcal{P}_E$ with respect to L. Then all the tensor powers of the polarization $\mathcal{L}_{r,m}$ are not Chow polystable, $\mathcal{L}_{r,m}$ is not asymptotically Chow polystable and not K-polystable.*
2. *Assume that E is not Gieseker semistable and F is a destabilizing subbundle. Then $\mathcal{L}_{r,m}$ is not K-semistable and thus not asymptotically Chow semistable for $m \gg 0$.*

3. *If $\mathcal{L}_{r,m}$ is K-stable (resp. K-polystable, resp. K-semistable) for all $m \gg 0$ then E is Gieseker stable (resp. Gieseker polystable, resp. strictly Gieseker semistable) with respect to L.*

Proof For (1), we consider the test configuration $\mathcal{T}$ of the deformation to the normal cone of $\mathbb{P}F$ described as before. From our assumption of Gieseker semistability we have $\delta_L = \Delta = 0$ while the assumption on the first Chern class gives $\delta_{K_B^*} = 0$ since $c_1(B) = 0$ or $c_1(B) = \lambda c_1(L)$. Therefore from Propositions 2.1 and 2.2, one has $F_1(\mathcal{T}) = \mathrm{Chow}_s(\mathcal{T}) = 0$ while the test configuration $\mathcal{T}$ is not a product test configuration. The point (2) can be treated in a similar way using the proof of Proposition 2.1. Actually the destabilizing subbundle leads to $C_1 = 0$ and $C_2 < 0$ or $C_1 < 0$ and thus $F_1(\mathcal{T}) < 0$. Remark that (2) strengthens a result of [17, Theorem 5.12] where it is shown that if E is not Mumford stable then $\mathcal{L}_{r,m}$ is not K-semistable.

Note that under the assumptions of (1) or (2), there is no Kähler metric with constant scalar curvature in the class $c_1(\mathcal{L}_{r,m})$ as a consequence of [4, 14, 19].

Now let us assume that $\mathcal{L}_{r,m}$ is K-stable. Then $C_1 \geq 0$ in the proof of Proposition 2.1 for all subbundles F of E. If the inequality is strict for any subbundle then E is Mumford stable. Actually, for a rank 2 bundle over a surface, it is sufficient to test stability with respect to subbundles. For any rank 1 torsion free subsheaf $\mathcal{F}$ of E, $\mathcal{F}^{**}$ is a reflexive rank 1 sheaf on the surface B and thus a line bundle. Now, if $C_1 = 0$ for a subbundle F of E, one has necessarily $C_2 \geq 0$. If $C_2 > 0$ then $\mathcal{P}_E > \mathcal{P}_F$. Now given $\mathcal{F}$ rank 1 torsion free subsheaf of E, one has $\mathcal{F} = F \otimes \mathcal{I}$ where F is a line bundle and $\mathcal{I}$ is an ideal sheaf with 0-dimensional support, the inequality $\mathcal{P}_E > \mathcal{P}_F$ only improves if F is replaced by $\mathcal{F}$ since $c_2(\mathcal{F})$ is the length of the support of $\mathcal{I}$ and thus is non-negative. Eventually if the inequality $C_2 > 0$ holds for all subbundles of E, then we have obtained that E is Gieseker stable. Consider now that $C_2 = 0$. Then we have $\delta_L = \delta_{K_B^*} = \Delta = 0$ and by Proposition 2.1, $F_1(\mathcal{T})$ vanishes. But the test configuration is not trivial so this leads to a contradiction. Therefore one has necessarily $C_2 > 0$ and we obtain Gieseker stability. The case of K-semistability is obtained by contraposition of (2).

In the case of K-polystability, the only case for which $C_2 = 0$ is when the rank 2 bundle E splits as a direct sum of two line bundles of same slope so is necessarily Mumford polystable. Since $C_3 \geq 0$, one has moreover Gieseker semistability. □

Remark that the case of K-unstability in (3) cannot be included since the base manifold B may be K-unstable which would induce a destabilizing test configuration for the projectivisation $\mathbb{P}E$.

Non simple semi-homogeneous rank 2 vector bundles over an abelian surface are Gieseker semistable and thus provide concrete examples of applications of our theorem, see [15, Sect. 6].

Conjecture 1 *Consider E an irreducible rank 2 holomorphic vector bundle on a K-stable polarized surface (B, L) with $c_1(B)$ proportional to $c_1(L)$. For $m \gg 0$, the polarization $\mathcal{L}_{r,m}$ is K-stable (resp. K-polystable, resp. K-semistable) if and only E is Gieseker stable (resp. Gieseker polystable, resp. Gieseker semistable).*

The conjecture is wrong if one removes the assumption on the first Chern class of B: in [11] it is constructed an example of a Gieseker stable bundle with $\mathcal{L}_{1,m}$ not K-semistable for $m \gg 0$. The hard sense of the conjecture is true under stronger assumption: on a surface with a constant scalar curvature Kähler metric and no non trivial holomorphic vector field, a Mumford stable bundle gives rise to a polarization $\mathcal{L}_{r,m}$ that admits a constant scalar curvature Kähler metric and thus is K-stable, see [8–10].

One can now wonder when the Futaki invariant as computed in Proposition 2.1 may vanish. We cannot say much for a fixed couple (r, m) but at the fiber or base limit we obtain the following result.

Proposition 2.3 *Let (B, L) be a polarized surface such that its first Chern class satisfies $c_1(B) = 0$ or $c_1(B)c_1(L) \neq 0$ and E a rank 2 holomorphic vector bundle on B. Then, for the test configuration as in Proposition 2.1,*

- *the Futaki invariant $F_1(\mathcal{T})$ vanishes for all $m \gg 0$ (or all $r \gg 0$) if and only if the Chow weight* $\mathrm{Chow}_s(\mathcal{T})$ *vanishes for all $m \gg 0$ and any fixed $s > 0$ (or all $r \gg 0$ and $s \gg 0$).*
- *the Futaki invariant $F_1(\mathcal{T})$ is positive for all $m \gg 0$ if and only if the Chow weight* $\mathrm{Chow}_s(\mathcal{T})$ *is positive for all $m \gg 0$ and $s \gg 0$.*

Proof This comes from the computations of the Futaki invariant and Chow weight. Imposing $C_1 = C_2 = C_3 = 0$ in Proposition 2.1 implies firstly that $\delta_L = 0$, then $\Delta = \frac{1}{4}\delta_{K_B^*}$ and finally $\delta_{K_B^*}c_1(L)c_1(B) = 0$. Under our assumptions one gets in all the cases

$$\delta_L = \delta_{K_B^*} = \Delta = 0. \tag{2.2}$$

This forces obviously the Chow weight to vanish, see Proposition 2.2.

Conversely, if the Chow weight vanishes seen as a polynomial in the variables m, one gets from Proposition 2.2 that $\Delta = \frac{kr-2}{4kr}\delta_{K_B^*}$ and $\delta_{K_B^*}c_1(L)c_1(B) = 0$ and thus (2.2) holds which implies the vanishing of the Futaki invariant. Computations in the variables r are similar but slightly more involved. The second part of the result is using the same reasoning. □

Next we compute the Hilbert weight for the test configuration $\mathcal{T}$ for the deformation to the normal cone of $\mathbb{P}F$ where F is a subbundle of E. We remark that the Hilbert weight has a similar expression to the Chow weight and the Futaki invariant.

Proposition 2.4 *In the same setting as in Proposition 2.1 and with Notations 2, the Hilbert weight associated to the test configuration $\mathcal{T}$ is given by*

$$\begin{aligned}\mathrm{Hilb}_{s,k}(\mathcal{T}) = &\frac{r(rs-1)(rk+1)}{36}\beta_1(s,r)\delta_{K_B^*}^2 \\ &+ \frac{1}{72}\big(\beta_1(s,r)B_1 - \beta_2(s,r)A_1\big)\delta_{K_B^*}\end{aligned}$$

$$+\left(\frac{\beta_2(s,r)}{24}A_2 - \frac{(rk+2)\beta_1(s,r)}{6}\,\mathrm{Todd}_2(B)\right)(m\delta_L + r\Delta)$$

with $\beta_1(s,r) = rks(rs+1)(k-s)(rk+1)$, $\beta_2(s,r) = rs^3(rs+1)^2(k-s)$, *and*

$$\begin{aligned} B_1 = {} & kr^2\big(c_1(E)^2 + 2c_1(F)^2 + 4\Delta + 6\,\mathrm{Todd}_2(B)\big) \\ & + 6krm\deg_L(E) + 6km^2c_1(L)^2 \\ & + r\big(-c_1(E)^2 + 6\,\mathrm{Todd}_2(B) + 8\Delta + 6c_1(F)c_1(B) + 4c_1(F)^2\big) \\ & + 6mc_1(L)c_1(B) \end{aligned}$$

Proof The result is obtained by a computation of the weight $\mathrm{Hilb}_{s,k}(\mathcal{T}) = \tilde{w}(s,k)$ using (1.1) and the computations of a_i, b_i in Proposition 2.2. □

Proposition 2.3 can also be extended to Hilbert weights. We have also another obvious consequence.

Proposition 2.5 *In the same setting as in Proposition* 2.1, *let us assume that* $c_1(B) = 0$. *Then the Chow weight* $\mathrm{Chow}_s(\mathcal{T})$ *and the Hilbert weight* $\mathrm{Hilb}_{s,k}$ *are proportional to the Futaki invariant* $F_1(\mathcal{T})$, *and have same sign when one takes* $k, s > 0$ *large enough.*

Proof This comes from the fact that when $c_1(B) = 0$ one has $\delta_{K_B^*} = 0$ and both quantities Γ_2 and A_2 do not depend on the bundle F. □

3 Strictly Semistable Examples

Inspired from [2], we construct a new example of a threefold which is Asymptotically Chow semistable and not Asymptotically Chow stable.

Let (B, L) be a polarized surface such that $c_1(L)$ admits a Kähler metric with constant scalar curvature and $Aut(B, L)/\mathbb{C}^\times$ is trivial and assume that the torus $\mathrm{Pic}^0(B) = H^1(B, \mathcal{O})/H^1(B, \mathbb{Z})$ parametrizing line bundles with trivial first Chern class is not trivial. Consider $E_0 = G_1 \oplus G_2$ a direct sum of two line bundles with $c_1(G_1) = c_1(G_2)$ over B. Then E_0 is Mumford polystable. On the polarized ruled manifold

$$\big(X_0, \mathcal{L}^0_{r,m}\big) = \big(\mathbb{P}E_0, \mathcal{O}_{\mathbb{P}E_0}(r) \otimes \pi_0^* L^m\big)$$

there exists under our assumptions a Kähler metric with constant scalar curvature for all $m \gg 0$. Actually, the Futaki character associated to the Lie algebra $Lie(Aut(E_0)/\mathbb{C}^\times)$ vanishes thanks to Proposition 2.1, and one can apply [9, Corollary B]. Therefore, $(X_0, \mathcal{L}^0_{r,m})$ is K-polystable for all $m \gg 0$ from the work of Donaldson, Stoppa and Mabuchi [4, 14, 19].

Next, we do a small deformation of the trivial line bundle $T_0 = \mathbb{C} \times B$ in order to obtain a line bundle T over B such that T^2 is non trivial. We can consider the following induced extension

$$0 \to G_1 \otimes T \to E \to G_2 \otimes T^* \to 0. \tag{3.1}$$

Using Riemann–Roch formula we have $h^0(B, G_1 \otimes G_2^* \otimes T^2) - h^1(B, G_1 \otimes G_2^* \otimes T^2) + h^2(B, G_1 \otimes G_2^* \otimes T^2) = \mathrm{Todd}_2(B)$ since $c_1(G_1) = c_1(G_2)$. Now, if we assume $\mathrm{Todd}_2(B) < 0$, the space $Ext^1(G_2 \otimes T^*, G_1 \otimes T) = H^1(B, G_1 \otimes G_2^* \otimes T^2)$ has positive dimension and our extension (3.1) does not split. The ruled manifold

$$(X, \mathcal{L}_{r,m}) = \left(\mathbb{P}E, \mathcal{O}_{\mathbb{P}E}(r) \otimes \pi^* L^m\right)$$

is not K-polystable for $m \gg 0$. Actually for the choice $F = G_1 \otimes T$ one checks that the Futaki invariant $F_1(\mathcal{T})$ associated to the test configuration to the normal cone of $\mathbb{P}F$ vanishes for $m \gg 0$. Furthermore one obtains $\delta_L = \delta_{K_B^*} = \Delta = 0$. These relationships impose that the Chow weight Chow_s vanishes by Proposition 2.2. Therefore, $(X, \mathcal{L}_{r,m})$ cannot be asymptotically Chow stable.

On another hand, from the fact that all the higher Futaki invariants $F_2(\mathcal{T})$, $F_3(\mathcal{T})$ vanish simultaneously we can apply Mabuchi's main result in [12] (see also [3, Proposition 3.2, Theorem 3.5]). One concludes that $(X_0, \mathcal{L}^0_{r,m})$ is asymptotically Chow polystable. By openness of the semistability condition in G.I.T, its small deformations are asymptotically Chow semistable and consequently $(X, \mathcal{L}_{r,m})$ is asymptotically Chow semistable.

Finally, in order to construct base manifolds that satisfy the assumptions as above, it is sufficient to consider for B a ruled surface as the projectivisation of a rank 2 Mumford stable bundle over a curve of genus > 1, see [11]. We have proved the following result.

Corollary 3.1 *There are some ruled threefolds (projectivisation of rank 2 bundles over a surface endowed with a constant scalar curvature Kähler metric) that are asymptotically Chow semistable, but not asymptotically Chow stable.*

One can also compare Corollary 3.1 with [22, Sect. 5] where other examples of non asymptotically Chow stable threefolds are discussed.

Since $(X_0, \mathcal{L}^0_{r,m})$ is asymptotically Chow polystable, for the test configurations that have positive Chow weight asymptotically, the main result of [13] shows that they have also positive Hilbert weight asymptotically. Thanks to our assumptions on B, the product test configurations that have vanishing Chow weight Chow_s for $s \gg 0$ are associated to the splitting of E_0 and the deformation to the normal cone of $\mathbb{P}G_1$ or $\mathbb{P}G_2$. Thus one gets in both case for $m \gg 0$ that $\delta_L = \Delta = \delta_{K_B^*} = 0$. Proposition 2.4 shows that the Hilbert weight also vanishes. Consequently, $(X_0, \mathcal{L}^0_{r,m})$ is asymptotically Hilbert polystable and thus its small deformation $(X, \mathcal{L}_{r,m})$ is also asymptotically Hilbert semistable. On another hand, considering the subbundle $F = G_1 \otimes T$ of E, one has for the test configuration associated to the deformation to the normal cone of $\mathbb{P}F$ that $\delta_L = \delta_{K_B^*} = \Delta = 0$ and so $\mathrm{Hilb}_{s,k} = 0$ for all s, k. Finally, $(X, \mathcal{L}_{r,m})$ for $m \gg 0$ cannot be asymptotically Hilbert stable since $\mathcal{T}$ is not a product test configuration.

Corollary 3.2 *There are some ruled threefolds (projectivisation of rank 2 bundles over a surface endowed with a constant scalar curvature Kähler metric) that are asymptotically Hilbert semistable, but not asymptotically Hilbert stable.*

Note that using [3, Proposition 4.1 and Corollary 4.4] our reasoning could also be applied to the case of Mumford semistable vector bundle over a curve of genus ≥ 2 to produce other similar examples to Corollaries 3.1 and 3.2. This will be discussed in more details in a forthcoming paper since one can be a little bit more precise in dimension one. For instance the following conjecture is true if the base manifold is a curve of genus $g > 1$.

Conjecture 2 *Consider E a holomorphic vector bundle on a base manifold B polarized by L with $c_1(B) = 0$ or $c_1(B)c_1(L) \neq 0$. Then for $m \gg 0$, the following assertions are equivalent:*

- *the polarization $\mathcal{L}_{r,m}$ on $\mathbb{P}E$ is asymptotically Hilbert semistable,*
- *the polarization $\mathcal{L}_{r,m}$ on $\mathbb{P}E$ is asymptotically Chow semistable,*
- *the polarization $\mathcal{L}_{r,m}$ on $\mathbb{P}E$ is K-semistable.*

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Atiyah Classes of Lie Algebroids

Francesco Bottacin

Abstract Given a smooth morphism of analytic spaces $\pi : X \to Y$, we introduce the notion of a relative Lie algebroid $(\mathcal{A}, \sharp)$ over X. By replacing the relative tangent sheaf $\mathcal{T}_{X/Y}$ with the Lie algebroid $\mathcal{A}$, we define the notion of a relative $(\mathcal{A}, \sharp)$-connection on a quasi-coherent $\mathcal{O}_X$-module $\mathcal{E}$. Then, we define the $(\mathcal{A}, \sharp)$-Atiyah class of $\mathcal{E}$ as the obstruction to the existence of a holomorphic $(\mathcal{A}, \sharp)$-connection on $\mathcal{E}$. Many results of the classical theory of connections can be restated in the more general setting of Lie algebroid connections. As an application we prove the following result.

Let X be a complex manifold and $(A, \sharp)$ a Lie algebroid over X. For any quasi-coherent sheaf of commutative $\mathcal{O}_X$-algebras $\mathcal{F}$, let us write $\mathfrak{g}_i = H^{i-1}(X, A \otimes \mathcal{F})$. The $(A, \sharp)$-Atiyah class of A yields maps $\mathfrak{g}_i \otimes \mathfrak{g}_j \to \mathfrak{g}_{i+j}$. These maps define a graded Lie algebra structure on the graded vector space $\mathfrak{g}^\bullet = \bigoplus_i \mathfrak{g}_i$. In a similar way, for any holomorphic vector bundle E over X, let us write $V_j = H^{j-1}(X, E \otimes \mathcal{F})$. Then, for any i and j, the $(A, \sharp)$-Atiyah class of E yields a map $\mathfrak{g}_i \otimes V_j \to V_{i+j}$, and these maps define a structure of graded module on the graded vector space $V^\bullet = \bigoplus_j V_j$, over the graded Lie algebra $\mathfrak{g}^\bullet$. This generalizes a similar result proved by Kapranov in Compos. Math. 115:71–113, 1999. Similar results have been obtained by Chen, Stiénon and Xu in From Atiyah classes to homotopy Leibniz algebras 2012, by using different techniques.

Keywords Atiyah classes · Connections · Differential forms · Jets · Lie algebroids

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1 Introduction

The theory of connections is a central topic in differential geometry. A rather natural generalization of the classical notion of connection on a vector or principal bundle over a differentiable manifold X is obtained by replacing the tangent bundle of X with a Lie algebroid $(A, \sharp)$ over X; this leads to the notion of a *Lie algebroid connection*.

Most of the results of the classical theory of connections (e.g., the Chern–Weil theory of characteristic classes) extend to Lie algebroid connections. We refer to [10] for an introduction to Lie algebroids and to [5] for a detailed account on Lie algebroid connections.

While Lie algebroid connections on a smooth vector bundle over a differentiable manifold X always exist (this is a consequence of the existence of partitions of unity on X), when X is a complex manifold there is an obstruction to the existence of a global holomorphic Lie algebroid connection on a holomorphic vector bundle E over X. This obstruction is given by a cohomology class that is the analogue of the Atiyah class of E; we call it the $(A, \sharp)$-Atiyah class of E. As a special case, if we take $E = A$, we may look at the $(A, \sharp)$-Atiyah class of A itself.

As happens for their classical counterparts, the new Atiyah classes arising from Lie algebroid connections present very interesting features.

In the classical case, i.e., when the Lie algebroid $(A, \sharp)$ is the tangent bundle of a complex manifold X, M. Kapranov [8] (inspired by ideas of M. Kontsevich) discovered the fundamental role played by the Atiyah class of T_X in the construction of the topological invariants of 3-dimensional manifolds, previously introduced by L. Rozansky and E. Witten. One of the main results contained in Kapranov's paper may be restated as follows. Let $T_X[-1]$ denotes the shifted tangent sheaf of X, considered as an object in the derived category $D^+(X)$ of bounded below complexes of sheaves of $\mathcal{O}_X$-modules with coherent cohomology. Then the Atiyah class of the tangent bundle of X determines a map $T_X[-1] \otimes T_X[-1] \to T_X[-1]$, which makes $T_X[-1]$ into a Lie algebra object in $D^+(X)$.

As an application of the general theory of Lie algebroid connections, we prove that similar results hold if we replace the tangent bundle of a complex manifold X with a Lie algebroid A over X. In this case the role of the Atiyah class of T_X is played by the $(A, \sharp)$-Atiyah class of A.

More precisely, we prove that, given a Lie algebroid $(A, \sharp)$ over X and a quasi-coherent sheaf of commutative $\mathcal{O}_X$-algebras $\mathcal{F}$, there is a map

$$H^i(X, A \otimes \mathcal{F}) \otimes H^j(X, A \otimes \mathcal{F}) \to H^{i+j+1}(X, A \otimes \mathcal{F})$$

obtained by composing the cup-product of two cohomology classes with the $(A, \sharp)$-Atiyah class of A. If we set $\mathfrak{g}_i = H^{i-1}(X, A \otimes \mathcal{F})$, the collection of maps $\mathfrak{g}_i \otimes \mathfrak{g}_j \to \mathfrak{g}_{i+j}$ defines a graded Lie algebra structure on the graded vector space $\mathfrak{g}^\bullet = \bigoplus_i \mathfrak{g}_i$.

In a similar way, for any holomorphic vector bundle E over X, we can define a map

$$H^i(X, A \otimes \mathcal{F}) \otimes H^j(X, E \otimes \mathcal{F}) \to H^{i+j+1}(X, E \otimes \mathcal{F}),$$

by using the $(A, \sharp)$-Atiyah class of E. If we write $V_j = H^{j-1}(X, E \otimes \mathcal{F})$, we get a collection of maps $\mathfrak{g}_i \otimes V_j \to V_{i+j}$, for any i and j, defining a structure of graded module on the graded vector space $V^\bullet = \bigoplus_j V_j$, over the graded Lie algebra $\mathfrak{g}^\bullet$.

We remark that, in a recent paper, Z. Chen, M. Stiénon and P. Xu [4] developed a general theory of Atiyah classes relative to pairs consisting of a Lie algebroid A over X and a Lie subalgebroid of A, over the same base manifold. They also proved a generalization of Kapranov's results by using different techniques.

This paper is organized as follows. In Sect. 1 we develop the basic theory of holomorphic Lie algebroids and Lie algebroid connections in a relative setting. More precisely, we introduce the notion of a relative Lie algebroid over X, where $\pi: X \to Y$ is a smooth morphism of analytic spaces. Then we define relative $(\mathcal{A}, \sharp)$-connections on a quasi-coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{E}$ and study their basic properties.

In Sect. 2 we introduce the sheaf of first $(\mathcal{A}, \sharp)$-jets of $\mathcal{E}$ and define the $(\mathcal{A}, \sharp)$-Atiyah class of $\mathcal{E}$ as the obstruction to the existence of a global holomorphic $(\mathcal{A}, \sharp)$-connection on $\mathcal{E}$. We also prove that the $(\mathcal{A}, \sharp)$-Atiyah class of $\mathcal{A}$ is symmetric.

In Sects. 3 and 4 we define the sheaves of higher $(\mathcal{A}, \sharp)$-jets and the sheaf of $(\mathcal{A}, \sharp)$-differential operators. Then, in Sect. 5, we prove a version of the so-called 'cohomological Bianchi identity,' originally proved in [8] for the usual Atiyah class of a vector bundle.

Finally, in the last section, we show how Kapranov's results can be generalized to the framework of Lie algebroid connections. The proofs are obtained by following Kapranov's original argument, with suitable modifications. Note that the basic tool needed for proving that the composition with the $(A, \sharp)$-Atiyah class of A defines a graded Lie algebra structure on the graded vector space $\mathfrak{g}^\bullet = \bigoplus_i H^{i-1}(X, A \otimes \mathcal{F})$ is precisely the cohomological Bianchi identity, which implies the graded Jacobi identity for the graded Lie bracket.

2 Preliminaries

2.1 $(\mathcal{A}, \sharp)$-*Connections*

Let $\pi: X \to Y$ be a smooth morphism of analytic spaces (or a smooth morphism of schemes, defined over a field of characteristic 0). We denote by $\mathcal{T}_{X/Y} = \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_{X/Y}, \mathcal{O}_X)$ the relative tangent sheaf (which is locally free, since π is smooth).

Definition 2.1 A *relative Lie algebroid* over X is a locally free sheaf of $\mathcal{O}_X$-modules $\mathcal{A}$, with a $\pi^{-1}\mathcal{O}_Y$-linear morphism $[\cdot,\cdot]\colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ which defines a Lie algebra structure on the spaces of sections, together with a homomorphism of $\mathcal{O}_X$-modules $\sharp\colon \mathcal{A} \to \mathcal{T}_{X/Y}$, called the *anchor map*, such that the induced map on the spaces of sections $\sharp\colon \Gamma(\mathcal{A}) \to \Gamma(\mathcal{T}_{X/Y})$ is a homomorphism of Lie algebras, and for any sections $a_1, a_2 \in \Gamma(\mathcal{A})$ and $f \in \Gamma(\mathcal{O}_X)$, the following Leibniz identity holds:

$$[a_1, f a_2] = f[a_1, a_2] + \sharp a_1(f) a_2. \tag{2.1}$$

Remark 2.2 Let us denote by X_y the fiber of $\pi : X \to Y$ over a point $y \in Y$. If $(\mathcal{A}, \sharp)$ is a relative Lie algebroid over X we shall denote by $\mathcal{A}_y$ the restriction of $\mathcal{A}$ to X_y and by $\sharp_y : \mathcal{A}_y \to \mathcal{T}_{X_y}$ the map induced by $\sharp$. The previous definition implies that, for any $y \in Y$, $(\mathcal{A}_y, \sharp_y)$ is a Lie algebroid over X_y. Thus a relative Lie algebroid over X may be thought as a family of Lie algebroids over the fibers X_y, parametrized by the points $y \in Y$.

Let $\flat : \Omega^1_{X/Y} \to \mathcal{A}^* = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{A}, \mathcal{O}_X)$ be the dual of the anchor map, and let $d_{\mathcal{A}} : \mathcal{O}_X \to \mathcal{A}^*$ be the $\pi^{-1}\mathcal{O}_Y$-derivation defined by $d_{\mathcal{A}} = \flat \circ d_{X/Y}$

$$\begin{array}{ccc} \mathcal{O}_X & \xrightarrow{d_{X/Y}} & \Omega^1_{X/Y} \\ & \searrow^{d_{\mathcal{A}}} & \downarrow \flat \\ & & \mathcal{A}^* \end{array}$$

where $d_{X/Y} : \mathcal{O}_X \to \Omega^1_{X/Y}$ is the usual relative differential.

Let now $\mathcal{E}$ be a quasi-coherent $\mathcal{O}_X$-module.

Definition 2.3 A *relative* $(\mathcal{A}, \sharp)$*-connection* on $\mathcal{E}$ is a $\pi^{-1}\mathcal{O}_Y$-linear morphism

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes \mathcal{A}^*$$

such that

$$\nabla(fs) = f\nabla(s) + s \otimes d_{\mathcal{A}}(f),$$

for any local sections s of $\mathcal{E}$ and f of $\mathcal{O}_X$.

For any section $a \in \Gamma(\mathcal{A})$, we define

$$\nabla_a : \mathcal{E} \to \mathcal{E}$$

by setting $\nabla_a(s) = \langle \nabla s, a \rangle$. The map ∇_a is $\pi^{-1}\mathcal{O}_Y$-linear and satisfies the following identity:

$$\nabla_a(fs) = f\nabla_a(s) + \sharp a(f)s.$$

We also have

$$\nabla_{f_1 a_1 + f_2 a_2} = f_1 \nabla_{a_1} + f_2 \nabla_{a_2}.$$

Remark 2.4 If ∇ and ∇' are two relative $(\mathcal{A}, \sharp)$-connections on $\mathcal{E}$, their difference $\nabla' - \nabla$ is $\mathcal{O}_X$-linear, hence $\nabla' - \nabla \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E} \otimes \mathcal{A}^*)$. It follows that the space $\mathsf{Conn}_{(\mathcal{A},\sharp)}(\mathcal{E})$ of relative $(\mathcal{A}, \sharp)$-connections on $\mathcal{E}$ is an affine space modeled on the vector space $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E} \otimes \mathcal{A}^*)$.

2.2 Extension of a Relative $(\mathcal{A}, \sharp)$-Connection

We can extend the $\pi^{-1}\mathcal{O}_Y$-derivation $d_{\mathcal{A}}: \mathcal{O}_X \to \mathcal{A}^*$ to an operator

$$d_{\mathcal{A}}: \ \wedge^p \mathcal{A}^* \to \wedge^{p+1}\mathcal{A}^*$$

by setting, for any section α of $\wedge^p\mathcal{A}^*$,

$$\begin{aligned}(d_{\mathcal{A}}\alpha)(a_1, \dots, a_{p+1}) = &\sum_{i=1}^{p+1}(-1)^{i+1}\sharp a_i\alpha(a_1, \dots, \hat{a}_i, \dots, a_{p+1}) \\ &+ \sum_{i<j}(-1)^{i+j}\alpha\big([a_i, a_j], a_1, \dots, \hat{a}_i, \dots, \hat{a}_j, \dots, a_{p+1}\big)\end{aligned}$$

where $[\cdot, \cdot]: \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is the Lie bracket of the relative Lie algebroid $\mathcal{A}$ (the Leibniz identity (2.1) implies that $d_{\mathcal{A}}(\alpha)$ is actually a section of $\wedge^{p+1}\mathcal{A}^*$).

The fact that $\sharp: \mathcal{A} \to \mathcal{T}_{X/Y}$ induces a homomorphism of Lie algebras, together with the Jacobi identity for the Lie bracket on $\mathcal{A}$, imply that $d_{\mathcal{A}} \circ d_{\mathcal{A}} = 0$, hence we have a complex

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{d_{\mathcal{A}}} \mathcal{A}^* \xrightarrow{d_{\mathcal{A}}} \wedge^2\mathcal{A}^* \xrightarrow{d_{\mathcal{A}}} \cdots \tag{2.2}$$

called the $(\mathcal{A}, \sharp)$-de Rham complex.

Let now $\nabla: \mathcal{E} \to \mathcal{E} \otimes \mathcal{A}^*$ be a $(\mathcal{A}, \sharp)$-connection on $\mathcal{E}$. As in the classical case, we shall extend ∇ to an operator

$$\nabla: \mathcal{E} \otimes \wedge^p\mathcal{A}^* \to \mathcal{E} \otimes \wedge^{p+1}\mathcal{A}^*$$

by requiring that

$$\nabla(s \otimes \alpha) = (\nabla s) \wedge \alpha + s \otimes d_{\mathcal{A}}(\alpha),$$

for any sections s of $\mathcal{E}$ and α of $\wedge^p\mathcal{A}^*$.

Then we can define the $(\mathcal{A}, \sharp)$-curvature of ∇ by setting

$$R = \nabla \circ \nabla: \mathcal{E} \to \mathcal{E} \otimes \wedge^2\mathcal{A}^*.$$

It is immediate to check that R is $\mathcal{O}_X$-linear, hence it is a section of $\mathcal{E}nd(\mathcal{E}) \otimes \wedge^2\mathcal{A}^*$. A $(\mathcal{A}, \sharp)$-connection is called *flat* if its $(\mathcal{A}, \sharp)$-curvature vanishes. The $(\mathcal{A}, \sharp)$-curvature R satisfies an analogue of the classical Bianchi identity.

3 $(\mathcal{A}, \sharp)$-Jets and Atiyah Classes

For a quasi-coherent $\mathcal{O}_X$-module $\mathcal{E}$ let us consider the standard 1-jet exact sequence (also called Atiyah sequence)

$$0 \longrightarrow \mathcal{E} \otimes \Omega^1_{X/Y} \longrightarrow J^1_{X/Y}(\mathcal{E}) \longrightarrow \mathcal{E} \longrightarrow 0 \tag{3.1}$$

(which is split as a sequence of $\pi^{-1}\mathcal{O}_Y$-modules but not, in general, as a sequence of $\mathcal{O}_X$-modules).

We can define the sheaf of first $(\mathcal{A}, \sharp)$-jets of $\mathcal{E}$ by pushing forward the previous exact sequence via the map $\mathrm{id}_{\mathcal{E}} \otimes \flat \colon \mathcal{E} \otimes \Omega^1_{X/Y} \to \mathcal{E} \otimes \mathcal{A}^*$. Hence, by definition, we have a commutative diagram (morphism of extensions)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{E} \otimes \Omega^1_{X/Y} & \longrightarrow & J^1_{X/Y}(\mathcal{E}) & \longrightarrow & \mathcal{E} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathcal{E} \otimes \mathcal{A}^* & \longrightarrow & J^1_{(\mathcal{A},\sharp)}(\mathcal{E}) & \longrightarrow & \mathcal{E} & \longrightarrow & 0 \end{array} \tag{3.2}$$

Note that the exact sequence

$$0 \longrightarrow \mathcal{E} \otimes \mathcal{A}^* \longrightarrow J^1_{(\mathcal{A},\sharp)}(\mathcal{E}) \longrightarrow \mathcal{E} \longrightarrow 0 \tag{3.3}$$

is split as a sequence of $\pi^{-1}\mathcal{O}_Y$-modules but not, in general, as a sequence of $\mathcal{O}_X$-modules.

Remark 3.1 As sheaves of $\pi^{-1}\mathcal{O}_Y$-modules, we have

$$J^1_{(\mathcal{A},\sharp)}(\mathcal{E}) = \mathcal{E} \oplus \big(\mathcal{E} \otimes \mathcal{A}^*\big).$$

Note that $J^1_{(\mathcal{A},\sharp)}(\mathcal{E})$ has two structures of $\mathcal{O}_X$-module: one is given by

$$f \cdot (s, \sigma) = (fs, f\sigma),$$

for sections $f \in \Gamma(\mathcal{O}_X)$, $s \in \Gamma(\mathcal{E})$ and $\sigma \in \Gamma(\mathcal{E} \otimes \mathcal{A}^*)$; we shall call this the *left* $\mathcal{O}_X$-module structure.

The other one is defined by setting

$$(s, \sigma) \cdot f = (fs, f\sigma + s \otimes d_{\mathcal{A}} f),$$

and is called the *right* $\mathcal{O}_X$-module structure.

Unless otherwise stated, we shall always consider $J^1_{(\mathcal{A},\sharp)}(\mathcal{E})$ as an $\mathcal{O}_X$-module with its right module structure.

It is well known that the data of a relative connection on $\mathcal{E}$ is equivalent to a splitting of the exact sequence (3.1). A similar result holds for relative $(\mathcal{A}, \sharp)$-connections:

Lemma 3.2 *A splitting of the sequence* (3.3) *is equivalent to a relative* $(\mathcal{A},\sharp)$-*connection on* $\mathcal{E}$.

Proof As sheaves of $\pi^{-1}\mathcal{O}_Y$-modules, we have

$$J^1_{(\mathcal{A},\sharp)}(\mathcal{E}) = \mathcal{E} \oplus \left(\mathcal{E} \otimes \mathcal{A}^*\right),$$

hence a splitting of (3.3) is given by a homomorphism of $\mathcal{O}_X$-modules

$$\phi\colon \mathcal{E} \to J^1_{(\mathcal{A},\sharp)}(\mathcal{E}), \quad s \mapsto \phi(s) = \left(s, \nabla(s)\right),$$

for some map $\nabla\colon \mathcal{E} \to \mathcal{E} \otimes \mathcal{A}^*$. Since ϕ is $\mathcal{O}_X$-linear, we have $\phi(fs) = \phi(s)f$, for sections $f \in \Gamma(\mathcal{O}_X)$ and $s \in \Gamma(\mathcal{E})$. But $\phi(fs) = (fs, \nabla(fs))$ and $\phi(s)f = (s, \nabla(s))f = (fs, f\nabla(s) + s \otimes d_{\mathcal{A}}(f))$, hence the map ∇ must satisfy the identity

$$\nabla(fs) = f\nabla(s) + s \otimes d_{\mathcal{A}}(f).$$

So, requiring that ϕ be a homomorphism of $\mathcal{O}_X$-modules is equivalent to requiring that ∇ be a $(\mathcal{A},\sharp)$-connection on $\mathcal{E}$. □

Definition 3.3 The $(\mathcal{A},\sharp)$-Atiyah class of $\mathcal{E}$ is the class

$$a_{(\mathcal{A},\sharp)}(\mathcal{E}) \in \operatorname{Ext}^1\left(\mathcal{E}, \mathcal{E} \otimes \mathcal{A}^*\right)$$

corresponding to the extension (3.3).

From Lemma 3.2 we obtain the following result:

Corollary 3.4 *A relative* $(\mathcal{A},\sharp)$-*connection on* $\mathcal{E}$ *exists if and only if the* $(\mathcal{A},\sharp)$-*Atiyah class of* $\mathcal{E}$ *vanishes.*

Remark 3.5 The construction of the relative Atiyah class, and its relation with the usual notion of a relative connection, can be found in [7], where it is discussed in great generality.

Let us now compare the $(\mathcal{A},\sharp)$-Atiyah class of $\mathcal{E}$ with its usual Atiyah class. The usual Atiyah class of $\mathcal{E}$ is the class $a(\mathcal{E}) \in \operatorname{Ext}^1(\mathcal{E}, \mathcal{E} \otimes \Omega^1_{X/Y})$ corresponding to the extension (3.1). The morphism of extensions (3.2) induces a morphism

$$\operatorname{Ext}^1\left(\mathcal{E}, \mathcal{E} \otimes \Omega^1_{X/Y}\right) \to \operatorname{Ext}^1\left(\mathcal{E}, \mathcal{E} \otimes A^*\right).$$

It is now immediate to verify that the $(\mathcal{A},\sharp)$-Atiyah class of $\mathcal{E}$ is the image of the usual Atiyah class $a(\mathcal{E})$ under the previous map.

Remark 3.6 Exactly as the usual Atiyah class can be used to define the Chern classes of $\mathcal{E}$, we could use the $(\mathcal{A},\sharp)$-Atiyah class to define what we may call $(\mathcal{A},\sharp)$-Chern classes.

If we consider the morphism $\mathrm{Ext}^1(\mathcal{E}, \mathcal{E} \otimes \Omega^1_{X/Y}) \to \mathrm{Ext}^1(\mathcal{E}, \mathcal{E} \otimes \mathcal{A}^*)$, induced by the map $\flat \colon \Omega^1_{X/Y} \to \mathcal{A}^*$, and we apply the trace maps, we obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}^1(\mathcal{E}, \mathcal{E} \otimes \Omega^1_{X/Y}) & \xrightarrow{\mathrm{id}_{\mathcal{E}} \otimes \flat} & \mathrm{Ext}^1(\mathcal{E}, \mathcal{E} \otimes \mathcal{A}^*) \\ \downarrow{\scriptstyle \mathrm{tr}} & & \downarrow{\scriptstyle \mathrm{tr}} \\ H^1(X, \Omega^1_{X/Y}) & \xrightarrow[\flat]{} & H^1(X, \mathcal{A}^*) \end{array}$$

Since the first Chern class of $\mathcal{E}$ is given by $c_1(\mathcal{E}) = \mathrm{tr}(a(\mathcal{E}))$, we find that $\flat(c_1(\mathcal{E})) = \mathrm{tr}(a_{(\mathcal{A},\sharp)}(\mathcal{E}))$ (and a similar statement holds for all higher Chern classes). It follows that the $(\mathcal{A}, \sharp)$-Chern classes that we could define using a $(\mathcal{A}, \sharp)$-connection on $\mathcal{E}$ are not particularly interesting because they are the image of the usual Chern classes of $\mathcal{E}$ under the maps $H^i(X, \Omega^i_{X/Y}) \to H^i(X, \wedge^i \mathcal{A}^*)$ induced by the morphism $\flat \colon \Omega^1_{X/Y} \to \mathcal{A}^*$.

Remark 3.7 As we mentioned in the introduction, in [4] the authors introduced a notion of Atiyah class of a vector bundle E over a smooth manifold M, relative to a pair consisting of a Lie algebroid L over M and a Lie subalgebroid $A \subset L$. More precisely, given a pair (L, A) as above, we say that E is a vector bundle over (L, A) if E is a vector bundle over M endowed with an A-module structure. Under these assumptions, Chen, Stiénon and Xu defined the notion of a Lie algebroid connection on E, relative to the Lie algebroid pair (L, A). When $A = 0$ this definition coincides with the definition of a Lie algebroid connection we gave in Sect. 2.1.

The construction of the Atiyah class of E, relative to the pair (L, A), is carried out as follows. Let ∇ be any smooth L-connection on E that extends the action of A on E (E is assumed to be an A-module, as before). The curvature of such a connection defines a section $R \in \Gamma(A^* \otimes A^\perp \otimes \mathcal{E}nd(E))$. This is a 1-cocycle for the Lie algebroid A with values in the A-module $A^\perp \otimes \mathcal{E}nd(E)$. The cohomology class of this cocycle in $H^1(A; A^\perp \otimes \mathcal{E}nd(E))$ is the Atiyah class of the A-module E. When $A = 0$ the Atiyah class of E defined in this way coincides with the one we have introduced in Sect. 3, computed by using the Dolbeault model for cohomology.

3.1 $(\mathcal{A}, \sharp)$-Connections on $\mathcal{A}$

Let us consider now the special case $\mathcal{E} = \mathcal{A}$. Let $\nabla \colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}^*$ be a $(\mathcal{A}, \sharp)$-connection on $\mathcal{A}$.

For any section $a \in \Gamma(\mathcal{A})$ we define the derivation $\nabla_a \colon \mathcal{A} \to \mathcal{A}$ by setting

$$\nabla_a(b) = \langle \nabla b, a \rangle.$$

Then we define the $(\mathcal{A}, \sharp)$-torsion of ∇ by setting

$$T(a, b) = \nabla_a(b) - \nabla_b(a) - [a, b],$$

for sections a, b of $\mathcal{A}$. It is easy to see that $T \in \mathrm{Hom}_{\mathcal{O}_X}(\wedge^2\mathcal{A}, \mathcal{A})$. A $(\mathcal{A}, \sharp)$-connection on $\mathcal{A}$ is said to be *torsion-free* if its $(\mathcal{A}, \sharp)$-torsion vanishes.

The following result, proved in [8], carries over into this more general setting (with a similar proof).

Theorem 3.8 *Let* $(\mathcal{A}, \sharp)$ *be a relative Lie algebroid over* X *and let*

$$a_{(\mathcal{A},\sharp)}(\mathcal{A}) \in \mathrm{Ext}^1\big(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^*\big) = \mathrm{Ext}^1(\mathcal{A} \otimes \mathcal{A}, \mathcal{A})$$

be its $(\mathcal{A}, \sharp)$*-Atiyah class. Then* $a_{(\mathcal{A},\sharp)}(\mathcal{A})$ *is symmetric, i.e., it belongs to* $\mathrm{Ext}^1(\mathsf{S}^2\mathcal{A}, \mathcal{A})$.

Proof Let $\mathsf{Conn}_{(\mathcal{A},\sharp)}(\mathcal{A})$ be the sheaf whose sections over $U \subset X$ are the holomorphic $(\mathcal{A}, \sharp)$-connections defined on $\mathcal{A}|_U$. As seen in Remark 2.4, this is an affine space over $\Gamma(U, \mathcal{E}nd(\mathcal{A}) \otimes \mathcal{A}^*)$. Then $\mathsf{Conn}_{(\mathcal{A},\sharp)}(\mathcal{A})$ is a sheaf of torsors over $\mathcal{E}nd(\mathcal{A}) \otimes \mathcal{A}^*$. Sheaves of torsors over $\mathcal{E}nd(\mathcal{A}) \otimes \mathcal{A}^*$ are classified by elements of $H^1(X, \mathcal{E}nd(\mathcal{A}) \otimes \mathcal{A}^*) = \mathrm{Ext}^1(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^*)$, and $a_{(\mathcal{A},\sharp)}(\mathcal{A})$ is precisely the element that classifies $\mathsf{Conn}_{(\mathcal{A},\sharp)}(\mathcal{A})$.

Similarly, let $\mathsf{Conn}^{\mathrm{tf}}_{(\mathcal{A},\sharp)}(\mathcal{A})$ be the sheaf whose sections over $U \subset X$ are the torsion free $(\mathcal{A}, \sharp)$-connections on $\mathcal{A}|_U$. Then $\mathsf{Conn}^{\mathrm{tf}}_{(\mathcal{A},\sharp)}(\mathcal{A})$ is a sheaf of torsors over $\mathsf{S}^2(\mathcal{A}^*) \otimes \mathcal{A}$. Since the sheaf of torsors $\mathsf{Conn}_{(\mathcal{A},\sharp)}(\mathcal{A})$ is obtained from $\mathsf{Conn}^{\mathrm{tf}}_{(\mathcal{A},\sharp)}(\mathcal{A})$ by "change of scalars" (i.e., from $\mathsf{S}^2(\mathcal{A}^*) \otimes \mathcal{A}$ to $\mathcal{A}^* \otimes \mathcal{A}^* \otimes \mathcal{A}$), it follows that the classifying element $a_{(\mathcal{A},\sharp)}(\mathcal{A}) \in H^1(X, \mathcal{A}^* \otimes \mathcal{A}^* \otimes \mathcal{A})$ actually belongs to the summand $H^1(X, \mathsf{S}^2(\mathcal{A}^*) \otimes \mathcal{A}) = \mathrm{Ext}^1(\mathsf{S}^2\mathcal{A}, \mathcal{A})$. □

4 Higher $(\mathcal{A}, \sharp)$-Jets

In this section we shall briefly describe how it is possible to define sheaves of higher order $(\mathcal{A}, \sharp)$-jets.

For a quasi-coherent $\mathcal{O}_X$-module $\mathcal{E}$ we have already seen that $J^1_{(\mathcal{A},\sharp)}(\mathcal{E}) = \mathcal{E} \oplus (\mathcal{E} \otimes \mathcal{A}^*)$, with the structure of $\mathcal{O}_X$-module (on the right) given by

$$(s, \sigma) \cdot f = (fs, f\sigma + s \otimes d_{\mathcal{A}} f).$$

We can now define the sheaf of 2nd $(\mathcal{A}, \sharp)$-jets of $\mathcal{E}$ by setting

$$J^2_{(\mathcal{A},\sharp)}(\mathcal{E}) = J^1_{(\mathcal{A},\sharp)}(\mathcal{E}) \oplus \big(\mathcal{E} \otimes \mathsf{S}^2\mathcal{A}^*\big) = \mathcal{E} \oplus \big(\mathcal{E} \otimes \mathcal{A}^*\big) \oplus \big(\mathcal{E} \otimes \mathsf{S}^2\mathcal{A}^*\big),$$

as $\pi^{-1}\mathcal{O}_Y$-modules, where $\mathsf{S}^2\mathcal{A}^*$ denotes the symmetric square of $\mathcal{A}^*$.

Let $d^{(2)}_{X/Y}\colon \mathcal{O}_X \to \mathsf{S}^2\Omega^1_{X/Y}$ be the quadratic differential expressed, in suitable local coordinates z_i, by

$$d^{(2)}_{X/Y} f = \frac{1}{2!}\sum_{i,j} \frac{\partial^2 f}{\partial z_i \partial z_j} dz_i \odot dz_j,$$

where $\odot$ denotes the symmetric product.

Let us define the quadratic derivation $d^{(2)}_{\mathcal{A}}\colon \mathcal{O}_X \to \mathsf{S}^2\mathcal{A}^*$ as the composition $d^{(2)}_{\mathcal{A}} = (\flat \odot \flat) \circ d^{(2)}_{X/Y}$

$$\begin{array}{ccc} \mathcal{O}_X & \xrightarrow{d^{(2)}_{X/Y}} & \mathsf{S}^2\Omega^1_{X/Y} \\ & {\scriptstyle d^{(2)}_{\mathcal{A}}}\searrow & \downarrow {\scriptstyle \flat\odot\flat} \\ & & \mathsf{S}^2\mathcal{A}^* \end{array}$$

The structure of (right) $\mathcal{O}_X$-module on $J^2_{(\mathcal{A},\sharp)}(\mathcal{E})$ is defined by setting

$$(s, \sigma, \tau) \cdot f = \big(fs,\, f\sigma + s \otimes d_{\mathcal{A}} f,\, f\tau + \sigma \otimes d_{\mathcal{A}} f + s \otimes d^{(2)}_{\mathcal{A}} f\big),$$

for sections $f \in \Gamma(\mathcal{O}_X)$, $s \in \Gamma(\mathcal{E})$, $\sigma \in \Gamma(\mathcal{E} \otimes \mathcal{A}^*)$ and $\tau \in \Gamma(\mathcal{E} \otimes \mathsf{S}^2\mathcal{A}^*)$ (here, by $\sigma \otimes d_{\mathcal{A}} f$ we mean the image of $\sigma \otimes d_{\mathcal{A}} f \in \mathcal{E} \otimes \mathcal{A}^* \otimes \mathcal{A}^*$ in $\mathcal{E} \otimes \mathsf{S}^2\mathcal{A}^*$ under the symmetrization map $\mathcal{E} \otimes \mathcal{A}^* \otimes \mathcal{A}^* \to \mathcal{E} \otimes \mathsf{S}^2\mathcal{A}^*$).

There is an exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{E} \otimes \mathsf{S}^2\mathcal{A}^* \to J^2_{(\mathcal{A},\sharp)}(\mathcal{E}) \to J^1_{(\mathcal{A},\sharp)}(\mathcal{E}) \to 0 \tag{4.1}$$

(which is split as a sequence of $\pi^{-1}\mathcal{O}_Y$-modules but not, in general, as a sequence of $\mathcal{O}_X$-modules).

More generally, for any $r \geq 1$ we can define inductively the sheaf of r-th $(\mathcal{A}, \sharp)$-jets of $\mathcal{E}$ by setting

$$J^r_{(\mathcal{A},\sharp)}(\mathcal{E}) = J^{r-1}_{(\mathcal{A},\sharp)}(\mathcal{E}) \oplus \big(\mathcal{E} \otimes \mathsf{S}^r\mathcal{A}^*\big) = \bigoplus_{i=0}^{r} \big(\mathcal{E} \otimes \mathsf{S}^i\mathcal{A}^*\big),$$

as $\pi^{-1}\mathcal{O}_Y$-modules, where $\mathsf{S}^i\mathcal{A}^*$ denotes the i-th symmetric power of $\mathcal{A}^*$. The (right) $\mathcal{O}_X$-module structure of $J^r_{(\mathcal{A},\sharp)}(\mathcal{E})$ is defined as follows. For any $j \geq 0$, let

$d_{\mathcal{A}}^{(j)} : \mathcal{O}_X \to \mathsf{S}^j \mathcal{A}^*$ be the composition $d_{\mathcal{A}}^{(j)} = (\mathsf{S}^j \flat) \circ d_{X/Y}^{(j)}$

$$\begin{array}{ccc} \mathcal{O}_X & \xrightarrow{d_{X/Y}^{(j)}} & \mathsf{S}^j \Omega^1_{X/Y} \\ & {\scriptstyle d_{\mathcal{A}}^{(j)}} \searrow & \downarrow {\scriptstyle \mathsf{S}^j \flat} \\ & & \mathsf{S}^j \mathcal{A}^* \end{array}$$

where $d_{X/Y}^{(j)} : \mathcal{O}_X \to \mathsf{S}^j \Omega^1_{X/Y}$ is given locally by

$$d_{X/Y}^{(j)} f = \frac{1}{j!} \sum_{i_1,\dots,i_j} \frac{\partial^j f}{\partial z_{i_1} \cdots \partial z_{i_j}} dz_{i_1} \odot \cdots \odot dz_{i_j}.$$

Let $(s_0, s_1, \dots, s_r)$ be a section of $J^r_{(\mathcal{A},\sharp)}(\mathcal{E})$, with $s_i \in \Gamma(\mathcal{E} \otimes \mathsf{S}^i \mathcal{A}^*)$. Then, for any $f \in \Gamma(\mathcal{O}_X)$, we set $(s_0, s_1, \dots, s_r) \cdot f = (t_0, t_1, \dots, t_r)$ where, for each $h = 0, \dots, r$, the section $t_h \in \Gamma(\mathcal{E} \otimes \mathsf{S}^h \mathcal{A}^*)$ is given by the following expression:

$$t_h = \sum_{j=0}^{h} s_j \otimes d_{\mathcal{A}}^{(h-j)} f.$$

There is an exact sequence

$$0 \to \mathcal{E} \otimes \mathsf{S}^r \mathcal{A}^* \to J^r_{(\mathcal{A},\sharp)}(\mathcal{E}) \to J^{r-1}_{(\mathcal{A},\sharp)}(\mathcal{E}) \to 0 \tag{4.2}$$

(which is split as a sequence of $\pi^{-1}\mathcal{O}_Y$-modules but not, in general, as a sequence of $\mathcal{O}_X$-modules).

Finally, note that, for any r, there is a homomorphism of sheaves of abelian groups

$$d^r_{(\mathcal{A},\sharp),\mathcal{E}} : \mathcal{E} \to J^r_{(\mathcal{A},\sharp)}(\mathcal{E})$$

that is $\mathcal{O}_X$-linear for the *right* $\mathcal{O}_X$-module structure of $J^r_{(\mathcal{A},\sharp)}(\mathcal{E})$. All the verifications are left as exercises.

5 $(\mathcal{A}, \sharp)$-Differential Operators

Let us recall that $\mathcal{D} = \mathcal{D}_{X/Y}$, the sheaf of rings of finite-order (holomorphic) differential operators on X over Y, is generated, as an algebra, by $\mathcal{O}_X$ and by $\mathcal{T}_{X/Y}$.

In a similar way, we define $\mathcal{D}_{(\mathcal{A},\sharp)}$ to be the algebra generated by $\mathcal{O}_X$ and $\mathcal{A}$, with the commutation relations given by

$$af = \sharp(a)(f) + fa \quad \text{and} \quad a_1 a_2 = a_2 a_1 + [a_1, a_2], \tag{5.1}$$

where a, a_1, a_2 are sections of $\mathcal{A}$ and f is a section of $\mathcal{O}_X$.

The sheaf of non-commutative rings $\mathcal{D}_{(\mathcal{A},\sharp)}$ is endowed with a filtration

$$0 \subset \mathcal{O}_X = \mathcal{D}^{\leq 0}_{(\mathcal{A},\sharp)} \subset \mathcal{D}^{\leq 1}_{(\mathcal{A},\sharp)} \subset \cdots \subset \mathcal{D}^{\leq r}_{(\mathcal{A},\sharp)} \subset \cdots \subset \mathcal{D}_{(\mathcal{A},\sharp)}$$

such that

$$\mathcal{D}_{(\mathcal{A},\sharp)} = \bigcup_{r\geq 0} \mathcal{D}^{\leq r}_{(\mathcal{A},\sharp)},$$

where, for each r, the $\mathcal{O}_X$-module $\mathcal{D}^{\leq r}_{(\mathcal{A},\sharp)}$ is the dual of the sheaf of r-th $(\mathcal{A},\sharp)$-jets $J^r_{(\mathcal{A},\sharp)}(\mathcal{O}_X)$,

$$\mathcal{D}^{\leq r}_{(\mathcal{A},\sharp)} = \mathcal{H}om_{\mathcal{O}_X}\big(J^r_{(\mathcal{A},\sharp)}(\mathcal{O}_X), \mathcal{O}_X\big).$$

If $\mathcal{D}$ is the usual ring of differential operators on X over Y, the anchor map $\sharp\colon \mathcal{A} \to \mathcal{T}_{X/Y}$ induces a homomorphism of filtered rings

$$\sharp\colon \mathcal{D}_{(\mathcal{A},\sharp)} \to \mathcal{D}.$$

The map $\sigma : \mathcal{D}^{\leq r}_{(\mathcal{A},\sharp)} \to \mathsf{S}^r\mathcal{A}$, that associates to a $(\mathcal{A},\sharp)$-differential operator its highest order term, is well defined and is called the *principal symbol* map. For every $r > 0$, there is an exact sequence

$$0 \to \mathcal{D}^{\leq r-1}_{(\mathcal{A},\sharp)} \to \mathcal{D}^{\leq r}_{(\mathcal{A},\sharp)} \to \mathsf{S}^r\mathcal{A} \to 0$$

which is the dual of

$$0 \to \mathsf{S}^r\mathcal{A}^* \to J^r_{(\mathcal{A},\sharp)}(\mathcal{O}_X) \to J^{r-1}_{(\mathcal{A},\sharp)}(\mathcal{O}_X) \to 0.$$

The associated graded ring of the filtered ring $\mathcal{D}_{(\mathcal{A},\sharp)}$ is isomorphic to the symmetric algebra over $\mathcal{A}$

$$\mathrm{gr}(\mathcal{D}_{(\mathcal{A},\sharp)}) \cong \mathsf{S}^{\cdot}(\mathcal{A}).$$

Let us recall that a relative flat connection on a coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{E}$ is equivalent to a structure of $\mathcal{D}$-module on $\mathcal{E}$. In a similar way it is easy to prove that a relative flat $(\mathcal{A},\sharp)$-connection on $\mathcal{E}$ is equivalent to a structure of $\mathcal{D}_{(\mathcal{A},\sharp)}$-module on $\mathcal{E}$.

To end this section let us recall that if X is a complex manifold (or a smooth algebraic variety defined over a field of characteristic zero) and $\mathcal{E}$ is a coherent $\mathcal{O}_X$-module endowed with a (usual) connection, then it is well known that $\mathcal{E}$ is actually locally free, hence it is the sheaf of sections of a vector bundle E over X. This was proved in [9], under the assumption that the connection is flat, and in [1] in general.

For Lie algebroid connections a similar result holds under the assumption that the anchor map is surjective, as we shall now explain. The proof can be obtained by an easy adaptation of the proof of Theorem 1.4.10 in [6]. For the reader's convenience we shall report the argument here.

Theorem 5.1 *Let X be a complex manifold, $(A, \sharp)$ be a Lie algebroid over X and let $\mathcal{E}$ be a coherent $\mathcal{O}_X$-module endowed with a $(A, \sharp)$-connection ∇. Let us assume that the anchor map $\sharp\colon A \to T_X$ is surjective. Then $\mathcal{E}$ is locally free.*

Proof Since $\mathcal{E}$ is coherent, to prove that it is locally free it is enough to show that the stalk $\mathcal{E}_x$ is a free $\mathcal{O}_{X,x}$-module, for any $x \in X$.

Let us choose local coordinates $\{x_1, \dots, x_n\}$ in X, so that the maximal ideal $\mathfrak{m}$ of the local ring $\mathcal{O}_{X,x}$ is generated by $x_1, \dots, x_n$. From Nakayama's lemma it follows that there exist elements $s_1, \dots, s_m \in \mathcal{E}_x$ such that $\mathcal{E}_x$ is generated by $s_1, \dots, s_m$ as an $\mathcal{O}_{X,x}$-module and $\bar{s}_1, \dots, \bar{s}_m \in \mathcal{E}_x/\mathfrak{m}\mathcal{E}_x$ are free generators of the vector space $\mathcal{E}_x/\mathfrak{m}\mathcal{E}_x$ over $\mathbb{C} = \mathcal{O}_{X,x}/\mathfrak{m}$.

We shall now prove that $\{s_1, \dots, s_m\}$ is a free set of generators of the $\mathcal{O}_{X,x}$-module $\mathcal{E}_x$.

Let us assume that there exists a non-trivial relation $\sum_{i=1}^m f_i s_i = 0$, for some $f_i \in \mathcal{O}_{X,x}$. We define the order of f_i at $x \in X$ by setting

$$\operatorname{ord}(f_i) = \max\bigl\{h \bigm| f_i \in \mathfrak{m}^h\bigr\}.$$

For each $j = 1, \dots, n$ let us set $\partial_j = \partial/\partial x_j$ and choose a local section $a_j \in A$ such that $\sharp a_j = \partial_j \in T_X$ (this is possible because we are assuming $\sharp\colon A \to T_X$ to be surjective). Now let us apply ∇_{a_j} to the above relation. We obtain a new relation

$$0 = \sum_{i=1}^m (\partial_j f_i) s_i + \sum_{i=1}^m f_i \nabla_{a_j} s_i = \sum_{i=1}^m g_i s_i,$$

for some $g_i \in \mathcal{O}_{X,x}$.

Applying the operator ∂_j to f_i has the effect of lowering the order of f_i at x, hence we can choose a suitable j such that

$$\min\bigl\{\operatorname{ord}(f_i) \bigm| i = 1, \dots, m\bigr\} > \min\bigl\{\operatorname{ord}(g_i) \bigm| i = 1, \dots, m\bigr\}.$$

By repeating this procedure, after a finite number of steps we obtain a non-trivial relation $\sum_{i=1}^m h_i s_i = 0$, with some $h_i \notin \mathfrak{m}$. It follows that we get a non-trivial relation $\sum_{i=1}^m \bar{h}_i \bar{s}_i = 0$, with $\bar{h}_i \in \mathcal{O}_{X,x}/\mathfrak{m} = \mathbb{C}$, but this contradicts the choice of $s_1, \dots, s_m$. □

6 Kapranov's 'Cohomological Bianchi Identity'

In this section we shall generalize the so-called 'cohomological Bianchi identity,' proved by Kapranov in [8], to the setting of Lie algebroid connections.

Let X be a complex manifold, $(A, \sharp)$ a Lie algebroid over X, and $\mathcal{D}_{(A,\sharp)}$ the sheaf of rings of $(A, \sharp)$-differential operators.

Remark 6.1 Let E be a vector bundle over X and E^* its dual bundle. The exact sequence

$$0 \longrightarrow E^* \otimes A^* \longrightarrow J^1_{(A,\sharp)}\big(E^*\big) \longrightarrow E^* \longrightarrow 0$$

computes the $(A,\sharp)$-Atiyah class of E^*, $a_{(A,\sharp)}(E^*) = -a_{(A,\sharp)}(E)$.

By dualizing, we obtain the exact sequence

$$0 \longrightarrow E \longrightarrow \mathcal{D}^{\le 1}_{(A,\sharp)}(E) \longrightarrow E \otimes A \longrightarrow 0, \tag{6.1}$$

where $\mathcal{D}^{\le 1}_{(A,\sharp)}(E) = \mathcal{D}^{\le 1}_{(A,\sharp)} \otimes_{\mathcal{O}_X} E$, that also computes the class

$$-a_{(A,\sharp)}(E) \in \operatorname{Ext}^1(E \otimes A, E) = \operatorname{Ext}^1\big(E, E \otimes A^*\big).$$

Let M be a locally free left $\mathcal{D}_{(A,\sharp)}$-module, endowed with a good filtration M_i by vector bundles. The $\mathcal{D}_{(A,\sharp)}$-module structure on M is equivalent to a flat $(A,\sharp)$-connection $\nabla\colon M \to M \otimes A^*$. It follows that, for any j, we have an induced map $\nabla_j\colon M_j \to M_{j+1} \otimes A^*$.

The following result is a generalization of a similar statement, proved in [2]:

Lemma 6.2 ([2], (4.1.2.3)) *Let M be as before. Then*:

(a) *The class $-a_{(A,\sharp)}(M_i)$ is given by the following composition of maps*

$$M_i \xrightarrow{\pi_i} M_i/M_{i-1} \xrightarrow{\nabla_i} (M_{i+1}/M_i) \otimes A^* \xrightarrow{\alpha_i \otimes 1_{A^*}} M_i \otimes A^*[1],$$

where $\pi_i\colon M_i \to M_i/M_{i-1}$ is the projection, $\nabla_i\colon M_i/M_{i-1} \to (M_{i+1}/M_i) \otimes A^*$ is induced by the $(A,\sharp)$-connection ∇, and $\alpha_i \in \operatorname{Ext}^1(M_{i+1}/M_i, M_i)$ is the element that corresponds to the exact sequence

$$0 \longrightarrow M_i \longrightarrow M_{i+1} \longrightarrow M_{i+1}/M_i \longrightarrow 0.$$

(b) *The class $-a_{(A,\sharp)}(M_i/M_{i-1})$ is equal to the difference between the composition of morphisms*

$$M_i/M_{i-1} \xrightarrow{\nabla_i} (M_{i+1}/M_i) \otimes A^* \xrightarrow{\alpha_i \otimes 1_{A^*}} M_i \otimes A^*[1] \xrightarrow{\pi_i[1]} (M_i/M_{i-1}) \otimes A^*[1],$$

and the composition

$$M_i/M_{i-1} \xrightarrow{\alpha_{i-1}} M_{i-1}[1] \xrightarrow{\pi_{i-1}[1]} (M_{i-1}/M_{i-2})[1] \xrightarrow{\nabla_{i-1}[1]} (M_i/M_{i-1}) \otimes A^*[1].$$

Proof The proof is the same as in [2], since it follows from purely formal properties of extension classes. □

Remark 6.3 The map $\nabla_j\colon M_j/M_{j-1} \to M_{j+1}/M_j \otimes A^*$ induced by the $(A,\sharp)$-connection ∇ on M, corresponds to the so-called "symbol multiplication map"

$$\mu_j\colon A \otimes M_j/M_{j-1} \to M_{j+1}/M_j.$$

If we denote by $f_i \in \mathrm{Ext}^1(M_{i+1}/M_i, M_i/M_{i-1})$ the composition

$$M_{i+1}/M_i \xrightarrow{\alpha_i} M_i[1] \xrightarrow{\pi_i[1]} (M_i/M_{i-1})[1],$$

then part (b) of the previous lemma can be restated by saying that the class $-a_{(A,\sharp)}(M_i/M_{i-1})$ is given by the difference between the following two compositions of morphisms:

$$A \otimes M_i/M_{i-1} \xrightarrow{\mu_i} M_{i+1}/M_i \xrightarrow{f_i} (M_i/M_{i-1})[1],$$

and

$$A \otimes M_i/M_{i-1} \xrightarrow{1_A \otimes f_{i-1}} A \otimes (M_{i-1}/M_{i-2})[1] \xrightarrow{\mu_{i-1}[1]} (M_i/M_{i-1})[1],$$

i.e., we can write

$$-a_{(A,\sharp)}(M_i/M_{i-1}) = f_i \circ \mu_i - \mu_{i-1}[1] \circ (1_A \otimes f_{i-1}). \tag{6.2}$$

If E is a vector bundle over X, we can consider the $\mathcal{D}_{(A,\sharp)}$-module $M = \mathcal{D}_{(A,\sharp)} \otimes_{\mathcal{O}_X} E$, with the filtration given by $M_i = \mathcal{D}^{\leq i}_{(A,\sharp)} \otimes E$. The exact sequence

$$0 \longrightarrow \frac{M_1}{M_0} \longrightarrow \frac{M_2}{M_0} \longrightarrow \frac{M_2}{M_1} \longrightarrow 0$$

becomes

$$0 \longrightarrow A \otimes E \longrightarrow \frac{\mathcal{D}^{\leq 2}_{(A,\sharp)} \otimes E}{E} \longrightarrow \mathsf{S}^2(A) \otimes E \longrightarrow 0.$$

Let us denote by $\xi \in \mathrm{Ext}^1(\mathsf{S}^2(A) \otimes E, A \otimes E)$ the corresponding extension class. Let $\sigma : A \otimes A \to \mathsf{S}^2(A)$ be the symmetrization map. From Lemma 6.2 and the subsequent remark, it follows that:

Lemma 6.4 *With the above notations, we have*

$$a_{(A,\sharp)}(A \otimes E) = -\xi \circ (\sigma \otimes 1) - 1 \otimes a_{(A,\sharp)}(E).$$

Proof Since $M_i = \mathcal{D}^{\leq i}_{(A,\sharp)} \otimes E$, we have $M_1/M_0 = A \otimes E$ and $M_2/M_1 = \mathsf{S}^2(A) \otimes E$. From (6.2) we know that $-a_{(A,\sharp)}(M_1/M_0) = -a_{(A,\sharp)}(A \otimes E)$ is the difference between the following composition of morphisms:

$$A \otimes A \otimes E \xrightarrow{\sigma \otimes 1_E} \mathsf{S}^2(A) \otimes E \xrightarrow{\xi} A \otimes E[1]$$

and

$$A \otimes A \otimes E \xrightarrow{1 \otimes a_{(A,\sharp)}(E)} A \otimes E[1] \xrightarrow{-\mathrm{id}} A \otimes E[1].$$

Hence $-a_{(A,\sharp)}(A \otimes E) = \xi \circ (\sigma \otimes 1) + 1 \otimes a_{(A,\sharp)}(E)$. □

Now we introduce some notation in order to state the main result.

Let $a, b \in H^1(X, \mathcal{E}nd(E) \otimes A^*)$. Their cup-product is

$$a \smile b \in H^2\big(X, \mathcal{E}nd(E) \otimes \mathcal{E}nd(E) \otimes A^* \otimes A^*\big).$$

Consider the map

$$\begin{aligned}\mathcal{E}nd(E) \otimes \mathcal{E}nd(E) \otimes A^* \otimes A^* &\to \mathcal{E}nd(E) \otimes \mathsf{S}^2\big(A^*\big)\\ \phi \otimes \psi \otimes \alpha \otimes \beta &\mapsto [\phi, \psi] \otimes (\alpha \odot \beta)\end{aligned}$$

We denote by $[a \smile b] \in H^2(X, \mathcal{E}nd(E) \otimes \mathsf{S}^2(A^*))$ the image of $a \smile b$ under the induced map in cohomology.

Let $a \in H^1(X, \mathcal{E}nd(E) \otimes A^*) = \mathrm{Ext}^1(E, E \otimes A^*) = \mathrm{Ext}^1(A \otimes E, E)$, and let $c \in \mathrm{Ext}^1(A \otimes A, A)$. Let us consider the composition

$$\mathsf{S}^2(A) \otimes E \hookrightarrow A \otimes A \otimes E \xrightarrow{c \otimes 1} A \otimes E[1] \xrightarrow{a} E[2]$$

We denote by

$$\begin{aligned}a * c \in \mathrm{Hom}\big(\mathsf{S}^2(A) \otimes E, E[2]\big) &= \mathrm{Ext}^2\big(\mathsf{S}^2(A) \otimes E, E\big)\\ &= H^2\big(X, \mathcal{E}nd(E) \otimes \mathsf{S}^2\big(A^*\big)\big)\end{aligned}$$

the corresponding element.

Theorem 6.5 (Cohomological Bianchi identity) *Let $a_{(A,\sharp)}(E) \in \mathrm{Ext}^1(E, E \otimes A^*) = H^1(X, \mathcal{E}nd(E) \otimes A^*)$ be the $(A, \sharp)$-Atiyah class of a vector bundle E. Let $a_{(A,\sharp)}(A) \in \mathrm{Ext}^1(A, A \otimes A^*) = H^1(X, \mathcal{E}nd(A) \otimes A^*)$ be the $(A, \sharp)$-Atiyah class of A. Then we have the identity*

$$2\big[a_{(A,\sharp)}(E) \smile a_{(A,\sharp)}(E)\big] + a_{(A,\sharp)}(E) * a_{(A,\sharp)}(A) = 0$$

in $H^2(X, \mathcal{E}nd(E) \otimes \mathsf{S}^2(A^*))$.

Proof Let $M = \mathcal{D}_{(A,\sharp)} \otimes E$, with the filtration

$$0 \subset M_0 = E \subset M_1 = \mathcal{D}^{\le 1}_{(A,\sharp)} \otimes E \subset M_2 = \mathcal{D}^{\le 2}_{(A,\sharp)} \otimes E \subset \dots$$

The exact sequence $0 \to M_0 \to M_1 \to M_1/M_0 \to 0$ is

$$0 \to E \to \mathcal{D}^{\le 1}_{(A,\sharp)} \otimes E \to A \otimes E \to 0,$$

whose extension class is $-a_{(A,\sharp)}(E) \in \mathrm{Ext}^1(A \otimes E, A)$. The next exact sequence $0 \to M_1/M_0 \to M_2/M_0 \to M_2/M_1 \to 0$ is

$$0 \to A \otimes E \to M_2/M_0 \to \mathsf{S}^2(A) \otimes E \to 0,$$

whose extension class we have denoted by ξ.

Standard results (cf., for instance, [3]) tell us that the composition (Yoneda product) of these two extensions is zero: $a_{(A,\sharp)}(E) \circ \xi = 0$

$$\mathsf{S}^2(A) \otimes E \xrightarrow{\xi} A \otimes E[1] \xrightarrow{a_{(A,\sharp)}(E)} E[2].$$

From Lemma 6.4 we have

$$a_{(A,\sharp)}(A \otimes E) = -\xi \circ (\sigma \otimes 1) - 1 \otimes a_{(A,\sharp)}(E).$$

The $(A,\sharp)$-Atiyah class of a tensor product of vector bundles is given by

$$a_{(A,\sharp)}(A \otimes E) = a_{(A,\sharp)}(A) \otimes 1 + 1 \otimes a_{(A,\sharp)}(E),$$

hence

$$2\big(1 \otimes a_{(A,\sharp)}(E)\big) + a_{(A,\sharp)}(A) \otimes 1 = -\xi \circ (\sigma \otimes 1).$$

Now we take the Yoneda product of the previous expression with $a_{(A,\sharp)}(E)$ (on the left), and we recall that $a_{(A,\sharp)}(E) \circ \xi = 0$.

We get

$$2\big[a_{(A,\sharp)}(E) \smile a_{(A,\sharp)}(E)\big] + a_{(A,\sharp)}(E) * a_{(A,\sharp)}(A) = 0. \qquad \square$$

7 The Lie Algebra Structure

Let X and $(A,\sharp)$ be as before, and let $\mathcal{F}$ be a quasi-coherent sheaf of commutative $\mathcal{O}_X$-algebras. We consider the composition of the following maps: first we take the cup-product

$$H^i(X, A \otimes \mathcal{F}) \otimes H^j(X, A \otimes \mathcal{F}) \to H^{i+j}(X, A \otimes A \otimes \mathcal{F} \otimes \mathcal{F})$$

followed by the map

$$H^{i+j}(X, A \otimes A \otimes \mathcal{F} \otimes \mathcal{F}) \to H^{i+j}(X, A \otimes A \otimes \mathcal{F})$$

induced by the commutative multiplication $\mathcal{F} \otimes \mathcal{F} \to \mathcal{F}$. Then we take the Yoneda product with $a_{(A,\sharp)}(A) \in H^1(X, \mathrm{Hom}(\mathsf{S}^2(A), A))$:

$$H^{i+j}(X, A \otimes A \otimes \mathcal{F}) \to H^{i+j+1}(X, A \otimes \mathcal{F}).$$

So, for any i and j, we obtain maps

$$H^i(X, A \otimes \mathcal{F}) \otimes H^j(X, A \otimes \mathcal{F}) \to H^{i+j+1}(X, A \otimes \mathcal{F}).$$

Let us set $\mathfrak{g}_i = H^{i-1}(X, A \otimes \mathcal{F})$. Then we can rewrite the previous maps as follows:

$$\mathfrak{g}_i \otimes \mathfrak{g}_j \to \mathfrak{g}_{i+j}.$$

Theorem 7.1 *The maps above define a graded Lie algebra structure on the graded vector space* $\mathfrak{g}^\bullet = \bigoplus_i \mathfrak{g}_i$.

Proof Let $\alpha_i \in \mathfrak{g}_i$, $\alpha_j \in \mathfrak{g}_j$, and let us denote the bracket by $[\alpha_i, \alpha_j] \in \mathfrak{g}_{i+j}$. The bilinearity of the bracket is obvious. The (graded) antisymmetry is given by the following expression:

$$[\alpha_j, \alpha_i] = -(-1)^{ij}[\alpha_i, \alpha_j].$$

This follows immediately from the graded commutativity of the cup-product. It remains only to prove the (graded) Jacobi identity:

$$(-1)^{ik}\big[\alpha_i, [\alpha_j, \alpha_k]\big] + (-1)^{ij}\big[\alpha_j, [\alpha_k, \alpha_i]\big] + (-1)^{jk}\big[\alpha_k, [\alpha_i, \alpha_j]\big] = 0.$$

Let us denote the left-hand side by $\theta(\alpha_i, \alpha_j, \alpha_k)$. This defines an element $\theta \in \mathrm{Hom}(\wedge^3 \mathfrak{g}^\bullet, \mathfrak{g}^\bullet)$, and we can check that $\theta(\alpha_i, \alpha_j, \alpha_k)$ is obtained by taking the cup-product

$$\alpha_i \smile \alpha_j \smile \alpha_k \in H^{i+j+k-3}(X, A \otimes A \otimes A \otimes \mathcal{F})$$

followed by the composition with an element of $H^2(X, \mathcal{H}om(\mathsf{S}^3(A), A))$. This element turns out to be the symmetrization of

$$\big[a_{(A,\sharp)}(A) \smile a_{(A,\sharp)}(A)\big] \in H^2\big(X, \mathcal{H}om\big(A \otimes \mathsf{S}^2(A), A\big)\big).$$

Now we use the cohomological Bianchi identity (for $E = A$):

$$2\big[a_{(A,\sharp)}(A) \smile a_{(A,\sharp)}(A)\big] + a_{(A,\sharp)}(A) * a_{(A,\sharp)}(A) = 0.$$

From the definition, it follows that the symmetrization of $a_{(A,\sharp)}(A) * a_{(A,\sharp)}(A)$ is 0, hence the same is true for the symmetrization of $[a_{(A,\sharp)}(A) \smile a_{(A,\sharp)}(A)]$. This finally means that $\theta = 0$, which proves the Jacobi identity. □

Let X, $(A, \sharp)$, $\mathcal{F}$ be as before, and let E be a holomorphic vector bundle over X. We consider now the composition of the following maps: first we take the cup-product

$$H^i(X, A \otimes \mathcal{F}) \otimes H^j(X, E \otimes \mathcal{F}) \to H^{i+j}(X, A \otimes E \otimes \mathcal{F})$$

(where we have used the multiplication $\mathcal{F} \otimes \mathcal{F} \to \mathcal{F}$, as before). Then we take the Yoneda product with $a_{(A,\sharp)}(E) \in H^1(X, \mathrm{Hom}(A \otimes E, E))$:

$$H^{i+j}(X, A \otimes E \otimes \mathcal{F}) \to H^{i+j+1}(X, E \otimes \mathcal{F}).$$

If we set $\mathfrak{g}_i = H^{i-1}(X, A \otimes \mathcal{F})$ and $V_j = H^{j-1}(X, E \otimes \mathcal{F})$, for any i and j, we have maps $\mathfrak{g}_i \otimes V_j \to V_{i+j}$. We can now prove the following result:

Theorem 7.2 *The maps above define a structure of graded module on the graded vector space* $V^\bullet = \bigoplus_j V_j$, *over the graded Lie algebra* $\mathfrak{g}^\bullet$.

Proof Let $\alpha_i \in \mathfrak{g}_i$, $\alpha_j \in \mathfrak{g}_j$ and $v_k \in V_k$. We must prove that

$$[\alpha_i, \alpha_j]v_k - \alpha_i(\alpha_j v_k) + (-1)^{ij}\alpha_j(\alpha_i v_k) = 0.$$

The left-hand side defines an element $\phi \in \mathrm{Hom}(\wedge^2\mathfrak{g}^\bullet \otimes V^\bullet, V^\bullet)$, and we can check that ϕ is obtained by taking the cup-product

$$\alpha_i \smile \alpha_j \smile v_k \in H^{i+j+k-3}(X, A \otimes A \otimes E \otimes \mathcal{F})$$

followed by the Yoneda composition with an element of $H^2(X, \mathcal{H}om(\mathsf{S}^2(A) \otimes E, E))$. This element is precisely

$$2\big[a_{(A,\sharp)}(E) \smile a_{(A,\sharp)}(E)\big] + a_{(A,\sharp)}(E) * a_{(A,\sharp)}(A),$$

which vanishes by the cohomological Bianchi identity. □

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Kähler Metrics with Cone Singularities and Uniqueness Problem

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Abstract The Kähler metric with cone singularities has been the main subject which is being studied recently. In this expository note, we focus on the modular space of the Kähler metric with cone singularities. We first summary our work on the construction of the geodesic of the cone singularities. Then we apply the cone geodesic to obtain a uniqueness theorem of the constant scalar curvature Kähler metrics with cone singularities.

Keywords Kähler cone metrics

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1 Introduction

Let X be a smooth n-dimensional compact Kähler manifold without boundary and Ω be a Kähler class on X. We denote by ω_0 a smooth Kähler metric in X, and by $\mathcal{H}$ the space of Kähler metrics in Ω. In $\mathcal{H}$, the famous L^2 metric was defined independently by Mabuchi [23], Donaldson [17] and Semmes [24]. Under which, $\mathcal{H}$ becomes a non-positive curved infinite-dimensional symmetric space. Semmes [24] pointed out that the geodesic equation in $\mathcal{H}$ is a homogeneous complex Monge–Ampère (HCMA) equation,

$$\begin{cases} \left(\Omega_0 + \frac{\sqrt{-1}}{2}\partial\bar{\partial}\Psi\right)^{n+1} = 0 & \text{in } X \times R, \\ \sum_{1\le i,j\le n} \left(\Omega_0 + \frac{\sqrt{-1}}{2}\partial\bar{\partial}\Psi\right)_{i\bar{j}} dz^i \wedge dz^{\bar{j}} > 0 & \text{in } X \times \left\{z^{n+1}\right\}; \end{cases} \tag{1.1}$$

here R is a cylinder with boundary, and Ω_0 is the pull-back metric of ω_0 under the natural projection to X.

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Geodesics are basic geometric objects in modular space $\mathcal{H}$. The intensive relation between the geodesics of $\mathcal{H}$ and the existence and the uniqueness of the constant scalar curvature Kähler (cscK) metrics or the extremal Kähler metrics in general, was pointed out by Donaldson [17]. In the same paper, he conjectured that $\mathcal{H}$ endowed with the L^2 metric is geodesically convex and is a metric space. The geodesic equation (1.1) is solved in Chen [13] under smooth Dirichlet conditions, then the existence of $C^{1,1}$ geodesic segments (of bounded mixed derivatives) is established. Thus he verified that the space of Kähler metrics is a metric space. Later in [11], Chen and Tian improved the partial regularity of the $C^{1,1}$ geodesic, then proved the uniqueness of the extremal metrics. In general, a $C^{1,1}$ geodesic does not need to be smooth that was showed by Donaldson [18], Darvas–Lempert [15] and Lempert–Vivas [22].

The Kähler metrics with cone singularities (Kähler cone metrics for short) have been extensively studied recently. They are the main objects in Donaldson's programme [19], in which a new continuity method is invented. The cone angle is used as parameter of deforming the Kähler–Einstein cone metrics to the smooth ones. There are many beautiful works around the Kähler cone metrics, while many new articles are also coming out, it is impossible to make the list of the references of this topic complete, however more references could be found in our paper [10].

In this expository paper, we focus on the geometry of the space of Kähler cone metrics and the uniqueness problem of the cscK cone metrics. We will summary the results in Calamai–Zheng [10], where we constructed the geodesics with cone singularity in a proper subspace of the space of the Kähler cone metrics and further proved that it is a metric space. Then we will apply the cone geodesic to prove a uniqueness theorem of the cscK metrics with cone singularity.

2 Preliminary

In the local holomorphic coordinates $(z_1, z_2, \ldots, z_n)$, the Kähler form ω_0 is written as

$$\omega_0 = \frac{\sqrt{-1}}{2}\sum_{i=1}^{n}(g_0)_{i\bar{j}}dz^i \wedge dz^{\bar{j}}.$$

The Riemannian metric corresponding to ω is given by $g = \sum_{i=1}^{n}(g_0)_{i\bar{j}}dz^i \otimes dz^{\bar{j}}$ on $T^{\mathbb{C}}(M)$. Written in this form, the metric g_0 is Kähler if and only if

$$(g_0)_{ij} = (g_0)_{\bar{i}\bar{j}} = 0 \quad \text{and} \quad \frac{\partial (g_0)_{i\bar{j}}}{\partial z^k} = \frac{\partial (g_0)_{k\bar{j}}}{\partial z^i}.$$

The volume form is the (n, n) form

$$dV = \frac{\omega_0^n}{n!} = \left(\frac{\sqrt{-1}}{2}\right)^n \det\bigl((g_0)_{i\bar{j}}\bigr)dz^1 \wedge dz^{\bar{1}} \wedge \cdots \wedge dz^n \wedge dz^{\bar{n}}.$$

The volume of M with respect different Kähler metrics within a fixed Kähler class is a topological invariant. For each $\omega \in \Omega$, the corresponding Ricci form

$$Ric = \frac{\sqrt{-1}}{2} \sum_{i=1}^{n} R_{i\bar{j}} dz^i \wedge dz^{\bar{j}} = \frac{\sqrt{-1}}{2} \bar{\partial}\partial \log \det(g_0)$$

is a closed form. The first Chern class is $C_1(M) = \frac{[Ric]}{\pi}$. The scalar curvature is the contraction of the Ricci curvature

$$S = g_0^{i\bar{j}} R_{i\bar{j}}.$$

Furthermore, from the formulae

$$S\omega_0^n = nRic \wedge \omega_0^{n-1},$$

we obtain that the average of the scalar curvature is

$$\underline{S} = \frac{\int_M SdV}{V} = \frac{1}{(n-1)!V} \int_M Ric \wedge \omega_0^{n-1} = \frac{Ric \cdot \Omega^{n-1}}{n \cdot \Omega^n} = \frac{\pi C_1(M)\Omega^{n-1}}{n \cdot \Omega^n}.$$

Thus $\underline{S}$ is also a topological invariant. According to the $\partial\bar{\partial}$ lemma, the space of Kähler potentials is formulated as

$$\mathcal{H} = \left\{ \varphi \in C^\infty(M, \mathbb{R}) \,\middle|\, \omega_0 + \frac{\sqrt{-1}}{2} \partial\bar{\partial}\varphi > 0 \right\} / \mathbb{R}.$$

In [8, 9], Calabi suggest minimizing the L^2-norm of the scalar curvature in $\mathcal{H}$,

$$\int_M S^2 \omega_0^n,$$

which is now known as Calabi functional. Its critical points are called the extremal Kähler metrics which contain both the Kähler–Einstein metrics

$$Ric = l\omega,$$

in which, the constant l is the sign of the first Chern class and more general, the constant scalar curvature Kähler (cscK) metrics

$$S = \underline{S}.$$

In $\mathcal{H}$, a Riemannian metric is defined in Donaldson [17], Mabuchi [23] and Semmes [24] as

$$\frac{1}{n!} \int_M f_1 f_2 \omega_\varphi^n \tag{2.1}$$

for any $f_1, f_2 \in T_\varphi \mathcal{H}$. Under this metric, $\mathcal{H}$ becomes a non-positive curved infinite dimensional symmetric space. The $C^{1,1}$ geodesics with respect to this L^2 metric are constructed by Chen [13].

Theorem 2.1 (Chen [13]) *Any two points in $\mathcal{H}$ can be connected by a unique $C^{1,1}$ geodesic and $\mathcal{H}$ is a metric space.*

3 Space of Kähler Cone Metric

Let us detail now the concept of Kähler cone metric. Let

$$D = \sum_{i=1}^{m} (1 - \beta_i) V_i$$

be a normal crossing, effective smooth divisor of X with $0 < \beta_i \leq 1$ for $1 \leq i \leq m$, where $V_i \subset X$ are irreducible smooth hypersurfaces. Set $\beta := (\beta_1, \ldots, \beta_m)$ and call the β_i's the *cone angles*.

Definition 3.1 Given a point p in D, label a local chart (U_p, z^i) centered at p as *local cone chart* when $z^1, \ldots, z^k$ are the local defining functions of the hypersurfaces where p locates. A *Kähler cone metric* ω of cone angle $2\pi\beta_i$ along V_i, for $1 \leq i \leq m$, is a closed positive $(1,1)$-current and a smooth Kähler metric on the regular part $M := X \setminus D$. In a local cone chart U_p its Kähler form is quasi-isometric to the cone flat metric, which is

$$\omega_{cone} := \frac{\sqrt{-1}}{2} \sum_{i=1}^{k} \beta_i^2 \left|z^i\right|^{2(\beta_i - 1)} dz^i \wedge dz^{\bar{i}} + \sum_{k+1 \leq j \leq n} dz^j \wedge dz^{\bar{j}}. \tag{3.1}$$

Let $\mathcal{H}_\beta$ be the space of Kähler cone metrics of cone angle $2\pi\beta_i$ along V_i in the cohomology class Ω. An example of the Kähler cone metric in $\mathcal{H}_\beta$ is constructed in Donaldson [20] as following. Let s be a global meromorphic section of $[D]$. Let h be an Hermitian metric on $[D]$. It is shown in [20] that, for sufficiently small $\delta > 0$,

$$\omega = \omega_0 + \delta \sum_{i=1}^{m} \frac{\sqrt{-1}}{2} \partial\bar{\partial} |s_i|_{h_\Lambda}^{2\beta_i} \tag{3.2}$$

is a Kähler cone metric. Moreover, ω is independent of the choices of ω_0, h_Λ, δ up to quasi-isometry. We call it model metric in this paper.

We denote $\mathcal{H}_\beta^{2,\alpha}$ be the space of $C_\beta^{2,\alpha}$ ω_0-psh-functions (see next section for details). In $\mathcal{H}_\beta^{2,\alpha}$, we could also first define the L^2 Riemannian metric like (2.1). The delicate part is to verify the integral is well-defined by the integration by part. Then we could compute the Levi–Civita connection and the geodesic equation.

$$\varphi'' - \left(\partial\varphi', \partial\varphi'\right)_{g_\varphi} = 0 \quad \text{on } M. \tag{3.3}$$

This part has been done in Calamai–Zheng [10].

Let $R = [0,1] \times S^1$ be a cylinder and let $z^{n+1} = t + \sqrt{-1}y^{n+1}$ be the coordinate on R. We extend the functions on X to the product manifold $X \times R$,

$$\varphi(z', z^{n+1}) = \varphi(z^1, \dots, z^n, t).$$

Let π be the natural projection form $X \times R$ to X and Ω_0 and Ω be the pulling-back metrics of ω_0 and ω respectively. We also denote

$$\Psi = \varphi(z^1, \dots, z^n, z^{n+1}) - |z^{n+1}|^2.$$

Using the definition of the determinant, one could show that a path $\varphi(t)$ with end-points φ_0, φ_1 satisfies the geodesic equation (3.3) on X if and only if Ψ satisfies the following Dirichlet problem of a degenerate complex Monge–Ampère equation

$$\begin{cases} \det(\Omega_{i\bar{j}} + \Psi_{i\bar{j}}) = 0 & \text{in } M \times R, \\ \Psi(z) = \Psi_0 & \text{on } X \times \partial R, \\ \sum_{1 \le i,j \le n} (\Omega_{i\bar{j}} + \Psi_{i\bar{j}}) dz^i dz^{\bar{j}} > 0 & \text{in } X \times \{z^{n+1}\}. \end{cases} \tag{3.4}$$

4 Weighted Hölder Spaces

Let U_p a local cone chart as in Definition 3.1. Let $W : U_p \setminus D \to U_p \setminus D$ be the quasi-isometric mapping given by

$$W(z^1, \dots, z^n) := \left(w^1 = |z^1|^{\beta_1 - 1} z^1, \dots, w^k = |z^k|^{\beta_k - 1} z^k, z^{k+1}, \dots, z^n\right). \tag{4.1}$$

We also denote

$$\begin{aligned} \varepsilon_i &:= dr_i + \sqrt{-1}\beta_i r_i d\theta_i = \beta_i |w^i|^{1-\mu_i} (w^i)^{-1} dz^i \\ &= \beta_i \left[\left(1 + \frac{\mu_i}{2}\right)|w^i|(w^i)^{-1} dw^i + \frac{\mu_i}{2}|w^i|^{-1} w^i dw^{\bar{i}}\right] \end{aligned}$$

and notice that it is not a holomorphic 1-form. The weighted Hölder spaces $C^{2,\alpha}_\beta$ is introduced in Donaldson [20]. We need more preparation before this space is defined. A function is said to be in C^α_β if it is Hölder continuous with respect to a Kähler cone metric. While, $C^\alpha_{\beta,0}$ denotes the subspace of C^α_β for which the limit of functions is zero along each component V_i for any $1 \le i \le m$. The Hölder continuous $(1,0)$-forms, in the local cone chart U_p, is of the shape

$$\xi = f_i \varepsilon_i + f_j dz^j, \tag{4.2}$$

where $f_i \in C^\alpha_0$ and $f_j \in C^\alpha$. Meanwhile, a Hölder $(1,1)$-form η in the local cone chart U_p is expressed as

$$\eta = f_{i_1 \bar{i}_2} \varepsilon_{i_1} \varepsilon_{\bar{i}_2} + f_{i\bar{j}} \varepsilon_i dz^{\bar{j}} + f_{\bar{i}j} \varepsilon_{\bar{i}} dz^j + f_{j_1 \bar{j}_2} dz^{j_1} dz^{\bar{j}_2}; \tag{4.3}$$

here the coefficients satisfy $f_{i\bar{j}}, f_{\bar{i}j} \in C_0^\alpha$ and $f_{i_1\bar{i}_2}, f_{j_1\bar{j}_2} \in C^\alpha$.

Definition 4.1 (Donaldson [20]) The Hölder space $C_\beta^{2,\alpha}$ is defined by

$$C_\beta^{2,\alpha} = \{ f \mid f, \partial f, \partial\bar{\partial} f \in C_\beta^\alpha \}.$$

Note that the $C_\beta^{2,\alpha}$ space, since it concerns only the mixed derivatives, is different from the usual $C^{2,\alpha}$ space. The definitions of the higher order space $C_\beta^{k,\alpha}$ require the covariant derivatives of the background metrics which result in the restriction of the cone angle. In order to overcome this difficulty, in Calamai–Zheng [10], we choose the flat cone metric ω_{cone} (3.1) to define $C_\beta^{k,\alpha}$. Notice that under the quasi-isometric mapping W, $\partial\bar{\partial} f \in C_\beta^\alpha$ is equivalent to $\frac{\partial^2}{\partial w^i \partial w^{\bar{j}}} \in C^\alpha$ for any $1 \le i, j \le n$ under the coordinate $\{w^i\}$. So a nature way define the third derivative of a function belongs to C_β^α is to require

$$\frac{\partial^3}{\partial w^k \partial w^i \partial w^{\bar{j}}} f \in C^\alpha$$

for any $1 \le i, j, k \le n$. In particular,

Definition 4.2 (Calamai–Zheng [10]) The Hölder space C^3 is defined by

$$C_\beta^3 = \{ f \mid f \in C_\beta^{2,\alpha} \text{ and the third derivative of } f \text{ w.r.t } \omega_{cone} \text{ is bounded} \}.$$

Following the same spirit, the higher order spaces are defined by induction on the index k.

In order to solve the geodesic equation (3.4), we need a weighted Hölder space on the whole product manifold $\mathfrak{X} = X \times R$. In the interior of $\mathfrak{X}$, we could define the same as the ones above. On the boundary of the product manifold, it is sufficient to defined a weighted Hölder space in the cone coordinates which contain the points of the divisor. We first note that the solution of geodesic equation is independent of the variable y^{n+1}, so the partial derivative on the variable x^{n+1} is the same to the one on the variable z^{n+1}. Next, the quasi-isometric mapping W is still well defined in U_p^+ as follows,

$$W(z^1, \ldots, z^{n+1}) := (w^1 = |z^1|^{\beta_1 - 1} z^1, \ldots, w^k = |z^k|^{\beta_k - 1} z^k, z^{k+1}, \ldots, z^{n+1}).$$

So we could define the Hölder space $C_\beta^\alpha(U_p^+)$ to be the set of functions which are Hölder continuous under $\{z^i\}_{i=1}^{n+1}$ with respect to a Kähler cone metric. Also, $C_{\beta,0}^\alpha(U_p^+)$ denotes the subspace of those functions in $C_\beta^\alpha(U_p^+)$ for which their limit is zero along V_i for any $1 \le i \le m$. The Hölder continuous $(1,0)$-forms, in local boundary cone coordinates U_p^+, can be expressed as (4.2), in which the coefficients $f_i \in C_0^\alpha(U_p^+)$ and $f_j \in C^\alpha(U_p^+)$. Meanwhile, a Hölder $(1,1)$-form η in local boundary cone coordinates U_p^+ is of the shape as (4.3) with the coefficients satisfying

$f_{i\bar{j}}, f_{\bar{i}j} \in C_0^\alpha(U_p^+)$ and $f_{i_1\bar{i}_2}, f_{j_1\bar{j}_2} \in C^\alpha(U_p^+)$. The Hölder space $C_\beta^{2,\alpha}(U_p^+)$ is parallelly defined by

$$C_\beta^{2,\alpha}(U_p^+) = \{f \mid f, \partial f, \partial\bar{\partial} f \in C_\beta^\alpha(U_p^+)\}.$$

Then we use the flat cone metric ω_{cone} (3.1) to define the higher order space $C_\beta^{k,\alpha}(U_p^+)$. The boundary C_β^3 space is defined in the same manner.

Definition 4.3 (Calamai–Zheng [10]) The Hölder space $C_\beta^3(U_p^+)$ is defined by

$$C_\beta^3(U_p^+) = \{f \mid f \in C_\beta^{2,\alpha}(U_p^+) \text{ and the 3nd derivative of } f \text{ w.r.t } \omega_{cone} \text{ is bounded}\}.$$

Thus the higher order spaces are also defined by induction on the index k in the same way.

5 Geodesics with Cone Singularities

When the ω_0-plurisubharmonic potentials of the geodesic equation are not smooth, which are merely bounded. The weak solution was constructed by Berndtsson.

Theorem 5.1 (Berndtsson [3]) *Given two bounded ω_0-plurisubharmonic potentials, there is a bounded geodesic connecting them.*

The cone geodesic we construct in Calamai–Zheng [10] has more regularity across the divisor in a subspace $\mathcal{H}_C$ (see the Definition 5.3 below) which still contains the critical metrics. The regularity of the cone geodesic across the divisor are not only important to prove the metric structure, but also to our further application on existence and uniqueness of cscK cone metrics.

We first explain the construct of the background metric and its geometric properties. Let $\tilde{\Psi}_0 = t\varphi_1 + (1-t)\varphi_0$ be the linear combination of the two boundary potentials. After choosing a sufficient convex function Φ which depends only on z^{n+1} and vanishing on the end point, we denote

$$\Psi_1 := \tilde{\Psi}_0 + m\Phi.$$

It is verified directly that the corresponding Kähler metric

$$\Omega_1 := \Omega + \frac{\sqrt{-1}}{2}\sum_{i,j=1}^{n+1} \partial_i \partial_{\bar{j}} \Psi_1$$

is a Kähler cone metric with the same cone angle to Ω on the product manifold $\mathfrak{X}$. In general, the Kähler cone metrics do not have bounded geometry. The following estimates are computed in Calamai–Zheng [10].

Proposition 5.2 (Calamai–Zheng [10]) *The following estimates hold.*

- *The connection of* ω *is bounded for* $0 < \beta < \frac{2}{3}$ *under the coordinate chart* $\{w^i\}$. *So is the connection of* Ω.
- *When* $0 < \beta < \frac{2}{3}$ *and* $\varphi_0, \varphi_1 \in C^3_\beta$, *the connection of* Ω_1 *is bounded.*
- *Suppose that* $\varphi_0, \varphi_1 \in C^3_\beta$ *have curvature lower (upper) bound. Then* Ω_1 *has also curvature lower (resp. upper) bound.*
- *Suppose that* $0 < \beta < \frac{2}{3}$, $\varphi_0, \varphi_1 \in C^3_\beta$ *and their Ricci curvature have lower (upper) bound. Then the Ricci curvature of* Ω_1 *also has lower (resp. upper) bound.*

Definition 5.3 (Calamai–Zheng [10]) Assume D are disjoint smooth hypersurfaces and the cone angles β belong to the interval $(0, \frac{1}{2})$. Then, we denote as $\mathcal{H}^3_\beta$ the space of C^3_β ω_0-plurisubharmonic potentials. Moreover, we label as $\mathcal{H}_C \subset \mathcal{H}^3_\beta$ one of the following spaces;

$$\mathfrak{I}_1 = \left\{\varphi \in \mathcal{H}^3_\beta \text{ such that } \sup Ric(\omega_\varphi) \text{ is bounded}\right\};$$

$$\mathfrak{I}_2 = \left\{\varphi \in \mathcal{H}^3_\beta \text{ such that } \inf Ric(\omega_\varphi) \text{ is bounded}\right\}.$$

In particular, when the cone angle is 1, i.e., the potentials are smooth. The geometric conditions of the endpoints of the geodesic which we require is the Ricci curvature. That improved Chen's theorem, where the uniform $C^{1,1}$ norm of the geodesic depends on the lower bound of the bisectional curvature of the endpoints (c.f. [5, 13, 14]). This advantage makes $\mathcal{H}_C$ contain more critical metrics with cone singularities. For example, the Kähler–Einstein cone metrics with cone angle less than $\frac{1}{2}$ are included in $\mathcal{H}_C$. This is specified in the last section.

In general, we could approximate the geodesic equation (3.4) by replacing the right hand side by a function $F(\epsilon)$ with a parameter ϵ and $\lim_{\epsilon\to 0} F(\epsilon) = 0$. We denote

$$\tilde{\Psi} = \Psi - \Psi_1.$$

In Calamai–Zheng [10], we considered the family of equations with parameter $a \in \mathbb{R}$ as

$$\begin{cases} \det(\Omega_{1i\bar{j}} + \tilde{\Psi}_{i\bar{j}}) = \epsilon \cdot e^{a\tilde{\Psi}} \det(\Omega_{1i\bar{j}}) & \text{in } \mathfrak{M}, \\ \tilde{\Psi}(z) = 0 & \text{on } \partial\mathfrak{X}. \end{cases} \tag{5.1}$$

We solved the approximation equation (5.1) with the boundary potentials in $\mathcal{H}_C$. In the following proposition we still use the particular family considered by Chen [13] with $a = 0$.

Proposition 5.4 (Calamai–Zheng [10]) *Let* $\mathcal{H}_C \subset \mathcal{H}^{2,\alpha}_\beta$ *be as in Definition 5.3. Also, let* $C_i := \varphi_i(s) : [0, 1] \to \mathcal{H}^{2,\alpha}_\beta$, *for* $i = 1, 2$, *be two smooth curves. Then, for a small enough* ϵ_0, *there is a two-parameter family of curves*

$$C(s, \epsilon) : \varphi(t, s, \epsilon) : [0, 1] \times [0, 1] \times (0, \epsilon_0] \to \mathcal{H}$$

such that the following properties hold:

1. *Fixed* s, ϵ, *then* $C(s,\epsilon) \in C^{2,\alpha}_{\beta}$ *is an* ϵ*-approximate geodesic from* $\varphi_1(s)$ *to* $\varphi_2(s)$.
2. *There exists a uniform constant* C *such that*

$$|\varphi| + \left|\frac{\partial \varphi}{\partial t}\right| + \left|\frac{\partial \varphi}{\partial s}\right| < C; \qquad 0 \le \frac{\partial^2 \varphi}{\partial t^2} < C; \qquad \frac{\partial^2 \varphi}{\partial s^2} < C.$$

3. *Fixed any* s, *the limit in* $C^{1,1}_{\beta}$ *of* $C(s,\epsilon)$ *as* $\epsilon \to 0$ *is the unique geodesic arc from* $\varphi_1(s)$ *to* $\varphi_2(s)$.
4. *There exists an uniform constant* C *such that*, *about the energy* $E(t,s,\epsilon)$ *along the curve* $C(s,\epsilon)$, *there holds*

$$\sup_{t,s}\left|\frac{\partial E}{\partial t}\right| \le \epsilon \cdot C \cdot Vol.$$

In which, E *is the energy defined by* $E := \int_0^1 \int_M |\varphi'(t)|^2 \omega^n_{\varphi(t)} dt$.

Theorem 5.5 (Calamai–Zheng [10]) *Any two Kähler cone metrics in* $\mathcal{H}_C$ *are connected by a unique* $C^{1,1}_{\beta}$ *cone geodesic. More precisely, it is the limit under the* $C^{1,1}_{\beta}$*-norm by a sequence of* $C^{2,\alpha}_{\beta}$ *approximate geodesics.*

As an application, we prove the following structure theorem of $\mathcal{H}_C$.

Theorem 5.6 (Calamai–Zheng [10]) $\mathcal{H}_C$ *is a metric space.*

6 Uniqueness of the cscK Metrics with Cone Singularities

In the smooth situation, Calabi [7] first proved the uniqueness of the Kähler–Einstein metrics when $C_1(X) < 0$ or $= 0$. When $C_1(X) > 0$, Bando–Mabuchi [1] proved that the Kähler–Einstein metrics are unique up to holomorphic diffeomorphisms.

On Fano manifold, a generalization of the Kähler–Einstein metric is the Kähler–Ricci soliton. A Kähler metric ω is called Kähler–Ricci soliton, if there is a holomorphic vector field X such that

$$L_X \omega = Ric - \omega. \tag{6.1}$$

Bando–Mabuchi's method was generalized to prove the uniqueness of Kähler–Ricci soliton.

Theorem 6.1 (Tian–Zhu [26]) *If* (ω, X) *and* (ω', X') *are two Kähler–Ricci solitons*, *then there is a holomorphic transformation group* $\sigma \in Aut_0(M)$ *such that* $\omega = \sigma^*\omega'$ *and* $X = \sigma_*^{-1} X'$.

In a general Kähler class, the (modified) K-energy plays an important role. It is discovered in Donaldson [17] that if there is a smooth geodesic between two extremal Kähler metrics, the convexity of the modified K-energy along this geodesic implies the uniqueness of the extremal Kähler metrics. Chen's $C^{1,1}$ geodesic is not enough for the computation of the convexity of the modified K-energy. After improving the partial regularity of the $C^{1,1}$ geodesic, Chen–Tian [11] prove the uniqueness of the extremal metric.

Theorem 6.2 (Chen–Tian [11]) *There is at most one extremal Kähler metric with Kähler class Ω modulo holomorphic transformations. Namely, if ω_1 and ω_2 are two extremal Kähler metrics with the same Kähler class, then there is a holomorphic transformation σ such that $\omega_1 = \sigma^*\omega_2$.*

Now we turn to the Kähler cone metrics. A Kähler–Einstein cone metric is defined to be a Kähler cone metric which satisfies

$$Ric(\omega) = \lambda\omega + 2\pi[D],$$

for a real number λ. In which, $[D]$ is the current defined by the divisor D. Let L_D be the associated line bundle of D. The necessary topological condition is

$$2\pi\lambda\big(C_1(X) - C_1(L_D)\big) > 0.$$

When $\lambda < 0$, the uniqueness of the Kähler–Einstein cone metrics are proved by Jeffres [21] by using the maximum principle for the Kähler cone metrics.

In the canonical Kähler class, Ding [16] defined a functional which is the Euler–Lagrangian function of the Kähler–Einstein equation. Compared with the K-energy, the Ding functional requires less regularity of the Kähler potentials. Moreover, Berndtsson [4] observed that the Ding functional is convex along the bounded geodesic. This is used to prove the uniqueness of twisted Kähler–Einstein metrics including the Kähler–Ricci solitons (Theorem 6.1) and also the Kähler–Einstein cone metrics. Berndtsson's method was further generalized to Kähler–Einstien metrics on a Q-Fano variety with log terminal singularities in Berman–Boucksom–Eyssidieux–Guedj–Zeriahi [2].

Generally, the uniqueness of the cscK cone metrics is not well understood yet. We considered this problem by using the convexity of the K-energy of the Kähler cone metrics. So we constructed the geodesic segment with cone singularities and applied it to obtain the uniqueness of the cscK cone metrics. From the following definition of the cscK cone metrics, we see that the Kähler–Einstein cone metrics are also included. Unaware of Berndtsson's method, we developed the cone geodesic to study the uniqueness problem of the Kähler–Einstein cone metrics. However, it would be more interesting to understand how to apply Berndtsson's method to the cscK cone metrics. Now we explain a natural definition of the cscK metrics with cone singularities and apply our cone geodesic to obtain a uniqueness theorem.

Definition 6.3 A Kähler cone metric in $\mathcal{H}_\beta$ is called a cscK metric with cone angle $2\pi\beta$ if its scalar curvature $S(\omega) = const$ outside D.

The constant appearing in the definition could be computed explicitly and depends on the Kähler class and the divisor, i.e.

$$\underline{S}_\beta = \underline{S} - \frac{Vol(D)}{Vol(X)}.$$

The Futaki invariant and the K-energy for the Kähler cone metrics were both introduced in Donaldson [20]. Assume X carries a circle action. Let H be the Hamiltonian of the circle action and D is invariant under the circle action. The Futaki invariant for the Kähler cone metrics is defined to be

$$Fut(X, D) = Fut(X) + \int_D H\omega^{n-1} - \frac{Vol(D)}{Vol(X)} \int_X H\omega^n.$$

While, the corresponding K-energy is its integral

$$\nu(\varphi) = E_\omega(\varphi) + \underline{S}_\beta \cdot D_\omega(\varphi) + j_\omega(\varphi) + \frac{n}{V} \int_0^1 \int_D \dot{\varphi}\omega_\varphi^{n-1} dt.$$

Here we use the decomposition formulae of the K-energy, which was discovered in [12, 25] in smooth case. In which, the first functional is the entropy of the Kähler metrics

$$E_\omega(\varphi) = \frac{1}{V} \int_X \log \frac{\omega_\varphi^n}{\omega^n} \omega_\varphi^n.$$

Recall Aubin's J-functional

$$J_\omega(\varphi) = \frac{1}{V} \sum_{i=0}^{n-1} \frac{i+1}{n+1} \int_X \partial\varphi \wedge \bar{\partial}\varphi \wedge \omega^i \wedge \omega_\varphi^{n-1-i}.$$

Then the second term is the Lagrangian functional of the Monge–Ampère operator which was defined in Ding [16],

$$D_\omega(\varphi) = \frac{1}{V} \int_X \varphi\omega^n - J_\omega(\varphi).$$

The third term is the j-functional given by

$$j_\omega(\varphi) = -\frac{1}{V} \sum_{i=0}^{n-1} \frac{n!}{(i+1)!(n-i-1)!} \int_X \varphi Ric(\omega) \wedge \omega^{n-1-i} \wedge \left(\frac{\sqrt{-1}}{2} \partial\bar{\partial}\varphi \right)^i.$$

The appropriate asymptotic behavior of the Kähler cone metrics is required to make all these formulas well-defined. In particular, we could choose $\varphi \in \mathcal{H}_\beta^{1,1}$ with appropriate $Ric(\omega)$. Meanwhile, when applying our cone geodesic, we also need to specify the appropriate asymptotic behavior of the ending points of the cone geodesic. This is our space $\mathcal{H}_C$. Here we restrict ourself to a special situation in order to illustrate the mechanism.

Theorem 6.4 *Suppose that the divisor is disjoint smooth hypersurfaces and the cone angle stay in* $(0, \frac{1}{2})$. *When* $C_1(X) = C_1(L_D)$, *the* C^{α}_{β} *cscK cone metrics (if exist) is unique.*

Proof The proof is divided into five steps.

Step 1: We first need to prove the asymptotic behavior of the C^{α}_{β} cscK cone metrics. We decompose the equation of the cscK cone metrics into a second order system,

$$\begin{cases} \det \omega_{\varphi} = e^{P} \det \omega, \\ \triangle_{\varphi} P = g^{i\bar{j}}_{\varphi} R_{i\bar{j}}(\omega) - \underline{S}_{\beta}. \end{cases}$$

From the second equation, applying Donaldson's Schauder estimate (Theorem 1 in [20]), we obtain that P belongs to $C^{2,\alpha}_{\beta}$. Then we use the first equation to prove that $\varphi \in C^{3,\alpha}_{\beta}$ and have bounded Riemannian curvature, then thus it lies in $\mathcal{H}_C$. The proof follows close to the argument in Brendle [6]. The estimate of the tangent direction holds by differentiating the first equation directly and using Theorem 1 in [20] repeatedly. The estimate along the normal direction follows from differentiating the equation in this direction and using the tangent estimates.

Step 2: Assume that φ_0 and φ_1 are two $C^{2,\alpha}_{\beta}$ cscK cone potentials. According to the Step 1, we apply Theorem 5.5 and connect them by our cone geodesic. Thus we have a family of the $C^{2,\alpha}_{\beta}$ approximation geodesic from Lemma 5.4. We also need to know the asymptotic behavior of this approximation geodesic. That is used in the computation of the convexity of the K-energy in the next step. Recall the approximation equation

$$\begin{cases} \det(\Omega_{1 i\bar{j}} + \tilde{\Psi}_{i\bar{j}}) = \epsilon \det(\Omega_{1 i\bar{j}}) & \text{in } \mathfrak{M}, \\ \tilde{\Psi}(z) = 0 & \text{on } \partial\mathfrak{X}. \end{cases}$$

The proof of the asymptotic behavior in the interior of $X \times R$ follows the same line to the first equation in Step 1. The boundary asymptotic behavior need Proposition 5.18 from Calamai–Zheng [10].

Step 3: According to Brendle [6], there exists a Ricci flat Kähler cone metrics when D is disjoint smooth and $0 < \beta < \frac{1}{2}$. Choose $h_{i\bar{j}}$ be the Ricci flat Kähler cone metric, so from its equation

$$Ric(h) = [D]$$

we simplify the third term $j_{\omega}(\varphi) = -\frac{n}{V} \int_0^1 \int_D \dot{\varphi} \omega^{n-1}_{\varphi} dt$ which cancels the last term in the formula of the K-energy. With the asymptotic behavior in the second step, using a lemma of the integration by part (Lemma 2.1 in Calamai–Zheng [10]), we could write down the second derivative of the K-energy along the approximation geodesic.

$$\nu'' = \int_X |\mathcal{D}\varphi'|^2 \omega^n_{\varphi} - \int_X S\big(\varphi'' - |\partial\varphi'|^2\big)\omega^n_{\varphi} + \frac{\underline{S}_{\beta}}{V} \int_X \big(\varphi'' - |\partial\varphi'|^2\big)\omega^n_{\varphi}.$$

Here the operator $\mathcal{D}$ is defined by $\mathcal{D}u = u_{,ij}$ in the local holomorphic coordinate.

Step 4: Let $f = \varphi'' - |\partial\varphi'|^2$, the second term becomes by integration by part

$$\int_X \big(|\partial \log f|^2 - g^{i\bar{j}} Ric(h)_{i\bar{j}}\big) f \omega_\varphi^n.$$

Since $h_{i\bar{j}}$ is the Ricci flat Kähler cone metric, so the second term vanishes as $\epsilon \to 0$. Then integrating along the geodesic, using the fact that the end points of the cone geodesic are cscK cone metrics and taking $\epsilon \to 0$, we arrive at

$$\int_0^1 \int_X \big(\big|\mathcal{D}\varphi'\big|^2 + |\partial f|^2 f^{-1}\big)\omega_\varphi^n dt = 0.$$

That implies that $g_\varphi^{i\bar{j}} \partial_{\bar{j}} \varphi' \partial_i$ is a holomorphic vector field on M. We remark that the local formula of $\mathcal{D}$ involves the gradient of the g_φ which is bounded w.r.t itself by Step 2.

Step 5: We need to prove when $C_1(X) = C_1(L_D)$, there exists only trivial holomorphic vector field. We first choose a ϵ-neighborhood D_ϵ of D and let $M_\epsilon = X \setminus D_\epsilon$. On M_ϵ, we have the Ricci identity,

$$\varphi'_{ij\bar{i}} = \varphi'_{i\bar{i}j} + R(h)_{j\bar{i}}\varphi'_i.$$

Since h is a Ricci flat Kähler cone metric, the second term on the right hand side vanishes. Multiplying the both sides of the formula above with $\varphi'_{\bar{j}}$, we have by integration by part

$$-\int_{M_\epsilon} \big|\varphi'_{ij}\big|^2 + \int_{\partial M_\epsilon} \varphi'_{ij}\varphi'_{\bar{j}} = -\int_{M_\epsilon} \big|\varphi'_{i\bar{j}}\big|^2 + \int_{\partial M_\epsilon} \varphi'_{i\bar{i}}\varphi'_{\bar{j}}.$$

Then from Step 2, we have $\varphi' \in C_\beta^{2,\alpha}$, therefore, both the boundary terms vanish when $\epsilon \to 0$. Furthermore, from Step 4, the first term on the left hand side also vanishes. This forces φ' to be zero. □

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Part VII
Didactics and Education

Organizers: Ewa Swoboda, Vladimir Mityushev

Teaching of Mathematics in Vocational Schools Upon 1951 Reorganisation

Ryszard Ślęczka

Abstract The radical reform in 1951 of organisation and policy in the vocational schooling in Poland is analyzed. The conclusive points for such a deep transformation of the vocational schooling were political and economical reasons discussed in the paper.

The radical reform of organisation and policy in the vocational schooling in Poland was conducted in 1951. It was prepared and carried out by Central Department of Professional Training which had been established in 1949. Solutions assumed in that time outlasted almost half a century, until the early nineties.[1] The conclusive points for such a deep transformation of the vocational schooling were political and economical reasons. Deep changes in many realms of life started when Polish Labour Party and Polish Socialist Party took over power in 1948. In the vocational schooling pre-war organisational structure, i.e. continuation and elementary schools (lower schools, professional junior high schools and professional secondary schools) and professional training schools, was abandoned. The new organisation included only three types of schools: six-month professional training schools, two-year elementary vocational schools and mostly three-year vocational technical colleges.[2] New solution was to base qualified workers education (elementary vocational schools) and technicians (vocational technical colleges) on programme fundamentals of seven-year primary school. Dynamical industrial development and new industry branches extorted shortening of the education cycle for the period between six months and three years. It was a time of great demand for high qualified employees. Professional education, by preparing adequate staff, was supposed to play significant role

[1]Polish vocational schooling based on two acts: *Statute on Education System* of 7th of September 1991 (Journal of Laws of the Republic of Poland 1991, No. 95, Ref. 425) and *Regulations Introducing School System Reform* of 08/01/1991 (Journal of Laws of the Republic of Poland 1999, No. 12, Ref. 96).

[2]Resolution of the Government Presiding Board of 23rd of June 1951 On Vocational Schooling System (Official Journal of the Republic of Poland 1951, No. A-59, Ref. 776).

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Table 1 One year learning plan in the 6.5-month professional training school. Source: *Appendix to the President of the Central Department of Professional Training of 25th of February 1952* [in:] *Zbiór przepisów prawnych*, p. 223

No.	Subjects	Weeks		Total number of hours
		1–14	15–27	
		Number of hours		
1.	Practicals	32	40	968
2.	Information on Production	8	4	164
3.	Study on Poland	2	2	54
4.	Polish Language	4	2	82
5.	Physical Education	2	0	28
6.	**Total**	**48**	**48**	**1296**
7.	Optionally			
8.	Sports	2	2	54

in industrialisation of the country, modernisation of agriculture and communication system building.

Discussion on other vocational schooling solutions, especially on its programmes, started however much earlier. Ministry of Education and 3rd Department of Vocational Schooling started working on the new form of vocational school organisation in March 1945. Their proposals assumed creating of a common programme and organisation base for all schools in order to provide unobstructed system of professional education. The Section for the Vocational Schools of Polish Teaching Society took significant part in a discussion on the future of the vocational schooling as well. The opinions of its members were diversified: some of them were for existent solutions, while the group with Jerzy Witkowski considered that the necessary solution was to establish three-year elementary vocational schools for all comprehensive school graduates. The conclusions of The Comission for the Vocational Schools in the frame of The Polish Education Convention in March 1945 in Łódź were of the most importance for the future of Polish vocational schooling. The Comission proposed organisational and programme unification of eight-class elementary school for the three-, four-, and five-year vocational schools which were supposed to prepare pupils to the future employment and enable them to learn in higher schools. The vocational schooling network basing on such standards was expected to take into consideration industrial and craft development needs as well as to facilitate each pupil to change the course of their education.[3]

In the professional training schools teaching plan subjects as: Practicals, Information on Production, Study on Poland, Polish Language, Physical Education and optionally Sports were introduced (see Tables 1 and 2).[4] In these plans mathematics

[3]The Polish Education Convention, Łódź, 18th–22nd of June 1945, Warsaw 1945, pp. 173–195.

[4]*Appendix to the President of the Central Department of Professional Training of 25th of February 1952* [in:] *Zbiór przepisów prawnych obowiązujących w szkolnictwie zawodowym w zakre-*

Table 2 Frame teaching plan for elementary vocational schools. Source: C. Siwiński, *Zmiany strukturalno-organizacyjne w szkolnictwie zawodowym PRL*, p. 107

Subjects	Classes		Total
	I	II	
A. Practicals	**18**	**21**	**39**
B. Professional and supporting			
Professional Technology	3	4	7
Study on Material	2	0	2
Professional Design	2	2	4
Physics	2	2	4
Mathematics	4	2	6
Total B	**13**	**10**	**23**
C. Comprehensive			
Polish Language	3	3	6
Study on Poland and Contemporary World	2	2	4
Russian Language	2	2	4
Physical Education	2	2	4
Total C	**9**	**9**	**18**
D. Religious Study	**1**	**1**	**2**
Total	**41**	**41**	**82**

as a distinct subject was not taken into account. Elementary mathematical contents related with physics, chemistry, geography and natural science were introduced in the frame of Information on Production.

For the elementary vocational school unified frame teaching plans were prepared at the beginning of 1952. It was considered as a model while creating teaching plans for various professional courses and specializations.[5] All of the subjects were divided into three groups: practicals, professional and supporting subjects and comprehensive subjects. Religious study was an optional subject.

In the frame teaching plan for elementary vocational schools 39 hours were devoted to practicals. Professional and supporting subjects, among which there was mathematics, were taught for 23 hours. It was presumed that 4 hours in the first and 2 in the second class would be sufficient. It is worth to emphasize that professional and supporting subjects were free of propaganda content according to practical and utilitary principles, as knowledge gained during the lesson was supposed to be used by performing some production assignments. The aim of teaching mathematics was

sie organizacji i administracji szkolnictwa zawodowego, organizacji wychowania i nauczania w szkolnictwie zawodowym oraz stosunków służbowych nauczycieli, J. Wójcie (ed.), Warszawa 1953, p. 223.

[5]C. Siwiński, *Zmiany strukturalno-organizacyjne w szkolnictwie zawodowym PRL*, Poznań 1981, p. 107.

to broaden and deepen information gained in primary school. Integers, decimal fractions, proportions, percentage, involutions, fractions and equations were taught. Depending on the teaching profile, mathematics was deliberately oriented. In economic schools the greatest attention was paid to proficient counting, addition, subtracting and the simple mathematics operations as well as cash register accounting, striking a day-, month- and year-balance. An integral part of mathematics was geometry with information on lines, triangles, squares, rectangles and figures. Accountancy exercises on calculating of figures area were also important part of teaching.[6]

New programmes and teaching plans for the technical colleges were introduced in the school year 1951/1952. In the teaching plans, in the elementary vocational schools likewise, all subjects were divided into three groups: professional and supporting subjects, comprehensive and optional subjects. In those schools the programme basis were professional and supporting subjects which had the crucial influence on the type of education. In economical technical colleges for instance such subjects were as follows: trade organisation and technique, commodity competence, planning, accountancy, political economy, construction organisation and economics, legal aspects of building industry. In this group of subjects, apart from Russian language, chemistry and physics, mathematics was also mentioned.[7] The fact that mathematics did not belong to the group of comprehensive subjects led in effect to the lowering of its meaning in the whole process of education in the technical colleges of different profiles and specialisations. The numbers of teaching hours of this subject was also diversified, for instance in the technical colleges there were 15 hours total per week and construction schools—18 hours total per week. Transferring mathematics from one group of subjects to another was also a common practice. In the teaching plans for the technical colleges for example mathematics belonged to the group of supporting, not professional subjects. Mathematics, the other teaching subjects alike, was strongly embedded in current social and political issues. Its role was to provide pupils with knowledge and skills which would allow them to understand organisation and techniques of work in an enterprise as well as to recognize modern industrial technologies. Mathematics content, as an universal matter, played a significant role during qualifying exams, as they allowed to verify various knowledge.

Analysis of the vocational schools teaching plans and programmes from 1951 to 1955 shows that mathematics was not included to the main teaching subjects at that time.[8] In the elementary vocational schools teaching was concentrated on practical skills. It emerged from the development of specialisation of production and diversification of production plants. For that reason education should have been

[6]Instruction on applying of the teaching hours plans and programmes in elementary vocational schools in the school year 1952/53, No. VIII PT, 4062/52.

[7]Schedule No. 445 from 1951, submitted and introduced by the Central Department of Professional Training.

[8]Conclusions based on analysis of the vocational schools teaching programmes and plans from 1951 to 1955.

strongly connected with narrow professional specialisation and oriented to practical knowledge. The principle of unobstructed education system which could have been supported by mathematics teaching however was forgotten. Similar situation was observed in the technical colleges. Teaching content was connected with social, political and economical reality. Great importance was attached to the detailed assignments which led new profiles and professional specialisations come into being. A number of the new teaching subjects was singled out and mathematics role was being decreased. In the teaching plans of different specialisations there were over thirty subjects. The professional profile of the graduate was more important than their education. In the technical colleges where the medium level technical staff was educated, mathematics had lower importance in the process of education. Great attention was paid to the professional subjects and practical training necessary for work in the fields of industry, trade and services, according to the education profile. The aim of technical college education in that period was not to prepare to academic level study but to precipitate education of the medium technical supervising staff according to the current economical needs.

Arithmetic in Polish Parish Schools in the Period of the Commission of National Education

Ryszard Ślęczka and Jan Ryś

Abstract The paper is devoted to historical description and analysis of the Polish primers in 16th–18th centuries and arithmetic in Polish primers until 1795. Conceptions of various schools and famous educators are discussed.

1 Introduction

The purpose of school education and preparation of the pupil acknowledged by the Commission of National Education was *so as to he was well and was well with him*. Education was about to make the ward happy and useful for others. This could be done through practical preparation for life and by providing knowledge appropriate for each state. It could not therefore lack in school teaching mathematical and natural sciences among which mathematics was in the first place. Members of the Commission of Education unanimously emphasized that learning this subject is needed at all levels of education, due to its usability and capacity of developing reasoning. Everyone agreed that it must be taught in a way so that the theory is closely associated with the practice. The Commission paid particular attention to the study of mathematics in secondary schools for which special textbook was developed. The subject was not neglected in parochial schools either where it was taught as arithmetic. This was reflected in the legislation of the Commission of National Education. The problem of teaching arithmetic was already widely reflected in the *Teacher's duties* of G. Piramowicz which became the official guide for parochial school teachers. Piramowicz advised teachers of arithmetic to start reckoning *from things belonging to senses* and express it in signs afterwards. He was supposed to use examples taken from rural or urban life. Studies should be conducted systematically and thoroughly. The teacher should not move to the new party of material if students have not mastered the current material. Teaching should avoid formalism and should be done in a manner so that the student comprehends in *rational*

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way and *notices the relationship between one truth and another*. He also warned a teacher against ambitions and desire to teach mathematics at a higher level than is intended for parochial schools.[1] The second document regulating the teaching of arithmetic in the parish schools were *Acts* of the Commission from 1783.[2] Chapter XXII contains a record that the programs of parochial schools should include *accounts, introduction to the sizes with the information on measures, weights and coins*.

Acts of Commission officially carried into effect what had already been practised in the Polish school. That included teaching arithmetic at the elementary level. Its scope and methodological solutions found place in Primer for the national parochial schools (1775) based on which we will present the substantive scope of teaching this subject and methodological solutions which were applied in it.

2 Polish Primers 16th–18th Century

Documented origins of Polish primers date back to the end of the first half of the 16th century. Several prints containing teaching reading have survived from this period which however does not exclude the possibility of existence the earlier period works of similar content. The first preserved Polish "primer" is considered to be the work of the Lutheran theologian John Seklucjan *Catechism is the finest teaching and necessary for salvation about the Christian faith* released in Königsberg in 1547. The title itself suggests that little work of Seklucjan was used to study the truths of faith and at the occasion reading as well to which the author devoted a few cards only. In the next edition of the Catechism already in the title there was a reference to the attached guidance on reading which indicates that the purpose of his was twofold.[3] Seklucjan joined these hints to his works of theological content in order to propagate the art of reading at least among his co-religionists.[4] Relatively late information about primers we get from the center of Polish science and education which in the 16th century was Krakow. However there are reasonable suspicions that such prints could have come out here in the early 16th century.[5]

[1] G. Piramowicz, *Duties of the teacher in parish schools and methods of their completion.* /in:/G. Piramowicz. *Duties of the teacher and selection of speeches and letters*, coll. and introduction K. Mrozowska, Wrocław 1959, 77–78.

[2] *Acts of the Commission of National Education written for academic state and for schools in Polish Republic*, /in:/*Commission of National Education (Commission writings and about the Commission)* coll. S. Tync, Wrocław 1954, 700.

[3] J. Seklucjan, *Catechismus…*, *Who needs short Writing and Reading course*, Königsberg 1549.

[4] J. Seklucjan, *First Part of the New Testament*, Königsberg 1551; J. Seklucjan, *New Testament Complete*, Königsberg 1553; J. Seklucjan, *Holy Gospel of Jesus Christ according to Saint Matthew*, Königsberg 1555.

[5] J. Pirożyński, *About Poznan's printer Peter Sextilis of Oborzyck and Polish primers from 16th century, Historical Studies*, XXVIII, nr 1, 8.

Noteworthy is the fact that in 1549 Hungarian[6] primer was released in Krakow. In 1550 *Polish–Latin Primer* appeared preserved unfortunately only in a few cards likewise *Short learning towards reading Polish handwriting* released at about the same time. Primer entirely preserved to our times entitled *Teaching toddlers how to read Polish handwriting* was released in Poznan in 1556.[7] It is small because only a four-page little book containing patterns of different typeface. Noteworthy is the primer of Krakow's printer and publisher Maciej Wirzbięta which was issued in his own print shop.[8] This bilingual primer was also used for religious education as evidenced by its extensive part of the catechism. The last Polish primer from the 16th century is *Learning to reading* published in Lviv with some interesting woodcuts of the content of the Bible.[9] The same content and layout has a primer released at the beginning of the 17th century in Krakow. It is however more diligent and graphically more elaborated.[10] The 17th century also brought several Polish–German primers.[11] Also Vilnius[12] got its first primer and interestingly appeared a primer for girls which unfortunately has not survived to our times.[13] The turning point in the history of education was the appearance of the work of J.A. Komeński *JanuaLinguarumreserata* and *Orbissensualiumpictus*.[14] These works as well as minor part of his writings had a great influence on the development of European teaching.[15]

In the first half of the 18th century primers were pressed in Poland only ten times. The most common primer was at that time *Elementapuerilis* issued four times.[16] This manual was used to teach reading in Polish and Latin. In the second half of this century, there are also bilingual Polish–German and Polish–French primers.[17] The most perfect work in the history of Polish primers to the end of the 18th cen-

[6] F. Pilarczyk, *Elementarze polskie od ich XVI—wiecznych początków do II wojny światowej*, Zielona Góra 2003, s. 71.

[7] See. J. Pirożyński, op. cit.

[8] M. Wirzbięta, *Elementaria institutio latini sermonis et pietatis Christianae*, Cracoviae 1575.

[9] *Learning reading Polish writing with figures.* Are applied devotional prayers and psalms, Lviv 1599.

[10] *For little children teaching Polish writing*, Kraków 1611.

[11] P. Glodius, *Catechism is a summa of Christian faith shortly in German and Polish collected for the exercise of young children*, Wrocław 1605, 167, 1615.

[12] *For little children teaching Polish writing*, Wilno 1633.

[13] *Studies to read Polish writing for young girls*, Kraków 1657.

[14] Works of Komeński were printed very often. The first edition of Janua... was in Leszno in 1631. During Komeński lifetime it was issued approximately 100 times similarly as *Orbissensualium...*, first edition in Nuremberg in 1658.

[15] See. Last edition: A. Fijałkowski, *Tradition and novelty in Orbissensualiumpictus of Jan Amos Komeński*, Warszawa 2012.

[16] *Elementa puerilis institutionis oluribus in locis reformata piisque orationibus Ac doctrina Christiana recenteraucta*, Cracoviae 1713.

[17] *New book to syllabify and read in Polish and German. NeuesBuchstabier Und Lesebuchleinpolnischunddeutsch*. Warszawa 1770; *New book to syllabify and read in Polish and French*, Warszawa 1770.

tury was already mentioned *Primer for the national parochial schools* containing: *I Teaching reading and writing; II Catechism; III Citizen Science; IV Teaching accounts*, Kraków 1875. Work was created under the patronage and at the request of the Commission of National Education. To the end of the Republic had eight releases. It was collective elaboration in which the teaching of reading expounded Onufry Kopczyński, Andrze Gawroński expounded bills, and the other two parts were the work of Gregory Piramowicz. On the background of contemporary teaching it was the work at European level.

One of the most entertaining elaborations from the end of the period which interests us should be mentioned the work of F. Paprocki, *Primer for male laid in a new useful way* (Łowicz 1777). The author, a follower of physiocratism, posted in his work outside of teaching reading a number of practical information useful in human life. In the same year appeared in Krakow work of M.D. Krajewski *Games of sciences for children*. Although this handbook contained a lot of interesting methodological solutions did not found wider recognition and went into oblivion. In contrast a reference to the sixteenth-century primers was *Primer białogşowski or information for learners to read the catechism of prayers for femalechildren* (Łowicz 1789). More progressive and innovative was the primer of S. Stawski designed for St. Elizabeth Secondary School in Wroclaw.[18] The primer contains a lot of practical information and a number of moralizing stories. The final primer issued prior to the third partition of Poland was primer of A.M. Prokopowicz.[19] The novelty in this work was to fit methodological guidelines for female teachers.

3 Arithmetic in Polish Primers Until 1795

Elementary school, regardless the historical period was supposed to teach how to read, write and count. This program was usually accompanied by religion classes. In the period of antiquity and the early Middle Ages the study of reading was taking place on literary texts which teacher had. In the course of time in school practice special school textbooks—primers (Latin.*elementarius*-initial) appeared. Primers were written mainly for purpose of teaching reading. Reading did not have to be related to the art of writing hence the first primers contained only typefaces. The closer to our time the more often there were tips in primers on how to write correctly. Even exercise books to learn calligraphy were introduced. Most primers also functioned as a catechism and a small prayer book hence the religious texts were occupying the greater part of the printing and could be used as material for exercises in reading. Nevertheless first primers lack tips in terms of arithmetic. We know certainly that the subject was taught from the beginning since schools existed. It was

[18] S. Stawski, *Booklet to syllabify and read for Polish class in St. Elizabeth Secondary School in Wroclaw*, Wrocław 1790.

[19] A.M. Prokopowicz, *The new easiest way to write and read together for the girls with footnotes for female teachers*, Kraków 1790.

taught in Greek schools and Plato made it the basis of teaching in the ideal state. It was taught in Rome where in the 3rd century BC appeared special teacher of the subject—calculator. It was taught in medieval schools often under the name komputu understood as bills and the art of Church calendar calculation.[20] The level of teaching this subject was variant. Italian schools usually hired a second teacher of mathematics called abbachista or maestro d'abbaco.[21] In Polish medieval schools arithmetic was often ignored or reduced to four mathematical operations.[22] We do not know what learning of arithmetic in Polish medieval schools looked like. We do not have information on whether teachers used some methodological materials, whether apart from traditional abacus and blackboard other teaching aids were used. It seems that they based on their own creativity and knowledge. This may be shown by the fact that none of mentioned primers published until the end of 17th century included guidance on teaching arithmetic. J.A. Komeński didn't post it either in *Janua Linguarum*. In *Orbis Sensualium Pictus* in part LXXIII he confined himself to the calculation of the benefits of learning arithmetic and geometry. Only in released in 1710 in Königsberg *Primer or the beginnings of science* there were *Latin and Arabic numbers* on the last card. The same solution contains *Elementa puerilis Institutionum* from 1713. Whereas mentioned Polish German and Polish French primers contain multiplication table on the last page.[23] Modest chapter *About the number* included F. Paprocki in his primer but given information does not go beyond the four basic operations. Dymitr M. Krajewski identified small chapter (13 pages) in his primer—*Arithmetic games*. The substantive scope of the subject also does not go beyond the four basic operations. Krajewski most of the chapter devoted to the reflection on ways of enjoyable learning. Learning arithmetic has to be fun. He therefore proposes to engrave *arithmetic letters* (numbers) on the ankles and assemble numbers from them creating rows of unity tens, hundreds and thousands.

Teaching of arithmetic was only treated wider by A.M. Prokopowicz[24] who certainly knew *Primer for national parochial schools* and adopted solutions. In his primer of arithmetic he devoted a chapter *Accounting for the ladies*. It starts with an explanation of *accounting science thus numerical* reaching into Greek word *arithmein*—count. Then shows record of numbers in Arabic and Roman system and the method of reading numbers consisting of several digits. Devotes separate place to each of four mathematical operations. Addition and subtraction in written system is supported by simple writing exercises in the form of word problems. In multiplication he distinguishes *multiplicand, multiplier and ratio*. He explains multiplication on the example of adding as it was explained in the primer for the national parochial

[20] O. Kanfer, *Teaching accounts in collegiate and convent schools* /in:/Report of Private Secondary E. Orzeszkowa School in Brody for 1930/1931, 4.

[21] P.F. Grendler, *Schooling in Renaissance Italy, Literacy and Learning 1300–1600*, Baltimore 1989, 22.

[22] J. Ryś, *Parish education in the cities of Little Poland in the 15th century*, Warszawa 1995, 69.

[23] See. Footnote 15.

[24] See. Footnote 17.

schools. After examples of written multiplication he goes to *the Pythagoras plate* so multiplication table. In division he distinguishes *dividend, divisor and quotient*. He publishes a table of division until the number 1118, much more difficult to use than multiplication. On the occasion of division he also discusses fractions arising as a result of division sharing with the remainder going further than it was in the aforementioned primer of the Commission of National Education. Closing the mathematical part of the primer is the discussion over *the rule of three* or *the golden rule* which was introduced to the practice of merchants in Europe between 15th and 16th century.

4 Arithmetic in Primer for National Parochial Schools

Primer issued under the patronage of the Commission of National Education was one of the best Polish primers therefore some more attention should be paid to him. Primer in the section of reading was based on analytic-synthetic method which in elementary education has been popularized by J.H. Pestalozzi. The fourth part of the primer on arithmetic by A. Gawroński is most extensive and consists of six chapters relating respectively to counting, addition, subtraction, multiplication, division, and the rule of three. The chapter on counting has been divided into four formulas. Formula I for counting is based on the principle of sight. The author recommended the use of peas, cereals and other items in counting. Students also had sets of dots from 1–9 in the primer. Under each set the appropriate digit was given. When they mastered the art of counting to nine the teacher walked to the formula II and introduced ten units of the order explaining that if each of the numbers will add up “circle” to the right then its value will increase ten times. Students should understand it using the tens of peas or corn. Subsequently the teacher discussed the mechanism of number formation consisting of tens and units based on illustrative material and writing numbers on the board. Further mathematic formulas relate to the creation of hundreds and thousands of rows and their multiplication. Chapters about adding and subtracting numbers are constructed similarly. Understanding the nature of these activities was based on the use in teaching already mentioned peas or corn. After the appropriate recording exercises the teacher walked to the written action, first using complex numbers consisting of tens later on adding units. In the following formulas he introduced a record of adding a few numbers composed of units, tens, hundreds and thousands. The next chapter concerns the multiplication of numbers. Explanation of the whole mechanism of multiplication is based on the addition. The teacher had to provide a written record of adding the same number, e.g. $5+5+5+5$ and then explain that this is equivalent to multiplying number five four times and immediately present it in written way. Once the basics of multiplication are mastered they are followed by the examples of written multiplication where the *number of multiplying* is initially in the form of unit then it takes more complex form. Gawroński does not use the term *multiplier* and *multiplicand* but only *the number of multiplying* and *ratio*. Also publishes a simple multiplication table in the text. At the end of the chapter provides a number of practical information on the various systems of weights, measures and currency. Chapter V about dividing the number is more complicated. Division is explained by subtraction that is what is happening by *subtracting one number from another as many times as may be deducted*. Next the author explains that in order to understand how many times the smaller number is located in the greater one we would have to subtract it as many times as possible. This would be a very bothersome and lengthy process and therefore the action of division was introduced. As in the case of multiplication sharing plate with a detailed description on how to use it is provided. Further examples concerned the division of more complex numbers after which students had to move to "share with the remainder" and write the remainder of the division created using a *numerical mark* or fraction number line. The author however does not introduce the concept of fractions. The last chapter was devoted to the rule of three and is very short. The rule of three, as Gawroñski explains, is the ability to search fourth number out of three obvious numbers. As he says: *Judgement of human nature itself gave way to conduct yourself in this action*.[25] He explains the method for calculating the proportion on practical examples from real life.

Primer as a whole met with positive response of the Commission and G. Piramowicz emphasized its practicality and affordability which was supposed to be aimed at the true enlightenment of the people and help them realize their obligations towards lords, peasants and themselves. As for arithmetic part he emphasized its clarity and accessibility for children by selecting appropriate *formula*[26] to their level. In Polish literature this work is not explicitly evaluated. Favorable opinions about it were expressed by M. Baraniecki, the author of several successful textbooks in mathematics in the 19th century.[27] It was negatively assessed by Z. Iwaszkiewiczowa[28] and T. Mizia.[29] However, as rightly pointed Cz. Majorek Primer has good and bad sides and its assessment should be centered.[30] Reservations can be aroused by graphic side of the study. An example might be the first chapter in which a small space has accumulated too many characters and images to organize for the child easily and accordingly.[31] In this case the transfer of individual images on the school board which was suggested by Gawroñski was necessary and could facilitate the pupils to master the material. Primer was also written with a view of a teacher and was supposed to suggest him some methodological solutions. Whereas

[25] *Primer for national parochial schools*, 118.

[26] G. Piramowicz, *Mowy miane w Towarzystwie do Ksiąg Elementarnych...*, wyd. W. Wisłocki, Kraków 1889, 141.

[27] M. Baraniecki, *Arithmetic*, Warszawa 1884.

[28] Z. Iwaszkiewiczowa, *Teaching arithmetic in schools of the Commission of National Education, /in:/The great reform epoch*, edited by S. Łempicki, Lviv 1923, 53–55.

[29] T. Mizia, *Parish education in the times of the Commission of National Education*, Wrocław 1964, 126–129.

[30] Cz. Majorek, *School books of the Commission of National Education*, Warszawa 1975, 152–153.

[31] Cz. Majorek, *School books...*, Warszawa 1975, 152–153.

students used it mainly as homework. It is not known whether it is typesetter's or Gawroński's fault that in case of unity multiplication unities are not under unities and tens are not under tens. Also the chapter about division is chaotically arranged and the rule of three is not supported by any examples of specific solutions. It is worth emphasizing the visual method, grading difficulties and Gawroński's aspirations to eliminate formalizing from arithmetic studies as well as rules avoidance and basing it instead on the natural rational child premises. The positive side of this part of the *Primer* was also an appropriate choice of tasks that involved the practice of everyday life and thus had to encourage children to learn arithmetic as extremely useful science in every area of life.

5 Conclusion

Above presented substantive range and ways of teaching elementary arithmetic during the times of the Commission of National Education are based on the *Primer…* and are rather visionary. We cannot say anything closer on school practice in this regard. Inspectors' reports are extremely scarce and conventional. Parochial schools were on the margin of CNE's activities although *Acts…* confirmed its authority over all the schools. Because of lack of funds they stayed under superiority and maintenance of Church. Introduction of Ordinal Commissions in 1789 did not change anything here as they were focusing on quantitative development of parochial schools instead of improving their educational level. Taking into account the fact that level of teaching mathematics in secondary schools especially in the early years of CNE was not the best, we cannot be too optimistic about the teaching of this subject in elementary schools. On the other hand if we consider the attempts to give the teaching useful form and elementary arithmetic knowledge, which did not require any special preparation to be taught, we can draw a conclusion about universality of teaching this subject in the Polish parochial schools.

Life as an Example. S.M. Nikolskij

Alexandr Rusakov

Abstract This paper is devoted to the outstanding Russian mathematician Sergey Mikhailovich Nikolskij. His biography and his famous results are outlined. A list of his pupils with last known positions is given.

Keywords Sergey Mikhailovich Nikolskij

Mathematics Subject Classification (2010) Primary 99Z99 · Secondary 00A00

Sergey Mikhailovich was born in Zavod Talitsa of Perm Governorate (now, the district center Talitsa of Yekaterinburg Oblast) 30 April 1905. His father, Mikhail Dmitrievich Nikolskij was a forester, a member of the Russian forest nomenclature. By bureaucratic report card, forester is a court counselor. In 1906 M.D. Nikolskij got a boost; he was nominated by a forester of Schebro-Olshansky forestry of Suwalki Province on the border with Prussia (now it belongs to Poland) where Sergei spent his childhood.

At age 14, S.M. Nikolskij and his family moved to the Voronezh region and he had to work. At the same time, he continued his education under the guidance of his father, who taught him mathematics, physics and natural sciences.

In 1921, the family of S.M. Nikolskij returned to Chernigov, where S.M. Nikolskij worked in Gubpolitprosvet and studied in college.

In 1925, he entered Ekaterynoslavsky University of Education at the Faculty of Mathematics and Physics. Soon, having went to come to a technical university considered more prestigious, he was so imbued creative atmosphere at the faculty that unconditionally decided to become a mathematician.

After graduation of the Faculty of Physics and Mathematics of the Ekaterinoslavskij University of National Education in 1930, Sergey Nikolskij got the assistant position. As the best lecturer, since 1932 he was the Head of the Department of Mathematics of the Transport Institute. He also worked in the Mine, pharmaceutical Institutions of Dnepropetrovsk.

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Andrei Nikolaevich Kolmogorov[1] involved Sergey Nikolskij into the scientific work. A.N. Kolmogorov came from Moscow to give lectures with Academician Pavel Sergeevich Alexandrov and Ivan Georgievich Petrovsky. Sergey Nikolskij was a PhD student at Moscow State University named after M.V. Lomonosov in 1934–1935 (MGU) and defended his thesis titled “Linear equations in Banach spaces”.

In 1940, S.M. Nikolskij entered the doctoral study of the Steklov Institute of Mathematics of the USSR Academy of Sciences and at the beginning of 1942 successfully defended his doctoral thesis (habilitation) on the theory of approximation of functions by polynomials. After, he was a senior fellow of the Institute of Mathematics. In 1947 he became a professor in the Department of Mathematics at the same institute, and from 1950 to 1954 he was the Head of the Department. From 1953 to 1961, he was deputy Head and from 1961 to 1989 Head of the Department of the theory of functions. In 1968, S.M. Nikolskij was elected a corresponding member of the Academy of Sciences of the USSR, and a full member in 1972.

The first studies of S.M. Nikolskij on the theory of linear operators (criterion discrete and continuous spectra, conditions for the Fredholm alternative, singular integral equations) were made at that time when the functional analysis of the Soviet Union only started to develop.

He established criteria for discrete and continuous spectra of the linear operator through the expansion of the resolvent operator corresponding to the sum of invertible and completely continuous operator, proved the presentation of such sums as a sum of the reversible and finite operators, found the necessary and sufficient conditions for the Fredholm alternative of the resolvent operator. These results due to Nikolskij had significant applications in the theory of singular integral equations, and subsequently served as the basis for the development of a branch of functional analysis for many other authors.

The Rector of MGU Academician V.A. Sadovnichij said in his report at the conference devoted to the 100th anniversary of Sergei Mikhailovich, that he completed a series of studies (which began back in the PhD thesis) related to linear equations in Banach spaces. Nikolskij generalized the theory of Banach equation $X - \lambda BX = Y$ to completely continuous linear operators. Banach’s theory is a generalization of the theory for continuous functions due to Riesz (1918) to arbitrary Banach spaces.

[1]Great Scientist of Russia, one of the greatest mathematicians of the twentieth century, recognized by almost all reputable scientific communities of the world, member of the US National Academy of Sciences and the American Academy of Arts and Sciences, a member of the Royal Netherlands Academy of Sciences and the Academy of Finland, member of the French Academy of Sciences and the German Academy of Naturalists “Leopoldina”, member of the International Academy of the History of Sciences and the National Academy of Romania, Hungary and Poland, an honorary member of the Royal Statistical Society of Great Britain and the London Mathematical Society, an honorary member of the International Statistical Institute and the Mathematical Society of India, a foreign member of the American Philosophical and the American Meteorological Society; winner of the most respected science prizes: Prize Chebyshev and N.I. Lobachevskij of the USSR Academy of Sciences, the International Balzan Prize Foundation International Award and the Wolf Foundation, as well as State and Lenin prizes, awarded seven Orders of Lenin and the Gold Medal of the Hero of Socialist Labor, Academician Andrei Nikolaevich Kolmogorov himself simply called professor at Moscow University.

S.M. Nikolskij set himself the task to carry out such a transfer from the space C in the framework of the theory due to Radon to the Banach spaces. It was done immediately when a space has a basis. These results were included in his PhD thesis. An analogous proof met difficulties when a space has no basis. S.M. Nikolskij overcame these difficulties later in 1940. Because of the World War II these results were published in 1943.

It is easily proved that every compact linear operator V in the Banach space with basis can be approximated by the norm with an arbitrary accuracy by a finite-rank operator K. This yields the representation

$$A + V = A' + K, \tag{0.1}$$

where A and A' are linear invertible operators.

Sergey Mikhailovich had masterly removed the additional condition about the basis and proved that any sum of the operators $A + V$, where A is invertible and V is compact, can be represented in the form (0.1). This Nikolskij's theorem (about Fredholm type operators) was published in Izvestia Akademii Nauk SSSR in 1943 in time of the World War II. Subtlety and significance of this result can be confirmed by a counterexample due to Per Enflo (live goose award in 1972) that there exists a compact operator not approximated by a limit of finite-rank operators. Nikolskij's theorem has a lucky continuation. It yielded a new approach on generalized Fredholm elements in operator rings in functional analysis (F.V. Atkinson and I.C. Gokhberg) and in the structural theory of rings with applications in the theory of elasticity.

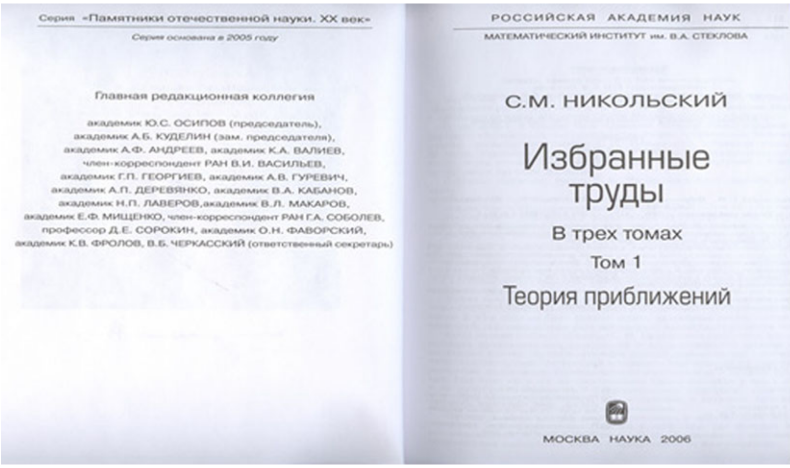
Nikolskij's theorem took its place in handbooks on functional analysis. In 1959, L.V. Kantorovich and G.P. Akilov includes it in the handbook "Functional analysis in normed spaces" and after in "Functional analysis", Moscow, Nauka, 1977. Nikolskij's theorem beautifies modern courses on functional analysis (see for example A.Ya. Khelemskij "Lectures on functional analysis", Moscow, MCNMO, 2004).

Other cycle of the deep works by Nikolskij on the theory of approximations (from forties to the present time) contains solution to difficult problems concerning asymptotic exact estimations for function approximations by trigonometric and algebraic polynomials.

The third cycle of his works is addressed to the theory of differentiable functions of several variables and their applications to partial differential equations. S.M. Nikolskij was first who obtained exact direct and inverse embedding theorems.

S.M. Nikolskij justified the variational method to solve the first boundary value problem for a class of equations generalizing (hypo) elliptic type equations. He had created the best quadrature formulas for some classes of functions and obtained exact estimations in certain cases.

S.M. Nikolskij also found conditions to continue continuously differentiable functions from sets to the whole space. Many such results were summarised in his monograph "Approximation of functions of several variables and imbedding theorems" Nauka, Moscow, 1969. The second edition was in 1975 and it was translated by Springer.



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Том 1

Теория приближений

МОСКВА НАУКА 2006

Fig. 1 Nikolskij's book from the series "Heritage of national sciences of the 20th century"

In recognition of the outstanding contributions to mathematics by S.M. Nikolskij (see Figs. 1–2 and works [1–5]), its objects, studied them, named after him. Mathematics all over the world are exploring space and inequality by Nikolskij.

In 2006, the three volumes of works by Nikolskij were published by the Academy of Sciences in the series "Heritage of national sciences of the 20th century". The most significant his results in the fundamental mathematics are selected in his three monographs "Approximations theory" (2006), "Functional spaces" (2007) and "Equations in functional spaces" (2009).

S.M. Nikolskij "There are theorems, people said that Nikolskij proved them. I remember that an idea how to prove it flashed during a ski trip or near Dnepr, or on the island where I swum, rested and thought."

S.M. Nikolskij prepared about fifty candidates physical and mathematical sciences, fifteen of his pupils became the doctor of physical and mathematical sciences (habilitation). Among his students there are such famous scientists as Corresponding Member of the Russian Academy of Sciences (RAS), member of the European Academy of Sciences O.V. Besov; Corresponding Member of the RAS, full member of the European Academy of Sciences L.D. Kudryavtsev, who was nominated in his 30 years by the Head of the Department of Mathematics of the leading university of the country, MFTI, due to the bold decision of his teacher justified by Kudryavtsev's seminal works (The example of Kudryavtsev demonstrates the power of the scientific school by S.M. Nikolskij); Prof. A.F. Timman, Corresponding Member of the Academy of Sciences of the USSR; V.K. Dzyadyk, NAS academician; N.P. Kornejchuk, Corresponding Member of NAS; V.P. Motornyj, Corresponding Member of the Academy of Sciences of the Kazakh SSR; T.I. Omanov, Professor S.V. Uspenskij; Professor V.I. Burenkov; emeritus professor of the Moscow State University, M.K. Potapov and others.

Fig. 2 Academician S.M. Nikolskij

PUPILS
of the Adviser of the Russian Academy of Sciences, Chief Scientific Researcher of the Mathematical Institute of the Russian Academy of Sciences Academician Sergey Nikolskij

Valentina Alkhimova, Dr., Dnepropetrovskiy gosuniversitet
Tyuleubay Amanov, Corresponding Member of the Academy of Sciences Kazakhstan, Dr., Professor of Institute of Mathematics and Mechanics, Semipalatinsk
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A. Gazar'yants, Dr., associate professor of the Baku University
I. Ginzburg, Dr., associate professor of the Dnepropetrovskiy University
I. Globenko, Dr., Novosibirsk
Gurevich, Dr., Moscow Automobile University
Vladislav Dzyadyk, Dr., Corresponding Member of the NAS of Ukraine, Head of the Institute of Mathematics of the NAS
Yuriy Doronin, Dr., associate professor of the Dnepropetrovsk Building University
Yuriy Kashchenko, Dr., associate professor of the Moscow Power Engineering University
Kiselev, Dr., associate professor of the Moscow University of Railway Engineers
A. Konyushkov, Dr., associate professor of the Moscow Engineering Physics University
Vladimir Kopchenov, Dr., associate professor of the Moscow Forestry University
Nicholas Kornejchuk, Dr., Prof., Academician of the NAS of Ukraine, Kiev
Lev Kudryavtsev, Corresponding Member of RAS
Ivan Matveev, Dr., associate professor of Moscow State University
Nina Mozzherova, Dr., associate professor of the Moscow Engineering Physics University
Vitaly Motor, Dr., Professor of Dnepropetrovsk State University
Kabduzh Nazryzbaev, Dr., Professor, Almaty
Victor Olovyanishnikov, Dr., associate professor of the Moscow Automobile and Road University
Petr Pilika, Dr., Head of Department at the University of Albania
V. Pinkevich, Dr., associate professor of Dnepropetrovsk State University
Mikhail Potapov, Dr., Professor of Moscow State University
Yusif Salmanov, Dr., Professor of Baku Pedagogical Institute
S. Selivanova, Dr., associate professor of the Moscow Engineering Physics University
Alexander Timman, Dr., Prof. Dnepropetrovsk State University
Stanislav Uspenskij, Dr., Prof., Head of the Moscow State University of Environmental Engineering
Alexander Foht Dr., associate professor of the Moscow Physical-Technical Institute
Vladimir Fufaev, Dr., associate Professor of the Moscow Physical-Technical Institute
Vladimir Shan'kov, Dr., associate professor of the Moscow Physical-Technical Institute
Shcherbin Alexander, Dr., Head of Dnepropetrovsk Medical University
Vladimir Yanchak, Dr., associate professor of the Lviv State University

Scientific achievements and activity of S.M. Nikolskij were highly recognised by the state. He was awarded orders and medals and state awards. S.M. Nikolskij was awarded by a gold medal named after Vinoradov of the USSR Academy of Sciences (1991), by the Chebyshev Prize of the USSR (1972), by Kolmogorov Prize of the

Fig. 3 School handbooks by S.M. Nikolskij

RAS (2000), by Gold Medal of the Bolzano Czech Academy of Sciences (1978), Medal, by Copernicus Polish Academy of Sciences (1992), by the Ostrogradskij Prize of the National Academy of Sciences of Ukraine (2000). He was a foreign member of the Hungarian and Polish Academy of Sciences, Honorary Member of the Moscow Mathematical and Kiev Mathematical Societies, professor emeritus of the MGU, professor emeritus of the Moscow Physical-Technical Institute.

Sergei Mikhailovich Nikolskij payed great interest in the problems of school education in mathematics and informatics. Last years, S.M. Nikolskij was actively engaged into development of school teaching. Great discussions took place after his talk at the International mathematical Congress on Mathematical Education in 2004 in Copenhagen.

Sergei Mikhailovich believed that good mathematics in schools can be based only on a solid foundation of freely use of arithmetic; that arithmetic is the basic of the logical science, and that its proper study forms not only to calculate, but also the ability to think logically, and thus gives perspective to other disciplines, algebra and geometry.

His is an author of many handbooks for secondary and high school (see Fig. 3).

It was unexpected to hear the words: "April 30, 2005 Sergei Mikhailovich has 100 years." In his congratulatory speech devoted to Sergei Mikhailovich, the President of the Polish Academy of Sciences Andrzej B. Legocki said: "You, in particular, supported new ideas developed by Stefan Banach, and had the opportunity to meet with him until his death. It was lucky that you played a remarkable role in the creation of an International Center in Warsaw named after Stefan Banach, well-known today in the world mathematics. It was also your personal achievement. You were undoubtedly many years a central figure in the Scientific Council of the Centre Banach. I can assure you that the Polish Academy of Sciences is proud that you are its foreign member. Hundredth anniversary of the birth, this is quite an exceptional event, the privilege of those who, like you enthralls environment unprecedented vigor, cheerfulness and kindness. Sure, it's a big celebration not only for you, I assure you, that also for our Academy. We are happy that you are a foreign member of our Academy. We are well aware of your scientific achievements in mathematical analysis and its applications. On the occasion of your hundredth anniversary of the birth we want to stress your contribution to the teaching of mathematics in recent years. Teaching for you is so important as research."

Accuracy and commitment in his work were motor force of Sergei Mikhailovich, which give freedom and freshness of his thoughts. His organization, allow him to carry out clear and accurate generalization. It should be added that the S.M. Nikolskij was a man with a strong character; the power of his spirit never leaves him. His characteristic feature was ability to give himself to the affair that he intended and performed.

We touch here few events of the century way of life of Sergei Mikhailovich. We always astonished his active life position, unbending character, personal charm.

With him, we were much younger.

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The Area Method and Proving Plane Geometry Theorems

Martin Billich

Abstract The process of proving, deriving and discovering theorems is important in mathematics investigation. In this paper, we will use the elimination technique which is based on the theory of the area method. The main idea of this method will be illustrated through an example from plane geometry. In addition, we look at the application possibilities of using GCLC geometry system with built-in theorem prover in verification and proving constructive geometric statements.

Keywords Area method · Theorem proving · Geometric constructions

Mathematics Subject Classification (2010) Primary 03F99 · Secondary 97N80

1 Introduction

The unique feature that sets mathematics apart from other sciences, from philosophy, and indeed from all other forms of intellectual discourse, is the use of rigorous proof. It is the proof concept that makes the subject coheres, and that gives it its timelessness. There is relatively clear definition of what a proof is. A proof of a theorem is a finite sequence of claims, each claim being derived logically (i.e. by substituting in some tautology) from the previous claims, as well as theorems whose truth has been already established. The last claim in the sequence is the statement of the theorem, or a statement that clearly implies the theorem (see for example [2]).

In this paper we will look at the application possibilities of the area method, an efficient synthetic (semi-algebraic) method for a fragment of Euclidean geometry in proving of constructive geometric statements.

2 The Area Method and Constructive Geometric Statements

The area method for plane geometry was developed by Chou, Gao, and Zhang [1]. The basic idea of the method is to express hypotheses of a theorem using a set of

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Table 1 Common geometric notions

Properties	Formalizations
Points A, B, C are collinear	$\|ABC\| = 0$
AB is parallel to CD	$\|ACD\| = \|BCD\|$
M is the midpoint of AB	$\|ABC\| = 0 \wedge \frac{\|AB\|}{\|AM\|} = 2$
Points A, B, C, D are harmonic	$\frac{\|AC\|}{\|CB\|} = \frac{\|DA\|}{\|DB\|}$

constructive statements, each of them introducing a new point (as intersection of lines, etc.), and to express a conclusion by an equality of arithmetic expressions, without considering Cartesian coordinates, using only three geometric quantities: *the ratio of directed segments*, *the signed area of a triangle* and *the Pythagoras difference* (for details see [5, 6]). In the rest of the paper, we denote by $\|AB\|$ *the signed length* of a segment AB (the length $|AB|$, endowed by the sign + or −, i.e. $\|AB\| = -\|BA\|$) and we denote by $\|ABC\|$ *the signed area*[1] of a triangle ABC. Expressing some common geometric notions using ratios and signed area is given in Table 1.

2.1 Geometric Constructions

The area method is used for proving constructive geometry conjectures, i.e. for statements about properties of objects constructed by some fixed set of elementary constructions, which have a specific form $(C_1, C_2, \ldots, C_m; G)$, where C_i, for $1 \le i \le m$, are elementary construction steps, and the conclusion G of statements is of the form $E_1 = E_2$, where E_1 and E_2 are arithmetical expressions containing only geometric quantities (signed areas and ratios) [6]. For each constructed point there is some construction C_i stating how it has been constructed.

Then the proof is based on eliminating (in reverse order) the points introduced before from the goal using a set of appropriate *elimination lemmas*. After eliminating all introduced points (and changing the goal into an expression containing only *independent* geometric quantities), the current goal becomes a trivial equality that can be simply tested for validity.

2.2 Basic Lemmas

Here we present only two lemmas those are the base for the area method and will be used in the next section of the paper (see, for instance, [1, 6] for a survey).

Lemma 2.1 *Let A, B, and C be three collinear points such that $\|AB\| = r\|AC\|$ $(r \in \mathbb{R})$. Then for any point P, we have $\|PAB\| = r\|PAC\|$.*

[1]The signed area of a triangle is the area of a triangle with a sign depending on its orientation in the plane. We have anticlockwise, positive sign, and clockwise, negative sign.

Remark The signed area of a quadrilateral $ABCD$ can be defined in the following way:

$$\|ABCD\| = \|ABC\| + \|ACD\|.$$

Lemma 2.2 (The Co-side Theorem) *Let M be the intersection of two non-parallel lines AB and PQ and $M \neq Q$. Then it holds that*

$$\frac{\|PM\|}{\|QM\|} = \frac{\|PAB\|}{\|QAB\|}; \qquad \frac{\|PM\|}{\|PQ\|} = \frac{\|PAB\|}{\|PAQB\|}. \tag{EL1}$$

The Co-side Theorem is one of the most important elimination lemmas for the area method. From (EL1) it follows that the point M can by eliminated by the substitution from the ratio of directed parallel segments by ratio of two signed areas, not involving M.

3 Computer Proof Checking

There is a range of geometry software tools, covering different geometries and geometry problems. Dynamic geometry software (DGS) visualizes geometric objects and link formal, axiomatic nature of geometry with its standard models (e.g., Cartesian model) and corresponding illustrations. Ones can use it to construct geometric objects, observe their changes by moving free points or applying Euclidean transformations, and then discover some conjectures. We can see with our own eyes that this or that geometrical theorem is true.

Some DGS provide proof feature by combining with automated geometry theorem proving (AGTP), allow users to verify conjectures. They relies on several efficient automatic proof methods, such as *Gröbner bases method*, *Wu's method*, *Area method* and *Full-angles method* (see [1, 4, 7]). The first two ones are algebraic methods which use polynomials to solve problems, that is, they first transform geometric properties into equations in coordinates of the related points and then deal with these equations. The last two ones can produce human-readable proofs, that is, each step of the generated proof has clear geometric meanings.

We will not give an overview of AGTP here, instead we present only some informations of the GCLC system as a tool for visualizing geometrical (and not only geometrical) objects and notions, and for producing traditional proofs with the geometry theorem prover (*GCLCprover*) which is built in GCLC and based on the area method. For more details about the prover and GCLC system, see [3, 5].

3.1 GCLC System

GCLC is a tool for visualizing objects and notions of geometry. GCLC uses the specific GC language for describing figures. GC language consists of the following groups of commands: *definitions, basic constructions, transformations, drawing* *commands, marking and printing commands, low level commands, Cartesian commands, commands for describing animations, commands for the geometry theorem prover*. These descriptions are compiled by the processor and can be exported to different output formats. There is also support for symbolic expressions, for parametric curves and surfaces, for drawing functions, graphs, and trees.

3.2 GCLCprover

The GCLCprover, as an integral part of GCLC system, can be very useful in testing conjectures. This means that one can use the prover to reason about objects introduced in GCLC construction, without changing for verification, i.e. we need to add only the conclusion that will be proved.

The theorem prover can prove any geometry theorem expressed in terms of geometry quantities, and involving only points introduced by using the commands: `point`, `line`, `intersec`, `midpoint`, `med`, `perp`, `foot`, `parallel`, `translate`, `towards`, `online`.

4 Worked Example

In this section we give a detailed description of how the elimination technique for area method works on the following example:

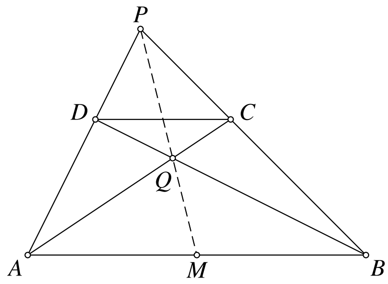
Example Show that the line joining the point of intersection of the extension of the non-parallel sides of a trapezoid to the point of intersection of its diagonals bisects the base of the trapezoid.

Construction Let $ABCD$ be a trapezoid with the parallel sides AB and CD (Fig. 1). The points A, B, and C are free points (points not defined by construction steps) and D is the point on the line passing through C and parallel to AB. The point P is the intersection of the extension of sides BC and AD and Q is the intersection of AC and BD. Let M be the intersection of the line PQ with the side AB. The GCLC code for our construction is shown in Fig. 2. We need to show that M is the midpoint of AB, that is equivalent to the condition

$$\frac{\|AM\|}{\|BM\|} = -1. \tag{4.1}$$

Proof We can eliminate the constructed points P, Q and M (in reverse order), using for that purpose the properties of the geometric quantities, until an equality in only

Fig. 1 Trapezoid $ABCD$



the points A, B, C, and D is reached. We first eliminate M using the co-side theorem (Lemma 2.2):

$$\frac{\|AM\|}{\|BM\|} = \frac{\|APQ\|}{\|BPQ\|}.$$

By the same theorem, we can eliminate the point Q:

$$\frac{\|APQ\|}{\|APC\|} = \frac{\|AQ\|}{\|AC\|} = \frac{\|ABD\|}{\|ABCD\|}$$

$$\Rightarrow \|APQ\| = \frac{\|APC\| \cdot \|ABD\|}{\|ABCD\|}$$

and

$$\frac{\|BPQ\|}{\|BPD\|} = \frac{\|BQ\|}{\|BD\|} = \frac{\|BCA\|}{\|BCDA\|} = \frac{\|ABC\|}{\|ABCD\|}$$

$$\Rightarrow \|BPQ\| = \frac{\|BPD\| \cdot \|ABC\|}{\|ABCD\|}.$$

The new goal is

$$\frac{\|AM\|}{\|BM\|} = \frac{\|APC\|}{\|BPD\|} \cdot \frac{\|ABD\|}{\|ABC\|}.$$

Applying Lemma 2.1, we can eliminate P:

$$\frac{\|APC\|}{\|ADC\|} = \frac{\|AP\|}{\|AD\|} = \frac{\|BP\|}{\|BC\|} = \frac{\|BPD\|}{\|BCD\|}$$

$$\Rightarrow \frac{\|APC\|}{\|BPD\|} = \frac{\|ADC\|}{\|BCD\|}.$$

It obvious, that $\|ABD\| = \|ABC\|$ (from $AB\|CD$) and $\|ADC\| = -\|BCD\|$, and we have the goal in the form

$$\frac{\|AM\|}{\|BM\|} = \frac{-\|BCD\|}{\|BCD\|} \cdot \frac{\|ABC\|}{\|ABC\|} = -1$$

and the proof is completed. □

Fig. 2 GCLC code for the example with trapezoid

```
point A 10 10
point B 60 10
point C 40 30
towards X A B 0.6
translate D B X C

intersection P A D B C
intersection Q A C B D
intersection M A B P Q

cmark_lb A
cmark_rb B
cmark_rt C
cmark_lt D
cmark_t P
cmark_lb Q
cmark_b M

drawsegment A B
drawsegment B P
drawsegment C D
drawsegment P A
drawsegment A C
drawsegment B D
drawsegment P M
```

Theorem Prover It can be checked (with GCLCprover) that M is the midpoint of AB. This statement can be given to the prover by adding line

```
prove { equal { sratio A M B M } {-1} }
```

to the code given in Fig. 2. The prover produced a short report of information on number of steps performed, on CPU time spent and whether or not the conjecture has been proved (Fig. 3).

The prover generate also proof in LaTEX form and in XML format. We can control the level of details given in generated proof. The proof consists of proof steps. For each step, there is an explanation and its semantic counterpart. This semantic information is calculated for concrete points used in the construction. In addition, at the end of the main proof all non-degenerative conditions are listed.

5 Conclusion

Theorems with their proofs are at the core of mathematics and play an important role in the working of mathematicians. The area method is a synthetic method providing short and human-readable proofs for one class of geometry statements. This method works for constructive geometric statements and is one of the most successful methods for automated geometry theorem proving. We presented some advantages of the area method and GC language for explicit describing construction in Euclidean plane. The theorem prover which is implemented in GCLC system, can

```
The theorem prover based on the area method used.
    Number of elimination proof steps:    10
    Number of geometric proof steps:      19
    Number of algebraic proof steps:      68
    Total number of proof steps:          97
Time spent by the prover: 0.002 seconds
The conjecture successfully proved.
The prover output is written in the file trapezoid.tex.
```

Fig. 3 Report for the example with trapezoid automatically prove a number of geometry theorems in plane geometry. This system provides an environment for modern ways of studying and teaching geometry.

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Part VIII
Clifford, Quaternion and Wavelet Analysis

Organizers: Keiko Fujita, Akira Morimoto, Swanhild Bernstein, Uwe Kahler, Irene Sabadini, Frank Sommen

Redundant Multiscale Haar Wavelet Transforms

Kensuke Fujinoki

Abstract We consider a redundant lifting scheme for Haar wavelet transform that does not use the polyphase decomposition. We also extend the method to a two-dimensional triangular lattice, and define a nonseparable two-dimensional redundant Haar wavelet transform on the lattice.

Keywords Wavelets · Lifting scheme · Redundant transform

Mathematics Subject Classification (2010) Primary 42C40 · Secondary 65T00

1 Introduction

The lifting scheme of Sweldens [1, 2] has been widely used in a range of applications, as it can provide a particularly easy way to construct perfect reconstruction filters that are defined even on general domains such as irregular grids over arbitrary surfaces [3, 4]. In particular, any discrete wavelet transform with finite impulse response filters can be decomposed into a finite sequence of simple lifting steps [5]. The computation of the wavelet decomposition or reconstruction implemented via the lifting is efficient and thus fast, because it uses the polyphase decomposition that divides a one-dimensional discrete signal into even and odd components, which is also called decimation or downsampling by a factor of two [6]. However, due to the nature of the polyphase decomposition, this leads to a large number of artifacts when the signal is reconstructed after modification of its wavelet coefficients. In this paper, we consider a redundant lifting scheme for Haar wavelet transform that does not use the polyphase decomposition in both one and two dimensions.

2 Haar Wavelet Transform with Lifting

For a discrete one-dimensional signal $c_j[k]$, $k, j \in \mathbb{Z}$ with a resolution level $j \geq 0$, the lifting first splits the signal into even indexed samples $c_j[2k]$ and odd indexed

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samples $c_j[2k+1]$, which are completely disjoint components each other. The odd indexed component $c_j[2k+1]$ is predicted by the constant prediction that only uses the even indexed component $c_j[2k]$, and $c_j[2k+1]$ is replaced as a detail component $d_{j+1}[k]$ which is the error of the prediction:

$$c_j[2k+1] \to d_{j+1}[k] = c_j[2k+1] - c_j[2k]. \tag{2.1}$$

Then the even indexed component $c_j[2k]$ is updated to a coarse component $c_{j+1}[k]$ by using $c_j[2k]$ and results of the prediction $d_{j+1}[k]$:

$$c_j[2k] \to c_{j+1}[k] = c_j[2k] + \frac{d_{j+1}[k]}{2}. \tag{2.2}$$

Finally the outputs $c_{j+1}[k]$ and $d_{j+1}[k]$ are rescaled by $\sqrt{2}$ and $1/\sqrt{2}$ respectively, if we need the energy normalization of the coefficients $\|c_j\|^2 = \|c_{j+1}\|^2 + \|d_{j+1}\|^2$.

These lifting steps are easily inverted by undoing each update and predict step with flipping the sign:

$$\begin{aligned} c_j[2k] &= c_{j+1}[k] - \frac{d_{j+1}[k]}{2}, \\ c_j[2k+1] &= d_{j+1}[k] + c_j[2k]. \end{aligned} \tag{2.3}$$

One can see the efficiency of the lifting against the direct implementation by the Mallat algorithm [7],

$$\begin{aligned} c_{j+1}[k] &= \sum_l h[l-2k]c_j[l], \\ d_{j+1}[k] &= \sum_l g[l-2k]c_j[l], \end{aligned} \tag{2.4}$$

$$c_j[k] = \sum_l \big(\tilde{h}[k-2l]c_{j+1}[l] + \tilde{g}[k-2l]d_{j+1}[l]\big), \tag{2.5}$$

with filters $h[k] = \tilde{h}[k] = \{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\}$ and $g[k] = \tilde{g}[k] = \{-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\}$.

3 Redundant Haar Lifting

The redundant lifting scheme that we consider here does not use the split step, which means that there is no distinction between even and odd indexed components. In our redundant scheme, the $c_j[k]$ is treated as an even-like component and $c_j[k+2^j]$ as an odd-like component, in stead of the even $c_j[2k]$ and the odd $c_j[2k+1]$. The redundant version of the constant prediction (2.1) now becomes simply

$$d_{j+1}[k] = c_j\big[k+2^j\big] - c_j[k], \tag{3.1}$$

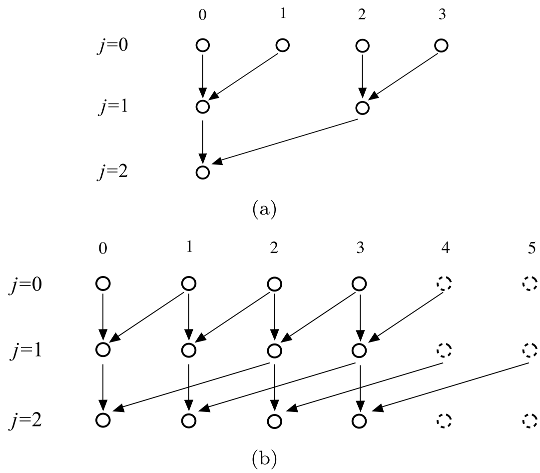


Fig. 1 A schematic illustration of constant prediction steps of (**a**) normal lifting and (**b**) redundant lifting

where j starts from 0. Note that the index of the odd component $c_j[k+2^j]$ varies depending on the resolution level j. A similar setting that we use here for the odd component $c_j[k+2^j]$ can also be seen in the à trous algorithm [8, 9] that realizes the undecimated Mallat transform. The redundant lilting described here is different from that of [10], which apply twice the step of predict and update to implement the redundant transform.

The update step (2.2) is then written as

$$c_{j+1}[k] = c_j[k] + \frac{d_{j+1}[k]}{2}. \tag{3.2}$$

In the case of the normal lifting, undoing each lifting step (2.3) yields the original signal of $c_j[2k]$ and $c_j[2k+1]$. However, in the redundant case, since the even-like component $c_j[k]$ also contains the odd-like component $c_j[k+2^j]$, we can recover the original signal $c_j[k]$ only reverting the update step:

$$c_j[k] = c_{j+1}[k] - \frac{d_{j+1}[k]}{2}, \tag{3.3}$$

which implies that we have a higher degree of freedom when designing the synthesis filters.

Figure 1 shows the redundant lifting for the constant prediction compared with the standard lifting. Unlike in the standard prediction, one can recognize that the resolution of each decomposed signal $c_{j+1}[k]$ and $d_{j+1}[k]$ is maintained in the redundant lifting. Furthermore, this redundancy enables to build the discrete wavelet transform with translation-invariant property. It also makes the reconstruction robust to the ringing artifact, which can be very important in some applications such as feature analysis [11].

4 Haar Lifting on the Triangular Lattice

The triangular wavelets [12, 13] are defined on a triangular lattice Λ generated by a linear combination of two vectors $\boldsymbol{t}_1 = (1\ 0)^T$ and $\boldsymbol{t}_2 = (-\frac{1}{2}\ \frac{\sqrt{3}}{2})^T$. The reciprocal lattice $\widetilde{\Lambda}$ that corresponds to the Fourier domain is similarly generated with vectors $\boldsymbol{\lambda}_1 = (0\ \frac{2}{\sqrt{3}})^T$ and $\boldsymbol{\lambda}_2 = (1\ \frac{1}{\sqrt{3}})^T$. We also define $\boldsymbol{t}_0 = \boldsymbol{0}$, $\boldsymbol{t}_3 = -\boldsymbol{t}_1 - \boldsymbol{t}_2$, and $\boldsymbol{\lambda}_3 = \boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2$ for notational convenience.

A discrete signal, $c_j[\boldsymbol{t}], \boldsymbol{t} \in \Lambda$ defined on the lattice, is represented with its four polyphase components in the Fourier domain as

$$\hat{c}_{m,j}(\boldsymbol{\omega}) = \sum_{\boldsymbol{t} \in \Lambda} c_j[2\boldsymbol{t} + \boldsymbol{t}_m] e^{-i\boldsymbol{\omega}\cdot\boldsymbol{t}}, \quad m = 0, 1, 2, 3,\ \boldsymbol{\omega} \in \mathbb{R}^2.$$

This implies that we have one even component $c_j[2\boldsymbol{t}]$ and three odd components $c_j[2\boldsymbol{t} + \boldsymbol{t}_k]$, $k = 1, 2, 3$. Note that in one dimension there only exist one even $c_j[2k]$ and one odd $c_j[2k+1]$ components.

Correspondingly, a straightforward generalization of the wavelet transform shows that a signal $c_j[\boldsymbol{t}]$ is decomposed into a coarse component $c_{j+1}[\boldsymbol{t}]$ and three detail components $d_{k,j+1}[\boldsymbol{t}]$, $k = 1, 2, 3$, of half a resolution. This can be written via the lifting form generalized to two dimension; in the Haar case, three detail components are obtained using three constant predictions for each direction $\boldsymbol{t}_k$:

$$d_{k,j+1}[\boldsymbol{t}] = c_j[2\boldsymbol{t} + \boldsymbol{t}_k] - c_j[2\boldsymbol{t}], \quad k = 1, 2, 3, \tag{4.1}$$

and a coarse component is given by using the results of each prediction:

$$c_{j+1}[\boldsymbol{t}] = c_j[2\boldsymbol{t}] + \frac{1}{4} \sum_{k=1}^{3} d_{k,j+1}[\boldsymbol{t}], \tag{4.2}$$

which preserves the average of a two-dimensional signal. Finally the normalization steps are applied for energy normalization. Repeating the procedures of the predict (4.1) and the update (4.2) steps up to a resolution level $L > j$ produces the following multiscale coefficients

$$c_j[\boldsymbol{t}] \to \left\{ d_{k,j+1}[\boldsymbol{t}], d_{k,j+2}[\boldsymbol{t}], \ldots, d_{k,L}[\boldsymbol{t}], c_L[\boldsymbol{t}] \right\}, \quad k = 1, 2, 3.$$

Analogous to the one-dimensional case, we can also build the update and predict for the inverse transform, which can be written respectively as

$$c_j[2\boldsymbol{t}] = c_{j+1}[\boldsymbol{t}] - \frac{1}{4} \sum_{k=1}^{3} d_{k,j+1}[\boldsymbol{t}],$$

and

$$c_j[2\boldsymbol{t} + \boldsymbol{t}_k] = d_{k,j+1}[\boldsymbol{t}] + c_j[2\boldsymbol{t}], \quad k = 1, 2, 3.$$

This lifting representation of the two-dimensional wavelet decomposition on the lattice Λ corresponds to the Mallat algorithm (2.4) and (2.5) generalized to two dimension, which is the convolutions with four filters followed by downsampling:

$$c_{j+1}[\boldsymbol{t}] = \sum_{\boldsymbol{s}\in\Lambda} h[\boldsymbol{s} - 2\boldsymbol{t}]c_j[\boldsymbol{s}],$$

$$d_{k,j+1}[\boldsymbol{t}] = \sum_{\boldsymbol{s}\in\Lambda} g_k[\boldsymbol{s} - 2\boldsymbol{t}]c_j[\boldsymbol{s}].$$

The reconstruction of the signal is

$$c_j[\boldsymbol{t}] = \sum_{\boldsymbol{s}\in\Lambda}\left(\tilde{h}[\boldsymbol{t} - 2\boldsymbol{s}]c_{j+1}[\boldsymbol{s}] + \sum_{k=1}^{3}\tilde{g}_k[\boldsymbol{t} - 2\boldsymbol{s}]d_{k,j+1}[\boldsymbol{s}]\right),$$

where the filter coefficients of the system $(h, g_k, \tilde{h}, \tilde{g}_k)$, $k = 1, 2, 3$ are given in [12].

5 Redundant Haar Lifting on the Lattice

The original lifting on the lattice, which is described before, uses the polyphase decomposition that classifies a signal $c_j[\boldsymbol{t}]$ into even $c_j[2\boldsymbol{t}]$ and three odd components $c_j[2\boldsymbol{t} + \boldsymbol{t}_k]$, $k = 1, 2, 3$. As a result, each polyphase component has half a resolution compared with the original signal. As we see in one dimension, the redundant lifting does not use the polyphase decomposition, or decimation. We now define an even-like component $c_j[\boldsymbol{t}]$ and three odd-like components $c_j[\boldsymbol{t} + 2^j\boldsymbol{t}_k]$, $k = 1, 2, 3$, in a similar manner to the one dimensional case. Thus, in the case of the two-dimensional redundant Haar transform on the lattice Λ, which uses the constant prediction, the steps can be rewritten in a straightforward way:

$$d_{k,j+1}[\boldsymbol{t}] = c_j\left[\boldsymbol{t} + 2^j\boldsymbol{t}_k\right] - c_j[\boldsymbol{t}], \quad k = 1, 2, 3. \tag{5.1}$$

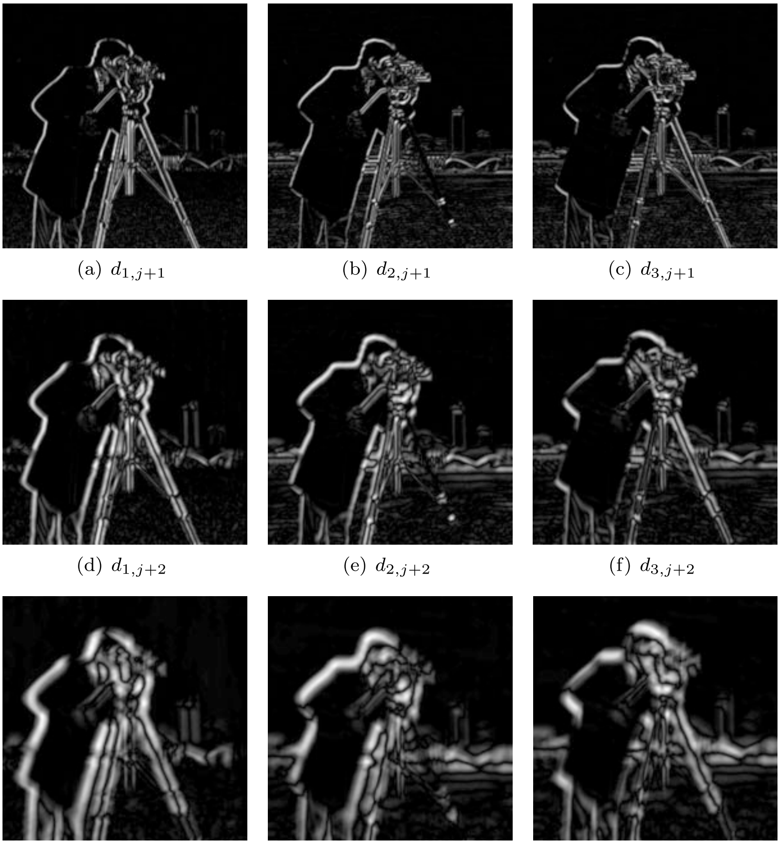
The update is similarly written as

$$c_{j+1}[\boldsymbol{t}] = c_j[\boldsymbol{t}] + \frac{1}{4}\sum_{k=1}^{3} d_{k,j+1}[\boldsymbol{t}]. \tag{5.2}$$

In the one-dimensional redundant lifting, the nature of the lifting has not changed without dropping the split operation that decomposes a signal into even and odd components. Due to the redundancy, the reconstruction of the signal becomes much more easier as in (3.3). Thus, we can still build the inverse transform in two dimension by undoing the update lifting steps:

$$c_j[\boldsymbol{t}] = c_{j+1}[\boldsymbol{t}] - \frac{1}{4}\sum_{k=1}^{3} d_{k,j+1}[\boldsymbol{t}].$$

Here we describe how the redundant lifting decomposes an image. Note that the original squared sampled image data has been mapped to the triangular lattice Λ by


(a) $d_{1,j+1}$ (b) $d_{2,j+1}$ (c) $d_{3,j+1}$

(d) $d_{1,j+2}$ (e) $d_{2,j+2}$ (f) $d_{3,j+2}$

(g) $d_{1,j+3}$ (h) $d_{2,j+3}$ (i) $d_{3,j+3}$

Fig. 2 Decomposed images of Cameraman with the redundant Haar lifting on the triangular lattice Λ. From the left, detail components $d_{1,j}$, $d_{2,j}$, and $d_{3,j}$ are shown. Images on the *upper row* have resolution level $j+1$, and *middle* and *lower ones* have $j+2$ and $j+3$, respectively, while all of the images have the same resolution density

using the half-shift pixel method [14]. By the redundant lifting, the original image has been decomposed into a coarse approximation $c_{j+1}[\boldsymbol{t}]$ and three oriented detail components $d_{k,j+1}[\boldsymbol{t}]$, $k = 1, 2, 3$ at each resolution level. Due to the nature of the redundant transform, all the decomposed images have the same resolution density. This redundancy would offer several advantages against the conventional wavelet decomposition for some applications such as edge detection or feature analysis of an image, where translation-invariant property plays a key role (Fig. 2).

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Gabor Transform of Analytic Functional on the Sphere

Keiko Fujita

Abstract We studied the Fourier–Borel transform of analytic functional on the complex sphere. In this paper, we will consider the Gabor transformation which is a windowed Fourier transformation whose window function is the Gaussian function. Following our previous results we will represent the Gabor transform of analytic functional on the sphere using a series expansion by means of the Bessel functions.

Keywords Series expansion · Analytic functional on the sphere · Gabor transformation

Mathematics Subject Classification (2010) Primary 41A58 · Secondary 46F15

1 Introduction

We denote by $L^2(X)$ the space of square integrable functions on X. The Fourier transform of $f \in L^2(\mathbf{R}^{n+1})$ is defined by

$$\hat{f}(\omega) = \int_{R^{n+1}} e^{-ix\cdot\omega}\overline{f(x)}dx,$$

where $x \cdot y = x_1 y_1 + \cdots + x_{n+1} y_{n+1}$ for $x, y \in \mathbf{R}^{n+1}$. Let w be a window function on $\mathbf{R}^{n+1}$; that is, $w(t), tw(t), \omega\hat{w}(\omega) \in L^2(\mathbf{R}^{n+1})$. The windowed Fourier transform with respect to w is defined by

$$\mathcal{WF}f(y,\omega) = \int_{R^{n+1}} w(x-y)e^{-ix\cdot\omega}\overline{f(x)}dx.$$

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We denote by $x^2 = x \cdot x$. Take $w(x) = e^{-x^2/2}$, then we call

$$\mathcal{W}_G \mathcal{F} f(y,\omega) = \int_{R^{n+1}} e^{-(x-y)^2/2} e^{-ix\cdot\omega} \overline{f(x)} dx$$

the Gabor transform of f.

Let $\mathbf{R}_+ = \{x \in \mathbf{R}; x > 0\}$. Put $G_{\omega_0}(x) = e^{-x^2/2} e^{-ix\cdot\omega_0}$ for $\omega_0 \in \mathbf{R}^{n+1} \setminus \{0\}$. For $a \in \mathbf{R}_+$, we define the Gabor wavelet transform by

$$\mathcal{G}_{\omega_0} f(y,a) = a^{-(n+1)/2} \int_{\mathbf{R}^{n+1}} G_{\omega_0}\left(\frac{x-y}{a}\right) \overline{f(x)} dx.$$

Let S^n be the unit sphere in $\mathbf{R}^{n+1}$. For $f \in L^2(S^n)$, we define the Fourier transform by

$$\mathcal{F} f(\omega) = \int_{S^n} e^{-ix\cdot\omega} \overline{f(x)} d\Omega,$$

where $d\Omega$ is the normalized invariant measure on S^n, and define the Gabor transform by

$$\mathcal{W}_G \mathcal{F} f(y,\omega) = \int_{S^n} e^{-(x-y)^2/2} e^{-ix\cdot\omega} \overline{f(x)} d\Omega.$$

Similarly we define the Gabor wavelet transform of $f \in L^2(S^n)$ by

$$\mathcal{G}_{\omega_0} f(y,a) = a^{-n/2} \int_{S^n} G_{\omega_0}\left(\frac{x-y}{a}\right) \overline{f(x)} d\Omega.$$

We remark that in [2], for a wavelet function g, which is a summable function on $(0,\infty)$ and satisfies

$$\int_0^\infty g(t)dt = 0, \qquad \int_0^\infty \left|g(t)\log t\right| dt < \infty,$$

the "spherical" wavelet transform of f generated by g is defined by

$$\mathcal{W} f(y,a) = \frac{2^{(1-n/2)}}{a} \int_{S^n} (1 - y\cdot x)^{(1-n/2)} g\left(\frac{1-y\cdot x}{a}\right) f(x) dx.$$

In this paper, we will represent $\mathcal{W}_G \mathcal{F} f(y,\omega)$ or $\mathcal{G}_{\omega_0} f(y,a)$ using a series expansion by means of the Bessel functions following our previous results summarized in the book [1], and references can be found in [1].

2 Gabor Transform of Analytic Functional on the Sphere

We denote by $\mathcal{A}(S^n)$ the space of real analytic functions on S^n, by $\mathcal{A}'(S^n)$ the space of analytic functionals (or hyperfunctions) on S^n. Let $\langle T, g\rangle$ be the canonical bilinear form of duality on $\mathcal{A}'(S^n) \times \mathcal{A}(S^n)$. For continuous functions f and g on S^n, we define a sesquilinear form $(g, f)_{S^n}$ by

$$(g, f)_{S^n} \equiv \int_{S^n} g(\omega)\overline{f(\omega)}d\Omega.$$

We know the volume of S^n; $\mathrm{vol}(S^n) = 2\pi^{(n+1)/2}/\Gamma((n+1)/2)$, where $\Gamma(\cdot)$ is the Gamma function.

From now on, $L^2(S^n)$ denotes the space of square integrable functions on S^n with the inner product $(f, g)_{L^2(S^n)} = (f, g)_{S^n}$ and we introduce the norm by $\|f\|_{L^2(S^n)} = \sqrt{(f, f)_{S^n}}$. Since $\mathcal{A}(S^n) \subset L^2(S^n) \subset \mathcal{A}'(S^n)$, for $f \in L^2(S^n)$ we define $T_f \in \mathcal{A}'(S^n)$ by

$$\langle T_f, g\rangle = (g, f)_{S^n} = \int_{S^n} g(x)\overline{f(x)}d\Omega, \quad g \in \mathcal{A}\big(S^n\big). \tag{2.1}$$

Note the mapping $f \mapsto T_f$ is a continuous antilinear injection.

2.1 Gabor Transformation

Let $T \in \mathcal{A}'(S^n)$. We will define the Fourier–Borel transform of T by

$$\mathcal{F}T(\omega) = \big\langle T_x, \exp(-ix\cdot\omega)\big\rangle, \quad \omega \in \mathbf{C}^{n+1}.$$

Note that we defined $\mathcal{F}T(\omega) = \langle T_x, \exp(x\cdot\omega)\rangle$ in [1].

For $\omega, \tau \in \mathbf{C}^{n+1}$, we define the Gabor transformation $\mathcal{W}_G\mathcal{F}$ by

$$\mathcal{W}_G\mathcal{F} : T \mapsto \mathcal{W}_G\mathcal{F}T(\tau, \omega) = \big\langle T_x, \exp\big(-(x-\tau)^2/2\big)\exp(-ix\cdot\omega)\big\rangle.$$

Since $T \in \mathcal{A}'(S^n)$,

$$\mathcal{W}_G\mathcal{F}T(\tau, \omega) = \exp\big(\big(-1-\tau^2\big)/2\big)\big\langle T_x, \exp\big(x\cdot(\tau - i\omega)\big)\big\rangle \tag{2.2}$$

$$= \exp\big(\big(-1-\tau^2\big)/2\big)\big\langle e^{x\cdot\tau}T_x, \exp(-ix\cdot\omega)\big\rangle. \tag{2.3}$$

As a function in ω, $\mathcal{W}_G\mathcal{F}T(\tau, \omega)$ satisfies the following differential equation:

$$(\Delta_\omega + 1)\mathcal{W}_G\mathcal{F}T(\tau, \omega) = 0,$$

where $\Delta_z = \partial^2/\partial z_1^2 + \cdots + \partial^2/\partial z_{n+1}^2$ for $z = (z_1, \dots, z_{n+1}) \in \mathbf{C}^{n+1}$ is the complex Laplacian. Similarly, as a function in τ,

$$(\Delta_\tau - 1)\big(e^{(1+\tau^2)/2}\mathcal{W}_G\mathcal{F}T(\tau, \omega)\big) = 0.$$

We denote by $\mathcal{F}_S^{-1}$ (resp., $(\mathcal{W}_G\mathcal{F})_S^{-1}$) the inverse mapping of $\mathcal{F}$ (resp., $\mathcal{W}_G\mathcal{F}$) for $\mathcal{A}'(S^n)$. Then we have

$$e^{(1+\tau^2)/2}\mathcal{F}_S^{-1}\big(\mathcal{W}_G\mathcal{F}T(\tau,\omega)\big)(x)=e^{x\cdot\tau}T_x.$$

Therefore we have

$$(\mathcal{W}_G\mathcal{F})_S^{-1}\big(\mathcal{W}_G\mathcal{F}T(\tau,\omega)\big)(x)=e^{-x\cdot\tau+(1+\tau^2)/2}\mathcal{F}_S^{-1}\big(\mathcal{W}_G\mathcal{F}T(\tau,\omega)\big)(x).$$

That is, the inverse Gabor transformation on S^n is given by

$$(\mathcal{W}_G\mathcal{F})_S^{-1}:\mathcal{W}_G\mathcal{F}T(\tau,\omega)\mapsto e^{-x\cdot\tau+(1+\tau^2)/2}\big(\mathcal{F}_S^{-1}\big(\mathcal{W}_G\mathcal{F}T(\tau,\cdot)\big)(x)\big),\quad x\in S^n.$$

3 Expansion Formula

Let $P_{k,n}(t)$ be the Legendre polynomial of degree k and of dimension $n+1$:

$$P_{k,n}(t)=\left(\frac{-1}{2}\right)^2\frac{\Gamma(n/2)}{\Gamma(k+n/2)}\big(1-t^2\big)^{(2-n)/2}\frac{d^k}{dt^k}\big(1-t^2\big)^{k+(n-2)/2}.$$

Note that $P_{k,1}(t)=\cos(k\cos^{-1}t)$ is the Chebyshev polynomial of degree k.

We define the extended Legendre polynomial by

$$P_{k,n}(z,w)=\big(\sqrt{z^2}\big)^k\big(\sqrt{w^2}\big)^kP_{k,n}\left(\frac{z}{\sqrt{z^2}}\cdot\frac{w}{\sqrt{w^2}}\right),\quad z,w\in\mathbf{C}^{n+1}.$$

$P_{k,n}(z,w)$ is a homogeneous harmonic polynomial of degree k in z and in w. Thus $\Delta_zP_{k,n}(z,w)=\Delta_wP_{k,n}(z,w)=0$. The dimension $N(k,n)$ of the space of k-homogeneous harmonic polynomials is given by

$$N(k,n)=\frac{(2k+n-1)(k+n-2)!}{k!(n-1)!}.$$

For $f\in L^2(S^n)$, define

$$f_k(x)=N(k,n)\int_{S^n}f(\omega)P_{k,n}(\omega,x)d\omega. \tag{3.1}$$

Then we have

$$f(x)=\sum_{k=0}^{\infty}f_k(x), \tag{3.2}$$

in the sense of $L^2(S^n)$. If $f\in\mathcal{A}(S^n)$, then the convergence of the right-hand side of (3.2) is in the sense of $\mathcal{A}(S^n)$.

Let $g \in \mathcal{A}(S^n)$ and define g_k of g by (3.1). For $T \in \mathcal{A}'(S^n)$ we consider

$$\langle T, g\rangle = \left\langle T, \sum_{k=0}^{\infty} g_k \right\rangle = \sum_{k=0}^{\infty} \langle T, g_k\rangle. \tag{3.3}$$

By (3.1), we have

$$\begin{aligned} \langle T, g_k\rangle &= \left\langle T_\zeta, N(k,n) \int_{S^n} g(\omega) P_{k,n}(\omega,\zeta) d\omega \right\rangle \\ &= N(k,n) \int_{S^n} \big\langle T_\zeta, P_{k,n}(\omega,\zeta)\big\rangle g(\omega) d\Omega \\ &= \int_{S^n} g(\omega) \overline{S_k(T;\omega)} d\Omega, \end{aligned} \tag{3.4}$$

where we put

$$S_k(T;\omega) = N(k,n) \overline{\big\langle T_\zeta, P_{k,n}(\omega,\zeta)\big\rangle}.$$

For $f \in L^2(S^n)$, by (2.1) we have

$$\begin{aligned} S_k(T_f;\omega) &= N(k,n) \overline{\int_{S^n} P_{k,n}(\omega,x) \overline{f(x)} d\Omega} \\ &= N(k,n) \int_{S^n} P_{k,n}(\omega,x) f(x) d\Omega = f_k(\omega). \end{aligned}$$

Note that $\overline{S_k(T;\omega)}$ is a homogeneous harmonic polynomial of degree k. By (3.3) and (3.4), we have

$$\langle T, g\rangle = \sum_{k=0}^{\infty} \int_{S^n} g(\omega) \overline{S_k(T;\omega)} d\Omega;$$

that is,

$$T_\omega = \sum_{k=0}^{\infty} S_k(T;\omega) = \sum_{k=0}^{\infty} N(k,n) \overline{\big\langle T_\zeta, P_{k,n}(\omega,\zeta)\big\rangle}$$

in the sense of $\mathcal{A}'(S^n)$.

3.1 Expansion Formula of the Exponential Function

For $\nu \neq -1, -2, \ldots$, we define the Bessel function of order ν by

$$J_\nu(t) = \left(\frac{t}{2}\right)^{\nu} \sum_{l=0}^{\infty} \frac{1}{l!\Gamma(\nu+l+1)} \left(\frac{it}{2}\right)^{2l}.$$

We put

$$\tilde{J}_\nu(t) = \Gamma(\nu+1)\left(\frac{t}{2}\right)^{-\nu} J_\nu(t), \qquad \tilde{j}_k(t) = \tilde{J}_{k+(n-1)/2}(t),$$

$$C(k,n) = \frac{\Gamma(\frac{n+1}{2})}{2^k \Gamma(k+\frac{n+1}{2})}.$$

Using these notation we have

$$\exp(z \cdot w) = \sum_{k=0}^{\infty} C(k,n) N(k,n) \tilde{j}_k\big(i\sqrt{z^2}\sqrt{w^2}\big) P_{k,n}(z,w). \tag{3.5}$$

Thus for $T \in \mathcal{A}'(S^n)$, we have

$$\mathcal{F}T(\omega) = \big\langle T_x, \exp(-ix\cdot\omega)\big\rangle = \sum_{k=0}^{\infty} C(k,n)(-i)^k \tilde{j}_k\big(\sqrt{\omega^2}\big)\overline{S_k(T;\omega)}. \tag{3.6}$$

3.2 Expansion Formula of Gabor Transform

By (2.3) and (3.6), for $T \in \mathcal{A}'(S^n)$ we have

$$\mathcal{W}_G\mathcal{F}T(\tau,\omega) = e^{(-1-\tau^2)/2} \sum_{k=0}^{\infty} C(k,n)(-i)^k \tilde{j}_k\big(\sqrt{\omega^2}\big)\overline{S_k\big(e^{x\cdot\tau}T;\omega\big)}. \tag{3.7}$$

On the other hand, by (2.2) and (3.5), we also have

$$\mathcal{W}_G\mathcal{F}T(\tau,\omega) = e^{\frac{-1-\tau^2}{2}} \sum_{k=0}^{\infty} C(k,n)(-i)^k \tilde{j}_k\big(\sqrt{(\tau-i\omega)^2}\big)\overline{S_k(T;\tau-i\omega)}. \tag{3.8}$$

For $f \in L^2(S^n)$, we have

$$\mathcal{W}_G\mathcal{F}T_f(\tau,\omega) = e^{\frac{-1-\tau^2}{2}} \sum_{k=0}^{\infty} C(k,n)(-i)^k \tilde{j}_k\big(\sqrt{(\tau-i\omega)^2}\big)\overline{f_k}(\tau-i\omega). \tag{3.9}$$

Put

$$\big(e^{x\cdot\tau}\overline{f}(x)\big)_k(\omega) = N(k,n)\int_{S^n} e^{x\cdot\tau}\overline{f(x)} P_{k,n}(\omega,x)\,d\Omega.$$

Then we also have

$$\mathcal{W}_G\mathcal{F}T_f(\tau,\omega) = e^{\frac{-1-\tau^2}{2}} \sum_{k=0}^{\infty} C(k,n)(-i)^k \tilde{j}_k\big(\sqrt{\omega^2}\big)\big(e^{x\cdot\tau}\overline{f}(x)\big)_k(\omega). \tag{3.10}$$

Thus (3.7)–(3.10) give series expansions of Gabor transform of analytic functional on S^n by means of the Bessel functions.

For example, consider the case that $f_{\zeta_0}(x) = e^{ix\cdot\zeta_0} \in L^2(S^n)$. Then

$$\begin{aligned}
&\mathcal{F}T_{f_{\zeta_0}}(\omega) \\
&\quad = \int_{S^n} \exp(-ix\cdot\omega)\overline{\exp(ix\cdot\zeta_0)}d\Omega \\
&\quad = \sum_{k=0}^{\infty} C(k,n)^2(-1)^k N(k,n)\tilde{j}_k\big(\sqrt{\omega^2}\big)\overline{\tilde{j}_k\Big(\sqrt{\zeta_0^2}\Big)}P_{k,n}(\overline{\zeta_0},\omega), \\
&\mathcal{W}_G\mathcal{F}T_{f_{\zeta_0}}(\tau,\omega) \\
&\quad = e^{(-1-\tau^2)/2}\int_{S^n} \exp(x\cdot\omega)\exp(x\cdot\tau)\overline{\exp(x\cdot\zeta_0)}d\Omega \\
&\quad = e^{\frac{-1-\tau^2}{2}}\sum_{k=0}^{\infty} C(k,n)^2 N(k,n)\tilde{j}_k\big(i\sqrt{\omega^2}\big)\tilde{j}_k\Big(i\sqrt{\tau+\overline{\zeta_0}^2}\Big)P_{k,n}(\tau+\overline{\zeta_0},\omega).
\end{aligned}$$

3.3 Gabor Wavelet Transformation and Expansion Formula

Let $\omega_0 \neq 0 \in \mathbf{R}^{n+1}$ be fixed. For $T \in \mathcal{A}'(S^n)$, $a \in \mathbf{R}_+$ and $\tau \in \mathbf{R}^{n+1}$, we define a modified Gabor transformation $\mathcal{G}_{\omega_0}$ by

$$\begin{aligned}
\mathcal{G}_{\omega_0} : T \mapsto \mathcal{G}_{\omega_0}T(\tau,a) &= \left\langle T_x, a^{-\frac{n}{2}}G_{\omega_0}\left(\frac{x-\tau}{a}\right)\right\rangle \\
&= \left\langle T_x, a^{-\frac{n}{2}}\exp\left(-i\omega_0\cdot\frac{x-\tau}{a}\right)\exp\left(-\frac{1}{2}\left(\frac{x-\tau}{a}\right)^2\right)\right\rangle.
\end{aligned}$$

Since $T \in \mathcal{A}'(S^n)$, we have

$$\begin{aligned}
\mathcal{G}_{\omega_0}T(\tau,a) &= a^{-\frac{n}{2}}\exp\left(-\frac{1+\tau^2-2ai\tau\cdot\omega_0}{2a^2}\right)\left\langle T_x, \exp\left(\frac{x}{a}\cdot\frac{\tau-ai\omega_0}{a}\right)\right\rangle \\
&= a^{-\frac{n}{2}}\exp\left(-\frac{\tau^2-2ai\tau\cdot\omega_0+1}{2a^2}\right)\left\langle \exp\left(\frac{x}{a}\cdot\frac{\tau}{a}\right)T_x, \exp\left(\frac{-ix\cdot\omega_0}{a}\right)\right\rangle.
\end{aligned}$$

By (3.5), we have

$$\begin{aligned}
\mathcal{G}_{\omega_0}T(a,\tau) = a^{-\frac{n}{2}}&\exp\left(-\frac{1+\tau^2-2ai\tau\cdot\omega_0}{2a^2}\right) \\
&\times\sum_{k=0}^{\infty}\frac{(-i)^k C(k,n)}{a^k}\tilde{j}_k\left(\frac{\sqrt{\omega_0^2}}{a^2}\right)\overline{S_k\big(e^{x\cdot\tau/a^2}T_x;\omega_0\big)}
\end{aligned}$$

$$= a^{-\frac{n}{2}} \exp\left(-\frac{1+\tau^2-2ai\tau\cdot\omega_0}{2a^2}\right)$$
$$\times \sum_{k=0}^{\infty} \frac{C(k,n)}{a^{2k}} \tilde{j}_k\left(\frac{i\sqrt{(\tau-ai\omega_0)^2}}{a^2}\right) \overline{S_k(T_x;\tau-ia\omega_0)}.$$

For $f \in L^2(S^n)$, we call

$$\mathcal{G}_{\omega_0} T_f(\tau,a) = a^{-\frac{n}{2}} e^{-(1+\tau^2-2ai\tau\cdot\omega_0)/(2a^2)} \int_{S^n} \exp\left(\frac{x}{a}\cdot\frac{\tau-ai\omega_0}{a}\right)\overline{f(x)}d\Omega$$

the Gabor wavelet transform of f. Then we have

$$\mathcal{G}_{\omega_0} T_f(a,\tau) = a^{-\frac{n}{2}} \exp\left(-\frac{1+\tau^2-2ai\tau\cdot\omega_0}{2a^2}\right)$$
$$\times \sum_{k=0}^{\infty} \frac{C(k,n)}{a^{2k}} \tilde{j}_k\left(\frac{i\sqrt{(\tau-ai\omega_0)^2}}{a^2}\right) \overline{f_k}(\tau-ia\omega_0).$$

This is the series expansions of the Gabor wavelet transform of $f \in L^2(S^n)$ by means of the Bessel functions.

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On the Interpolation of Orthonormal Wavelets with Compact Support

Naohiro Fukuda and Tamotu Kinoshita

Abstract The N-th order Daubechies wavelet is obtained with the spectral decomposition method from modulus of associated low-pass filters $M(\xi) = |m_0(\xi)|^2$. Meanwhile, we can denote $M(\xi)$ with the integration. In this paper, we focus on integrands and construct some wavelets by changing them. Moreover, we construct some kind of fractional order wavelets and give regularity estimates of them.

Keywords Wavelets · B-splines · Fractional order

Mathematics Subject Classification (2010) 65T60

1 Introduction

The wavelet theory is applied in many areas such as signal or image processing. Completion of the multiresolution analysis (MRA) makes the construction of wavelets easy and yields wavelets with good properties.

The compact support property of wavelets is very important for some applications. If a wavelet ψ has a compact support (i.e., associated filter $\{h_n\}_n$ is finite), we can perform computations with high accuracy. The simplest type of such a wavelet is the Haar wavelet. Ingrid Daubechies [1] constructed a family of compactly supported wavelets ψ_n^D, which are called the Daubechies wavelets and have n vanishing moments. A wavelet whose scaling function also has vanishing moments is called coiflet.

There are two approaches for constructing orthonormal wavelets: start with a low-pass filter m_0, or start with a scaling function φ. In particular, the Daubechies family (including coiflet or symlet) is constructed from $M(\xi) = |m_0(\xi)|^2$, and $m_0(\xi)$

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are derived from $M(\xi)$ by using the spectral decomposition. In the case when N-th order Daubechies wavelet, $M(\xi) = M_N(\xi)$ is given by

$$M_N(\xi) = \left(\cos^2 \frac{\xi}{2}\right)^N L_N(\xi), \tag{1.1}$$

where $L_N(\xi) = P_N(\sin^2 \xi/2)$ with

$$P_N(y) = \sum_{k=0}^{N-1} \binom{N-1+k}{k} y^k.$$

There are several ways for the spectral decomposition to get a low-pass filter m_0. Especially, Daubechies wavelets have minimum-phase filters. In this paper, we treat another expression of $M(\xi)$ and construct various wavelets.

2 Construction Through the Integration

For the construction of Daubechies wavelets, (1.1) has the simpler form

$$M_N(\xi) = \frac{\int_{-1}^{\cos \xi} (1-t^2)^{N-1} dt}{\int_{-1}^{1} (1-t^2)^{N-1} dt}. \tag{2.1}$$

Now we shall deal with a more general form. We set function $M(\xi)$ with a real-valued function $p(t) \in L^1(-1, 1)$ by

$$M(\xi) = \frac{\int_{-1}^{\cos \xi} p(t)dt}{\int_{-1}^{1} p(t)dt}. \tag{2.2}$$

Then, the next proposition shows that $M(\xi)$ is a "good candidate" for a low-pass filter m_0 such that $|m_0(\xi)|^2 = M(\xi)$.

Proposition 2.1 *$M(\xi)$ defined by (2.2) with a real-valued even function $p \in L^1(-1, 1)$ satisfies*

$$M(0) = 1 \tag{2.3}$$

and

$$M(\xi) + M(\xi + \pi) = 1. \tag{2.4}$$

Proof Equation (2.3) is obvious. Noting that $p(t)$ is even, we see that

$$\begin{aligned} M(\xi) + M(\xi + \pi) &= \frac{\int_{-1}^{\cos \xi} p(t)dt + \int_{-1}^{-\cos \xi} p(t)dt}{\int_{-1}^{1} p(t)dt} \\ &= \frac{\int_{-1}^{\cos \xi} p(t)dt + \int_{\cos \xi}^{1} p(t)dt}{\int_{-1}^{1} p(t)dt} = 1. \end{aligned}$$

□

Proposition 2.1 means that a function m_0 such that $|m_0(\xi)|^2 = M(\xi)$ satisfies $|m_0(0)| = 1$ and $|m_0(\xi)|^2 + |m_0(\xi + \pi)|^2 = 1$ which are necessary conditions for an orthonormal wavelet. To construct an orthonormal wavelet, $m_0(\xi)$ should satisfy an additional condition as the next lemma [5].

Lemma 2.2 *Assume that $m_0(\xi) \in C^1(\mathbb{R})$ is a 2π-periodic function and satisfies $m_0(0) = 1$ and $|m_0(\xi)|^2 + |m_0(\xi + \pi)|^2 = 1$. If $m_0(\xi) \neq 0$ for all $\xi \in [-\pi/3, \pi/3]$, then m_0 is a low-pass filter for an orthonormal wavelet.*

For instance, in the case when $\int_{-1}^{1} p(t)dt > 0$, if $p(t) \geq 0$ for all $t \in [1/2, 1]$, $M(\xi)$ satisfies all the conditions of Lemma 2.2.

Example 1 The simplest choice $p(t) = 1$ gives the Haar wavelet. This corresponds to first order Daubechies wavelet (i.e., $N = 1$ in (2.1)).

Example 2 Let us set $p(t) = \cos(\frac{\pi}{2}t)$. We then obtain $M(\xi) = \cos^2(\frac{\pi}{2}\sin^2\frac{\xi}{2})$. Resulting low-pass filter $m_0(\xi) = \cos(\frac{\pi}{2}\sin^2\frac{\xi}{2})$ is a low-pass filter for an orthonormal wavelet. Reference [3] derived this wavelet by another idea.

3 Fractional Order Wavelets

Let $p_1, p_2 \in L^1(-1, 1)$ be integrands for M_1 and M_2, respectively, i.e., we put

$$M_1(\xi) = \frac{\int_{-1}^{\cos\xi} p_1(t)dt}{\int_{-1}^{1} p_1(t)dt} \quad \text{and} \quad M_2(\xi) = \frac{\int_{-1}^{\cos\xi} p_2(t)dt}{\int_{-1}^{1} p_2(t)dt}.$$

It is easy to check that for $a, b \in \mathbb{R}$ such that $a + b \neq 0$, $M(\xi) = \frac{aM_1(\xi)+bM_2(\xi)}{a+b}$ also satisfies (2.3) and (2.4). Putting $a = 1 - \alpha$ and $b = \alpha$ for $0 < \alpha < 1$ and define

$$M = (1 - \alpha)M_1 + \alpha M_2. \tag{3.1}$$

In some sense, M can be regarded to divide internally M_1 and M_2 in ratio $\alpha : 1 - \alpha$.

3.1 Fractional Order Daubechies Wavelets

As we already noted that the integrand $p(t) = (1 - t)^{N-1}$ gives the Daubechies wavelet of order N. We therefore define for $N \in \mathbb{N}$ and $0 < \alpha < 1$

$$M_{N+\alpha}^{\mathrm{D}}(\xi) = (1 - \alpha)\frac{\int_{-1}^{\cos\xi}(1 - t^2)^{N-1}dt}{\int_{-1}^{1}(1 - t^2)^{N-1}dt} + \alpha\frac{\int_{-1}^{\cos\xi}(1 - t^2)^{N}dt}{\int_{-1}^{1}(1 - t^2)^{N}dt}.$$

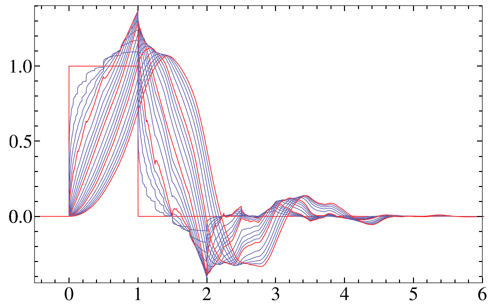


Fig. 1 The fractional order Daubechies scaling functions ($1 \le N+\alpha \le 5$)

Similarly as the standard Daubechies wavelets, $M^{\mathrm{D}}_{N+\alpha}(\xi)$ is a finite-degree polynomial of $\cos\xi$. This means that it is possible to apply the Riesz lemma for the spectral decomposition. In the same way as the standard Daubechies wavelets, we obtain an appropriate low-pass filter and call it the low-pass filter associated with the $(N+\alpha)$-th fractional order Daubechies wavelet.

Generally, the order of vanishing moments of a wavelet ψ can be calculated from its low-pass filter m_0. If $m_0(\xi+\pi) = \mathcal{O}(\xi^n)$, then ψ has n vanishing moments. Since $\frac{d}{d\xi}\int_{-1}^{\cos(\xi+\pi)}(1-t^2)^{N-1}dt = \sin^{2N-2}\xi$, $(N+\alpha)$-th fractional order Daubechies wavelet has N vanishing moments. In addition, $M(\xi)$ is a $(2N+1)$-th degree polynomial of $\cos\xi$ and thus the support of the scaling function is equal to $[0, 2N+1]$. We show the graphs of fractional order Daubechies scaling functions in Fig. 1.

3.2 Fractional Order B-Splines

B-spline functions having simple structures are often applied to numerical analysis. Let us denote N_n the n-th order B-spline, i.e., $N_1 = \chi_{[0,1)}$ and $N_m = N_{m-1} * N_1$ for $m \ge 2$. They have many good properties such as the convolution property $N_{k_1} * N_{k_2} = N_{k_1+k_2}$, $k_1, k_2 \in \mathbb{Z}$, compact support, and positivity ($N_m(x) \ge 0$ for all $m \in \mathbb{N}$, $x \in \mathbb{R}$).

Uncer et al. [6] generalized the conventional B-splines N_n and constructed fractional order B-splines $\phi^{\text{B-sp}}_\alpha$ for $\alpha \ge 1$. They succeeded to keep the convolution property, i.e., it holds that $\phi^{\text{B-sp}}_\alpha * \phi^{\text{B-sp}}_\beta = \phi^{\text{B-sp}}_{\alpha+\beta}$ for $\alpha, \beta \ge 1$. Meanwhile, $\phi^{\text{B-sp}}_\alpha$, $\alpha \notin \mathbb{N}$ does not have a compact support and positivity. In this section, we introduce another version of fractional order B-spline functions by using (3.1).

For $n \in \mathbb{N}$, the low-pass filter corresponding to the n-th order B-spline N_n is given by $m_0(\xi) = (\frac{1+e^{-i\xi}}{2})^n$ and therefore $M_n(\xi) = |m_0(\xi)|^2 = (\frac{1+\cos\xi}{2})^n$. This leads to the definition of $M^{\text{B-sp}}_{N+\alpha}$ by

$$\begin{aligned} M^{\text{B-sp}}_{N+\alpha}(\xi) &= (1-\alpha)M_N(\xi) + \alpha M_{N+1}(\xi) \\ &= \left(\frac{1+\cos\xi}{2}\right)^N \left(1 - \frac{\alpha}{2} + \frac{\alpha}{2}\cos\xi\right). \end{aligned}$$

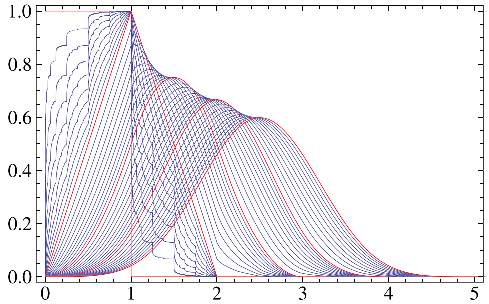


Fig. 2 The fractional order B-splines $\varphi_{N+\alpha}^{\text{B-sp}}$ $(1 \le N+\alpha \le 5)$

By using Riesz lemma, we obtain a low-pass filter

$$m_0(\xi) = \left(\frac{1+\sqrt{1-\alpha}}{2}\right)\left(\frac{1+e^{-i\xi}}{2}\right)^N \left(1 + \frac{2-\alpha-2\sqrt{1-\alpha}}{\alpha} e^{-i\xi}\right). \tag{3.2}$$

We call resulting scaling function $\varphi_\alpha^{\text{B-sp}}$ defined by $\widehat{\varphi_\alpha^{\text{B-sp}}}(\xi) = \prod_{j=1}^\infty m_0(2^{-j}\xi)$ $(N+\alpha)$-th order fractional order B-spline function. The graphs of fractional order B-spline function $\varphi_\alpha^{\text{B-sp}}$ with $1 \le N+\alpha \le 5$ are shown in Fig. 2.

Remark 3.1 The integral expression of n-th order B-spline N_n is given by

$$M_n(\xi) = \frac{\int_{-1}^{\cos\xi}(1+t)^{n-1}dt}{\int_{-1}^{1}(1+t)^{n-1}dt}.$$

In this case, since N_n is not an orthonormal scaling function, the integrand $(1+t)^{n-1}$ is not an even function and M_n does not satisfy (2.4).

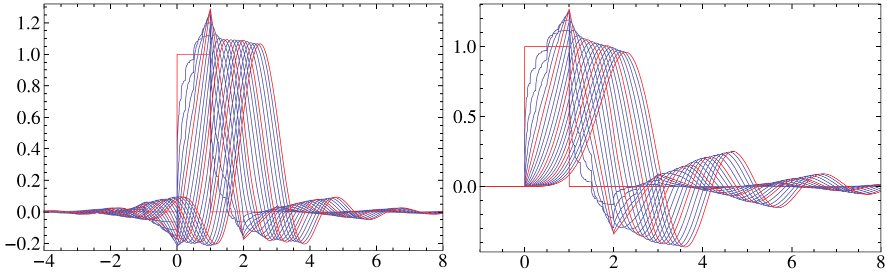
Here we note that fractional order B-splines of our scheme do not keep the convolution property. However, for all $N+\alpha$, $\varphi_{N+\alpha}^{\text{B-sp}}$ has a compact support with support size $N+1$ for $\alpha \ne 0$. Moreover, the positivity is also valid, since all filter coefficients corresponding to $\varphi_{N+\alpha}^{\text{B-sp}}$ are positive (see (3.2)) and $\varphi_{N+\alpha}^{\text{B-sp}}$ is the limit function of the cascade algorithm. Fractional order B-splines $\phi_{N+\alpha}^{\text{B-sp}}$ in [6] are in the opposite situation; $\phi_{N+\alpha}^{\text{B-sp}}$ has the convolution property, but fails to have a compact support and the positivity.

There are a few types of wavelets derived from B-splines $\varphi_n = N_n$. Setting a dual scaling function $\tilde{\varphi}_n$ as

$$\widehat{\tilde{\varphi}_n}(\xi) = \frac{\hat{\varphi}_n(\xi)}{\sum_{k=-\infty}^{\infty}|\hat{\varphi}_n(\xi+2k\pi)|^2}$$

forms a semi-orthogonal wavelet, and

$$\widehat{\phi_n^{\text{BT}}}(\xi) = \frac{\hat{\varphi}_n(\xi)}{\sqrt{\sum_{k=-\infty}^{\infty}|\hat{\varphi}_n(\xi+2k\pi)|^2}}$$



(a) $\phi^{\mathrm{BL}}_{N+\alpha}$ (b) $\phi^{\mathrm{St}}_{N+\alpha}$

Fig. 3 Fractional order Battle–Lemarié (**a**) and Strömberg scaling (**b**) scaling functions $(1 \le N+\alpha \le 5)$

gives an orthonormal (Battle–Lemarié) wavelet. Moreover, by a suitable choice of $A(\xi)$ such that $|A_n(\xi)|^2 = \sum_{k=-\infty}^{\infty} |\hat{\varphi}_n(\xi + 2k\pi)|^2$, we are able to get the n-th order Strömberg scaling function ϕ_n^{St} as

$$\widehat{\phi_n^{\mathrm{St}}}(\xi) = \frac{\hat{\varphi}_n(\xi)}{A_n(\xi)}.$$

For the choice of $A(\xi)$, refer to [4].

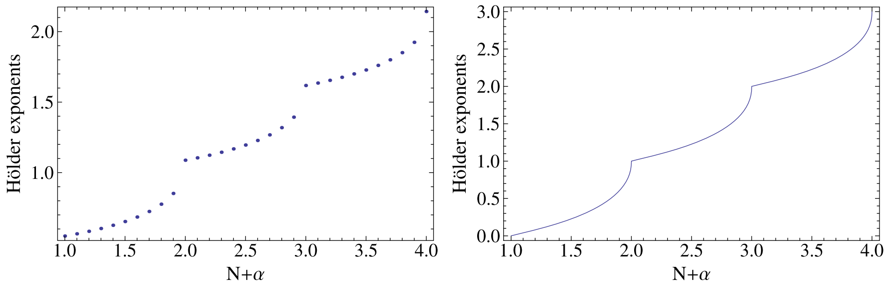
With a natural generalization, we can define fractional order semi-orthogonal wavelets, Battle–Lemarié wavelets $\phi^{\mathrm{BL}}_{N+\alpha}$ and Strömberg wavelets $\phi^{\mathrm{St}}_{N+\alpha}$ in a uniform manner. We show in Fig. 3 fractional order Battle–Lemarié and Strömberg scaling functions.

4 Regularity Estimate of Fractional Order Wavelets

There are some wavelet families with an order parameter n. In most cases, the smoothness of resulting wavelet increases with increasing n. It is well-known that Daubechies wavelets and Battle–Lemarié wavelets converge to the Shannon wavelet as order n tends to infinity. Daubechies [2] derived that n-th order Daubechies wavelet ψ_n^{D} belongs to $C^{0.2n}$, asymptotically. Moreover, sharp regularity estimates give $\psi_1^{\mathrm{D}} \in C^{0.5500}$, $\psi_2^{\mathrm{D}} \in C^{1.0878}$, $\psi_3^{\mathrm{D}} \in C^{1.6179}$, and so on. These regularity estimate methods can also be applied to the case of fractional order wavelets. Figure 4(a) shows the regularity estimates for fractional order Daubechies wavelets. Additionally, in the case of fractional order B-splines, it is possible to obtain the Hölder exponents for any $1 \le N+\alpha$; $\varphi^{\text{B-sp}}_{N+\alpha}$ has the Hölder exponents $N - 1 - \log_2(1+\sqrt{1-\alpha})$ (see Fig. 4(b)).

5 Conclusion

In this paper, we introduced the construction of wavelets with integral expressions. If we set an integrand $p(t)$ as a polynomial, resulting scaling function and wavelet



(a) Daubechies scaling functions (b) B-spline functions

Fig. 4 Regularity of $(N + \alpha)$-th order scaling functions

have a compact support. Moreover, by changing $p(t)$, various types of wavelets are constructed. By interpolating two functions M_1 and M_2, and using the spectral decomposition, we can construct fractional order wavelets. As an example, we treated fractional order Daubechies wavelets and B-spline wavelets. In the same way as conventional B-splines, from fractional order B-spline $\varphi_\alpha^{\text{B-sp}}$, we can also obtain dual, Battle–Lemarié, Strömberg scaling functions. Furthermore, we have seen that the smoothness of fractional order wavelets interpolates that of standard wavelets.

From these observations, we ensure that new wavelets are also useful for applications.

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An Estimation Method of Shift Parameters in Image Separation Problem

Ryuichi Ashino, Takeshi Mandai, and Akira Morimoto

Abstract The simplest spatio-temporal mixing model of blind source separation for images is discussed. Shift parameters are estimated by total correlation functions of continuous wavelet transforms. An image separation algorithm using an annular sector multiwavelet is proposed.

Keywords Image separation · Continuous multiwavelet transform · Shift parameter

Mathematics Subject Classification (2010) Primary 42C40 · Secondary 65T60

1 Introduction

The cocktail party effect [1] is a challenging problem in auditory perception, it is also called blind source separation in engineering. A mathematical background for the blind source separation by the quotient signal decomposition was discussed in [2]. Generalizations of the quotient signal decomposition for speech signals and for images were given in [3] and [4, 5], respectively. We dealt with the simplest spatio-temporal mixing model of blind source separation for speech signals in [6].

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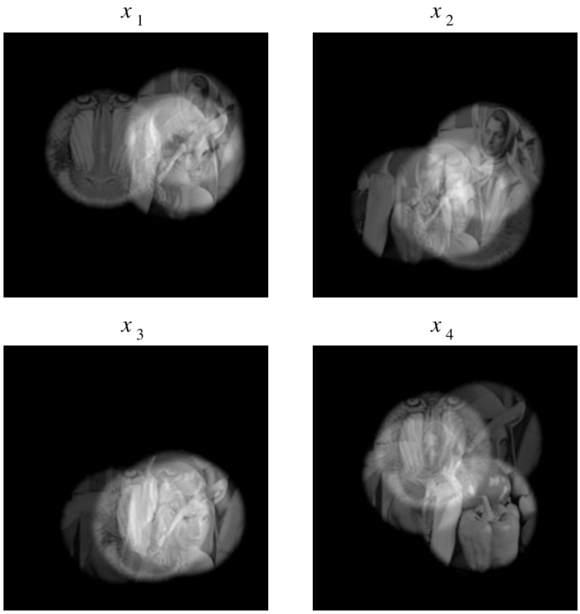

Fig. 1 An example of observed images

In this paper, we consider the simplest spatio-temporal mixing model of blind source separation for images. Images are supposed to be real-valued square integrable functions on $\mathbb{R}^2$. Let N be the number of sources and J, $J \geq N$, be the number of observed images. In the simplest spatio-temporal mixing model, we assume that observed images $x_j(t)$, $j = 1, \ldots, J$, are the following mixtures of original source images $s_n(t) \in L^2(\mathbb{R}^2)$, $n = 1, \ldots, N$:

$$x_j(t) = \sum_{n=1}^{N} d_{j,n} s_n(t - c_{j,n}), \tag{1.1}$$

where $d_{j,n} \in \mathbb{R}$ are mixing coefficients and $c_{j,n} \in \mathbb{R}^2$ are shift parameters. An example of observed images is illustrated in Fig. 1. In this paper, we focus on a new estimating method of shift parameters.

2 Estimation by Total Correlation Functions

We use an annular sector multiwavelet $\Psi = (\psi_1, \ldots, \psi_P)^T \in (L^2(\mathbb{R}^2))^P$, $P \in \mathbb{N}$, $P \geq 2$, generated by annular sector tiling in the Fourier domain [4]. We illustrate $\widehat{\psi}_p(a\xi)$ in the middle column of Fig. 2. A continuous wavelet transform $W_{\psi_p} f(b, a)$ of an image $f \in L^2(\mathbb{R}^2)$ with respect to wavelet functions $\psi_p(t)$, $p = 1, \ldots, P$ is

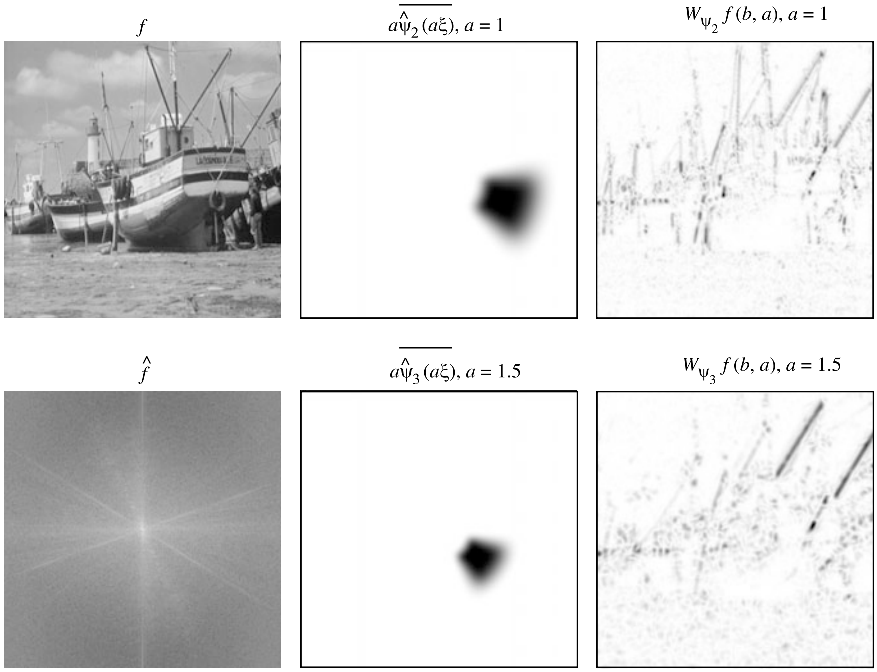

Fig. 2 An example of annular sector multiwavelet transform ($P = 20$). *Top left*: image f, *bottom left*: the Fourier transform $\widehat{f}$. *Top middle*: $a\overline{\widehat{\psi}_2(a\xi)}$, $a = 1$, *bottom middle*: $a\overline{\widehat{\psi}_3(a\xi)}$, $a = 1.5$. *Top right*: $W_{\psi_2} f(b, 1)$, *bottom right*: $W_{\psi_3} f(b, 1.5)$

defined by

$$W_{\psi_p} f(b, a) = \int_{\mathbb{R}^2} f(t) a^{-1} \overline{\psi_p\left(\frac{t-b}{a}\right)} dt$$
$$= (2\pi)^{-2} \int_{\mathbb{R}^2} \widehat{f}(\xi) a \overline{\widehat{\psi}_p(a\xi) e^{-ib\cdot\xi}} d\xi = \mathcal{F}^{-1}_{\xi\to b}\left[\widehat{f}(\xi) a \overline{\widehat{\psi}_p(a\xi)}\right](b),$$

where $b \in \mathbb{R}^2$ and $a \in \mathbb{R}_+ := \{a \in \mathbb{R} \mid a > 0\}$ are called position and scale parameters, respectively. An example of $W_{\psi_p} f(b, a)$ is sketched in the right column of Fig. 2.

For simplicity, the continuous wavelet transforms of an observed image $x_j(t)$ and the original image $s_n(t)$ with respect to a wavelet function $\psi_p(t)$ are denoted by

$$X_j^p(t, a) = W_{\psi_p} x_j(t, a), \qquad S_n^p(t, a) = W_{\psi_p} s_n(t, a),$$

respectively. Since a continuous wavelet transform is linear and commutes with a translation operator, from the model equation (1.1), we have

$$X_j^p(t, a) = \sum_{n=1}^{N} d_{j,n} S_n^p(t - c_{j,n}, a), \quad j = 1, \ldots, J.$$

For fixed j and k, we define the complex-valued correlation function $R^p_{j,k}(y,a)$ between $x_j(t)$ and $x_k(t)$ with respect to ψ_p by

$$R^p_{j,k}(y,a) = \frac{1}{\|X^p_j(\cdot,a)\|\,\|X^p_k(\cdot,a)\|}\int_{\mathbb{R}^2} X^p_j(t,a)\overline{X^p_k(t-y,a)}dt.$$

Fix a scale parameter a. In order to calculate $R^p_{j,k}(y,a)$ numerically, let us calculate the correlation function $R^p_{j,k}(y,a)$ in the Fourier domain. By Parseval's theorem, we have

$$\begin{aligned}
R^p_{j,k}(y,a) &= \frac{1}{(2\pi)^2\|X^p_j(\cdot,a)\|\,\|X^p_k(\cdot,a)\|}\int_{\mathbb{R}^2} \widehat{X^p_j}(\xi,a)\overline{\widehat{X^p_k}(\xi,a)e^{-iy\cdot\xi}}d\xi \\
&= \frac{1}{(2\pi)^2\|X^p_j(\cdot,a)\|\,\|X^p_k(\cdot,a)\|}\int_{\mathbb{R}^2} \widehat{x_j}(\xi)a\overline{\widehat{\psi}_p(a\xi)}\overline{\widehat{x_k}(\xi)a\overline{\widehat{\psi}_p(a\xi)}e^{-iy\cdot\xi}}d\xi \\
&= \frac{a^2}{(2\pi)^2\|X^p_j(\cdot,a)\|\,\|X^p_k(\cdot,a)\|}\int_{\mathbb{R}^2} \widehat{x_j}(\xi)\overline{\widehat{x_k}(\xi)}\big|\widehat{\psi}_p(a\xi)\big|^2 e^{iy\cdot\xi}d\xi \\
&= \frac{a^2}{\|X^p_j(\cdot,a)\|\,\|X^p_k(\cdot,a)\|}\mathcal{F}^{-1}\Big[\widehat{x_j}(\xi)\overline{\widehat{x_k}(\xi)}\big|\widehat{\psi}_p(a\xi)\big|^2\Big](y).
\end{aligned}$$

Recall that the observed images $x_j(t)$ and $x_k(t-y)$ are

$$x_j(t) = \sum_{n=1}^{N} d_{j,n}s_n(t-c_{j,n}),$$

$$x_k(t-y) = \sum_{n=1}^{N} d_{k,n}s_n(t-y-c_{k,n}).$$

If we choose $y = c_{j,q} - c_{k,q}$, then the shift parameter of the original image s_q in $x_k(t-y)$ coincides with the shift parameter $c_{j,q}$ of s_q in $x_j(t)$. Moreover, if the original image s_q contains edges which can be accessed by a wavelet ψ_p with a scale a, then $|R^p_{j,k}(y,a)|$ is large. Let us discretize a and denote by A the set of discretized scales. For fixed j and k, we define the total correlation function $R_{j,k}(y)$ by

$$R_{j,k}(y) = \sum_{a\in A}\sum_{p=1}^{P}\sqrt{\big|R^p_{j,k}(y,a)\big|}. \tag{2.1}$$

Here we take the square root of $|R^p_{j,k}(y,a)|$ in (2.1) to enhance the sensitivity. For $P = 20$ and $A = \{2^{-1/2}, 1, 2^{1/2}\}$, we illustrate total correlation functions $R_{1,2}(y)$, $R_{2,3}(y)$, $R_{3,4}(y)$ and $R_{4,1}(y)$ in Fig. 3. Note that the square roots enhance contrast of peaks.

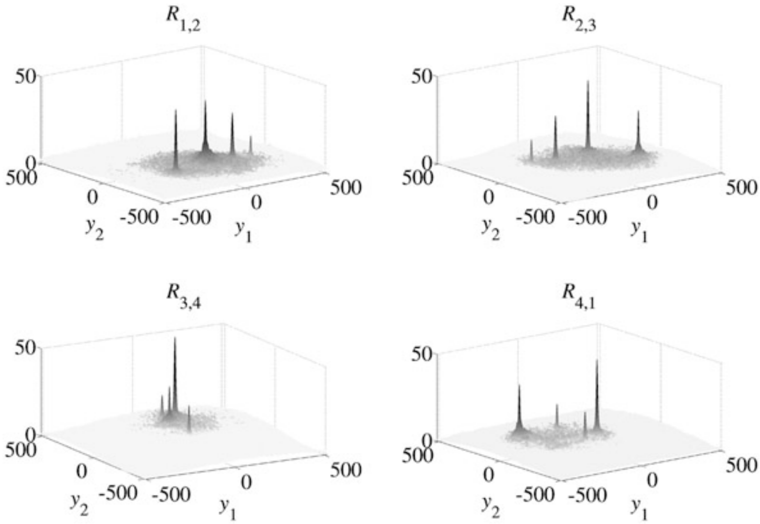

Fig. 3 Total correlation functions of the observed images in Fig. 1. *Top left*: $R_{1,2}$, *top right*: $R_{2,3}$, *bottom left*: $R_{3,4}$, *bottom right*: $R_{4,1}$. $P = 20$, $A = \{2^{-1/2}, 1, 2^{1/2}\}$

We estimate the number of sources, $\widetilde{N}$, by the number of peaks in Fig. 3. Then, we have $\widetilde{N} = 4$. The coordinates where $R_{j,k}(y)$ attain peaks correspond to relative shift parameters $\{c_{j,q} - c_{k,q}\}_{q=1,\ldots,4}$.

3 Image Separation Algorithm

We propose Algorithm 3.1 with the annular sector multiwavelets [4].

Algorithm 3.1 We estimate model parameters using the following five steps.

1. Illustrate total correlation functions $R_{j,j+1}(y)$, $j = 1, \ldots, J-1$ and $R_{J,1}(y)$. The number of sources, N, is estimated by the number of peaks, $\widetilde{N}$, in each total correlation function.
2. Take sets of coordinates where $R_{j,j+1}(y)$ and $R_{J,1}(y)$ attain peaks. These coordinates correspond to relative shift parameters
$$V_j = \left\{v_j^q \approx c_{j,q} - c_{j+1,q}\right\}_{q=1,\ldots,\widetilde{N}}, \quad j = 1, \ldots, J-1,$$
$$V_J = \left\{v_J^q \approx c_{J,q} - c_{1,q}\right\}_{q=1,\ldots,\widetilde{N}}.$$
3. Since $\sum_{j=1}^{J-1}(c_{j,q} - c_{j+1,q}) + (c_{J,q} - c_{1,q}) = 0$, select the vector of relative shift parameters associated with each source image $\widetilde{s}_q$
$$\left\{\vec{v}_q := \left(v_1^q, \ldots, v_J^q\right) \in V_1 \times \cdots \times V_J \,\middle|\, \sum_{j=1}^{J} v_j^q = 0\right\}_{q=1,\ldots,\widetilde{N}}.$$

4. It is enough to estimate the relative shift parameters $c_{j+1,q} - c_{1,q}$ with respect to the source image $\widetilde{s}_q$ by

$$\widetilde{c}_{j+1,q} = -\sum_{k=1}^{j} v_k^q, \quad j = 1, \ldots, J-1, \ q = 1, \ldots, \widetilde{N},$$

where $\widetilde{c}_{j+1,q} \approx c_{j+1,q} - c_{1,q}$. Set $\widetilde{c}_{1,q} = (0,0)$, $q = 1, \ldots, \widetilde{N}$.

5. For the set of shifted observed images

$$\left\{x_1(t + \widetilde{c}_{1,q}), x_2(t + \widetilde{c}_{2,q}), \ldots, x_J(t + \widetilde{c}_{J,q})\right\},$$

use the algorithm [4, Algorithm 12, §6] and take the J-dimensional vector, $(\widetilde{d}_{1,q}, \ldots, \widetilde{d}_{J,q})^T$, corresponding to the highest peak. Estimate the mixing matrix by $\widetilde{D} = (\widetilde{d}_{j,q})_{1 \le j \le J, 1 \le q \le \widetilde{N}}$.

In the step 5 of Algorithm 3.1, the source image $\widetilde{s}_q$ lies in the same position in each shifted observed image of

$$\left\{x_1(t + \widetilde{c}_{1,q}), x_2(t + \widetilde{c}_{2,q}), \ldots, x_J(t + \widetilde{c}_{J,q})\right\}.$$

Therefore, we can apply the algorithm [4, Algorithm 12, §6], which solves the blind source separation problem with temporal mixing model, that is, all $c_{j,n} = 0$, to the set of shifted observed images.

After estimating all model parameters, we have

$$x_j(t) = \sum_{n=1}^{\widetilde{N}} \widetilde{d}_{j,n} \widetilde{s}_n(t - \widetilde{c}_{j,n}), \quad j = 1, \ldots, J. \tag{3.1}$$

Taking the Fourier transform of (3.1), we have the following linear system:

$$\vec{\hat{x}}(\xi) = D(\xi) \vec{\hat{\widetilde{s}}}(\xi), \tag{3.2}$$

where

$$\vec{\hat{x}}(\xi) = \left(\widehat{x}_1(\xi), \ldots, \widehat{x}_J(\xi)\right)^T,$$
$$\vec{\hat{\widetilde{s}}}(\xi) = \left(\widehat{\widetilde{s}}_1(\xi), \ldots, \widehat{\widetilde{s}}_{\widetilde{N}}(\xi)\right)^T,$$
$$D(\xi) = \left(\widetilde{d}_{j,n} e^{-i\widetilde{c}_{j,n} \cdot \xi}\right)_{j=1,\ldots,J,\, n=1,\ldots,\widetilde{N}}.$$

When $D(\xi)$ is invertible for each ξ, we can solve (3.2) for each ξ. Taking the inverse Fourier transform of $\vec{\hat{\widetilde{s}}}(\xi)$, we have estimated original images. We separate the observed images in Fig. 1 into the estimated source images in Fig. 4.

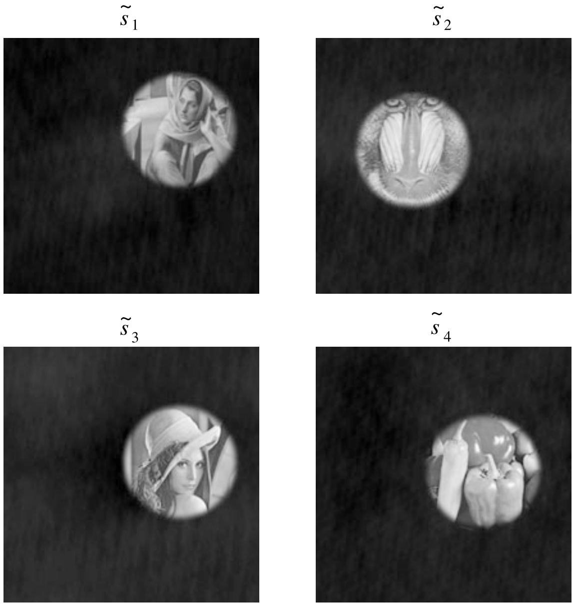

Fig. 4 The estimated original images

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Slice Functional Calculus in Quaternionic Hilbert Spaces

R. Ghiloni, V. Moretti, and A. Perotti

Abstract We propose a continuous functional calculus in quaternionic Hilbert spaces. The class of continuous functions considered is the one of slice quaternionic functions. Slice functions generalize the concept of slice regular function, which comprises power series with quaternionic coefficients on one side and that can be seen a generalization to quaternions of holomorphic functions of one complex variable. The notion of slice function allows to introduce suitable classes of real, complex and quaternionic C^*-algebras and to define, on each of these C^*-algebras, a functional calculus for quaternionic normal operators.

Keywords Quaternionic Hilbert space · Functional calculus · Slice functions · Spectral map theorem

Mathematics Subject Classification (2010) Primary 46S10 · Secondary 47A60 · 47C15 · 30G35 · 32A30 · 81R15

1 Introduction

We start from basic issues regarding the general notion of *spherical spectrum* of an operator on a (right) quaternionic Hilbert space. For the definition of the spectrum we follow the viewpoint adopted in [3] for quaternionic Banach modules. A pivotal tool in our investigation is the notion of *slice function* [8]. That notion allows one to introduce suitable classes of real, complex and quaternionic C^*-algebras of functions and to define, on each of these C^*-algebras, a functional calculus for normal

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operators. In particular, we establish several versions of the *spectral map theorem*. For quaternionic Hilbert spaces, a formulation of the spectral theorem already exists [9] without any systematic investigation of the continuous functional calculus. We also show that our continuous functional calculus, when restricted to slice regular functions, coincides with the functional calculus developed in [3] as a generalization of the classical holomorphic functional calculus. We refer to [6] for complete proofs of the stated result.

2 Quaternionic Hilbert Spaces

We recall some basic notions about quaternionic Hilbert spaces (see e.g. [1]). Let $\mathbb{H}$ denote the skew field of quaternions. Let H be a *right* $\mathbb{H}$-module. H is called a *quaternionic pre-Hilbert space* if there exists a Hermitian quaternionic scalar product $\mathsf{H} \times \mathsf{H} \ni (u, v) \mapsto \langle u|v\rangle \in \mathbb{H}$ satisfying the following three properties:

- *Right linearity*: $\langle u|vp + wq\rangle = \langle u|v\rangle p + \langle u|w\rangle q$ if $p, q \in \mathbb{H}$ and $u, v, w \in \mathsf{H}$.
- *Quaternionic Hermiticity*: $\langle u|v\rangle = \overline{\langle v|u\rangle}$ if $u, v \in \mathsf{H}$.
- *Positivity*: If $u \in \mathsf{H}$, then $\langle u|u\rangle \in \mathbb{R}^+$ and $u = 0$ if $\langle u|u\rangle = 0$.

We can define the *quaternionic norm* by setting

$$\|u\| := \sqrt{\langle u|u\rangle} \in \mathbb{R}^+ \quad \text{if } u \in \mathsf{H}.$$

Definition 2.1 A quaternionic pre-Hilbert space H is said to be a *quaternionic Hilbert space* if it is complete with respect to its natural distance $d(u, v) := \|u - v\|$.

Example The space $\mathbb{H}^n$ with scalar product $\langle u, v\rangle = \sum_{i=1}^n \bar{u}_i v_i$ is a finite-dimensional quaternionic Hilbert space.

Definition 2.2 A *right* $\mathbb{H}$*-linear operator* is a map $T : D(T) \longrightarrow \mathsf{H}$ such that:

$$T(ua + vb) = (Tu)a + (Tv)b \quad \text{if } u, v \in D(T) \text{ and } a, b \in \mathbb{H},$$

where the *domain* $D(T)$ of T is a (not necessarily closed) right $\mathbb{H}$-linear subspace of H.

It can be shown that an operator $T : D(T) \longrightarrow \mathsf{H}$ is continuous if and only if it is bounded, i.e. there exists $K \geq 0$ such that

$$\|Tu\| \leq K\|u\| \quad \text{for each } u \in D(T).$$

The set $\mathfrak{B}(\mathsf{H})$ of all bounded operators $T : \mathsf{H} \longrightarrow \mathsf{H}$ is a complete metric space w.r.t. the metric $D(T, S) := \|T - S\|$, where $\|T\| := \sup_{u \in D(T)\setminus\{0\}} \frac{\|Tu\|}{\|u\|} = \inf\{K \in \mathbb{R} \mid \|Tu\| \leq K\|u\| \ \forall u \in D(T)\}$.

Many assertions that are valid in the complex Hilbert spaces case, continue to hold for quaternionic operators. We mention the uniform boundedness principle, the open map theorem, the closed graph theorem, the Riesz representation theorem and the polar decomposition of operators.

Left Scalar Multiplications It is possible to equip a (right) quaternionic Hilbert space H with a *left* multiplication by quaternions. It is a non-canonical operation relying upon a choice of a preferred Hilbert basis. So, pick out a Hilbert basis $\mathcal{N}$ of H and define the *left scalar multiplication of* H *induced by* $\mathcal{N}$ as the map $\mathbb{H} \times \mathsf{H} \ni (q, u) \mapsto qu \in \mathsf{H}$ given by

$$qu := \sum_{z \in \mathcal{N}} zq\langle z|u\rangle \quad \text{if } u \in \mathsf{H} \text{ and } q \in \mathbb{H}.$$

For every $q \in \mathbb{H}$, the map $L_q : u \mapsto qu$ belongs to $\mathfrak{B}(\mathsf{H})$. Moreover, the map $\mathcal{L}_{\mathcal{N}} : \mathbb{H} \longrightarrow \mathfrak{B}(\mathsf{H})$, defined by setting $\mathcal{L}_{\mathcal{N}}(q) := L_q$ is a norm-preserving real algebra homomorphism.

The set $\mathfrak{B}(\mathsf{H})$ is always a *real Banach C^*-algebra with unity*. It suffices to consider the right scalar multiplication $(Tr)(u) = T(u)r$ for real r and the adjunction $T \mapsto T^*$ as *-involution. By means of a left scalar multiplication, it can be given the richer structure of *quaternionic Banach C^*-algebra*.

Theorem 2.1 *Let* H *be a quaternionic Hilbert space equipped with a left scalar multiplication. Then the set* $\mathfrak{B}(\mathsf{H})$, *equipped with the pointwise sum*, *with the scalar multiplications defined by*

$$(qT)u := q(Tu) \quad \textit{and} \quad (Tq)(u) := T(qu),$$

with the composition as product, *with* $T \mapsto T^*$ *as* **-involution*, *is a* quaternionic two-sided Banach C^*-algebra with unity.

Observe that the map $\mathcal{L}_{\mathcal{N}}$ gives a *-representation of $\mathbb{H}$ in $\mathfrak{B}(\mathsf{H})$.

3 Resolvent and Spectrum

It is not clear how to extend the definitions of spectrum and resolvent in quaternionic Hilbert spaces. Let us focus on the simpler case of eigenvalues of a bounded right $\mathbb{H}$-linear operator T. Without fixing any left scalar multiplication of H, the equation determining the eigenvalues reads as follows:

$$Tu = uq.$$

Here a drawback arises: if $q \in \mathbb{H} \setminus \mathbb{R}$ is fixed, the map $u \mapsto uq$ is not right $\mathbb{H}$-linear. Consequently, the eigenspace of q cannot be a right $\mathbb{H}$-linear subspace. Indeed, if $\lambda \neq 0$, $u\lambda$ is an eigenvector of $\lambda^{-1}q\lambda$ instead of q itself. As a second guess, one could decide to deal with quaternionic Hilbert spaces equipped with a left scalar multiplication and require that

$$Tu = qu.$$

Now both sides are right $\mathbb{H}$-linear. However, this approach is not suitable for physical applications, where self-adjoint operators should have real spectrum. We come back to the former approach and accept that each eigenvalue q brings a whole conjugation class of the quaternions, the *eigensphere*

$$\mathbb{S}_q := \left\{\lambda^{-1} q \lambda \in \mathbb{H} \;\middle|\; \lambda \in \mathbb{H} \setminus \{0\}\right\}.$$

We adopt the viewpoint introduced in [3] for quaternionic two-sided Banach modules. Given an operator $T : D(T) \longrightarrow \mathsf{H}$ and $q \in \mathbb{H}$, let

$$\Delta_q(T) := T^2 - T(q + \overline{q}) + \mathrm{I}|q|^2.$$

Definition 3.1 The *spherical resolvent set* of T is the set $\rho_S(T)$ of $q \in \mathbb{H}$ such that:

(a) $Ker(\Delta_q(T)) = \{0\}$.
(b) $Range(\Delta_q(T))$ is dense in H.
(c) $\Delta_q(T)^{-1} : Range(\Delta_q(T)) \longrightarrow D(T^2)$ is bounded.

The *spherical spectrum* $\sigma_S(T)$ of T is defined by $\sigma_S(T) := \mathbb{H} \setminus \rho_S(T)$. It decomposes into three disjoint *circular* (i.e. invariant by conjugation) subsets:

(i) The *spherical point spectrum* of T (the set of *eigenvalues*):

$$\sigma_{pS}(T) := \left\{q \in \mathbb{H} \;\middle|\; Ker\big(\Delta_q(T)\big) \neq \{0\}\right\}.$$

(ii) The *spherical residual spectrum* of T:

$$\sigma_{rS}(T) := \left\{q \in \mathbb{H} \;\middle|\; Ker\big(\Delta_q(T)\big) = \{0\}, \overline{Range\big(\Delta_q(T)\big)} \neq \mathsf{H}\right\}.$$

(iii) The *spherical continuous spectrum* of T:

$$\sigma_{cS}(T) := \left\{q \in \mathbb{H} \;\middle|\; \Delta_q(T)^{-1} \text{ is densely defined but not bounded}\right\}.$$

The *spherical spectral radius* of T is defined as

$$r_S(T) := \sup\left\{|q| \;\middle|\; q \in \sigma_S(T)\right\} \in \mathbb{R}^+ \cup \{+\infty\}.$$

In our context, the subspace $Ker(\Delta_q(T))$ has the role of an "eigenspace". In particular, $Ker(\Delta_q(T)) \neq \{0\}$ if and only if $\mathbb{S}_q$ is an eigensphere of T.

3.1 Spectral Properties

The spherical resolvent and the spherical spectrum can be defined for bounded right $\mathbb{H}$-linear operators on quaternionic two-sided Banach modules in a form similar to that introduced above (see [3]). Several spectral properties of bounded operators on complex Banach or Hilbert spaces remain valid in that general context. Here we recall some of these properties in the quaternionic Hilbert setting.

Theorem 3.1 *Let* H *be a quaternionic Hilbert space and let* $T \in \mathfrak{B}(\mathsf{H})$. *Then*

(a) $r_S(T) \leq \|T\|$.
(b) $\sigma_S(T)$ *is a non-empty compact subset of* $\mathbb{H}$.
(c) *Let* $P \in \mathbb{R}[X]$. *Then, if* T *is self-adjoint, the following* spectral map property *holds*:

$$\sigma_S\big(P(T)\big) = P\big(\sigma_S(T)\big).$$

(d) Gelfand's spectral radius formula *holds*:

$$r_S(T) = \lim_{n\to+\infty} \left\|T^n\right\|^{1/n}.$$

In particular, if T *is* normal (*i.e.* $TT^* = T^*T$), *then* $r_S(T) = \|T\|$.

Regardless different definitions with respect to the complex Hilbert space case, the notions of spherical spectrum and resolvent set enjoy some properties which are quite similar to those for complex Hilbert spaces. Other features, conversely, are proper to the quaternionic Hilbert space case. First of all, it turns out that the spherical point spectrum coincides with the set of eigenvalues of T.

Proposition 3.2 *Let* H *be a quaternionic Hilbert space and let* $T : D(T) \longrightarrow \mathsf{H}$ *be an operator. Then* $\sigma_{pS}(T)$ *coincides with the set of all eigenvalues of* T.

Theorem 3.2 *Let* T *be an operator with dense domain on a quaternionic Hilbert space* H.

(a) $\sigma_S(T) = \sigma_S(T^*)$.
(b) *If* $T \in \mathfrak{B}(\mathsf{H})$ *is normal, then*
   (i) $\sigma_{pS}(T) = \sigma_{pS}(T^*)$.
   (ii) $\sigma_{rS}(T) = \sigma_{rS}(T^*) = \emptyset$.
   (iii) $\sigma_{cS}(T) = \sigma_{cS}(T^*)$.
(c) *If* T *is self-adjoint, then* $\sigma_S(T) \subset \mathbb{R}$ *and* $\sigma_{rS}(T)$ *is empty.*
(d) *If* T *is anti self-adjoint, then* $\sigma_S(T) \subset \mathrm{Im}(\mathbb{H})$ *and* $\sigma_{rS}(T)$ *is empty.*
(e) *If* $T \in \mathfrak{B}(\mathsf{H})$ *is unitary, then* $\sigma_S(T) \subset \{q \in \mathbb{H} \mid |q| = 1\}$.
(f) *If* $T \in \mathfrak{B}(\mathsf{H})$ *is anti self-adjoint and unitary, then* $\sigma_S(T) = \sigma_{pS}(T) = \mathbb{S}$ (*the sphere of quaternionic imaginary units*).

It can be shown that, differently from operators on complex Hilbert spaces, a normal operator T on a quaternionic space is unitarily equivalent to T^*.

4 Slice Functions

The concept of *slice regularity* has been introduced by Gentili and Struppa [4, 5] for functions of one quaternionic variable and then extended to other real *-algebras (e.g. Clifford algebras [2] and alternative *-algebras [7, 8]). This function theory comprises polynomials and power series with quaternionic coefficients on one side. At the base of the definition there is the "slice" character of $\mathbb{H}$:

- $\mathbb{H} = \bigcup_{J \in \mathbb{S}} \mathbb{C}_J$ where $\mathbb{C}_J$ is the real subalgebra $\langle J \rangle \simeq \mathbb{C}$.
- $\mathbb{C}_J \cap \mathbb{C}_K = \mathbb{R}$ for every $J, K \in \mathbb{S}$ with $J \neq \pm K$.

The original definition requires that, for every $J \in \mathbb{S}$, the restriction $f_{|\mathbb{C}_J}$ is holomorphic with respect to the complex structures given by left multiplication by J. Another approach (see [7, 8]) starts from the embedding of the space of slice regular functions into a larger class, that of continuous *slice functions*. Given $\mathcal{K} \subset \mathbb{C}$, consider the *circular set* $\Omega_{\mathcal{K}}$ defined by

$$\Omega_{\mathcal{K}} = \{\alpha + J\beta \in \mathbb{H} \mid \alpha, \beta \in \mathbb{R},\ \alpha + i\beta \in \mathcal{K},\ J \in \mathbb{S}\}.$$

Let $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} = \{x + iy \mid x, y \in \mathbb{H}\}$, with complex conjugation $w = x + iy \mapsto \overline{w} = x - iy$. A function $F : \mathcal{K} \longrightarrow \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$ satisfying $F(\overline{z}) = \overline{F(z)}$ for every $z \in \mathcal{K}$, is called a *stem function* on $\mathcal{K}$. Any stem function induces a (left) *slice function* $f = \mathcal{I}(F) : \Omega_{\mathcal{K}} \to \mathbb{H}$: if $q = \alpha + J\beta \in \Omega_{\mathcal{K}} \cap \mathbb{C}_J$, with $J \in \mathbb{S}$, we set

$$f(q) := F_1(\alpha + i\beta) + J F_2(\alpha + i\beta)$$

where F_1, F_2 are the two $\mathbb{H}$-valued components of F. A quaternionic function f turns out to be *slice regular* if and only if it is the slice function induced by a *holomorphic* stem function F.

4.1 C-Algebras of Slice Functions*

Given two slice functions $f = \mathcal{I}(F)$ and $g = \mathcal{I}(G)$ on $\Omega_{\mathcal{K}}$, their pointwise product is not necessarily a slice function. However, we can define their *slice product* by means of the multiplication in $\mathbb{H} \otimes \mathbb{C}$:

$$f \cdot g = \mathcal{I}(FG) = \mathcal{I}\big((F_1G_1 - F_2G_2) + i(F_1G_2 + F_2G_1)\big).$$

Theorem 4.1 *The set $\mathcal{S}(\Omega_{\mathcal{K}}, \mathbb{H})$ of continuous slice functions on $\Omega_{\mathcal{K}}$ is a quaternionic two-sided Banach C^*-algebra with unity the constant function $1_{\Omega_{\mathcal{K}}}$ w.r.t. the slice product, the *-involution defined by $f^* := \mathcal{I}(\overline{F_1} - i\overline{F_2})$ and the supremum norm.*

Remark 4.1 The scalar multiplications on $\mathcal{S}(\Omega_{\mathcal{K}}, \mathbb{H})$ are defined by $f \cdot q := f \cdot c_q$ and $q \cdot f := c_q \cdot f$, where c_q denotes the constant slice function with value q on $\Omega_{\mathcal{K}}$. $f \cdot q$ coincides with the pointwise scalar multiplication fq for every $q \in \mathbb{H}$. If $q \in \mathbb{R}$, then also $q \cdot f$ is equal to the pointwise scalar multiplication qf. Otherwise, qf is not, in general, a slice function and hence is different from $q \cdot f$.

5 Slice Functional Calculus

5.1 Slice Nature of Normal Operators

The definition of a continuous slice function of a normal operator on a quaternionic Hilbert space is based on the "operatorial" counterpart of the slice character of $\mathbb{H}$.

Theorem 5.1 *Given any normal operator $T \in \mathfrak{B}(\mathsf{H})$, there exist three operators $A, B, J \in \mathfrak{B}(\mathsf{H})$ such that*:

(i) $T = A + JB$.
(ii) *A is self-adjoint and B is positive.*
(iii) *J is anti self-adjoint and unitary.*
(iv) *A, B and J commute mutually.*

Furthermore, it holds:

- *A and B are uniquely determined by T: $A = (T + T^*)\frac{1}{2}$ and $B = |T - T^*|\frac{1}{2}$.*
- *J is uniquely determined by T on $Ker(T - T^*)^{\perp}$.*

(*where for $S \in \mathfrak{B}(\mathsf{H})$, $|S|$ denotes the operator defined as the square root of the positive operator S^*S*).

This parallelism suggests a natural way to define the operator $f(T)$ for the class of $\mathbb{H}$-*intrinsic* continuous slice functions, i.e. functions satisfying $f(\bar{q}) = \overline{f(q)}$ for every $q \in \Omega_{\mathcal{K}}$ or, equivalently, $f(\Omega_{\mathcal{K}} \cap \mathbb{C}_J) \subset \mathbb{C}_J$ $\forall J \in \mathbb{S}$. If $f = \mathcal{I}(F) = \mathcal{I}(F_1 + iF_2)$ is a *polynomial* slice function, with components F_1, $F_2 \in \mathbb{R}[X, Y]$, we define the normal operator $f(T) \in \mathfrak{B}(\mathsf{H})$ by setting

$$f(T) := F_1(A, B) + JF_2(A, B)$$

and then extend the definition to $\mathbb{H}$-intrinsic *continuous* slice functions on $\sigma_S(T)$ by density.

Remark 5.1

(i) f is $\mathbb{H}$-intrinsic if and only if the components F_1, F_2 of the stem function F are real valued.
(ii) Even when J is not uniquely determined, $f(T)$ does *not* depend on the choice of the operator J.

Continuous Slice Functional Calculus for Normal Operators: The $\mathbb{H}$-Intrinsic Functions Case Consider the commutative real Banach C^*-subalgebra $\mathcal{S}_{\mathbb{R}}(\sigma_S(T), \mathbb{H})$ of $\mathcal{S}(\sigma_S(T), \mathbb{H})$ consisting of $\mathbb{H}$-intrinsic slice functions. The functional calculus $f \mapsto f(T)$ defined above has the following properties.

Theorem 5.2 *The mapping $f \mapsto f(T)$ is the unique continuous *-homomorphism*

$$\Psi_{\mathbb{R},T} : \mathcal{S}_{\mathbb{R}}\big(\sigma_S(T), \mathbb{H}\big) \to \mathfrak{B}(\mathsf{H})$$

of real Banach unital C^*-algebras such that:

(i) $\Psi_{\mathbb{R},T}$ *is unity-preserving*; *that is*, $\Psi_{\mathbb{R},T}(1_{\sigma_S(T)}) = \mathrm{I}$.
(ii) $\Psi_{\mathbb{R},T}(id) = T$.

Moreover, the following facts hold true:

(a) $f(T)$ *is normal.*
(b) $\Psi_{\mathbb{R},T}$ *is isometric*: $\|f(T)\| = \|f\|_\infty$ *for every* $f \in \mathcal{S}_{\mathbb{R}}(\sigma_S(T), \mathbb{H})$.
(c) *The* spectral map property $\sigma_S(f(T)) = f(\sigma_S(T))$ *holds.*

Continuous Slice Functional Calculus for Normal Operators: The $\mathbb{C}_J$-Slice Functions Case The definition of $f(T)$ can be extended to other classes of continuous slice functions. The set $\mathcal{S}_{\mathbb{C}_J}(\Omega_{\mathcal{K}}, \mathbb{H})$ of functions which leave only *one* slice $\mathbb{C}_J$ invariant is a commutative $\mathbb{C}_J$-Banach unital C^*-subalgebra of $\mathcal{S}(\Omega_{\mathcal{K}}, \mathbb{H})$. The space $\mathfrak{B}(\mathsf{H})$ has a similar structure of complex C^*-algebra depending on the choice of the anti self-adjoint operator J such that $T = A + JB$ and J commutes with T, T^*.

Theorem 5.3 *There exists a unique continuous *-homomorphism*

$$\Psi_{\mathbb{C}_J,T} : \mathcal{S}_{\mathbb{C}_J}\big(\sigma_S(T), \mathbb{H}\big) \to \mathfrak{B}(\mathsf{H})$$

of $\mathbb{C}_J$-Banach C^*-algebras such that

(i) $\Psi_{\mathbb{C}_J,T}$ *is unity-preserving*; *that is*, $\Psi_{\mathbb{R},T}(1_{\sigma_S(T)}) = \mathrm{I}$.
(ii) $\Psi_{\mathbb{C}_J,T}(id) = T$.

Moreover, the following facts hold true:

(a) $f(T)$ *is normal.*
(b) *For every* $f \in \mathcal{S}_{\mathbb{C}_J}(\sigma_S(T), \mathbb{H})$, *the following* $\mathbb{C}_J$-slice spectral map property *holds*:

$$\sigma_S\big(f(T)\big) = \Omega_{f(\sigma_S(T)\cap\mathbb{C}_J^+)}.$$

(c) $\Psi_{\mathbb{C}_J,T}$ *is norm decreasing*: $\|f(T)\| \leq \|f\|_\infty$ *if* $f \in \mathcal{S}_{\mathbb{C}_J}(\sigma_S(T), \mathbb{H})$. *More precisely, it holds*:

$$\big\|f(T)\big\| = \|f|_{\sigma_S(T)\cap\mathbb{C}_J^+}\|_\infty$$

for every $f \in \mathcal{S}_{\mathbb{C}_J}(\sigma_S(T), \mathbb{H})$.

Continuous Slice Functional Calculus for Normal Operators: The Circular Case Fix an anti self-adjoint and unitary operator $J \in \mathfrak{B}(\mathsf{H})$ such that $T = A + JB$ and J commutes with T, T^*. Choose $\jmath \in \mathbb{S}$ and a left scalar multiplication $q \mapsto L_q$ with $L_\jmath = J$ and $L_q A = A L_q$ and $L_q B = B L_q$ for each $q \in \mathbb{H}$. Then $\mathfrak{B}(\mathsf{H})$ get a structure of quaternionic two-sided Banach unital C^*-algebra.

The set $\mathcal{S}_c(\Omega_{\mathcal{K}}, \mathbb{H})$ of *circular* slice functions, those which satisfy the condition $f(\bar{q}) = f(q)$ for every q, is a non-commutative quaternionic Banach C^*-subalgebra of $\mathcal{S}(\Omega_{\mathcal{K}}, \mathbb{H})$.

Theorem 5.4 *There exists a unique continuous (isometric) *-homomorphism*

$$\Psi_{c,T} : \mathcal{S}_c\big(\sigma_S(T), \mathbb{H}\big) \to \mathfrak{B}(\mathsf{H})$$

of quaternionic Banach C^*-algebras such that

(i) $\Psi_{c,T}$ *is unity-preserving; that is,* $\Psi_{\mathbb{R},T}(1_{\sigma_S(T)}) = \mathrm{I}$.
(ii) $\Psi_{c,T}(id) = T$.

Continuous Slice Functional Calculus for Normal Operators: The General Case The previous definitions of $f(T)$ can be extended to a generic continuous slice function $f \in \mathcal{S}(\sigma_S(T), \mathbb{H})$. We get a map $f \mapsto f(T)$ that is $\mathbb{R}$-linear and continuous: there exists $C > 0$ such that

$$\big\| f(T) \big\| \le C \| f \|_\infty$$

for every $f \in \mathcal{S}(\sigma_S(T), \mathbb{H})$. In the general case the *-homomorphism property is necessarily lost. However, if e.g. $f \in \mathcal{S}_{\mathbb{C}_\jmath}(\sigma_S(T), \mathbb{H})$ or $g \in \mathcal{S}_c(\sigma_S(T), \mathbb{H})$, then the multiplicative property

$$(f \cdot g)(T) = f(T) g(T)$$

remains true.

5.2 *The Slice Regular Case*

A functional calculus for slice regular functions of a bounded operator T on a quaternionic two-sided Banach module V has been developed in [3] as a generalization of the holomorphic functional calculus. Let $S_L^{-1}(s, x)$ denote the *Cauchy kernel* for slice regular functions (cf. [3] or [8]).

Definition 5.2 [3, Definition 4.10.4] Let f be slice regular on $\Omega_D \supset \sigma_S(T)$. Fix any $\jmath \in \mathbb{S}$ and define

$$f(T)_{reg} := \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_\jmath)} S_L^{-1}(s, T) ds_\jmath{}^{-1} f(s) \in \mathfrak{B}(V).$$

It can be shown that on slice regular functions our continuous calculus for normal operators on a quaternionic Hulbert space H coincides with the one defined by means of the Cauchy integral.

Proposition 5.3 *Let* $T \in \mathfrak{B}(\mathsf{H})$ *be normal and let* $f : U \longrightarrow \mathbb{H}$ *be a slice regular function defined on a circular open neighborhood of* $\sigma_S(T)$ *in* $\mathbb{H}$. *Then* $f(T)_{reg} = f|_{\sigma_S(T)}(T)$, *that is, the two functional calculi coincide if* T *is normal and* f *is slice regular.*

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Recent Progress on Spheroidal Monogenic Functions

Hung Manh Nguyen

Abstract Monogenic function theories are considered as generalizations of the holomorphic function theory in the complex plane to higher dimensions and are refinements of the harmonic analysis based on the Laplace operator's factorizations. The construction of spherical monogenic functions has been studied for decades with different methods. Recently, orthogonal monogenic bases are developed for spheroidal reference domains, first by J. Morais and later by others. This survey will go through the construction of spheroidal monogenic functions and discuss up-to-date results.

Keywords Harmonic functions · Monogenic functions · Quaternion analysis · Spheroidal functions

Mathematics Subject Classification (2010) 30G35 · 42C05 · 33E10

1 Introduction

The theory of harmonic functions plays an important role in many fields, both in pure and applied aspects. It can be seen, for example, in gravitational potential problems or approximation of the Earth's gravity and magnetic fields (see [18, 23, 31]). Spherical harmonic functions are used frequently because of their simple form and easy calculation. It is preferred for (almost) symmetric geometries. For asymmetric cases, it is inappropriate as shown in [31]. Simple generalizations of spherical domains are ellipsoidal domains. Garabedian introduced in [14] sets of orthogonal harmonic polynomials over prolate and oblate spheroids taken in several different norms. It is the root of the construction of orthogonal spheroidal monogenic functions, since monogenic functions can be obtained by applying the hypercomplex derivative to harmonic functions. The construction of Green's function for the Laplace equation on an ellipsoid of revolution has been studied by means of

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ellipsoidal harmonic functions in several articles (see [12, 22]). As refinements of harmonic functions, these spheroidal monogenic functions could play a role to solve some Dirichlet problems in spheroids.

There are several methods to construct a complete system of monogenic functions. Fueter variables $z_i = x_i - x_0\mathbf{e_i}$ $(i = 1, \dots, n)$, named after R. Fueter [13], were introduced as an idea to construct bases of homogeneous monogenic polynomials (see [6, 15, 21]). The construction is completely independent of endowed inner products. In general, the obtained sets of monogenic functions are not orthogonal. For spherical domains, one can obtain a complete orthogonal system of monogenic functions by the Gelfand–Tsetlin procedure which calculates functions by induction and then it costs time and memory. More information about the method can be found in [5, 20]. The harmonic function approach was developed based on factorizations of the Laplace operator in terms of Cauchy–Riemann or Dirac operators. To the best of our knowledge, I. Cação firstly used it to construct orthogonal bases for L_2-spaces of reduced quaternion ($\mathcal{A}$)—or quaternion ($\mathbb{H}$)-valued monogenic functions which are solutions of Riesz or Moisil–Theodorescu systems on the unit ball (cf. [7, 9]). The hypercomplex derivative and the monogenic primitive are also studied in [8, 10, 11]. Later on, S. Bock modified the $\mathbb{H}$-valued elements of the basis with respect to the Riesz system to obtain the Appell property. This property was introduced by Appell [1] by generalizing $\frac{d}{dx}x^n = nx^{n-1}$ to more general polynomial systems. Also, S. Bock proved recurrence formulae, an explicit representation formula for polynomials [2–4] and applied it to solve a boundary value problem for the equations of linear elasticity in spherical domains. $\mathcal{A}$-valued solutions of the Riesz system were also researched by J. Morais in the quaternionic setting in a similar way. Properties were investigated such as real part theorems, Bohr's type theorem and local mapping properties by means of spherical monogenic functions (cf. [16, 17, 24]).

The aim of this paper is to give a brief survey about the construction of complete orthogonal monogenic systems on spheroidal domains. In Sect. 3, inner prolate and oblate spheroidal monogenic functions will be revisited. The recurrence formulae and the explicit presentation will be discussed therein. In applications, we also need information on the exterior domain. That is the reason why in Sect. 4, outer spheroidal monogenic functions in the exterior domain of a prolate spheroid are described. Conclusions will be given in the last section.

2 Preliminaries

Let $\mathbb{H}$ be the algebra of real quaternions generated by the basis $\{\mathbf{1}, \mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ subjected to the multiplication rules

$$\mathbf{e_i}\mathbf{e_j} + \mathbf{e_j}\mathbf{e_i} = -2\delta_{ij}, \quad i, j = 1, 2, 3; \qquad \mathbf{e_1}\mathbf{e_2} = \mathbf{e_3}.$$

Each quaternion can be represented in the form $q = q_0 + q_1\mathbf{e_1} + q_2\mathbf{e_2} + q_3\mathbf{e_3}$ where q_j $(j = 0, \dots, 3)$ are real numbers. Like in the complex case, the conjugate of q is $\overline{q} = q_0 - q_1\mathbf{e_1} - q_2\mathbf{e_2} - q_3\mathbf{e_3}$ and the norm $|q|$ of q is defined by $|q|^2 = q\overline{q} = \overline{q}q = \sum_{j=0}^{3}(q_j)^2$. The real vector space $\mathbb{R}^3$ will be embedded in $\mathbb{H}$ by identifying the element $x = (x_0, x_1, x_2) \in \mathbb{R}^3$ with the reduced quaternion $x := x_0 + x_1\mathbf{e_1} + x_2\mathbf{e_2}$. Denote by $\mathcal{A}$ the real space of all reduced quaternions. The operator $\overline{\partial} = \frac{\partial}{\partial x_0} + \mathbf{e_1}\frac{\partial}{\partial x_1} + \mathbf{e_2}\frac{\partial}{\partial x_2}$ is called generalized Cauchy–Riemann (C–R) operator. Given a domain Ω in $\mathbb{R}^3$, a function $\mathbf{f}$ is called monogenic in Ω if satisfying $\overline{\partial}\mathbf{f}(x) = 0$ for all $x \in \Omega$. The hypercomplex derivative is simply denoted by $\frac{1}{2}\partial$, where ∂ is the conjugate C–R operator. $\mathcal{M}(\Omega, \mathcal{A})$ and $\mathcal{M}(\Omega, \mathbb{H})$ stand for the Hilbert spaces of square integrable $\mathcal{A}$—or $\mathbb{H}$-valued monogenics in Ω respectively, endowed with the inner products

$$\langle \mathbf{f}, \mathbf{g}\rangle_{L^2(\Omega;\mathbb{R})} = \int_\Omega \mathrm{Sc}(\bar{\mathbf{f}}\mathbf{g})dV, \qquad \langle \mathbf{f}, \mathbf{g}\rangle_{L^2(\Omega;\mathbb{H})} = \int_\Omega \bar{\mathbf{f}}\mathbf{g}dV. \tag{2.1}$$

The induced norm is in both cases $\|\mathbf{f}\|_{L^2(\Omega)} = \langle \mathbf{f}, \mathbf{f}\rangle^{\frac{1}{2}}_{L^2(\Omega)}$. In this paper, let Γ be a spheroid with x_0-axis as the symmetry axis. The equation of Γ is given by

$$\frac{x_0^2}{a^2} + \frac{x_1^2 + x_2^2}{b^2} = 1,$$

where $a = c\cosh\mu_0$, $b = c\sinh\mu_0$ (prolate spheroid) or $a = c\sinh\mu_0$, $b = c\cosh\mu_0$ (oblate spheroid). For the sake of simplicity, it is assumed that $c = 1$. We adopt the notations Ω^+ and Ω^- for the interior and exterior domains of Γ, respectively. In particular, $x \in \Omega^-$ can be given by the spheroidal coordinate

$$x_0 = \cosh\mu\cos\theta, \qquad x_1 = \sinh\mu\sin\theta\cos\varphi, \qquad x_2 = \sinh\mu\sin\theta\sin\varphi,$$

for prolate cases or

$$x_0 = \sinh\mu\cos\theta, \qquad x_1 = \cosh\mu\sin\theta\cos\varphi, \qquad x_2 = \cosh\mu\sin\theta\sin\varphi,$$

for oblate cases, with $\mu \in (\mu_0, \infty)$, $\theta \in [0, \pi)$, $\varphi \in [0, 2\pi)$.

3 Inner Spheroidal Monogenics Revisited

Since 2010, J. Morais has intensively investigated sets of prolate spheroidal monogenic functions which play a role for constructing bases in L_2-spaces of monogenic functions in a prolate spheroid over $\mathbb{R}$ and $\mathbb{H}$ in [25, 26] with applications in [19, 27, 28]. An analogous monogenic system can be constructed for oblate spheroids as shown in [30]. In general, spheroidal monogenic functions have the structure as follows:

$$
\begin{aligned}
X_{n,m} &= \frac{n+m+1}{2} A_{n,m}(\mu,\theta)\cos(m\varphi) \\
&\quad + \frac{\delta}{4(n-m+1)} A_{n,m+1}(\mu,\theta)\big\{\cos\big[(m+1)\varphi\big]\mathbf{e_1} + \sin\big[(m+1)\varphi\big]\mathbf{e_2}\big\} \\
&\quad - \frac{\delta(n+m+1)(n+m)(n-m+2)}{4} A_{n,m-1}(\mu,\theta) \\
&\quad \times \big\{\cos\big[(m-1)\varphi\big]\mathbf{e_1} - \sin\big[(m-1)\varphi\big]\mathbf{e_2}\big\} \\
Y_{n,m} &= \frac{(n+m+1)}{2} A_{n,m}(\mu,\theta)\sin(m\varphi) \\
&\quad + \frac{\delta}{4(n-m+1)} A_{n,m+1}(\mu,\theta)\big\{\sin\big[(m+1)\varphi\big]\mathbf{e_1} - \cos\big[(m+1)\varphi\big]\mathbf{e_2}\big\} \\
&\quad - \frac{\delta(n+m+1)(n+m)(n-m+2)}{4} A_{n,m-1}(\mu,\theta) \\
&\quad \times \big\{\sin\big[(m-1)\varphi\big]\mathbf{e_1} + \cos\big[(m-1)\varphi\big]\mathbf{e_2}\big\}
\end{aligned}
$$

where

$$
A_{n,m}(\mu,\theta) = \sum_{k=0}^{[(n-m)/2]} \delta^{k+1} \frac{(2n+1-4k)(n+m-2k+1)_{2k}}{(n-m-2k+1)_{2k+1}} U_{n-2k,m},
$$

with

$$
A_{n,-1}(\mu,\theta) := \begin{cases} -\frac{1}{n(n+1)^2(n+2)} A_{n,1}(\mu,\theta), & n = 1,2,\dots \\ 0, & n = 0, \end{cases}
$$

$m = 0,\dots,n+1$ and $(a)_r = a(a+1)(a+2)\dots(a+r-1)$ with $(a)_0 = 1$, denotes the Pochhammer symbol. The notations δ and $U_{n-2k,m}$ take values $\delta = 1$, $U_{n-2k,m} = P^m_{n-2k}(\cosh\mu)P^m_{n-2k}(\cos\theta)$ in cases of prolate monogenic functions and $\delta = -1$, $U_{n-2k,m} = \mathbf{i}^{n-2k-m} P^m_{n-2k}(\mathbf{i}\sinh\mu)P^m_{n-2k}(\cos\theta)$ in cases of oblate monogenic functions. The first were studied in [25, 26] and the second were studied in [30]. Spheroidal monogenic functions $X_{n,m}$ and $Y_{n,m}$ are $\mathcal{A}$-valued and they form a complete orthogonal system in the space $\mathcal{M}(\Omega^+,\mathcal{A})$. A complete orthogonal system of the space $\mathcal{M}(\Omega^+,\mathbb{H})$ of $\mathbb{H}$-valued monogenic functions can be constructed by functions of the form

$$
\Phi^m_n := X_{n,m} - Y_{n,m}\mathbf{e_3},
$$

with $m = 0,\dots,n$ and $n = 0,1,\dots$ That is similar to the spherical case, investigated by I. Cação [8] and then by S. Bock [3].

A common property of those functions is that they are inhomogenous polynomials. That fact can be seen in [30] as well as in the underlying theorems. That makes it difficult to calculate them numerically. In [30], the authors found several recurrence formulae and their explicit representation in terms of spherical monogenic polynomials. Precisely, one has the following theorems.

Theorem 3.1 *The four-step recurrence formula for* Φ_n^m *is given by*

$$\begin{aligned}\Phi_{n+1}^m = &-\frac{2n+3}{2(n-m+2)(n-m+1)}\big[(2n+3)x+(2n+1)\overline{x}\big]\Phi_n^m \\ &-\frac{(2n+3)(2n+1)(n+m+1)}{(n-m+2)(n-m+1)^2}\overline{x}x\Phi_{n-1}^m \\ &+\frac{(2n+1)(n+m+1)}{2(n-m+2)(n-m+1)^2}\left[2n+3+\frac{(2m+1)^2}{2n-1}\right]\Phi_{n-1}^m \\ &+\frac{(2n+3)(n+m+1)(n+m)}{2(n-m+2)(n-m+1)^2(n-m)}\big[(2n+1)\overline{x}+(2n-1)x\big]\Phi_{n-2}^m \\ &-\frac{(2n+3)(n+m+1)(n+m)^2(n+m-1)}{(2n-1)(n-m+2)(n-m+1)^2(n-m)}\Phi_{n-3}^m.\end{aligned}$$

Theorem 3.2 *The relation between* $\{\Phi_n^m\}$ *and* $\{\widetilde{A}_n^m\}$ *can be described as follows*:

$$\Phi_{n+k}^n = (-1)^{k+1}\sum_{j=0}^{[k/2]}\frac{(2n+k-2j+1)!(2n+1)!!}{2^{n+j}\cdot(k+1)!j!(n+k-2j)!}a_{k,j}^n\widetilde{A}_{n+k-2j}^n,$$

where

$$a_{k,j}^n := \frac{(2n+2)_{2k-2j}}{2^{k-j}(n+1)_{k-j}}\cdot(2n+k+2-2j)_{2j},$$

and $\{\widetilde{A}_n^m\}$ *is the Appell system in* [3].

Notice that the Appell functions for spherical domains $\{\widetilde{A}_n^m\}$ are homogeneous polynomials and they satisfy the two step recurrence formula (cf. [3])

$$\widetilde{A}_{n+1}^l = \frac{n+1}{2(n-l+1)(n+l+2)}\Big[\big((2n+3)x+(2n+1)\overline{x}\big)\widetilde{A}_n^l - 2nx\overline{x}\widetilde{A}_{n-1}^l\Big],$$

with initial polynomials

$$\widetilde{A}_{l+1}^l = \frac{1}{4}\big[(2l+3)x+(2l+1)\overline{x}\big]\widetilde{A}_l^l; \quad \widetilde{A}_l^l = (x_1 - x_2\mathbf{e_3})^l.$$

Theorem 3.1 shows the analogy between spherical and spheroidal cases. The other terms in the formula express the asymmetry of oblate spheroids. These results help to reduce computational time for those functions. Especially, it is shown in [30] that there does not exist a complete system for spheroidal domains with respect to the standard inner product satisfying both orthogonal and Appell properties.

4 Outer Prolate Spheroidal Monogenics

Initially, inner spheroidal monogenic functions were described by means of associated Legendre functions of the first kind. With the help of associated Legendre functions of the second kind, J. Morais tried to construct a complete orthogonal system for the exterior domain of a prolate spheroid. This work becomes more complicated since the latter contains logarithmic functions so that a simple substitution is not enough. In [4], the Kelvin transform was applied for the construction of $\mathbb{H}$-valued outer spherical monogenic functions from inner spherical monogenic functions. That keeps properties such as orthogonality invariant. However for $\mathcal{A}$-valued functions in a spheroid, the Kelvin transform is not directly applicable. The method, based on the decomposition of a function space into subspaces of homogeneous functions to prove the completeness of a function system (see [2, 24]), fails because of the appearance of logarithmic functions. To this end, we firstly pay attention to the asymptotic behavior of the constructed functions compared with spherical cases. The extra term in the coefficient function is dealt with to prove the orthogonal property which will be discussed later. Finally, by using the harmonic extension to the outer domain of a function defined on the boundary of a prolate spheroid, one can prove the completeness of such a system. This research can be found in [29]. Here it is summarized briefly.

4.1 A System of Outer Prolate Spheroidal Monogenics

A system of outer prolate spheroidal monogenic functions is obtained by applying $\frac{1}{2}\partial$ to outer spheroidal harmonic functions

$$V_{n,l}(\mu,\theta)\cos(l\varphi), \qquad V_{n,l}(\mu,\theta)\sin(l\varphi),$$

where $V_{n,l}(\mu,\theta) := Q_n^l(\cosh\mu)P_n^l(\cos\theta)$, $(n = 0, 1, \dots;\ l = 0, \dots, n)$. Denote $\widehat{\mathcal{E}_{n-1,l}} := \frac{1}{2}\partial[V_{n,l}(\mu,\theta)\cos(l\varphi)]$ and $\widehat{\mathcal{F}_{n-1,l}} := \frac{1}{2}\partial[V_{n,l}(\mu,\theta)\sin(l\varphi)]$, one gets

$$\widehat{\mathcal{E}_{-1,0}}(\mu,\theta,\varphi) := \frac{-\sinh\mu\cos\theta + \cosh\mu\sin\theta(\cos\varphi\mathbf{e_1} + \sin\varphi\mathbf{e_2})}{\sinh\mu(\sin^2\theta + \sinh^2\mu)}, \tag{4.1}$$

$$\begin{aligned}\widehat{\mathcal{E}_{0,0}}(\mu,\theta,\varphi) &:= \frac{1}{4}\ln\left(\frac{\cosh\mu+1}{\cosh\mu-1}\right) - \frac{1}{2}\frac{\cosh\mu}{\sin^2\theta+\sinh^2\mu} \\ &\quad + \frac{1}{2}\frac{\sin\theta\cos\theta}{\sinh\mu(\sin^2\theta+\sinh^2\mu)}(\cos\varphi\mathbf{e_1} + \sin\varphi\mathbf{e_2}),\end{aligned} \tag{4.2}$$

$$\begin{aligned}\widehat{\mathcal{E}_{n,l}}(\mu,\theta,\varphi) &:= \frac{(n+l+1)}{2}B_{n,l}(\mu,\theta)\cos(l\varphi) \\ &\quad + \frac{1}{4(n-l+1)}B_{n,l+1}(\mu,\theta)\big[\cos\big((l+1)\varphi\big)\mathbf{e_1} + \sin\big((l+1)\varphi\big)\mathbf{e_2}\big]\end{aligned}$$

$$+\frac{1}{4}(n+1+l)(n+l)(n-l+2)B_{n,l-1}(\mu,\theta)$$
$$\times\big[-\cos\big((l-1)\varphi\big)\mathbf{e_1}+\sin\big((l-1)\phi\big)\mathbf{e_2}\big], \tag{4.3}$$

$$\widehat{\mathcal{F}_{n,l}}(\mu,\theta,\varphi) := \frac{(n+l+1)}{2}B_{n,l}(\mu,\theta)\sin(l\varphi)$$
$$+\frac{1}{4(n-l+1)}B_{n,l+1}(\mu,\theta)\big[\sin\big((l+1)\phi\big)\mathbf{e_1}-\cos\big((l+1)\varphi\big)\mathbf{e_2}\big]$$
$$-\frac{1}{4}(n+1+l)(n+l)(n-l+2)B_{n,l-1}(\mu,\theta)\big[\sin\big((l-1)\varphi\big)\mathbf{e_1}$$
$$+\cos\big((l-1)\varphi\big)\mathbf{e_2}\big], \tag{4.4}$$

(for $l=0,\dots,n;\ n=1,2,\dots$)

$$\widehat{\mathcal{E}_{n,n+1}}(\mu,\theta,\varphi) := (n+1)B_{n,n+1}(\mu,\theta)\cos\big((n+1)\varphi\big)$$
$$-\frac{\cosh\mu\,P_{n+2}^{n+2}(\cos\theta)\,Q_{n+1}^{n+2}(\cosh\mu)}{4(2n+3)(\sin^2\theta+\sinh^2\mu)}$$
$$\times\big[\cos\big((n+2)\phi\big)\mathbf{e_1}+\sin\big((n+2)\varphi\big)\mathbf{e_2}\big]$$
$$+\frac{(2n+2)(2n+1)}{4}B_{n,n}(\mu,\theta)\big[-\cos(n\varphi)\mathbf{e_1}+\sin(n\phi)\mathbf{e_2}\big], \tag{4.5}$$

$$\widehat{\mathcal{F}_{n,n+1}}(\mu,\theta,\varphi) := (n+1)B_{n,n+1}(\mu,\theta)\sin\big((n+1)\varphi\big)$$
$$-\frac{\cosh\mu\,P_{n+2}^{n+2}(\cos\theta)\,Q_{n+1}^{n+2}(\cosh\mu)}{4(2n+3)(\sin^2\theta+\sinh^2\mu)}$$
$$\times\big[\sin\big((n+2)\phi\big)\mathbf{e_1}-\cos\big((n+2)\varphi\big)\mathbf{e_2}\big]$$
$$-\frac{(2n+2)(2n+1)}{4}B_{n,n}(\mu,\theta)\big[\sin(n\varphi)\mathbf{e_1}+\cos(n\phi)\mathbf{e_2}\big], \tag{4.6}$$

(for $n=0,1,\dots$). The coefficients are given by

$$B_{n,l}(\mu,\theta) := \frac{1}{\sin^2\theta+\sinh^2\mu}\big[\cosh\mu\,P_n^l(\cos\theta)\,Q_{n+1}^l(\cosh\mu)$$
$$-\cos\theta\,P_{n+1}^l(\cos\theta)\,Q_n^l(\cosh\mu)\big], \tag{4.7}$$

where

$$B_{n,-1}(\mu,\theta) := -\frac{1}{n(n+1)^2(n+2)}B_{n,1}(\mu,\theta)\quad \text{for } n=1,2,\dots$$

It can be proved that $B_{n,l}(\mu,\theta)$ has the explicit presentation

$$B_{n,l}(\mu,\theta) = \sum_{k=0}^{[\frac{n-l}{2}]-1} \frac{(2n+1-4k)(n+l-2k+1)_{2k}}{(n-l-2k+1)_{2k+1}} P^l_{n-2k}(\cos\theta) Q^l_{n-2k}(\cosh\mu)$$
$$+ \begin{cases} \frac{(2l+1)_{n-l}}{(n-l+1)!} B_{l,l}(\mu,\theta) & \text{if } n-l \text{ even} \\ \frac{2(2l+2)_{n-l-1}}{(n-l+1)!} B_{l+1,l}(\mu,\theta) & \text{if } n-l \text{ odd.} \end{cases}$$

Because the terms $Q^l_{n+1}(\cosh\mu)$ contain logarithmic functions, the question of their behavior at infinity arises and we will see that they are completely similar to the outer spherical monogenic functions.

4.2 Outer Spherical Monogenics Revisited

To compare, we firstly revisit the spherical case. The construction of outer spherical monogenic functions has been studied in parallel with the construction of inner spherical monogenics. In [6], they are constructed based on the Cauchy kernel function and its derivatives. Spherical monogenics can be obtained also by applying the Kelvin transform as in [4]. Different methods we apply, different representations we get. For $\mathcal{A}$-valued monogenic functions, these methods do not lead directly to what we need. Hence, the harmonic function approach is again used together with spherical harmonic functions. Let $\mathbb{B}(R)$ be a ball with radius $R > 0$. Denote by $\mathcal{H}(\mathbb{R}^3\backslash\overline{\mathbb{B}(R)}, -(n+1))$ the space of real-valued homogeneous harmonic functions of degree $-(n+1)$ in $\mathbb{R}^3\backslash\overline{\mathbb{B}(R)}$ with $n \geq 0$. A basis of $\mathcal{H}(\mathbb{R}^3\backslash\overline{\mathbb{B}(R)}, -(n+1))$ is given by

$$\left\{ \frac{1}{r^{n+1}} P_n(\cos\theta),\ \frac{1}{r^{n+1}} P^m_n(\cos\theta)\cos(m\varphi),\ \frac{1}{r^{n+1}} P^m_n(\cos\theta)\sin(m\varphi) \right\}$$

where $m = 1,\ldots,n$. By applying the hypercomplex derivative $\frac{1}{2}\partial$, one obtains a system of monogenic functions defined in $\mathbb{R}^3\backslash\overline{\mathbb{B}(R)}$ as follows:

$$X^0_{-(n+2)} = -\frac{n+1}{2}\frac{P_{n+1}(\cos\theta)}{r^{n+2}} - \frac{1}{2}\frac{P^1_{n+1}(\cos\theta)}{r^{n+2}}[\cos\varphi\mathbf{e_1} + \sin\varphi\mathbf{e_2}]$$

$$\begin{aligned} X^m_{-(n+2)} = &-\frac{n-m+1}{2}\frac{P^m_{n+1}(\cos\theta)}{r^{n+2}}\cos(m\varphi) \\ &- \frac{1}{4}\frac{P^{m+1}_{n+1}(\cos\theta)}{r^{n+2}}\big[\cos\big((m+1)\varphi\big)\mathbf{e_1} + \sin\big((m+1)\varphi\big)\mathbf{e_2}\big] \\ &+ \frac{(n-m+1)(n-m+2)}{4}\frac{P^{m-1}_{n+1}(\cos\theta)}{r^{n+2}} \\ &\times \big[\cos\big((m-1)\varphi\big)\mathbf{e_1} - \sin\big((m-1)\varphi\big)\mathbf{e_2}\big] \end{aligned}$$

$$
\begin{aligned}
Y^m_{-(n+2)} = &-\frac{n-m+1}{2}\frac{P^m_{n+1}(\cos\theta)}{r^{n+2}}\sin(m\varphi) \\
&-\frac{1}{4}\frac{P^{m+1}_{n+1}(\cos\theta)}{r^{n+2}}\big[\sin\big((m+1)\varphi\big)\mathbf{e_1}-\cos\big((m+1)\varphi\big)\mathbf{e_2}\big] \\
&+\frac{(n-m+1)(n-m+2)}{4}\frac{P^{m-1}_{n+1}(\cos\theta)}{r^{n+2}} \\
&\times\big[\sin\big((m-1)\varphi\big)\mathbf{e_1}+\cos\big((m-1)\varphi\big)\mathbf{e_2}\big].
\end{aligned}
$$

Note that $\frac{1}{2}\partial$ establishes an isomorphism between $\mathcal{H}(\mathbb{R}^3\backslash\overline{\mathbb{B}(R)}, -(n+1))$ and $\mathcal{M}(\mathbb{R}^3\backslash\overline{\mathbb{B}(R)}, \mathcal{A}, -(n+2))$. The latter consists of all homogeneous monogenic polynomials of degree $-(n+2)$. Due to the orthogonal decomposition

$$
\mathcal{M}\big(\mathbb{R}^3\backslash\overline{\mathbb{B}(R)}, \mathcal{A}\big) = \bigoplus_{n=0}^{\infty}\mathcal{M}\big(\mathbb{R}^3\backslash\overline{\mathbb{B}(R)}, \mathcal{A}, -(n+2)\big),
$$

the system $\{X^0_{-(n+2)}, X^m_{-(n+2)}, Y^m_{-(n+2)}\}_{n=0,1,\ldots;\, m=1,\ldots,n}$ forms an orthogonal basis of $\mathcal{M}(\mathbb{R}^3\backslash\overline{\mathbb{B}(R)}, \mathcal{A})$.

4.3 Asymptotic Behavior

The behavior at infinity of outer spheroidal monogenic functions is related closely to the behavior of $Q^l_n(z)$. When z tends to infinity

$$
Q^l_n(z) = \frac{(n+l)!}{(2n+1)!!}\frac{1}{z^{n+1}} + O\left(\frac{1}{z^{n+3}}\right).
$$

Now let $z=\cosh\mu \simeq \sinh\mu \simeq r = |x|$ if μ is large enough, it leads to

$$
B_{n,l}(\mu,\theta) = -\frac{(n-l+2)(n+l)!}{(2n+3)!!}\frac{P^l_{n+2}(\cos\theta)}{r^{n+3}} + O\left(\frac{1}{r^{n+5}}\right).
$$

As a result, we obtain the asymptotic behavior of $\widehat{\mathcal{E}_{n,l}}$ and $\widehat{\mathcal{F}_{n,l}}$ for $l=0,\ldots,n+1$; $n=0,1,\ldots$

$$
\widehat{\mathcal{E}_{n,l}} = \frac{(n+l+1)!}{(2n+3)!!}X^l_{-(n+3)} + O\left(\frac{1}{|x|^{n+5}}\right),
$$

$$
\widehat{\mathcal{F}_{n,l}} = \frac{(n+l+1)!}{(2n+3)!!}Y^l_{-(n+3)} + O\left(\frac{1}{|x|^{n+5}}\right).
$$

Particularly, when $|x|\to\infty$

$$
\widehat{\mathcal{E}_{-1,0}} = -\frac{\overline{x}}{|x|^3} + O\left(\frac{1}{|x|^4}\right),
$$

and it behaves like *the Cauchy kernel* in a neighborhood of infinity.

4.4 Orthogonality

It could be easy to see that each following pair of functions is orthogonal with respect to the inner product (2.1) whenever $l_1 \neq l_2$

- $\{\widehat{\mathcal{E}_{n_1,l_1}}, \widehat{\mathcal{E}_{n_2,l_2}}\}$.
- $\{\widehat{\mathcal{F}_{n_1,l_1}}, \widehat{\mathcal{F}_{n_2,l_2}}\}$.
- $\{\widehat{\mathcal{E}_{n_1,l_1}}, \widehat{\mathcal{F}_{n_2,l_2}}\}$.

The assertion is based on the orthogonalities of $\sin(l\varphi)$ and $\cos(k\varphi)$ on $[0, 2\pi]$. In the other cases, one can decompose coefficient functions $B_{n,l}(\mu, \theta)$ into summands of the form

$$Q^l_{n-2k}(\cosh\mu)\, P^l_{n-2k}(\cos\theta), \tag{4.8}$$

except one extra term

$$\frac{\cosh\mu\, P^l_l(\cos\theta)\, Q^l_{l-1}(\cosh\mu)}{\sin^2\theta + \sinh^2\mu} \quad \text{or} \quad \frac{\cos\theta\, P^l_l(\cos\theta)\, Q^l_{l-1}(\cosh\mu)}{\sin^2\theta + \sinh^2\mu}. \tag{4.9}$$

Consequently, orthogonality holds for the terms of the form (4.8) according to equalities

$$\int_0^\pi P^l_n(\cos\theta)\, P^l_s(\cos\theta)\sin\theta d\theta = 0,$$
$$\int_0^\pi P^l_{n+1}(\cos\theta)\cos\theta\, P^l_s(\cos\theta)\sin\theta d\theta = 0$$

for $s < n$. Besides, we can prove by induction the following proposition.

Proposition 4.1 *Let $B_{n,l}(\mu, \theta)$ be functions as in* (4.7), *then with $l = 0, 1, \ldots$ the following equalities hold when n, k are equal to l or $l + 1$*

$$\int_0^\pi B_{n,l}(\mu, \theta)\, P^l_k(\cos\theta)\sin\theta d\theta = 0.$$

The proposition is applied to deal with the extra term (4.9) in expansions of $B_{n,l}(\mu, \theta)$ and it results in the following theorem.

Theorem 4.2 *The constructed functions* (4.1)–(4.6) *form an orthogonal system in $\mathcal{M}(\Omega^-, \mathcal{A})$ with respect to the inner product* (2.1).

The proof can be found in [29].

4.5 Completeness

Any function $\mathbf{f} \in \mathcal{M}(\Omega^-, \mathcal{A})$ has a Fourier series expansion related to the function system (4.1)–(4.6). The question is whether the Fourier series expansion converges to $\mathbf{f}$ in L_2-norm. In order to find the answer, one needs the following result.

Theorem 4.3 *Let* $\mathbf{f}$ *be a function in* $\mathcal{M}(\Omega^-, \mathcal{A}) \cap C^1(\Omega^- \cup \Gamma)$. *Then the Fourier series expansion of* $\mathbf{f}$ *converges to* $\mathbf{f}$ *in the sense of the* $L^2(\Omega^-)$*-norm.*

Notice that Theorem 4.3 considers only the case of smooth functions in $\mathcal{M}(\Omega^-, \mathcal{A})$. For the L_2-case, the analogous result is obtained by applying the following corollary.

Corollary 4.4 *Any outer spherical monogenic functions*

$$\left\{X^0_{-(n+2)}, X^m_{-(n+2)}, Y^m_{-(n+2)}\right\}_{n\geq 0;\, m=1,\ldots,n}$$

can be presented by its Fourier series expansion with respect to the system (4.1)–(4.6).

To this end, we give the completeness theorem.

Theorem 4.5 *The function system* (4.1)–(4.6) *forms a complete orthogonal system of the space* $\mathcal{M}(\Omega^-, \mathcal{A})$ *with respect to the inner product* (2.1) *in the exterior domain* Ω^-.

Details can be found in [29].

5 Conclusion

Ellipsoidal harmonic functions have attracted the attention of several researchers and shown their importance in many fields. By means of the hypercomplex derivative, ellipsoidal monogenic functions are currently being developed. In accordance with advantages of Clifford analysis, it will become a helpful tool for solving problems in ellipsoidal domains. Further applications of such systems hopefully will be announced in the near future.

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Clifford Algebras with Induced (Semi)-Riemannian Structures and Their Compactifications

Craig A. Nolder and John A. Emanuello

Abstract Identifying the Clifford algebra $C\ell_{r,s}$ with the semi-Riemannian manifold $\mathbb{R}^{p,q}$, one is afforded an opportunity to examine the conformal geometry of the associated compact manifold, in a manner similar to the case of the Riemann sphere in complex analysis. In this work we consider some low-dimensional examples and provide conjectures to inspire further research.

Keywords Clifford analysis · Conformal mappings

Mathematics Subject Classification (2010) 30G35 · 30C20

1 Introduction

In 1878, famed geometer William Kingdon Clifford introduced *geometric algebras*, which are special cases of what are now called Clifford algebras [3]. In many ways, Clifford algebras may be thought as higher dimensional analogues of the complex numbers.

A great deal of work has been done to develop a theory of functions of a Clifford variable. However, there is still much to be done to fully generalize the results of many analogues from complex analysis.

One of the great results from complex variables is the development of the Riemann sphere, which proved to have some advantages as a domain for functions of a complex variable. As a compact space, it possesses some desirable topological properties. In terms of the analysis, one finds that many nice results are true on the sphere which are not true in the plane. For example the class of meromorphic functions on the sphere are merely the rational functions, while in the plane the

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meromorphic functions are a larger class of functions. Also, the sphere has a rich conformal geometry.

A natural question arises: *Can these ideas be extended to the general Clifford case?* The literature demonstrates that this question may be answered in the affirmative, at least in low dimensions, [4–6, 9–12]. In this paper, we provide some explanation of the literature and motivating examples which we believe will inspire further research in this area.

2 Induced (Semi)-Riemannian Structure on a Clifford Algebra

We write $\mathbb{R}^{p,q}$ for the quadratic space with (the possibly definite) semi-Riemannian metric $g^{p,q}$, which is defined at each $a \in \mathbb{R}^{p+q}$ and $X, Y \in T_a\mathbb{R}^{p+q}$ by

$$g^{p,q}(X,Y) = \sum_{i=1}^{p} X_i Y_i - \sum_{i=p+1}^{p+q} X_i Y_i.$$

We shall denote the corresponding quadratic form by $\langle \cdot, \cdot \rangle_{p,q} = \sum_{i=1}^{p} x_i^2 - \sum_{i=p+1}^{p+q} x_i^2$.

Some attention has been paid toward compactifying $\mathbb{R}^{p,q}$ and their conformal mappings. In particular, Schottenloher describes the following compactification.

First, we embed $\mathbb{R}^{p,q}$ in projective space:

$$\phi : \mathbb{R}^{p,q} \to \mathbb{P}_{p+q+1}(\mathbb{R})$$

$$\zeta = (x_1, \dots, x_{p+q}) \mapsto \left(\frac{1 - \langle \zeta, \zeta \rangle_{p,q}}{2} : x_1 : \dots : x_{p+q} : \frac{1 + \langle \zeta, \zeta \rangle_{p,q}}{2} \right).$$

It is easy to check that ϕ is a conformal mapping.

To borrow notation from Schottenloher, we define the compactification of $\mathbb{R}^{p,q}$ by

$$N^{p,q} := \overline{\phi(\mathbb{R}^{p,q})}.$$

It is important to note that the product of spheres $S^p \times S^q \subseteq \mathbb{R}^{p+1,q+1}$ is a 2-to-1 covering of $N^{p,q}$, whose covering map is the restriction of quotient map $\pi : \mathbb{R}^{p+q+2} \setminus \{0\} \to \mathbb{P}_{p+q+1}(\mathbb{R})$ [12].

We consider the real Clifford algebra $C\ell_{r,s}$. We write $n = r + s$. This is an associative algebra of dimension 2^n generated over $\mathbb{R}$ by $\{1, e_1, e_2, \dots, e_n\}$ and is subject to relations

$$e_j^2 = 1 \quad \text{if } j = 1, \dots, r \quad \text{and} \quad e_j^2 = -1 \quad \text{if } j = r+1, \dots, n$$

$$e_i e_j + e_j e_i = 2\delta_{i,j}.$$

In general elements of the algebra are of the form

$$\zeta = \sum_A x_A e_A,$$

where $A = \{i_1, \ldots, i_k\} \subseteq \{1, 2, \ldots, n\}$ and $e_A = e_{i_1} e_{i_2} \cdots e_{i_k}$. By convention $e_0 = 1$.

A conjugation is defined by $\overline{e_0} = e_0$, $\overline{e_j} = -e_j$, $j = 1, \ldots, n$. We define an involution which is an analogue of complex conjugation:

$$\bar{\zeta} = x_0 + \sum_{A \neq 0} x_A \overline{e_A}.$$

Here $\overline{e_A e_B} = \overline{e_B e_A}$.

We are now interested in classifying the Möbius transformations on a compactification of $C\ell_{r,s}$ (that is compositions of dilations, translations, rotations, and reflections) as in the complex case. We assume throughout the rest of this section that $r + s \leq 2$. Then the product $\zeta\bar{\zeta}$ defines a quadratic form with corresponding bilinear form $\zeta\bar{\omega}$. Identifying $C\ell_{r,s}$ with $\mathbb{R}^{2^n}$, this bilinear form is the standard bilinear form of signature (p, q), where $p + q = 2^n$.

Passing this form to the tangent spaces of the manifold $\mathbb{R}^{2^n}$, means we can then identify $C\ell_{r,s}$ with the semi-Riemannian manifold $\mathbb{R}^{p,q}$. In the course of our investigation we have strong evidence to suggest the following.

Conjecture 2.1 *In the case that $r + s \leq 2$, the group of orientation preserving Möbius transformations on $C\ell_{r,s}$ is isomorphic to $SO_o(p+1, q+1)$, which is the connected component containing the identity of the Lie group $SO(p+1, q+1)$.*

This conjecture is supported by results appearing in the literature, see [1, 5, 6]. Finding Lie group homomorphisms is usually a non-trivial task. The associated Lie algebras tend to be easier objects to understand. Under certain circumstances, a Lie algebra isomorphism determines an isomorphism between the associated Lie groups [7].

In the next section we shall provide examples where the conjecture is true.

3 Examples

3.1 Example 1: The Real Line

By convention, the Clifford algebra $C\ell_{0,0}$ is identified with the real line $\mathbb{R}$. Using the standard Riemannian structure we think of $\mathbb{R}$ as $\mathbb{R}^{1,0}$.

We know that $S^1 \times S^0 = S^1 \times \{-1, 1\}$ forms a 2-to-1 cover of $N^{1,0}$. Of course, this means that $N^{1,0}$ is homeomorphic to S^1, which is the one point compactification of $\mathbb{R}$. The conformal mappings here are the diffeomorphisms of the circle.

An important subclass of these mappings are the real Möbius transformations

$$x \mapsto \frac{ax+b}{cx+d},$$

which can be described as $PGL(2,\mathbb{R})$ acting on $\mathbb{R}$. If we consider the Möbius transformations induced by $PSL(2,\mathbb{R})$, we find that we see that Conjecture 2.1 is true for $\mathbb{R}$:

$$PSL(2,\mathbb{R}) \cong SO_o(2,1),$$

see [1]. Incidentally, we have the isomorphism of Lie algebras [8]:

$$\mathfrak{sl}(2,\mathbb{R}) \cong \mathfrak{so}(2,1).$$

3.2 Example 2: The Complex Plane

Recall, that the Riemann sphere was our motivating example. It is prudent to check that our construction coincides with the Riemann sphere.

The complex plane is identified with the Clifford algebra $C\ell_{0,1}$. Of course the bilinear form on $\mathbb{R}^2$ is the one induced from $\zeta\overline{\varpi}$ as above. Hence we identify $\mathbb{C} \sim \mathbb{R}^{2,0}$.

Since $S^2 \times \{-1,1\}$ is a double cover of $N^{2,0}$, we know that

$$N^{2,0} \cong S^2,$$

which is the Riemann sphere.

Results from complex analysis tell us that the conformal mappings $S^2 \to S^2$ are the usual orientation preserving Möbius transformations

$$z \mapsto \frac{az+b}{cz+d},$$

where $ad-bc=1$.

The orientation preserving transformations are isomorphic to $PSL(2,\mathbb{C})$. It so happens that

$$PSL(2,\mathbb{C}) \cong SO_o(3,1),$$

as needed [1]. Again, the associated Lie algebras are isomorphic [8]:

$$\mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{so}(3,1).$$

3.3 Example 3: The Minkowski Plane

The case of the Minkowski plane is considered in our previous work, whose bibliography contains several resources outlining the historical significance of this semi-Riemannian manifold [4].

The Clifford algebra we consider in this case is $C\ell_{1,0}$, which we also refer to as the *split complex numbers* and is associated with the quadratic space $\mathbb{R}^{1,1}$. Unlike $\mathbb{C}$, this algebra has zero divisors.

In our previous work, we show that Möbius transformations

$$\zeta \mapsto \frac{a\zeta + b}{c\zeta + d},$$

where $a, b, c, d \in C\ell_{1,0}$ and $ad - bc$ is an invertible element, form a subclass of the conformal mappings on $N^{1,1}$. For reasons made clear in our paper, the Möbius transformations are isomorphic to $PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$ [4]. In this case we have

$$PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R}) \cong SO_o(2,1) \times SO_o(2,1) \cong SO_o(2,2).$$

Moreover, $\mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{R}) \cong \mathfrak{so}(2,2)$ [8].

3.4 Example 4: Quaternions

The quaternions $\mathbb{H}$ are the Clifford algebra $C\ell_{0,2}$. The corresponding quadratic space is $\mathbb{R}^{4,0}$ which is embedded as above into $S^4 \times S^0$ in $\mathbb{R}^{5,1}$. The Lie algebra here is $\mathfrak{sl}(2,\mathbb{H})$ consisting of 2×2 traceless matrices with entries from $\mathbb{H}$. By considering the fundamental representation of this Lie algebra as real linear operators on $\mathbb{H}^2$ one finds that $PSL(2,\mathbb{H})$ is isomorphic to $SO_0(5,1)$. See [1, 5, 6].

3.5 Example 5: Split Quaternions

The Clifford algebra $C\ell_{1,1}$, often referred to as the split quaternions, induce the semi-Riemannian structure $\mathbb{R}^{2,2}$. Here $SL(2, C\ell_{1,1})$ is isomorphic to $SL(4,\mathbb{R})$, see [5, 6]. We have the local isomorphism $\mathfrak{sl}(4,\mathbb{R}) \cong \mathfrak{so}(3,3)$ [8]. As such it is likely that $PSL(4,\mathbb{R})$ is isomorphic to $SO_0(3,3)$. We remark that $C\ell_{2,0}$ is isomorphic to $C\ell_{1,1}$.

3.6 Example 6: $C\ell_{0,3}$

This Clifford algebra is generated over $\mathbb{R}$ by generators $\{e_1, e_2, e_3\}$ with a basis $\{e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ satisfying the relations

$$e_0 e_i = e_i e_0 = e_i, \quad i = 0, 1, \ldots, 7,$$

$$e_1 e_2 = e_4, \qquad e_1 e_3 = e_5, \qquad e_2 e_3 = e_6, \qquad e_1 e_6 = e_7,$$

$$e_i^2 = -e_0, \qquad e_7^2 = e_0, \quad i = 1, \dots, 6,$$
$$e_i e_j + e_j e_i = 0, \quad i \neq j,\ i, j = 1, 2, \dots, 6,\ i + j \neq 7,$$
$$e_i e_j = e_j e_i, \quad i = 0, 1, \dots, 7,\ i \neq j,\ i + j = 7.$$

Here $\zeta = \sum_{i=0}^{7} x_i e_i$ with conjugate $\overline{\zeta} = x_0 e_0 - \sum_{i=1}^{6} x_i e_i + x_7 e_7$. Hence in this case $\zeta\overline{\zeta} = (\sum_{i=0}^{7} x_i^2) e_0 + (x_0 x_7 - x_1 x_6 + x_2 x_5 - x_3 x_4) e_7$ which is an element of $C\ell_{1,0}$. For more details in this case, see [2].

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Part IX
Integral Transforms and Reproducing Kernels

Organizers: Saburou Saitoh, Juri Rappoport

Generalized Shift Operators Generated by Convolutions of Integral Transforms

Lyubov Y. Britvina

Abstract In this work we discuss a problem of the equivalence of two main approaches to introducing of generalized convolution operators. The first of them is based on the constructing of a generalized shift operator. The idea of the second approach is based on the works by Valentin Kakichev. On this problem we demonstrate on the examples of classical and nonclassical convolution constructions of integral transforms. In particular, we consider the shift operators defined by the convolutions for Hankel integral transform with the function $j_\nu(xt) = (2xt)^\nu \Gamma(\nu + 1) J_\nu(xt)$ in the kernel. Here $J_\nu(xt)$ is the Bessel function of the first kind of order ν, $\operatorname{Re}\nu > -1/2$.

Keywords Shift (translation) operator · Convolution · Fourier integral transform · Hankel integral transform

Mathematics Subject Classification (2010) Primary 44A35 · Secondary 42B10

1 Convolutions and Generalized Shift Operators

Nowadays, there are two main approaches to the constructing and generalizing of convolutions for integral transforms.

The first of them is based on the constructing of a generalized shift operator (also called generalized translation operator, or generalized displacement operator). Then the classical translation operator (ordinary translation) into convolution is replaced on the generalized shift operator, and we get a generalized convolution. Usually, the generalized translation operators of the Delsarte–Levitan–Povzner type are used in these constructions.

In 1938 Jean Delsarte [1] (see also [2, 3]) formulated an entirely new generalization of the notion of translation operator. He had been interested in finding a formal

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generalization of the Taylor expansion formula. This research allowed him to obtain the first results in the theory of generalized translation operator.

Generalized translation in the sense of Jean Delsarte has later been considered from various points of view by many authors. An important step in the direction of the Delsarte's works has been made by Aleksandr Povzner [4]. He determined the translation operators by the Riemann's method for solving the hyperbolic equation. It makes possible to define the translation operators over a very large family of functions.

The main results in the theory of generalized translation operators belong to Boris Levitan [5, 6]. He gave an axiomatic definition of generalized translation operators, studied their structure, considered the application of spectral analysis of generalized translation operators. The operators associating with Sturm–Liouville differential operators have been studied by him in detail.

Later S. Bochner, N. Weinberger, I.I. Hirschman, N. Leblanc, N. Dunford and J.T. Schwartz, V. Hutson and J.S. Pym, B. Fishel and others authors have discussed generalized translation operators.

The idea of the second approach is based on the works by Valentin Kakichev [7]. In 1967 he introduced the definition of polyconvolution, or generalized convolution.

Definition 1.1 Let A_i, $i = 1, 2, 3$ be linear operators mapping linear spaces $U_i(T_i)$ to an algebra $W(X)$. The polyconvolution (generalized convolution) of function $f(t)$ and $g(t)$ generated by these operators with weighted function $\alpha(x)$, is the function $h(t)$ denoted by $(f_{A_1} \overset{\alpha}{*} g_{A_2})_{A_3}(t)$ for which the following factorization property is valid:

$$(A_3 h)(x) = A_3\big[(f_{A_1} \overset{\alpha}{*} g_{A_2})_{A_3}\big](x) = \alpha(x) \cdot (A_1 f)(x) \cdot (A_2 g)(x). \tag{1.1}$$

Here the sympol "·" denotes a multiplication in the algebra $W(X)$.

Using this definition we can construct the polyconvolutions generated by various linear operators, in particular we can construct the convolutions for integral transforms. And these convolutions define translation operators.

Now we discuss a problem of the equivalence of these approaches to introducing of generalized convolution operators. We take the Levitan's definition of generalized translation operator (see his paper [5, 6]).

We will deal throughout the paper with weighted Lebesgue spaces $L_p(\mathbb{R}_+; \omega(t)dt)$, $1 \le p < \infty$ with respect to a positive measure $\omega(t)dt$ equipped with the norm for which

$$\|f\|_\omega = \left(\int_0^\infty \big|f(t)\big|^p \omega(t)dt \right)^{1/p} < \infty.$$

Let T^τ be a family of operators depending on the parametre $\tau \in \mathbb{R}_+$.

Thus, to every function $f(t) \subset L_p(\mathbb{R}_+; \omega(t)dt)$ there corresponds a function $T_t^\tau f(t)$ of two variable points.

Denote by $\tilde{T}_t^\tau f(t)$ the adjoined operator which is defined by the relation

$$\int T_t^\tau f(t)\overline{g(t)}\omega(t)dt = \int f(t)\overline{\tilde{T}_t^\tau g(t)}\omega(t)dt.$$

Definition 1.2 Let then the operators T^τ satisfy the following conditions, which we will call the conditions of generalized translation:

I. Linearity:

$$T_t^\tau\big[af(t)+bg(t)\big] = aT_t^\tau f(t) + bT_t^\tau g(t),$$

a, b are real numbers.

II. The single element $e \in \mathbb{R}_+$ exists such that

$$T_t^e f(t) = f(t), \qquad T_e^\tau f(t) = f(\tau).$$

III. Associativity:

$$T_s^\tau T_t^s f(t) = T_t^s T_t^\tau f(t).$$

IV. Boundedness:

$$\left[\int \left|T_t^\tau f(t)\right|^p \omega(t)dt\right]^{1/p} \le A_p(\tau)\left[\int \left|f(t)\right|^p \omega(t)dt\right]^{1/p},$$

$$\left[\int \left|\tilde{T}_t^\tau f(t)\right|^p \omega(t)dt\right]^{1/p} \le A_p^*(\tau)\left[\int \left|f(t)\right|^p \omega(t)dt\right]^{1/p},$$

where $A_p(\tau), A_p^*(\tau)$ are positive functions bounded on every compact set $T \subset \mathbb{R}_+$.

V. Continuity: if $f(t) \subset L_p$ then for each $\epsilon > 0$ there is the neighborhood U such that if $s, r \in U$ then

$$\int \left|T_t^s f(t) - T_t^r f(t)\right|^p \omega(t)dt < \epsilon^p, \qquad \int \left|\tilde{T}_t^s f(t) - \tilde{T}_t^r f(t)\right|^p \omega(t)dt < \epsilon^p.$$

In the next section we consider certain convolution operators and translation operators which are generated by them.

2 Examples of Convolutions and Generalized Shift Operators

2.1 The Convolutions of the Fourier Cosine and Sine Transforms

Polyconvolutions which are associated with the Fourier cosine and sine transformations are well studied. It is widely known, that the Fourier transform is well-defined on the space $L_1(\mathbb{R}_+; dt)$

$$\mathrm{V}_{\{{c \atop s}\}}[f](x) = \int_0^\infty f(t) \begin{Bmatrix} \cos xt \\ \sin xt \end{Bmatrix} dt, \quad f(t) \in L_1(\mathbb{R}_+). \tag{2.1}$$

Moreover, if $g(x) = (F_c f)(x) \in L_1(\mathbb{R}_+; dx)$ we have the reciprocal inversion formula $f(x) = (F_c g)(x)$.

In the case of $L_2(\mathbb{R}_+; dt)$-space we should define the cosine Fourier transform in the mean-square convergence sense, namely

$$\mathrm{V}_{\{{c \atop s}\}}[f](x) = \lim_{N\to\infty} \int_{1/N}^{N} f(t) \begin{Bmatrix} \cos xt \\ \sin xt \end{Bmatrix} dt, \quad f(t) \in L_2(\mathbb{R}_+), \tag{2.2}$$

and familiar Plancherel's theorem says that $F_c : L_2(\mathbb{R}_+; dt) \to L_2(\mathbb{R}_+; dt)$ is an isometric isomorphism and Parseval's equality

$$\|F_c f\|_{L_2(\mathbb{R}_+;dt)} = \|f\|_{L_2(\mathbb{R}_+;dt)}. \tag{2.3}$$

In these function spaces we can introduce various convolutions which are generated by the Fourier cosine and sine transforms. For example, the following constructions without weight functions [8–10]

$$(f_c * g_c)_c(t) = \frac{1}{2} \int_0^\infty f(\tau)\big[g(t+\tau) + g\big(|t-\tau|\big)\big] d\tau, \tag{2.4}$$

$$\mathrm{V}_c\big[(f_c * g_c)_c(t)\big](x) = \mathrm{V}_c[f](x)\mathrm{V}_c[g](x); \tag{2.5}$$

$$(f_c * g_s)_s(t) = \frac{1}{2} \int_0^\infty g(\tau)\big[f\big(|t-\tau|\big) - f(t+\tau)\big] d\tau \tag{2.6}$$

$$= \frac{1}{2} \int_0^\infty f(\tau)\big[g(t+\tau) + \operatorname{sign}(t-\tau) g\big(|t-\tau|\big)\big] d\tau, \tag{2.7}$$

$$\mathrm{V}_s\big[(f_c * g_s)_s(t)\big](x) = \mathrm{V}_c[f](x)\mathrm{V}_s[g](x); \tag{2.8}$$

$$(f_s * g_s)_c(t) = \frac{1}{2} \int_0^\infty f(\tau)\big[g(t+\tau) - \operatorname{sign}(t-\tau) g\big(|t-\tau|\big)\big] d\tau, \tag{2.9}$$

$$\mathrm{V}_c\big[(f_s * g_s)_c(t)\big](x) = \mathrm{V}_s[f](x)\mathrm{V}_s[g](x). \tag{2.10}$$

The convolutions (2.4), (2.6), (2.7) and (2.9) generate the following translation operators:

$$\begin{aligned}
{}_1T_t^\tau f(t) &= \frac{1}{2}\big[f(t+\tau) + f\big(|t-\tau|\big)\big], \\
{}_1T_t^0 f(t) &= f(t), \qquad {}_1T_0^\tau f(t) = f(\tau), \\
\frac{\partial}{\partial\tau} {}_1T_t^\tau f(t)\Big|_{\tau=0} &= {}_3T_t^0 f'(t) = 0, \\
\frac{\partial}{\partial t} {}_1T_t^\tau f(t)\Big|_{t=0} &= {}_2\tilde{T}_0^\tau f'(t) = 0,
\end{aligned} \tag{2.11}$$

$$
\begin{aligned}
{}_2T_t^\tau f(t) &= \frac{1}{2}\big[f\big(|t-\tau|\big) - f(t+\tau)\big],\\
&{}_2T_t^0 f(t) = 0, \qquad {}_2T_0^\tau f(t) = 0,\\
&\frac{\partial}{\partial\tau}{}_2T_t^\tau f(t)\bigg|_{\tau=0} = -{}_2\tilde{T}_t^0 f'(t) = -f'(t),\\
&\frac{\partial}{\partial t}{}_2T_t^\tau f(t)\bigg|_{t=0} = -{}_3T_0^\tau f'(t) = -f'(\tau),
\end{aligned} \tag{2.12}
$$

$$
\begin{aligned}
{}_2\tilde{T}_t^\tau f(t) &= \frac{1}{2}\big[f(t+\tau) + \operatorname{sign}(t-\tau) f\big(|t-\tau|\big)\big],\\
&{}_2\tilde{T}_t^0 f(t) = f(t), \qquad {}_2\tilde{T}_0^\tau f(t) = 0,\\
&\frac{\partial}{\partial\tau}{}_2\tilde{T}_t^\tau f(t)\bigg|_{\tau=0} = -{}_2T_t^0 f'(t) = 0,\\
&\frac{\partial}{\partial t}{}_2\tilde{T}_t^\tau f(t)\bigg|_{t=0} = {}_1T_0^\tau f'(t) = f'(\tau),
\end{aligned} \tag{2.13}
$$

$$
\begin{aligned}
{}_3T_t^\tau f(t) &= \frac{1}{2}\big[f(t+\tau) - \operatorname{sign}(t-\tau) f\big(|t-\tau|\big)\big],\\
&{}_3T_t^0 f(t) = 0, \qquad {}_3T_0^\tau f(t) = f(\tau),\\
&\frac{\partial}{\partial\tau}{}_3T_t^\tau f(t)\bigg|_{\tau=0} = {}_1T_t^0 f'(t) = f'(t),\\
&\frac{\partial}{\partial t}{}_3T_t^\tau f(t)\bigg|_{t=0} = -{}_2T_0^\tau f'(t) = 0.
\end{aligned} \tag{2.14}
$$

The translation operator (2.11) is well known. It is the operator of the Levitan's type. The other operators (2.12)–(2.14) and polyconvolutions with weight functions for the Fourier cosine and sine transforms are not the operators of Levitan's type because they do not satisfy the conditions II and III, in particular.

2.2 *The Convolutions of the Hankel Transform*

The Hankel integral transform is defined by

$$h_\nu(f)(s) = \tilde{f}(s) = \int_0^\infty f(t) j_\nu(st) t^{2\nu+1} dt, \quad \nu > -1/2, \tag{2.15}$$

where the function

$$j_\nu(st) = \frac{2^\nu \Gamma(\nu+1)}{(st)^\nu} J_\nu(st) = \sum_{m=0}^\infty \frac{(-1)^m \Gamma(\nu+1)(st)^{2m}}{2^{2m} m! \Gamma(m+\nu+1)} \tag{2.16}$$

is associated with the Bessel function J_ν of the first kind of order ν.

The function $j_\nu(st)$ is the solution of the equation

$$\frac{d^2y}{dt^2}+\frac{2\nu+1}{t}\frac{dy}{dt}+s^2t=0 \tag{2.17}$$

under conditions $y(0)=1$ and $y'(0)=0$.

The inversion formula for (2.15) is given by

$$h_\nu^{-1}(\tilde{f})(t)=f(t)=\left[2^{2\nu}\Gamma^2(\nu+1)\right]^{-1}\int_0^\infty \tilde{f}(s)j_\nu(st)s^{2\nu+1}ds. \tag{2.18}$$

The classical convolution for the Hankel transform (2.15) is well studied and is defined by

$$(f_\nu * g_\nu)_\nu(t)=\frac{\Gamma(\nu+1)}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})}\int_0^\pi \sin^{2\nu}\varphi \times\int_0^\infty f(\tau)g\left(\sqrt{t^2+\tau^2-2t\tau\cos\varphi}\right)\tau^{2\nu+1}d\tau d\varphi. \tag{2.19}$$

This convolution can be rewritten as

$$(f_\nu * g_\nu)_\nu(t)=\int_0^\infty f(\tau){}_0T_t^\tau g(t)\tau^{2\nu+1}d\tau \tag{2.20}$$

if we introduce the translation operator:

$${}_0T_t^\tau f(t)=\frac{\Gamma(\nu+1)}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})}\int_0^\pi f\left(\sqrt{t^2+\tau^2-2t\tau\cos\varphi}\right)\sin^{2\nu}\varphi d\varphi,$$
$${}_0T_t^0 f(t)=f(t),\qquad {}_0T_0^\tau f(t)=f(\tau). \tag{2.21}$$

This translation operator is the generalized translation operator of the Levitan's type. It was first introduced and studied by B.M. Levitan in 1949 [5] (see also [11]). In 1955 Ya.I. Zhitomirskii [12] constructed the convolution for the Hankel transform using the translation operator and its properties. Also this classical convolution and corresponding translation operator were introduced and investigated by I.I. Hirschman, D.T. Haimo, F.M. Cholowinski, V.A. Kakichev, Vu Kim Tuan and Megumi Saigo [7, 13–16].

The Kakichev's approach allows us to introduce some convolutions for the Hankel transform. For example, we can construct the following convolution

$$(f_{\nu+m} * g_\nu)_{\nu+m}(t)=\frac{2^{2\nu-1}m!\Gamma(\nu+1)\Gamma(\nu)}{\pi\Gamma(2\nu+m)t^m}\int_0^\pi C_m^\nu(\cos\varphi)\sin^{2\nu}\varphi \times\int_0^\infty f(\tau)g\left(\sqrt{t^2+\tau^2-2t\tau\cos\varphi}\right)\tau^{2\nu+m+1}d\tau d\varphi \tag{2.22}$$

which generate the translation operator

$$
{}_mT_t^\tau f(t) = \frac{c_{\nu,m}}{(t\tau)^m}\int_0^\pi f\big(\sqrt{t^2+\tau^2-2t\tau\cos\varphi}\big)C_m^\nu(\cos\varphi)\sin^{2\nu}\varphi d\varphi,
$$

$$
\tau^m{}_mT_t^\tau f(t)\big|_{\tau=0} = 0, \quad m>0, \qquad t^m{}_mT_t^\tau f(t)\big|_{t=0} = 0, \quad m>0, \tag{2.23}
$$

where

$$
c_{\nu,m} = \frac{2^{2(\nu+m)-1}m!\Gamma^2(\nu+m+1)\Gamma(\nu)}{\pi\Gamma(2\nu+m)\Gamma(\nu+1)}. \tag{2.24}
$$

The convolution (2.22) can be rewritten as

$$
(f_{\nu+m} * g_\nu)_{\nu+m}(t) = \frac{\Gamma^2(\nu+1)}{2^{2m}\Gamma^2(\nu+m+1)}\int_0^\infty f(\tau)_mT_t^\tau g(t)\tau^{2(\nu+m)+1}d\tau. \tag{2.25}
$$

If $m=0$ then we get the classical convolution (2.19) and the corresponding translation operator.

The translation operator (2.23) is not the operator of the Levitan's type.

If we will consider the other polyconvolutions for integral transforms, for example, the various convolution constructions for Hankel transform [17–20] then only convolutions generated by one linear operator without weight function can get the translation operators of the Levitan's type.

The presented examples show us that these two approaches are not equivalent if we use the Levitan's notion for translation operator.

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An Approach for Developing Fourier Convolutions and Applications

Nguyen Minh Tuan

Abstract Based on the papers published recently, this talk presents a concept of convolution so-called pair-convolution which is a generalization of known convolutions, and considers applications for solving integral equations.

Keywords Generalized convolution · Integral equation of convolution type · Banach algebra

Mathematics Subject Classification (2010) Primary 42A85 · 45E10 · Secondary 44A20 · 46J10

1 Introduction

It is well-known that the transform

$$(f \underset{\mathcal{F}}{*} g)(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} f(x-y)g(y)dy \tag{1.1}$$

is called the Fourier convolution of two functions g and f, and the following factorization identity holds

$$\mathcal{F}(f \underset{\mathcal{F}}{*} g)(x) = (\mathcal{F}f)(x)(\mathcal{F}g)(x).$$

The above-mentioned convolution was found most early, and nowadays it has been applying widely in both theoretical and practical problems.

We can say that many convolutions, generalized convolutions, and polyconvolutions of the well-known integral transforms as Fourier's, Hankel's, Mellin's,

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Laplace's and their applications have been published. Loosely speaking, the theory of convolutions has been strongly developing and vigorously discussing in many research groups (see [1, 2, 4, 5, 7, 10, 11, 14, 15], and references therein).

In fact, convolutions are considered as a powerful tool in many fields of mathematics such as numerical computing, digital, image and signal processing, partial-differential equations, and other fields of mathematics (see [3, 6–16, 20]). Other reason for which the theory of convolutions attracts attention of many mathematicians is that each of convolutions is a new integral transform, therefore it could be a new object of study.

1.1 Present Studies of Convolution Operators

It is easy to show a long list of authors and their works concerning convolution operators such as: A. Böttcher, L.E. Britvina, Yu. Brychkov, L. Castro, I. Feldman, H.J. Glaeske, I. Gohberg, N. Krupnik, O.I. Marichev, S. Saitoh, B. Silbermann, H.M. Srivastava, V.K. Tuan, S.B. Yakubovich… Among those listed, there are many mathematicians leading the potential and strong groups in the worldwide, they have been creating significant discoveries, namely: A. Böttcher (Germany), L.E. Britvina (Ukraina), Yu. Brychkov (Russia), L. Castro and S. Saitoh (Aveiro-Portugal and Gunma-Japan), I. Gohberg (Israel), B. Silbermann (Germany), H.M. Srivastava (Canada), V.K. Tuan (USA), S.B. Yakubovich (Porto, Portugal).

2 An Approach to Developing Convolutions

The nice idea of convolution focuses on the factorization identity. We now deal with the concept of convolutions. Let U_1, U_2, U_3 be the linear spaces on the field of scalars $\mathcal{K}$, and let V be a commutative algebra on $\mathcal{K}$. Suppose that $K_1 \in L(U_1, V)$, $K_2 \in L(U_2, V)$, $K_3 \in L(U_3, V)$ are linear operators from U_1, U_2, U_3 to V respectively. Let δ denote an element in algebra V. We recall the definition of convolutions.

Definition 2.1 (see also [4]) A bilinear map $* : U_1 \times U_2 :\longrightarrow U_3$ is called a convolution associated with K_3, K_1, K_2 (in that order) if the following identity holds

$$K_3\big(*(f, g)\big) = \delta K_1(f) K_2(g),$$

for any $f \in U_1$, $g \in U_2$. Above identity is called the factorization identity of the convolution.

We now deal with several approaches to developing convolutions.

2.1 Using Eigenfunctions

Let Φ_α denote the Hermite function (see [14]).

Theorem 2.2 ([18, 19]) *The following transform defines a convolution*

$$(f \overset{\Phi_\alpha}{\underset{\mathcal{F}}{*}} g)(x) = \frac{i^{|\alpha|}}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(u)g(v)\Phi_\alpha(x-u-v)dudv.$$

Let $r_0 \in \{0, 1, 2, 3\}$ be given, and let

$$\Psi(x) = \sum_{|\alpha|=r_0 \pmod 4} a_\alpha \Phi_\alpha(x) \quad (a_\alpha \in \mathbb{C}) \tag{2.1}$$

be a finite linear combination of the Hermite functions ($|\alpha| \le N$ for some $N \in \mathbb{N}$). The following theorem is an immediate consequence of Theorem 2.2.

Theorem 2.3 *The following transform defines a convolution*

$$(f \overset{\Psi}{\underset{\mathcal{F}}{*}} g)(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(u)g(v)\Psi(x-u-v)dudv.$$

2.2 Trigonometric Weight Functions

Let T_c, T_s denote the Fourier-cosine and Fourier-sine integral transforms. Let $h \in \mathbb{R}^d$ be fixed. Put $\theta_1(x) = \cos xh := \cos(\langle x, h\rangle)$, $\theta_2(x) = \sin xh := \sin(\langle x, h\rangle)$ as there is no danger of confusion.

Theorem 2.4 (see [9, 18, 19]) *Each of the integral transforms* (2.2)–(2.5) *below defines a convolution*:

$$(f \overset{\theta_1}{\underset{T_c}{*}} g)(x) = \frac{1}{4(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \big[f(x-u+h) + f(x-u-h) + f(x+u+h) + f(x+u-h)\big]g(u)du, \tag{2.2}$$

$$(f \overset{\theta_1}{\underset{T_c,T_s,T_s}{*}} g)(x) = \frac{1}{4(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \big[-f(x-u+h) - f(x-u-h) + f(x+u+h) + f(x+u-h)\big]g(u)du, \tag{2.3}$$

$$(f \overset{\theta_2}{\underset{T_c,T_s,T_c}{*}} g)(x) = \frac{1}{4(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \big[f(x-u+h) - f(x-u-h) + f(x+u+h) - f(x+u-h)\big]g(u)du, \tag{2.4}$$

$$(f \overset{\theta_2}{\underset{T_c,T_c,T_s}{*}} g)(x) = \frac{1}{4(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \big[f(x-u+h) - f(x-u-h) - f(x+u+h) + f(x+u-h) \big] g(u)du. \qquad (2.5)$$

Comparison 2.5 Different from the convolutions presented by other authors, what above does not require the invertibility of associated transforms. Indeed, according to our point of view as showed in Definition 2.1 the condition about the invertibility of transforms is not needed for constructing convolutions; namely, three operators K_1, K_2, K_3 may be un-injective. As we know that the Fourier-cosine and Fourier-sine transforms T_c and T_s are not injective, but there are still many infinitely many convolutions associated with them as presented in [9, 17–19]. In our point of view, that is a main reason why no convolution for un-invertible transforms appears until this moment. Most of convolution multiplications published are not commutative and not associative.

3 New Concept: Pair-Convolution

In this section we propose a new concept so-called *pair-convolution* which is a considerable generalization of convolution and generalized convolutions.

For any given multi-index $\alpha \in \mathbb{N}^d$, consider the transform

$$D_1(f,g)(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Phi_\alpha(x+u+v) f(u) g(v) du dv.$$

Using the convolutions in [18] we get

- *The case* $|\alpha| = 0 \pmod 4$:

$$T_c\big(D_1(f,g)\big)(x) = \Phi_\alpha(x)\big((T_c f)(x)(T_c g)(x) - (T_s f)(x)(T_s g)(x)\big),$$
$$T_s\big(D_1(f,g)\big)(x) = -\Phi_\alpha(x)\big((T_c f)(x)(T_s g)(x) + (T_s f)(x)(T_c g)(x)\big).$$

- *The case* $|\alpha| = 1 \pmod 4$:

$$T_c\big(D_1(f,g)\big)(x) = \Phi_\alpha(x)\big((T_c f)(x)(T_s g)(x) + (T_s f)(x)(T_c g)(x)\big),$$
$$T_s\big(D_1(f,g)\big)(x) = \Phi_\alpha(x)\big((T_c f)(x)(T_c g)(x) - (T_s f)(x)(T_s g)(x)\big).$$

- *The case* $|\alpha| = 2 \pmod 4$:

$$T_c\big(D_1(f,g)(x)\big) = \Phi_\alpha(x)\big((T_s f)(x)(T_s g)(x) - (T_c f)(x)(T_c g)(x)\big),$$
$$T_s\big(D_1(f,g)\big)(x) = \Phi_\alpha(x)\big((T_c f)(x)(T_s g)(x) + (T_s f)(x)(T_c g)(x)\big).$$

- *The case* $|\alpha| = 3 \pmod 4$:

$$T_c\big(D_1(f,g)\big)(x) = -\Phi_\alpha(x)\big((T_c f)(x)(T_s g)(x) + (T_s f)(x)(T_c g)(x)\big),$$
$$T_s\big(D_1(f,g)\big)(x) = \Phi_\alpha(x)\big((T_s f)(x)(T_s g)(x) - (T_c f)(x)(T_c g)(x)\big).$$

Motivated by the operational identities above, we introduce the following concept. Let U be a linear space, and let V be a commutative algebra on the complex field $\mathbb{C}$. Let $T_1, T_2 \in L(U, V)$ be the linear operators from U to V.

Definition 3.1 A bilinear map $* : \ U \times U :\longrightarrow U$ is called a *pair-convolution* associated with T_1, T_2, if there exist eight elements $\delta_k \in V$, $k = 1, \dots, 8$ so that the following identities hold for any $f, g \in U$:

$$T_1\big(*(f,g)\big) = \delta_1 T_1 f T_1 g + \delta_2 T_1 f T_2 g + \delta_3 T_2 f T_1 g + \delta_4 T_2 f T_2 g,$$
$$T_2\big(*(f,g)\big) = \delta_5 T_1 f T_1 g + \delta_6 T_1 f T_2 g + \delta_7 T_2 f T_1 g + \delta_8 T_2 f T_2 g.$$

Example 3.2 The above-mentioned bilinear transform $D_1(\ ,\ )$ is the pair-convolution for T_c, T_s. Note that this transform is not the generalized convolution associated with T_c, T_s.

Example 3.3 Consider the transform

$$D_2(f,g)(x) := \frac{1}{4(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \big[a\Phi_\alpha(x+u+v) + b\Phi_\beta(x+u-v) + c\Phi_\gamma(x-u+v) + d\Phi_\delta(x-u-v)\big] f(u)g(v)dudv,$$

where $a, b, c, d \in \mathbb{C}$, and $\alpha, \beta, \gamma, \delta$ are the multi-indexes. As (2.2), $D_2(\ ,\ )$ is a pair-convolution associated with $\mathcal{F}, \mathcal{F}^{-1}$.

Example 3.4 Let Ψ be the Hermite-type function as defined by (2.1). Write

$$\Psi(x) := \sum_{|\alpha|=0 \pmod 4} a_\alpha \Phi_\alpha(x) + \sum_{|\alpha|=1 \pmod 4} a_\alpha \Phi_\alpha(x) + \sum_{|\alpha|=2 \pmod 4} a_\alpha \Phi_\alpha(x) + \sum_{|\alpha|=3 \pmod 4} a_\alpha \Phi_\alpha(x). \tag{3.1}$$

Using the operational identities of $D_1(\ ,\ )$, we can prove that the transform

$$D_3(f,g)(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Psi(x \pm u \pm v) f(u)g(v)dudv \tag{3.2}$$

is a pair-convolution associated with T_c, T_s.

Example 3.5 Suppose that a_1, a_2, a_3, a_4 are any complex numbers. By Theorem 2.4 we can prove that the transform

$$D_4(f,g)(x) := \frac{1}{4(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \big[a_1 f(x-u+h) + a_2 f(x-u-h) + a_3 f(x+u+h) + a_4 f(x+u-h)\big] g(u)du$$

defines a pair-convolution associated with the transforms T_c, T_s.

We now consider the general integral equation

$$\lambda\varphi(x) + \int_E K(x, y)\varphi(y)dy = f(x). \tag{3.3}$$

We suppose that, by way of decomposing the kernel as

$$K(x, y) = \sum_{n=1}^{N} k_n(x, y) \tag{3.4}$$

such that each one of the transforms

$$(K_n\varphi)(x) = \int_E k_n(x, y)\varphi(y)dy \tag{3.5}$$

is a pair-convolution for specific operators T_1, T_2, then (3.3) may be solved by convolution approach. The main key of this approach is that we can reduce integral equations to a linear algebraic system of functional equations, and then apply an inverse transform of the transform $aT_1 + bT_2$ for some $a, b \in \mathbb{C}$. Thanks to pair-convolutions this approach could be more flexible, and realizable for a larger class of equations.

4 Final Remarks

To summary Sect. 3, we can interpret in other words as: the generalized convolution transforms, and the pair-convolution transforms might be called the *factorisable integrals, and pair-factorisable integrals* respectively by means of two specific transforms.

Problems for Further Studying Construct more pair-convolutions for the well-known integral transforms such as Hilbert, Mellin, Laplace, ..., and look for their applications.

Finally, since the set of all Hermite functions is a normally orthogonal basic of $L^2(\mathbb{R}^d)$, and thanks to the infinitely many pair-convolutions concerning the Hermite functions as presented, we propose the following conjecture.

Conjecture 4.1 *For any function $k \in L^2(\mathbb{R}^d)$, there exists a function $f \in L^2(\mathbb{R}^d)$ sufficiently closed to k such that each one of the transforms*

$$\int_E \int_E k(x \pm u \pm v) f(u) g(v) du dv$$

is either convolution or pair-convolution for specific operators K_1, K_2.

If this fact would be proved, we would have an approximately solvable manner called convolution one which could be different from that of the Galerkin method for Fredholm integral equations.

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Whittaker Differential Equation Associated to the Initial Heat Problem

M.M. Rodrigues and S. Saitoh

Abstract In this paper, by using the theory of reproducing kernels, we investigate integral transforms with kernels related to the solutions of the initial Whittaker heat problem.

Keywords Integral transform · Isometric mapping · Inversion formula · Initial value problem · Linear integral equation · Regularization · Eigenfunction · Eigenvalue · Reproducing kernel · Whittaker function

Mathematics Subject Classification (2010) Primary 42A38 · Secondary 32A30 · 4E22

1 Introduction

The Whittaker functions are closely related to the confluent hypergeometric functions which play an important role in various branches of applied mathematics and theoretical physics, for instance, fluid mechanics scalar and electromagnetic diffraction theory, atomic structure theory, input–output situations and storage consumption situations in economic problems, and so on. Moreover, they have acquired an ever increasing significance due to their frequent use in applications of mathematics to physical and technical problems [3, 4]. This justifies the continuous effort in studying properties of these functions and in gathering information about them.

Let consider the general method given in [6] for the existence and construction of the solution of the following initial problem

$$(\partial_t + L_x)u_f(t,x) = 0, \quad t > 0 \tag{1.1}$$

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satisfying the initial value condition

$$u_f(0,x) = f(x), \tag{1.2}$$

for some general linear operator L_x on some function space, and on some domain, by using the theory of reproducing kernels.

We shall investigate the integral transforms with kernels related to the solutions of the equations by using the theory of reproducing kernels, for the Whittaker heat equations.

The Whittaker functions denoted by $M_{\mu,\nu}(x)$, $W_{\mu,\nu}(x)$ arise as solutions to the Whittaker differential equation, i.e., they are solutions of the linear homogeneous ordinary differential equation of the second order

$$\frac{d^2W}{dx^2} + \left(-\frac{1}{4} + \frac{\mu}{x} + \frac{\frac{1}{4} - \nu^2}{x^2}\right)W = 0. \tag{1.3}$$

Here, we will deal with the Whittaker function $W_{i\tau,k-1/2}$ which is an eigenfunction of a second order differential operator

$$A_x W_{i\tau,k-1/2}(x) = \tau W_{i\tau,k-1/2}(x) \tag{1.4}$$

with an eigenvalue τ, and where

$$A_x = ix\frac{d^2}{dx^2} - \frac{ix}{4} - \frac{i(k^2-k)}{4x}. \tag{1.5}$$

We shall consider the case that τ are reals and positive, and then the functions

$$\exp(-\tau t) W_{i\tau,k-1/2}(x) \tag{1.6}$$

are the solutions of the operator equation

$$(\partial_t + A_x)u(t,x) = 0. \tag{1.7}$$

We shall consider some general solutions of (1.7) by a suitable sum of the solutions (1.6). In order to consider a fully general sum, we shall consider the kernel form for a nonnegative continuous function ρ,

$$\mathcal{K}_t(x,y;\rho) = \int_0^{+\infty} \exp\{-\tau t\} W_{i\tau,k-1/2}(x)\overline{W_{i\tau,k-1/2}(y)}\rho(\tau)d\tau. \tag{1.8}$$

Of course, here, we are considering the integral with absolutely convergence for the kernel form.

The fully general solutions of (1.7) may be represented in the integral form

$$u(t,x) = \int_0^{+\infty} \exp\{-\tau t\} W_{i\tau,k-1/2}(x)F(\tau)\rho(\tau)d\tau, \tag{1.9}$$

for the functions F satisfying

$$\int_0^{+\infty} \exp\{-\tau t\}\left|F(\tau)\right|^2 \rho(\tau)d\tau < \infty. \tag{1.10}$$

Then, the solution $u(t,x)$ of (1.7) satisfying the initial condition

$$u(0,x) = F(x) \tag{1.11}$$

will be obtained by $t \to 0$ in (1.9) with a natural meaning. However, this point will be very delicate and we will need to consider some deep and beautiful structure. Here, (1.8) is a reproducing kernel and in order to analyze the logic above, we will need the theory of reproducing kernels, essentially and beautiful ways. Indeed, in order to construct natural solutions (1.8) we will need a new framework and function space.

2 Main Results

In order to analyze the integral transform (1.9), we will need the essence of the theory of reproducing kernels. We are interesting in the integral transforms (1.9) in the framework of Hilbert spaces. Of course, we are interesting in the characterization of the image functions, the isometric identity like the Parseval identity and the inversion formula, basically. For these general and fundamental problems, we have a unified and fundamental method and concept in the general situation in [8–10], where we can find the general theory for linear mappings in the framework of Hilbert spaces.

Moreover, recently, we obtained a very general image identification method and inversion formula based on the Aveiro Discretization Method in Mathematics [5] by using the ultimate realization of reproducing kernel Hilbert spaces. Following the general theory, we shall build our results.

We form the reproducing kernel

$$\mathcal{K}(x,y;\rho) = \int_0^{+\infty} W_{i\tau,k-\frac{1}{2}}(x)\overline{W_{i\tau,k-\frac{1}{2}}(y)}\rho(\tau)d\tau, \quad t>0, \tag{2.1}$$

and consider the reproducing kernel Hilbert space $H_{\mathcal{K}(\rho)}(\mathbb{R}^+)$ admitting the kernel $\mathcal{K}(x,y;\rho)$. In particular, note that

$$\mathcal{K}_t(x,y;\rho) \in H_{\mathcal{K}(\rho)}\left(\mathbb{R}^+\right), \quad y>0.$$

Then, we obtain the main theorem in this paper:

Theorem 2.1 (Main Theorem) *For any member $f \in H_{\mathcal{K}(\rho)}(\mathbb{R}^+)$, the solution $u_f(t,x)$ of the initial value problem, for $t>0$*

$$(\partial_t + A_x)u_f(t,x) = 0 \tag{2.2}$$

satisfying the initial value condition

$$u_f(0,x) = f(x), \tag{2.3}$$

exists and it is given by

$$u_f(t,x) = \big(f(\cdot), \mathcal{K}_t(\cdot, x; \rho)\big)_{H_{\mathcal{K}(\rho)}(\mathbb{R}^+)}. \tag{2.4}$$

Here, the meaning of the initial value (2.3) *is given by*

$$\begin{aligned}\lim_{t\to+0} u_f(t,x) &= \lim_{t\to+0}\big(f(\cdot), \mathcal{K}_t(\cdot, x; \rho)\big)_{H_{\mathcal{K}(\rho)}(\mathbb{R}^+)}\\ &= \big(f(\cdot), \mathcal{K}(\cdot, x; \rho)\big)_{H_{\mathcal{K}(\rho)}(\mathbb{R}^+)}\\ &= f(x),\end{aligned} \tag{2.5}$$

whose existence is, in general, ensured and the limit is the uniformly convergence on any subset of $\mathbb{R}^+$ *such that* $\mathcal{K}(x,x;\rho)$ *is bounded.*

The uniqueness property of the initial value problem is valid.

In our theorem, the complete property of the solutions $u_f(t,x)$ of (2.2) and (2.3) satisfying the initial value f may be derived by the reproducing kernel Hilbert space admitting the kernel

$$k(x,t;y,\tau;\rho) := \big(\mathcal{K}_\tau(\cdot, y; \rho), \mathcal{K}_t(\cdot, x; \rho)\big)_{H_{\mathcal{K}(\rho)}(\mathbb{R}^+)}. \tag{2.6}$$

In our method, we see that the existence problem of the initial value problem is based on the eigenfunctions and we are constructing the desired solution satisfying the desired initial condition. For a larger knowledge for the eigenfunctions we can consider a more general initial value problem.

Furthermore, by considering the linear mapping of (2.4) with various situations, we will be able to obtain various inverse problems looking for the initial values f from the various out put data of $u_f(t,x)$.

3 Proof of the Main Theorem

The first, note that the kernel $\mathcal{K}_t(x,y;\rho)$ satisfies the operator equation (2.2) for any fixed y, because the functions

$$\exp(-\tau t) W_{i\tau, k-1/2}(x)$$

satisfy the operator equation and it is the summation. Similarly, the function $u_f(t,x)$ defined by (2.4) is the solution of the operator equation (2.2).

In order to see the initial value problem, we note the important general properties

$$\mathcal{K}_t(x, y; \rho) \ll \mathcal{K}(x, y; \rho); \tag{3.1}$$

that is, $\mathcal{K}(x, y; \rho) - \mathcal{K}_t(x, y; \rho)$ is a positive definite quadratic form function and we have

$$H_{\mathcal{K}_t(\rho)} \subset H_{\mathcal{K}(\rho)}(\mathbb{R}^+)$$

and for any function $f \in H_{\mathcal{K}_t(\rho)}$

$$\|f\|_{H_{\mathcal{K}(\rho)}(\mathbb{R}^+)} = \lim_{t \to +0} \|f\|_{H_{\mathcal{K}_t(\rho)}}$$

in the sense of non-decreasing norm convergence [2]. In order to see the crucial point in (2.5), note that

$$\begin{aligned}
&\left\|\mathcal{K}(x, y; \rho) - \mathcal{K}_t(x, y; \rho)\right\|^2_{H_{\mathcal{K}(\rho)}(\mathbb{R}^+)} \\
&= \mathcal{K}(y, y; \rho) - 2\mathcal{K}_t(y, y; \rho) + \left\|\mathcal{K}_t(x, y; \rho)\right\|^2_{H_{\mathcal{K}(\rho)}(\mathbb{R}^+)} \\
&\leq \mathcal{K}(y, y; \rho) - 2\mathcal{K}_t(y, y; \rho) + \left\|\mathcal{K}_t(x, y; \rho)\right\|^2_{H_{\mathcal{K}_t(\rho)}} \\
&= \mathcal{K}(y, y; \rho) - \mathcal{K}_t(y, y; \rho),
\end{aligned}$$

that converses to zero as $t \to +0$. We thus obtain the desired limit property in the theorem.

The uniqueness property of the initial value problem follows from (2.4) easily, by using the below completeness Theorem 4.2.

In the main theorem, the realization of the reproducing kernel Hilbert space $H_{\mathcal{K}(\rho)}(\mathbb{R}^+)$ is a crucial point, for this purpose, we are interested in the calculation of the kernel $\mathcal{K}(x, y; \rho)$.

4 Realizations of the Reproducing Kernel Hilbert Spaces

As the theory of reproducing kernels, their realizations will give interesting research topics that are requested separate papers. So, here, we shall discuss the following concrete form of the reproducing kernels. From [3], we have

$$\begin{aligned}
&\int_0^{+\infty} \left|\Gamma(k + i\tau)\right|^2 W_{i\tau, k-\frac{1}{2}}(x) \overline{W_{i\tau, k-\frac{1}{2}}(y)} d\tau \\
&= \sqrt{\pi}\,\Gamma(2k)(xy)^k (x + y)^{-2k+1} K_{2k-\frac{1}{2}}\left(\frac{x + y}{2}\right),
\end{aligned} \tag{4.1}$$

where $K_{2k-\frac{1}{2}}(z)$ denotes the modified Bessel function. For the realization of the reproducing kernel Hilbert space H_K admitting the kernel (4.1), we can use the following representations for the modified Bessel function:

$$K_\alpha(x) = \frac{\sqrt{\pi}}{\Gamma(\alpha+1/2)}\left(\frac{x}{2}\right)^\alpha \int_1^{+\infty} e^{-xt}\left(t^2-1\right)^{\alpha-1/2} dt, \quad x>0, \tag{4.2}$$

$$K_\alpha(x) = \frac{2^\alpha \Gamma(\alpha+1/2)}{x^\alpha\sqrt{\pi}} \int_0^{+\infty} \frac{\cos xt}{(1+t^2)^{\alpha+1/2}} dt, \quad x>0,\ \alpha > -\frac{1}{2}, \tag{4.3}$$

and

$$K_\alpha(x) = \frac{1}{2}\left(\frac{x}{2}\right)^\alpha \int_0^{+\infty} e^{-t-x^2/(4t)} t^{-\alpha-1} dt, \quad |\arg x| < \frac{\pi}{4}. \tag{4.4}$$

See [1], pp. 236–347, for example. By using the structure theorems and techniques of reproducing kernels, we can realize the space H_K, however, in this case, the realizations are not so simple. See, [9].

However, we can apply quite general formula by the Aveiro discretization method in the sense of numerical in [5]. In the method, numerical experiments were also given.

Proposition 4.1 (Ultimate realization of reproducing kernel Hilbert spaces) *In our general situation and for a uniqueness set $\{p_j\}$ for the reproducing kernel Hilbert space H_K of the set E satisfying the linearly independence of $K(\cdot, p_j)$ for any finite number of the points p_j, we obtain*

$$\|f\|_{H_K}^2 = \lim_{n\to\infty} \sum_{j=1}^n \sum_{j'=1}^n f(p_j)\widetilde{a_{jj'}}\overline{f(p_{j'})}. \tag{4.5}$$

Here, $\widetilde{a_{jj'}}$ is the element of the complex conjugate inverse of the positive definite Hermitian matrix formed by

$$a_{jj'} = K(p_j, p_{j'}).$$

For applying Proposition 4.1, we need only—see [5] for the details:

Theorem 4.2 *In the integral transform induced from* (4.1),

$$\left\{W_{i\tau,k-1/2}(x); x\in\mathbb{R}^+\right\}$$

is complete in $L^2(\mathbb{R}^+, |\Gamma(k+i\tau)|^2 d\tau)$. For any different points $\{x_j\}_{j=1}^n (x_j>0)$,

$$\left\{W_{i\tau,k-1/2}(x_j)\right\}_{j=1}^n$$

are linearly independent.

First we recall the identity for the Whittaker function

$$W_{k,m}(x) = \frac{e^{-x/2}x^k}{\Gamma(\frac{1}{2}-k+m)} \int_0^{+\infty} e^{-t} t^{-k-(1/2)+m} (1+t/x)^{k-(1/2)+m} dt$$

for

$$\Re\big(k-(1/2)+m\big)<0, \quad x>0$$

([1], p. 197, (4.4.3)). By the transform $t/x=\xi$ and in our notation, we have

$$W_{i\tau,k-1/2}(x)=\frac{e^{-x/2}x^k}{\Gamma(-i\tau+k)} \times \int_0^{+\infty} e^{-x\xi}\xi^{i\tau+k-1}(1+\xi)^{i\tau+k-1}d\xi.$$

By using this formula, we can derive the completeness.

Meanwhile, by using the asymptotic expansion

$$W_{\lambda,\mu}(z)\sim -\left(\frac{4z}{\lambda}\right)^{\frac{1}{4}} e^{-\lambda+\lambda\log\lambda}\left(\sin 2\sqrt{\lambda z}-\lambda\pi-\frac{\pi}{4}\right)$$

([7], p. 1075), we can see the linearly independence.

In Proposition 4.1, for the uniqueness set of the space, if the reproducing kernel is analytical as in the present case, then, the criteria will be very simple by the *identity theorem of analytic functions*. For the Sobolev space cases, we have to consider some dense subset of E for the uniqueness set. Meanwhile, the linearly independence will be easily derived from the integral representations of the kernels.

We can realize the important reproducing kernel Hilbert space concretely and analytically. Meanwhile, we are also interested in the kernel forms $\mathcal{K}_t$ and k. These calculations will create a new and large field in integral formulas.

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Recovery of Holomorphic Functions and Taylor Coefficients by Sampling

Vu Kim Tuan and Amin Boumenir

Abstract We introduce an interpolation formula for holomorphic functions and prove its convergence pointwise under very general condition. We obtain also a recovery formula for Taylor coefficients from discrete samples.

1 Problems

Let f be a holomorphic function on a connected domain $\Omega \subset \mathbb{C}$, and $z_0 \in \Omega$. Then if $\mathbb{D}_{z_0} \subset \Omega$ is a disk with center at z_0, then $f(z)$ has a Taylor series

$$f(z) = \sum_{m=0}^{\infty} F_m (z - z_0)^m,$$

that converges, at least inside $\mathbb{D}_{z_0}$. Let $\{z_n\}_1^\infty$ be a sequence of distinct points in Ω that converges to some $w_0 \in \mathbb{D}_{z_0}$. The classical interior uniqueness theorem for holomorphic functions says that f is uniquely determined by its values at $\{z_n\}_1^\infty$. However, to our best knowledge, in the general case there is no formula to recover $f(z)$ from $\{f(z_n)\}_1^\infty$. Also, no practical formula to determine the Taylor coefficients $\{F_m\}_0^\infty$ from $\{f(z_n)\}_1^\infty$ is known. For recovery of holomorphic functions from special sampling sequences see [1, 4, 5]. In this paper we introduce a formula, that recovers $f(z)$ from $\{f(z_n)\}_1^\infty$, and another formula that determines its Taylor coefficients $\{F_m\}_0^\infty$ from $\{f(z_n)\}_1^\infty$. We will discuss also the problem what would be a necessary and sufficient condition on a sequence of complex pairs $\{(z_k, u_k)\}_1^\infty$ such that there exists a holomorphic function f with $f(z_k) = u_k$ for any $k \geq 1$. The tools to be used here are the Hardy space and the reproducing kernel Hilbert spaces.

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2 Interpolation Formula in Hardy Space $\mathcal{H}^2(\mathbb{D})$

The Hardy space $\mathcal{H}^2(\mathbb{D})$ is defined as the space of all holomorphic functions f in the open unit disk $\mathbb{D} = \{z \in \mathbb{C}, |z| < 1\}$ for which the norm

$$\|f\|_{\mathcal{H}^2} = \sup_{0\le r<1} \sqrt{\frac{1}{2\pi}\int_0^{2\pi} \left|f\left(re^{it}\right)\right|^2 dt}$$

is finite. Consider the MacLaurin series for f

$$f(z) = \sum_{m=0}^{\infty} F_m z^m, \quad z \in \mathbb{D},$$

it is well-known that $f \in \mathcal{H}^2(\mathbb{D})$ if, and only if, the sequence of its MacLaurin coefficients $\{F_m\}_0^\infty$ belongs to the space of square summable sequences $l^2(\mathbb{N}_0)$

$$\mathcal{F} := \{F_m\}_0^\infty \in l^2(\mathbb{N}_0) \quad \text{if } \|\mathcal{F}\|_{l^2} = \sqrt{\sum_{m=0}^{\infty} |F_m|^2} < \infty.$$

With the endowed inner product

$$\langle f, g\rangle = \sum_{m=0}^{\infty} F_m \overline{G}_m, \quad g(z) = \sum_{m=0}^{\infty} G_m z^m \in \mathcal{H}^2(\mathbb{D}),$$

$\mathcal{H}^2(\mathbb{D})$ becomes a Hilbert space and the transform $L : l^2(\mathbb{N}_0) \to \mathcal{H}^2(\mathbb{D})$ defined by

$$L(\mathcal{F}) = f(z) = \sum_{m=0}^{\infty} F_m z^m$$

is a linear inner product preserving isomorphism. $\mathcal{H}^2(\mathbb{D})$ is also a reproducing kernel Hilbert space (RKHS) with the reproducing kernel

$$K(z, w) = \frac{1}{1 - \overline{w}z}.$$

We recall the following general result for interpolation in RKHS, [2].

Theorem 2.1 *Let $\mathcal{H}$ be a RKHS on X with reproducing kernel, K, and let $\Gamma = \{x_1, \dots, x_n\} \subset X$ be distinct. If the matrix $(K(x_i, x_j))$ is invertible, then for any $\{\lambda_1, \dots, \lambda_n\} \subset \mathbb{C}$ there exists a function interpolating these values, and the unique interpolating function of minimum norm is given by the formula $g(x) = \sum_{j=1}^n \alpha_j K(x, x_j)$, where $w = (\alpha_1, \dots, \alpha_n)^T$ is given by $w = (K(x_i, x_j))^{-1} v$, with $v = (\lambda_1, \dots, \lambda_n)^T$.*

Let $\{z_k\}_{k=1}^n$ be n arbitrary distinct complex points in $\mathbb{D}$. The following theorem is a practical interpolation formula for functions in $\mathcal{H}^2(\mathbb{D})$.

Theorem 2.2 *Let*

$$f_n(z) = \sum_{k,l=1}^{n} \frac{\prod_{j=1}^{n}[(1-z_k\overline{z}_j)(1-\overline{z}_l z_j)]}{(1-z_k\overline{z}_l)\prod_{j=1,j\neq k}^{n}(z_k-z_j)\prod_{j=1,j\neq l}^{n}(\overline{z}_l-\overline{z}_j)} \frac{f(z_k)}{1-z\overline{z}_l}. \tag{2.1}$$

Then $f_n(z) \in \mathcal{H}^2(\mathbb{D})$, and $f_n(z_k) = f(z_k)$ for $k = 1, \ldots, n$. Moreover, the function $f_n(z)$ is the minimal approximation of $f(z)$ in the sense that among all functions in the Hardy space $\mathcal{H}^2(\mathbb{D})$ that take the values $f(z_k)$ at z_k, $k = 1, 2, \ldots, n$, it has the minimal norm in $\mathcal{H}^2(\mathbb{D})$.

Proof Let

$$\tilde{a}_{lk} = \frac{\prod_{j=1}^{n}[(1-z_k\overline{z}_j)(1-\overline{z}_l z_j)]}{(1-z_k\overline{z}_l)\prod_{j=1,j\neq k}^{n}(z_k-z_j)\prod_{j=1,j\neq l}^{n}(\overline{z}_l-\overline{z}_j)}. \tag{2.2}$$

Then formula (2.1) can be rewritten in the form

$$f_n(z) = \sum_{k,l=1}^{n} f(z_k)\frac{\widetilde{a}_{lk}}{1-z\overline{z}_l}. \tag{2.3}$$

Let A be the matrix

$$A = (a_{kl})_{n\times n} := \left(\frac{1}{1-z_k\overline{z}_l}\right)_{n\times n}.$$

Since $K(z,w) = \frac{1}{1-\overline{w}z}$ is the reproducing kernel of the RKHS $\mathcal{H}^2(\mathbb{D})$, and $a_{kl} = K(p_k, p_l)$, then the matrix A is positive definite. Since $\frac{1}{1-z\overline{z}_l} \in \mathcal{H}^2(\mathbb{D})$, then $f_n(z)$, as a linear combination of functions $\frac{1}{1-z\overline{z}_l}$ from $\mathcal{H}^2(\mathbb{D})$, belongs to $\mathcal{H}^2(\mathbb{D})$.

Now we prove that $B = (\widetilde{a}_{kl})_{n\times n}$ with $\tilde{a}_{lk}$ being defined by (2.2) is the inverse of A. Consider the matrix $C = (c_{kl})_{n\times n} = (\frac{1}{1-x_k y_l})_{n\times n}$. It is well-known that (see [3], Problem 1.9)

$$\det(C) = \det\left(\frac{1}{1-x_k y_l}\right)_{n\times n} = \frac{\prod_{1\le j<i\le n}[(x_i-x_j)(y_i-y_j)]}{\prod_{1\le i,j\le n}(1-x_i y_j)}. \tag{2.4}$$

Moreover, the entry $\tilde{c}_{kl}$ of the inverse matrix $C^{-1} = (\tilde{c}_{kl})_{n\times n}$ of C is defined by

$$\tilde{c}_{kl} = (-1)^{k+l}\frac{\det(C_{lk})}{\det(C)}, \tag{2.5}$$

here C_{lk} is the $(n-1)\times(n-1)$ submatrix of A obtained by deleting the lth row and the kth column of A. Since C_{lk} has the same structure as C, we have

$$\det(C_{lk}) = \frac{\prod_{\substack{1\le j<i\le n\\ i,j\ne l}}(x_i - x_j)\prod_{\substack{1\le j<i\le n\\ i,j\ne k}}(y_i - y_j)}{\prod_{\substack{1\le j<i\le n\\ i\ne l,\, j\ne k}}(1 - x_i y_j)}. \tag{2.6}$$

Consequently, from (2.4), (2.5) and (2.6) we obtain

$$\begin{aligned}\tilde{c}_{kl} &= \frac{(-1)^{k+l}\prod_{j=1}^n(1-x_l y_j)\prod_{i=1}^n(1-x_i y_k)}{(1-x_l y_k)\prod_{j=1}^{l-1}(x_l - x_j)\prod_{i=l+1}^n(x_i - x_l)\prod_{j=1}^{k-1}(y_k - y_j)\prod_{i=k+1}^n(y_i - y_k)}\\ &= \frac{\prod_{j=1}^n[(1-x_l y_j)(1-x_j y_k)]}{(1-x_l y_k)\prod_{j=1,j\ne l}^n(x_l - x_j)\prod_{j=1,j\ne k}^n(y_k - y_j)}.\end{aligned} \tag{2.7}$$

Let $x_k = \overline{y}_k = z_k$, $k = 1,2,\ldots,n$. Then $C = A$, and comparing (2.7) with (2.2) we see $\tilde{c}_{kl} = \tilde{a}_{kl}$. Hence, $A^{-1} = (\tilde{a}_{kl})_{n\times n}$. In particular,

$$\sum_{l=1}^n \widetilde{a}_{lk} a_{jl} = \delta_{kj},$$

where $\delta_{kj} = 0$ if $k \ne j$, and $\delta_{kj} = 1$ if $k = j$. We have

$$\begin{aligned} f_n(z_j) &= \sum_{k,l=1}^n f(z_k)\frac{\widetilde{a}_{lk}}{1 - z_j\overline{z}_l} = \sum_{k=1}^n f(z_k)\sum_{l=1}^n \frac{\widetilde{a}_{lk}}{1 - z_j\overline{z}_l}\\ &= \sum_{k=1}^n f(z_k)\sum_{l=1}^n \widetilde{a}_{lk}a_{jl} = \sum_{k=1}^n f(z_k)\delta_{kj} = f(z_j).\end{aligned} \tag{2.8}$$

Hence, $f_n(z)$ interpolates $f(z)$ at $z_1, z_2, \ldots, z_n$. From RKHS theory (see Corollary 4.5 from [2]) it follows that $f_n(z)$ is the orthogonal projection of f on the subspace spanned by the functions $\{\frac{1}{1-z\overline{z}_1}, \ldots, \frac{1}{1-z\overline{z}_n}\}$, and therefore, is the unique interpolation function of minimum norm. Theorem is proved. □

3 Convergence

The following theorem deals with the convergence of the interpolation formula.

Theorem 3.1 *Let $\{z_k\}_{k=1}^\infty$ be a sequence of distinct complex numbers on $\mathbb{D}$, that converges to $w_0 \in \mathbb{D}$. Then*

$$\lim_{n\to\infty} f_n(z) = f(z), \quad z \in \mathbb{D}, \tag{3.1}$$

where the convergence is both pointwise and in $\mathcal{H}^2(\mathbb{D})$ norm.

Proof Let $\mathcal{F} = \{F_m\}_0^\infty$ be the sequence of MacLaurin coefficients of $f(z)$. Denote

$$\mathcal{F}_n = \{F_{n,m}\}_{m=0}^\infty, \quad \text{where } F_{n,m} = \sum_{k,l=1}^{n} f(z_k)\widetilde{a}_{lk}\overline{z}_l^m. \tag{3.2}$$

Then

$$\begin{aligned}\sum_{m=0}^{\infty} F_{n,m} z^m &= \sum_{k,l=1}^{n} f(z_k)\widetilde{a}_{lk} \sum_{m=0}^{\infty} z^m \overline{z}_l^m \\ &= \sum_{k,l=1}^{n} f(z_k)\frac{\widetilde{a}_{lk}}{1 - z\overline{z}_l} = f_n(z).\end{aligned} \tag{3.3}$$

Hence, $\mathcal{F}_n$ is the sequence of MacLaurin coefficients of $f_n(z)$ and so

$$\left\| f_n(z) \right\|_{\mathcal{H}^2} = \|\mathcal{F}_n\|_{l^2}.$$

Since function $f_n(z)$ is the minimal approximation of $f(z)$ in the sense that among functions of the space $\mathcal{H}^2(\mathbb{D})$ that interpolates $f(z)$ at $z_1, z_2, \ldots, z_n$, $f_n(z)$ has the minimal norm in $\mathcal{H}^2(\mathbb{D})$, then the sequence $\mathcal{F}_n$ is the minimal approximation of the sequence $\mathcal{F}$ in the sense that among sequences of the space $l^2(\mathbb{N}_0)$ whose L transform is $f(z_k)$ at z_k, $k = 1, 2, \ldots, n$, it has the minimal norm in $l^2(\mathbb{N}_0)$. As both $\mathcal{F}_k$ with $k > n > 0$ and $\mathcal{F}$ also have L transforms $f(z_k)$ at z_k, $k = 1, 2, \ldots, n$, the minimal norm property of $\mathcal{F}_n$ yields

$$\|\mathcal{F}_n\|_{l^2} \le \|\mathcal{F}_k\|_{l^2}, \quad \text{if } 0 < n < k,$$

and

$$\|\mathcal{F}_n\|_{l^2} \le \|\mathcal{F}\|_{l^2}, \quad \text{if } n > 0.$$

The sequence $\{\mathcal{F}_n\}_{n=0}^\infty$ is uniformly bounded in norm by $\|\mathcal{F}\|_{l^2}$ in the Hilbert space $l^2(\mathbb{N}_0)$, therefore, there exists a subsequence $\{\mathcal{F}_{n_j}\}_{j=0}^\infty$ that converges weakly to some $\mathcal{F}^* \in l^2(\mathbb{N}_0)$ and $\|\mathcal{F}^*\|_{l^2} \le \liminf_{j\to\infty} \|\mathcal{F}_{n_j}\|_{l^2}$. Let

$$\mathcal{G}_z = \left\{\overline{z}^m\right\}_{m=0}^\infty, \quad z \in \mathbb{D}.$$

Then $\mathcal{G}_z \in l^2(\mathbb{N}_0)$, and the weak convergence yields

$$\lim_{j\to\infty} \langle \mathcal{F}_{n_j}, \mathcal{G}_z \rangle = \left\langle \mathcal{F}^*, \mathcal{G}_z \right\rangle,$$

for any $z \in \mathbb{D}$. We have

$$\langle \mathcal{F}_n, \mathcal{G}_z \rangle = \sum_{m=0}^{\infty} F_{n,m} z^m = f_n(z).$$

Denote $f^*(z) = \langle \mathcal{F}^*, \mathcal{G}_z \rangle$, then $f^* \in \mathcal{H}^2(\mathbb{D})$, and

$$f^*(z) = \lim_{j\to\infty} f_{n_j}(z), \quad z \in \mathbb{D}.$$

In particular, since $f_{n_j}(z_k) = f(z_k)$ when $n_j \geq k$, we have

$$f^*(z_k) = \lim_{j\to\infty} f_{n_j}(z_k) = f(z_k), \quad k > 0.$$

If the sequence $\{z_k\}$ has a limit $w_0 \in \mathbb{D}$, then by the classical interior uniqueness theorem $f^*(z) = f(z)$.

The sequence $\{\|\mathcal{F}_{n_j}\|_{l^2}\}_{j=0}^\infty$ is monotone increasing, and bounded above by $\|\mathcal{F}\|_{l^2}$, hence,

$$\|\mathcal{F}\|_{l^2} = \left\|\mathcal{F}^*\right\|_{l^2} \leq \liminf_{j\to\infty} \|\mathcal{F}_{n_j}\|_{l^2} \leq \|\mathcal{F}\|_{l^2}.$$

Consequently,

$$\lim_{j\to\infty} \|\mathcal{F}_{n_j}\|_{l^2} = \|\mathcal{F}\|_{l^2}.$$

The sequence $\{\mathcal{F}_{n_j}\}_{j=0}^\infty$ converges weakly to $\mathcal{F}$, and the sequence of their norms $\{\|\mathcal{F}_{n_j}\|_{l^2}\}_{j=0}^\infty$ converges to $\|\mathcal{F}\|_{l^2}$, therefore, the subsequence $\{\mathcal{F}_{n_j}\}_{j=0}^\infty$ converges strongly to $\mathcal{F}$.

We claim now that the whole sequence $\{\mathcal{F}_n\}_{n=0}^\infty$ converges strongly to $\mathcal{F}$. If not, there would exist an $\epsilon > 0$ and a subsequence $\{\mathcal{F}_{n_s}\}_{s=0}^\infty$ such that

$$\|\mathcal{F} - \mathcal{F}_{n_s}\|_{l^2} \geq \epsilon, \quad s > 0.$$

By the same techniques as above, from the sequence $\{\mathcal{F}_{n_s}\}_{s=0}^\infty$ one can choose a subsequence convergent strongly to $\mathcal{F}$, that is a contradiction. Hence, $\mathcal{F}_n$ converges in norm to $\mathcal{F}$, and therefore, f_n converges in norm to f. Moreover, $\mathcal{F}_n$ converges weakly to $\mathcal{F}$, then $f_n(z)$, as inner product of $\mathcal{F}_n$ with $\mathcal{G}_z$, converges pointwise to $f(z)$ in $\mathbb{D}$. □

Theorem 3.2 *The MacLaurin coefficients F_m of $f(z)$ can be determined from $\{f(z_k)\}_1^\infty$ by formula*

$$F_m = \lim_{n\to\infty} \sum_{k,l=1}^{n} f(z_k)\widetilde{a}_{lk}\overline{z}_l^m, \quad m = 0, 1, 2, \ldots.$$

Proof In fact, as $n \to \infty$

$$\left| F_m - \sum_{k,l=1}^{n} f(z_k)\widetilde{a}_{lk}\overline{z}_l^m \right| = |F_m - F_{n,m}|$$

$$\leq \sqrt{\sum_{j=0}^{\infty} |F_j - F_{n,j}|^2} = \|\mathcal{F} - \mathcal{F}_n\|_{l^2} \to 0.$$

□

4 General Case

Let $h(w)$ be a holomorphic function in the domain Ω, and $\{p_n\}_1^\infty$ be a sequence in Ω that has a limit point $p_0 \in \Omega$. We will explain how to recover $h(w)$ from $\{h(p_n)\}_1^\infty$. Let $\mathbb{D}_R(w_0)$ be a disc inside Ω, that contains p_0

$$p_0 \in \mathbb{D}_R(w_0) = \big\{w, |w - w_0| < R\big\} \subset \Omega.$$

The Taylor series

$$h(w) := \sum_{m=0}^{\infty} H_m (w - w_0)^m$$

converges in $\mathbb{D}_R(w_0)$, therefore, for any $0 < R_1 < R$,

$$|H_m| R_1^m \to 0, \quad \text{as } m \to \infty.$$

Consequently,

$$\big\{H_m r^m\big\}_0^\infty \in l^2(\mathbb{N}_0) \quad \text{for } 0 < r < R_1 < R.$$

Thus,

$$f(z) := \sum_{m=0}^{\infty} \big(H_m r^m\big) z^m \in \mathcal{H}^2(\mathbb{D}), \quad z \in \mathbb{D}.$$

Let $\{w_k\}_1^\infty$ be a subsequence of $\{p_n\}_1^\infty$, that is inside $\mathbb{D}_r(w_0)$ and convergent to p_0. Then

$$f(z) = h(w), \qquad f(z_k) = h(w_k), \quad z = \frac{w - w_0}{r},\ z_k = \frac{w_k - w_0}{r}.$$

Consequently, the function

$$h_n(w) = \sum_{k,l=1}^{n} \frac{h(w_k)}{r^2 - (w - w_0)(\overline{w}_l - \overline{w}_0)} \times \frac{\prod_{j=1}^n [(r^2 - (w_k - w_0)(\overline{w}_j - \overline{w}_0))(r^2 - (\overline{w}_l - \overline{w}_0)(w_j - w_0))]}{(r^2 - (w_k - w_0)(\overline{w}_l - \overline{w}_0)) \prod_{j=1, j\neq k}^n (w_k - w_j) \prod_{j=1, j\neq l}^n (\overline{w}_l - \overline{w}_j)}, \tag{4.1}$$

converges pointwise to $h(w)$ for any $w \in \mathbb{D}_r(w_0)$. Its Taylor coefficients H_m at w_0 can be found by formula for $m = 0, 1, \ldots,$

$$H_m = \lim_{n\to\infty} \sum_{k,l=1}^{n} \frac{h(w_k)}{r^{2m+2}} (\overline{w}_l - \overline{w}_0)^m \times \frac{\prod_{j=1}^n [(r^2 - (w_k - w_0)(\overline{w}_j - \overline{w}_0))(r^2 - (\overline{w}_l - \overline{w}_0)(w_j - w_0))]}{(r^2 - (w_k - w_0)(\overline{w}_l - \overline{w}_0)) \prod_{j=1, j\neq k}^n (w_k - w_j) \prod_{j=1, j\neq l}^n (\overline{w}_l - \overline{w}_j)}. \tag{4.2}$$

Let w' be any point of Ω at which we want to determine the value of h, as well as its Taylor coefficients. Since Ω is connected, there is a chain of disks $\mathbb{D}_1, \mathbb{D}_2, \dots, \mathbb{D}_t$, with the following properties:

(a) All $\mathbb{D}_j$, $j = 1, \dots, t$, are inside Ω.
(b) Two consecutive discs have no empty intersection $\mathbb{D}_j \cap \mathbb{D}_{j+1} \neq \emptyset$.
(c) $\mathbb{D}_1$ contains p_0.
(d) $\mathbb{D}_k$ is centered at w'.

Function h is given on a sequence in $\mathbb{D}_1$, therefore, h can be determined in the whole disc $\mathbb{D}_1$. In particular, h can be recovered at a sequence of points in $\mathbb{D}_1 \cap \mathbb{D}_2 \subset \mathbb{D}_2$, that is convergent inside $\mathbb{D}_2$. Once h is found at a convergent sequence in $\mathbb{D}_2$, it can be found on a convergent sequence in $\mathbb{D}_3$. By the end we can recover the function and its Taylor coefficients in $\mathbb{D}_t$, in particular, at w'.

5 Characterization of the Holomorphic Functions

It is well-known (Sect. 2) that to determine whether a function belongs to the Hardy space $\mathcal{H}^2(\mathbb{D})$ or not, we need the values of the function in the whole disc $\mathbb{D}$, or all its MacLaurin coefficients. In this section first we will determine whether a function belongs to the Hardy space $\mathcal{H}^2(\mathbb{D})$ based on its values at a sequence of points. Let $\{z_k\}_{k=1}^{\infty}$ be a sequence of distinct complex numbers on $\mathbb{D}$, that converges to $w_0 \in \mathbb{D}$, and let $\{u_k\}_{k=1}^{\infty}$ be a sequence of complex numbers. The following theorem answers the question under which conditions there exists a Hardy function $f \in \mathcal{H}^2(\mathbb{D})$ such that $f(z_k) = u_k$ for any $k = 1, 2, \dots$.

Theorem 5.1 *The necessary and sufficient condition for existence of a Hardy function $f \in \mathcal{H}^2(\mathbb{D})$ such that $f(z_k) = u_k$ for any $k = 1, 2, \dots,$ is*

$$\sup_{n\geq 1} \sum_{m=0}^{\infty} \left| \sum_{k,l=1}^{n} u_k \widetilde{a}_{lk} \overline{z}_l^m \right|^2 < \infty. \tag{5.1}$$

Proof Let condition (5.1) hold. Denote

$$f_n(z) = \sum_{k,l=1}^{n} u_k \frac{\widetilde{a}_{lk}}{1 - z\overline{z}_l}. \tag{5.2}$$

Since $\frac{1}{1-z\overline{z}_l} \in \mathcal{H}^2(\mathbb{D})$, then $f_n(z)$, defined by (5.2), is a linear combination of functions $\frac{1}{1-z\overline{z}_l}$ from $\mathcal{H}^2(\mathbb{D})$, and therefore belongs to $\mathcal{H}^2(\mathbb{D})$. We have

$$\begin{aligned} f_n(z_j) &= \sum_{k,l=1}^{n} u_k \frac{\widetilde{a}_{lk}}{1 - z_j\overline{z}_l} = \sum_{k=1}^{n} u_k \sum_{l=1}^{n} \frac{\widetilde{a}_{lk}}{1 - z_j\overline{z}_l} \\ &= \sum_{k=1}^{n} u_k \sum_{l=1}^{n} \widetilde{a}_{lk} a_{jl} = \sum_{k=1}^{n} u_k \delta_{kj} = u_j, \quad j = 1, 2, \dots, n. \end{aligned} \tag{5.3}$$

Denote

$$\mathcal{F}_n = \{F_{n,m}\}_{m=0}^{\infty}, \quad \text{where } F_{n,m} = \sum_{k,l=1}^{n} u_k \widetilde{a}_{lk} \overline{z}_l^m. \tag{5.4}$$

The condition (5.1) yields that the sequence $\{\|\mathcal{F}_n\|_{l^2}\}_{n=1}^{\infty}$ is bounded. Moreover,

$$\begin{aligned}\sum_{m=0}^{\infty} F_{n,m} z^m &= \sum_{k,l=1}^{n} u_k \widetilde{a}_{lk} \sum_{m=0}^{\infty} z^m \overline{z}_l^m \\ &= \sum_{k,l=1}^{n} u_k \frac{\widetilde{a}_{lk}}{1 - z\overline{z}_l} = f_n(z).\end{aligned} \tag{5.5}$$

Hence, $\mathcal{F}_n$ is the sequence of MacLaurin coefficients of $f_n(z)$ and so

$$\left\| f_n(z) \right\|_{\mathcal{H}^2} = \|\mathcal{F}_n\|_{l^2}.$$

Since the sequence $\{\|\mathcal{F}_n\|_{l^2}\}_{n=1}^{\infty}$ is bounded, therefore, there exists a subsequence $\{\mathcal{F}_{n_j}\}_{j=1}^{\infty}$ that converges weakly to some $\mathcal{F} \in l^2(\mathbb{N}_0)$ and $\|\mathcal{F}\|_{l^2} \le \liminf_{j\to\infty} \|\mathcal{F}_{n_j}\|_{l^2}$. Let

$$\mathcal{G}_z = \left\{\overline{z}^m\right\}_{m=0}^{\infty}, \quad z \in \mathbb{D}.$$

Then $\mathcal{G}_z \in l^2(\mathbb{N}_0)$, and the weak convergence yields

$$\lim_{j\to\infty} \langle \mathcal{F}_{n_j}, \mathcal{G}_z \rangle = \langle \mathcal{F}, \mathcal{G}_z \rangle,$$

for any $z \in \mathbb{D}$. We have

$$\langle \mathcal{F}_n, \mathcal{G}_z \rangle = \sum_{m=0}^{\infty} F_{n,m} z^m = f_n(z).$$

Denote $f(z) = \langle \mathcal{F}, \mathcal{G}_z \rangle$, then $f \in \mathcal{H}^2(\mathbb{D})$, and

$$f(z) = \lim_{j\to\infty} f_{n_j}(z), \quad z \in \mathbb{D}.$$

In particular, since $f_{n_j}(z_k) = u_k$ when $n_j \ge k$, we have

$$f(z_k) = \lim_{j\to\infty} f_{n_j}(z_k) = u_k, \quad k > 0.$$

Conversely, suppose there exists a function $f \in \mathcal{H}^2(\mathbb{D})$ such that $f(z_k) = u_k$ for any $k = 1, 2, \ldots$. Then $\{\| f_n\|_{\mathcal{H}^2}\}_{n=1}^{\infty}$ is bounded, therefore, $\{\|\mathcal{F}_n\|_{l^2}\}_{n=1}^{\infty}$ is bounded, that is equivalent to (5.1). The theorem is proved. □

Now we consider the general case. Let $\{p_n\}_1^\infty$ be a sequence of distinct complex numbers in the disc $\mathbb{D}_R(w_0)$ that has a limit point $p_0 \in \mathbb{D}_R(w_0)$, and let $\{u_n\}_1^\infty$ be a sequence of complex numbers. Denote

$$R_2 = \sup_{n\geq 1} |p_n - w_0|,$$

then $0 < R_2 < R$. Suppose there exists a function $h(w)$ holomorphic in the disc $\mathbb{D}_R(w_0)$, such that $h(p_n) = u_n$ for all $n \geq 1$. Then, for $R_2 < r < R$, the sequence of Taylor coefficients $\{H_m\}_0^\infty$ satisfies the condition

$$\left\{H_m r^m\right\}_0^\infty \in l^2(\mathbb{N}_0), \tag{5.6}$$

where H_m is defined by, for $m = 0, 1, \ldots,$

$$H_m = \lim_{n\to\infty} H_{n,m}, \quad \text{where}$$

$$H_{n,m} = \sum_{k,l=1}^{n} \frac{u_k}{r^{2m+2}} (\overline{w}_l - \overline{w}_0)^m \times \frac{\prod_{j=1}^n [(r^2 - (w_k - w_0)(\overline{w}_j - \overline{w}_0))(r^2 - (\overline{w}_l - \overline{w}_0)(w_j - w_0))]}{(r^2 - (w_k - w_0)(\overline{w}_l - \overline{w}_0)) \prod_{j=1, j\neq k}^n (w_k - w_j) \prod_{j=1, j\neq l}^n (\overline{w}_l - \overline{w}_j)}. \tag{5.7}$$

Condition (5.6) can be rewritten in the form, for $0 < R_2 < r < R$,

$$\sum_{m=0}^{\infty} \left| \sum_{k,l=1}^{n} \frac{u_k}{r^m} (\overline{w}_l - \overline{w}_0)^m \times \frac{\prod_{j=1}^n [(r^2 - (w_k - w_0)(\overline{w}_j - \overline{w}_0))(r^2 - (\overline{w}_l - \overline{w}_0)(w_j - w_0))]}{(r^2 - (w_k - w_0)(\overline{w}_l - \overline{w}_0)) \prod_{j=1, j\neq k}^n (w_k - w_j) \prod_{j=1, j\neq l}^n (\overline{w}_l - \overline{w}_j)} \right|^2 < \infty. \tag{5.8}$$

Thus, the necessary and sufficient condition for existence of a holomorphic function h in $\mathbb{D}_R(w_0)$ with $h(p_n) = u_n$ for all $n \geq 1$ is (5.8) for an arbitrary, but fixed r on the interval (R_2, R).

6 Special Cases

For certain choices of the sequence z_k, we can simplify the sums in (2.1) to obtain computable approximations for the interpolated function. Recall that as $n \to \infty$, $f_n(z)$ converges to $f(z)$ for any $z \in \mathbb{D}$, and also its MacLaurin coefficients $F_{n,m}$ converges to F_m for any $m \in \mathbb{N}_0$.

Example 1 Let $f \in \mathcal{H}^2(\mathbb{D})$, and $z_k = q^k$, $k = 1, 2, \dots, 0 < q < 1$. Using the q-Pochhammer symbol

$$(a;q)_n = \prod_{j=0}^{n-1} \big(1 - aq^j\big),$$

we obtain

$$f_n(z) = \sum_{k,l=1}^{n} f\big(q^k\big) \frac{q^{(k+l)n+k(k+1)/2+l(l+1)/2}(q^{k+1};q)_n(q^{l+1};q)_n}{(1-q^{k+l})(q;q)_{k-1}(q;q)_{n-k}(q;q)_{l-1}(q;q)_{n-l}} \frac{(-1)^{k+l}}{1-zq^l}$$

and

$$F_{n,m} = \sum_{k,l=1}^{n} f\big(q^k\big) \frac{(-1)^{k+l} q^{(k+l)n+k(k+1)/2+l(l+1)/2+lm}(q^{k+1};q)_n(q^{l+1};q)_n}{(1-q^{k+l})(q;q)_{k-1}(q;q)_{n-k}(q;q)_{l-1}(q;q)_{n-l}}. \tag{6.1}$$

Example 2 Let $f \in \mathcal{H}^2(\mathbb{D})$, and $z_k = \frac{1}{k+1}$, $k = 1, 2, \dots$. Using the Pochhammer symbol

$$[a]_n = \prod_{j=1}^{n} (a + j - 1),$$

we obtain

$$f_n(z) = \sum_{k,l=1}^{n} f\left(\frac{1}{k+1}\right) \frac{(-1)^{k+l}(k+1)^{n-1}(l+1)^{n-1}[\frac{k+2}{k+1}]_n[\frac{l+2}{l+1}]_n}{(kl+k+l)(k-1)!(n-k)!(l-1)!(n-l)!} \frac{1}{1-\frac{z}{l+1}}, \tag{6.2}$$

and

$$F_{n,m} = \sum_{k,l=1}^{n} f\left(\frac{1}{k+1}\right) \frac{(-1)^{k+l}(k+1)^{n-1}(l+1)^{n-1}[\frac{k+2}{k+1}]_n[\frac{l+2}{l+1}]_n}{(kl+k+l)(k-1)!(n-k)!(l-1)!(n-l)!} \frac{1}{(l+1)^m}. \tag{6.3}$$

Example 3 Let $f \in \mathcal{H}^2(\mathbb{D})$, and $z_k = a + \frac{b}{k}$, $k = 1, 2, \dots$ where $a, b \in \mathbb{C}$ and $|a| + |b| < 1$, so all $z_k \in \mathbb{D}$. If we denote the product

$$p(k,n) = \prod_{j=1}^{n} \big[kj - (ak+b)(\overline{a}j + \overline{b})\big]$$

then

$$f_n(z) = \sum_{k,l=1}^{n} f\left(a + \frac{b}{k}\right) \frac{(-1)^{k+l} p(k,n)p(l,n)kl}{(kl-(ak+b)(\overline{a}l+\overline{b}))l!k!(n-l)!(n-k)!} \frac{l}{l-(\overline{a}l+\overline{b})z},$$

$$F_{n,m} = \sum_{k,l=1}^{n} f\left(a + \frac{b}{k}\right) \frac{(-1)^{k+l} p(k,n)p(l,n)kl}{(kl-(ak+b)(\overline{a}l+\overline{b}))l!k!(n-l)!(n-k)!} \left(\overline{a} + \frac{\overline{b}}{k}\right)^m.$$

Numerical Tests:

On the numerical side we can use formula (6.2) to interpolate few values of the holomorphic function

$$f(z) = \frac{1}{4+z^2}.$$

Observe that since $f_n(x)$ is a finite sum, it can be computed in an exact manner, with no round off error. In what follows, we use $=$ to mean exact and $\simeq$ to mean approximation.

We first check the interpolation formula (6.2) at the interpolation point $x = 1/30$. The formula $f_n(1/30)$ should give us the exact value $f(1/30) = 900/3601$, for $n \geq k = 29$ since there is no interpolation in (6.2). For example the interpolated values are

$$\begin{aligned} f_{10}(1/30) &\simeq 0.2499305748402793976 \\ f_{28}(1/30) &\simeq 0.2499305748403221327 \\ f_{29}(1/30) &= 900/3601 \\ f_{35}(1/30) &= 900/3601. \end{aligned}$$

The values $f_{10}(1/30)$ and $f_{28}(1/30)$ cannot be exact because they are extrapolations. Note that $f_{29}(1/30)$ and $f_{35}(1/30)$ the series terminates and contains only 29 terms and so the values are exact as expected.

Next we check the formula outside sampling points, and so the series in (6.2) does not terminates before the nth term. We use formula (6.2) to interpolate the values $f(0) = \frac{1}{4}$, $f(\frac{\mathrm{i}}{2})$, $f(\frac{1}{2} + \frac{\mathrm{i}}{2})$ where $\mathrm{i} = \sqrt{-1}$

$$\begin{aligned} f(0) &= \frac{1}{4} \\ f_{10}(0) &= \frac{695457128012177047}{2781828512056818000} \\ &\simeq 0.2499999999992711797 \\ f_{20}(0) &= \frac{118570457496285733575429081726030244652410166865 7}{474281829985142934301716348738551221007849616000 0} \\ &\simeq 0.24999999999999999999999988491. \end{aligned}$$

Next

$$\begin{aligned} f\left(\frac{\mathrm{i}}{2}\right) &= 4/15 \\ f_{35}\left(\frac{\mathrm{i}}{2}\right) &\simeq 0.2666666666666666667 + 0.3283497872982402614\ 10^{-21}\mathrm{i} \\ f\left(\frac{1}{2} + \frac{\mathrm{i}}{2}\right) &= \frac{16}{65} - \frac{2}{65}\mathrm{i} \\ f_{35}\left(\frac{1}{2} + \frac{\mathrm{i}}{2}\right) &\simeq 0.2461538461538461560 - 0.03076923076923079497\mathrm{i} \end{aligned}$$

which compares well with the exact values

$$\frac{16}{65} = 0.246153\overline{846153} \quad \text{and} \quad \frac{2}{65} = 0.3\overline{076923}.$$

As for the Taylor coefficients we have for

$$F_{20,6} \simeq -0.0039062500000007548185$$

and similarly

$$F_{20,8} \simeq 0.0009765624997488025053 3.$$

The exact values are

$$\frac{1}{6!} f^{(6)}(0) = \frac{-1}{256} = -0.003\,906\,250$$
$$\frac{1}{8!} f^{(8)}(0) = \frac{1}{1024} = 0.000\,976\,562\,500.$$

Note that because the formula (6.2) mainly uses integers, the results shown above are free from roundoff error.

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On Approximation of Lebedev Type Transforms

Juri Rappoport

Abstract New modification of KONTOROVITCH–LEBEDEV and LEBEDEV–SKALSKAYA integral transforms was introduced by YAKUBOVICH. These transforms contain modified BESSEL functions $K_{\frac{1}{4}+i\tau}(x)$ and $K_{\frac{3}{4}+i\tau}(x)$ and their real and imaginary parts as kernels. The vector Tau method approach is used for the approximation and calculation of these functions. This approach is based on the general Tau method's computational scheme and canonical vector-polynomial notion. We obtain the system of two differential equations and then the system of two Volterra integral equations for the determination of the polynomial approximation of the kernels. These results may be used for the application of YAKUBOVICH transforms to the solution of boundary value problems of mathematical physics.

Keywords KONTOROVITCH–LEBEDEV integral transform · LEBEDEV–SKALSKAYA integral transform · Modified BESSEL function · Tau method · CHEBYSHEV polynomial

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1 Some Properties of the Functions Re $K_{\alpha+i\beta}(x)$ and Im $K_{\alpha+i\beta}(x)$

It is possible to write the modified BESSEL functions in the form Re $K_{\alpha+i\beta}(x) = \frac{K_{\alpha+i\beta}(x)+K_{\alpha-i\beta}(x)}{2}$ and Im $K_{\alpha+i\beta}(x) = \frac{K_{\alpha+i\beta}(x)-K_{\alpha-i\beta}(x)}{2i}$, where $K_\nu(x)$ is the modified BESSEL function of the second kind (also called MACDONALD function).

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The functions $K_{i\beta}(x)$, $\operatorname{Re} K_{\alpha+i\beta}(x)$ and $\operatorname{Im} K_{\alpha+i\beta}(x)$ have integral representations [1, 2]

$$K_{i\beta}(x) = \int_0^\infty e^{-x\cosh t}\cos(\beta t)dt,$$

$$\operatorname{Re} K_{\alpha+i\beta}(x) = \int_0^\infty e^{-x\cosh t}\cosh(\alpha t)\cos(\beta t)dt, \tag{1.1}$$

$$\operatorname{Im} K_{\alpha+i\beta}(x) = \int_0^\infty e^{-x\cosh t}\sinh(\alpha t)\sin(\beta t)dt. \tag{1.2}$$

It follows from (1.1)–(1.2) that it is possible to write $\operatorname{Re} K_{\alpha+i\beta}(x)$ in the form of the FOURIER cosinus-transform

$$\operatorname{Re} K_{\alpha+i\beta}(x) = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} F_C\left[e^{-x\cosh t}\cosh(\alpha t); t \to \beta\right], \tag{1.3}$$

and $\operatorname{Im} K_{\alpha+i\beta}(x)$ in the form of the FOURIER sinus-transform

$$\operatorname{Im} K_{\alpha+i\beta}(x) = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} F_S\left[e^{-x\cosh t}\sinh(\alpha t); t \to \beta\right]. \tag{1.4}$$

The inversion formulas have the respective forms

$$\begin{aligned} F_C\left[\operatorname{Re} K_{\alpha+i\beta}(x); \beta \to t\right] &= \left(\frac{\pi}{2}\right)^{\frac{1}{2}} e^{-x\cosh t}\cosh(\alpha t), \\ F_S\left[\operatorname{Im} K_{\alpha+i\beta}(x); \beta \to t\right] &= \left(\frac{\pi}{2}\right)^{\frac{1}{2}} e^{-x\cosh t}\sinh(\alpha t) \end{aligned} \tag{1.5}$$

or, in integral form,

$$\int_0^\infty \operatorname{Re} K_{\alpha+i\beta}(x)\cos(t\beta)d\beta = \frac{\pi}{2}e^{-x\cosh t}\cosh(\alpha t), \tag{1.6}$$

$$\int_0^\infty \operatorname{Im} K_{\alpha+i\beta}(x)\sin(t\beta)d\beta = \frac{\pi}{2}e^{-x\cosh t}\sinh(\alpha t). \tag{1.7}$$

For the computation of certain integrals of the functions $\operatorname{Re} K_{\frac{1}{2}+i\beta}(x)$ and $\operatorname{Im} K_{\frac{1}{2}+i\beta}(x)$ integral identities are useful which reduce this problem to the computation of some other integrals over elementary functions.

Proposition 1.1 *If f is absolutely integrable on $[0,\infty)$, then the following identities hold,*

$$\int_0^\infty \operatorname{Re} K_{\alpha+i\beta}(x) f(\beta)d\beta = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \int_0^\infty e^{-x\cosh t}\cosh(\alpha t)F_C(t)dt, \tag{1.8}$$

$$\int_0^\infty \operatorname{Im} K_{\alpha+i\beta}(x) f(\beta) d\beta = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \int_0^\infty e^{-x\cosh t} \sinh(\alpha t) F_S(t) dt, \quad (1.9)$$

where $F_C(t)$ *is the* FOURIER *cosinus-transform of* $f(\beta)$, *and* $F_S(t)$ *the* FOURIER *sinus-transform of* $f(\beta)$.

Proposition 1.2 *If* f *is absolutely integrable on* $[0, \infty)$, *then the following identities hold*

$$\int_0^\infty \operatorname{Re} K_{\alpha+i\beta}(x) F_C(\beta) d\beta = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \int_0^\infty e^{-x\cosh t} \cosh(\alpha t) f(t) dt, \quad (1.10)$$

$$\int_0^\infty \operatorname{Im} K_{\alpha+i\beta}(x) F_S(\beta) d\beta = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \int_0^\infty e^{-x\cosh t} \sinh(\alpha t) f(t) dt. \quad (1.11)$$

Equations (1.8)–(1.11) are useful for the simplification and the calculation of different integrals containing $\operatorname{Re} K_{\alpha+i\beta}(x)$ and $\operatorname{Im} K_{\alpha+i\beta}(x)$.

It follows from (1.1) that for all $\beta \in [0, \infty)$

$$\left|\operatorname{Re} K_{\alpha+i\beta}(x)\right| \le K_\alpha(x), \qquad \left|\operatorname{Re} K_{\frac{1}{2}+i\beta}(x)\right| \le K_{\frac{1}{2}}(x) = \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} e^{-x}, \quad (1.12)$$

and it follows from (1.2) that for all $\beta \in [0, \infty)$

$$\left|\operatorname{Im} K_{\frac{1}{2}+i\beta}(x)\right| \le \int_0^\infty e^{-x\cosh t} \sinh\frac{t}{2} dt = \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} e^x \left[1 - \phi\left((2x)^{\frac{1}{2}}\right)\right] \le B\frac{e^{-x}}{x}, \quad (1.13)$$

where B is some positive constant.

2 Yakubovich Integral Transforms

Integral transforms containing integration with respect to the index of the BESSEL function play an important role for the solution of some classes of the problems in mathematical physics [3–5]. In particular, for the solution of mixed boundary value problems for the HELMHOLTZ equation in wedge-shaped and conic domains, the KONTOROVITCH–LEBEDEV and LEBEDEV–SKALSKAYA transforms [1, 2] are used.

The proofs of the inversion formulas and PARSEVAL equalities for these transforms are investigated by the author in [6–11]. The problem of the evaluation of the KONTOROVITCH–LEBEDEV integral transforms is simplified by means of their decompositions in the form of compositions of simpler integral transformations, in particular, FOURIER and LAPLACE transforms.

The KONTOROVITCH–LEBEDEV integral transforms may be expressed in terms of general MEYER integral transforms of special index and argument.

YAKUBOVICH [12] considered the LEBEDEV type integral transforms with arbitrary complex order, i.e. with $\operatorname{Re} K_{\alpha+i\beta}(x)$ and $\operatorname{Im} K_{\alpha+i\beta}(x)$ in the kernel. The inversion formulas were obtained for the general case. But they have the more simplest form for the case $\alpha = \frac{1}{4}$. Let's rewrite these formulas for the real parts [12]

$$F_{\frac{1}{4}+}(\beta) = \int_0^\infty \operatorname{Re} K_{\frac{1}{4}+i\beta}(x) f(x) dx,$$

$$f(x) = \frac{2\sqrt{2}}{\pi^2} \int_0^\infty \Big[\cosh(\pi\beta) \operatorname{Re}\big(K_{\frac{3}{4}+i\beta}(x) - K_{\frac{1}{4}+i\beta}(x)\big) + \sinh(\pi\beta) \operatorname{Im}\big(K_{\frac{3}{4}+i\beta}(x) + K_{\frac{1}{4}+i\beta}(x)\big)\Big] F_{\frac{1}{4}+}(\beta) d\beta.$$

3 Tau Method Approximation

The questions of the approximation of the solutions of the linear differential equations with polynomial coefficients by means of polynomials and construction of approximations of the KONTOROVITCH–LEBEDEV integral transforms kernels are considered.

The numerical scheme of the Tau method application is proposed for the solution of the second order linear differential equations systems with the second order polynomial coefficients of the following kind:

$$\big(a_0^{(j)} y^2 + a_1^{(j)} y\big) v_j''(y) + \sum_{i=1}^{k} \Big[\big(a_{3i-1}^{(j)} y - a_{3i}^{(j)}\big) v_i'(y) + a_{3i+1}^{(j)} v_i(y)\Big] = 0,$$

$$v_j(0) = a_{3k+2}^{(j)}, \quad j = 1, \ldots, k, \;\; y \in [0, 1],$$

in the unknown vector-function $v(y) = (v_1(y), \ldots, v_k(y))$. It is assumed to have only one solution. Integrating twice and carrying an addition in the right part in the kind of the vector-polynomial $P_n(y)$, we derive for the determination of the n-th approximation of the solution $v(y) = (v_1(y), \ldots, v_k(y))$ the system of Volterra integral equations with polynomial kernels

$$\big(b_0^{(j)} y^2 + b_1^{(j)} y\big) v_j(y) = \int_0^y \left[\sum_{i=1}^{k} \big(b_{3i-1}^{(j)} x + b_{3i}^{(j)} y + b_{3i+1}^{(j)}\big) v_i(x)\right] dx + P_{jn+2}(y),$$

$$j = 1, \ldots, k,$$

where coefficients $b_i^{(j)}$ and $a_i^{(j)}$, $i = 0, \ldots, 3k+2$ and $j = 1, \ldots, k$, are connected in definite way and $P_{jn+2}(y)$, $j = 1, \ldots, k$—$n+2$-th degree polynomials. The different variables of the vector residue choice and its minimization are analyzed. The recurrent formulas for the canonical vector-polynomials coefficients convenient for calculations are given.

Consider the system of two second order differential equations ($k = 2$) in more detail. This case contain differential equations with complex coefficients.

The scheme of the integral form of the Tau Method described in this paper can be used for deriving polynomial approximations of hypergeometric and confluent hypergeometric functions of the first kind with complex parameters.

We find it necessary to compute $\operatorname{Re} K_{\alpha+i\beta}(x)$ and $\operatorname{Im} K_{\alpha+i\beta}(x)$ to use YAKUBOVICH transform in practice. Therefore we consider the second kind modified BESSEL function $K_{\alpha+i\beta}(x)$ in more detail.

We have a system of two second order differential equations

$$
\begin{aligned}
&y^2 v_1'' + 2(y+1)v_1' + \left(\frac{1}{4} - \alpha^2 + \beta^2\right) v_1 + 2\alpha\beta v_2 = 0,\\
&y^2 v_2'' + 2(y+1)v_2' - 2\alpha\beta v_1 + \left(\frac{1}{4} - \alpha^2 + \beta^2\right) v_2 = 0,\\
&\quad v_1(0) = 1,\ v_2(0) = 0,
\end{aligned}
$$

or the system of VOLTERRA integral equations

$$
\begin{aligned}
y^2 v_1(y) &= \int_0^y \left(\left(\frac{9}{4} - \alpha^2 + \beta^2\right)x - \left(2 + \left(\frac{1}{4} - \alpha^2 + \beta^2\right)y\right)\right) v_1(x)dx\\
&\quad + 2\alpha\beta \int_0^y (x-y) v_2(x)dx + 2y,\\
y^2 v_2(y) &= 2\alpha\beta \int_0^y (y-x) v_1(x)dx\\
&\quad + \int_0^y \left(\left(\frac{9}{4} - \alpha^2 + \beta^2\right)x - \left(2 + \left(\frac{1}{4} - \alpha^2 + \beta^2\right)y\right)\right) v_2(x)dx,\\
K_{\alpha+i\beta}(x) &= \big(\pi/(2x)\big)^{1/2} e^{-x} \big(v_1(1/x) + i v_2(1/x)\big), \quad x \ge 1.
\end{aligned}
$$

By means of computations is shown that the choice of the residue in the form $P_{jn+2}(y) = \tau_{jn+2} T_{n+2}[(1-\alpha_{n+2})y + \alpha_{n+2}]$, $j = 1, 2$, is optimal as compared with other known variants in this case too. Here $\alpha_{n+2} = \sin^2(\pi/(4(n+2)))$—the most left root of the shifted Chebyshev polynomial of the $n+2$-th degree $T^*_{n+2}(y)$ in the interval [0, 1], τ_{n+2}-undefined coefficient.

Application of the integral form of the Tau method for the construction of the approximate solution of the system consists in the following: We seek n-th approximation of the solution in the form of pair of polynomials n-th degree $v_{1n}(y)$ and

$v_{2n}(y)$, which are the solutions of (3.1)

$$\begin{cases} y^2 v_1(y) = \int_0^y \left(\left(\frac{9}{4} - \alpha^2 + \beta^2 \right) x - \left(2 + \left(\frac{1}{4} - \alpha^2 + \beta^2 \right) y \right) \right) v_1(x) dx \\ \qquad + 2\alpha\beta \int_0^y (x-y) v_2(x) dx \\ \qquad + 2y + \tau_1 T^*_{n+1}(y) + \tau_2 T^*_{n+2}(y), \\ y^2 v_2(y) = 2\alpha\beta \int_0^y (y-x) v_1(x) dx \\ \qquad + \int_0^y \left(\left(\frac{9}{4} - \alpha^2 + \beta^2 \right) x - \left(2 + \left(\frac{1}{4} - \alpha^2 + \beta^2 \right) y \right) \right) v_2(x) dx \\ \qquad + \tau_3 T^*_{n+1}(y) + \tau_4 T^*_{n+2}(y), \end{cases} \tag{3.1}$$

$T^*_{n+i}(y)$, $i = 1, 2$—shifted CHEBYSHEV polynomials $n + i$-th degree, τ_i, $i = 1, \dots, 4$—undefined coefficients.

The use of canonical polynomials may lead in this case to additional computational difficulties connected with renormalization of big and small values. So let's describe the methods based on the direct solution (3.1).

Let's explain the process of the determination of $v_{1n}(y)$ and $v_{2n}(y)$. Method of undefined coefficients is used for the determination of undefined coefficients by means of substitution into the system (3.1) $v_{1n}(y) = \sum_{k=0}^{n} a_{1k} y^k$, $v_{2n}(y) = \sum_{k=0}^{n} a_{2k} y^k$ and equating coefficients under the identical degrees y. We obtain the system from $n + 3$ pair of equations according $2n + 6$ unknowns

$$a_{1k}, \quad k = 0, \dots, n, \qquad a_{2k}, \quad k = 0, \dots, n, \qquad \tau_i, \quad i = 1, \dots, 4.$$

We find from the last pair of equations

$$\tau_1 = -\frac{c(0, n+2)}{c(0, n+1)} \tau_2 = \tau_2,$$
$$\tau_3 = -\frac{c(0, n+2)}{c(0, n+1)} \tau_4 = \tau_4$$

and substitute derived expression in all other equations.

We find consequently a_{1k}, a_{2k}, $k = n, \dots, 0$ in the form of linear combinations and recurrent relations. It's possible to obtain that for every fixed β the number N exists that for $n > N$ denominator's values are different from zero. Conducted computations show that denominators are different from zero for $0.1 \leq \beta \leq 10$, $n = 16$.

Further by formulas (3.1) we find values $a_{1k}, a_{2k}, k = 0, \ldots, n$, and by GORNER scheme compute expressions

$$v_{1n}(y) = \sum_{k=0}^{n} a_{1k} y^k,$$

$$v_{2n}(y) = \sum_{k=0}^{n} a_{2k} y^k.$$

We obtain on the interval $(1 \leq x < \infty)$ convenient for computations for small β expansions

$$\operatorname{Re} K_{\alpha+i\beta}(x) = \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} e^{-x} \left(\sum_{m=0}^{n} a_{1m} \left(\frac{1}{x}\right)^m + R_{1n}\right),$$

$$\operatorname{Im} K_{\alpha+i\beta}(x) = \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} e^{-x} \left(\sum_{m=0}^{n} a_{2m} \left(\frac{1}{x}\right)^m + R_{2n}\right),$$

where R_{1n} and R_{2n}—reminder terms.

The applications for the solution of mixed boundary value problems in wedge domains, dual integral equations, numerical algorithms, approximation and computation of the kernels of the modified LEBEDEV type integral transforms are described in [13–17].

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Reproducing Kernels and Discretization

L.P. Castro, H. Fujiwara, M.M. Rodrigues, S. Saitoh, and V.K. Tuan

Abstract We give a short survey of a general discretization method based on the theory of reproducing kernels. We believe our method will become the next generation method for solving analytical problems by computers. For example, for solving linear PDEs with general boundary or initial value conditions, independently of the domains. Furthermore, we give an ultimate sampling formula and a realization of reproducing kernel Hilbert spaces.

Keywords Reproducing kernel · Aveiro discretization · Linear operator equation · Approximate solution · Numerical problem

Mathematics Subject Classification (2010) Primary 30C40 · 44A05 · Secondary 44A15 · 35K05

1 The Inverse by Using a Finite Number of Data

Let $\mathcal{H}$ be a Hilbert (possibly finite-dimensional) space, and consider E to be an abstract set and $\mathbf{h}$ a Hilbert $\mathcal{H}$-valued function on E. Then, we consider the linear

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transform

$$f(p) = \big(\mathbf{f}, \mathbf{h}(p)\big)_{\mathcal{H}}, \quad \mathbf{f} \in \mathcal{H}, \tag{1.1}$$

from $\mathcal{H}$ into the linear space $\mathcal{F}(E)$ comprising all the complex valued functions on E. We form a positive definite quadratic form function $K(p,q)$ on $E \times E$ defined by

$$K(p,q) = \big(\mathbf{h}(q), \mathbf{h}(p)\big)_{\mathcal{H}} \quad \text{on } E \times E. \tag{1.2}$$

Proposition 1.1

(I) *The range of the linear mapping* (1.1) *on* $\mathcal{H}$ *is characterized as the reproducing kernel Hilbert space* H_K *admitting the reproducing kernel* $K(p,q)$.

(II) *In general, we have the inequality* $\|f\|_{H_K} \leq \|\mathbf{f}\|_{\mathcal{H}}$. *Here, for any member* f *of* H_K *there exists a uniquely determined* $\mathbf{f}^* \in \mathcal{H}$ *satisfying* $f(p) = (\mathbf{f}^*, \mathbf{h}(p))_{\mathcal{H}}$ *on* E *and*

$$\|f\|_{H_K} = \big\|\mathbf{f}^*\big\|_{\mathcal{H}}. \tag{1.3}$$

(III) *In general, we have the inversion formula in* (1.1) *in the form*

$$f \mapsto \mathbf{f}^* \tag{1.4}$$

in (II) *by using the reproducing kernel Hilbert space* H_K.

The typical ill-posed problem (1.1) in $\mathcal{H}$ will become a well-posed problem in H_K, see the details [7–9].

Our idea is based on the approximate realization of the abstract Hilbert space H_K by taking a finite number of points of E. This is done because, in general, the reproducing kernel Hilbert space H_K has a complicated structure.

By taking a finite number of points $\{p_j\}_{j=1}^n$, we set

$$K(p_j, p_{j'}) := a_{jj'}. \tag{1.5}$$

Then, if the matrix $A_n := \|a_{jj'}\|$ is positive definite, then, the corresponding norm in H_{A_n} comprising the vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ is determined by $\|\mathbf{x}\|_{H_{A_n}}^2 = \mathbf{x}^* \widetilde{A_n} \mathbf{x}$, where $\widetilde{A_n} = \overline{A_n^{-1}} = \|\widetilde{a_{jj'}}\|$ (see [8], p. 250).

Proposition 1.2 *In the linear mapping*

$$f(p) = \big(\mathbf{f}, \mathbf{h}(p)\big)_{\mathcal{H}}, \quad \mathbf{f} \in \mathcal{H} \tag{1.6}$$

for A_n, *the minimum norm inverse* $\mathbf{f}_{A_n}^*$ *satisfying*

$$f(p_j) = \big(\mathbf{f}, \mathbf{h}(p_j)\big)_{\mathcal{H}}, \quad \mathbf{f} \in \mathcal{H} \tag{1.7}$$

is given by

$$\mathbf{f}^*_{A_n} = \sum_j \sum_{j'} f(p_j)\widetilde{a_{jj'}}\mathbf{h}(p_{j'}), \tag{1.8}$$

where $\widetilde{a_{jj'}}$ are assumed the elements of the complex conjugate inverse of the positive definite Hermitian matrix A_n constituted by the elements $a_{jj'} = (\mathbf{h}(p_{j'}), \mathbf{h}(p_j))_{\mathcal{H}}$. Here, the positive definiteness of A_n is a basic assumption.

2 Convergence of the Approximate Inverses

The following proposition deals with the convergence of our approximate inverses in Proposition 1.2. See [1, 2] for the details.

Proposition 2.1 *Let $\{p_j\}_{j=1}^{\infty}$ be a sequence of distinct points on E, that is the positive definiteness of A_n for any n and a uniqueness set for the space H_K. Then, in the space $\mathcal{H}$*

$$\lim_{n\to\infty} \mathbf{f}^*_{A_n} = \mathbf{f}^*. \tag{2.1}$$

Proposition 2.2 (Ultimate realization of reproducing kernel Hilbert spaces) *In our general situation and for a uniqueness set $\{p_j\}$ of the set E satisfying the linearly independence in Proposition 1.2, we obtain*

$$\|f\|^2_{H_K} = \|\mathbf{f}^*\|^2_{\mathcal{H}} = \lim_{n\to\infty} \sum_j \sum_{j'} f(p_j)\widetilde{a_{jj'}}\overline{f(p_{j'})}. \tag{2.2}$$

Proposition 2.3 (Ultimate sampling theory) *In our general situation and for a uniqueness set $\{p_j\}$ of the set E satisfying the linearly independence, we obtain*

$$\begin{aligned} f(p) &= \lim_{n\to\infty} \left(\mathbf{f}^*_{A_n}, \mathbf{h}(p)\right)_{\mathcal{H}} = \lim_{n\to\infty} \left(\sum_j \sum_{j'} f(p_j)\widetilde{a_{jj'}}\mathbf{h}(p_{j'}), \mathbf{h}(p)\right)_{\mathcal{H}} \\ &= \lim_{n\to\infty} \sum_j \sum_{j'} f(p_j)\widetilde{a_{jj'}} K(p, p_{j'}). \end{aligned} \tag{2.3}$$

3 Ordinary Linear Differential Equations

In view to have a concrete exemplification of the method, let us consider a prototype differential operator

$$Ly := \alpha y'' + \beta y' + \gamma y. \tag{3.1}$$

Here, we shall consider a very general situation that the coefficients are arbitrary functions (no continuity requirement) and on a general interval I. We wish to construct some natural solution of

$$Ly = g \tag{3.2}$$

for a very general function g on a general interval I.

Proposition 3.1 ([1, 3]) *Let us fix a positive number h and take a finite number of points $\{t_j\}_{j=1}^n$ of I such that $|\alpha(t_j)|^2 + |\beta(t_j)|^2 + |\gamma(t_j)|^2 \neq 0$ for each j. Then, the optimal solution $y_h^{A_n}$ of (3.2) is given by*

$$y_h^{A_n}(t) = \frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} F_h^{A_n}(\xi)e^{-it\xi}d\xi$$

in terms of the function $F_h^{A_n} \in L_2(-\pi/h, +\pi/h)$ in the sense that $F_h^{A_n}$ has the minimum norm in $L_2(-\pi/h, +\pi/h)$ among the functions $F \in L_2(-\pi/h, +\pi/h)$ satisfying, for the characteristic function $\chi_h(t)$ of the interval $(-\pi/h, +\pi/h)$:

$$\frac{1}{2\pi}\int_{\mathbb{R}} F(\xi)\big[\alpha(t)\big(-\xi^2\big) + \beta(t)(-i\xi) + \gamma(t)\big]\chi_h(\xi)\exp(-it\xi)d\xi = g(t) \tag{3.3}$$

for all $t = t_j$ and for the function space $L_2(-\pi/h, +\pi/h)$.

The minimal norm function $F_h^{A_n}$ is given by

$$F_h^{A_n}(\xi) = \sum_{j,j'=1}^n g(t_j)\widetilde{a_{jj'}}\overline{\big(\alpha(t_{j'})\big(-\xi^2\big) + \beta(t_{j'})(-i\xi) + \gamma(t_{j'})\big)}\exp(it_{j'}\xi).$$

Here, the matrix $A_n = \{a_{jj'}\}_{j,j'=1}^n$ formed by the elements $a_{jj'} = K_{hh}(t_j, t_{j'})$ with

$$\begin{aligned} K_{hh}&\big(t, t'\big) \\ &= \frac{1}{2\pi}\int_{\mathbb{R}}\big[\alpha(t)\big(-\xi^2\big) + \beta(t)(-i\xi) + \gamma(t)\big]\overline{\big[\alpha\big(t'\big)\big(-\xi^2\big) + \beta\big(t'\big)(-i\xi) + \gamma\big(t'\big)\big]} \\ &\quad \times \chi_h(\xi)\exp\big(-i\big(t - t'\big)\xi\big)d\xi \end{aligned}$$

is positive definite and the $\widetilde{a_{jj'}}$ are the elements of the inverse of $\overline{A_n}$ (the complex conjugate of A_n).

The minimal norm solution $y_h^{A_n}$ of (3.2) is given by

$$\begin{aligned} y_h^{A_n}(t) = \sum_{j,j'=1}^n g(t_j)\widetilde{a_{jj'}}\frac{1}{2\pi}\bigg[&-\overline{\alpha(t_{j'})}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}}\xi^2 e^{-i(t-t_{j'})\xi}d\xi \\ &+ i\overline{\beta(t_{j'})}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}}\xi e^{-i(t-t_{j'})\xi}d\xi + \overline{\gamma(t_{j'})}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{-i(t-t_{j'})\xi}d\xi\bigg]. \end{aligned}$$

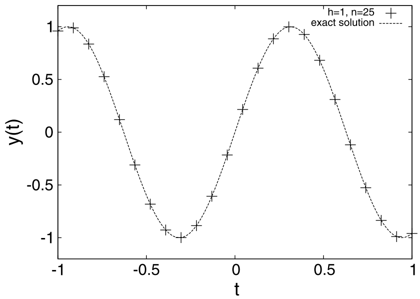


Fig. 1 Numerical results for the initial value problem for $n = 25$ with 100 decimal digits. Maximum error is approximately 10^{-13}

As about general linear operator equations, we consider the equations in some reproducing kernel Hilbert spaces. These spaces can be considered as the images of some Hilbert spaces as in Proposition 1.1 (see [8, 9]). Then, the linear operator equation may be reduced to Proposition 1.2 by *the backward transformation* as in Proposition 3.1. So, we will be able to consider our method as *a fundamental theory for linear operator equations in the framework of Hilbert spaces*.

4 Numerical Examples

We set $h = 1$; we seek our solution in the Paley–Wiener space $W(\pi)$ with equi-spaced collocation points.

Example We consider an initial value problem

$$t^3 y''(t) + t y'(t) = -25t^3 \sin(5t) + 5t \cos(5t) \quad (-1 < t \le 1),$$
$$y(-1) = \sin(5), \qquad y'(-1) = 5\cos(5),$$

and we set collocation points to $t_j = -1 + 2j/(n-2)$, $j = 1, 2, \ldots, n-2$.

Numerical results shown in Fig. 1 have a good coincide with the exact solution $y(t) = \sin(5t)$.

Example We consider an initial value problem

$$y''(t) = g(t) \quad (-1 < t \le 1), \qquad y(-1) = y'(-1) = 0,$$

where

$$g(t) = \begin{cases} 0, & t < 0; \\ t, & t \ge 0. \end{cases}$$

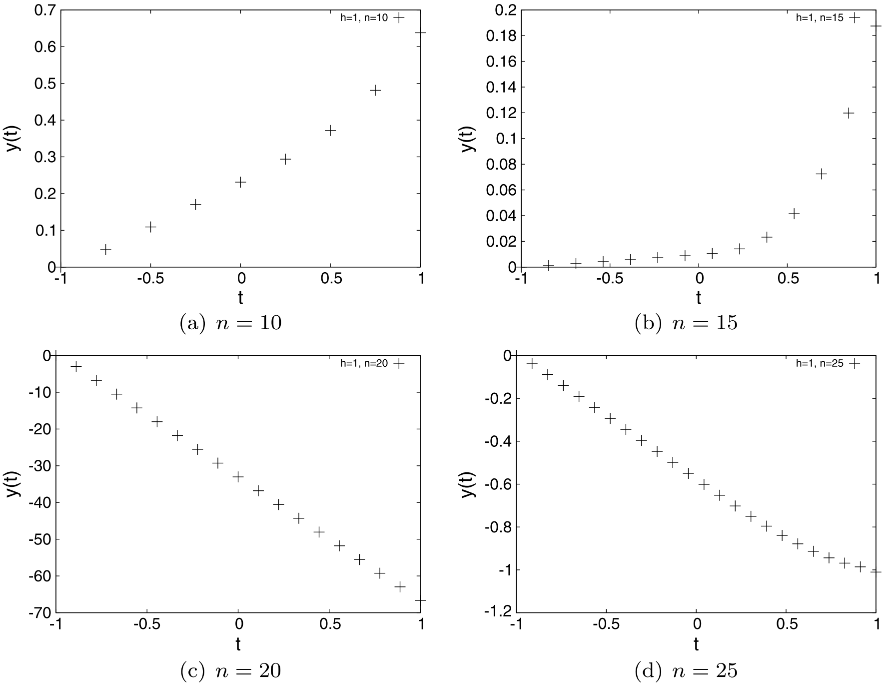


Fig. 2 Numerical results by 500 decimal digits precision, $h = 1$

We know that there exists a unique solution

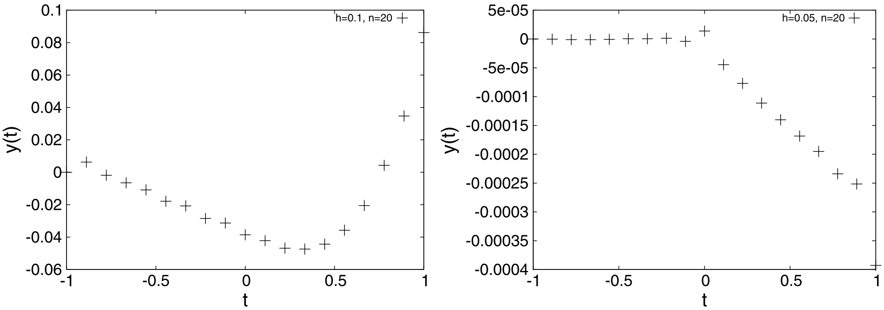
$$y(t) = \begin{cases} 0, & t < 0; \\ \frac{t^3}{6}, & t \geq 0. \end{cases}$$

The proposed method assumes that the solution belongs to the Paley–Wiener space $W(\frac{\pi}{h})$, where h is an approximation parameter. Our numerical results (Figs. 2–3) imply that the Paley–Wiener space seems to show the need of application of suitable Sobolev spaces as basic approximate function spaces.

In our new discretization method we will need the precision in some deep way and huge computer resources. However, these both requirements were prepared by Fujiwara already (e.g., recall the case of the inverse Laplace transform). See [4–6] for the details.

We are looking for some optimal solutions satisfying the differential equations at the given discrete points and so, we are free from important restrictions on the domains which occur on ordinary methods. For instance, this is not the case of the *Finite Element Method* and the *Difference Method* which are depending seriously on the domains. In our case, we can consider the problems on any domains. See [1, 2] for the details.

Anyhow, error estimates for our approximate solutions are entirely new open problems.



(a) $h = 0.1$ (b) $h = 0.05$

Fig. 3 Numerical results by 500 decimal digits precision, $n = 20$

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Dirichlet's Problem by Using Computers with the Theory of Reproducing Kernels

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Abstract In this paper we shall give practical and numerical solutions of the Laplace equation on multidimensional spaces and show their numerical experiments by using computers. Our method is based on the Dirichlet principle by combinations with generalized inverses, Tikhonov's regularization and the theory of reproducing kernels.

Keywords Laplace equation · Dirichlet problem · Reproducing kernel · Inverse problem · Tikhonov regularization

Mathematics Subject Classification (2010) Primary 30C40 · 31A05 · 35J05

1 Introduction

Depending on the power of computers, we shall propose a new algorithm for constructing of approximate solutions for the Laplace equation

$$\Delta u = 0 \tag{1.1}$$

on a regular domain D on $\mathbf{R}^n$ satisfying a boundary condition on ∂D in the class of the functions of the s order Sobolev Hilbert space H^s on the whole real space $\mathbf{R}^n$ ($n \geq 1, s \geq 2, s > n/2$). Our method and approach are new and general concepts, but we are interested in numerical experiments by using computers and so, we shall restrict our problems to this prototype case as the first step.

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We shall use the m order Sobolev Hilbert space H^m comprising functions F on $\mathbf{R}^n$ with the norm

$$\|F\|_{H^m}^2 = \sum_{\nu=0}^{m} {}_mC_\nu \sum_{r_1,r_2,\ldots,r_n\geq 0}^{\nu} \frac{\nu!}{r_1!r_2!\cdots r_n!} \int_{\mathbf{R}^n} \left(\frac{\partial^\nu F(x)}{\partial x_1^{r_1} \partial x_2^{r_2} \cdots \partial x_n^{r_n}} \right)^2 dx. \quad (1.2)$$

Here, of course, $r_1 + r_2 + \cdots + r_n = \nu$.

This Hilbert space admits the reproducing kernel

$$K(x, y) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \frac{1}{(1 + |\xi|^2)^m} e^{i(x-y)\cdot\xi} d\xi \quad (1.3)$$

as we see easily by using Fourier's transform (cf. [3], p. 58).

We shall recall the Dirichlet principle that the harmonic function $u(x)$ satisfying the boundary condition

$$u(x) = g(x) \quad \text{on } \partial D \quad (1.4)$$

is the extremal function minimizing the Dirichlet integral on D among a class of functions on D satisfying the boundary condition (1.4). Here, the famous historical fact is the existence problem of the extremal function. Now, we would like to clear the Dirichlet principle by modifying and simplifying it from the viewpoint of numerical analysis as follows:

In order to use the theory of Hilbert spaces, as a function space we shall use the Sobolev Hilbert space H^s and we shall consider the extremal problem

$$\inf_{F\in H^s} \int_D |\Delta F|^2 dx, \quad (1.5)$$

satisfying the boundary condition (1.4).

Intuitively, we can see that the harmonic function $u(x)$ satisfying the boundary condition (1.4) will be the extremal function of this extremal problem.

We wish to discuss clearly and simply the existence of the extremal functions and furthermore, we wish to obtain some good representation of the extremal functions when they exist. For this purpose, we shall apply the Tikhonov regularization and the theory of reproducing kernels based on the recent methods in [1, 2, 4] as follows:

We shall consider the extremal problem, for fixed $\lambda > 0$ and for any $g \in L_2(\partial D)$

$$\inf_{F\in H^s} \left\{ \lambda \|F\|_{H^s}^2 + \|\Delta F\|_{L_2(\mathbf{R}^n)}^2 + \|F - g\|_{L_2(\partial D)}^2 \right\}. \quad (1.6)$$

In order to simplify the problem, we consider here as follows:

(1) As a function space, we use the Sobolev Hilbert space on the whole space; in this case the space admits the reproducing kernel (1.3). In the extremal problem, for the flexibility of the Sobolev space, we will be able to use the Sobolev space. In this case the reproducing kernel which is used essentially in our method is extremely simple.

(2) For the integral of the ΔF, we shall consider it on the whole space, not on the domain D. Then, the extremal problem will become very simple. Furthermore, by this setting we can consider the inner Dirichlet problem on D and the outer Dirichlet problem on $\overline{D}^c$ at the same time.

For these simplifications, we see that the reproducing kernel $K_\lambda(x, y)$ of the Hilbert space H_{K_λ} with the norm square

$$\lambda \|F\|_{H^s}^2 + \|\Delta F\|_{L_2(\mathbf{R}^n)}^2 \tag{1.7}$$

is given by

$$K_\lambda(x, y) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \frac{e^{ip\cdot(x-y)}}{\lambda(|p|^2+1)^s + |p|^4} dp. \tag{1.8}$$

2 Boundary Conditions

Our strategy is first to represent the extremal function $F^*_{s,\lambda,g}(x)$ in (1.6) in an explicit form and second to consider the limit of this extremal function as $\lambda \to 0$. Exactly, these procedures are given as follows:

We look for the reproducing kernel $K_{s,\lambda,\Delta}(x, y)$ for the Hilbert space $H_{K_{s,\lambda,\Delta}}$ with the norm square, for $F \in H^s$

$$\left\{\lambda \|F\|_{H^s}^2 + \|\Delta F\|_{L_2(\mathbf{R}^n)}^2 + \|F\|_{L_2(\partial D)}^2\right\}. \tag{2.1}$$

The reproducing kernel $K_{s,\lambda,\Delta}(x, y)$ is determined by the functional equation

$$K_\lambda(x, y) = \big(\lambda I + L^* L\big) K_{s,\lambda,\Delta}(x, y), \tag{2.2}$$

where L is the bounded linear operator from H_{K_λ} into $L_2(\partial D)$ and L^* is its adjoint operator.

In this case, the functional equation is the Fredholm integral equation of the second kind containing the reproducing kernel $K_\lambda(x, y)$. Then the extremal function is represented by g directly as follows [1, 2, 4]:

$$F^*_{s,\lambda,g}(x) = \big(g, LK_{s,\lambda,\Delta}(\cdot, x)\big)_{L_2(\partial D)}. \tag{2.3}$$

Indeed, in general, if the functional equation of type (2.2) may be solved effectively, then we can solve many problems in our situation.

Later, we let λ tend to zero, and so, we cannot apply the Neumann expansion for the equation, because we need the assumption $\|L^* L\| < \lambda$ in the Neumann expansion.

When the operator L is compact, we apply the spectral theory to solve the functional equation (2.2) and we must look for singular values and singular functions of the operator $L^* L$.

They will be abstract in a sense for a general domain D. So, in this paper, we would like to propose a new approach for the present problem based on the power of computers.

3 New Algorithm

In order to consider the boundary value problem in (1.4), we shall consider it in (1.6) as follows:

For any fixed points $\{x_j\}_{j=1}^N$ of the boundary ∂D and for any given values $\{A_j\}_{j=1}^N$, we consider the extremal problem, for any fixed $\{\lambda_j\}_{j=1}^N$ $(\lambda_j > 0)$

$$\inf_{F \in H^s} \left\{ \lambda \|F\|_{H^s}^2 + \|\Delta F\|_{L_2(\mathbf{R}^n)}^2 + \sum_{j=1}^N \lambda_j \left|F(x_j) - A_j\right|^2 \right\}; \tag{3.1}$$

that is, we approximate the integral in (1.6) by the summation.

This translation will be reasonable in the sense: $\|F - g\|_{L_2(\partial D)}^2$ is replaced by $\sum_{j=1}^N \lambda_j |F(x_j) - A_j|^2$.

Then, the reproducing kernel $K_{\lambda_j}(x, y)$ of the Hilbert space $H_{K_{\lambda_j}}$ with the norm square

$$\lambda \|F\|_{H^s}^2 + \|\Delta F\|_{L_2(\mathbf{R}^n)}^2 + \sum_{j=1}^N \lambda_j \left|F(x_j)\right|^2 \tag{3.2}$$

is given in terms of $K_\lambda(x, y)$ as follows:

$$K_{\lambda_j}(x, y) = K_\lambda(x, y) - \sum_{j, j'=1}^N \lambda_j K_\lambda(x, x_j) A_{jj'} K_\lambda(x_{j'}, y), \tag{3.3}$$

where $\|A_{jj'}\|$ is the inverse of the positive matrix $\|\delta_{jj'} + \lambda_j K_\lambda(x_{j'}, x_j)\|$ [3].

For this direct representation of the reproducing kernel $K_{\lambda_j}(x, y)$ which we need, for many points $\{x_j\}_{j=1}^N$ of the boundary ∂D, the size of the matrix $\|A_{jj'}\|$ is a large one and so, this direct representation will not be effective.

In order to overcome this difficulty, we shall propose a new approach.

First we shall start with one point.

The reproducing kernel $K_\lambda^{(1)}(x, y)$ of the Hilbert space with the norm square

$$\lambda \|F\|_{H^s}^2 + \|\Delta F\|_{L_2(\mathbf{R}^n)}^2 + \sum_{j=1}^1 \lambda_j \left|F(x_j)\right|^2 \tag{3.4}$$

is given by

$$K_\lambda^{(1)}(x, y) = K_\lambda(x, y) - \frac{\lambda_1 K_\lambda(x, x_1) K_\lambda(x_1, y)}{1 + \lambda_1 K_\lambda(x_1, x_1)}. \tag{3.5}$$

For two points x_1, x_2, the reproducing kernel $K_\lambda^{(2)}(x, y)$ of the Hilbert space with the norm square

$$\lambda \|F\|_{H^s}^2 + \|\Delta F\|_{L_2(\mathbf{R}^n)}^2 + \sum_{j=1}^{2} \lambda_j \left|F(x_j)\right|^2 \tag{3.6}$$

is given by

$$K_\lambda^{(2)}(x, y) = K_\lambda^{(1)}(x, y) - \frac{\lambda_2 K_\lambda^{(1)}(x, x_2) K_\lambda^{(1)}(x_2, y)}{1 + \lambda_2 K_\lambda^{(1)}(x_2, x_2)} \tag{3.7}$$

by using the reproducing kernel $K_\lambda^{(1)}(x, y)$.

In this way, we can proceed many steps, by the similar calculation. For this procedure, to write the computer program is very easy

$$\lambda \|F\|_{H^s}^2 + \|\Delta F\|_{L_2(\mathbf{R}^n)}^2 + \sum_{j=1}^{k} \lambda_j \left|F(x_j)\right|^2 \tag{3.8}$$

is given by

$$K_\lambda^{(k)}(x, y) = K_\lambda^{(k-1)}(x, y) - \frac{\lambda_k K_\lambda^{(k-1)}(x, x_k) K_\lambda^{(k-1)}(x_k, y)}{1 + \lambda_k K_\lambda^{(k-1)}(x_k, x_k)} \tag{3.9}$$

by using the reproducing kernel $K_\lambda^{(k-1)}(x, y)$.

When we have the N-th kernel $K_\lambda^{(N)}(x, y)$, it is the reproducing kernel of the Hilbert space with the norm square (3.2) and it coincides with $K_{\lambda_j}(x, y)$.

The extremal function in the minimum problem in (3.1) is given by

$$F_{\lambda,s,x_j,A_j}^*(x) = \sum_{j=1}^{N} A_j \lambda_j K_\lambda^{(N)}(x, x_j). \tag{3.10}$$

By taking a small λ, we will be able to obtain the approximate solution of the problem:

$$\Delta u \sim 0, \tag{3.11}$$

and

$$u(x_j) \sim A_j \quad j = 1, 2, 3, \ldots, N. \tag{3.12}$$

Letting λ tend to zero, we obtain mathematically the solution u of the problem:

$$\Delta u = 0 \tag{3.13}$$

and

$$u(x_j) = A_j \quad j = 1, 2, 3, \ldots, N, \tag{3.14}$$

for any finite points $\{x_j\}_{j=1}^N$ and for any values $\{A_j\}_{j=1}^N$.

We considered a general extremal problem in (3.1) by considering a general weight $\{\lambda_j\}$. This means that for a larger λ_{j_0}, the speed of the convergence $u(x_{j_0}) \to A_{j_0}$ is higher.

If the power of our computers is great, then our algorithm seems to be effective and we will be able to solve many concrete problems by using computers by our algorithm.

Note that analytical theory depends seriously on the domain D—for example, recall the Poisson integral formula—however, our method does essentially not depend on the domain D.

4 Numerical Experiments

At first we calculate the kernel $K_\lambda(x, y)$ in (1.8) for $n = 2$ and $s = 2$. It is a function in $x - y$ and so we set $y = 0$ and further we set $x_2 = 0$. Then, by integrating (1.8) in p_2 we obtain

$$K_\lambda(x_1, 0; 0, 0) = \frac{1}{4\sqrt{\lambda}\pi} \int_{\mathbf{R}} \left(\frac{1+\lambda}{p_1^2 + \lambda(1+p_1^2)^2} \right)^{1/4} \cos(p_1 x_1) \cdot \sin\left(\frac{1}{2} \arctan\left(\frac{\sqrt{\lambda}}{(\lambda+1)p_1^2 + \lambda} \right) \right) dp_1. \tag{4.1}$$

In the following numerical experiments in Figs. 1(b)–(f), we calculate in (3.10) on the boundary of the square $[-1, 1] \times [-1, 1]$ with 0.05 span.

For the 160 points of the boundary of the square, first, we put the numbers from 0 to 159 from the point $(1, 0)$ to $(1, -0.05)$ in a counterclockwise direction on the boundary with span 0.05.

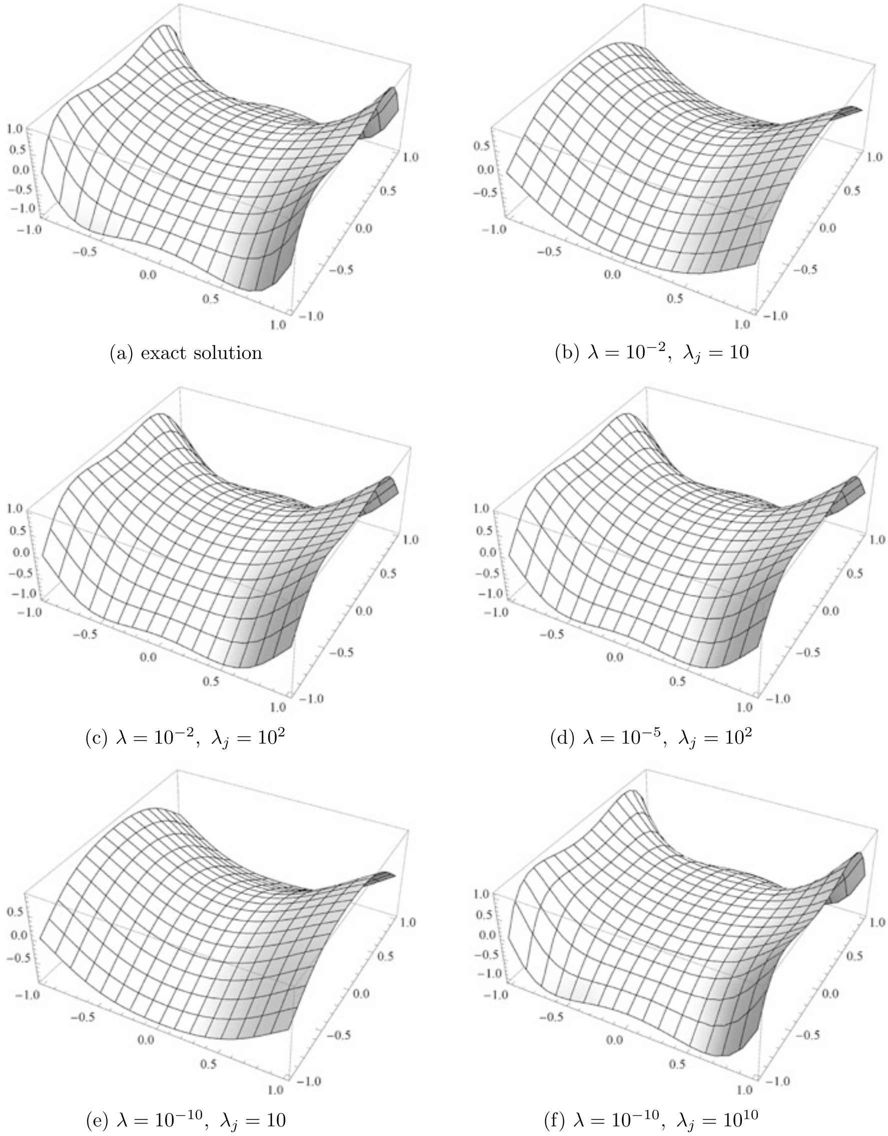
Then, in order to take the boundary points uniformly over the boundary, we take the sequence of the points $\{x_j\}_{j=1}^{160}$ in (3.14) as follows:

0, 80, 40, 120, 20, 60, 100, 140, 10, 30, 50, 70, 90, 110, 130, 150, 5, 15, 25, 35, 45, 55, 65, 75, 85, 95, 105, 115, 125, 135, 145, 155, 2, 7, 12, 17, 22, 27, 32, 37, 42, 47, 52, 57, 62, 67, 72, 77, 82, 87, 92, 97, 102, 107, 112, 117, 122, 127, 132, 137, 142, 147, 152, 157, 1, 3, 6, 8, 11, 13, 16, 18, 21, 23, 26, 28, 31, 33, 36, 38, 41, 43, 46, 48, 51, 53, 56, 58, 61, 63, 66, 68, 71, 73, 76, 78, 81, 83, 86, 88, 91, 93, 96, 98, 101, 103, 106, 108, 111, 113, 116, 118, 121, 123, 126, 128, 131, 133, 136, 138, 141, 143, 146, 148, 151, 153, 156, 158, 4, 9, 14, 19, 24, 29, 34, 39, 44, 49, 54, 59, 64, 69, 74, 79, 84, 89, 94, 99, 104, 109, 114, 119, 124, 129, 134, 139, 144, 149, 154, 159.

From these numerical experiments, we see that in order to adjust the boundary conditions, to take large weights λ_j is very effective for a small λ.

In these calculations we used $2825761 (= 41^4)$ data for each kernel on $[-1, 1]^4$ with 0.05 span. Therefore, for a large domain or a domain in a higher dimensional space, we need a computer with a great power.

However, we hope we will be able to have such supper computers, soon. Then, may we say that we can solve the Dirichlet problem using computers?



(a) exact solution

(b) $\lambda = 10^{-2},\ \lambda_j = 10$

(c) $\lambda = 10^{-2},\ \lambda_j = 10^2$

(d) $\lambda = 10^{-5},\ \lambda_j = 10^2$

(e) $\lambda = 10^{-10},\ \lambda_j = 10$

(f) $\lambda = 10^{-10},\ \lambda_j = 10^{10}$

Fig. 1 For $g(x_1, x_2) = \cosh(2x_1x_2)\sin(x_1^2 - x_2^2)$ on the boundary of $\chi_{[-1,1]}(x_1) \times \chi_{[-1,1]}(x_2)$ in $\mathbf{R}^2$

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Part X
Toeplitz Operators and Their Applications

Organizers: Sergei Grudsky, Nikolai Vasilevski

C^*-Algebras of Two-Dimensional Singular Integral Operators with Shifts

Y.I. Karlovich and V.A. Mozel

Abstract We construct a Fredholm symbol calculus for the C^*-algebra $\mathfrak{B}$ generated by the C^*-algebra $\mathfrak{A}$ of two-dimensional singular integral operators with continuous coefficients on a bounded closed simply connected domain $\overline{U} \subset \mathbb{R}^2$ with Liapunov boundary and by all unitary shift operators W_g where g runs through a discrete solvable group $G = F \rtimes H$ of diffeomorphisms of $\overline{U}$ onto itself, where F is a commutative group of conformal mappings, $H = \{e, \gamma\}$ and γ is similar to the shift $z \mapsto \overline{z}$. As a result, we establish a Fredholm criterion for the operators $B \in \mathfrak{B}$.

Keywords Two-dimensional singular integral operator · Shift operator · Solvable group · C^*-algebra · Representation · Fredholmness

Mathematics Subject Classification (2010) Primary 47G10 · Secondary 31A10 · 47A53 · 47A67 · 47B33

1 Introduction

Given a domain $U \subset \mathbb{R}^2$, let $\mathcal{B} := \mathcal{B}(L^2(U))$ be the C^*-algebra of all bounded linear operators on the Hilbert space $L^2(U)$ with Lebesgue area measure $dA(z) = dxdy$, let $\mathcal{K} := \mathcal{K}(L^2(U))$ be the closed two-sided ideal of all compact operators in $\mathcal{B}$, and let $\mathcal{B}^\pi := \mathcal{B}/\mathcal{K}$ denote the quotient C^*-algebra consisting of the cosets $A^\pi := A + \mathcal{K}$ $(A \in \mathcal{B})$.

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Suppose now that U is a bounded simply connected domain in $\mathbb{C}$ with Liapunov boundary Γ, $\overline{U} = U \cup \Gamma$, $G = F \rtimes H$ is a solvable group of diffeomorphisms $g : \overline{U} \to \overline{U}$, which is the semidirect product of F and H, where F is a commutative group of conformal mappings similar, respectively, to elliptic, hyperbolic or parabolic maps of the closed unit disc $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \le 1\}$ onto itself, $H = \{e, \gamma\}$, e is the unit of G, γ is similar to the shift $z \mapsto \overline{z}$, and $(g_1 h_1)(g_2 h_2) = (g_1(h_1 g_2 h_1^{-1}))(h_1 h_2)$ for all $g_1, g_2 \in F$ and all $h_1, h_2 \in H$, where $\gamma \circ g \circ \gamma = g^{-1}$ for all $g \in F$ and $g_1 g_2 = g_2 \circ g_1$ for all $g_1, g_2 \in G$. By the Kellogg–Warschawski theorem (see, e.g., [10, Theorem 3.6]), for all $g \in G$ the partial derivatives

$$\frac{\partial g}{\partial z} = \frac{1}{2}\left(\frac{\partial g}{\partial x} - i\frac{\partial g}{\partial y}\right), \qquad \frac{\partial g}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial g}{\partial x} + i\frac{\partial g}{\partial y}\right)$$

satisfy a Hölder condition on $\overline{U}$. Such group G acts on $\overline{U}$ topologically freely (see Sect. 3), and the set Φ_g of all fixed points of g on $\overline{U}$ has empty interior for every shift $g \in G \setminus \{e\}$. With every $g \in G$ we associate a unitary weighted shift operator W_g acting on the Lebesgue space $L^2(U)$ by

$$W_g f = |J_g|^{1/2}(f \circ g) \quad \text{for all } f \in L^2(U), \tag{1.1}$$

where $J_g(z) = |\frac{\partial g}{\partial z}|^2 - |\frac{\partial g}{\partial \overline{z}}|^2$ is the Jacobian of g.

Let S_U, S_U^* be the two-dimensional singular integral operators given by

$$(S_U f)(z) = -\frac{1}{\pi}\int_U \frac{f(w)}{(w-z)^2}dA(w), \qquad \left(S_U^* f\right)(z) = -\frac{1}{\pi}\int_U \frac{f(w)}{(\overline{w}-\overline{z})^2}dA(w)$$

for all $z \in U$ and bounded on the space $L^2(U)$. We denote by

$$\mathfrak{A} := \operatorname{alg}\left\{cI, S_U, S_U^* : c \in C(\overline{U})\right\} \tag{1.2}$$

the C^*-subalgebra of $\mathcal{B}(L^2(U))$ generated by all multiplication operators cI with $c \in C(\overline{U})$ and by the operators S_U and S_U^*.

The aim of this paper is to study the Fredholmness of operators B (equivalently, the invertibility of cosets $B^\pi = B + \mathcal{K}$) in the C^*-algebra

$$\mathfrak{B} := C^*(\mathfrak{A}, W_G) \subset \mathcal{B}\big(L^2(U)\big) \tag{1.3}$$

generated by all operators $A \in \mathfrak{A}$ and all shift operators W_g $(g \in G)$. By [8, Lemma 2.6], which remains valid for arbitrary domains $U \subset \mathbb{C}$, the C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ contain the ideal $\mathcal{K}$ of all compact operators in $\mathcal{B}(L^2(U))$.

Applying results of [6], where the C^*-algebra $\mathfrak{B}$ was studied for discrete amenable groups of quasiconformal shifts $g : \overline{U} \to \overline{U}$, and using the local-trajectory method elaborated in [4, 5], in the present paper we construct a Fredholm symbol calculus for the C^*-algebra $\mathfrak{B}$ and establish a Fredholm criterion for the operators $B \in \mathfrak{B}$ provided that $G = F \rtimes H$ is a solvable group containing analytic and anti-analytic shifts, and F consists, respectively, of elliptic, hyperbolic and parabolic conformal maps $g : \overline{U} \to \overline{U}$.

2 Solvable Groups of Shifts on $\overline{U}$

As is known (see, e.g., [2, Chap. 2]), any non-identical conformal map of the extended complex plane onto itself has two fixed points different or coincident. Consider now conformal mappings of the open unit disc $\mathbb{D}$ onto itself. They have the form

$$w = \theta \frac{z - z_0}{1 - \overline{z_0} z} \quad \text{for } z \in \mathbb{C}, \tag{2.1}$$

where $z_0 \in \mathbb{D}$, $\theta \in \mathbb{T}$, and $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle. If z_1, z_2 are fixed points of the conformal map (2.1), then either $z_0 = 0$ and then $z_1 = 0$ and $z_2 = \infty$, or $z_0 \neq 0$ and then

$$z_1 + z_2 = (1 - \theta)/\overline{z_0}, \qquad z_1 z_2 = -\theta z_0/\overline{z_0}. \tag{2.2}$$

In the latter case $|z_1 z_2| = 1$. Thus, either $|z_1| < 1 < |z_2|$ or $|z_1| = |z_2| = 1$.

Consider the shift $\lambda : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ given by $\lambda(z) = \overline{z}$ for all $z \in \overline{\mathbb{D}}$.

Given a bounded simply connected domain U in $\mathbb{C}$ with Liapunov boundary $\Gamma = \partial U$, we now define three solvable groups of the form $G = F \rtimes H$ mentioned in the introduction.

Let F_1 be the group of conformal mappings of the closed domain $\overline{U}$ onto itself with a common fixed point $z_1 \in U$. Fix a conformal mapping $\varphi_1 : \overline{\mathbb{D}} \to \overline{U}$ such that $\varphi_1(0) = z_1$. Then any $g \in F_1$ is of the form $g = \varphi_1 \circ f_\theta \circ \varphi_1^{-1}$ where $\theta \in \mathbb{T}$ and $f_\theta(z) = \theta z$ for $z \in \overline{\mathbb{D}}$. Rotations f_θ are elliptic maps of $\overline{\mathbb{D}}$ onto itself. Thus F_1 is a commutative one-parameter group that is similar to the group of rotations f_θ of the closed unit disc $\overline{\mathbb{D}}$.

Let F_2 be the group of conformal mappings of the closed domain $\overline{U}$ onto itself with two different common fixed point $z_1, z_2 \in \Gamma$. Fix a conformal mapping $\varphi_2 : \overline{\mathbb{D}} \to \overline{U}$ such that $\varphi_2(i) = z_1$ and $\varphi_2(-i) = z_2$. Hence for any $g \in F_2$ the conformal map $\varphi_2^{-1} \circ g \circ \varphi_2 : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ has two fixed points $\pm i \in \mathbb{T}$. Then from (2.2) it follows that $\theta = 1$ and $z_0 = -\overline{z_0}$. Hence any $g \in F_2$ is of the form $g = \varphi_2 \circ f_x \circ \varphi_2^{-1}$ where $x \in (-1, 1) \subset \mathbb{R}$ and f_x is a hyperbolic map of $\overline{\mathbb{D}}$ onto itself given by $f_x(z) = (z - ix)/(1 + ixz)$ for $z \in \overline{\mathbb{D}}$. It is easily seen [7] that for every $x, y \in (-1, 1)$ we have $f_x \circ f_y = f_y \circ f_x = f_{x \circ y}$ where

$$x \circ y = (x + y)/(1 + xy) \tag{2.3}$$

is a group operation on the interval $(-1, 1)$. Hence $(-1, 1)$ becomes a commutative group with operation (2.3), unit 0 and the inverse $-x$ for $x \in (-1, 1)$. Thus F_2 is a commutative one-parameter group that is similar to the group of hyperbolic maps f_x of the closed unit disc $\overline{\mathbb{D}}$ onto itself.

Let F_3 be the group of conformal mappings of the closed domain $\overline{U}$ onto itself with a common double fixed point $z_1 \in \Gamma$. Fix a conformal mapping $\varphi_3 : \overline{\mathbb{D}} \to \overline{U}$ such that $\varphi_3(1) = z_1$. Hence for any $g \in F_3$ the conformal map $\varphi_3^{-1} \circ g \circ \varphi_3 : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ has the double fixed point $1 \in \mathbb{T}$. Writing this map in the form (2.1), we infer from (2.2) that 1 is a double fixed point of this map if and only if $\overline{z_0} = (1 - \theta)/2$. Since

$|\overline{z_0}| < 1$, we need to exclude $\theta = -1$. For every $\theta \in \mathbb{T} \setminus \{-1\}$, we consider parabolic maps ψ_θ of $\overline{\mathbb{D}}$ onto itself given by

$$\psi_\theta(z) = \theta \frac{z - (1-\overline{\theta})/2}{1 - z(1-\theta)/2} = \frac{2\theta z + 1 - \theta}{2 - (1-\theta)z} \quad \text{for } z \in \overline{\mathbb{D}}. \tag{2.4}$$

On the set $\mathbb{T} \setminus \{-1\}$ we introduce the new multiplication (see [7]) by

$$\theta \circ \vartheta = \theta\vartheta \frac{4 - (1-\overline{\theta})(1-\overline{\vartheta})}{4 - (1-\theta)(1-\vartheta)} \quad \text{for } \theta, \vartheta \in \mathbb{T} \setminus \{-1\}. \tag{2.5}$$

Then $\mathbb{T} \setminus \{-1\}$ becomes a commutative group with operation (2.5), unit 1 and the inverse $\theta^{-1} = \overline{\theta}$ for $\theta \in \mathbb{T} \setminus \{-1\}$, and

$$\psi_\theta \circ \psi_\vartheta = \psi_\vartheta \circ \psi_\theta = \psi_{\theta\circ\vartheta} \quad \text{for every } \theta, \vartheta \in \mathbb{T} \setminus \{-1\}.$$

Thus any $g \in F_3$ is of the form $g = \varphi_3 \circ \psi_\theta \circ \varphi_3^{-1}$ where $\theta \in \mathbb{T} \setminus \{-1\}$ and ψ_θ is a parabolic map of the form (2.4). Hence F_3 is a commutative one-parameter group that is similar to the group of parabolic maps ψ_θ of the closed unit disc $\overline{\mathbb{D}}$ onto itself.

For every $j = 1, 2, 3$, let $H_j = \{e, \gamma_j\}$, where $\gamma_j = \varphi_j \circ \lambda \circ \varphi_j^{-1}$ and $\lambda(z) = \overline{z}$ for all $z \in \overline{\mathbb{D}}$. Consider the solvable groups $G_j = F_j \rtimes H_j$ being the semidirect product of groups F_j and H_j. Then $G_j = \{g, g\gamma_j : g \in F_j\}$ and $\gamma_j g = g^{-1}\gamma_j$ for all $g \in F_j$ and all $j = 1, 2, 3$.

Let Φ_g be the set of all fixed points for the shift g on $\overline{U}$. Then $\Phi_g = \{z_1\}$ if $g \in F_1 \setminus \{e\}$, $\Phi_{\gamma_1} = \varphi_1([-1, 1])$ and $\Phi_{g\gamma_1} = \{z_1\}$ for $g \in F_1 \setminus \{e\}$; $\Phi_g = \{z_1, z_2\}$ if $g \in F_2 \setminus \{e\}$, $\Phi_{\gamma_2} = \varphi_2([-1, 1])$ and $\Phi_{g\gamma_2} = \emptyset$ for $g \in F_2 \setminus \{e\}$; $\Phi_g = \{z_1\}$ if $g \in F_3 \setminus \{e\}$, $\Phi_{\gamma_3} = \varphi_3([-1, 1])$ and $\Phi_{g\gamma_3} = \{z_1\}$ for $g \in F_3 \setminus \{e\}$. Thus, for each $j = 1, 2, 3$ and each $g \in G_j \setminus \{e\}$ the set Φ_g has empty interior.

Let $G \in \{G_j : j = 1, 2, 3\}$ and let $\gamma \in G$ mean that $\gamma = \gamma_j$ if $G = G_j$.

For every $g \in G$ we consider unitary weighted shift operators $W_g \in \mathcal{B}(L^2(U))$ defined by (1.1). Observe that if $g \in G$ is a conformal map of U onto itself, then $J_g = |g'|^2$, while $J_{g\gamma} = -|g'|^2$.

Lemma 2.1 *If U is a bounded simply connected domain in $\mathbb{C}$ with Liapunov boundary Γ, $G \in \{G_j : j = 1, 2, 3\}$ and $g \in G$ is a conformal mapping of U onto itself, then the operators*

$$W_g S_U W_g^{-1} - \big(\overline{g'}/g'\big) S_U, \qquad W_g S_U^* W_g^{-1} - \big(g'/\overline{g'}\big) S_U^*, \tag{2.6}$$

$$W_{g\gamma} S_U W_{g\gamma}^{-1} - \big(g'/\overline{g'}\big) S_U^*, \qquad W_{g\gamma} S_U^* W_{g\gamma}^{-1} - \big(\overline{g'}/g'\big) S_U \tag{2.7}$$

are compact on the space $L^2(U)$.

Proof The compactness of operators (2.6) follows from [6, Corollary 4.2 and Remark 4.3]. Consider the operator $W_\lambda : L^2(\mathbb{D}) \to L^2(\mathbb{D})$ given by $W_\lambda f = f \circ \lambda$, where $\lambda(z) = \overline{z}$ for $z \in \mathbb{D}$. Since $W_\lambda S_{\mathbb{D}} W_\lambda^{-1} = S_{\mathbb{D}}^*$ and $W_\lambda S_{\mathbb{D}}^* W_\lambda^{-1} = S_{\mathbb{D}}$, we infer from [6, Corollary 4.2] that the operators

$$W_\gamma S_U W_\gamma^{-1} - S_U^*, \qquad W_\gamma S_U^* W_\gamma^{-1} - S_U \tag{2.8}$$

are compact on the space $L^2(U)$. Applying then (2.8) and (2.6), we obtain the compactness of operators (2.7) on the space $L^2(U)$. □

3 The Local-Trajectory Method

Let $\mathcal{A}$ be a unital C^*-algebra, $\mathcal{Z}$ a central C^*-subalgebra of $\mathcal{A}$ with the same unit I, G a discrete group with unit e, $U : g \mapsto U_g$ a homomorphism of the group G onto a group $U_G = \{U_g : g \in G\}$ of unitary elements such that $U_{g_1 g_2} = U_{g_1} U_{g_2}$ and $U_e = I$. Suppose $\mathcal{A}$ and U_G are contained in a C^*-algebra $\mathcal{D}$ and assume that

(A1) *for every $g \in G$ the mappings $\alpha_g : a \mapsto U_g a U_g^*$ are *-automorphisms of the C^*-algebras $\mathcal{A}$ and $\mathcal{Z}$;*
(A2) *G is an amenable* [3, § 1.2] *discrete group.*

Let $\mathcal{B} := C^*(\mathcal{A}, U_G)$ be the minimal C^*-subalgebra of $\mathcal{D}$ containing the unital C^*-algebra $\mathcal{A}$ and the group U_G. Let $M := M(\mathcal{Z})$ be the maximal ideal space of the (commutative) C^*-algebra $\mathcal{Z}$. Under assumption (A1), identifying characters φ_m of the algebra $\mathcal{Z} \cong C(M)$ and the maximal ideals $m = \operatorname{Ker}\varphi_m \in M$, we obtain the homomorphism $g \mapsto \beta_g(\cdot)$ of the group G into the homeomorphism group of M according to the rule

$$z\big(\beta_g(m)\big) = \big(\alpha_g(z)\big)(m), \quad z \in \mathcal{Z},\ m \in M,\ g \in G, \tag{3.1}$$

where $z(\cdot) \in C(M)$ is the Gelfand transform of the element $z \in \mathcal{Z}$.

Let $P_{\mathcal{A}}$ be the set of all pure states [9] on the C^*-algebra $\mathcal{A}$ equipped with the induced weak* topology, and let the following version of *topologically free* action of the group G hold (see [1, 5]):

(A3) *there is a set $M_0 \subset M$ such that for every finite set $G_0 \subset G \setminus \{e\}$ and every nonempty open set $V \subset P_{\mathcal{A}}$ there exists a state $\nu \in V$ such that $\beta_g(m_\nu) \neq m_\nu$ for all $g \in G_0$, where the point $m_\nu := \mathcal{Z} \cap \operatorname{Ker}\nu \in M$ belongs to the G-orbit $G(M_0) := \{\beta_g(m) : g \in G, m \in M_0\}$ of M_0.*

For every $m \in M$, let J_m be the closed two-sided ideal of the algebra $\mathcal{A}$ generated by the maximal ideal m of the algebra $\mathcal{Z}$, and let $\mathcal{H}_m$ be the Hilbert space of an isometric representation $\widetilde{\pi}_m : \mathcal{A}/J_m \to \mathcal{B}(\mathcal{H}_m)$. We also consider the canonical *-homomorphism $\varrho_m : \mathcal{A} \to \mathcal{A}/J_m$ and the representation

$$\pi'_m : \mathcal{A} \to \mathcal{B}(\mathcal{H}_m), \qquad a \mapsto (\widetilde{\pi}_m \circ \varrho_m)(a).$$

Since $\alpha_g(J_{\beta_g(m)}) = J_m$ for all $g \in G$ and all $m \in M$ in view of (A1), the quotient algebras $\mathcal{A}/J_{\beta_g(m)}$ and $\mathcal{A}/J_m$ are *-isomorphic. Then the spaces $\mathcal{H}_{\beta_g(m)}$ can be chosen equal for all $g \in G$.

Given $X \subset M$, let $\Omega(X)$ be the set of G-orbits of all points $m \in X$, let $H_\omega = \mathcal{H}_m$ where $m = m_\omega$ is any point of an orbit $\omega \in \Omega$ and $\Omega = \Omega(M)$, and let $l^2(G, H_\omega)$ be the Hilbert space of all functions $f : G \mapsto H_\omega$ such that $f(g) \neq 0$ for at most countable set of $g \in G$ and $\sum \|f(g)\|^2_{H_\omega} < \infty$. For every $\omega \in \Omega$ we consider the representation $\pi_\omega : \mathcal{B} \to \mathcal{B}(l^2(G, H_\omega))$ defined by

$$\big[\pi_\omega(a) f\big](g) = \pi'_m\big(\alpha_g(a)\big) f(g), \qquad \big[\pi_\omega(U_h) f\big](g) = f(gh) \tag{3.2}$$

for all $a \in \mathcal{A}$, all $g, h \in G$, and all $f \in l^2(G, H_\omega)$.

Theorem 3.1 [5, Theorem 4.1] *If assumptions* (A1)–(A3) *are satisfied, then an element $b \in \mathcal{B}$ is invertible in $\mathcal{B}$ if and only if for every orbit $\omega \in \Omega$ the operator $\pi_\omega(b)$ is invertible on the space $l^2(G, H_\omega)$ and, for infinite Ω,*

$$\sup\big\{\big\|\big(\pi_\omega(b)\big)^{-1}\big\|_{\mathcal{B}(l^2(G,H_\omega))} : \omega \in \Omega\big\} < \infty. \tag{3.3}$$

The next result following from [5, Theorems 4.8, 4.2] gives a sufficient condition that allows us to remove the uniform boundedness condition (3.3) for norms of inverse operators. Let $\overline{\omega}$ be the closure of an orbit $\omega \in \Omega$, and let ω' be the set of all limit points of ω.

Theorem 3.2 [6, Theorem 2.2] *If conditions* (A1)–(A3) *are satisfied, the C^*-algebra $\mathcal{Z}$ is separable, and $\bigcap_{m\in\omega} J_m = \bigcap_{m\in\overline{\omega}} J_m$ for every G-orbit $\omega \in \Omega$ such that $\overline{\omega} = \omega'$, then an element $b \in \mathcal{B}$ is invertible in $\mathcal{B}$ if and only if for every orbit $\omega \in \Omega$ the operator $\pi_\omega(b)$ is invertible on the space $l^2(G, H_\omega)$.*

4 C^*-Algebra $\mathfrak{A}$ of Two-Dimensional Singular Integral Operators

Given a C^*-algebra $\mathfrak{S}$, let $\widehat{\mathfrak{S}}$ be the spectrum of $\mathfrak{S}$, i.e., the compact topological space of all unitary equivalence classes of non-zero irreducible representations of $\mathfrak{S}$ in Hilbert spaces (see, e.g., [9, Sect. 5.4]).

Consider the orthogonal projection $P = (I + S_\mathbb{T})/2$ on the Lebesgue space $L^2(\mathbb{T})$, where I is the identity operator and $S_\mathbb{T}$ is the Cauchy singular integral operator on the unit circle $\mathbb{T}$,

$$(S_\mathbb{T}\varphi)(t) = \lim_{\varepsilon\to 0} \frac{1}{2\pi} \int_{\{z\in\mathbb{T}:|z-t|\geq\varepsilon\}} \frac{f(\tau)}{\tau - t} d\tau, \quad t \in \mathbb{T}.$$

As is well known, the Toeplitz operators $T_a := PaP$ with symbols $a \in L^\infty(\mathbb{T})$ are bounded on the Hilbert space $H^2(\mathbb{T}) := PL^2(\mathbb{T})$.

Below we use the following result from [11, Theorem 5] (cf. also [6, Theorem 5.7]). In our notations it says

Theorem 4.1 *Let U be a bounded simply connected domain in $\mathbb{C}$ with Liapunov boundary Γ. Then the spectrum $\widehat{\mathfrak{A}^\pi}$ of the quotient C^*-algebra $\mathfrak{A}^\pi$ can be parameterized by the points $(w, z) \in (\overline{U} \times \mathbb{T}) \cup (\Gamma \times \{\pm 2\})$ where the one-dimensional non-zero irreducible representations $\pi_{w,z} : \mathfrak{A}^\pi \to \mathbb{C}$ for every $(w, z) \in \overline{U} \times \mathbb{T}$ and infinite dimensional non-zero irreducible representations $\pi_{w,z} : \mathfrak{A}^\pi \to \mathcal{B}(H^2(\mathbb{T}))$ for every $(w, z) \in \Gamma \times \{\pm 2\}$ are given on the generators of the C^*-algebra $\mathfrak{A}^\pi$ by*

$$\begin{aligned}
&\pi_{w,z}\big([cI]^\pi\big) = c(w), \qquad \pi_{w,z}\big([S_U]^\pi\big) = z, \qquad \pi_{w,z}\big(\big[S_U^*\big]^\pi\big) = \overline{z} \\
&\quad \textit{if } (w,z) \in \overline{U} \times \mathbb{T}, \\
&\pi_{w,z}\big([cI]^\pi\big) = c(w)I, \qquad \pi_{w,z}\big([S_U]^\pi\big) = T_z, \qquad \pi_{w,z}\big(\big[S_U^*\big]^\pi\big) = T_{\overline{z}} \\
&\quad \textit{if } (w,z) \in \Gamma \times \{2\}, \\
&\pi_{w,z}\big([cI]^\pi\big) = c(w)I, \qquad \pi_{w,z}\big([S_U]^\pi\big) = T_{\overline{z}}, \qquad \pi_{w,z}\big(\big[S_U^*\big]^\pi\big) = T_z \\
&\quad \textit{if } (w,z) \in \Gamma \times \{-2\},
\end{aligned}$$

where T_z and $T_{\overline{z}}$ are Toeplitz operators with symbols z and $\overline{z}$ on $H^2(\mathbb{T})$.

Identifying numbers $b \in \mathbb{C}$ with multiplication operators bI acting on the Hilbert space $H = \mathbb{C}$ and using Theorem 4.1, we obtain the continuity of the functions

$$\eta_A : \overline{U} \times \mathbb{T} \to \mathbb{C}, \qquad (w,z) \mapsto \pi_{w,z}\big(A^\pi\big), \tag{4.1}$$

$$\eta_A^\pm : \Gamma \to \mathcal{B}\big(H^2(\mathbb{T})\big), \qquad w \mapsto \pi_{w,\pm 2}\big(A^\pi\big), \tag{4.2}$$

for every $A \in \mathfrak{A}$, where $\eta_A(w,z) = \pi_{w,z}(A^\pi)$,

$$\begin{aligned}
\pi_{w,2}\big(A^\pi\big) &= \eta_A(w, T_z) = T_{\eta_A(w,z)} + K_1, \\
\pi_{w,-2}\big(A^\pi\big) &= \eta_A(w, T_{\overline{z}}) = T_{\eta_A(w,\overline{z})} + K_2,
\end{aligned} \tag{4.3}$$

and K_1, K_2 are compact operators on the space $H^2(\mathbb{T})$.

5 C^*-Algebra $\mathfrak{B} := C^*(\mathfrak{A}, W_G)$

Let U be a bounded simply connected domain in $\mathbb{C}$ with Liapunov boundary Γ, and let $G \in \{G_j : j = 1, 2, 3\}$ be a discrete solvable (and therefore amenable) group of diffeomorphisms $g : \overline{U} \to \overline{U}$, with $G_j = \{g, g\gamma : g \in F_j\}$.

Consider the C^*-subalgebra $\mathfrak{B} := C^*(\mathfrak{A}, W_G)$ of $\mathcal{B}(L^2(U))$ generated by all operators $A \in \mathfrak{A}$ and all operators W_g $(g \in G)$ given by (1.1), where the C^*-algebra $\mathfrak{A}$ is defined by (1.2). Then $\mathcal{Z}^\pi = \{cI + \mathcal{K} : c \in C(\overline{U})\}$ is a central subalgebra of the C^*-algebra $\mathfrak{A}^\pi = \mathfrak{A}/\mathcal{K}$, and $M(\mathcal{Z}^\pi) = C(\overline{U})$.

By Lemma 2.1, for every $g \in G$ the mappings $\alpha_g : A \mapsto W_g A W_g^*$ are *-automorphisms of the C^*-algebras $\mathfrak{A}^\pi$ and $\mathcal{Z}^\pi$. Thus, assumptions (A1) and (A2) of Sect. 3 are satisfied. The set $P_{\mathfrak{A}^\pi}$ of all pure states of the C^*-algebra $\mathfrak{A}^\pi$ consists of all functionals $\pi_{w,z}(A^\pi)$ for $(w,z) \in \overline{U} \times \mathbb{T}$ (these functionals simultaneously are one-dimensional representations of $\mathfrak{A}^\pi$) and all vector states $(\pi_{w,z}(A^\pi)\xi, \xi)$ for $(w,z) \in \Gamma \times \{\pm 2\}$ where $\xi \in H^2(\mathbb{T})$ are vectors of norm 1.

Since the interior of each set Φ_g $(g \in G \setminus \{e\})$ is empty, we infer from the continuity of the functions $\eta_A : \overline{U} \times \mathbb{T} \to \mathbb{C}$ given by (4.1) that for every finite set $G_0 \subset G \setminus \{e\}$ and every open neighborhood $V_{w_0,z_0} \subset \overline{U} \times \mathbb{T}$ of any point $(w_0, z_0) \in (\bigcup_{g \in G_0} \Phi_g) \times \mathbb{T}$ there exists a point $(w, z_0) \in V_{w_0,z_0}$ such that $g(w) \neq w$ for all $g \in G_0$. In that case $M_0 = \overline{U}$, and for every $\varepsilon > 0$, every $(w_0, z_0) \in (\bigcup_{g \in G_0} \Phi_g) \times \mathbb{T}$ and every $A \in \mathfrak{A}$ there is a $\delta > 0$ such that

$$\left|\pi_{w,z_0}\left(A^\pi\right) - \pi_{w_0,z_0}\left(A^\pi\right)\right| = \left|\eta_A(w, z_0) - \eta_A(w_0, z_0)\right| < \varepsilon \quad \text{if } |w - w_0| < \delta.$$

On the other hand, from the continuity of functions $\eta_A^\pm : \Gamma \to \mathcal{B}(H^2(\mathbb{T}))$ given by (4.2) it follows that for every finite set $G_0 \subset G \setminus \{e\}$ and every open neighborhood $V_{w_0} \subset \Gamma$ of any point $w_0 \in \bigcup_{g \in G_0} \Phi_g$ there exists a point $w \in V_{w_0}$ such that $g(w) \neq w$ for all $g \in G_0$. In that case again $M_0 = \overline{U}$, and for every $\varepsilon > 0$, every $w_0 \in \bigcup_{g \in G_0} \Phi_g$ and every $A \in \mathfrak{A}$ there is a $\delta > 0$ such that for any vector $\xi \in H^2(\mathbb{T})$ of norm 1,

$$\left|\left(\pi_{w,2}\left(A^\pi\right)\xi, \xi\right) - \left(\pi_{w_0,2}\left(A^\pi\right)\xi, \xi\right)\right| \le \left\|\eta_A(w, T_z) - \eta_A(w_0, T_z)\right\|_{\mathcal{B}(H^2(\mathbb{T}))} < \varepsilon,$$
$$\left|\left(\pi_{w,-2}\left(A^\pi\right)\xi, \xi\right) - \left(\pi_{w_0,-2}\left(A^\pi\right)\xi, \xi\right)\right| \le \left\|\eta_A(w, T_{\overline{z}}) - \eta_A(w_0, T_{\overline{z}})\right\|_{\mathcal{B}(H^2(\mathbb{T}))} < \varepsilon$$

if $|w - w_0| < \delta$. Thus, condition (A3) is also fulfilled along with (A1)–(A2).

Hence, we can obtain an invertibility criterion for the cosets $B^\pi \in \mathfrak{B}^\pi$ on the basis of Theorem 3.1.

Since $\mathcal{Z}^\pi \cong C(\overline{U})$, we conclude from (3.1) that $\beta_g = g$ for every $g \in G$. With each point $w \in \overline{U}$ we associate its G-orbit $G(w) = \{g(w) : g \in G\} \subset \overline{U}$. Consider the representations $\pi'_w := \bigoplus_{z \in \mathbb{T}} \pi_{w,z} : \mathfrak{A}^\pi \to \mathcal{B}(\mathcal{H}_w)$ for $w \in U$ and $\pi'_w := (\bigoplus_{z \in \mathbb{T}} \pi_{w,z}) \oplus \pi_{w,2} \oplus \pi_{w,-2} : \mathfrak{A}^\pi \to \mathcal{B}(\mathcal{H}_w)$ for $w \in \Gamma$, where $\mathcal{H}_w = \bigoplus_{z \in \mathbb{T}} \mathbb{C}$ if $w \in U$, and $\mathcal{H}_w = (\bigoplus_{z \in \mathbb{T}} \mathbb{C}) \oplus H^2(\mathbb{T}) \oplus H^2(\mathbb{T})$ if $w \in \Gamma$.

Let $\Omega := \Omega(\overline{U})$ be the set of all G-orbits of points $w \in \overline{U}$. Fix a point $w = w_\omega$ on every G-orbit $\omega \in \Omega$ and put $H_\omega = \mathcal{H}_w$. For every $\omega \in \Omega$ we consider the representation $\pi_\omega : \mathfrak{B} \to \mathcal{B}(l^2(G, H_\omega))$ defined in view of (3.2) by

$$\left[\pi_\omega\left(A^\pi\right) f\right](g) = \pi'_w\left(\left[\alpha_g(A)\right]^\pi\right) f(g), \qquad \left[\pi_\omega\left([W_h]^\pi\right) f\right](g) = f(gh) \tag{5.1}$$

for all $A \in \mathfrak{A}$, all $g, h \in G$, and all $f \in l^2(G, H_\omega)$.

We infer from Lemma 2.1 that

$$\left[\alpha_g(aI)\right]_w^\pi = \left[a\left(g(w)\right)I\right]_w^\pi, \qquad \left[\alpha_{g\gamma}(aI)\right]_w^\pi = \left[a\left((g\gamma)(w)\right)I\right]_w^\pi \quad \left(a \in C(\overline{U})\right),$$
$$\left[\alpha_g(S_U)\right]_w^\pi = \left[\left(\overline{g'(w)}/g'(w)\right)S_U\right]_w^\pi, \qquad \left[\alpha_{g\gamma}(S_U)\right]_w^\pi = \left[\left(g'(w)/\overline{g'(w)}\right)S_U^*\right]_w^\pi,$$
$$\left[\alpha_g\left(S_U^*\right)\right]_w^\pi = \left[\left(g'(w)/\overline{g'(w)}\right)S_U^*\right]_w^\pi, \qquad \left[\alpha_{g\gamma}\left(S_U^*\right)\right]_w^\pi = \left[\left(\overline{g'(w)}/g'(w)\right)S_U\right]_w^\pi,$$

where $g \in F$. Hence, setting

$$\begin{aligned} \sigma_g(w,z) &:= \big(\overline{g'(w)}/g'(w)\big)z, & \sigma_g(w,T_z) &:= \big(\overline{g'(w)}/g'(w)\big)T_z, \\ \sigma_{g\gamma}(w,z) &:= \big(g'(w)/\overline{g'(w)}\big)\overline{z}, & \sigma_{g\gamma}(w,T_z) &:= \big(g'(w)/\overline{g'(w)}\big)T_{\overline{z}}, \end{aligned} \tag{5.2}$$

we conclude from (5.2) that

$$\sigma_{g\gamma}(w,z) = \big(g'(w)/\overline{g'(w)}\big)\overline{z} = \overline{\sigma_g(w,z)},$$
$$\sigma_{g\gamma}(w,T_z) = \big(g'(w)/\overline{g'(w)}\big)T_{\overline{z}} = \big[\sigma_g(w,T_z)\big]^*,$$

and therefore, taking into account (4.1) and (4.3), for all $g \in G$ we obtain

$$\pi_{w,z}\big(\big[\alpha_g(A)\big]^\pi\big) = \eta_A\big(g(w),\sigma_g(w,z)\big) \quad \text{if } (w,z) \in \overline{U} \times \mathbb{T}, \tag{5.3}$$

$$\pi_{w,2}\big(\big[\alpha_g(A)\big]^\pi\big) = \eta_A\big(g(w),\sigma_g(w,T_z)\big) \quad \text{if } w \in \Gamma, \tag{5.4}$$

$$\pi_{w,-2}\big(\big[\alpha_g(A)\big]^\pi\big) = \eta_A\big(g(w),\big[\sigma_g(w,T_z)\big]^*\big) \quad \text{if } w \in \Gamma. \tag{5.5}$$

Thus every *-automorphism α_g ($g \in G$) of the C^*-algebra $\mathfrak{A}^\pi$ induces the homeomorphism λ_g of the compact $\overline{U} \times \mathbb{T}$ onto itself by the rule:

$$\lambda_g(w,z) = \big(g(w),\sigma_g(w,z)\big) \quad \text{for all } (w,z) \in \overline{U} \times \mathbb{T}.$$

It is easily seen that if G acts topologically freely on $M = \overline{U}$, then the group $\{\lambda_g : g \in G\}$ acts topologically freely on $\overline{U} \times \mathbb{T}$.

Setting $\Omega_U := \{G(w) : w \in U\}$, $\Omega_\Gamma := \{G(w) : w \in \Gamma\}$, taking points $w = w_\omega$ on each G-orbit $\omega \in \Omega$ and representing the spaces $l^2(G,H_\omega)$ (to within isometric isomorphisms) as $l^2(G,H_\omega) = \bigoplus_{z\in\mathbb{T}} l^2(G)$ if $\omega \in \Omega_U$, and

$$l^2(G,H_\omega) = \Big(\bigoplus_{z\in\mathbb{T}} l^2(G)\Big) \oplus l^2\big(G,H^2(\mathbb{T})\big) \oplus l^2\big(G,H^2(\mathbb{T})\big) \quad \text{if } \omega \in \Omega_\Gamma,$$

we infer from (5.1) that the representation $\pi_\omega : \mathfrak{B}^\pi \to \mathcal{B}(l^2(G,H_\omega))$ can be given by $\pi_\omega = \bigoplus_{z\in\mathbb{T}} \widetilde{\pi}_{\omega,z}$ if $\omega \subset U$, and

$$\pi_\omega = \Big(\bigoplus_{z\in\mathbb{T}} \widetilde{\pi}_{\omega,z}\Big) \oplus \widetilde{\pi}_{\omega,2} \oplus \widetilde{\pi}_{\omega,-2} \quad \text{if } \omega \subset \Gamma,$$

where the representations $\widetilde{\pi}_{\omega,z} : \mathfrak{B}^\pi \to l^2(G)$ for $\omega \subset \overline{U}$ and $z \in \mathbb{T}$ are defined for all $A \in \mathfrak{A}$, all $g,h \in G$ and all $f \in l^2(G)$ in view of (5.3) by

$$\begin{aligned} \big[\widetilde{\pi}_{\omega,z}\big(A^\pi\big)f\big](g) &= \pi_{w,z}\big(\big[\alpha_g(A)\big]^\pi\big)f(g) = \eta_A\big(g(w),\sigma_g(w,z)\big)f(g), \\ \big[\widetilde{\pi}_{\omega,z}\big([W_h]^\pi\big)f\big](g) &= f(gh); \end{aligned} \tag{5.6}$$

and the representations $\widetilde{\pi}_{\omega,\pm 2} : \mathfrak{B}^\pi \to l^2(G, H^2(\mathbb{T}))$ for $\omega \subset \Gamma$ are defined for all $A \in \mathfrak{A}$, all $g, h \in G$ and all $f \in l^2(G, H^2(\mathbb{T}))$ in view of (5.4)–(5.5) by

$$\begin{aligned}
&\big[\widetilde{\pi}_{\omega,2}\big(A^\pi\big) f\big](g) = \pi_{w,2}\big(\big[\alpha_g(A)\big]^\pi\big) f(g) = \eta_A\big(g(w), \sigma_g(w, T_z)\big) f(g),\\
&\big[\widetilde{\pi}_{\omega,-2}\big(A^\pi\big) f\big](g) = \pi_{w,-2}\big(\big[\alpha_g(A)\big]^\pi\big) f(g) = \eta_A\big(g(w), \big[\sigma_g(w, T_z)\big]^*\big) f(g),\\
&\big[\widetilde{\pi}_{\omega,\pm 2}\big([W_h]^\pi\big) f\big](g) = f(gh).
\end{aligned} \tag{5.7}$$

Theorem 5.1 *Let U be a bounded simply connected domain in $\mathbb{C}$ with Liapunov boundary Γ and let $G \in \{G_j : j = 1, 2, 3\}$. Then an operator $B \in \mathfrak{B}$ is Fredholm on the space $L^2(U)$ if and only if for every $(\omega, z) \in \Omega \times \mathbb{T}$ the operators $\widetilde{\pi}_{\omega,z}(B^\pi)$ are invertible on the space $l^2(G)$, for every $\omega \in \Omega_\Gamma$ the operators $\widetilde{\pi}_{\omega,\pm 2}(B^\pi)$ are invertible on the space $l^2(G, H^2(\mathbb{T}))$, and*

$$\sup_{(\omega,z)\in\Omega\times\mathbb{T}} \big\| \big(\widetilde{\pi}_{\omega,z}\big(B^\pi\big)\big)^{-1} \big\|_{\mathcal{B}(l^2(G))} < \infty,$$

$$\sup_{(\omega,z)\in\Omega_\Gamma\times\{\pm 2\}} \big\| \big(\widetilde{\pi}_{\omega,z}\big(B^\pi\big)\big)^{-1} \big\|_{\mathcal{B}(l^2(G,H^2(\mathbb{T})))} < \infty.$$

Note that Theorem 5.1 remains valid under the replacement of $\Omega \times \mathbb{T}$ by $\Omega_U \times \mathbb{T}$. Indeed, for every $(w, z) \in \Gamma \times \mathbb{T}$, the maps

$$\begin{aligned}
\nu^+_{w,z} &: \pi_{w,2}\big(A^\pi\big) = \eta_A(w, T_z) \mapsto \pi_{w,z}\big(A^\pi\big) = \eta_A(w, z),\\
\nu^-_{w,z} &: \pi_{w,-2}\big(A^\pi\big) = \eta_A(w, T_{\overline{z}}) \mapsto \pi_{w,z}\big(A^\pi\big) = \eta_A(w, z)
\end{aligned}$$

are C^*-algebra homomorphisms of the C^*-algebra $\pi_{w,\pm 2}(\mathfrak{A}^\pi) \subset \mathcal{B}(H^2(\mathbb{T}))$ onto $\mathbb{C}$, respectively. Consequently, for every $(\omega, z) \in \Omega_\Gamma \times \mathbb{T}$, the maps

$$\mu^\pm_{\omega,z} : \mathcal{B}\big(l^2\big(G, H^2(\mathbb{T})\big)\big) \to \mathcal{B}\big(l^2(G)\big), \qquad \widetilde{\pi}_{\omega,\pm 2}\big(B^\pi\big) \mapsto \widetilde{\pi}_{\omega,z}\big(B^\pi\big)$$

are *-homomorphisms of the C^*-algebra $\widetilde{\pi}_{\omega,\pm 2}(\mathfrak{B}^\pi) \subset \mathcal{B}(l^2(G, H^2(\mathbb{T})))$ onto the C^*-algebra $\widetilde{\pi}_{\omega,z}(\mathfrak{B}^\pi) \subset \mathcal{B}(l^2(G))$, whence the invertibility of operators $\widetilde{\pi}_{\omega,\pm 2}(\mathfrak{B}^\pi)$ on the space $l^2(G, H^2(\mathbb{T}))$ for all $\omega \in \Omega_\Gamma$ implies the invertibility of operators $\widetilde{\pi}_{\omega,z}(B^\pi)$ on the space $l^2(G)$ for all $(\omega, z) \in \Omega_\Gamma \times \mathbb{T}$.

Taking this into account and since $\overline{\omega} \cap \mathbb{T} = \emptyset$ for every G_1-orbit $\omega \in \Omega_U$, we infer the following from Theorem 3.2 by analogy with [6, Theorem 6.1].

Theorem 5.2 *Let U be a bounded simply connected domain in $\mathbb{C}$ with Liapunov boundary Γ and let $G = G_1$. Then an operator $B \in \mathfrak{B}$ is Fredholm on the space $L^2(U)$ if and only if for every $\omega \in \Omega_U$ and every $z \in \mathbb{T}$ the operators $\widetilde{\pi}_{\omega,z}(B^\pi)$ are invertible on the space $l^2(G)$ and for every $\omega \in \Omega_\Gamma$ the operators $\widetilde{\pi}_{\omega,\pm 2}(B^\pi)$ are invertible on the space $l^2(G, H^2(\mathbb{T}))$.*

By Theorem 5.1, the operator function $\widetilde{\Psi}(B) : (\omega, z) \mapsto \widetilde{\pi}_{\omega,z}(B^\pi)$ given for $(\omega, z) \in (\Omega \times \mathbb{T}) \cup (\Omega_\Gamma \times \{\pm 2\})$ by (5.6)–(5.7) and equipped with the norm

$$\max\Big\{ \sup_{(\omega,z)\in\Omega\times\mathbb{T}} \big\|\widetilde{\pi}_{\omega,z}\big(B^\pi\big)\big\|_{\mathcal{B}(l^2(G))}, \sup_{(\omega,z)\in\Omega_\Gamma\times\{\pm 2\}} \big\|\widetilde{\pi}_{\omega,z}\big(B^\pi\big)\big\|_{\mathcal{B}(l^2(G,H^2(\mathbb{T})))} \Big\}$$

can be considered as a Fredholm symbol for operators $B \in \mathfrak{B}$. Thus, the map $\widetilde{\Psi} : \mathfrak{B} \to \widetilde{\Psi}(\mathfrak{B})$ gives a Fredholm symbol calculus for the C^*-algebra $\mathfrak{B}$ defined by (1.3), because the Fredholmness of an operator $B \in \mathfrak{B}$ is equivalent to the invertibility of its Fredholm symbol.

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Uncertainty and Analyticity

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Abstract We describe a connection between minimal uncertainty states and holomorphy-type conditions on the images of the respective wavelet transforms. The most familiar example is the Fock–Segal–Bargmann transform generated by the Gaussian, however, this also occurs under more general assumptions.

Keywords Quantum mechanics · Classical mechanics · Heisenberg commutation relations · Observables · Heisenberg group · Fock–Segal–Bargmann space · SU(1, 1) · Hardy space

Mathematics Subject Classification (2010) Primary 81P05 · Secondary 22E27

1 Introduction

There are two and a half main examples of reproducing kernel spaces of analytic function. One is the Fock–Segal–Bargmann (FSB) space and others (one and a half)—the Bergman and Hardy spaces. The first space is generated by the Heisenberg group [2, § 1.6; 5, § 7.3], two others—by the group SU(1, 1) [5, § 4.2] (this explains our way of counting).

Those spaces have the following properties, which make their study particularly pleasant and fruitful:

i. There is a group, which acts transitively on functions' domain.
ii. There is a reproducing kernel.
iii. The space consists of holomorphic functions.

Furthermore, for FSB space there is the following property:

iv. The reproducing kernel is generated by a function, which minimises the uncertainty for coordinate and momentum observables.

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It is known, that a transformation group is responsible for the appearance of the reproducing kernel [1, Theorem 8.1.3]. This paper shows that the last two properties are equivalent and connected to the group as well.

2 The Uncertainty Relation

In quantum mechanics [2, § 1.1], an observable (self-adjoint operator on a Hilbert space $\mathcal{H}$) A produces the expectation value $\bar{A}$ on a state (a unit vector) $\phi \in \mathcal{H}$ by $\bar{A} = \langle A\phi, \phi\rangle$. Then, the dispersion is evaluated as follow:

$$\Delta_\phi^2(A) = \left\langle (A-\bar{A})^2\phi, \phi\right\rangle = \left\langle (A-\bar{A})\phi, (A-\bar{A})\phi\right\rangle = \left\|(A-\bar{A})\phi\right\|^2. \qquad (1)$$

The next theorem links obstructions of exact simultaneous measurements with non-commutativity of observables.

Theorem 1 (The uncertainty relation) *If A and B are self-adjoint operators on a Hilbert space $\mathcal{H}$, then*

$$\left\|(A-a)u\right\|\left\|(B-b)u\right\| \geq \frac{1}{2}\left|\left\langle (AB-BA)u, u\right\rangle\right|, \qquad (2)$$

for any $u \in \mathcal{H}$ from the domains of AB and BA and $a, b \in \mathbb{R}$. Equality holds precisely when u is a solution of $((A-a) + \mathrm{i}r(B-b))u = 0$ for some real r.

Proof The proof is well-known [2, § 1.3], but it is short, instructive and relevant for the following discussion, thus we include it in full. We start from simple algebraic transformations:

$$\begin{aligned}\left\langle (AB-BA)u, u\right\rangle &= \left\langle (A-a)(B-b)-(B-b)(A-a)u, u\right\rangle\\ &= \left\langle (B-b)u, (A-a)u\right\rangle - \left\langle (A-a)u, (B-b)u\right\rangle\\ &= 2\mathrm{i}\Im\left\langle (B-b)u, (A-a)u\right\rangle. \qquad (3)\end{aligned}$$

Then by the Cauchy–Schwartz inequality:

$$\frac{1}{2}\left\langle (AB-BA)u, u\right\rangle \leq \left|\left\langle (B-b)u, (A-a)u\right\rangle\right| \leq \left\|(B-b)u\right\|\left\|(A-a)u\right\|.$$

The equality holds if and only if $(B-b)u$ and $(A-a)u$ are proportional by a *purely imaginary* scalar. □

The famous application of the above theorem is the following fundamental relation in quantum mechanics. Recall [2, § 1.2], that the one-dimensional Heisenberg group $\mathbb{H}^1$ consists of points $(s, x, y) \in \mathbb{R}^3$, with the group law:

$$(s, x, y) * (s', x', y') = \left(s + s' + \frac{1}{2}(xy' - x'y), x + x', y + y'\right). \qquad (4)$$

This is a nilpotent step two Lie group. By the Stone–von Neumann theorem [2, § 1.5], any infinite-dimensional unitary irreducible representation of $\mathbb{H}^1$ is unitary equivalent to the Schrödinger representation $\rho_\hbar$ in $\mathcal{L}_2(\mathbb{R})$ parametrised by the Planck constant $\hbar \in \mathbb{R} \setminus \{0\}$. A physically consistent form of $\rho_\hbar$ is [6, (3.5)]:

$$\big[\rho_\hbar(s,x,y)f\big](q) = e^{-2\pi \mathrm{i}\hbar(s+xy/2)-2\pi \mathrm{i}xq} f(q+\hbar y). \tag{5}$$

Elements of the Lie algebra $\mathfrak{h}_1$, corresponding to the infinitesimal generators X and Y of one-parameters subgroups $(0, t/(2\pi), 0)$ and $(0, 0, t)$ in $\mathbb{H}^1$, are represented in (5) by the (unbounded) operators M and D on $\mathcal{L}_2(\mathbb{R})$:

$$M = -\mathrm{i}q, \qquad D = \hbar \frac{d}{dq}, \quad \text{with the commutator } [M, D] = \mathrm{i}\hbar I. \tag{6}$$

In the Schrödinger model of quantum mechanics, $f(q) \in \mathcal{L}_2(\mathbb{R})$ is interpreted as a wave function (a state) of a particle, with M and D are the observables of its coordinate and momentum.

Corollary 2 (Heisenberg–Kennard uncertainty relation) *For the coordinate M and momentum D observables we have the* Heisenberg–Kennard uncertainty *relation*:

$$\Delta_\phi(M) \cdot \Delta_\phi(D) \geq \frac{h}{2}. \tag{7}$$

The equality holds if and only if $\phi(q) = e^{-cq^2}$, $c \in \mathbb{R}_+$ is the vacuum state in the Schrödinger model.

Proof The relation follows from the commutator $[M, D] = \mathrm{i}\hbar I$, which, in turn, is the representation of the Lie algebra $\mathfrak{h}_1$ of the Heisenberg group. The minimal uncertainty state in the Schrodinger representation is a solution of the differential equation: $(M - \mathrm{i}rD)\phi = 0$ for some $r \in \mathbb{R}$, or, explicitly:

$$(M - \mathrm{i}rD)\phi = -\mathrm{i}\left(q + r\hbar \frac{d}{dq}\right)\phi(q) = 0. \tag{8}$$

The solution is the Gaussian $\phi(q) = e^{-cq^2}$, $c = \frac{1}{2r\hbar}$. For $c > 0$, this function is in the state space $\mathcal{L}_2(\mathbb{R})$. □

It is common to say that the Gaussian $\phi(q) = e^{-cq^2}$ represents the ground state, which minimises the uncertainty of coordinate and momentum.

3 Wavelet Transform and Analyticity

3.1 Induced Wavelet Transform

The following object is common in quantum mechanics [4], signal processing, harmonic analysis [8], operator theory [7, 9] and many other areas [5]. Therefore, it has various names [1]: coherent states, wavelets, matrix coefficients, etc. In the most fundamental situation [1, Chap. 8], we start from an irreducible unitary representation ρ of a Lie group G in a Hilbert space $\mathcal{H}$. For a vector $f \in \mathcal{H}$ (called mother wavelet, vacuum state, etc.), we define the map $\mathcal{W}_f$ from $\mathcal{H}$ to a space of functions on G by:

$$[\mathcal{W}_f v](g) = \tilde{v}(g) := \langle v, \rho(g) f \rangle. \tag{9}$$

Under the above assumptions, $\tilde{v}(g)$ is a bounded continuous function on G. The map $\mathcal{W}_f$ intertwines $\rho(g)$ with the left shifts on G:

$$\mathcal{W}_f \circ \rho(g) = \Lambda(g) \circ \mathcal{W}_f, \quad \text{where } \Lambda(g) : \tilde{v}(g') \mapsto \tilde{v}(g^{-1} g'). \tag{10}$$

Thus, the image $\mathcal{W}_f \mathcal{H}$ is invariant under the left shifts on G. If ρ is square integrable and f is admissible [1, § 8.1], then $\tilde{v}(g)$ is square-integrable with respect to the Haar measure on G. At this point, none of admissible vectors has an advantage over others.

It is common [5, § 5.1], that there exists a closed subgroup $H \subset G$ and a respective $f \in \mathcal{H}$ such that $\rho(h) f = \chi(h) f$ for some character χ of H. In this case, it is enough to know values of $\tilde{v}(\mathsf{s}(x))$, for any continuous section s from the homogeneous space $X = G/H$ to G. The map $v \mapsto \tilde{v}(x) = \tilde{v}(\mathsf{s}(x))$ intertwines ρ with the representation ρ_χ in a certain function space on X induced by the character χ of H [3, § 13.2]. We call the map $\mathcal{W}_f : v \mapsto \tilde{v}(x)$ the *induced wavelet transform* [5, § 5.1].

For example, if $G = \mathbb{H}^1$, $H = \{(s, 0, 0) \in \mathbb{H}^1 : s \in \mathbb{R}\}$ and its character $\chi_\hbar(s, 0, 0) = e^{2\pi \mathrm{i} \hbar s}$, then any vector $f \in \mathcal{L}_2(\mathbb{R})$ satisfies $\rho_\hbar(s, 0, 0) f = \chi_\hbar(s) f$ for the representation (5). Thus, we still do not have a reason to prefer any admissible vector to others.

3.2 Right Shifts and Analyticity

To discover some preferable mother wavelets, we use the following a general result from [5, § 5]. Let G be a locally compact group and ρ be its representation in a Hilbert space $\mathcal{H}$. Let $[\mathcal{W}_f v](g) = \langle v, \rho(g) f \rangle$ be the wavelet transform defined by a vacuum state $f \in \mathcal{H}$. Then, the right shift $R(g) : [\mathcal{W}_f v](g') \mapsto [\mathcal{W}_f v](g' g)$ for $g \in G$ coincides with the wavelet transform $[\mathcal{W}_{f_g} v](g') = \langle v, \rho(g') f_g \rangle$ defined by the vacuum state $f_g = \rho(g)f$. In other words, the covariant transform intertwines right shifts on the group G with the associated action ρ on vacuum states, cf. (10):

$$R(g) \circ \mathcal{W}_f = \mathcal{W}_{\rho(g)f}. \tag{11}$$

Although, the above observation is almost trivial, applications of the following corollary are not.

Corollary 3 (Analyticity of the wavelet transform, [5, § 5]) *Let G be a group and dg be a measure on G. Let ρ be a unitary representation of G, which can be extended by integration to a vector space V of functions or distributions on G. Let a mother wavelet $f \in \mathcal{H}$ satisfy the equation*

$$\int_G a(g)\rho(g) f \, dg = 0,$$

for a fixed distribution $a(g) \in V$. Then any wavelet transform $\tilde{v}(g) = \langle v, \rho(g) f\rangle$ obeys the condition:

$$D\tilde{v} = 0, \quad \text{where } D = \int_G \bar{a}(g) R(g) dg, \tag{12}$$

with R being the right regular representation of G.

Some applications (including discrete one) produced by the $ax + b$ group can be found in [8, § 6]. We turn to the Heisenberg group now.

Example 4 (Gaussian and FSB transform) The Gaussian $\phi(x) = e^{-cq^2/2}$ is a null-solution of the operator $\hbar c M - \mathrm{i}D$. For the centre $Z = \{(s, 0, 0) : s \in \mathbb{R}\} \subset \mathbb{H}^1$, we define the section $\mathsf{s} : \mathbb{H}^1/Z \to \mathbb{H}^1$ by $\mathsf{s}(x, y) = (0, x, y)$. Then, the corresponding induced wavelet transform is:

$$\tilde{v}(x, y) = \big\langle v, \rho\big(\mathsf{s}(x, y)\big) f\big\rangle = \int_{\mathbb{R}} v(q) e^{\pi \mathrm{i}\hbar xy - 2\pi \mathrm{i} xq} e^{-c(q+\hbar y)^2/2} dq. \tag{13}$$

The infinitesimal generators X and Y of one-parameters subgroups $(0, t/(2\pi), 0)$ and $(0, 0, t)$ are represented through the right shift in (4) by

$$R_*(X) = -\frac{1}{4\pi} y \partial_s + \frac{1}{2\pi}\partial_x, \qquad R_*(Y) = \frac{1}{2} x \partial_s + \partial_y.$$

For the representation induced by the character $\chi_\hbar(s, 0, 0) = e^{2\pi \mathrm{i}\hbar s}$ we have $\partial_s = 2\pi \mathrm{i}\hbar I$. Corollary 3 ensures that the operator

$$\hbar c \cdot R_*(X) + \mathrm{i} \cdot R_*(Y) = -\frac{\hbar}{2}(2\pi x + \mathrm{i}\hbar c y) + \frac{\hbar c}{2\pi}\partial_x + \mathrm{i}\partial_y \tag{14}$$

annihilate any $\tilde{v}(x, y)$ from (13). The integral (13) is known as Fock–Segal–Bargmann (FSB) transform and in the most common case the values $\hbar = 1$ and

$c = 2\pi$ are used. For these, operator (14) becomes $-\pi(x+\mathrm{i}y)+(\partial_x+\mathrm{i}\partial_y) = -\pi z + 2\partial_{\bar{z}}$ with $z = x + \mathrm{i}y$. Then the function $V(z) = e^{\pi z\bar{z}/2}\tilde{v}(z) = e^{\pi(x^2+y^2)/2}\tilde{v}(x, y)$ satisfies the Cauchy–Riemann equation $\partial_{\bar{z}} V(z) = 0$.

This example shows, that the Gaussian is a preferred vacuum state (as producing analytic functions through FSB transform) exactly for the same reason as being the minimal uncertainty state: the both are derived from the identity $(\hbar c M + \mathrm{i}D)e^{-cq^2/2} = 0$.

3.3 Uncertainty and Analyticity

The main result of this paper is a generalisation of the previous observation, which bridges together Corollary 3 and Theorem 1. Let G, H, ρ and $\mathcal{H}$ be as before. Assume, that the homogeneous space $X = G/H$ has a (quasi-)invariant measure $d\mu(x)$ [3, § 13.2]. Then, for a function (or a suitable distribution) k on X we can define the integrated representation:

$$\rho(k) = \int_X k(x)\rho\bigl(\mathsf{s}(x)\bigr)d\mu(x), \tag{15}$$

which is (possibly, unbounded) operators on (possibly, dense subspace of) $\mathcal{H}$. In particular, $R(k)$ denotes the integrated right shifts, for $H = \{e\}$.

Theorem 5 *Let k_1 and k_2 be two distributions on X with the respective integrated representations $\rho(k_1)$ and $\rho(k_2)$. The following are equivalent*:

i. *A vector $f \in \mathcal{H}$ satisfies the identity*

$$\Delta_f\bigl(\rho(k_1)\bigr) \cdot \Delta_f\bigl(\rho(k_2)\bigr) = \bigl|\bigl\langle\bigl[\rho(k_1), \rho(k_1)\bigr]f, f\bigr\rangle\bigr|.$$

ii. *The image of the wavelet transform $\mathcal{W}_f : v \mapsto \tilde{v}(g) = \langle v, \rho(g)f\rangle$ consists of functions satisfying the equation $R(k_1 + \mathrm{i}rk_2)\tilde{v} = 0$ for some $r \in \mathbb{R}$, where R is the integrated form (15) of the right regular representation on G.*

Proof This is an immediate consequence of a combination of Theorem 1 and Corollary 3. □

Example 4 is a particular case of this theorem with $k_1(x, y) = \delta'_x(x, y)$ and $k_2(x, y) = \delta'_y(x, y)$ (partial derivatives of the delta function), which represent vectors X and Y from the Lie algebra $\mathfrak{h}_1$. The next example will be of this type as well.

3.4 Hardy Space

Let SU(1, 1) be the group of 2×2 complex matrices of the form $\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$ with the unit determinant $|\alpha|^2 - |\beta|^2 = 1$. A standard basis in the Lie algebra $\mathfrak{su}_{1,1}$ is

$$A = \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad B = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}.$$

The respective one-dimensional subgroups consist of matrices:

$$e^{tA} = \begin{pmatrix} \cosh\frac{t}{2} & -\mathrm{i}\sinh\frac{t}{2} \\ \mathrm{i}\sinh\frac{t}{2} & \cosh\frac{t}{2} \end{pmatrix}, \quad e^{tB} = \begin{pmatrix} \cosh\frac{t}{2} & \sinh\frac{t}{2} \\ \sinh\frac{t}{2} & \cosh\frac{t}{2} \end{pmatrix}, \quad e^{tZ} = \begin{pmatrix} e^{\mathrm{i}t} & 0 \\ 0 & e^{-\mathrm{i}t} \end{pmatrix}.$$

The last subgroup—the maximal compact subgroup of SU(1, 1)—is usually denoted by K. The commutators of the $\mathfrak{su}_{1,1}$ basis elements are

$$[Z, A] = 2B, \qquad [Z, B] = -2A, \qquad [A, B] = -\frac{1}{2}Z. \tag{16}$$

Let $\mathbb{T}$ denote the unit circle in $\mathbb{C}$ with the rotation-invariant measure. The mock discrete representation of SU(1, 1) [10, § VI.6] acts on $\mathcal{L}_2(\mathbb{T})$ by unitary transformations

$$\left[\rho_1(g) f\right](z) = \frac{1}{(\bar{\beta}z + \bar{\alpha})} f\left(\frac{\alpha z + \beta}{\bar{\beta}z + \bar{\alpha}}\right), \qquad g^{-1} = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}. \tag{17}$$

The respective derived representation ρ_{1*} of the $\mathfrak{su}_{1,1}$ basis is:

$$\rho_{1*}^{A} = \frac{\mathrm{i}}{2}\left(z + \left(z^2 + 1\right)\partial_z\right), \qquad \rho_{1*}^{B} = \frac{1}{2}\left(z + \left(z^2 - 1\right)\partial_z\right), \qquad \rho_{1*}^{Z} = -\mathrm{i}I - 2\mathrm{i}z\partial_z.$$

Thus, $\rho_{1*}^{B+\mathrm{i}A} = -\partial_z$ and the function $f_+(z) \equiv 1$ satisfies $\rho_{1*}^{B+\mathrm{i}A} f_+ = 0$. Recalling the commutator $[A, B] = -\frac{1}{2}Z$ we note that $\rho_1(e^{tZ}) f_+ = e^{\mathrm{i}t} f_+$. Therefore, there is the following identity for dispersions on this state:

$$\Delta_{f_+}\left(\rho_{1*}^{A}\right) \cdot \Delta_{f_+}\left(\rho_{1*}^{B}\right) = \frac{1}{2},$$

with the minimal value of uncertainty among all eigenvectors of the operator $\rho_1(e^{tZ})$.

Furthermore, the vacuum state f_+ generates the induced wavelet transform for the subgroup $K = \{e^{tZ} \mid t \in \mathbb{R}\}$. We identify SU(1, 1)/$K$ with the open unit disk $D = \{w \in \mathbb{C} \mid |w| < 1\}$ [5, § 5.5; 9]. The map $\mathsf{s} : \mathrm{SU}(1,1)/K \to \mathrm{SU}(1,1)$ is defined as $\mathsf{s}(w) = \frac{1}{\sqrt{1-|w|^2}}\begin{pmatrix} 1 & w \\ \bar{w} & 1 \end{pmatrix}$. Then, the induced wavelet transform is:

$$\begin{aligned}
\tilde{v}(w) = \left\langle v, \rho_1\left(\mathsf{s}(w)\right) f_+ \right\rangle &= \frac{1}{2\pi\sqrt{1-|w|^2}} \int_{\mathbb{T}} \frac{v(e^{\mathrm{i}\theta})\,d\theta}{1 - we^{-\mathrm{i}\theta}} \\
&= \frac{1}{2\pi\mathrm{i}\sqrt{1-|w|^2}} \int_{\mathbb{T}} \frac{v(e^{\mathrm{i}\theta})\,de^{\mathrm{i}\theta}}{e^{\mathrm{i}\theta} - w}.
\end{aligned}$$

Clearly, this is the Cauchy integral up to the factor $\frac{1}{\sqrt{1-|w|^2}}$, which presents the conformal metric on the unit disk. Similarly, we can consider the operator $\rho_{1*}^{B-\mathrm{i}A} = z + z^2\partial_z$ and the function $f_-(z) = \frac{1}{z}$ simultaneously solving the equations $\rho_{1*}^{B-\mathrm{i}A} f_- = 0$ and $\rho_1(e^{tZ}) f_- = e^{-\mathrm{i}t} f_-$. It produces the integral with the conjugated Cauchy kernel.

Finally, we can calculate the operator (12) annihilating the image of the wavelet transform. In the coordinates $(w, t) \in (\mathrm{SU}(1,1)/K) \times K$, the restriction to the induced subrepresentation is, cf. [10, § IX.5]:

$$\mathfrak{L}^{B-\mathrm{i}A} = e^{2\mathrm{i}t}\left(-\frac{1}{2}w + \left(1 - |w|^2\right)\partial_{\bar{w}}\right).$$

Furthermore, if $\mathfrak{L}^{B-\mathrm{i}A}\tilde{v}(w) = 0$, then $\partial_{\bar{w}}(\sqrt{1 - w\bar{w}} \cdot \tilde{v}(w)) = 0$. That is, $V(w) = \sqrt{1 - w\bar{w}} \cdot \tilde{v}(w)$ is a holomorphic function on the unit disk.

Similarly, we can treat representations of SU(1, 1) in the space of square integrable functions on the unit disk. The irreducible components of this representation are isometrically isomorphic [5, § 4–5] to the weighted Bergman spaces of (purely poly-)analytic functions on the unit, cf. [11].

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Toeplitz Operators on the Harmonic Bergman Space with Pseudodifferential Defining Symbols

Maribel Loaiza and Nikolai Vasilevski

Abstract We study the C^*-algebra $\mathcal{T}(\mathcal{R}(C(\overline{\mathbb{D}}); S_{\mathbb{D}}, S^*_{\mathbb{D}}))$ generated by Toeplitz operators acting on the harmonic Bergman space on the unit disk whose pseudodifferential defining symbols belong to the algebra $\mathcal{R} = \mathcal{R}(C(\overline{\mathbb{D}}); S_{\mathbb{D}}, S^*_{\mathbb{D}})$. The algebra $\mathcal{R}$ is generated by the multiplication operators aI, where $a \in C(\overline{\mathbb{D}})$, and the following two operators

$$S_{\mathbb{D}}(\varphi)(z) = \frac{-1}{\pi}\int_{\mathbb{D}} \frac{\varphi(\zeta)}{(\zeta - z)^2} d\nu(\zeta) \quad \text{and} \quad S^*_{\mathbb{D}}(\varphi)(z) = \frac{-1}{\pi}\int_{\mathbb{D}} \frac{\varphi(\zeta)}{(\overline{\zeta} - \overline{z})^2} d\nu(\zeta).$$

We describe the Fredholm symbol algebra of $\mathcal{T}(\mathcal{R}(C(\overline{\mathbb{D}}); S_{\mathbb{D}}, S^*_{\mathbb{D}}))$ and the index formula for its Fredholm elements.

Keywords Harmonic function · Bergman space · Bergman projection · Anti-Bergman projection · Algebras of Toeplitz operators

Mathematics Subject Classification (2010) 31A05 · 32A36 · 32A99 · 47L80

1 Introduction

Denote by $\mathbb{D}$ the complex unit disk with the area measure $d\nu(z) = dxdy$, $z = x + iy$. As usual, $L_2(\mathbb{D})$ denotes the space of all measurable and square integrable functions defined in $\mathbb{D}$. The harmonic Bergman space $b^2(\mathbb{D})$ is the closed subspace of $L_2(\mathbb{D})$ consisting of all complex-valued functions $f(z) = u(z) + iv(z)$, such that $u(z)$, $v(z)$ are real-valued harmonic functions. The harmonic Bergman space is the (non direct)

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sum of the Bergman and the anti-Bergman spaces. The orthogonal projection from $L_2(\mathbb{D})$ onto $b^2(\mathbb{D})$, denoted by Q, has the form

$$Q = B_{\mathbb{D}} + \widetilde{B}_{\mathbb{D}} + K,$$

where $B_{\mathbb{D}}$ is the Bergman projection, $\widetilde{B}_{\mathbb{D}}$ is the anti-Bergman projection, and K is a one-dimensional operator.

Toeplitz operators, acting on $b^2(\mathbb{D})$, have some properties that are just analogous to those for the Bergman space case. For example, the Calkin algebra of the C^*-algebra $\widehat{\mathcal{T}}(C(\overline{\mathbb{D}}))$, generated by Toeplitz operators with continuous symbols, is isomorphic to $C(\mathbb{T})$, where $\mathbb{T}$ is the boundary of $\mathbb{D}$. At the same time every Fredholm Toeplitz operator with continuous symbol has index zero (see [3] for details). Moreover, M. Loaiza and C. Lozano proved [5] that Fredholm Toeplitz operators with piecewise continuous symbols also have index zero.

Another substantial difference between Toeplitz operators acting on the Bergman space and Toeplitz operators acting on the harmonic Bergman space is that the spectrum of a Toeplitz operator with piecewise constant symbol, acting on the Bergman space, is always connected and it does not depend on the angles related to this discontinuity (see [11] for details) while for Toeplitz operators acting on the harmonic Bergman space the spectrum of such kind of operators is not necessarily connected, and it does depend on the angles between the curves supporting the symbol discontinuities (see [5]).

Denote by $\mathcal{R} = \mathcal{R}(C(\overline{\mathbb{D}}); S_{\mathbb{D}}, S_{\mathbb{D}}^*)$ the C^*-algebra which is generated by the multiplication operators $a(z)I$, where $a(z) \in C(\overline{\mathbb{D}})$, and the following two singular integral (pseudodifferential) operators

$$(S_{\mathbb{D}}\varphi)(z) = -\frac{1}{\pi}\int_{\mathbb{D}} \frac{\varphi(\zeta)}{(\zeta - z)^2} d\nu(\zeta) \quad \text{and} \quad \left(S_{\mathbb{D}}^*\varphi\right)(z) = -\frac{1}{\pi}\int_{\mathbb{D}} \frac{\varphi(\zeta)}{(\overline{\zeta} - \overline{z})^2} d\nu(\zeta), \tag{1.1}$$

and let $\mathcal{T}(C(\overline{\mathbb{D}}))$ be the C^*-algebra generated by Toeplitz operators T_a with defining symbols $a(z) \in C(\overline{\mathbb{D}})$.

A. Sánchez-Nungaray and N. Vasilevski studied [7] the C^*-algebra generated by Toeplitz operators acting on the Bergman over the unit disk, whose pseudodifferential defining symbols belong to $\mathcal{R} = \mathcal{R}(C(\overline{\mathbb{D}}); S_{\mathbb{D}}, S_{\mathbb{D}}^*)$, and showed that both algebras $\mathcal{T}(C(\overline{\mathbb{D}}))$ and $\mathcal{T}(\mathcal{R}(C(\overline{\mathbb{D}}); S_{\mathbb{D}}, S_{\mathbb{D}}^*))$, which is generated by Toeplitz operators T_A with defining symbols $A \in \mathcal{R}(C(\overline{\mathbb{D}}); S_{\mathbb{D}}, S_{\mathbb{D}}^*)$, are, in fact, the same; and that the Fredholm symbol algebra for both of them is isomorphic and isometric to $C(\mathbb{T})$.

In this paper we show that, for the harmonic Bergman space, the C^*-algebra generated by Toeplitz operators $\widehat{T}_A$ with $A \in \mathcal{R}(C(\overline{\mathbb{D}}); S_{\mathbb{D}}, S_{\mathbb{D}}^*)$ is not anymore isomorphic to the algebra generated by all Toeplitz operators with symbol in $C(\overline{\mathbb{D}})$. We give a complete characterization of this algebra as well as the index formula for its Fredholm operators.

2 Preliminaries

Let $\widehat{\mathcal{T}}(C(\overline{\mathbb{D}}))$ be the C^*-algebra generated by all Toeplitz operators with continuous symbols acting on the harmonic Bergman space, and denote by $\mathcal{K}$ the closed ideal of all compact operators on the space $L^2(\mathbb{D})$. It is known [3] that the quotient algebra $\widehat{\mathcal{T}}(C(\overline{\mathbb{D}}))/\mathcal{K}$ is $*$-isometrically isomorphic to $C(\mathbb{T})$, and that this isomorphism is given by $T_\phi + K \mapsto \phi|_{\mathbb{T}}$.

Recall also that the singular integral operators

$$(S_{\mathbb{D}}\varphi)(z) = -\frac{1}{\pi}\int_{\mathbb{D}} \frac{\varphi(\zeta)}{(\zeta - z)^2} d\nu(\zeta) \quad \text{and} \quad \big(S_{\mathbb{D}}^*\varphi\big)(z) = -\frac{1}{\pi}\int_{\mathbb{D}} \frac{\varphi(\zeta)}{(\overline{\zeta} - \overline{z})^2} d\nu(\zeta)$$

are bounded on $L_2(\mathbb{D})$ and mutually adjoint. It is known as well that, for each $a(z) \in C(\overline{\mathbb{D}})$, both commutators $[S_{\mathbb{D}}, a(z)I]$ and $[S_{\mathbb{D}}^*, a(z)I]$ are compact, and being considered in the whole complex plane, these operators obey the relation $S_{\mathbb{C}}^* = S_{\mathbb{C}}^{-1}$.

The description of the Fredholm symbol (Calkin) algebra $\operatorname{Sym}\mathcal{R} = \mathcal{R}(C(\overline{\mathbb{D}}), S_{\mathbb{D}}, S_{\mathbb{D}}^*)/\mathcal{K}$ of the algebra $\mathcal{R}(C(\overline{\mathbb{D}}), S_{\mathbb{D}}, S_{\mathbb{D}}^*)$ is well known. Following [7] we recall here this description.

Denote by $\mathcal{T}(C(\mathbb{T}))$ the C^*-algebra generated by all Toeplitz operators T_a, $a \in C(\mathbb{T})$, acting on the Hardy space $H^2(\mathbb{T})$. Let $\mathcal{M} = \overline{\mathbb{D}} \times \mathbb{T} \sqcup \mathbb{T} \times \{0, \infty\}$, and denote by $\mathfrak{S}$ the set of all vector functions σ, continuous on $\mathcal{M}$ and having the form

$$\sigma(m) = \begin{cases} c(z,t) \in \mathbb{C}, & \text{if } m = (z,t) \in \overline{\mathbb{D}} \times \mathbb{T}, \\ T_{c(z,t)} + K_0(z) \in \mathcal{T}(C(\mathbb{T})), & \text{if } m = (z,0) \in \mathbb{T} \times \{0,\infty\}, \\ T_{c(z,\overline{t})} + K_\infty(z) \in \mathcal{T}(C(\mathbb{T})), & \text{if } m = (z,\infty) \in \mathbb{T} \times \{0,\infty\}, \end{cases}$$

where K_0 and K_∞ are compact operators.

Theorem 2.1 *The Fredholm symbol algebra* $\operatorname{Sym}\mathcal{R}$ *of the algebra* $\mathcal{R} = \mathcal{R}(C(\overline{\mathbb{D}}), S_{\mathbb{D}}, S_{\mathbb{D}}^*)$ *is isomorphic and isometric to the algebra* $\mathfrak{S}$. *Under their identification, the symbol homomorphism*

$$\operatorname{sym}: \mathcal{R} \to \operatorname{Sym}\mathcal{R} = \mathfrak{S}$$

is generated by the following mapping: if $A = a_1(z)I + a_2(z)S_{\mathbb{D}} + a_3(z)S_{\mathbb{D}}^*$ *and* $c(z,t) = a_1(z) + a_2(z)t + a_3(z)\overline{t}$, *then*

$$(\operatorname{sym} A)(m) = \begin{cases} c(z,t) \in \mathbb{C}, & \text{if } m = (z,t) \in \overline{\mathbb{D}} \times \mathbb{T}, \\ T_{c(z,t)} \in \mathcal{T}(C(\mathbb{T})), & \text{if } m = (z,0) \in \mathbb{T} \times \{0,\infty\}, \\ T_{c(z,\overline{t})}(z) \in \mathcal{T}(C(\mathbb{T})), & \text{if } m = (z,\infty) \in \mathbb{T} \times \{0,\infty\}. \end{cases}$$

3 The Bergman Type Spaces on the Upper Half Plane and the Action of the Operators S_Π and S_Π^*

Denote by Π the upper half plane with the area measure $d\nu(z) = dxdy$, $z = x + iy$, and denote by $L_2(\Pi)$ the space of all measurable and square integrable functions defined in Π. The Bergman space $\mathcal{A}^2(\Pi)$ is the closed subspace of $L_2(\Pi)$ consisting of all analytic functions. The orthogonal Bergman projection B_Π from $L_2(\Pi)$ onto $\mathcal{A}^2(\Pi)$ has the following integral form

$$B_\Pi f(z) = -\frac{1}{\pi}\int_\Pi \frac{f(w)}{(z-\bar{w})^2}d\nu(w).$$

The anti-Bergman space is the closed subspace of $L_2(\Pi)$, denoted by $\widetilde{\mathcal{A}}^2(\Pi)$, consisting of all measurable and anti-analytic functions. The orthogonal projection from $L_2(\Pi)$ onto $\widetilde{\mathcal{A}}^2(\Pi)$ is called the anti-Bergman projection, is denoted by $\widetilde{B}_\Pi$, and its integral form is as follows

$$\widetilde{B}_\Pi f(z) = -\frac{1}{\pi}\int_\Pi \frac{f(w)}{(\bar{z}-w)^2}d\nu(w).$$

The space $\widetilde{\mathcal{A}}_n^2(\Pi)$ of n-anti-analytic functions is the subspace of $L_2(\Pi)$ consisting of all functions $\varphi(z,\overline{z}) = \varphi(x,y)$, which satisfy the equation

$$\left(\frac{\partial}{\partial z}\right)^n \varphi = \frac{1}{2^n}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)^n \varphi = 0.$$

The space $\mathcal{A}_n^2(\Pi)$ of n-analytic functions is the subspace of $L_2(\Pi)$ consisting of all functions $\varphi(z,\overline{z}) = \varphi(x,y)$, which satisfy the equation

$$\left(\frac{\partial}{\partial \overline{z}}\right)^n \varphi = \frac{1}{2^n}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)^n \varphi = 0.$$

The space $\mathcal{A}_{(n)}^2(\Pi)$ of true-n-analytic functions is the subspace of $L_2(\Pi)$ consisting of all functions in the set

$$\mathcal{A}_{(n)}^2(\Pi) = \mathcal{A}_n^2(\Pi) \ominus \mathcal{A}_{n-1}^2(\Pi), \quad \text{for } n > 1, \quad \text{and} \quad \mathcal{A}_{(1)}^2(\Pi) = \mathcal{A}_1^2(\Pi).$$

Analogously, the space of $\widetilde{\mathcal{A}}_{(n)}^2(\Pi)$ of true-n-anti-analytic functions is the subspace of $L_2(\Pi)$ consisting of all functions in the set

$$\widetilde{\mathcal{A}}_{(n)}^2(\Pi) = \widetilde{\mathcal{A}}_n^2(\Pi) \ominus \widetilde{\mathcal{A}}_{n-1}^2(\Pi), \quad \text{for } n > 1, \quad \text{and} \quad \widetilde{\mathcal{A}}_{(1)}^2(\Pi) = \widetilde{\mathcal{A}}_1^2(\Pi).$$

It is well known, see for example [9], that the Hilbert space $L_2(\Pi)$ admits the following direct sum decomposition

$$L_2(\Pi) = \bigoplus_{k=1}^{\infty} \mathcal{A}_{(k)}^2(\Pi) \oplus \bigoplus_{k=1}^{\infty} \widetilde{\mathcal{A}}_{(k)}^2(\Pi).$$

Theorem 3.1 ([4, 10]) *For all admissible indexes, we have*

$$(S_\Pi)^k|_{\mathcal{A}^2_{(n)}(\Pi)} : \mathcal{A}^2_{(n)}(\Pi) \to \mathcal{A}^2_{(n+k)}(\Pi),$$
$$\big(S^*_\Pi\big)^k|_{\mathcal{A}^2_{(n)}(\Pi)} : \mathcal{A}^2_{(n)}(\Pi) \to \mathcal{A}^2_{(n-k)}(\Pi),$$
$$(S_\Pi)^k|_{\widetilde{\mathcal{A}}^2_{(n)}(\Pi)} : \widetilde{\mathcal{A}}^2_{(n)}(\Pi) \to \widetilde{\mathcal{A}}^2_{(n-k)}(\Pi),$$
$$\big(S^*_\Pi\big)^k|_{\widetilde{\mathcal{A}}^2_{(n)}(\Pi)} : \widetilde{\mathcal{A}}^2_{(n)}(\Pi) \to \widetilde{\mathcal{A}}^2_{(n+k)}(\Pi).$$

These operators are unitary and the corresponding adjoint operators are the restrictions of the adjoint operators to the corresponding spaces. Besides $(S_\Pi)^k(\widetilde{\mathcal{A}}^2_{(j)}) = \{0\}$, $(S^*_\Pi)^k(\mathcal{A}^2_{(j)}) = \{0\}$ *if* $k \geq j$.

Furthermore, for all $n \in \mathbb{N}$ *we have*

$$(S_\Pi)^n\big(S^*_\Pi\big)^n(S_\Pi)^n = (S_\Pi)^n \quad \textit{and} \quad \big(S^*_\Pi\big)^n(S_\Pi)^n\big(S^*_\Pi\big)^n = \big(S^*_\Pi\big)^n.$$

The relations between the Bergman projection, anti-Bergman projection, and the powers of the singular integral operators S_Π and S^*_Π are given in the next corollary.

Corollary 3.2 *For* $n, m \in \mathbb{Z}_+$ *we have*

$$B_\Pi\big(S^*_\Pi\big)^m(S_\Pi)^n B_\Pi = \begin{cases} B_\Pi, & \text{if } m = n, \\ 0, & \text{if } m \neq n, \end{cases}$$
$$\widetilde{B}_\Pi(S_\Pi)^m\big(S^*_\Pi\big)^n \widetilde{B}_\Pi = \begin{cases} \widetilde{B}_\Pi, & \text{if } m = n, \\ 0, & \text{if } m \neq n, \end{cases} \tag{3.1}$$

while for $n, m \in \mathbb{N}$, $B_\Pi(S_\Pi)^m(S^*_\Pi)^n B_\Pi = 0$, $\widetilde{B}_\Pi(S^*_\Pi)^m(S_\Pi)^n\widetilde{B}_\Pi = 0$.

Furthermore, for $n, m \in \mathbb{Z}_+$

1. $\widetilde{B}_\Pi(S_\Pi)^m\big(S^*_\Pi\big)^n B_\Pi = 0$,
2. $B_\Pi(S_\Pi)^m\big(S^*_\Pi\big)^n \widetilde{B}_\Pi = 0$,
3. $\widetilde{B}_\Pi\big(S^*_\Pi\big)^m(S_\Pi)^n B_\Pi = 0$,
4. $B_\Pi\big(S^*_\Pi\big)^m(S_\Pi)^n \widetilde{B}_\Pi = 0$.

Proof To prove (3.1) we use Theorem 3.1 and the representation $\widetilde{B}_\Pi = I - S^*_\Pi S_\Pi$. Then, for $n > 0$:

$$\begin{aligned}\widetilde{B}_\Pi\big(S^*_\Pi\big)^m(S_\Pi)^n\widetilde{B}_\Pi &= \widetilde{B}_\Pi\big(S^*_\Pi\big)^m(S_\Pi)^n\big(I - S^*_\Pi S_\Pi\big) \\ &= \widetilde{B}_\Pi\big(\big(S^*_\Pi\big)^m(S_\Pi)^n - \big(S^*_\Pi\big)^m(S_\Pi)^{n-1}S_\Pi S^*_\Pi S_\Pi\big) \\ &= \widetilde{B}_\Pi\big(\big(S^*_\Pi\big)^m(S_\Pi)^n - \big(S^*_\Pi\big)^m(S_\Pi)^{n-1}S_\Pi\big) = 0.\end{aligned}$$

If $n = 0$ and $m > 0$, we have

$$\widetilde{B}_\Pi\big(S^*_\Pi\big)^m\widetilde{B}_\Pi = \big(I - S^*_\Pi S_\Pi\big)\big(S^*_\Pi\big)^m\widetilde{B}_\Pi = \big(\big(S^*_\Pi\big)^m - S^*_\Pi S_\Pi S^*_\Pi\big(S^*_\Pi\big)^{m-1}\big)\widetilde{B}_\Pi = 0.$$

On the other hand if $m > n$

$$\widetilde{B}_\Pi (S_\Pi)^m \big(S_\Pi^*\big)^n \widetilde{B}_\Pi = \widetilde{B}_\Pi (S_\Pi)^{m-n} (S_\Pi)^n \big(S_\Pi^*\big)^n \widetilde{B}_\Pi.$$

According to Theorem 3.1, $(S_\Pi)^n (S_\Pi^*)^n = I$ when it is restricted to $\widetilde{\mathcal{A}}^2(\Pi)$. Then,

$$\begin{aligned}\widetilde{B}_\Pi (S_\Pi)^m \big(S_\Pi^*\big)^n \widetilde{B}_\Pi &= \widetilde{B}_\Pi (S_\Pi)^{m-n} \widetilde{B}_\Pi = \widetilde{B}_\Pi (S_\Pi)^{m-n} \big(I - S_\Pi^* S_\Pi\big)\\ &= \widetilde{B}_\Pi \big((S_\Pi)^{m-n} - (S_\Pi)^{m-n-1} S_\Pi S_\Pi^* S_\Pi\big)\\ &= \widetilde{B}_\Pi \big((S_\Pi)^{m-n} - (S_\Pi)^{m-n-1} S_\Pi\big) = 0.\end{aligned}$$

If $m = n$, then $\widetilde{B}_\Pi (S_\Pi)^n (S_\Pi^*)^n \widetilde{B}_\Pi = \widetilde{B}_\Pi I \widetilde{B}_\Pi = \widetilde{B}_\Pi$.

The proofs of (1), (2), (3) and (4) are quite similar. We prove here only (1). To do so we use Theorem 3.1 and the representations $B_\Pi = I - S_\Pi S_\Pi^*$, $\widetilde{B}_\Pi = I - S_\Pi^* S_\Pi$ to get, in the case $n > 0$:

$$\begin{aligned}\widetilde{B}_\Pi (S_\Pi)^m \big(S_\Pi^*\big)^n B_\Pi &= \widetilde{B}_\Pi (S_\Pi)^m \big(S_\Pi^*\big)^n \big(I - S_\Pi S_\Pi^*\big)\\ &= \widetilde{B}_\Pi (S_\Pi)^m \big(\big(S_\Pi^*\big)^n - \big(S_\Pi^*\big)^{n-1} S_\Pi^* S_\Pi S_\Pi^*\big)\\ &= \widetilde{B}_\Pi (S_\Pi)^m \big(\big(S_\Pi^*\big)^n - \big(S_\Pi^*\big)^n\big) = 0.\end{aligned}$$

For $n = 0$ it is enough to observe that the image of the operator $(S_\Pi)^m B_\Pi$ is contained in $\mathcal{A}^2_{(m+1)}(\Pi)$, which is orthogonal to $\widetilde{\mathcal{A}}^2(\Pi)$. □

4 Toeplitz Operators on the Harmonic Bergman Space on the Upper Half Plane

The harmonic Bergman space $b^2(\Pi)$ is the closed subspace of $L_2(\Pi)$ which consists of all complex-valued functions $f(z) = u(z) + i v(z)$, with u and v being real-valued harmonic functions.

Since $\mathcal{A}^2(\Pi)$ and $\widetilde{\mathcal{A}}^2(\Pi)$ are mutually orthogonal spaces, we have that $b^2(\Pi) = \mathcal{A}^2(\Pi) \oplus \widetilde{\mathcal{A}}^2(\Pi)$ (see [6] for details), and that the orthogonal projection Q from $L_2(\Pi)$ onto $b^2(\Pi)$ is of the form

$$Q = B_\Pi + \widetilde{B}_\Pi.$$

A Toeplitz operator $\widehat{T}_A$, with symbol $A \in \mathcal{R}(\mathbb{C}, S_\Pi, S_\Pi^*)$, is defined on $b^2(\Pi)$ by the standard formula,

$$\widehat{T}_A = Q_\Pi A Q_\Pi = (B_\Pi + \widetilde{B}_\Pi) A (B_\Pi + \widetilde{B}_\Pi).$$

Thus, with respect to the decomposition $b^2(\Pi) = \mathcal{A}^2(\Pi) \oplus \widetilde{\mathcal{A}}^2(\Pi)$, it has the following matrix form

$$\widehat{T}_A = \begin{pmatrix} B_\Pi A B_\Pi & B_\Pi A \widetilde{B}_\Pi \\ \widetilde{B}_\Pi A B_\Pi & \widetilde{B}_\Pi A \widetilde{B}_\Pi \end{pmatrix}.$$

4.1 Toeplitz Operators on the Harmonic Bergman Space on the Upper Half Plane with Defining Symbol in $\mathcal{R}(\mathbb{C}, S_\Pi, S^*_\Pi)$

In this section we analyze the interaction between S_Π, S^*_Π and the harmonic Bergman space on the upper half plane.

Lemma 4.1 *Given $k_1, \dots, k_N, n_1, \dots, n_N \in \mathbb{Z}_+$ such that*

$$\sum_{i=1}^{m} n_i \le \sum_{i=1}^{m} k_i \quad \textit{for } m = 1, \dots, N-1, \tag{4.1}$$

$$\sum_{i=1}^{N} n_i = \sum_{i=1}^{N} k_i, \tag{4.2}$$

then

1. *there exists $s \in \mathbb{Z}_+$ such that*

$$\left(S^*_\Pi\right)^{n_N} (S_\Pi)^{k_N} \dots \left(S^*_\Pi\right)^{n_1} (S_\Pi)^{k_1} = \left(S^*_\Pi\right)^{s} (S_\Pi)^{s},$$

2. *there exists $s \in \mathbb{Z}_+$ such that*

$$(S_\Pi)^{n_N} \left(S^*_\Pi\right)^{k_N} \dots (S_\Pi)^{n_1} \left(S^*_\Pi\right)^{k_1} = (S_\Pi)^{s} \left(S^*_\Pi\right)^{s}.$$

Proof The proof of (1) was already done in [7]. To prove (2) observe that, for $n_1, \dots, n_N$ and $k_1, \dots, k_N$ such that (4.1) and (4.2) hold, there exists $j \in \{1, \dots, N-1\}$ such that

(a) $k_j \ge n_j$ and $k_{j+1} \ge n_j$ or
(b) $n_j \ge k_{j+1}$ and $n_{j+1} \ge k_{j+1}$.

If (a) holds then

$$\begin{aligned}
&(S_\Pi)^{n_N} \left(S^*_\Pi\right)^{k_N} \dots (S_\Pi)^{n_1} \left(S^*_\Pi\right)^{k_1} \\
&\quad = (S_\Pi)^{n_N} \left(S^*_\Pi\right)^{k_N} \dots (S_\Pi)^{n_{j+1}} \left(S^*_\Pi\right)^{k_{j+1}} \\
&\qquad (S_\Pi)^{n_j} \left(S^*_\Pi\right)^{k_j} \dots (S_\Pi)^{n_1} \left(S^*_\Pi\right)^{k_1} \\
&\quad = (S_\Pi)^{n_N} \left(S^*_\Pi\right)^{k_N} \dots (S_\Pi)^{n_{j+1}} \left(S^*_\Pi\right)^{k_{j+1}-n_j} \\
&\qquad \left(S^*_\Pi\right)^{n_j} (S_\Pi)^{n_j} \left(S^*_\Pi\right)^{n_j} \left(S^*_\Pi\right)^{k_j-n_j} \dots (S_\Pi)^{n_1} \left(S^*_\Pi\right)^{k_1} \\
&\quad = (S_\Pi)^{n_N} \left(S^*_\Pi\right)^{k_N} \dots (S_\Pi)^{n_{j+1}} \left(S^*_\Pi\right)^{k_{j+1}+k_j-n_j} \\
&\qquad (S_\Pi)^{n_{j-1}} \left(S^*_\Pi\right)^{k_{j-1}} \dots (S_\Pi)^{n_1} \left(S^*_\Pi\right)^{k_1}.
\end{aligned}$$

If (b) holds then

$$\begin{aligned}
&(S_\Pi)^{n_N}\left(S_\Pi^*\right)^{k_N}\dots(S_\Pi)^{n_1}\left(S_\Pi^*\right)^{k_1}\\
&\quad=(S_\Pi)^{n_N}\left(S_\Pi^*\right)^{k_N}\dots(S_\Pi)^{n_{j+1}}\left(S_\Pi^*\right)^{k_{j+1}}(S_\Pi)^{n_j}\left(S_\Pi^*\right)^{k_j}\dots(S_\Pi)^{n_1}\left(S_\Pi^*\right)^{k_1}\\
&\quad=(S_\Pi)^{n_N}\left(S_\Pi^*\right)^{k_N}\dots(S_\Pi)^{n_{j+1}-k_{j+1}}(S_\Pi)^{k_{j+1}}\left(S_\Pi^*\right)^{k_{j+1}}(S_\Pi)^{k_{j+1}}\\
&\qquad(S_\Pi)^{n_j-k_{j+1}}\dots(S_\Pi)^{n_1}\left(S_\Pi^*\right)^{k_1}\\
&\quad=(S_\Pi)^{n_N}\left(S_\Pi^*\right)^{k_N}\dots(S_\Pi)^{n_{j+1}+n_j-k_{j+1}}\left(S_\Pi^*\right)^{k_j}\dots(S_\Pi)^{n_1}\left(S_\Pi^*\right)^{k_1}.
\end{aligned}$$

Observe that in both cases, (a) and (b), we can apply the above argument inductively $N-1$ times. In each step we use the new family of indexes gotten in the preceding step. □

Given a multi-index $J=(n_1,k_1,\dots,n_N,k_N)$, where $n_i,k_i\in\mathbb{Z}_+$, the non-commutative monomial $m_J(x,y)$ is defined by

$$m_J(x,y)=y^{n_N}x^{k_N}\dots y^{n_1}x^{k_1},$$

and set its degree by

$$\deg m_J=|J|=n_N+k_N+\dots+n_1+k_1.$$

Denote by $I_{0,0}$ the set of indexes $J=(n_1,k_1,\dots,n_N,k_N)$ that satisfy conditions (4.1) and (4.2) and such that the monomial $m_J(S_\Pi,S_\Pi^*)$ starts with S_Π. Denote by $I_{0,0}^*$ the set of indexes $J=(n_1,k_1,\dots,n_N,k_N)$ that satisfy conditions (4.1) and (4.2) and such that the monomial $m_J(S_\Pi,S_\Pi^*)$ starts with S_Π^*.

Corollary 4.2 *Let $m_J(x,y)$ be a non-commutative monomial, then*

$$B_\Pi m_J\left(S_\Pi,S_\Pi^*\right)B_\Pi=\begin{cases}B_\Pi, & \text{if } J\in I_{0,0}^*,\\ 0, & \text{otherwise},\end{cases}$$

$$\widetilde{B}_\Pi m_J\left(S_\Pi,S_\Pi^*\right)\widetilde{B}_\Pi=\begin{cases}\widetilde{B}_\Pi, & \text{if } J\in I_{0,0},\\ 0, & \text{otherwise}.\end{cases}$$

Furthermore,

$$B_\Pi m_J\left(S_\Pi,S_\Pi^*\right)\widetilde{B}_\Pi=0,\ \widetilde{B}_\Pi m_J\left(S_\Pi,S_\Pi^*\right)B_\Pi=0.$$

Proof Follows from Lemma 4.1 and Corollary 3.2. □

The following lemma is an immediate consequence of last corollary.

Lemma 4.3 *Let $P(x, y)$ be a non-commutative polynomial of degree k*

$$P(x, y) = \sum_{|J|\le k} a_J m_J(x, y),$$

where $a_J \in \mathbb{C}$. Then

$$B_\Pi P\big(S_\Pi, S_\Pi^*\big) B_\Pi = b_p B_\Pi,$$

where $b_p = \sum_{|J|\le k, J\in I_{0,0}^*} a_J$.

Analogously,

$$\widetilde{B}_\Pi P\big(S_\Pi, S_\Pi^*\big) \widetilde{B}_\Pi = \widetilde{b}_p \widetilde{B}_\Pi,$$

where $\widetilde{b}_p = \sum_{|J|\le k, J\in I_{0,0}} a_J$.

Furthermore,

$$B_\Pi P\big(S_\Pi, S_\Pi^*\big) \widetilde{B}_\Pi = 0, \qquad \widetilde{B}_\Pi P\big(S_\Pi, S_\Pi^*\big) B_\Pi = 0.$$

Theorem 4.4 *Let A be an element of $\mathcal{R}(\mathbb{C}, S_\Pi, S_\Pi^*)$. Then the Toeplitz operator $\widehat{T}_A$, acting on $b^2(\Pi)$, is equal to*

$$b_A B_\Pi + \widetilde{b}_A \widetilde{B}_\Pi,$$

where $b_A = \langle Af_0, f_0\rangle$, $\widetilde{b}_A = \langle Af_1, f_1\rangle$, and $f_0 \in \mathcal{A}^2(\Pi)$, $f_1 \in \widetilde{\mathcal{A}}^2(\Pi)$ are such that $\|f_0\| = \|f_1\| = 1$.

Proof The set of noncommutative polynomials $P(S_\Pi, S_\Pi^*)$ is dense in the algebra $\mathcal{R}(\mathbb{C}, S_\Pi, S_\Pi^*)$. Then

$$\begin{aligned}
\widehat{T}_{P(S_\Pi, S_\Pi^*)} &= (B_\Pi + \widetilde{B}_\Pi) P\big(S_\Pi, S_\Pi^*\big)(B_\Pi + \widetilde{B}_\Pi) \\
&= B_\Pi P\big(S_\Pi, S_\Pi^*\big) B_\Pi + B_\Pi P\big(S_\Pi, S_\Pi^*\big) \widetilde{B}_\Pi \\
&\quad + \widetilde{B}_\Pi P\big(S_\Pi, S_\Pi^*\big) \widetilde{B}_\Pi + \widetilde{B}_\Pi P\big(S_\Pi, S_\Pi^*\big) B_\Pi.
\end{aligned}$$

Using Lemma 4.3 we get

$$\widehat{T}_{P(S_\Pi, S_\Pi^*)} = b_p B_\Pi + \widetilde{b}_p \widetilde{B}_\Pi.$$

On the other hand

$$\begin{aligned}
b_p &= \langle b_p B_\Pi f_0, f_0\rangle = \langle T_{P(S_\Pi, S_\Pi^*)} f_0, f_0\rangle = \big\langle B_\Pi P\big(S_\Pi, S_\Pi^*\big) B_\Pi f_0, f_0\big\rangle \\
&= \big\langle P\big(S_\Pi, S_\Pi^*\big) f_0, f_0\big\rangle.
\end{aligned}$$

Analogously,

$$\widetilde{b}_p = \big\langle P\big(S_\Pi, S_\Pi^*\big) f_1, f_1\big\rangle.$$

Since the functionals

$$A \longmapsto \langle Af_0, f_0 \rangle \quad \text{and} \quad A \longmapsto \langle Af_1, f_1 \rangle$$

are continuous on $\mathcal{R}(\mathbb{C}, S_\Pi, S_\Pi^*)$, the result follows. □

The next theorem is the main result of this section.

Theorem 4.5 *Given a symbol $A \in \mathcal{R}(\mathbb{C}, S_\Pi, S_\Pi^*)$, the Toeplitz operator $\widehat{T}_A$, acting on the harmonic Bergman space $b^2(\Pi)$, is represented, with respect to the decomposition $b^2(\Pi) = \mathcal{A}^2(\Pi) \oplus \widetilde{\mathcal{A}}^2(\Pi)$, by the diagonal operator*

$$\widehat{T}_A = \begin{pmatrix} T_A & 0 \\ 0 & \widetilde{T}_A \end{pmatrix} = \begin{pmatrix} b_A I & 0 \\ 0 & \widetilde{b}_A I \end{pmatrix}.$$

Here $T_A : \mathcal{A}^2(\Pi) \to \mathcal{A}^2(\Pi)$ is the Toeplitz operator with pseudodifferential symbol A and $\widetilde{T}_A : \widetilde{\mathcal{A}}^2(\Pi) \to \widetilde{\mathcal{A}}^2(\Pi)$ is the corresponding Toeplitz operator acting on the anti-Bergman space $\widetilde{\mathcal{A}}^2(\Pi)$.

Proof Recall that the harmonic Bergman projection has the form $Q_\Pi = B_\Pi + \widetilde{B}_\Pi$, thus

$$\begin{aligned} \widehat{T}_A &= Q_\Pi A Q_\Pi = (B_\Pi + \widetilde{B}_\Pi) A (B_\Pi + \widetilde{B}_\Pi) \\ &= B_\Pi A B_\Pi + \widetilde{B}_\Pi A \widetilde{B}_\Pi + B_\Pi A \widetilde{B}_\Pi + \widetilde{B}_\Pi A B_\Pi. \end{aligned}$$

Using Corollary 3.2, we get the result for every polynomial $P(S_\Pi, S_\Pi^*)$. For a general case we only need to approximate each element in $\mathcal{R}(\mathbb{C}, S_\Pi, S_\Pi^*)$ by noncommutative polynomials. □

Corollary 4.6 *The C^*-algebra $\widehat{\mathcal{T}}(\mathcal{R}(\mathbb{C}, S_\Pi, S_\Pi^*))$ is isomorphic to $\mathbb{C} \oplus \mathbb{C}$ with the isomorphism defined on the generators by*:

$$\widehat{T}_A \to (b_A, \widetilde{b}_A).$$

To give alternative, to those of Theorem 4.4, formulas for b_A and $\widetilde{b}_A$ we proceed as follows. Theorems 3.1, 3.2, and Remark 3.3 of [10] stay that both operators S_Π and S_Π^* are unitary equivalent to the direct sum of two unilateral shifts, forward and backward, both taken with the infinite multiplicity. This implies [1] the isometric isomorphism of the next C^*-algebras

$$\kappa : \mathcal{R}\big(\mathbb{C}, S_\Pi, S_\Pi^*\big) \longrightarrow \mathcal{T}\big(C(\mathbb{T})\big) \oplus \mathcal{T}\big(C(\mathbb{T})\big),$$

which is generated by the following assignment

$$\kappa(S_\Pi) = (T_t, T_{\bar{t}}), \qquad \kappa\big(S_\Pi^*\big) = (T_{\bar{t}}, T_t).$$

Recall, that $\mathcal{T}(C(\mathbb{T}))$ denotes the C^*-algebra generated by all Toeplitz operators T_a with continuous defining symbols $a(t) \in C(\mathbb{T})$ and acting on the Hardy space $H^2(\mathbb{T})$. We use the standard normalized measure on $\mathbb{T}$, so that the system of functions $\{t^k\}_{k\in\mathbb{Z}_+}$ forms an orthonormal basis in $H^2(\mathbb{T})$.

The Bergman projection $B_\Pi = I - S_\Pi S_\Pi^*$ and the anti-Bergman projection $\widetilde{B}_\Pi = I - S_\Pi^* S_\Pi$ belong to the algebra $\mathcal{R}(\mathbb{C}, S_\Pi, S_\Pi^*)$, and

$$\kappa(B_\Pi) = (I - T_t T_{\bar{t}}, 0), \qquad \kappa(\widetilde{B}_\Pi) = (0, I - T_t T_{\bar{t}}),$$

with $I - T_t T_{\bar{t}}$ being the one-dimensional projection of $H^2(\mathbb{T})$ onto the subspace generated by 1. The next alternative formulas for b_A and $\widetilde{b}_A$ thus follow

$$b_A = \langle \sigma_0(A)1, 1\rangle, \qquad \widetilde{b}_A = \langle \sigma_\infty(A)1, 1\rangle,$$

where $(\sigma_0(A), \sigma_\infty(A)) = \kappa(A)$.

5 The Algebra $\widehat{\mathcal{T}}(\mathcal{R}(C(\overline{\mathbb{D}}); S_\mathbb{D}, S_\mathbb{D}^*))$

Consider now the C^*-algebra $\widehat{\mathcal{T}}(\mathcal{R}(C(\overline{\mathbb{D}}), S_\mathbb{D}, S_\mathbb{D}^*))$ generated by Toeplitz operators $\widehat{T}_A$, with $A \in \mathcal{R}(C(\overline{\mathbb{D}}), S_\mathbb{D}, S_\mathbb{D}^*)$, and acting on the harmonic Bergman space $b^2(\mathbb{D})$ on the unit disk.

Given an operator $A \in \mathcal{R}(C(\overline{\mathbb{D}}), S_\mathbb{D}, S_\mathbb{D}^*)$, consider its Fredholm symbol (see Theorem 2.1)

$$\operatorname{sym} A = \begin{cases} c(z,t) \in \mathbb{C}, & (z,t) \in \overline{\mathbb{D}} \times S^1, \\ \sigma_0(A,z) = T_{c(z,t)} + K_0(z) \in \mathcal{T}(C(S^1)), & (z,0) \in S^1 \times \{0,\infty\}, \\ \sigma_\infty(A,z) = T_{c(z,\bar{t})} + K_\infty(z) \in \mathcal{T}(C(S^1)), & (z,\infty) \in S^1 \times \{0,\infty\}. \end{cases}$$

Theorem 5.1 *The Fredholm symbol algebra*

$$\operatorname{Sym} \widehat{\mathcal{T}}\big(\mathcal{R}\big(C(\overline{\mathbb{D}}), S_\mathbb{D}, S_\mathbb{D}^*\big)\big) = \widehat{\mathcal{T}}\big(\mathcal{R}\big(C(\overline{\mathbb{D}}), S_\mathbb{D}, S_\mathbb{D}^*\big)\big)/\mathcal{K}$$

of the algebra $\widehat{\mathcal{T}}(\mathcal{R}(C(\overline{\mathbb{D}}), S_\mathbb{D}, S_\mathbb{D}^*))$ is isomorphic and isometric to $C(\mathbb{T}) \oplus C(\mathbb{T})$. The isomorphism is given on the generators by the following formula: for any $A \in \mathcal{R}(C(\overline{\mathbb{D}}), S_\mathbb{D}, S_\mathbb{D}^*)$,

$$\begin{aligned} \operatorname{sym} \widehat{T}_A &= \big(b_A(z), \widetilde{b}_A(z)\big) \\ &= \big(\langle \sigma_0(A,z)1, 1\rangle_{H^2(\mathbb{T})}, \langle \sigma_\infty(A,z)1, 1\rangle_{H^2(\mathbb{T})}\big). \end{aligned} \tag{5.1}$$

Proof The algebra $\widehat{\mathcal{T}}(\mathcal{R}(C(\overline{\mathbb{D}}), S_\mathbb{D}, S_\mathbb{D}^*))$ contains the ideal $\mathcal{K}$ of all compact operators. To describe its Calkin algebra we use the standard Douglas–Varela local principle [2, 8] with $\widehat{\mathcal{T}}(C(\overline{\mathbb{D}}))/\mathcal{K} \cong C(\mathbb{T})$ as a central commutative subalgebra of

$\widehat{\mathcal{T}}(\mathcal{R}(C(\overline{\mathbb{D}}), S_{\mathbb{D}}, S_{\mathbb{D}}^*))/\mathcal{K}$. We localize thus by points $z_0 \in \mathbb{T}$. Given $z_0 \in \mathbb{T}$, introduce the unitary operator $V_{z_0} : L_2(\Pi) \to L_2(\mathbb{D})$,

$$(V_{z_0}\varphi)(z) = \frac{2iz_0}{(z_0+z)^2}\varphi\left(i\frac{z_0-z}{z_0+z}\right).$$

It is easy to check that

$$V_{z_0}^{-1} S_{\mathbb{D}} V_{z_0} = (i\overline{z_0})^2 S_\Pi h(w) I, \qquad V_{z_0}^{-1} S_{\mathbb{D}}^* V_{z_0} = (iz_0)^2 \overline{h(w)} S_\Pi^*,$$

where $h(w) = \frac{(i+w)^4}{|i+w|^4}$. As $h(0) = 1$ we have that $V_{z_0}^{-1} S_{\mathbb{D}} V_{z_0}$ is locally equivalent at the point 0 to $(i\overline{z_0})^2 S_\Pi$ and $V_{z_0}^{-1} S_{\mathbb{D}}^* V_{z_0}$ is locally equivalent at the point 0 to $(iz_0)^2 S_\Pi^*$. The operator $V_{z_0}^{-1} a(z) V_{z_0}$ is locally equivalent at the point 0, to the operator $a(z_0)I$. By Corollary 2.3 of [7] the operators $(i\overline{z_0})^2 S_\Pi$ and $(iz_0)^2 S_\Pi^*$ are also unitary equivalent to the direct sum of two unilateral shifts, forward and backward, both taken with the infinite multiplicity. Thus the arguments of the last part of the previous section imply the theorem assertion. □

Let T be an arbitrary operator from $\widehat{\mathcal{T}}(\mathcal{R}(C(\overline{\mathbb{D}}), S_{\mathbb{D}}, S_{\mathbb{D}}^*))$, and let $\operatorname{sym} T = (b(z), \widetilde{b}(z)) \in C(\mathbb{T}) \oplus C(\mathbb{T})$ be its Fredholm symbol. In particular, if $T = \widehat{T}_A$ for some $A \in \mathcal{R}(C(\overline{\mathbb{D}}), S_{\mathbb{D}}, S_{\mathbb{D}}^*)$, then its Fredholm symbol is given by (5.1). We extend then the functions $b(z)$ and $\widetilde{b}(z)$, being continuous on $\mathbb{T}$, to functions $b^\#(z)$ and $\widetilde{b}^\#(z)$ that are continuous on the closed unit disk $\overline{\mathbb{D}}$. Then, with respect to the representation $b^2(\mathbb{D}) = \mathcal{A}^2(\mathbb{D}) + \widetilde{\mathcal{A}}^2(\mathbb{D})$, we have

$$T = \begin{pmatrix} T_{b^\#} & 0 \\ 0 & \widetilde{T}_{\widetilde{b}^\#} \end{pmatrix} + K,$$

where K is a compact operator.

Corollary 5.2 *With respect to the representation $b^2(\mathbb{D}) = \mathcal{A}^2(\mathbb{D}) + \widetilde{\mathcal{A}}^2(\mathbb{D})$, the algebra $\widehat{\mathcal{T}}(\mathcal{R}(C(\overline{\mathbb{D}}), S_{\mathbb{D}}, S_{\mathbb{D}}^*))$ coincides with the next matrix algebra*

$$\begin{pmatrix} \mathcal{T}(C(\overline{\mathbb{D}})) & \mathcal{K}_{1,2} \\ \mathcal{K}_{2,1} & \widetilde{\mathcal{T}}(C(\overline{\mathbb{D}})) \end{pmatrix}.$$

Here $\mathcal{T}(C(\overline{\mathbb{D}}))$ denotes the C^*-algebra generated by all Toeplitz operators with continuous symbol acting on $\mathcal{A}^2(\mathbb{D})$, $\widetilde{\mathcal{T}}(C(\overline{\mathbb{D}}))$ denotes the C^*-algebra generated by all Toeplitz operators with continuous symbol acting on $\widetilde{\mathcal{A}}^2(\mathbb{D})$, $\mathcal{K}_{1,2}$ is a set of compact operators from $\widetilde{\mathcal{A}}^2(\mathbb{D})$ to $\mathcal{A}^2(\mathbb{D})$, and $\mathcal{K}_{2,1}$ is a set of compact operators from $\mathcal{A}^2(\mathbb{D})$ to $\widetilde{\mathcal{A}}^2(\mathbb{D})$.

The proof of the next corollary follows from the formula for the index of a Fredholm Toeplitz operator acting on the Bergman space and from the fact that every Fredholm Toeplitz operator with continuous symbol, acting on the harmonic Bergman space, has index equal to zero.

Corollary 5.3 *An operator* $T \in \widehat{\mathcal{T}}(\mathcal{R}(C(\overline{\mathbb{D}}), S_{\mathbb{D}}, S_{\mathbb{D}}^*))$ *is Fredholm if and only if its Fredholm symbol* $\operatorname{sym} T = (b(z), \widetilde{b}(z))$ *does not vanish, i.e.,* $b(z) \neq 0, \widetilde{b}(z) \neq 0,$ *for all* $z \in \mathbb{T}$*. The index of a Fredholm operator* T *is given by*

$$\operatorname{Ind} T = \frac{1}{2\pi} \arg \left\{ \frac{\widetilde{b}(z)}{b(z)} \right\}_{\mathbb{T}}.$$

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Theorems of Paley–Wiener Type for Spaces of Polyanalytic Functions

Luís V. Pessoa and Ana Moura Santos

Abstract We prove Paley–Wiener theorems for the true poly-Bergman and poly-Bergman spaces based on properties of the compression of the Beurling–Ahlfors transform to the upper half-plane. An isometric isomorphism between j copies of the Hardy space and the poly-Bergman space of order j is constructed.

Keywords Beurling–Ahlfors transform · Paley–Wiener theorems · Polyanalytic functions · Poly-Bergman space

Mathematics Subject Classification (2010) Primary 46E22 · 32A25 · Secondary 31A10

1 Introduction

The so-called classical Paley–Wiener Theorem states that the complex Fourier transform defines an isometric isomorphism between the L^2 Lebesgue space on the positive real line $L^2(\mathbb{R}^+)$ and the Hardy space over the upper half-plane. N. Vasilevski in [11, Theorem 2.4] proved a Paley–Wiener type theorem for the Bergman space by showing that the composition of the complex Fourier transform and the multiplication operator by the square root defined on $\mathbb{R}^+$ gives a unitary operator from $L^2(\mathbb{R}^+)$ onto the Bergman space over the upper half-plane. More recently, in [2, Theorem 1] P. Duren et al. proved a Paley–Wiener theorem for weighted Bergman spaces, and L. Abreu in [1] showed that the poly-Bergman space of order j over the upper half-plane is isometrically isomorphic to j copies of the Hardy space. This last result could be easily obtained from [11, Theorem 4.5], even if the fruitful techniques of this paper were not applied to construct isometric isomorphisms between $L^2(\mathbb{R}^+)$ and true poly-Bergman or poly-Bergman spaces.

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In the present paper, we will establish Paley–Wiener theorems for the true poly-Bergman and poly-Bergman spaces based on properties of the compression S_Π of the Beurling–Ahlfors transform to the upper half-plane. We will prove that the action of the singular integral operator S_Π on the Bergman space coincides with a differential operator. This allows us to construct an isometric isomorphism from $L^2(\mathbb{R}^+)$ onto the true poly-Bergman and onto the poly-Bergman spaces, where its images can be represented as sums of both y^k and Laguerre polynomials $L_k(y)$, for $y := \operatorname{Im} z$ and $k = 0, 1, \dots$, with analytic coefficients given in terms of the complex Fourier transform. From the classical Paley–Wiener theorem it then follows that the poly-Bergman space of order j is isometrically isomorphic to j copies of the Hardy space.

2 The Paley–Wiener Theorem for Bergman Spaces

The Hardy space over the upper half-plane $\Pi := \{z : \operatorname{Im} z > 0\}$ is a Hilbert space of functions denoted here by $\mathcal{A}^2_\partial(\Pi)$, that consists of those analytic functions on Π which have finite norm given by

$$\|f\|^2_\partial = \sup_{y>0} \int_{\mathbb{R}} \bigl|f(x+iy)\bigr|^2 dx.$$

The classical isomorphism between the one-dimensional Lebesgue space and the Hardy space over Π is given by the complex Fourier transform

$$F^c : L^2\bigl(\mathbb{R}^+, dt\bigr) \to \mathcal{A}^2_\partial(\Pi), \qquad F^c f(z) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} f(t) e^{izt} dt. \tag{2.1}$$

Here, $L^2(\mathbb{R}^+, dt)$ or simpler $L^2(\mathbb{R}^+)$ denote the one-dimensional Lebesgue space on the positive real line $\mathbb{R}^+$. The unitary character of the operator in (2.1) is usually known as the Paley–Wiener theorem for Hardy spaces.

The Bergman space $\mathcal{A}^2(\Pi, dA_\alpha)$ consists of those analytic functions on Π which also belong to the Lebesgue space $L^2(\Pi, dA_\alpha)$ endowed with the weighted measure $dA_\alpha(z) := y^\alpha dA(z)$, where $z := x + iy$ are Cartesian coordinates and $dA(z) := dxdy$ denotes the area Lebesgue measure. In the case $\alpha = 0$, the unweighted Bergman space is usually denoted by $\mathcal{A}^2(\Pi)$. In this paper, the space $\bar{\mathcal{A}}^2(\Pi)$ represents the unweighted anti-Bergman space of integrable anti-analytic functions on Π. It is well known that $\mathcal{A}^2(\Pi, dA_\alpha)$, for $\alpha > -1$, is a Hilbert space endowed with the inner product induced by that of $L^2(\Pi, dA_\alpha)$. The following result can be called a Paley–Wiener type theorem for the weighted Bergman space.

Theorem 2.1 [2, Theorem 1] *For $\alpha > -1$, the complex Fourier transform*

$$F^c_\alpha f(z) = \frac{2^{\alpha/2}}{\sqrt{\pi \Gamma(\alpha+1)}} \int_0^{+\infty} f(t) e^{izt} dt, \quad z \in \Pi,$$

defines an isometric isomorphism from $L^2(\mathbb{R}^+, dt/t^{\alpha+1})$ onto $\mathcal{A}^2(\Pi, dA_\alpha)$.

It is clear that the following operator is a unitary operator

$$T : L^2(\mathbb{R}^+, dt) \to L^2(\mathbb{R}^+, dt/t), \qquad Tg(t) = \sqrt{t}g(t).$$

Hence, from this fact and Theorem 2.1, we prove that two related operators, one for the Bergman (see, also [11, Theorem 2.4]) and the other for the anti-Bergman case, are unitary.

Theorem 2.2 (see, also [11, Theorem 2.4]) *The following operators are unitary operators*

$$R : L^2(\mathbb{R}^+, dt) \to \mathcal{A}^2(\Pi), \qquad Rf(z) = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \sqrt{t} f(t) e^{izt} dt, \tag{2.2}$$

$$\widetilde{R} : L^2(\mathbb{R}^+, dt) \to \bar{\mathcal{A}}^2(\Pi), \qquad \widetilde{R}f(z) = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \sqrt{t} f(t) e^{-i\bar{z}t} dt. \tag{2.3}$$

Proof It is clear that $R = F_0^c T$. Then the fact that the operator (2.2) is unitary follows straightforward. On the other hand, observe that the unitary operator

$$V : L^2(\Pi) \to L^2(\Pi), \qquad Vf(z) = f(-\overline{z}) \tag{2.4}$$

transforms $\mathcal{A}^2(\Pi)$ onto $\bar{\mathcal{A}}^2(\Pi)$. Since $\widetilde{R} = VR$, this concludes the proof. □

3 A Paley–Wiener Theorem for the True Poly-Bergman Space

Let j be a nonzero integer. Then a complex smooth function f defined on a domain $U \subset \mathbb{C}$ (non-empty, bounded and connected) and satisfying

$$\partial_{\bar{z}}^{j} f = 0, \quad j > 0 \quad \text{or} \quad \partial_z^{-j} f = 0, \quad j < 0,$$

respectively, is said to be a j-analytic function on U. The poly-Bergman space $\mathcal{A}_j^2(U)$ consists of j-analytic functions on U which also belong to the Lebesgue space $L^2(U, dA)$. It is clear that $f \in \mathcal{A}_{-j}^2(U)$ if and only if $\bar{f} \in \mathcal{A}_j^2(U)$. Therefore, the space $\mathcal{A}_{-j}^2(U)$ will also here be denoted by $\bar{\mathcal{A}}_j^2(U)$. It is known that evaluation functionals on $\mathcal{A}_j^2(U)$ are uniformly bounded within U (see, e.g. [5, 7]). Then, it easily follows that poly-Bergman spaces are Hilbert spaces. Moreover, based on [7, Proposition 2.3] we know that, for $n, m = 0, 1, \ldots$ and $z \in U$, one has

$$\left|\partial_z^n \partial_{\bar{z}}^m f(z)\right| \le \frac{M}{d_z^{n+m+1}} \|f\|, \quad f \in \mathcal{A}_j^2(U), \tag{3.1}$$

where $M > 0$ is a constant only depending on n, m and j, and d_z is defined as the distance $d_z = \text{dist}(z, \partial U)$.

Let S denote the (unitary) Beurling–Ahlfors transform (see, e.g. [6]), i.e. S is defined to be the following two-dimensional singular integral operator

$$Sf(z) := -\frac{1}{\pi}\int_{\mathbb{C}} \frac{f(w)}{(w-z)^2} dA(w), \quad f \in L^2(\mathbb{C}).$$

The compression of the Beurling–Ahlfors transform to $L^2(\Pi)$ is denoted by S_Π, i.e. $S_\Pi := \chi_\Pi S \chi_\Pi I$, where $\chi_\Pi I$ is the multiplication operator by the characteristic function χ_Π of Π. It is known that the poly-Bergman projection, which is defined to be the orthogonal projection of $L^2(\Pi)$ onto $\mathcal{A}_j^2(\Pi)$, is related to the singular integral operator S_Π through the so-called Dzhuraev's formulas. More precisely, if j is a positive integer, then we know (see [8] and [12]) that $(S_\Pi)^j$ is a partial isometry with initial and final spaces given by the orthogonal spaces of $\bar{\mathcal{A}}_j^2(\Pi)$ and of $\mathcal{A}_j^2(\Pi)$, respectively. Based on this fact, the following result (see Theorem 3.1 below) on unitary operators acting between the Bergman and the true poly-Bergman spaces was established in [3] by the first author jointly with Yu.I. Karlovich. N. Vasilevski gave a different proof in [12] based on methods that can be found in [11] (see also [10]). The true poly-Bergman spaces $\mathcal{A}_{(j)}^2(\Pi)$ were introduced by N. Vasilevski in [11] and are defined by

$$\mathcal{A}_{(\pm 1)}^2(\Pi) := \mathcal{A}_{\pm 1}^2(\Pi) \quad \text{and} \quad \mathcal{A}_{(j)}^2(\Pi) := \mathcal{A}_j^2(\Pi) \ominus \mathcal{A}_{j-\operatorname{sgn} j}^2(\Pi), \quad |j| > 1.$$

Clearly $f \in \mathcal{A}_{(-j)}^2(U)$ if and only if $\bar{f} \in \mathcal{A}_{(j)}^2(U)$. Hence, as for the poly-Bergman space, the space $\mathcal{A}_{(-j)}^2(U)$ will also here be denoted by $\bar{\mathcal{A}}_{(j)}^2(U)$.

Theorem 3.1 [3, Theorem 2.4] (see also [12, Theorem 3.5] and [13]) *Let j be a positive integer. Then the following operators are isometric isomorphisms*

$$(S_\Pi)^{j-1} : \mathcal{A}^2(\Pi) \to \mathcal{A}_{(j)}^2(\Pi) \quad \textit{and} \quad \left(S_\Pi^*\right)^{j-1} : \bar{\mathcal{A}}^2(\Pi) \to \bar{\mathcal{A}}_{(j)}^2(\Pi).$$

In the following result we will prove that the natural powers of S_Π acting on the Bergman space coincide with a nice differential operator. For this propose we will use a special property of Π according to which the iterations of S_Π can be given as a singular integral operator with an explicit kernel. For a domain $U \subset \mathbb{C}$ and for a nonzero integer j, we consider the following two-dimensional singular integral operator acting on $L^2(U, dA)$

$$S_{U,j} f(z) := \frac{(-1)^j |j|}{\pi} \int_U \frac{(w-z)^{j-1}}{(\overline{w}-\overline{z})^{j+1}} f(w) dA(w).$$

From [8] (see also [4, Corollary 2.4]) we know that:

$$S_{\Pi,j} = \left(S_\Pi^*\right)^j \quad \text{and} \quad S_{\Pi,-j} = (S_\Pi)^j \quad (j = 1, 2, \ldots). \tag{3.2}$$

Lemma 3.2 *The set of functions which satisfy the following conditions*

$$\psi \in C^\infty(\overline{\Pi}) \cap \mathcal{A}^2(\Pi) \quad \textit{and} \quad \psi(z) = \mathcal{O}\big(1/|z|^2\big), \quad \textit{as } |z| \to +\infty \tag{3.3}$$

is dense in $\mathcal{A}^2(\Pi)$, where $\overline{\Pi}$ denotes the closure of the upper half-plane Π.

Proof Nonzero analytic functions do not admit a zero set with cluster points within Π. From the reproducing property of the Bergman kernel function $K_\Pi(z, w)$ it follows that the orthogonal space to the set $\{K_{\Pi,iy} : y > 1\}$ is the trivial space. Hence, its linear span is dense in $\mathcal{A}^2(\Pi)$. Since

$$K_{\Pi,w}(z) := K_\Pi(z, w) = -\frac{1}{\pi}\frac{1}{(z-\overline{w})^2},$$

for $y > 1$, the functions $K_{\Pi,iy}(z)$ satisfy the conditions in (3.3). □

Theorem 3.3 *Let j be a positive integer. Then it holds that*

$$(S_\Pi)^{j-1}\psi(z) = \frac{\partial_z^{j-1}[(\overline{z}-z)^{j-1}\psi(z)]}{(j-1)!}, \quad \psi \in \mathcal{A}^2(\Pi), \tag{3.4}$$

$$\left(S_\Pi^*\right)^{j-1}\psi(z) = \frac{\partial_{\overline{z}}^{j-1}[(z-\overline{z})^{j-1}\psi(z)]}{(j-1)!}, \quad \psi \in \bar{\mathcal{A}}^2(\Pi). \tag{3.5}$$

Proof If $j = 1$, then the assertions are evident. Then, first we assume that $j = 2, \ldots$ and that $\psi \in \mathcal{A}^2(\Pi)$ satisfies conditions (3.3). Let us define

$$D(z, r, \epsilon) := D(z, r)\backslash\overline{D}(z, \epsilon) \quad \text{and} \quad \Pi(z, r) := \Pi \cap D(z, r, 1/r),$$

where $0 < \epsilon < r$ and $D(z, r)$ denotes the disc centered at $z \in \Pi$ with radius $r > 0$. It is easily seen that the following equalities hold:

$$\lim_{\epsilon\to 0^+}\int_{|z-w|=\epsilon}\frac{(\overline{w}-\overline{z})^{j-1}}{(w-z)^j}\psi(w)dw = \lim_{r\to+\infty}\int_{\Pi\cap\partial D(z,r)}\frac{(\overline{w}-\overline{z})^{j-1}}{(w-z)^j}\psi(w)dw = 0.$$

Hence, from (3.2) together with the well known Green's formula

$$\int_U \partial_{\overline{w}}u(w)dA(w) = \frac{1}{2i}\int_{\partial U}u(w)dw, \quad u \in C^1(\overline{U}),$$

where U is a bounded finitely connected domain with piecewise smooth boundary, we obtain

$$\begin{aligned}(S_\Pi)^{j-1}\psi(z) &= \frac{(-1)^{j-1}}{\pi}\lim_{r\to+\infty}\int_{\Pi(z,r)}\partial_{\overline{w}}\left[\frac{(\overline{w}-\overline{z})^{j-1}\psi(w)}{(w-z)^j}\right]dA(w)\\ &= \frac{1}{2\pi i}\int_{\mathbb{R}}\frac{(\overline{z}-t)^{j-1}\psi(t)}{(t-z)^j}dt = \frac{\partial_z^{j-1}[(\overline{z}-z)^{j-1}\psi(z)]}{(j-1)!}.\end{aligned} \tag{3.6}$$

Let now $\psi \in \mathcal{A}^2(\Pi)$ be arbitrary. From Lemma 3.2 we know that there exists a sequence of functions $\psi_n \in \mathcal{A}^2(\Pi)$ that satisfy conditions (3.3) and such that ψ_n converges in the norm of $L^2(\Pi)$ to ψ. Convergence in the norm implies weak convergence, and by (3.1) we know that the latter implies the uniform convergence of all derivatives within Π. Now, we just apply the last equality in (3.6) to the sequence of functions ψ_n, and easily obtain that

$$(S_\Pi)^{j-1}\psi(z) = \lim_n (S_\Pi)^{j-1}\psi_n(z) = \lim_n \frac{\partial_z^{j-1}[(\overline{z}-z)^{j-1}\psi_n(z)]}{(j-1)!}$$
$$= \frac{\partial_z^{j-1}[(\overline{z}-z)^{j-1}\psi(z)]}{(j-1)!}.$$

We have proved (3.4). For the remaining part we note the following relations

$$\left(S_\Pi^*\right)^{j-1} = V(S_\Pi)^{j-1}V \quad \text{and} \quad \partial_z^{j-1}V\varphi = (-1)^{j-1}V\partial_{\overline{z}}^{j-1}\varphi,$$

where V is the unitary operator defined by (2.4) and φ is a smooth function. Since, the operator V transforms $\bar{\mathcal{A}}^2(\Pi)$ onto $\mathcal{A}^2(\Pi)$, then (3.5) follows straightforwardly from (3.4) and the above relations. □

In the next result we state a Paley–Wiener theorem for the true poly-Bergman spaces. In essence we will show that the operators

$$R_{(j)} := (S_\Pi)^{j-1}R \quad \text{and} \quad \widetilde{R}_{(j)} := \left(S_\Pi^*\right)^{j-1}\widetilde{R} \quad (j = 1, 2, \dots)$$

are unitary operators from $L^2(\mathbb{R}^+)$ onto $\mathcal{A}^2_{(j)}(\Pi)$ and onto $\bar{\mathcal{A}}^2_{(j)}(\Pi)$, respectively. We will manage this with the help of the Laguerre polynomials

$$L_n(z) := \sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{k!}z^k, \quad n = 0, 1, \dots.$$

Theorem 3.4 *Let j be a positive integer. Then the following operators*

$$R_{(j)} : L^2\left(\mathbb{R}^+\right) \to \mathcal{A}^2_{(j)}(\Pi), \qquad R_{(j)}f(z) = \frac{\partial_z^{j-1}[(\overline{z}-z)^{j-1}Rf(z)]}{(j-1)!}, \tag{3.7}$$

$$\widetilde{R}_{(j)} : L^2\left(\mathbb{R}^+\right) \to \bar{\mathcal{A}}^2_{(j)}(\Pi), \qquad \widetilde{R}_{(j)}f(z) = \frac{\partial_{\overline{z}}^{j-1}[(z-\overline{z})^{j-1}\widetilde{R}f(z)]}{(j-1)!} \tag{3.8}$$

are isometric isomorphisms. Furthermore, if $y := (z-\overline{z})/(2i)$ then

$$R_{(j)}f(z) = \sum_{k=0}^{j-1} y^k\varphi_k(z) = \sum_{k=0}^{j-1} L_k(y)\phi_k(z), \tag{3.9}$$

$$\widetilde{R}_{(j)}f(z) = \sum_{k=0}^{j-1} y^k\varphi_k(-\overline{z}) = \sum_{k=0}^{j-1} L_k(y)\phi_k(-\overline{z}), \tag{3.10}$$

where the analytic components φ_k and ϕ_k satisfy the following conditions

$$\varphi_k = F^c_{2k} f_k \in \mathcal{A}^2(\Pi, dA_{2k}) \quad \text{and} \quad \phi_k = F^c_{2j-2} h_k \in \mathcal{A}^2(\Pi, dA_{2j-2}), \tag{3.11}$$

and, for $k = 0, \dots, j-1$, the functions f_k and h_k are respectively given by

$$f_k(t) := (-1)^{j-1+k} \binom{j-1}{k} \frac{\sqrt{(2k)!}}{k!} \sqrt{t^{2k+1}} f(t), \tag{3.12}$$

$$h_k(t) := (-1)^k \binom{j-1}{k} \sqrt{(2j-2)!} \sqrt{t^{2k+1}} (t - 1/2)^{j-1-k} f(t). \tag{3.13}$$

Proof The assertion that the operators defined by (3.7) and (3.8) are unitary operators follows from Theorems 2.2, 3.1 and 3.3. Moreover, from (3.7), (2.2) and the derivative of the parametric integral, we obtain

$$R_{(j)} f(z) = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \sqrt{t} f(t) \frac{\partial_z^{j-1}[(\overline{z} - z)^{j-1} e^{izt}]}{(j-1)!} dt. \tag{3.14}$$

Then, since we can compute straightforwardly

$$\frac{\partial_z^{j-1}[(\overline{z} - z)^{j-1} e^{izt}]}{(j-1)!} = e^{izt} \sum_{k=0}^{j-1} \binom{j-1}{k} \frac{(-1)^{j+k-1}}{k!} \left[it(\overline{z} - z)\right]^k \tag{3.15}$$

$$= (-1)^{j-1} e^{izt} L_{j-1}\big(it(\overline{z} - z)\big), \tag{3.16}$$

where L_{j-1} is the Laguerre polynomial of degree $j-1$, the first equality at (3.9), the relations on the left of (3.11) and (3.12) follow from (3.15) together with (3.14). Moreover, based on the following special case of generating relations for Laguerre polynomials (see, e.g. [9, Chap. 12])

$$L_n(\lambda z) = \sum_{k=0}^{n} \binom{n}{k} (1-\lambda)^{n-k} \lambda^k L_k(z), \quad \lambda \neq 0,$$

we can conclude that the second equality at (3.9), the relations on the right of (3.11) and (3.13) follow from (3.16) together with (3.14). The case when j is negative follows from the above and from the following operator equalities

$$\widetilde{R}_{(j)} = \big(S_\Pi^*\big)^{j-1} \widetilde{R} = \big(S_\Pi^*\big)^{j-1} V R = V (S_\Pi)^{j-1} R = V R_{(j)}. \qquad \square$$

4 Paley–Wiener Type Theorems for the Poly-Bergman Space

Let j be a positive integer, $\mathcal{H}$ be a Hilbert space and $[\mathcal{H}]_j$ denote the Hilbert space of all $j \times 1$ matrices $(f_k)_k$ with entries in $\mathcal{H}$. Since the poly-Bergman space coincides with the direct sum of its true poly-Bergman spaces, i.e.

$$\mathcal{A}_j^2(\Pi) = \bigoplus_{k=1}^{j} \mathcal{A}_{(k)}^2(\Pi) \quad \text{and} \quad \bar{\mathcal{A}}_j^2(\Pi) = \bigoplus_{k=1}^{j} \bar{\mathcal{A}}_{(k)}^2(\Pi),$$

then the next result is a straightforward consequence of Theorem 3.4.

Theorem 4.1 *For* $j = 1, 2, \ldots,$ *the following operators are unitary operators*

$$R_j : \left[L^2(\mathbb{R}^+)\right]_j \to \mathcal{A}_j^2(\Pi), \qquad R_j(f_k)_k(z) = \sum_{k=1}^{j} R_{(k)} f_k(z),$$

$$\widetilde{R}_j : \left[L^2(\mathbb{R}^+)\right]_j \to \bar{\mathcal{A}}_j^2(\Pi), \qquad \widetilde{R}_j(f_k)_k(z) = \sum_{k=1}^{j} \widetilde{R}_{(k)} f_k(z).$$

It is well known that a function defined in $\mathcal{A}_\partial^2(\Pi)$ admits non-tangential limits almost at every point $t \in \mathbb{R}$, i.e. the following function of real variable

$$f(t) := \lim_{z \to t} f(z) \quad \text{whenever } \Pi \ni z \to t \in \mathbb{R} \text{ nontangentially}$$

is well defined, for almost every $t \in \mathbb{R}$, and $f(t) \in L^2(\mathbb{R})$. Moreover, the real Fourier transform $\mathcal{F} \in \mathcal{B}(L^2(\mathbb{R}))$ defined in the following way

$$\mathcal{F} f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-ixt} dt, \quad x \in \mathbb{R}$$

is isometric from $\mathcal{A}_\partial^2(\Pi)$ onto $L^2(\mathbb{R}^+)$, i.e. the following operator is unitary

$$\mathcal{F} : \mathcal{A}_\partial^2(\Pi) \to L^2(\mathbb{R}^+).$$

Thus, from Theorem 4.1 we can conclude the following result.

Theorem 4.2 *For* $j = 1, 2, \ldots,$ *the following operators are unitary operators*

$$W_j : \left[\mathcal{A}_\partial^2(\Pi)\right]_j \to \mathcal{A}_j^2(\Pi), \qquad W_j(f_k)_k(z) = \sum_{k=1}^{j} R_{(k)} \mathcal{F} f_k(z),$$

$$\widetilde{W}_j : \left[\mathcal{A}_\partial^2(\Pi)\right]_j \to \bar{\mathcal{A}}_j^2(\Pi), \qquad \widetilde{W}_j(f_k)_k(z) = \sum_{k=1}^{j} \widetilde{R}_{(k)} \mathcal{F} f_k(z).$$

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Fredholm Theory of Pseudodifferential Operators Acting in Variable Exponent Spaces of Bessel Potentials on Smooth Manifolds

Vladimir Rabinovich

Abstract The paper is devoted to the of Fredholm property of pseudodifferential operators acting in the spaces of Bessel potentials connected with variable exponent Lebesgue spaces on smooth compact manifolds and non compact manifolds with conical structure at infinity.

1 Introduction

The paper is devoted to an overview of some results of the papers [10, 11] concerning the local and global Fredholmness of pseudodifferential operators on $\mathbb{R}^n$ acting in the spaces of Bessel potentials connected with variable exponent Lebesgue spaces, and applications of these results to the investigation of the Fredholm property of pseudodifferential operators acting in the spaces of Bessel potentials with variable exponents on smooth compact and noncompact manifolds.

It should be noted that the last decade there arose a big interest to investigations of the classical operators of the analysis: singular and maximal operators, Hardy operators, pseudodifferential operators in the variable exponent Lebesgue spaces. See for instance the book [3], papers [4–7] and references cited there. See also the paper [10] devoted to boundedness and compactness of pseudodifferential operators in the L. Hörmander class $OPS^0_{1,0}(\mathbb{R}^n)$ and their Fredholm properties in weighted variable exponent Lebesgue spaces $L^{p(\cdot)}_w(\mathbb{R}^n)$ where w is an exponential weight, and the paper [11] where the results of [10] where applied to investigations of the Fredholm property of singular integral operators on composed Carleson curves Γ acting in weighted variable exponent Lebesgue spaces $L^{p(\cdot)}_w(\Gamma)$.

Our aim here is to show that the Fredholm theory of pseudodifferential operators on $\mathbb{R}^n$ obtained in [10] can be extended on pseudodifferential operators acting in Bessel potential spaces $H^{s,p(\cdot)}(M)$ where M are compact smooth manifolds or a non compact smooth manifolds of conical structure at infinity.

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2 Pseudodifferential Operators

- We will use the standard notations: $C^{\infty}(\mathbb{R}^n) \supset C_b^{\infty}(\mathbb{R}^n) \supset S(\mathbb{R}^n) \supset C_0^{\infty}(\mathbb{R}^n)$ for spaces of infinitely differentiable functions, infinitely differentiable functions bounded with all derivatives, infinitely differentiable functions decreasing at infinity rapidly than every function $|x|^{-N}$, $N \in \mathbb{N}$ with all derivatives, and for infinitely differentiable functions with compact supports on $\mathbb{R}^n$, respectively. We denote by $SO^{\infty}(\mathbb{R}^n)$ the subspace of $C_b^{\infty}(\mathbb{R}^n)$ of slowly oscillating functions, that is functions $u \in C_b^{\infty}(\mathbb{R}^n)$ such that

$$\lim_{x\to\infty} \partial_{x_j} u(x) = 0, \quad j = 1, \ldots, n.$$

- If X, Y are Banach spaces, we denote by $\mathcal{B}(X, Y)$ the space of all bounded linear operators acting from X into Y, and by $\mathcal{K}(X, Y)$ the subspace of $\mathcal{B}(X, Y)$ of all compact operators. If $X = Y$ we will write $\mathcal{B}(X), \mathcal{K}(X)$, respectively.
- We say that a function a is a symbol in the L. Hörmander class $S^m_{1,0}(\mathbb{R}^n)$ if $a \in C^{\infty}(\mathbb{R}^n_x \times \mathbb{R}^n_{\xi})$, and

$$|a|_{r,t} = \max_{|\alpha|\le r, |\beta|\le t} \sup_{\mathbb{R}^n\times\mathbb{R}^n} \left|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)\right| \langle\xi\rangle^{-(m-|\alpha|)} < \infty \tag{1}$$

for all $r, t \in \mathbb{N}\cup 0$. As usual, we associate with a symbol $a \in S^m_{1,0}(\mathbb{R}^n)$ a pseudodifferential operator defined on the space $C_0^{\infty}(\mathbb{R}^n)$ by the formula

$$Op(a)u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} d\xi \int_{\mathbb{R}^n} a(x,\xi)u(y)e^{i(x-y,\xi)}dy \tag{2}$$

and we denote the class of such operators by $OPS^m_{1,0}(\mathbb{R}^n)$. For the standard property of pseudodifferential operators see for instance the book [12].
- A symbol $a \in S^m_{1,0}(\mathbb{R}^n)$ is called slowly oscillating at infinity if for all multiindeces α, β

$$\left|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)\right| \le C_{\alpha\beta}(x)\langle\xi\rangle^{m-|\alpha|}, \tag{3}$$

where $\lim_{x\to\infty} C_{\alpha\beta}(x) = 0$ for every α and $\beta \neq 0$. We denote by $SO^m_{1,0}(\mathbb{R}^n)$, the class of slowly oscillating symbols, and by $\mathring{S}^m_{1,0}(\mathbb{R}^n)$ the class of symbols for which conditions (3) hold for all multiindices α, β. We use the notations $OPSO^m_{1,0}(\mathbb{R}^n)$, $OP\mathring{S}^m_{1,0}(\mathbb{R}^n)$ for the classes of operators with symbols in $SO^m_{1,0}(\mathbb{R}^n)$, $\mathring{S}^m_{1,0}(\mathbb{R}^n)$, respectively.
- Let p be a measurable function on $\mathbb{R}^n$ such that $p : \mathbb{R}^n \to (1, \infty)$. The generalized Lebesgue space with variable exponent is defined via the modular

$$I^p(f) := \int_{\mathbb{R}^n} \left|f(x)\right|^{p(x)} dx \tag{4}$$

by the norm

$$\|f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : I^p\left(\frac{f}{\lambda}\right) \le 1\right\}.$$

In what follows we assume that p satisfies the conditions

$$1 < p_- := \inf_{x\in\mathbb{R}^n} p(x) \le \sup_{x\in\mathbb{R}^n} p(x) =: p_+ < \infty,$$

$$\left|p(x) - p(y)\right| \le \frac{A}{\log\frac{1}{|x-y|}}, \quad x, y \in \mathbb{R}^n,\ |x-y| \le \frac{1}{2},$$

$$\left|p(x) - p(\infty)\right| \le \frac{A}{\log\frac{1}{1+|x|}}, \quad x \in \mathbb{R}^n.$$

The class of $p(\cdot)$ satisfying these properties is denoted by $\mathcal{P}(\mathbb{R}^n)$, and we set $p'(x) = \frac{p(x)}{p(x)-1}$.

Definition 1 We say that a distribution $u \in S'(\mathbb{R}^n)$ belongs to $H^{s,p(\cdot)}(\mathbb{R}^n)$, $s \in \mathbb{R}$, $p \in \mathcal{P}(\mathbb{R}^n)$ if

$$\|u\|_{H^{s,p(\cdot)}(\mathbb{R}^n)} = \left\|\langle D\rangle^s u\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} < \infty,$$

where $\langle D\rangle^s = Op(\langle\xi\rangle^s)$, $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$.

There are other definitions of the space $H^{s,p(\cdot)}(\mathbb{R}^n)$ (see for instance the book [3], and the papers [1, 2]).

Theorem 2 [10] *Let* $p \in \mathcal{P}(\mathbb{R}^n)$. *Then*: (i) *An operator* $Op(a) \in OPS^m_{1,0}$ *is bounded from* $H^{s,p(\cdot)}(\mathbb{R}^n)$ *into* $H^{s-m,p(\cdot)}(\mathbb{R}^n)$; (ii) *An operator* $Op(a) \in OP\mathring{S}^{m-\varepsilon}_{1,0}$, $\varepsilon > 0$ *is a compact operator from* $H^{s,p(\cdot)}(\mathbb{R}^n)$ *into* $H^{s-m,p(\cdot)}(\mathbb{R}^n)$.

3 Local and Global Fredholm Property of Operators in $OPS^m_{1,0}(\mathbb{R}^n)$ Acting at the Spaces $H^{s,p(\cdot)}(\mathbb{R}^n)$

3.1 Local Fredholmness

- Let U be a neighborhood of the point $x_0 \in \mathbb{R}^n$. We say that $\phi \in C_0^\infty(\mathbb{R}^n)$ is a smooth characteristic function of a neighborhood U, if there exists a neigbourhood U' of the point x_0 such that $\bar{U}' \subset U$, $0 \le \phi(x) \le 1$, and $\phi(x) = 1$ for all $x \in \bar{U}'$.
- We say that an operator $A \in \mathcal{B}(H^{s,p(\cdot)}(\mathbb{R}^n), H^{s-m,p(\cdot)}(\mathbb{R}^n))$ is locally Fredholm at the point $x_0 \in \mathbb{R}^n$ if there exists a neighborhood U_{x_0} of the point x_0, a smooth characteristic function ϕ of a neighborhood U_{x_0} and operators $\mathcal{R}_{x_0}, \mathcal{L}_{x_0} \in \mathcal{B}(H^{s-m,p(\cdot)}(\mathbb{R}^n), H^{s,p(\cdot)}(\mathbb{R}^n))$ such that

$$\mathcal{L}_{x_0} A\phi I = \phi I + T'_{x_0}, \qquad \phi A\mathcal{R}_{x_0} = \phi I + T''_{x_0}, \tag{5}$$

where $T'_{x_0} \in \mathcal{K}(H^{s,p(\cdot)}(\mathbb{R}^n))$, $T''_{x_0} \in \mathcal{K}(H^{s-m,p(\cdot)}(\mathbb{R}^n))$.

Theorem 3 *An operator $Op(a) \in OPS^m_{1,0}(\mathbb{R}^n)$ acting from $H^{s,p(\cdot)}(\mathbb{R}^n)$ into $H^{s-m,p(\cdot)}(\mathbb{R}^n)$ is locally Fredhom at the point $x_0 \in \mathbb{R}^n$ if and only if*

$$\liminf_{\xi\to\infty}\left|a(x_0,\xi)\right|\langle\xi\rangle^{-m} > 0. \tag{6}$$

- We denote by $\tilde{\mathbb{R}}^n$ the compactification of the space $\mathbb{R}^n$ obtained by joining an infinitely distant point η_ω to every ray $\{x \in \mathbb{R}^n : x = t\omega, t > 0, \omega \in S^{n-1}\}$. The fundamental system of neighborhoods at an infinitely distant point η_ω forms the conical neighborhoods

$$U_{\eta_\omega} = \left\{x \in \mathbb{R}^n : x = r\theta : r > a > 0, \theta \in \Omega_\omega\right\} \tag{7}$$

  where Ω_ω is a neighborhood of the point ω on the unit sphere. We denote by $\mathfrak{M} = \tilde{\mathbb{R}}^n \diagdown \mathbb{R}^n$ the set of all infinitely distant point of $\tilde{\mathbb{R}}^n$.
- Let U_{η_ω} be a neighborhood of the infinitely distant point η_ω. We say that $\phi \in C_b^\infty(\mathbb{R}^n)$ is a smooth characteristic function of a neighborhood U_{η_ω}, if there exists a neighborhood U'_{η_ω} of the point η_ω such that $\bar{U}'_{\eta_\omega} \subset U_{\eta_\omega}$, $0 \le \phi(x) \le 1$, and $\phi(x) = 1$ for all $x \in \bar{U}'$. As above we define the local Fredholmness of the operator $A \in \mathcal{B}(H^{s,p(\cdot)}(\mathbb{R}^n), H^{s-m,p(\cdot)}(\mathbb{R}^n))$ at the infinitely distant point η_ω.

Theorem 4 *An operator $Op(a) \in OPSO^m_{1,0}(\mathbb{R}^n)$ acting from $H^{s,p(\cdot)}(\mathbb{R}^n)$ into $H^{s-m,p(\cdot)}(\mathbb{R}^n)$ is locally Fredholm at infinitly distant point η_ω if and only if*

$$\liminf_{x\to\eta_\omega}\inf_{\xi\in\mathbb{R}^n}\left|a(x,\xi)\right|\langle\xi\rangle^{-m} > 0. \tag{8}$$

The main result on the Fredholmness of pseudodifferential operators in the class $OPSO^m_{1,0}(\mathbb{R}^n)$ on the variable exponent Bessel potential spaces is:

Theorem 5 *Let $Op(a) \in OPSO^m_{1,0}(\mathbb{R}^n)$. Then $Op(a)\colon H^{s,p(\cdot)}(\mathbb{R}^n) \to H^{s-m,p(\cdot)}(\mathbb{R}^n)$ is a Fredholm operator if and only if:*

(i) *for every point $x_0 \in \mathbb{R}^n$ condition* (6) *holds*;
(ii) *for every point $\eta_\omega \in \mathfrak{M}$ condition* (8) *holds.*

If conditions (6), (8) *hold then*

$$\ker\big(Op(a) : H^{s,p(\cdot)}\big(\mathbb{R}^n\big) \to H^{s-m,p(\cdot)}\big(\mathbb{R}^n\big)\big) \subset S\big(\mathbb{R}^n\big), \tag{9}$$

$$\ker\big(Op(a)^* : H^{-s,p'(\cdot)}\big(\mathbb{R}^n\big) \to H^{-s-m,p'(\cdot)}\big(\mathbb{R}^n\big)\big) \subset S\big(\mathbb{R}^n\big). \tag{10}$$

Hence the Fredholm index $Op(a) : H^{s,p(\cdot)}(\mathbb{R}^n) \to H^{s-m,p(\cdot)}(\mathbb{R}^n)$ coincides with a Fredholm index of $Op(a) : H^{s,2}(\mathbb{R}^n) \to H^{s-m,2}(\mathbb{R}^n)$.

Remark 6 Let $Op(a) \in OPSO^m_{1,0}(\mathbb{R}^n)$ be a differential operator and conditions (6), (8) hold. Then the functions $u \in \ker(Op(a))$ have subexponential decreasing at infinity, that is, for every $\delta > 0$, $r \in (0,1)$, and for every multiindex α

$$\left|\partial^\alpha u(x)\right| \le C_\alpha \exp\left(-\delta|x|^r\right).$$

4 Pseudodifferential Operators on Smooth Manifolds

4.1 Compact Manifolds

Let M be a C^∞ compact closed n-dimensional manifold. We denote by $C^\infty(M)$, $C_0^\infty(M)$ the spaces of all smooth functions on M and a subspace of all smooth functions with compact supports. If X is some chart in M and $\varkappa : X \to X' \subset \mathbb{R}^n$ is a diffeomorphism then $\varkappa^* : C_0^\infty(X') \to C_0^\infty(X)$, $\varkappa^* : C^\infty(X') \to C^\infty(X)$ are blow-up of the diffeomorphism $\varkappa$. We introduce also the operators of the natural embedding $i_X : C_0^\infty(X) \to C_0^\infty(M)$ and natural restrictions $r_X : C^\infty(M) \to C^\infty(X)$.

An operator $A : C_0^\infty(M) \to C^\infty(M)$ is called a pseudodifferential operator of the class $OPS^m_{1,0}(M)$ if for every *chart* diffeomorphism $\varkappa : X \to X'$ (X is not necessary connected set) the operator $\varkappa^{*-1} r_X A i_X \varkappa^* : C_0^\infty(X') \to C^\infty(X')$ is a restriction on X' by a pseudodifferential operator in the class $OPS^m_{1,0}(\mathbb{R}^n)$ (see for instance [12], p. 36). The theorem of the change of variable for pseudodifferential operators allows to define the main symbol σ_A of $A \in OPS^m_{1,0}(M)$ as a function on the cotangent bundle $T^*(M)$. Let $\{\eta_j\}_{j=1}^N$ be a partition of unity on M corresponding to a covering of M by the system $\{X_j\}_{j=1}^N$ of charts $\varkappa_j : X_j \to X_j^{'} \subset \mathbb{R}^n$. By means of the partition of unity $\{\eta_j\}_{j=1}^N$, $\eta_j \in C_0^\infty(M)$ on M one can introduce the spaces $H^{s,p(\cdot)}(M)$:

$$\|u\|_{H^{s,p(\cdot)}(M)} = \sum_{j=1}^N \left\|(\eta_j u) \circ \varkappa_j^{-1}\right\|_{H^{s,p(\cdot)}(\mathbb{R}^n)} < \infty.$$

Note that the different partitions of unity lead to equivalent norms.

Theorem 7 *An operator $A \in OPS^m_{1,0}(M)$ is Fredholm from $H^{s,p(\cdot)}(M)$ into $H^{s-m,p(\cdot)}(M)$ if and only if*

$$\sigma_A(x,\xi) \ne 0 \quad \textit{for all } (x,\xi) \in T^*(M) \backslash \left(M \times \{0\}\right).$$

If the previous condition holds, then

$$\ker\left(A : H^{s,p(\cdot)}(M) \to H^{s-m,p(\cdot)}(M)\right) \subset C^\infty(M),$$

$$\ker\left(A^* : H^{-s,p'(\cdot)}(M) \to H^{-s-m,p'(\cdot)}(M)\right) \subset C^\infty(M).$$

Hence the classical theory of pseudodifferential operators on compact smooth manifolds is transferred on pseudodifferential operators acting in $H^{s,p(\cdot)}(M)$.

The proof of Theorem 7 is based on a finite partition of unity, coordinate diffeomorphisms, and the local Fredholmness condition given in Theorem 3.

4.2 Non Compact Manifolds

Non compact manifolds and pseudodifferential operators on them considered below were introduced in [8] (see also [9]) where their Fredhom theory in the spaces $H^{s,p}$, $p \in (1,\infty)$ has been constructed.

- We say that a C^∞ manifold M belongs to the class $\mathfrak{N}(n)$ if M admits a compactification $\tilde{M} = M \cup M_\infty$ by a set M_∞ of infinitely distant points. We suppose that a fundamental system of neighborhoods of an infinitely distant point $\zeta \in M_\infty$ is given by open sets $\tilde{U} \subset \tilde{M}$ such that there exists a homeomorphism $\varkappa : \tilde{U} \to \tilde{U}'$ where U' is an open conical set ($U' = \{x \in \mathbb{R}^n : x = t\omega, t > R > 0, \omega \in \Omega\}$), Ω is an open set on the unit sphere $S^n \subset \mathbb{R}^{n+1}$, and $\tilde{U}'$ is the closure of U' in $\tilde{\mathbb{R}}^n$.
- We say that a homeomorphism $\tilde{\varkappa} : \tilde{X} \to \tilde{X}' \subset \tilde{\mathbb{R}}^n$ is an M_∞-chart if $\tilde{X} \cap M_\infty \neq \emptyset$, and $\tilde{X}'$ is a closure of an open conic set in $\tilde{\mathbb{R}}^n$, and $\tilde{\varkappa}|_X = \varkappa : X \to X'$ are diffeomorphisms with the following property: if $\varkappa_1 : X_1 \to X_1'$, $\varkappa_2 : X_2 \to X_2'$, where $X_1' \cap X_2' \neq \emptyset$ then

$$\begin{aligned} d\big(\varkappa_1 \circ \varkappa_2^{-1}\big) &\in SO^\infty\big(X_1' \cap X_2'\big) \otimes \mathcal{B}\big(\mathbb{R}^n\big), \\ d\big(\varkappa_2 \circ \varkappa_1^{-1}\big) &\in SO^\infty\big(X_1' \cap X_2'\big) \otimes \mathcal{B}\big(\mathbb{R}^n\big). \end{aligned} \tag{11}$$

One can see that $\tilde{M}$ is a Hausdorf compact space. By the definition of the manifolds $M \in \mathfrak{N}(n)$ there exists a finite covering of $\tilde{M}$ by charts of two types: (i) $\varkappa : X \to X'$, where $X' \subset \mathbb{R}^n$ is a bounded open set; (ii) $\varkappa : X \to X'$, where X' is an open conical set in $\mathbb{R}^n$ (not necessary connected).

A typical example of such manifold is a smooth n-dimensional surface $M \subset \mathbb{R}^{n+1}$ with a conical structure at infinity, that is such that $M \cap \{x \in \mathbb{R}^{n+1} : |x| > R > 0\} = \{x \in \mathbb{R}^{n+1} : x = t\omega, t > R, \omega \in \Omega\}$ for some $R > 0$, where Ω is a C^∞ closed $n-1$ dimensional surface on the unit sphere $S^n \subset \mathbb{R}^{n+1}$.

Definition 8 We say that A is a pseudodifferential operator of the class $OPSO^m_{1,0}(M)$, $M \in \mathfrak{N}(n)$ if for every above defined charts $\varkappa : X \to X'$ the operator $\varkappa^{*-1} r_X A i_X \varkappa^* : C_0^\infty(X') \to C^\infty(X')$ is a restriction on X' of a pseudodifferential operator in the class $OPSO^m_{1,0}(\mathbb{R}^n)$.

Theorem 9 *Let $A \in OPSO^m_{1,0}(M)$. Then $A : H^{s,p(\cdot)}(M) \to H^{s-m,p(\cdot)}(M)$ is a Fredholm operator if and only if:*

(i) *The main symbol $\sigma_A(x,\xi) \neq 0$ for every points $(x,\xi) \in T^*(M) \backslash (M \times \{0\})$;*

(ii) *Let* $\zeta \in M_\infty$, $\tilde{X}$ *be a coordinate neighborhood of* ζ, $\tilde{\varkappa} : \tilde{X} \to \tilde{X}'$ *be a coordinate homomorphism*, $\tilde{\varkappa}(\zeta) = \eta \in \tilde{\mathbb{R}}^n$, *and*

$$\varkappa^{*-1} r_X A i_X \varkappa^* = Op(a_\varkappa),$$

$a_\varkappa \in SO^m_{1,0}(\mathbb{R}^n)$.

Then for every point $\zeta \in M_\infty$

$$\liminf_{x \to \eta} \inf_{\xi \in \mathbb{R}^n} \left| a_\varkappa(x, \xi) \right| > 0. \tag{12}$$

Remark 10 One can proof that condition (12) independent of a choice of the chart.

The proof of Theorem 9 is based on the finite partition of unity and the local Fredholmness criteria of Theorems 3, 4.

Remark 11 Of course the results of Theorems 7, 9 are extended on pseudodifferential operators acting in sections of smooth vector bundles belonging to the variable exponent spaces of Bessel potentials over manifolds M.

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Part XI
Differential and Difference Equations with Applications

Organizers: Leonid Berezansky, Josef Diblik, Agacik Zafer, Miroslawa Zima

Boundary Value Problems for the Radiative Transfer Equation with Reflection and Refraction Conditions

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Abstract The paper contains a brief description of some new results about the boundary value problems for the radiative transfer equation with the reflection and refraction conditions.

Keywords Radiative transfer equation · Reflection and refraction conditions

Mathematics Subject Classification (2010) Primary 35Q60 · Secondary 35Q20

1 Introduction

We consider the monochromatic radiative transfer in the system $G = \bigcup_{j=1}^{m} G_j$ of semitransparent bodies G_j separated by the vacuum. Each body G_j is a bounded domain in $\mathbb{R}^3$ with boundary $\partial G_j \in C^1$. We assume that G_i and G_j are pairwise disjoint, whereas their boundaries can intersect for some $i \neq j$. Let $\Omega = \{\omega \in \mathbb{R}^3 \mid |\omega| = 1\}$ be a sphere of directions. The sought function $I(\omega, x)$ is defined on the set $D = \Omega \times G$ and is interpreted as the radiation intensity at a point $x \in G$ when the radiation propagates along the direction $\omega \in \Omega$. Assume that each G_j is occupied by a medium with constant absorption $\varkappa_j > 0$ and scattering $s_j \geq 0$ coefficients and the refraction exponent $k_j > 1$. We set $\varkappa(x) = \varkappa_j$, $s(x) = s_j$ and $k(x) = k_j$ for $x \in G_j$. To describe the radiation propagation in G, we use the radiative transfer equation

$$\omega \cdot \nabla I + (\varkappa + s)I = s\mathcal{S}(I) + k^2 F, \quad (\omega, x) \in D,$$

where $\omega \cdot \nabla I$ denotes the derivative of a function I along the direction ω and $\mathcal{S}$ denotes the scattering operator

$$\mathcal{S}(I)(\omega, x) = \frac{1}{4\pi} \int_{\Omega} \theta_j(\omega \cdot \omega') I(\omega', x) d\omega', \quad (\omega, x) \in D_j = \Omega \times G_j,\ 1 \leq j \leq m,$$

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with the scattering indicatrix possessing the following properties:

$$\theta_j \in L^1(-1,1),\ \theta_j \geq 0, \quad \frac{1}{2}\int_{-1}^{1} \theta_j(\mu)d\mu = 1.$$

The mathematical properties of the radiative transfer equation were studied by many authors; see for example [1–3]. The most studied problems in this relation are boundary value problems for this equation with the continuity condition for the radiation intensity imposed on the interface between media with different optic properties. In this case, the radiation passes through the interface without changing direction and intensity. In some applications (for example, the theory of neutron transfer), such conditions are justified from the physical point of view. However, in many applications (for example, optics, tomography, thermal physics), the reflection and refraction of radiation at the interface between media should be taken into account. The boundary value problems for the radiative transfer equation with the reflection and refraction conditions are still studied unsatisfactory, in spite of their importance. Some problems of such type were treated in [2, 4]. Problems for the radiative transfer equation with the reflection and refraction conditions in accordance with the Fresnel laws were first studied in [5, 6].

This paper contains a brief description of some new results [7–9] about the boundary value problems for the radiative transfer equation with the reflection and refraction conditions.

2 Notations and Function Spaces

Let $x \cdot y = \sum_{i=1}^{3} x_i y_i$ be the inner product in $\mathbb{R}^3$ and let n_j be the outward normal to the boundary ∂G_j of the domain G_j. We set

$$\Gamma = \Omega \times \partial G = \bigcup_{j=1}^{m} \Gamma_j, \quad \Gamma_j = \Omega \times \partial G_j,\ 1 \leq j \leq m,$$

$$\Gamma^- = \bigcup_{j=1}^{m} \Gamma_j^-, \quad \Gamma_j^- = \big\{(\omega, x) \in \Gamma_j \;\big|\; \omega \cdot n_j(x) < 0\big\},\ 1 \leq j \leq m,$$

$$\Gamma^+ = \bigcup_{j=1}^{m} \Gamma_j^+, \quad \Gamma_j^+ = \big\{(\omega, x) \in \Gamma_j \;\big|\; \omega \cdot n_j(x) > 0\big\},\ 1 \leq j \leq m.$$

Assume that the measure $d\Gamma(\omega, x) = d\omega d\sigma(x)$ is introduced on Γ. Here $d\omega$ and $d\sigma(x)$ are the measures induced by the Lebesgue measure in $\mathbb{R}^3$ on Ω and ∂G respectively.

Let $1 \leq p \leq \infty$ and $E^{\pm}$ be a subset of $\Gamma^{\pm}$, measurable with respect to the measure $d\Gamma$. We denote by $\widehat{L}^p(E^{\pm})$ and $\widehat{L}^{1,p}(E^{\pm})$ the Banach spaces of functions g

defined on $E^{\pm}$, measurable with respect to the measure $d\Gamma$ and (after extending them by zero to $\Gamma^{\pm} \setminus E^{\pm}$) possess the finite norms

$$\|g\|_{\widehat{L}^p(E^{\pm})} = \begin{cases} (\sum_{j=1}^{m} \int_{\Gamma_j^{\pm}} |g(\omega, x)|^p |\omega \cdot n_j(x)| d\omega d\sigma(x))^{1/p}, & 1 \le p < \infty, \\ \operatorname{ess\,sup}_{(\omega,x)\in E} |g(\omega, x)|, & p = \infty, \end{cases}$$

$$\|g\|_{\widehat{L}^{1,p}(E^{\pm})} = \begin{cases} (\sum_{j=1}^{m} \int_{\partial G_j} [\int_{\Omega_j^{\pm}(x)} |g(\omega, x)||\omega \cdot n_j(x)| d\omega]^p d\sigma(x))^{1/p}, \\ \quad 1 \le p < \infty, \\ \max_{1\le j\le m} \operatorname{ess\,sup}_{x\in\partial G_j} \int_{\Omega_j^{\pm}(x)} |g(\omega, x)||\omega \cdot n_j(x)| d\omega, \\ \quad p = \infty. \end{cases}$$

Hereinafter,

$$\Omega_j^{+}(x) = \{\omega \in \Omega \mid \omega \cdot n_j(x) > 0\}, \qquad \Omega_j^{-}(x) = \{\omega \in \Omega \mid \omega \cdot n_j(x) < 0\}.$$

We denote by $L^p(D)$ the Banach space of functions f defined on D and measurable with respect to the measure $d\omega dx$ with the finite norm

$$\|f\|_{L^p(D)} = \begin{cases} (\int_D |f(\omega, x)|^p d\omega dx)^{1/p}, & 1 \le p < \infty, \\ \operatorname{ess\,sup}_{(\omega,x)\in D} |f(\omega, x)|, & p = \infty. \end{cases}$$

We denote by $\mathcal{W}^p(D)$ the Banach space of functions $f \in L^p(D)$ possessing the weak derivative $\omega \cdot \nabla f \in L^p(D)$ equipped with the norm

$$\|f\|_{\mathcal{W}^p(D)} = \begin{cases} (\|f\|^p_{L^p(D)} + \|\omega \cdot \nabla f\|^p_{L^p(D)})^{1/p}, & 1 \le p < \infty, \\ \max\{\|f\|_{L^\infty(D)}, \|\omega \cdot \nabla f\|_{L^\infty(D)}\}, & p = \infty. \end{cases}$$

We will denote by $f|_{\Gamma^{\pm}}$ and $f|_{\Gamma_j^{\pm}}$ the traces of function $f \in \mathcal{W}^p(D)$ on $\Gamma^{\pm}$ and $\Gamma_j^{\pm}$ respectively. It is known that the traces $f|_{\Gamma^{\pm}}$ of a function $f \in \mathcal{W}^p(D)$ with $1 \le p < \infty$ do not necessarily belong to $\widehat{L}^p(\Gamma^{\pm})$. We introduce

$$\widehat{\mathcal{W}}^p(D) = \{f \in \mathcal{W}^p(D) \mid f|_{\Gamma^-} \in \widehat{L}^p(\Gamma^-),\ f|_{\Gamma^+} \in \widehat{L}^p(\Gamma^+)\}.$$

3 Boundary Value Problem for the Radiative Transfer Equation with Specular Reflection and Refraction Conditions in Accordance with the Fresnel Laws

3.1 Reflection and Refraction Laws

Remind that under our assumption, each body G_j is occupied by a homogeneous material with refraction exponent $k_j > 1$. We assume that the radiation is nonpolarized. We denote by $J(\omega', x)$ the intensity of radiation propagating in the vacuum and falling to a point x of the surface ∂G_j in the direction ω' under the incidence angle φ' with $\cos\varphi' = \mu'_j = \omega' \cdot n_j(x) < 0$. The incident radiation is partially mirror reflected by the surface ∂G_j in the direction

$$\omega = \omega' - 2\mu'_j n_j(x) \tag{3.1}$$

under reflection angle φ with $\cos\varphi = \mu_j = \omega \cdot n_j(x) = -\mu'_j$ and is partially refracted, entering the body G_j in the direction

$$\widehat{\omega} = \widehat{\mu}_j n_j(x) + \frac{1}{k_j}\big(\omega' - \mu'_j n_j(x)\big) \tag{3.2}$$

under the reflection angle ψ with $\cos\psi = \widehat{\mu}_j = \widehat{\omega} \cdot n_j(x) = -\nu_j^-(\mu'_j)$, where $\nu_j^-(\mu'_j) = \sqrt{1 - k_j^{-2}(1 - (\mu'_j)^2)}$. Note that $-1 \le \widehat{\mu}_j \le \widehat{\mu}_{j,\lim} = \sqrt{1 - 1/k_j^2}$.

The reflected radiation has intensity $r_j^+(\mu_j)J(\omega', x)$, whereas the refracted radiation has intensity $(1 - r_j^+(\mu_j))k_j^2 J(\omega', x)$. Here, r_j^+ and r_j^- are coefficients of the outer and inner reflections connected by

$$r_j^-(\widehat{\mu}_j) = r_j^+\big(\nu_j^+(\mu_j)\big), \qquad r_j^+(\mu_j) = r_j^-\big(-\nu_j^-(\mu_j)\big),$$

where $\nu_j^+(\widehat{\mu}_j) = \sqrt{1 - k_j^2(1 - (\widehat{\mu}_j)^2)}$.

In the classical geometric optics, according to the Fresnel formulas, the coefficient of the outer reflection for nonpolarized radiation has the form

$$r_j^+(\mu_j) = \frac{1}{2}\left[\left(\frac{\mu_j - k_j\nu_j^-(\mu_j)}{\mu_j + k_j\nu_j^-(\mu_j)}\right) + \left(\frac{k_j\mu_j - \nu_j^-(\mu_j)}{k_j\mu_j + \nu_j^-(\mu_j)}\right)\right], \quad 0 \le \mu_j \le 1.$$

From (3.1) we can find the value $\omega' = \omega - 2\mu_j n_j(x)$ for the direction of the incident radiation from the vacuum which is reflected in a given direction ω. From (3.2), where the direction of the refracted radiation is denoted by ω, we obtain the formula $\omega_{\mathcal{P}_j^-}(\omega, x) = -\nu_j^+(\mu_j)n_j(x) + k_j(\omega - \mu_j n_j(x))$ for the direction ω' of the incident radiation which is refracted in a prescribed direction ω.

Let $I|_{\Gamma_j^+}(\omega', x)$ be an intensity of the radiation propagating inside the domain G_j and falling to a point $x \in \partial G_j$ in the direction ω' under the incidence angle φ' with $\cos\varphi' = \mu'_j = \omega' \cdot n_j(x) > 0$. If $\mu'_j \le \widehat{\mu}_{j,\lim}$, then the effect of complete inner reflection holds: the radiation is completely reflected and propagates in the direction (3.1) under the reflection angle φ with $\cos\varphi = \mu_j = \omega \cdot n_j(x) = -\mu_j < 0$. If $\widehat{\mu}_{\lim,j} < \mu'_j$, then the radiation is partially reflected by the surface in the direction (3.1) and is partially refracted and goes from the body G_j to the vacuum along the direction

$$\widehat{\omega} = \widehat{\mu}_j n_j(x) + k_j\big(\omega' - \mu'_j n_j(x)\big) \tag{3.3}$$

under the refraction angle ψ with $\cos\psi = \widehat{\mu}_j = \widehat{\omega} \cdot n_j(x) = \nu_j^+(\mu'_j)$ The intensity of the reflected radiation is equal to $r_j^-(\mu_j)I|_{\Gamma_j^+}(\omega', x)$, whereas the intensity of the refracted radiation is equal to $(1 - r_j^-(\mu_j))\frac{1}{k_j^2} I|_{\Gamma_j^+}(\omega', x)$. From (3.1) we find the value $\omega' = \omega - \mu_j n_j(x)$ for the direction of the incident radiation which is reflected in a given direction ω. From (3.3), where the direction of the refracted radiation is denoted by ω, we have the direction $\omega_{\mathcal{P}_j^+}(\omega, x) = \nu_j^-(\mu_j) n_j(x) + \frac{1}{k_j}(\omega - \mu_j n_j(x))$ of the incident radiation refracted in a prescribed direction ω.

3.2 Boundary Operators

Let $\mu_j = \omega \cdot n_j(x)$ for $(\omega, x) \in \Gamma_j$, $1 \le j \le m$. We introduce the sets

$$\widehat{\Gamma}_j^- = \{(\omega, x) \in \Gamma_j^- \mid -\widehat{\mu}_{j,\lim} \le \mu_j < 0\},$$

$$\overset{\vee}{\Gamma}_j^- = \{(\omega, x) \in \Gamma_j^- \mid \mu_j < -\widehat{\mu}_{j,\lim}\}$$

and define the operators $\mathcal{R}^-$ and $\mathcal{R}^+$ of outer and inner reflections by

$$\mathcal{R}^-(I|_{\Gamma^+})(\omega, x) = r_j^-(\mu_j) I|_{\Gamma_j^+}(\omega - 2\mu_j n_j(x), x), \quad (\omega, x) \in \Gamma_j^-,\ 1 \le j \le m,$$

$$\mathcal{R}^+(J)(\omega, x) = r_j^+(\mu_j) J(\omega - 2\mu_j n_j(x), x), \quad (\omega, x) \in \Gamma_j^+,\ 1 \le j \le m.$$

Note that $r_j^-(\mu_j) = 1$ for $(\omega, x) \in \widehat{\Gamma}_j^-$ and, consequently,

$$\mathcal{R}^-(I|_{\Gamma^+})(\omega, x) = I|_{\Gamma_j^+}(\omega - 2\mu_j n_j(x), x), \quad (\omega, x) \in \widehat{\Gamma}_j^-,\ 1 \le j \le m.$$

We introduce the refraction operators $\mathcal{P}^-$ and $\mathcal{P}^+$ inside G and outside G by the formulas

$$\mathcal{P}^-(J)(\omega, x) = \begin{cases} (1 - r_j^-(\mu_j)) k_j^2 J(\omega_{\mathcal{P}_j^-}(\omega, x), x), & (\omega, x) \in \overset{\vee}{\Gamma}_j^-, \\ 0, & (\omega, x) \in \widehat{\Gamma}_j^-, \end{cases} \quad 1 \le j \le m,$$

$$\mathcal{P}^+(I|_{\Gamma^+})(\omega, x) = \left(1 - r_j^+(\mu_j)\right)\frac{1}{k_j^2} I|_{\Gamma_j^+}\left(\omega_{\mathcal{P}_j^+}(\omega, x), x\right),$$

$$(\omega, x) \in \Gamma_j^+,\ 1 \le j \le m.$$

Introduce the sets

$$S_j^- = \left\{(\omega, x) \in \Gamma_j^- \,\middle|\, x \in \partial G_j \setminus \bigcup_{i \ne j} \partial G_i\right\}, \quad S^- = \bigcup_{j=1}^m S_j^-,$$

$$\overset{*}{S}{}_j^- = \{(\omega, x) \in S_j^- \mid \{x - t\omega \mid t > 0\} \cap \overline{G} = \emptyset\}, \quad \overset{*}{S}{}^- = \bigcup_{j=1}^m \overset{*}{S}{}_j^-,$$

$$\widetilde{S}^- = \{(\omega, x) \in S^- \setminus \overset{*}{S}{}^- \mid (\omega, X^-(\omega, x)) \in \Gamma^+\},$$

where $X^-(\omega, x) = x - \tau^-(\omega, x)\omega$, $\tau^-(\omega, x) = \inf\{t > 0 \mid x - t\omega \in \overline{G}\}$.

We define the translation operator T by the formula

$$T\varphi(\omega, x) = \varphi(\omega, X^-(\omega, x)), \quad (\omega, x) \in \widetilde{S}^-.$$

Let $\partial G_j \cap \partial G_j \neq \emptyset$ for some $i \neq j$. We introduce $\Gamma_{ij}^- = \Gamma_i^- \cap \Gamma_j^+$ and define the operators $\mathcal{R}_{ij}^-$ and $\mathcal{P}_{ij}^-$ by the formulas

$$\mathcal{R}_{ij}^-(I|_{\Gamma_i^+})(\omega, x) = r_{ij}^-(\mu_i) I|_{\Gamma_i^+}(\omega - 2\mu_i n_i(x), x), \quad (\omega, x) \in \Gamma_{ij}^-,$$

$$\mathcal{P}_{ij}^-(I|_{\Gamma_j^+})(\omega, x) = \begin{cases} (1 - r_{ij}^-(\mu_i)) \frac{k_i^2}{k_j^2} I|_{\Gamma_j^+}(\omega_{\mathcal{P}_{ij}^-}(\omega, x), x), & (\omega, x) \in \overset{\vee}{\Gamma}{}_i^- \cap \Gamma_j^+, \\ 0, & (\omega, x) \in \widehat{\Gamma}_i^- \cap \Gamma_j^+, \end{cases}$$

where

$$\omega_{\mathcal{P}_{ij}^-}(\omega, x) = -\nu_{ij}^+(\mu_i) n_i(x) + \frac{k_i}{k_j}(\omega - \mu_i n_i(x)),$$

$$\nu_{ij}^+(\mu_i) = \sqrt{1 - k_i^2/k_j^2(1 - \mu_i^2)}.$$

If the surfaces ∂G_i and ∂G_j are separated by an infinitely thin vacuum layer at the points of tangency, then

$$r_{ij}^-(\mu_i) = \begin{cases} \frac{r_i^-(\mu_i) + r_j^+(\nu_i^+(\mu_i)) - 2r_i^-(\mu_i) r_j^+(\nu_i^+(\mu_i))}{1 - r_i^-(\mu_i) r_j^+(\nu_i^+(\mu_i))}, & -1 \le \mu_i < -\widehat{\mu}_{i,\lim}, \\ 1, & -\widehat{\mu}_{i,\lim} \le \mu_i \le 0. \end{cases}$$

In the general case, it is assumed that the reflection coefficients r_{ij}^- are continuous functions on $[-1, 0]$ possessing the following properties:

$$r_{ij}^-(\mu_i) = 1 \quad \text{for } -\widehat{\mu}_{i,\lim} \le \mu_i \le 0, \qquad 0 \le r_{ij}^-(\mu_i) < 1 \quad \text{for } -1 \le \mu_i < -\widehat{\mu}_{i,\lim},$$

$$r_{ji}^-(-\eta_{ij}^+(\mu_i)) = r_{ij}^-(\mu_i) \quad \text{for } \mu_i \in [-1, -\widehat{\mu}_{i,\lim}].$$

3.3 Statement of the Boundary Conditions

For $(\omega, x) \in \widetilde{S}^-$ the radiation J falling on ∂G from the vacuum goes from a point $X^-(\omega, x) \in \partial G$. This radiation is composed of the reflected and refracted radiations at the point $X^-(\omega, x)$:

$$J = T\mathcal{R}^+(J) + T\mathcal{P}^+(I|_{\Gamma^+}), \quad (\omega, x) \in \widetilde{S}^-.$$

For $(\omega, x) \in \overset{*}{S}^{-}$ the radiation J goes from outside, and we can assume that it is prescribed:

$$J = J_*, \quad (\omega, x) \in \overset{*}{S}^{-}.$$

For $(\omega, x) \in S^{-}$ the radiation $I|_{\Gamma^-}$ entering the body G is composed of the reflected and refracted radiations:

$$I|_{\Gamma^-} = \mathcal{R}^{-}(I|_{\Gamma^+}) + \mathcal{P}^{-}(J), \quad (\omega, x) \in S^{-}.$$

Finally, for $(\omega, x) \in \Gamma_{ij}^{-}$ the following condition is imposed:

$$I|_{\Gamma_i^-} = \mathcal{R}_{ij}^{-}(I|_{\Gamma_i^+}) + \mathcal{P}_{ij}^{-}(I|_{\Gamma_j^+}), \quad (\omega, x) \in \Gamma_{ij}^{-}.$$

3.4 Main Results

We consider the boundary value problem

$$\omega \cdot \nabla I + (\varkappa + s)I = s\mathcal{S}(I) + \varkappa k^2 F, \quad (\omega, x) \in D, \tag{3.4}$$

$$I|_{\Gamma^-} = \mathcal{R}^{-}(I|_{\Gamma^+}) + \mathcal{P}^{-}(J), \quad (\omega, x) \in S^{-}, \tag{3.5}$$

$$I|_{\Gamma_i^-} = \mathcal{R}_{ij}^{-}(I|_{\Gamma_i^+}) + \mathcal{P}_{ij}^{-}(I|_{\Gamma_j^+}), \quad (\omega, x) \in \Gamma_{ij}^{-},\ i \neq j, \tag{3.6}$$

$$J = T\mathcal{R}^{+}(J) + T\mathcal{P}^{+}(I|_{\Gamma^+}), \quad (\omega, x) \in \widetilde{S}^{-}, \tag{3.7}$$

$$J = J_*, \quad (\omega, x) \in \overset{*}{S}^{-}, \tag{3.8}$$

describing the radiative transfer in the system of semitransparent bodies with taking into account reflection and refraction on their boundaries.

By a solution to the problem (3.4)–(3.8) we mean a function $I \in \mathcal{W}^1(D)$ that satisfies (3.4) almost everywhere (a.e.) on D and the conditions (3.5), (3.6) a.e. on S^-, $\bigcup_{i \neq j} \Gamma_{ij}^{-}$ respectively. Moreover, the function J such that $(1 - r^+)J \in \widehat{L}^1(S^-)$ satisfies the conditions (3.7) and (3.8) a.e. on $\widetilde{S}^-$ and $\overset{*}{S}^-$ respectively.

Theorem 3.1 [7] *Let $F \in L^p(D)$, $J_* \in \widehat{L}^p(\overset{*}{S}^-)$ with some $p \in [1, \infty]$. Then the problem (3.4)–(3.8) has a unique solution $I \in \mathcal{W}^p(D)$. Moreover, the solution satisfies the estimates* (*where* $1/q = 1 - 1/p$)

$$\|\varkappa^{1/p} k^{-2/q} I\|_{L^p(D)} \leq \left(\|\varkappa^{1/p} k^{2/p} F\|_{L^p(D)}^p + \|J_*\|_{\widehat{L}^p(\overset{*}{S}^-)}^p\right)^{1/p},$$

$$\|\varkappa^{-1/q} k^{-2/q} \omega \cdot \nabla I\|_{L^p(D)} \leq \frac{2}{1 - \varpi_{\max}} \left(\|\varkappa^{1/p} k^{2/p} F\|_{L^p(D)}^p + \|J_*\|_{\widehat{L}^p(\overset{*}{S}^-)}^p\right)^{1/p}$$

for $1 \le p < \infty$ *and the estimates*

$$\left\|k^{-2} I\right\|_{L^\infty(D)} \le \max\left\{\|F\|_{L^\infty(D)}, \|J_*\|_{L^\infty(\overset{*}{S}^-)}\right\},$$

$$\left\|\varkappa^{-1} k^{-2} \omega \cdot \nabla I\right\|_{L^\infty(D)} \le \frac{2}{1-\varpi_{\max}} \max\left\{\|F\|_{L^\infty(D)}, \|J_*\|_{L^\infty(\overset{*}{S}^-)}\right\}$$

for $p = \infty$. *Here* $\varpi_{\max} = \max_{1 \le j \le m} \frac{s_j}{\varkappa_j + s_j}$.

The result about continuous dependence in $\mathcal{W}^1(D)$ norm of solutions of the problem (3.4)–(3.8) on the data is proved in [9]. Moreover it is shown that if $k \to 1$ then the solutions of the problem (3.4)–(3.8) tend to the solution of the following problem with "shooting conditions":

$$\omega \cdot \nabla I + (\varkappa + s) I = s\mathcal{S}(I) + \varkappa F, \quad (\omega, x) \in D,$$

$$I|_{\Gamma^-} = T(I|_{\Gamma^+}), \quad (\omega, x) \in \widetilde{S}^-,$$

$$I|_{\Gamma^-} = J_*, \quad (\omega, x) \in \overset{*}{S}^-,$$

$$I|_{\Gamma_i^-} = I|_{\Gamma_j^+}, \quad (\omega, x) \in \Gamma_{ij}^-, \ i \ne j.$$

4 Boundary Value Problem for the Radiation Transfer Equation with Diffuse Reflection and Refraction Conditions

4.1 Diffuse Reflection and Diffuse Refraction Laws. Boundary Operators

Let $J(\omega', x)$ be the intensity of the radiation propagating in the vacuum and falling on the surface ∂G_j at a point x in a direction $\omega' \in \Omega_j^-$. This radiation is partially diffusely reflected (i.e., it has the same intensity in all directions $\omega \in \Omega_j^+(x)$) by the surface and partially diffusely refracted (i.e., it has the same intensity in all directions $\omega \in \Omega_j^-(x)$), entering the body G_j. The intensities of reflected and refracted radiations are independent of the propagation direction and are equal, respectively,

$$\mathcal{R}_d^+(J)(\omega, x) \equiv \frac{\rho_j^+(x)}{\pi} \int_{\Omega_j^-(x)} J(\omega', x) \left|\omega' \cdot n_j(x)\right| d\omega,$$

$$(\omega, x) \in \Gamma_j^+, \ 1 \le j \le m,$$

$$\mathcal{P}_d^-(J)(\omega, x) \equiv \frac{1-\rho_j^+(x)}{\pi} \int_{\Omega_j^-(x)} J(\omega', x) \left|\omega' \cdot n_j(x)\right| d\omega,$$

$$(\omega, x) \in \Gamma_j^-, \ 1 \le j \le m,$$

where $\rho_j^+ > 0$ is the reflective ability of the surface ∂G_j. We assume that $\rho_j^+ \in L_\infty(\partial G_j)$ and $\|\rho_j^+\|_{L^\infty(\partial G_j)} < 1$ for all $1 \le j \le m$.

Let $I|_{\Gamma_j^+}(\omega', x)$ be an intensity of the radiation propagating in G_j and falling on the surface ∂G_j at a point x in a direction $\omega' \in \Omega_j^+(x)$. This radiation is partially diffusely reflected (i.e., it has the same intensity in all directions $\omega \in \Omega_j^-(x)$) and partially diffusely refracted (i.e., it has the same intensity in all directions $\omega \in \Omega_j^+(x)$), going out from the body to the vacuum. The intensities of the reflected and refracted radiations are given by

$$\mathcal{R}_d^-(I|_{\Gamma^+})(\omega, x) \equiv \frac{\rho_j^-(x)}{\pi} \int_{\Omega_j^+(x)} I|_{\Gamma_j^+}\big(\omega', x\big)\omega' \cdot n_j(x) d\omega,$$

$$(\omega, x) \in \Gamma_j^-, \; 1 \le j \le m,$$

$$\mathcal{P}_d^+(I|_{\Gamma^+})(\omega, x) \equiv \frac{1-\rho_j^-(x)}{\pi} \int_{\Omega_j^+(x)} I|_{\Gamma_j^+}\big(\omega', x\big)\omega' \cdot n_j(x) d\omega,$$

$$(\omega, x) \in \Gamma_j^+, \; 1 \le j \le m.$$

Here, $\rho_j^-(x) = 1 - \frac{1}{k_j^2}(1 - \rho_j^+(x))$.

Let x be a point of tangency of ∂G_i and ∂G_j for some $i \ne j$. Let $I|_{\Gamma_i^-}$ be the intensity of the radiation entering into G_i in a direction $\omega \in \Omega_i^-(x)$. This radiation is composed of diffusely reflected and refracted radiations:

$$I|_{\Gamma_i^-}(\omega, x) = \mathcal{R}_{d,ij}^-(I|_{\Gamma_i^+})(\omega, x) + \mathcal{P}_{d,ij}^-(I|_{\Gamma_j^+})(\omega, x), \tag{4.1}$$

where

$$\mathcal{R}_{d,ij}^-(I|_{\Gamma_i^+})(\omega, x) = \frac{\rho_{ij}^-(x)}{\pi} \int_{\Omega_i^+(x)} I|_{\Gamma_i^+}\big(\omega', x\big)\omega \cdot n_i(x) d\omega, \quad (\omega, x) \in \Gamma_{ij}^-,$$

$$\mathcal{P}_{d,ij}^-(I|_{\Gamma_j^+})(\omega, x) = \frac{1-\rho_{ij}^-(x)}{\pi} \int_{\Omega_j^+(x)} I|_{\Gamma_j^+}\big(\omega', x\big)\omega \cdot n_j(x) d\omega, \quad (\omega, x) \in \Gamma_{ij}^-,$$

$$\rho_{ij}^-(x) = 1 - \frac{(1-\rho_i^-(x))(1-\rho_j^+(x))}{1-\rho_i^+(x)\rho_j^+(x)}.$$

4.2 Statement of the Boundary Conditions

For $(\omega, x) \in S^-$ the radiation entering into G is composed of the diffusely reflected and diffusely refracted radiations:

$$I|_{\Gamma^-} = \mathcal{R}_d^-(I|_{\Gamma^+}) + \mathcal{P}_d^-(J), \quad (\omega, x) \in S^-.$$

For $(\omega, x) \in \overset{*}{S}{}^-$ the radiation J goes from outside, and we can assume that it is prescribed:

$$J = J_*, \quad (\omega, x) \in \overset{*}{S}{}^-.$$

For $(\omega, x) \in \widetilde{S}^-$ the radiation J falling on ∂G from the vacuum comes from a point $X^-(\omega, x) \in \partial G$. It is composed of the diffusely reflected and diffusely refracted radiations:

$$J = T\mathcal{R}_d^+(J) + T\mathcal{P}_d^+(I|_{\Gamma^+}), \quad (\omega, x) \in \widetilde{S}^-.$$

Finally, for $(\omega, x) \in \Gamma_{ij}^-$ the following condition is imposed:

$$I|_{\Gamma_i^-} = \mathcal{R}_{d,ij}^-(I|_{\Gamma_i^-}) + \mathcal{P}_{d,ij}^-(I|_{\Gamma_j^-}). \tag{4.2}$$

4.3 Main Results

We consider the boundary value problem

$$\omega \cdot \nabla I + (\varkappa + s)I = s\mathcal{S}(I) + \varkappa k^2 F, \quad (\omega, x) \in D, \tag{4.3}$$

$$I|_{\Gamma^-} = \mathcal{R}_d^-(I|_{\Gamma^+}) + \mathcal{P}_d^-(J), \quad (\omega, x) \in S^-, \tag{4.4}$$

$$I|_{\Gamma_i^-} = \mathcal{R}_{d,ij}^-(I|_{\Gamma_i^+}) + \mathcal{P}_{d,ij}^-(I|_{\Gamma_j^+}), \quad (\omega, x) \in \Gamma_{ij}^-, \tag{4.5}$$

$$J = T\mathcal{R}_d^+(J) + T\mathcal{P}_d^+(I|_{\Gamma^+}), \quad (\omega, x) \in \widetilde{S}^-, \tag{4.6}$$

$$J = J_*, \quad (\omega, x) \in \overset{*}{S}{}^-, \tag{4.7}$$

describing the radiation transfer in the system of semitransparent bodies with taking into account diffuse reflection and diffuse refraction on their boundaries.

By a solution to the problem (4.3)–(4.7) we mean a function $I \in \widehat{\mathcal{W}}^1(D)$ that satisfies (4.3) a.e. on D and the conditions (4.4), (4.5) a.e. on S^-, $\underset{i \neq j}{\Gamma}{}_{ij}^-$ respectively. Moreover, the function $J \in \widehat{L}^1(S^-)$ satisfies the conditions (4.6) and (4.7) a.e. on $\widetilde{S}^-$ and $\overset{*}{S}{}^-$ respectively.

Theorem 4.1 [8] *Let $F \in L^\infty(D)$, $J_* \in \widehat{L}^{1,\infty}(\overset{*}{S}{}^-)$. Then the problem (4.3)–(4.7) has a unique solution $I \in \mathcal{W}^\infty(D)$ Moreover, it satisfies the estimates*

$$\left\| k^{-2} I \right\|_{L^\infty(D)} \le \max\left\{ \|F\|_{L^\infty(D)}, \|J_*\|_{\widehat{L}^{1,\infty}(\overset{*}{S}{}^-)} \right\},$$

$$\left\| \varkappa^{-1} k^{-2} \omega \cdot \nabla I \right\|_{L^\infty(D)} \le \frac{2}{1 - \varpi_{\max}} \max\left\{ \|F\|_{L^\infty(D)}, \|J_*\|_{\widehat{L}^{1,\infty}(\overset{*}{S}{}^-)} \right\}.$$

Theorem 4.2 [8] *Let* $\partial G_j \in C^{1+\lambda}$ *for all* $1 \le j \le m$ *with some* $\lambda \in (0, 1)$. *Suppose that* $F \in L^p(D)$ *and* $J_* \in L^{1,p}(\overset{*}{S}{}^-)$, *where* $p \in (1 + (2\lambda)^{-1}, \infty)$. *Then the problem* (4.3)–(4.7) *has a unique solution* $I \in \widehat{\mathcal{W}}^p(D)$. *Moreover, it satisfies the estimates*

$$\big\|\varkappa^{1/p}k^{-2/q}I\big\|_{L^p(D)} \le \Big(\big\|\varkappa^{1/p}k^{2/p}F\big\|^p_{L^p(D)} + \|J_*\|^p_{\widehat{L}^{1,p}(\overset{*}{S}{}^-)}\Big)^{1/p},$$

$$\big\|\varkappa^{-1/q}k^{-2/q}\omega\cdot\nabla I\big\|_{L^p(D)} \le \frac{2}{1-\varpi_{\max}}\Big(\big\|\varkappa^{1/p}k^{2/p}F\big\|^p_{L^p(D)} + \|J_*\|^p_{\widehat{L}^{1,p}(\overset{*}{S}{}^-)}\Big)^{1/p}.$$

Theorem 4.2 does not cover the case $p = 1$. We note that this case is mathematically difficult, but is important from the physical point of view. We consider this case under two additional assumptions.

Theorem 4.3 [8] *Let* $\text{meas}(\partial G_i \cap \partial G_j; d\sigma) = 0$ *for all* $i \ne j$ *and there exists* $\overline{\alpha} \in [0, 1)$ *such that* $\frac{1}{\pi}\int_{\Omega_i^-(x)} \chi_{\widetilde{S}^-}(\omega, x)|\omega\cdot n(x)|d\omega \le \overline{\alpha}$ *for a.e.* $x \in \partial G_j$ *and all* $1 \le j \le m$, *where* $\chi_{\widetilde{S}^-}$ *is the characteristic function of the set* $\widetilde{S}^-$.

Let $F \in L^p(D)$, $J_* \in \widehat{L}^{1,p}(\overset{*}{S}{}^-)$, $1 \le p < \infty$. *Then the problem* (4.3)–(4.7) *has a unique solution* $I \in \widehat{\mathcal{W}}^p(D)$. *Moreover, it satisfies the same estimates as in Theorem* 4.2.

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Asymptotic Approximations of a Thin Elastic Beam with Auxiliary Coupled 1D System due to Robin Boundary Condition

Z. Bare, J. Orlik, and G. Panasenko

Abstract In Bare et al. (Appl. Anal., 2013, doi:10.1080/00036811.2013.823481), the dimension of a 3D linear elasticity boundary value problem with Robin boundary condition is asymptotically reduced. Assumption 1.4 in Bare et al. (Appl. Anal., 2013, doi:10.1080/00036811.2013.823481), leads to a 1D system in which the bending and tensile components are decoupled. With a generalization in this contribution, we obtain a coupled 1D system. We prove that the asymptotic error estimate in Bare et al. (Appl. Anal., 2013, doi:10.1080/00036811.2013.823481) remains true and illustrate the influence of the tension and torsion on the bending by a numerical example.

Keywords Asymptotic dimension reduction · Elasticity · Robin boundary condition · Coupled ordinary differential equations for beams · Solvability

Mathematics Subject Classification (2010) 41A60 · 35J25 · 74K10

1 Introduction

This work is an extension of [2]. In [2], the dimension of a 3D linear elasticity boundary value problem with Robin boundary condition in a thin beam was asymptotically reduced with respect to the relative thickness of the beam $0 < \varepsilon \ll 1$. The error between the truncated asymptotic expansion $\boldsymbol{u}^K$ and the exact 3D solution $\boldsymbol{u}^\varepsilon$

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was estimated as

$$\left\| \boldsymbol{u}^{\varepsilon} - \boldsymbol{u}^{K} \right\|_{H^1(\Omega^\varepsilon)} \leq C\varepsilon^{K+1}\sqrt{\left|\Omega^\varepsilon\right|}, \tag{1.1}$$

for a constant $C > 0$ independent of ε. The construction of $\boldsymbol{u}^K$ deals with a recursive chain of 1D limit systems of equations and with a recursive chain of boundary value problems in dilated domains, independent of ε. In [2], the Robin parameters and the Robin boundary condition were assumed in such a way that the recursive 1D limit consisted of decoupled problems, up to a unilateral coupling of the angle of twist and the bending, tangential to Robin condition surface. In this contribution, we show that with an alternative scaling we obtain a coupled 1D limit, generalizing certain assumptions in [2] (see I–II). We prove that the estimate (1.1) remains true. The main effort in this proof is to show that the coupled limit 1D problem is solvable. For a coupled 1D limit due to large deformations, non-symmetric cross-sections or anisotropic materials we refer the reader to [6].

We construct the asymptotic approximation for the following 3D problem: Find $\boldsymbol{u}^\varepsilon \in (H^1(\Omega^\varepsilon, \partial\Omega^\varepsilon_U))^3 := \{\boldsymbol{v} \in (H^1(\Omega^\varepsilon))^3 : \boldsymbol{v}|_{\partial\Omega^\varepsilon_U} = 0\}$ s.t. $\forall \boldsymbol{v} \in (H^1(\Omega^\varepsilon, \partial\Omega^\varepsilon_U))^3$

$$\sum_{i,j=1}^{3} \int_{\Omega^\varepsilon} \mathcal{A}_{ij} \frac{\partial \boldsymbol{u}^\varepsilon}{\partial x_j} \frac{\partial \boldsymbol{v}}{\partial x_i}\, \mathrm{d}\boldsymbol{x} + \sum_{k=1}^{3} s^\varepsilon_k \int_{S^\varepsilon_C} u^\varepsilon{}_k v_k \,\mathrm{d}\sigma = \int_{\Omega^\varepsilon} \boldsymbol{\Psi}^\varepsilon \boldsymbol{v}\, \mathrm{d}\boldsymbol{x}, \tag{1.2}$$

with $s^\varepsilon_1 = s_1 \geq 0$, $s^\varepsilon_j = \varepsilon^2 s_j \geq 0$, $j = 2, 3$, see [2, 3], $\mathcal{A} = (a^{kl}_{ij})_{i,j,k,l=1,2,3}$ and $a^{kl}_{ij} = (\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk})\mu + \lambda(\delta_{ik}\delta_{jl})$, $\lambda, \mu > 0$,

$$\boldsymbol{\Psi}^\varepsilon = \boldsymbol{\Phi}\left(\frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon}\right)\boldsymbol{\Pi}^\varepsilon \boldsymbol{f}^\varepsilon, \tag{1.3}$$

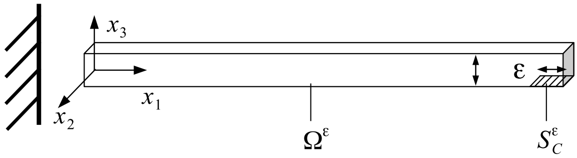
let $\boldsymbol{f}^\varepsilon = (f_1, \varepsilon^2 f_2, \varepsilon^2 f_3, f_4)^T$, $f_i \in \mathbb{R}$, $i = 1, 2, 3, 4$, be constants

$$\boldsymbol{\Phi}\left(\frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -(\frac{1}{|\omega|} I_p)^{-\frac{1}{2}} \frac{x_3}{\varepsilon} \\ 0 & 0 & 1 & (\frac{1}{|\omega|} I_p)^{-\frac{1}{2}} \frac{x_2}{\varepsilon} \end{pmatrix}, \qquad \boldsymbol{\Pi}^\varepsilon = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$I_p = \int_\omega \left(\left(\frac{x_2}{\varepsilon}\right)^2 + \left(\frac{x_3}{\varepsilon}\right)^2\right) d\left(\frac{\tilde{\boldsymbol{x}}}{\varepsilon}\right).$$

$\omega \in \mathbb{R}^2$ is an open, connected and symmetric (in the x_2, x_3-axes) polygon with non zero measure whose boundary $\partial\omega$ decomposes into two disjoint subsets, $\partial\omega = \partial\omega_N \cup \partial\omega_R$, where $\partial\omega_R = \{\frac{\tilde{\boldsymbol{x}}}{\varepsilon} \in \partial\omega : \frac{x_3}{\varepsilon} = \min_{\frac{\tilde{\boldsymbol{x}}}{\varepsilon} \in \partial\omega} \frac{x_3}{\varepsilon}\}$ is a segment. Let the beam be given by $\Omega^\varepsilon = \{\boldsymbol{x} \in \mathbb{R}^3 : \tilde{\boldsymbol{x}} \in \varepsilon\omega, x_1 \in (0, 1)\}$, $\varepsilon > 0$ denotes the relative thickness of Ω^ε. The beam is fixed on $\partial\Omega^\varepsilon_U = \{\boldsymbol{x} \in \overline{\Omega}^\varepsilon : x_1 = 0\}$ and the Robin boundary is denoted as $S^\varepsilon_C = \{\boldsymbol{x} \in \overline{\Omega}^\varepsilon : x_1 \in [1 - \varepsilon, 1], \frac{\tilde{\boldsymbol{x}}}{\varepsilon} \in \partial\omega_R\}$. The free boundary is denoted by $\partial\Omega^\varepsilon_N = \partial\Omega^\varepsilon \backslash (S^\varepsilon_C \cup \partial\Omega^\varepsilon_U)$, the coordinate system is chosen in such a way,

Fig. 1 Geometry



that on S_C^ε the outward unit normal $\boldsymbol{\nu} = (n_1, n_2, n_3)^T$ takes the form $\boldsymbol{\nu} = -\boldsymbol{e}_3$ (see Fig. 1).

In [2], we considered another scaling corresponding to $\boldsymbol{\Pi}^\varepsilon$ replaced by $\boldsymbol{Id}$ in (1.3), with an additional assumption, [2, Assumption 1.4], that either

I $s_1 = 0$, or
II $S_C^\varepsilon = \{\boldsymbol{x} \in \overline{\Omega}^\varepsilon : x_1 = 1, \frac{\tilde{\boldsymbol{x}}}{\varepsilon} \in \omega\}$, i.e. the Robin boundary condition is at the right extremity of the beam.

Then, estimate (1.1) was proven. In the present paper, we drop these assumptions and point out the modifications in the procedure of construction of the asymptotic expansion. Namely, now we seek an asymptotic solution in the form of the ansatz similar to [1, 2, 5], differing to the ansatz in [2] by the factor $\boldsymbol{\Pi}^\varepsilon$ below

$$\boldsymbol{u}^K = \boldsymbol{\Phi}(\tilde{\boldsymbol{y}})\boldsymbol{u}(x_1) + \sum_{k+s=1}^{K+1} \varepsilon^{k+s} \boldsymbol{N}_{k,s}(\boldsymbol{y}) \frac{\partial^k \boldsymbol{u}(x_1)}{\partial x_1^k} + \hat{\boldsymbol{\Phi}}\left(0, \frac{\tilde{\boldsymbol{x}}}{\varepsilon}\right)(1 - x_1)\boldsymbol{\rho}_{K+1}(x_1) \tag{1.4}$$

$$\boldsymbol{u}(x_1) = \boldsymbol{\Pi}^\varepsilon \sum_{i=0}^{K} \varepsilon^i \overline{\boldsymbol{u}}^i(x_1), \tag{1.5}$$

with $\boldsymbol{y} = (\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon})$, $\tilde{\boldsymbol{y}} = (\frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon})$, $\overline{\boldsymbol{y}} = \boldsymbol{y} + (\frac{1}{\varepsilon}, 0, 0)$, $\boldsymbol{N}_{k,s}(\boldsymbol{y}) = \boldsymbol{N}_{k,s}^{inner}(\tilde{\boldsymbol{y}}) + \chi(x_1)\boldsymbol{N}_{k,s}^{+}(\overline{\boldsymbol{y}}) + \chi(1 - x_1)\boldsymbol{N}_{k,s}^{-}(\boldsymbol{y})$, $\boldsymbol{N}_{k,s}^{inner}, \boldsymbol{N}_{k,s}^{\pm}$, are matrix valued functions with values in $\mathbb{R}^{3\times 4}$ (vanishing by convention if s or k is negative) and $\boldsymbol{N}_{k,s}^{inner}$, $\boldsymbol{N}_{k,s}^{+} = 0$ for $s \neq 0$, $\chi \in C^{K+2}([0, 1])$ is a cut-off function $\chi(x_1) = \begin{cases} 1 \text{ if } x_1 \in [0, \frac{1}{3}] \\ 0 \text{ if } x_1 \in [\frac{2}{3}, 1] \end{cases}$, see [2, 3]. $\hat{\boldsymbol{\Phi}}(0, \tilde{\boldsymbol{y}})(1 - x_1)\boldsymbol{\rho}_{K+1}(x_1)$ is a remainder that guarantees that $\boldsymbol{u}^K \in (H^1(\Omega^\varepsilon, \partial\Omega_U^\varepsilon))^3$. $\hat{\boldsymbol{\Phi}}(\boldsymbol{y}) = \begin{pmatrix} 1 & 0 & 0 & 0 & -y_2 & -y_3 \\ 0 & 1 & 0 & -(\frac{1}{|\omega|} I_p)^{-\frac{1}{2}} y_3 & y_1 & 0 \\ 0 & 0 & 1 & (\frac{1}{|\omega|} I_p)^{-\frac{1}{2}} y_2 & 0 & y_1 \end{pmatrix}$ is the extended matrix of rigid displacements, see [5, p. 66], $\boldsymbol{\rho}_{K+1}(x_1) = \sum_{l=K+1}^{2K+1} \varepsilon^l \sum_{0 \le i \le K, i+k=l, 0 \le k \le K+1} \boldsymbol{h}^{N_{k,0}^{+}} \frac{\partial^k \overline{\boldsymbol{u}}^i}{\partial x_1^k}$, where $\boldsymbol{h}^{N_{k,0}^{+}}$ are standard $\mathbb{R}^{6\times 4}$ matrices which can be computed as in [5]. It is reminded that a scaling of the tensile force and the bending forces f_i^ε, $i = 2, 3$, as in (1.3) and the corresponding scaling of the tensile and bending components of the ansatz (1.5) is also chosen in [6, p. 517].

We obtain a recursive chain of coupled 1D problems in [0, 1], for the components of the functions $\overline{\boldsymbol{u}}^i$, (1.5), following [2, 5]. We briefly remind that an important step of the construction of the recursive problems is the following requirement.

Requirement 1 (Key of the procedure) For

$$\boldsymbol{H}^{(\cdot_{k,s})} = -\sum_{i,j=1}^{3} \frac{\partial}{\partial y_i}\left(\mathcal{A}_{ij}\frac{\partial \cdot_{k,s}}{\partial y_j}\right) - \sum_{i=1}^{3}(\mathcal{A}_{i1}+\mathcal{A}_{1i})\frac{\partial \cdot_{k-1,s}}{\partial y_i} - \mathcal{A}_{11}\cdot_{k-2,s},$$

$$\boldsymbol{G}^{(\cdot_{k,s})} = \sum_{j=1}^{3}\left(\sum_{i=1}^{3}\mathcal{A}_{ji}\frac{\partial \cdot_{k,s}}{\partial y_i}\right)n_j + \sum_{i=1}^{3}A_{i1}(\cdot_{k-1,s})n_i,$$

require $\boldsymbol{H}^{\boldsymbol{N}^{inner}_{k,s}}(\tilde{\boldsymbol{y}}) = \boldsymbol{\Phi}(\tilde{\boldsymbol{y}})\boldsymbol{h}^{\boldsymbol{N}^{inner}_{k,s}}$ in ω, $\boldsymbol{N}_{k,s}(\boldsymbol{y}) = \hat{\boldsymbol{\Phi}}(0,\tilde{\boldsymbol{y}})\boldsymbol{h}^{\boldsymbol{N}^{+}_{k,s}}$ on $\{+0\}\times\omega$, $\boldsymbol{G}^{\boldsymbol{N}_{k,s}}(\boldsymbol{y}) = \hat{\boldsymbol{\Phi}}(0,\tilde{\boldsymbol{y}})\boldsymbol{h}^{\boldsymbol{N}^{-}_{k,s}}$ on $\{-0\}\times\omega$, $\boldsymbol{H}^{\boldsymbol{N}^{\pm}_{k,s}}(\boldsymbol{y}) = 0$ in Ω, $\boldsymbol{G}^{\boldsymbol{N}^{inner}_{k,s}}(\tilde{\boldsymbol{y}}) = 0$ on $\partial\omega$, $\boldsymbol{G}^{\boldsymbol{N}^{+}_{k,s}}(\boldsymbol{y}) = 0$ on $(((-\infty,0)\cup(0,\infty))\times\partial\omega)$, $\boldsymbol{G}^{\boldsymbol{N}^{-}_{k,s}}(\boldsymbol{y}) = 0$ on $(((-\infty,0)\cup(0,\infty))\times\partial\omega)\backslash S_C$, $\boldsymbol{G}^{\boldsymbol{N}^{-}_{k,s}}(\boldsymbol{y}) + \boldsymbol{S}_0\boldsymbol{N}_{k,s-1}(\boldsymbol{y}) + \boldsymbol{S}_2\boldsymbol{N}_{k,s-3}(\boldsymbol{y}) = 0$ on S_C, where $\boldsymbol{S}_1 = \mathrm{diag}(s_1,0,0)$, $\boldsymbol{S}_2 = \mathrm{diag}(0,s_2,s_3)$ and $\boldsymbol{h}^{\boldsymbol{N}^{inner}_{k,s}} \in \mathbb{R}^{4\times 4}$, $\boldsymbol{h}^{\boldsymbol{N}^{+}_{k,s}}, \boldsymbol{h}^{\boldsymbol{N}^{-}_{k,s}} \in \mathbb{R}^{6\times 4}$ are constant matrices.

The recursive chain of coupled 1D problems is obtained substituting the ansatz (1.4) into the problem (1.2) and calculating $\boldsymbol{h}^{\boldsymbol{N}^{inner}_{k,s}}$, $\boldsymbol{h}^{\boldsymbol{N}^{+}_{k,s}}, \boldsymbol{h}^{\boldsymbol{N}^{-}_{k,s}}$ from solvability and decay conditions in auxiliary problems for $\boldsymbol{N}^{inner}_{k,s}, \boldsymbol{N}^{\pm}_{k,s}$ resulting from Requirement 1. The problem for the fourth component, the angle of twist $\overline{u}^i_4$, can be solved independently from the other components, then it enters the right hand side of the problem for $\overline{u}^i_2$, see [2, Theorem 1.6]. At the left end of the interval [0, 1] all the four components satisfy generally non-homogeneous Dirichlet conditions. For the variational formulation of these 1D problems, as usual we subtract the right handside and change the unknown functions $\overline{\boldsymbol{u}}^k$ for $\boldsymbol{z}^k$

$$z^k_i = \overline{u}^k_i + g^{\mathrm{rec}(k)}_{i2} \quad i = 1,4 \tag{1.6}$$

$$z^k_j = \overline{u}^k_j + x_1 g^{\mathrm{rec}(k)}_{j3} + g^{\mathrm{rec}(k)}_{j4} \quad j = 2,3, \tag{1.7}$$

where

$$g^{\mathrm{rec}(k)}_{i2} = \sum_{l=0}^{k-1}\left(\boldsymbol{h}^{\boldsymbol{N}^{+}_{k-l,0}}\frac{\partial^{k-l}\boldsymbol{u}^{\mathrm{rec}(l)}}{\partial x_1^{k-l}}\right)_i\Bigg|_{x_1=0},$$

$$g^{\mathrm{rec}(k)}_{j3} = \sum_{l=0}^{k-1}\left(\boldsymbol{h}^{\boldsymbol{N}^{+}_{k-l,0}}\frac{\partial^{k-l}\boldsymbol{u}^{\mathrm{rec}(l)}}{\partial x_1^{k-l}}\right)_j\Bigg|_{x_1=0},$$

$$g^{\mathrm{rec}(k)}_{j4} = \sum_{l=0}^{k-1}\left(\boldsymbol{h}^{\boldsymbol{N}^{+}_{k-l+1,0}}\frac{\partial^{k-l+1}\boldsymbol{u}^{\mathrm{rec}(l)}}{\partial x_1^{k-l+1}}\right)_{j+3}\Bigg|_{x_1=0}, \quad \boldsymbol{u}^{\mathrm{rec}(l)} = \left(z^l_1, z^l_2, z^l_3, z^l_4\right)^T.$$

The variational formulation of the recursive coupled 1D system is, find $z_4^k \in \mathcal{V} := \{u \in H^1((0,1)) : u(0) = 0\}$ and $\boldsymbol{z}^k \in \mathcal{H} := \{\boldsymbol{u} = (u_1, u_2, u_3)^T : u_1 \in \mathcal{V}, u_j \in \mathcal{W}, j = 2, 3\}$, $\mathcal{W} = \{z \in H^2((0,1)) : z(0) = \frac{\partial z(0)}{\partial x_1} = 0\}$, s.t. $\forall v_4 \in \mathcal{V}$ and $\forall \boldsymbol{v} \in \mathcal{H}$

$$a_4\big(z_4^k, v_4\big) = \ell_4^k(v_4), \tag{1.8}$$

$$a_{1D}\big(\boldsymbol{z}^k, \boldsymbol{v}\big) = \ell^k(\boldsymbol{v}), \tag{1.9}$$

where $a_4(\cdot,\cdot) = \mu J \int_0^1 \frac{\partial \cdot}{\partial x_1} \frac{\partial \cdot}{\partial x_1} \,\mathrm{d}x_1$, $\ell_4^k(v_4) = \int_0^1 (\hat{f}_4 - \hat{f}_4^{\mathrm{rec}(k)}) v_4 \,\mathrm{d}x_1 - \hat{g}_4^{\mathrm{rec}(k)} v_4|_{x_1=1}$, $J = (A - \int_\omega (\frac{\partial (n_{1,0}^{inner})^{14}(\tilde{\boldsymbol{y}})}{\partial y_2})^2 + (\frac{\partial (n_{1,0}^{inner})^{14}(\tilde{\boldsymbol{y}})}{\partial y_3})^2 d\tilde{\boldsymbol{y}})$ is the torsional constant, where $A = |\omega|$ and the warping function or St. Venants function $(n_{1,0}^{inner})^{14}$ satisfies

$$\begin{aligned} -\Delta\big(n_{1,0}^{inner}\big)^{14}(\tilde{\boldsymbol{y}}) &= 0 \quad \text{in } \omega \\ \frac{\partial (n_{1,0}^{inner})^{14}(\tilde{\boldsymbol{y}})}{\partial \tilde{\boldsymbol{\nu}}} &= \left(\frac{1}{|\omega|} I_p\right)^{-\frac{1}{2}} y_2 n_3 - \left(\frac{1}{|\omega|} I_p\right)^{-\frac{1}{2}} y_3 n_2 \quad \text{on } \partial\omega, \\ \int_\omega \big(n_{1,0}^{inner}\big)^{14} \,\mathrm{d}\tilde{\boldsymbol{y}} &= 0. \end{aligned} \tag{1.10}$$

$$f_4^{\mathrm{rec}(k)} = \sum_{l=0}^{k-1} \left(\boldsymbol{h}^{N_{k-l+2,0}^{inner}} \frac{\partial^{k-l+2} \boldsymbol{u}^{\mathrm{rec}(l)}}{\partial x_1^{k-l+2}} \right)_4$$

and

$$g_{41}^{\mathrm{rec}(k)} = \sum_{l=0}^{k-1} \sum_{i+j=k-l+1}^{k-l+1} \left(\boldsymbol{h}^{N_{i,j}^-} \frac{\partial^i \boldsymbol{u}^{\mathrm{rec}(l)}}{\partial x_1^i} \right)_4 \Bigg|_{x_1=1},$$

$\hat{f}_i$, $\hat{g}_{\alpha\beta}$, $\hat{s}_i$ denote normalized values $|\omega| f_i$, $|\omega| g_{\alpha\beta}$, $|\omega| s_i$, $a_{1D}(\boldsymbol{z}^k, \boldsymbol{v}) = \sum_{i=1}^3 a_i(z_i^k, v_i) + \sum_{i=1}^3 b_i(z_i^k, v_i) + c(z_3^k, v_1) + c(v_3, z_1^k)$ with

$$a_1\big(z_1^k, v_1\big) := \int_0^1 EA \frac{\partial z_1^k}{\partial x_1} \frac{\partial v_1}{\partial x_1} \,\mathrm{d}x_1, \qquad a_j\big(z_j^k, v_j\big) := \int_0^1 EI_j \frac{\partial^2 z_j^k}{\partial x_1^2} \frac{\partial^2 v_j}{\partial x_1^2} \,\mathrm{d}x_1, \tag{1.11}$$

$$b_1\big(z_1^k, v_1\big) := \hat{s}_1 \frac{|S_C|}{|\omega|} z_1^k v_1|_{x_1=1}, \qquad c\big(z_3^k, v_1\big) := -\hat{s}_1 \frac{1}{|\omega|} \int_{S_C} y_3 \,\mathrm{d}\sigma_y \frac{\partial z_3^k}{\partial x_1} v_1 \Bigg|_{x_1=1}, \tag{1.12}$$

$$b_j\big(z_j^k, v_j\big) := \hat{s}_j \frac{|S_C|}{|\omega|} z_j^k v_j|_{x_1=1} + s_1 \int_{S_C} y_j^2 \,\mathrm{d}\sigma_y \frac{\partial z_j^k}{\partial x_1} \frac{\partial v_j}{\partial x_1} \Bigg|_{x_1=1}, \tag{1.13}$$

$I_j = \int_\omega y_j^2 \, \mathrm{d}\tilde{\boldsymbol{y}}$, $j = 2, 3$, $E = \mu \frac{3\lambda+2\mu}{\lambda+\mu}$, $S_C = [-1, 0] \times \partial\omega_R$ and

$$\ell^k(\boldsymbol{v}) = \sum_{i=1}^{3} \Big(\int_0^1 \big(\hat{f}_i - \hat{f}_i^{\mathrm{rec}(k)}\big) v_i \, \mathrm{d}x_1 - \hat{g}_{i1}^{\mathrm{rec}(k)} v_i|_{x_1=1} \Big) \tag{1.14}$$

$$+ s_1 \sum_{q=2}^{3} \int_{S_C} y_q \big(n_{1,0}^{inner}\big)^{14} \, \mathrm{d}\sigma_y \frac{\partial v_q}{\partial x_1} \frac{\partial z_4^k}{\partial x_1}\Big|_{x_1=1} - I_q g_{q2}^{\mathrm{rec}(k)} \frac{\partial v_q}{\partial x_1}\Big|_{x_1=1} \tag{1.15}$$

$$+ \hat{s}_2 \Big(\frac{1}{|\omega|} I_p \Big)^{-\frac{1}{2}} \frac{1}{|\omega|} \int_{S_C} y_3 \, \mathrm{d}\sigma_y z_4^k v_2|_{x_1=1}, \tag{1.16}$$

with

$$f_1^{\mathrm{rec}(k)} = \sum_{l=0}^{k-1} \Big(\boldsymbol{h}^{N_{k-l+2,0}^{inner}} \frac{\partial^{k-l+2} \boldsymbol{u}^{\mathrm{rec}(l)}}{\partial x_1^{k-l+2}} \Big)_1,$$

$$f_j^{\mathrm{rec}(k)} = \sum_{l=0}^{k-1} \Big(\boldsymbol{h}^{N_{k-l+4,0}^{inner}} \frac{\partial^{k-l+4} \boldsymbol{u}^{\mathrm{rec}(l)}}{\partial x_1^{i-l+4}} \Big)_j,$$

$$g_{11}^{\mathrm{rec}(k)} = \sum_{l=0}^{k-1} \sum_{i+j=k-l+1}^{k-l+1} \Big(\boldsymbol{h}^{N_{i,j}^-} \frac{\partial^i \check{\boldsymbol{u}}^{\mathrm{rec}(l)}}{\partial x_1^i} \Big)_1 \Big|_{x_1=1}$$

$$+ \sum_{l=0}^{k-1} \Big(\sum_{i+j=k-l+2}^{k-l+2} \boldsymbol{h}^{N_{i,j}^-} \frac{\partial^i \hat{\boldsymbol{u}}^{\mathrm{rec}(l)}}{\partial x_1^i} \Big)_1 \Big|_{x_1=1},$$

$$g_{j1}^{\mathrm{rec}(k)} = \sum_{l=0}^{k-1} \sum_{i+j=k-l+3}^{k-l+3} \Big(\boldsymbol{h}^{N_{i,j}^-} \frac{\partial^i \boldsymbol{u}^{\mathrm{rec}(l)}}{\partial x_1^i} \Big)_j \Big|_{x_1=1},$$

$$g_{32}^{\mathrm{rec}(k)} = \sum_{l=0}^{k-1} \sum_{i+j=k-l+1}^{k-l+1} \Big(\boldsymbol{h}^{N_{i,j}^-} \frac{\partial^i \check{\boldsymbol{u}}^{\mathrm{rec}(l)}}{\partial x_1^i} \Big)_6 \Big|_{x_1=1}$$

$$+ \sum_{l=0}^{k-1} \Big(\sum_{i+j=k-l+2}^{k-l+2} \boldsymbol{h}^{N_{i,j}^-} \frac{\partial^i \hat{\boldsymbol{u}}^{\mathrm{rec}(l)}}{\partial x_1^i} \Big)_6 \Big|_{x_1=1},$$

$$g_{22}^{\mathrm{rec}(k)} = \sum_{l=0}^{k-1} \Big(\sum_{i+j=k-l+2}^{k-l+2} \boldsymbol{h}^{N_{i,j}^-} \frac{\partial^i \hat{\boldsymbol{u}}^{\mathrm{rec}(l)}}{\partial x_1^i} \Big)_5 \Big|_{x_1=1},$$

$j = 2, 3$, $\hat{\boldsymbol{u}}^{\mathrm{rec}(i)} = (0, z_2^i, z_3^i, z_4^i)^T$ and $\check{\boldsymbol{u}}^{\mathrm{rec}(i)} = (z_1^i, 0, 0, 0)^T$.

2 Statement of the Results

Theorem 2.1 *Let $\boldsymbol{u}^\varepsilon$ be the exact solution to the* 3*D problem* (1.2) *and let z_i^k, $i = 1, 2, 3, 4$, be solutions to* (1.8)–(1.9) *and let $\boldsymbol{N}_{k,s}$ be computed from the recursive problems deduced from Requirement* 1. *Then the estimate* (1.1) *holds.*

Let us note, that the exact solution, $\boldsymbol{u}^\varepsilon$, converges to the leading term of the series (1.4)–(1.5), for $\varepsilon \to 0$, in the sense

$$\left\| \boldsymbol{u}^\varepsilon - \boldsymbol{\Phi}(\tilde{\boldsymbol{y}})\boldsymbol{\Pi}^\varepsilon \overline{\boldsymbol{u}}^0(x_1) \right\|_{L^2(\Omega^\varepsilon)} \le C\varepsilon\sqrt{|\Omega^\varepsilon|}, \tag{2.1}$$

for a constant $C > 0$ independent of ε. The 1D system for $\overline{\boldsymbol{u}}^0$, is stated in (4.2)–(4.8), Sect. 4 (in the interval $(-1, 0) \cup (0, 1)$, with Robin boundary condition at $x_1 = 0$). The proof of Theorem 2.1 is analogous to the proof of [2, Proposition 3.2] and (2.1) follows from an analog on of the proof of [2, Theorem 1.7]. The step that needs to be revisited, is the solvability of the coupled 1D system (1.9). The solvability of (1.8) is clear, since the angle of twist z_4^k is independent of the other components.

Proposition 2.2 *Let* (1.8) *be solved for $k \in [0, K]$. Then for $k \in [0, K]$ the recursive problem* (1.9) *has a unique solution.*

3 Proof of the Results

The proof of Theorem 2.1 is a consequence of [2, Proposition 3.2] and Proposition 2.2. The proof of Proposition 2.2 is based on the fact that the coupled recursive 1D system has constant right handsides and we can seek polynomial solutions. This permits us to obtain a linear algebraic system that is equivalent to (1.9). We show that the matrix in that system is regular.

Since the load f_i, $i = 1, 2, 3, 4$, is constant, see Sect. 1, following [2, p. 14], potential solutions $\boldsymbol{z}$ to (1.9) have a polynomial form, second order for z_1 and fourth order for z_i, $i = 2, 3$. Since $\boldsymbol{z} \in \mathcal{H}$ we can write

$$z_1(x_1) = az_1 x_1^2 + bz_1 x_1 \tag{3.1}$$

$$z_i(x_1) = az_i x_1^4 + bz_i x_1^3 + cz_i x_1^2 \tag{3.2}$$

or

$$\boldsymbol{z} = \boldsymbol{M}\boldsymbol{\eta}_z, \tag{3.3}$$

$$\boldsymbol{\eta}_z = (az_1, bz_1, az_2, bz_2, cz_2, az_3, bz_3, cz_3)^T, \ \boldsymbol{M} = \begin{pmatrix} x_1^2 & x_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_1^4 & x_1^3 & x_1^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_1^4 & x_1^3 & x_1^2 \end{pmatrix}.$$

Lemma 3.1 (On the stiffness matrix) *Let $\boldsymbol{z} = \boldsymbol{M}\boldsymbol{\eta}_z$ and $\boldsymbol{v} = \boldsymbol{M}\boldsymbol{\eta}_v$ with $\boldsymbol{M}$ as in (3.3). Then the bilinear form $a_{1D}(\boldsymbol{z}, \boldsymbol{v})$ can be written as*

$$a_{1D}(\boldsymbol{z}, \boldsymbol{v}) = \boldsymbol{\eta}_z^T \boldsymbol{K} \boldsymbol{\eta}_v, \tag{3.4}$$

where $\boldsymbol{K} \in \mathbb{R}^{8\times 8}$, $\boldsymbol{K} = \begin{pmatrix} \boldsymbol{K}_{11} & \boldsymbol{0}_{2,3} & \boldsymbol{K}_{13} \\ \boldsymbol{0}_{3,2} & \boldsymbol{K}_{22} & \boldsymbol{0}_{3,3} \\ \boldsymbol{K}_{31} & \boldsymbol{0}_{3,3} & \boldsymbol{K}_{33} \end{pmatrix}$, $\boldsymbol{0}_{i,j} = (0)_{l=1..i,k=1..j}$ and

$$\boldsymbol{K}_{11} = EA \begin{pmatrix} \frac{4}{3} & 1 \\ 1 & 1 \end{pmatrix} + s_1 |S_C| \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \tag{3.5}$$

$$\boldsymbol{K}_{22} = EI_2 \begin{pmatrix} \frac{144}{5} & 18 & 8 \\ 18 & 12 & 6 \\ 8 & 6 & 4 \end{pmatrix} + s_2 |S_C| \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$+ s_1 \int_{S_C} y_2^2 \,\mathrm{d}\sigma_y \begin{pmatrix} 16 & 12 & 8 \\ 12 & 9 & 6 \\ 8 & 6 & 4 \end{pmatrix}, \tag{3.6}$$

$$\boldsymbol{K}_{33} = EI_3 \begin{pmatrix} \frac{144}{5} & 18 & 8 \\ 18 & 12 & 6 \\ 8 & 6 & 4 \end{pmatrix} + s_3 |S_C| \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$+ s_1 \int_{S_C} y_3^2 \,\mathrm{d}\sigma_y \begin{pmatrix} 16 & 12 & 8 \\ 12 & 9 & 6 \\ 8 & 6 & 4 \end{pmatrix}, \tag{3.7}$$

$$\boldsymbol{K}_{13} = s_1 \left| \int_{S_C} y_3 \,\mathrm{d}\sigma_y \right| \begin{pmatrix} 4 & 3 & 2 \\ 4 & 3 & 2 \end{pmatrix}, \qquad \boldsymbol{K}_{31} = \boldsymbol{K}_{31}^T. \tag{3.8}$$

Proof Following (3.3), substitute $\boldsymbol{z} = \boldsymbol{M}\boldsymbol{\eta}_z$ and $\boldsymbol{v} = \boldsymbol{M}\boldsymbol{\eta}_v$ into $a_{1D}(\boldsymbol{z}, \boldsymbol{v})$, then a straight forward computation yields (3.4). □

Lemma 3.2 (On the symmetry and regularity of the stiffness matrix) *The matrix $\boldsymbol{K}$ form Lemma 3.1 is symmetric and regular.*

Proof From Lemma 3.1, we know that $\boldsymbol{K}_{ii}$, $i = 1, 2, 3$, are symmetric and that $\boldsymbol{K}_{31} = \boldsymbol{K}_{13}^T$, obviously $\boldsymbol{0}_{3,2} = \boldsymbol{0}_{3,2}^T$, hence $\boldsymbol{K} = \boldsymbol{K}^T$. A direct computation gives

$$\det(\boldsymbol{K}) = \frac{16}{75} E^4 A I_3 \Big(12 E A s_1 \int_{S_C} y_3^2 \,\mathrm{d}\sigma_y |S_C| I_3 + A s_1 \int_{S_C} y_3^2 \,\mathrm{d}\sigma_y |S_C|^2 s_3 \tag{3.9}$$

$$+ 12 E^2 A I_3^2 + 4 E A I_3 s_3 |S_C| + 12 E s_1 |S_C| I_3^2 + 4 s_1 |S_C|^2 I_3 s_3 \Big) factor, \tag{3.10}$$

$factor = I_2(12s_1 \int_{S_C} y_2^2 \,\mathrm{d}\sigma_y EI_2 + s_1 \int_{S_C} y_2^2 \,\mathrm{d}\sigma_y s_2|S_C| + 12E^2 I_2^2 + 4EI_2 s_2|S_C|)$. Since all the summands in $\det(\boldsymbol{K})$ are positive and E, A, I_2, I_3 are strictly positive $\det(\boldsymbol{K})$ is strictly positive. □

Note, that for the classical case $s_i = 0, i = 1, 2, 3$, we get $\det(\boldsymbol{K}) = \frac{768}{25} E^8 A^2 I_2^3 I_3^3$.

Lemma 3.3 (On an equivalent linear algebraic system) *Problem* (1.9) *is equivalent to the linear system*

$$\boldsymbol{K}\boldsymbol{\eta}_z = \boldsymbol{\Lambda}^k, \tag{3.11}$$

where $\boldsymbol{\Lambda}^k \in \mathbb{R}^8$ *is known.*

Proof Substitute $\boldsymbol{z} = \boldsymbol{M}\boldsymbol{\eta}_z$ and $\boldsymbol{v} = \boldsymbol{M}\boldsymbol{\eta}_v$ into $a_{1D}(\boldsymbol{z}, \boldsymbol{v})$, then with Lemma 3.1 (1.9) reads as

$$\boldsymbol{\eta}_z^T \boldsymbol{K}\boldsymbol{\eta}_v = \boldsymbol{\chi}^k \boldsymbol{\eta}_v, \quad \forall \boldsymbol{\eta}_v \in \mathbb{R}^8, \tag{3.12}$$

$\boldsymbol{\chi}^k \in \mathbb{R}^{1\times 8}$ is obtained from the factorization of ℓ^k. Since by Lemma 3.2, $\boldsymbol{K}$ is symmetric (3.12) is equivalent to

$$\boldsymbol{K}\boldsymbol{\eta}_z = \boldsymbol{\Lambda}^k, \tag{3.13}$$

with $\boldsymbol{\Lambda}^k = (\boldsymbol{\chi}^k)^T$. □

Proof of Proposition 2.2 By Lemmata 3.1 and 3.2, (1.9) is equivalent to a linear algebraic system with regular matrix. □

Proof With Proposition 2.2, the proof of Theorem 2.1 is analogous to the proof of [2, Proposition 3.2]. □

4 Numerical Example

We illustrate the coupling in the 1D system by a numerical example. We compare the leading term of the series (1.4), to a solution $\boldsymbol{u}^\varepsilon$ to the 3D problem (1.2), for a fixed small ε. Following [5, Sect. 2.3.5], [2, Remark 2, p. 4] we adapt the results of the previous sections to the geometry $\Omega^\varepsilon = \{\boldsymbol{x} \in \mathbb{R}^3 : x_1 \in (-1, 0) \cup (0, 1), \tilde{\boldsymbol{x}} \in \varepsilon\omega\}$ (see Fig. 2). The beam is fixed on $\partial\Omega_U^\varepsilon = \{\boldsymbol{x} \in \overline{\Omega}^\varepsilon : x_1 = -1\}$ and the Robin condition surface is denoted as $S_C^\varepsilon = \{\boldsymbol{x} \in \overline{\Omega}^\varepsilon : x_1 \in [-\varepsilon, \varepsilon], \frac{\tilde{\boldsymbol{x}}}{\varepsilon} \in \partial\omega_R\}$. $\Sigma^\varepsilon = \{\boldsymbol{x} \in \overline{\Omega}^\varepsilon : x_1 = 0, \tilde{\boldsymbol{x}} \in \varepsilon\overline{\omega}\backslash\partial\omega_R\}$ denotes an interface and $\partial\Omega_N^\varepsilon = \partial\Omega^\varepsilon\backslash(S_C^\varepsilon \cup \partial\Omega_U^\varepsilon)$ denotes a free boundary. To compute the leading term of the series (1.4),

$$\boldsymbol{u}^{\mathrm{Lead}}(\boldsymbol{x}) = \begin{pmatrix} \varepsilon\overline{u}_1^0(x_1) - x_2\frac{\partial\overline{u}_2^0}{\partial x_1} - x_3\frac{\partial\overline{u}_3^0}{\partial x_1} + \varepsilon(n_{1,0}^{inner})^{14}(\frac{\tilde{\boldsymbol{x}}}{\varepsilon})\frac{\partial\overline{u}_4^0}{\partial x_1} \\ \overline{u}_2^0 - \frac{x_3}{\varepsilon}(\frac{1}{|\omega|}I_p)^{-\frac{1}{2}}\overline{u}_4^0 \\ \overline{u}_3^0 + \frac{x_2}{\varepsilon}(\frac{1}{|\omega|}I_p)^{-\frac{1}{2}}\overline{u}_4^0 \end{pmatrix}, \tag{4.1}$$

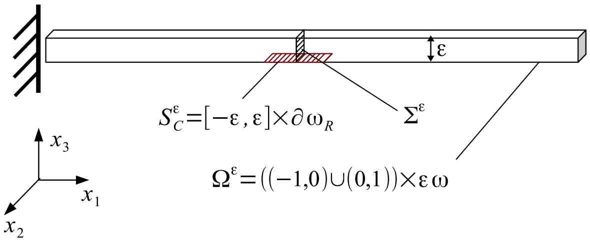


Fig. 2 Geometry with Robin boundary condition in the middle

we need to solve only the 1D system for $k = 0$ in the recursion (1.8)–(1.9). In particular, for $k = 0$, $z_i^0 = \overline{u}_i^0$, $i = 1, 2, 3, 4$. Adapted to the geometry of this section, (1.8)–(1.9) read as find $\overline{u}_4^0 \in \mathcal{V}$ and $(\overline{u}_1^0, \overline{u}_2^0, \overline{u}_3^0)^T \in \mathcal{H}$, s.t. $\forall v_4 \in \mathcal{V}$ and $\forall \boldsymbol{v} \in \mathcal{H}$

$$a_4\big(\overline{u}_4^0, v_4\big) = \ell_4^0(v_4), \tag{4.2}$$

$$a_{1D}\big(\big(\overline{u}_1^0, \overline{u}_2^0, \overline{u}_3^0\big)^T, \boldsymbol{v}\big) = \ell^0(\boldsymbol{v}), \tag{4.3}$$

where $a_4(\cdot,\cdot) = \mu J \int_{-1}^1 \frac{\partial\cdot}{\partial x_1}\frac{\partial\cdot}{\partial x_1}\,\mathrm{d}x_1$, $\ell_4^0(v_4) = \int_{-1}^1 \hat{f}_4 v_4\,\mathrm{d}x_1$, $a_{1D}((\overline{u}_1^0, \overline{u}_2^0, \overline{u}_3^0)^T, \boldsymbol{v}) = \sum_{i=1}^3 a_i(\overline{u}_i^0, v_i) + \sum_{i=1}^3 b_i(\overline{u}_i^0, v_i) + c(\overline{u}_3^0, v_1) + c(v_3, \overline{u}_1^0)$,

$$a_1\big(\overline{u}_1^0, v_1\big) = \int_{-1}^1 EA\frac{\partial \overline{u}_1^0}{\partial x_1}\frac{\partial v_1}{\partial x_1}\,\mathrm{d}x_1, \qquad a_j\big(\overline{u}_j^0, v_j\big) = \int_{-1}^1 EI_j\frac{\partial^2 \overline{u}_j^0}{\partial x_1^2}\frac{\partial^2 v_j}{\partial x_1^2}\,\mathrm{d}x_1, \tag{4.4}$$

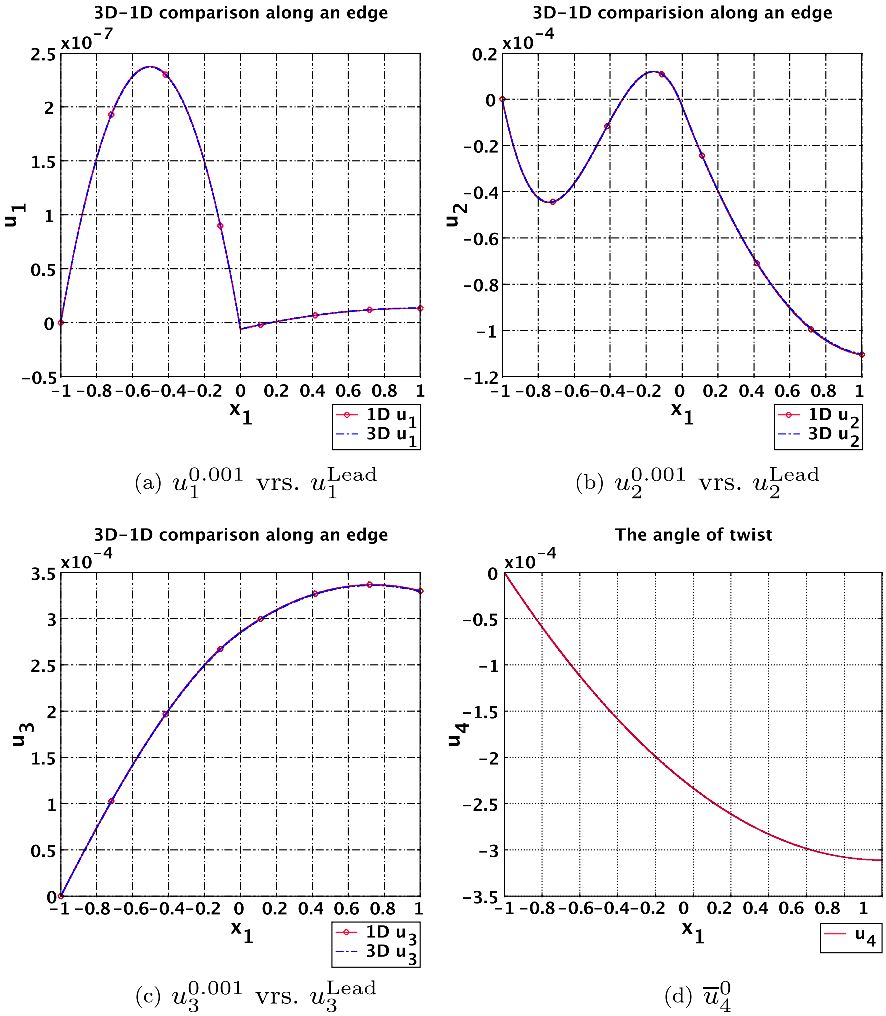
$$b_1\big(\overline{u}_1^0, v_1\big) = \hat{s}_1\frac{|S_C|}{|\omega|}\overline{u}_1^0 v_1|_{x_1=0}, \qquad c\big(\overline{u}_3^0, v_1\big) = -\hat{s}_1\frac{1}{|\omega|}\int_{S_C} y_3\,\mathrm{d}\sigma_y\frac{\partial \overline{u}_3^0}{\partial x_1}v_1\bigg|_{x_1=0}, \tag{4.5}$$

$$b_j\big(\overline{u}_j^0, v_j\big) = \hat{s}_j\frac{|S_C|}{|\omega|}\overline{u}_j^0 v_j|_{x_1=1} + s_1\int_{S_C} y_j^2\,\mathrm{d}\sigma_y\frac{\partial \overline{u}_j^0}{\partial x_1}\frac{\partial v_j}{\partial x_1}\bigg|_{x_1=0}, \tag{4.6}$$

$$\ell^0(\boldsymbol{v}) = \sum_{i=1}^3\int_{-1}^1 \hat{f}_i v_i\,\mathrm{d}x_1 + s_1\sum_{q=2}^3\int_{S_C} y_q\big(n_{1,0}^{inner}\big)^{14}\,\mathrm{d}\sigma_y\frac{\partial v_q}{\partial x_1}\frac{\partial \overline{u}_4^0}{\partial x_1}\bigg|_{x_1=0} \tag{4.7}$$

$$+\,\hat{s}_2\bigg(\frac{1}{|\omega|}I_p\bigg)^{-\frac{1}{2}}\frac{1}{|\omega|}\int_{S_C} y_3\,\mathrm{d}\sigma_y\overline{u}_4^0 v_2|_{x_1=0}, \quad j = 2, 3. \tag{4.8}$$

Let $\omega = (-\frac{1}{2}, \frac{1}{2})^2$, then the explicit geometric constants read as $|S_C| = 2$, $\int_{S_C} y_3\,\mathrm{d}\sigma_y = -1$, $I_j = \frac{1}{12}$, $j = 2, 3$, $I_p = \frac{1}{6}$, $A = 1$, $\int_{S_C} y_3^2\,\mathrm{d}\sigma_y = \frac{1}{2}$. Following



(a) $u_1^{0.001}$ vrs. u_1^{Lead} (b) $u_2^{0.001}$ vrs. u_2^{Lead}

(c) $u_3^{0.001}$ vrs. u_3^{Lead} (d) $\overline{u}_4^0$

Fig. 3 3D–1D comparison along the edge $(x_1, -\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$, $\varepsilon = 0.001$

[4, pp. 311–312], we approximate $(n_{1,0}^{inner})^{14}$ as

$$\left(n_{1,0}^{inner}\right)^{14} \tag{4.9}$$

$$\approx \left(\frac{1}{|\omega|} I_p\right)^{-\frac{1}{2}} \left(-y_2 y_3 + 4b^2 \left(\frac{2}{\pi}\right)^3 \sum_{k=0}^{10} \frac{(-1)^k}{(2k+1)^3}\right.$$

$$\left.\times \sin\left(\frac{(2k+1)\pi}{2b} y_3\right) \frac{\sinh(\frac{(2k+1)\pi}{2b} y_2)}{\cosh(\frac{(2k+1)\pi a}{2b})}\right), \tag{4.10}$$

$a = b = \frac{1}{2}$, with (4.10) we get $J \approx 0.14057 \cdot 6$, $\int_{S_C} y_2^2 \, d\sigma_y = 2I_2$, $\int_{S_C} y_2 (n_{1,0}^{inner})^{14}(\tilde{\boldsymbol{y}}) \, d\sigma_y \approx 0.01598$, $\int_{S_C} y_3 (n_{1,0}^{inner})^{14}(\tilde{\boldsymbol{y}}) \, d\sigma_y = 0$.

For $\varepsilon = 0.001$, $E = 2 \cdot 10^5$, $\frac{\lambda}{2(\lambda+\mu)} = 0.33$, $f_4 = -9.86$, $f_1 = 7.86$, $f_2 = f_3 = 0$ and $s_i = 10^7$, $i = 1, 2, 3$, the relative error $\frac{\|\boldsymbol{u}^\varepsilon - \boldsymbol{\Phi}(\tilde{\boldsymbol{y}})\boldsymbol{\Pi}^\varepsilon \overline{\boldsymbol{u}}^0(x_1)\|_{L^2(\Omega^\varepsilon)}}{|\boldsymbol{u}^\varepsilon\|_{L^2(\Omega^\varepsilon)}}$ is smaller than 10^{-4}. The components of $\boldsymbol{u}^{\text{Lead}}$ and $\boldsymbol{u}^{0.001}$ are plotted against each other along the edge $(x_1, -\frac{0.001}{2}, \frac{0.001}{2})$ in Fig. 3. We see numerically that the 3D problem does not decouple in the limit and that a tensile force f_1 and a torsional moment f_4 cause bending in x_3- and x_2-direction respectively, even if no bending forces f_i, $i = 2, 3$, act.

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Smooth Solution of an Initial Value Problem for a Mixed-Type Differential Difference Equation

Valentina Iakovleva and Judith Vanegas

Abstract In this paper, we show the construction of the solution to the mixed type differential difference equation:

$$x'(t) = Ax(t + a) + Bx(t - a) + Cx(t),$$

where $A, B, C \in \mathbb{C} \setminus \{0\}$, $a > 0$ and $t \in \mathbb{R}$. We use a step derivative method and a certain condition on the initial function $\varphi \in C^{\infty}[-a, a]$ to assure the existence, uniqueness and smoothness of the solution in $\mathbb{R}$.

Keywords Differential difference equations · Initial value problems

Mathematics Subject Classification (2010) Primary 34A12 · Secondary 34K05

1 Goal of This Article

Mixed functional differential equations (MFDE) are a class of functional differential equations where the time derivative depends on both past and future values of the variable. They are also known as forward-backward equations. At the end of the eighties appeared interesting papers on applications of these equations, such as the work of H. Chi et al. in nerve conduction [5], and the works of A. Rustichini in the context of optimal control problems [14] and economic dynamics [15]. These works helped to increase the study of these equations. From the beginning of this century, other applications of MFDE in Physics [1, 4] and Economy [11] and the study of other problems [3, 6–8, 12] have been developed. Topics like controllability [10, 13] and spectral analysis [9] on MFDE have been treated in this decade.

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MFDE are, in general, ill-posed as initial value problems (see for example [7, 14]), but there are also cases [6, 8, 12] where a unique solution exists.

In this work we show the construction of the solution to the mixed type differential difference equation:

$$x'(t) = Ax(t+a) + Bx(t-a) + Cx(t), \tag{1.1}$$

where $A, B, C \in \mathbb{C} \setminus \{0\}$, $a > 0$ and $t \in \mathbb{R}$.

We use a step derivative method (an analog to the step integration method [2]) so that the following condition

$$\varphi^{(n+1)}(0) = A\varphi^{(n)}(a) + B\varphi^{(n)}(-a) + C\varphi^{(n)}(0), \quad n = 0, 1, 2, \dots$$

on the initial function $\varphi \in C^\infty[-a, a]$ assures the existence, uniqueness and smoothness of the solution of (1.1) in $\mathbb{R}$.

2 Construction of the Solution

In this section we construct the solution of the differential difference equation (1.1), using the step derivative method.

We start rewriting (1.1) in the form:

$$x(t+a) = \frac{1}{A}x'(t) - \frac{B}{A}x(t-a) - \frac{C}{A}x(t). \tag{2.1}$$

Taking $t + a =: \tau$, then from (2.1) we get

$$x(\tau) = \frac{1}{A}x'(\tau - a) - \frac{B}{A}x(\tau - 2a) - \frac{C}{A}x(\tau - a). \tag{2.2}$$

Now we follow an induction process to construct the solution of (1.1) on the interval $[a, na]$, $n \in \mathbb{N}$. For $n = 2$, if $\tau \in [a, 2a]$, then $(\tau - a) \in [0, a]$ and $(\tau - 2a) \in [-a, 0]$. On the interval $[-a, a]$ the function $x(t)$ is known. In (2.2) the right side is determined uniquely for $\tau \in [a, 2a]$. Therefore the values of $x(t)$ are found for $\tau \in [a, 2a]$. Now suppose that we know $x(\tau)$ on $[a, ma]$, for an arbitrary fixed number $m \in \mathbb{N}$. Then in (2.2) the right side is determined for $\tau \in [ma, (m+1)a]$. Hence the values of $x(\tau)$ are determined on the interval $[a, (m+1)a]$. Therefore the function $x(t)$ is determined for all $\tau \geq a$ starting from the function $\varphi = x|_{[-a,a]}$.

Similarly, we rewrite (1.1) as

$$x(t-a) = \frac{1}{B}x'(t) - \frac{A}{B}x(t+a) - \frac{C}{B}x(t) \tag{2.3}$$

and changing the variable $s := t - a$ we obtain

$$x(s) = \frac{1}{B}x'(s+a) - \frac{A}{B}x(s+2a) - \frac{C}{B}x(s+a). \tag{2.4}$$

Applying the above procedure to (2.4) we construct the function $x(s)$ for all $s \le -a$, starting from the function $\varphi = x|_{[-a,a]}$.

Therefore the function $x \in C^\infty(ma,(m+1)a)$, $m \in \mathbb{Z}$, is constructed satisfying (1.1), i.e., we have constructed the smooth solution of (1.1) only in each open interval $(ma,(m+1)a)$, $m \in \mathbb{Z}$. However in the points $t = ka$, $k \in \mathbb{Z} \setminus \{0\}$, the function $x(t)$ and its derivatives can be discontinuous.

Since the solution constructed by this method contains the linear combination of the initial function and its derivatives, then the solution is unique in the intervals $(ma,(m+1)a)$, $m \in \mathbb{Z}$.

3 The Smoothness of the Solution

In this section we give necessary and sufficient conditions to assure the smoothness of the solution of (1.1) in $\mathbb{R}$ with the function $\varphi \in C^\infty[-a,a]$ as initial condition.

Theorem 3.1 *Let $\varphi \in C^\infty[-a,a]$. The solution $x(t)$ of (1.1) satisfying the initial condition $x|_{[-a,a]} = \varphi$ and constructed using the former step derivative method, belongs to $C^\infty(\mathbb{R})$ if and only if the following condition*

$$\varphi^{(n+1)}(0) = A\varphi^{(n)}(a) + B\varphi^{(n)}(-a) + C\varphi^{(n)}(0), \quad n = 0,1,2,\ldots \tag{3.1}$$

is satisfied.

Proof The necessary condition: Let $x \in C^\infty(\mathbb{R})$ be the solution of (1.1) and $x(t) = \varphi(t)$ for $t \in [-a,a]$. Taking the derivative of order n of (1.1) in $t = 0$, we obtain

$$x^{(n+1)}(0) = Ax^{(n)}(a) + Bx^{(n)}(-a) + Cx^{(n)}(0). \tag{3.2}$$

Since $x|_{[-a,a]} = \varphi$, (3.2) is exactly the condition (3.1).

The sufficient condition: We assume that (3.1) is satisfied. From (2.2) it follows

$$x^{(n)}(\tau) = \frac{1}{A}x^{(n+1)}(\tau - a) - \frac{B}{A}x^{(n)}(\tau - 2a) - \frac{C}{A}x^{(n)}(\tau - a) \tag{3.3}$$

or equivalently

$$x^{(n)}\left(a^+\right) = \frac{1}{A}\varphi^{(n+1)}(0) - \frac{B}{A}\varphi^{(n)}(-a) - \frac{C}{A}\varphi^{(n)}(0), \tag{3.4}$$

where $x^{(n)}(a^\pm) := \lim_{\varepsilon\to 0,\varepsilon>0} x^{(n)}(a \pm \varepsilon)$. Hence the equality $x^{(n)}(a^+) = \varphi^{(n)}(a)$ follows from (3.4) and condition (3.1). On the other hand, $x^{(n)}(a^-) = \varphi^{(n)}(a)$ because x coincides with φ on $[-a,a]$. Therefore, we have the continuity condition $x^{(n)}(a^+) = x^{(n)}(a^-)$. Similarly from (2.4) the continuity of $x^{(n)}(t)$ in $t = -a$ is given.

We finish the proof using mathematical induction on k. Suppose that for all $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ the identities $x^{(n)}(ka^+) = x^{(n)}(ka^-)$ are true for $k = \pm 1, \pm 2, \ldots, \pm m$. Then for (3.3) for any $n \in \mathbb{N}_0$ we have

$$x^{(n)}\big((m+1)a^+\big) = \frac{1}{A}x^{(n+1)}\big(ma^+\big) - \frac{B}{A}x^{(n)}\big((m-1)a^+\big) - \frac{C}{A}x^{(n)}(ma) \quad (3.5)$$

and also

$$x^{(n)}\big((m+1)a^-\big) = \frac{1}{A}x^{(n+1)}\big(ma^-\big) - \frac{B}{A}x^{(n)}\big((m-1)a^-\big) - \frac{C}{A}x^{(n)}(ma). \quad (3.6)$$

From the induction hypothesis $x^{(n+1)}(ma^+) = x^{(n+1)}(ma^-)$ and $x^{(n)}((m-1)a^+) = x^{(n)}((m-1)a^-)$. Hence the right sides of (3.5) and (3.6) are the same. Then $x^{(n)}((m+1)a^+) = x^{(n)}((m+1)a^-)$ for any $n \in \mathbb{N}_0$.

Similarly from (2.4), we have $x^{(n)}(-(m+1)a^-) = x^{(n)}(-(m+1)a^+)$ for any $n \in \mathbb{N}_0$.

Therefore, for any $n \in \mathbb{N}_0$ the equality $x^{(n)}(ka^+) = x^{(n)}(ka^-)$ is also correct for $k = \pm(m+1)$. It means that $x^{(n)}(ka^+) = x^{(n)}(ka^-)$ is correct for $k \in \mathbb{Z}$. Then it is proved that $x(t)$ and all of its derivatives are continuous in all $t = ka$, $k \in \mathbb{Z}$. Hence, $x \in C^\infty(\mathbb{R})$. □

Remark 3.2 Theorem 3.1 and the step derivative method applied in the former section give the existence and uniqueness of the solution of (1.1) with the initial condition φ in $\mathbb{R}$.

Remark 3.3 The functions $x_\lambda(t) = e^{\lambda t}$, $\lambda \in \mathbb{C}$ are the solutions of (1.1) if and only if $\lambda \in \Lambda$, where

$$\Lambda = \big\{\lambda \in \mathbb{C} : \lambda = Ae^{\lambda a} + Be^{-\lambda a} + C\big\}. \quad (3.7)$$

4 Outlook to Further Results

We might expect more general equations than (1.1) to be studied. We can suppose the coefficients A, B and C in (1.1) to be piecewise smooth and find conditions for the existence of the solution. Also a spectral theory, controllability theory or numerical solutions could be researched for these type of equations.

Mathematical models containing these equations could be used to study the effect of drugs as therapies. Radioactive therapies could also be modeled.

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On the Classical Lorenz System

Valery A. Gaiko

Abstract The classical Lorenz system is considered. For many years, this system has been the subject of study by numerous authors. However, until now the structure of the Lorenz attractor is not clear completely yet, and the most important question at present is to understand the bifurcation scenario of chaos transition in this system. Using some numerical results and our bifurcational geometric approach, we present a new scenario of chaos transition in the classical Lorenz system.

Keywords Lorenz system · Bifurcation · Singular point · Limit cycle · Chaos

Mathematics Subject Classification (2010) Primary 34C28 · Secondary 37D45 · 37G35

1 Introduction

We consider a three-dimensional dynamical system

$$\dot{x} = \sigma(y - x), \qquad \dot{y} = x(r - z) - y, \qquad \dot{z} = xy - bz \tag{1}$$

known as the Lorenz system. Historically, (1) was the first dynamical system for which the existence of an irregular attractor (chaos) was proved for $\sigma = 10$, $b = 8/3$, and $24{,}06 < r < 28$. For many years, the Lorenz system has been the subject of study by numerous authors; see, e.g., [1–8]. However, until now the structure of the Lorenz attractor is not clear completely yet, and the most important question at present is to understand the bifurcation scenario of chaos transition in system (1).

In Sect. 2 of this paper, we recall a relatively new scenario of chaos transition in the Lorenz system (1) proposed by N.A. Magnitskii and S.V. Sidorov [6]. In Sect. 3,

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we revise this scenario and present a different bifurcation scenario of chaos transition in system (1), where $\sigma = 10$, $b = 8/3$, and $r > 0$, using numerical results of [6] and our bifurcational geometric approach to the global qualitative analysis of three-dimensional dynamical systems which we applied earlier in the planar case [9–15].

2 The Magnitskii–Sidorov Scenario

There exists a contemporary point of view on the structure of the Lorenz attractor and chaos transition in system (1); see [1–8]. However, in [6], it is shown that absolutely another scenario of chaos transition is realized in the Lorenz system (1). It turns out that all cycles from infinite family of unstable cycles, generating the Lorenz attractor [6], have crossing with an one-dimensional unstable not invariant manifold V^u of the origin of system (1) (do not confuse with the invariant unstable manifold W^u of this point). This result follows from the theory of dynamical chaos stated in [6]. After the derivation of analytic formulas for the manifold V^u, it becomes possible to reduce the problem of establishing and proving the existence of unstable cycles in the Lorenz system to the one-dimensional case, namely, to finding stable points of the one-dimensional first return mapping defined on the unstable manifold [6]. By this method, it is shown that some items of the classical scenario of chaos transition in the Lorenz system (1) are invalid, while other require a more detailed investigation. The Magnitskii–Sidorov scenario is the following.

1. The Lorenz system (1) is dissipative and symmetric with respect to the z-axis. The origin $O(0, 0, 0)$ is a singular point of system (1) for any σ, b, and r. It is a stable node for $r < 1$. For $r = 1$, the origin becomes a triple singular point, and then, for $r > 1$, there are two more singular points in the system: $O_1(\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1)$ and $O_2(-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1)$ which are stable up to the parameter value $r_a = \sigma(\sigma + b + 3)/(\sigma - b - 1)$ ($r_a \approx 24{,}74$ for $\sigma = 10$ and $b = 8/3$). For all $r > 1$, the point O is a saddle-node. It has a two-dimensional stable manifold W^s and a one-dimensional unstable manifold W^u. If $1 < r < r_1 \approx 13{,}9$, then separatrices Γ_1 and Γ_2 issuing from the point O along its one-dimensional unstable manifold W^u are attracted by their nearest stable points O_1 and O_2, respectively.

2. If $r = r_1 \approx 13{,}9$, then the separatrices Γ_1 and Γ_2 do not form two separate homoclinic loops. Here we have a bifurcation with the generation of a single closed contour surrounding both stationary points O_1 and O_2; the end of the separatrix Γ_1 enters the beginning of the separatrix Γ_2, and vice versa, the end of Γ_2 enters the beginning of Γ_1. As r grows, from this contour, a closed cycle C_0 appears there first. It is an eight-shaped figure surrounding both points O_1 and O_2.

3. If $r_1 < r < r_2 \approx 24{,}06$, then cycles L_1 and L_2 surrounding the points O_1 and O_2, respectively, do not appear; but with further growth of r, pairs of cycles C_n^+, C_n^-, $n = 0, 1, \ldots,$ are successively generated. They determine the generation of the Lorenz attractor. The cycle C_n^+ makes n complete rotations in the half-space containing the point O_1 and one incomplete rotation around the point O_2.

Conversely, the cycle C_n^- makes n complete rotations around the point O_2 and one incomplete rotation around the point O_1.

For each r, $r_1 < r < r_2$, there exists the number $n(r)$ ($n(r) \to \infty$ as $r \to r_2$) such that in the phase-space of (1), there are unstable cycles C_0, C_k^+, C_k^-, $k = 0, \ldots, n$, and cycles C_{km}^+, C_{km}^-, $k, m < n$, which make k rotations around the point O_1 and m rotations around the point O_2 and are various combinations of the cycles C_n^+ and C_n^-, and many other cycles generated by bifurcations of the cycles C_n^+ and C_n^- [6]. Points of intersection of all these cycles with the manifold V_u have the following arrangement on the curve V_u for $0 \leq z_{min} \leq z \leq z_{max} < r - 1$. The point z_{min} corresponds to the right large single loop of the cycle C_n^-. This loop is the larger face of the right truncated cone of the set S. Further, the trajectory of the cycle passes into the left half-plane and makes n clockwise rotations around the point O_2. The smallest first loop around the point O_2 is the smaller face of the truncated cone of the set S. The point z_{max} corresponds to the smallest loop of the cycle C_n^+ around the point O_1. This loop is the smaller face of the right truncated cone. Further, the trajectory of this cycle makes n rotations around the point O_1 clockwise, passes into the left half-plane, and makes one large rotation around the point O_2. This rotation is the larger face of the left truncated cone. Between the points z_{min} and z_{max} there is a point z_0 corresponding to the main cycle C_0.

Boundaries of the attraction domains of the stable points O_1 and O_2 are given by the smallest loops of the cycles C_n^+ and C_n^-, whose size decay as r grows. Therefore, for some $r = r_m$, the attraction domain of the set B no longer intersects the attraction domains of points O_1 and O_2, and the set B becomes an attractor. Therefore, in the Lorenz system ($a = 10, b = 8/3$), metastable chaos exists only in the interval $r_1 < r < r_m$, and in the interval $r_m < r < r_2$, the system has three stable limit sets, namely, O_1 and O_2 and the Lorenz attractor.

If $r \to r_2$, then the eye size decreases as the number of rotations of the cycles C_n^+ and C_n^- around the points O_1 and O_2, respectively, grows. The value z_{max} grows, and z_{min} decays; moreover, $z_{min} \to 0$ as $r \to r_2$. The lengths of generatrices of truncated cones grow, since additional rotations are added to the cone vertex and diminish the size of the smaller face. Conversely, the larger face grows. If $r = r_2$, then $z_{min} = 0$, but $z_{max} < r - 1$; thus, the larger face of each cone achieves its maximal size, while the smaller face is not contracted into a point, the cone vertex. The following bifurcation takes place. In the limit as $n \to \infty$, each set of cycles C_n^+ (respectively, C_n^-) forms a point-cycle heteroclinic structure consisting of two separatrix contours of the point O. The first contour consists of a separatrix issuing from the point O along its unstable manifold and spinning on the appearing (only for $r = r_2$) saddle cycle L_1 (respectively, L_2) of the point O_1 (respectively, O_1). The second contour consists of the separatrix spinning out from the saddle cycle L_1 (respectively, L_2) and entering the point O along its stable manifold.

As mentioned above, the described bifurcation does not lead to generation of the Lorenz attractor for $r = r_2$. It is more correct to say that it is only a prerequisite of destruction of the attractor as r decays. The attractor itself, existing in the system for $r = r_2$, is formed from finitely many stable cycles $C_k^{\pm}$, $k = 0, \ldots, l$, for $r < 313$. It contains neither separatrices Γ_1 and Γ_2 of the point O nor infinitely many unstable cycles $C_n^{\pm}$ existing in the neighborhood of the point-cycle heteroclinic structure.

If $r_2 < r < r_3 = r_a$, then points O_1 and O_2 are still stable, and their attraction domains are bound by the appearing limit cycles L_1 and L_2 contracting to points as $r \to r_3$. But the Lorenz attractor B is not a set of integral curves going from L_1 to L_2 and back, and separatrices Γ_1 and Γ_2 of the saddle point O do not belong to the attractor. Cycles L_1 and L_2 have already made their job at $r = r_2$ and no longer have anything to do with the attractor. If $r_2 < r < r_3$, then, just as in the case of $r_1 < r < r_2$, the cycles C_n^+ and C_n^- appear again from separatrix contours. The attractor is determined by finitely many such cycles [6].

4. For $r = r_3 = r_a \approx 24{,}74$, the saddle cycles L_1 and L_2 disappear. In the system, there is a unique limit set, namely, the Lorenz attractor.

5. There exist one more important value of the parameter r which affects the formation of the Lorenz attractor. This is a point $r_4 \approx 30{,}485$. If r grows from r_3 to r_4, then the number of rotations of the cycles C_n^+ and C_n^- first rapidly decays, then grows again. In this case, eyes by separatrices of the point O are much smaller than attractor eyes and begin to grow as r increases. Therefore, almost heteroclinic and almost homoclinic contours exist in system (1) at the point r_4.

The process of generation of the Lorenz attractor in system (1) as r decays from the value 313 up to r_4 is referred to as the incomplete double homoclinic cascade [6]. The complete cascade occurs if the r-axis passes exactly through the point of existence of two homoclinic contours. Note that in systems with a single homoclinic contour, there can be a simple complete or incomplete homoclinic cascade of bifurcations of transition to chaos, and in [6], a detailed description of transition to chaos through the double homoclinic (complete or incomplete) cascade of bifurcations is given. Just as in item 6 of the classical scenario, if $r > 313$, then in the system, there exists a unique stable limit cycle C_0 surrounding both points. If $r \approx 313$, then the cycle C_0 becomes unstable and generates two stable cycles C_0^+ and C_0^- which also surround the points O_1 and O_2 but have deflections in the direction of corresponding halves of the unstable manifold V^u of the point O. This is the point where the double homoclinic cascade of bifurcations really begins. In case of an incomplete cascade, it consists of finitely many stages of appearance of stable cycles $C_k^{\pm}$, $k = 0, \dots, l$, and their infinitely many further bifurcations. But in case of a complete cascade, the number of stages is infinite, and at the limit of $l \to \infty$, cycles tend to homoclinic contours of the points O_1 and O_2, respectively. At the k-th stage of the cascade, originally stable cycles $C_k^{\pm}$ undergo a subharmonic cascade of bifurcations and form two band-form attractors that consist of infinitely many unstable limit cycles intersecting the respective domains of the unstable manifold V^u of the point O. Then these two bands merge and form a single attractor surrounding both the points O_1 and O_2, after which there is a cascade of bifurcations of cycles generated as a result of the merger and making rotations separately around the points O_1 and O_2 and simultaneously around both the points. The last cascade of bifurcations has the property of self-organization, since it is characterized by simplification of the structure of cycles and the generation of new stable cycles with a smaller number of rotations around the points O_1 and O_2 as r decays. Each cycle of the cascade of self-organization bifurcations undergoes its own subharmonic cascade of bifurcations, after which all cycles formed during infinitely many bifurcations of all subharmonic cascades and cascades of self-organization bifurcations of cycles become unstable and form some set B_k. After an incomplete homoclinic cascade of bifurcations, we obtain a set $B = \bigcup B_k$ consisting of infinitely many possible unstable cycles appearing at all stages of the cascade. These cycles generate an incomplete double homoclinic attractor, that is the classical Lorenz attractor.

6. If $r > 313$, then the unique stable limit cycle is an attractor in system (1).

3 The Bifurcational Geometric Scenario

Revising the above scenario, we present a new scenario of chaos transition in the Lorenz system (1) for $\sigma = 10$, $b = 8/3$, and $r > 0$.

1. If $r < 1$, the unique singular point O of system (1) is a stable node. For $r = 1$, it becomes a triple singular point, and then, for $r > 1$, there are two more singular points in the system: O_1 and O_2 which are stable up to the parameter value $r_a \approx 24{,}74$. For all $r > 1$, the point O is a saddle-node. It has a two-dimensional stable manifold W^s and an one-dimensional unstable manifold W^u. If $1 < r < r_l = r_1 \approx 13{,}9$, then the separatrices Γ_1 and Γ_2 issuing from the point O along its one-dimensional unstable manifold W^u are attracted by their nearest stable points O_1 and O_2, respectively.

2. If $r = r_l$, then each of the separatrices Γ_1 and Γ_2 becomes a closed homoclinic loop. In this case, two unstable homoclinic loops, C_0^+ and C_0^-, are formed around the points O_1 and O_2, respectively. They are tangent to each other and the z-axis at the point O and form together a homoclinic butterfly.

3. If $r_l < r < r_a \approx 24{,}74$, then, unfortunately, neither the classical scenario nor the Magnitskii–Sidorov scenario can be realized. The reason is that, in both cases, trajectories of system (1) should intersect the two-dimensional stable manifold W^s of the point O. Since this is impossible, the only way to overcome the contradiction is to suppose that a cascade of period-doubling bifurcations [6] will begin immediately in each of the half-spaces with respect to the manifold W^s, when $r > r_l$. In this case, each of the homoclinic loops C_0^+ and C_0^- generates an unstable limit cycle of period 2 which makes one rotation around the point O_1 and one rotation around the point O_2 but in the corresponding half-spaces containing the points O_1 and O_2, respectively, and a stable limit cycle of period 1 lying between the coils of the cycle of period 2. With further growth of r, each of the cycles of period 2 generates an unstable limit cycle of period 4 with a stable limit cycle of period 3 inside of it and each of the cycles of period 1 generates a stable limit cycle of period 2 with an unstable limit cycle of period 1 inside of it. Then, after next doubling, we will have in each of the half-spaces an unstable limit cycle of period 8 with an inserted stable limit cycle of period 7 and a stable limit cycle of period 6 with an inserted unstable limit cycle of period 5, and a stable limit cycle of period 4 with an inserted unstable limit cycle of period 3, and an unstable limit cycle of period 2 with

an inserted stable limit cycle of period 1. Continuing this process further, we will obtain limit cycles of all periods from one to infinity, and the space between these cycles will be filled by spirals issuing from unstable limit cycles and tending to stable limit cycles as $t \to +\infty$. These cycles are inserted into each other, they make various combinations of rotation around the points O_1 and O_2 in the corresponding half-spaces containing these points and form geometric constructions (limit periodic sets) which look globally like very flat truncated cones described in item 3 of the Magnitskii–Sidorov scenario [6].

4. For $r = r_a \approx 24{,}74$, the biggest unstable limit cycles of infinite period disappear through the Andronov–Shilnikiv bifurcation [4, 5] in each of the half-spaces containing the points O_1 and O_2 (the cone vertices are at these points), and these points become unstable saddle-foci.

5. If $r_a < r < +\infty$, then a cascade of period-halving bifurcations [6] occurs in each of the half-spaces with respect to the manifold W^s. We have got again two symmetric with respect to the z-axis limit periodic sets consisting of limit cycles of all periods which are inserted into each other and make various combinations of rotation around the points O_1 and O_2 in the corresponding half-spaces containing these points, and the space between the cycles is filled by spirals issuing from unstable limit cycles and tending to stable limit cycles as $t \to +\infty$. The biggest limit cycles of these sets are stable now, and with further growth of r, the period-halving process makes them and the whole limit periodic sets more and more flat. The obtained geometric constructions are the only stable limit sets of system (1). The spirals of the unstable saddle-foci O_1 and O_2 and the trajectories issuing from infinity tend to these limit periodic sets (more precisely, to their stable limit cycles) as $t \to +\infty$. Just these stable limit periodic sets form two symmetric parts of the so-called Lorenz attractor, and this really looks very chaotic.

6. If $r \to +\infty$ (numerically, when $r > 313$), then the period-halving process will be finishing and system (1) will have two stable limit cycles in two phase half-spaces containing the unstable saddle-foci O_1 and O_2 of (1). This completes our scenario of chaos transition in the Lorenz system (1).

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Discrete Singular Integrals in a Half-Space

Alexander V. Vasilyev and Vladimir B. Vasilyev

Abstract We consider Calderon–Zygmund singular integral in the discrete half-space $h\mathbf{Z}^m_+$, where $\mathbf{Z}^m$ is entire lattice ($h > 0$) in $\mathbf{R}^m$, and prove, that the discrete singular integral operator is invertible in $L_2(h\mathbf{Z}^m_+)$ iff such is its continual analogue. The key point for this consideration takes solvability theory of so-called periodic Riemann boundary problem, which is constructed by authors.

Keywords Calderon–Zygmund kernel · Discrete singular integral · Symbol

Mathematics Subject Classification (2010) Primary 42A50 · Secondary 42A85

1 Introduction

We consider simplest Calderon–Zygmund operators of convolution type

$$v.p.\int_{\mathbf{R}^m} K(x-y)u(y)dy = \lim_{\substack{\varepsilon\to 0\\ N\to+\infty}} \int_{\varepsilon<|x-y|<N} K(x-y)u(y)dy,$$

where the kernel $K(x)$ satisfies the following conditions:

(1) $K(tx) = t^{-m}K(x)$, $\forall x \neq 0$, $t > 0$;
(2) $\int_{S^{m-1}} K(\theta)d\theta = 0$, S^{m-1} is unit sphere in $\mathbf{R}^m$;
(3) $K(x)$ is differentiable on $\mathbf{R}^m \setminus \{0\}$.

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Let us consider a discrete operator generated by the Calderon–Zygmund kernel $K(x)$, and defined on functions $u_h(\tilde{x})$, $\tilde{x} \in h\mathbf{Z}^m$, where $\mathbf{Z}^m$ is entire lattice $(h > 0)$ in $\mathbf{R}^m$, and the corresponding equation

$$au_h(\tilde{x}) + \sum_{\tilde{y} \in h\mathbf{Z}^m_+} K(\tilde{x} - \tilde{y}) u_h(\tilde{y}) h^m = v_h(\tilde{x}), \quad \tilde{x} \in h\mathbf{Z}^m_+, \tag{1.1}$$

a is certain constant, in the discrete half-space $h\mathbf{Z}^m_+ = \{\tilde{x} \in h\mathbf{Z}^m : \tilde{x_m} > 0\}$, $u_h, v_h \in L_2(h\mathbf{Z}^m_+)$.

By definition we put $K(0) = 0$, and for the operator

$$u_h(\tilde{x}) \mapsto au(\tilde{x}) + \sum_{\tilde{y} \in h\mathbf{Z}^m} K(\tilde{x} - \tilde{y}) u_h(\tilde{y}) h^m, \quad \tilde{x} \in h\mathbf{Z}^m,$$

we introduce its symbol by the formula

$$\sigma_h(\xi) = a + \sum_{\tilde{x} \in h\mathbf{Z}^m} e^{-i\xi\tilde{x}} K(\tilde{x}) h^m;$$

it is periodic function with basic cube period $[-\pi h^{-1}; \ \pi h^{-1}]^m$.

The sum for $\sigma_h(\xi)$ is defined as a limit of partial sums over cubes Q_N

$$\lim_{N \to \infty} \sum_{\tilde{x} \in Q_N} e^{-i\xi\tilde{x}} K(\tilde{x}) h^m,$$

$$Q_N = \left\{ \tilde{x} \in h\mathbf{Z}^m : |\tilde{x}| \le N, \ |\tilde{x}| = \max_{1 \le k \le m} |\tilde{x}_k| \right\}.$$

It is very similar classical symbol of Calderon–Zygmund operator [4], which is defined as Fourier transform of the kernel $K(x)$ in principal value sense

$$\sigma(\xi) = \lim_{\substack{N \to \infty \\ \varepsilon \to 0}} \int_{\varepsilon < |x| < N} K(x) e^{i\xi x} dx.$$

Key point of our study is theorem proved in [6], asserting that images of σ and σ_h are the same.

We also introduce continual equation in a half-space

$$au(x) + \int_{\mathbf{R}^m_+} K(x - y) u(y) dy = v(x), \quad x \in \mathbf{R}^m_+, \tag{1.2}$$

and we'll prove, that (1.1) and (1.2) are uniquely solvable or unsolvable simultaneously for all $h > 0$ in corresponding spaces.

2 Discrete Calderon–Zygmund Operators

2.1 Symbol Properties

We recall some properties of symbols $\sigma(\xi)$ and $\sigma_h(\xi)$, which are needed for us [6].

Lemma 2.1 $\lim_{h\to 0}\sigma_h(\xi)=\sigma(\xi)$, $\forall\xi\neq 0$.

Proof Indeed, if we fix $\xi\neq 0$, then by definition of integral as a limit of integral sums, we finish the proof. □

Lemma 2.2 $\sigma_h(\xi)=\sigma_1(h\xi)$, $\forall h>0$, $\xi\in[-\pi h^{-1},\pi h^{-1}]^m$.

Proof

$$\sigma_h(\xi)=\sum_{\tilde{x}\in h\mathbf{Z}^m}K(\tilde{x})e^{-i\tilde{x}\cdot\xi}h^m$$
$$=\sum_{\tilde{y}\in\mathbf{Z}^m}K(h\tilde{y})e^{-i\tilde{y}\cdot h\xi}h^m=\sum_{\tilde{y}\in\mathbf{Z}^m}K(\tilde{y})e^{-i\tilde{y}\cdot h\xi}=\sigma_1(h\xi).$$
□

Lemma 2.3 *The images of* σ *and* σ_h *are the same, and their values are constant for any ray from origin.*

Proof It follows from previous lemmas immediately, because if we fix ξ, then $\sigma_1(0)=\sigma(\xi)\Longrightarrow\sigma_h(0)=\sigma(\xi)$. □

2.2 Symbols and Operators

We consider more general in whole space $\mathbf{R}^m$

$$(M_1P_+ + M_2P_-)U=V,$$

taking into account that M_1, M_2 are operators of type (1.2), and P_+, P_- are restriction operators on $\mathbf{R}^m_\pm=\{x=(x_1,\dots,x_m),\ \pm x_m>0\}$. It is easily verified that (1.2) is a special case for such equation, when $M_2\equiv I$, I is identity operator.

If we'll denote the Fourier transform by letter F, and use the notations [3]

$$FP_+=Q_{\xi'}F,\qquad FP_-=P_{\xi'}F,$$
$$P_{\xi'}=1/2(I+H_{\xi'}),\qquad Q_{\xi'}=1/2(I-H_{\xi'}),$$

where $H_{\xi'}$ is Hilbert transform on variable ξ_m for fixed $\xi'=(\xi_1,\dots,\xi_{m-1})$ [1]:

$$(H_{\xi'}u)\big(\xi',\xi_m\big)\equiv\frac{1}{\pi i}v.p.\int_{-\infty}^{+\infty}\frac{u(\xi',\tau)}{\tau-\xi_m}d\tau,$$

then the equation mentioned after applying the Fourier transform will be the following equation with the parameter ξ':

$$\frac{\sigma_{M_1}(\xi',\xi_m)+\sigma_{M_2}(\xi',\xi_m)}{2}\tilde{U}(\xi) + \frac{\sigma_{M_1}(\xi',\xi_m)-\sigma_{M_2}(\xi',\xi_m)}{2\pi i} v.p.\int_{-\infty}^{+\infty}\frac{\tilde{U}(\xi',\eta)}{\eta-\xi_m}d\eta = \tilde{V}(\xi)$$

($\tilde{}$ denotes the Fourier transform).

This equation is closely related to boundary Riemann problem with the parameter ξ' with coefficient [2, 5]

$$G(\xi',\xi_m) = \sigma_{M_1}(\xi',\xi_m)\sigma_{M_2}^{-1}(\xi',\xi_m).$$

3 Periodic Riemann Boundary Problem

The theory of periodic Riemann boundary problem was constructed by authors [8] (see also forthcoming paper with the same name in *Differential Equations*) with full details, and now we will use its general consequences.

Let's denote $\mathbf{Z}_+ = 0, 1, 2, \ldots, \mathbf{Z}_- = \mathbf{R} \setminus \mathbf{Z}_+$. The Fourier transform for function of discrete variable is the series

$$(Fu)(\xi) = \sum_{k=-\infty}^{+\infty} u(k)e^{-ik\xi}, \quad \xi \in [-\pi,\pi]. \tag{3.1}$$

Let's consider the Fourier transform (3.1) for the indicator of $\mathbf{Z}_+$:

$$\chi_{\mathbf{Z}_+}(x) = \begin{cases} 1, & x \in \mathbf{Z}_+ \\ 0, & x \notin \mathbf{Z}_+. \end{cases}$$

For summable functions their product transforms to convolution of their Fourier images on the segment $[-\pi,\pi]$ but for our case $F(\chi_{\mathbf{Z}_+}\cdot u)$ one of functions $\chi_{\mathbf{Z}_+}$ is not summable. Thus, first we introduce some regularizing multiplier and evaluate the following Fourier transform

$$F\big(e^{-\tau k}\cdot\chi_{\mathbf{Z}_+}\big)(\xi) = \frac{1}{2\pi}\sum_{k\in\mathbf{Z}_+} e^{-\tau k}e^{-ik\xi} = \frac{1}{2\pi}\sum_{k\in\mathbf{Z}_+} e^{-\tau k - ik\xi} = \frac{1}{2\pi}\sum_{k\in\mathbf{Z}_+} e^{-ik(\xi+i\tau)}$$

$$= \frac{1}{2\pi}\sum_{k\in\mathbf{Z}_+} e^{-ikz}, \quad \tau\to 0,\ z=\xi+i\tau,\ \tau>0.$$

The Fourier transform for the function $u(n)$ we'll denote $\hat{u}(\xi)$, it is left to find the sum for e^{-ikz},

$$\frac{1}{2\pi}\sum_{k\in\mathbf{Z}_+} e^{-ikz} = \frac{1}{2\pi}\left(1+e^{-iz}+e^{-2iz}+\cdots\right) = \frac{1}{2\pi}\frac{1}{1-e^{-iz}}.$$

After some transformations:

$$F(\chi_{\mathbf{Z}_+}\cdot u)(\xi) = \lim_{\tau\to 0+}\left(\frac{\hat{u}(\xi)}{4\pi} + \frac{1}{4\pi i}\int_{-\pi}^{\pi}\hat{u}(t)\cot\frac{z-t}{2}dt\right), \quad z=\xi+i\tau.$$

According to Sokhotskii formulas (these are almost same for periodic kernel $\cot(x)$) (see also classical books [2, 5])

$$F(\chi_{\mathbf{Z}_+}\cdot u)(\xi) = \frac{\hat{u}(\xi)}{4\pi} + \frac{1}{4\pi i}\int_{-\pi}^{\pi}\hat{u}(t)\cot\frac{\xi-t}{2}dt + \frac{\hat{u}(\xi)}{2}.$$

If we introduce the function $\chi_{\mathbf{Z}_-}(x)$ and consider the Fourier transform for the product $F(\chi_{\mathbf{Z}_-}\cdot u)$ with preliminary regularization, then we have

$$F\left(e^{-\tau k}\cdot\chi_{Z_-}\right) = \frac{1}{2\pi}\sum_{-\infty}^{-1}e^{\tau k}e^{-ik\xi} = \frac{1}{2\pi}\sum_{-\infty}^{-1}e^{\tau k-ik\xi}$$

$$= \frac{1}{2\pi}\sum_{-\infty}^{-1}e^{-ik(\xi+i\tau)} = \frac{1}{2\pi}\sum_{-\infty}^{-1}e^{-ikz}, \quad \tau\to 0,\ z=\xi+i\tau,\ \tau<0.$$

Further,

$$\frac{1}{2\pi}\sum_{-\infty}^{-1}e^{-ikz} = \frac{1}{2\pi}\left(-1+1+e^{iz}+e^{2iz}+\cdots\right) = -\frac{1}{2\pi}+\frac{1}{2\pi}\frac{1}{1-e^{iz}}.$$

With the help of some elementary calculations:

$$F(\chi_{\mathbf{Z}_-}\cdot u) =_{n\to\xi} \lim_{\tau\to 0}\left(-\frac{\hat{u}(\xi)}{4\pi} - \frac{1}{4\pi i}\int_{-\pi}^{\pi}\hat{u}(t)\cot\frac{z-t}{2}dt\right), \quad z=\xi+i\tau.$$

Applying Sokhotskii formulas, we have:

$$F(\chi_{\mathbf{Z}_-}\cdot u)(\xi) = \frac{\hat{u}(\xi)}{4\pi} + \frac{1}{4\pi i}\int_{-\pi}^{\pi}\hat{u}(t)\cot\frac{\xi-t}{2}dt + \frac{\hat{u}(\xi)}{2}.$$

To verify one can find the sum for $F(\chi_{\mathbf{Z}_+}\cdot u)$, $F(\chi_{\mathbf{Z}_-}\cdot u)$ and obtain:

$$F(\chi_{\mathbf{Z}_+}\cdot u) + F(\chi_{\mathbf{Z}_-}\cdot u) = \frac{\hat{u}(\xi)}{4\pi} + \frac{1}{4\pi i}\int_{-\pi}^{\pi}\hat{u}(t)\cot\frac{\xi-t}{2}dt + \frac{\hat{u}(\xi)}{2}$$

$$-\frac{\hat{u}(\xi)}{4\pi} - \frac{1}{4\pi i}\int_{-\pi}^{\pi}\hat{u}(t)\cot\frac{\xi-t}{2}dt + \frac{\hat{u}(\xi)}{2} = \hat{u}(\xi).$$

These calculations lead to certain periodic Riemann boundary value problem, for which the solvability conditions are defined by the index of its coefficient. The problem is formulated as following way: finding two functions $\Phi^{\pm}(t)$ which admit an analytical continuation into upper and lower half-strip in the complex plane $\mathbf{C}$, real part is the segment $[-\pi, \pi]$, and their boundary values satisfy the relation

$$\Phi^{+}(t) = G(t)\Phi^{-}(t) + g(t), \quad t \in [-\pi, \pi],$$

$G(t)$, $g(t)$ are given functions on $[-\pi, \pi]$, and such that $G(-\pi) = G(\pi)$, $g(-\pi) = g(\pi)$.

Index for such problem is called the integer number

$$æ = \frac{1}{2\pi} \int_{-\pi}^{\pi} d \arg G(t).$$

4 Solvability Conditions

Here we suppose additionally, that the symbol $\sigma(\xi', \xi_m)$ satisfies the condition

$$\sigma(0, \dots, 0, -1) = \sigma(0, \dots, 0, +1).$$

Theorem 4.1 (Main Theorem) *Equations* (1.1) *and* (1.2) *are uniquely solvable or unsolvable simultaneously for all* $h > 0$.

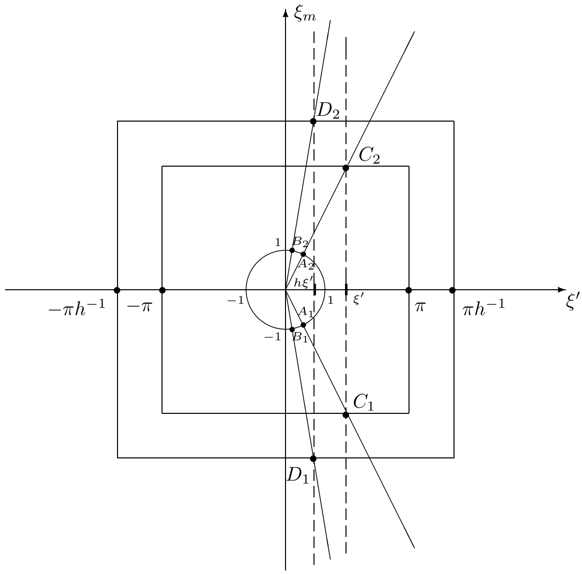
Proof We need to look our symbols $\sigma(\xi)$ and $\sigma_h(\xi)$ more exactly. We'll illustrate our consideration with the help of Fig. 1.

If we fix ξ' in the cube $[-\pi, \pi]^m$, then under varying ξ_m on $[-\pi, \pi]$ the argument of $\sigma_1(\xi)$ will vary along the curve on cubical surface of $[-\pi, \pi]^m$, which unites the points C_1 and C_2 (for the case $m \geq 3$ all such curves are homotopic, and for the case $m = 2$ there are two curves left and right one).

This varying corresponds to the varying of the argument of function $\sigma(\xi)$ along the curve from point A_1 to point A_2 on the unit sphere. Further, if we consider the symbol $\sigma_h(\xi)$ now on the cube $[-h^{-1}\pi, h^{-1}\pi]^m$, then according to Lemma 2.2 $h\xi_m$ will be varied on $[-\pi, \pi]$ also under fixed $h\xi'$. In other words, the argument of $\sigma_h(\xi)$ for fixed ξ' (we consider small $h > 0$) will be varied along curve on cubical surface of $[-h^{-1}\pi, h^{-1}\pi]^m$, which unites the points D_1 and D_2. It corresponds to varying argument of function $\sigma(\xi)$ from point B_1 to point B_2 on unit sphere. Obviously, under decreasing h the sequence $A_1, B_1, \dots$ will be convergent to south pole of the unit sphere $(0, \dots, 0, -1)$, and the sequence $A_2, B_2, \dots$ to north pole $(0, \dots, 0, +1)$. Thus, because the variation of argument of $\sigma_h(\xi)$ on ξ_m under fixed ξ' is $2\pi k$ (σ_h is periodic function), then under additional assumption

$$\sigma(0, \dots, 0, -1) = \sigma(0, \dots, 0, +1)$$

Fig. 1 Illustration to the proof of Theorem 4.1



(this property is usually called the transmission property) we'll obtain that variation of argument for the function $\sigma(\xi)$ under varying ξ_m from $-\infty$ to $+\infty$ under fixed ξ' (this variation of $\sigma(\xi)$ moves along the arc of big half-circumference on unit sphere) is also $2\pi k$. According to our assumptions on continuity of $\sigma(\xi)$ on the unit sphere, it will be the same number $2\pi k$, $k \in \mathbf{Z}$,

$$\lim_{h\to 0}\int_{-\pi h^{-1}}^{\pi h^{-1}} d\arg\sigma_h\big(\xi', \xi_m\big) = \int_{-\infty}^{+\infty} d\arg\sigma\big(\xi', \xi_m\big), \quad \forall \xi' \neq 0.$$

So, both for (1.1) and (1.2) the uniquely solvability condition is defined by the same number. This completes the proof. □

5 Conclusion

We see that both continual and discrete equations are solvable or unsolvable simultaneously, and then we need to find good finite approximation for infinite system of linear algebraic equations for computer calculations. First steps in this direction were done in the paper [7], where the authors suggested to use fast Fourier transform.

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Geometrical Features of the Soliton Solution

Zhanat Zhunussova

Abstract It is well known, that integrable equations are solvable by the inverse scattering method (Ablowitz and Clarkson in Solitons, Non-linear Evolution Equations and Inverse Scattering, 1992). Investigating of the integrable spin equations in (1 + 1), (2 + 1) dimensions are topical both from the mathematical and physical points of view (Lakshmanan and Myrzakulov in J. Math. Phys. 39:3765–3771, 1998; Gardner et al. in Phys. Rev. Lett. 19(19):1095–1097, 1967). Integrable equations admit different kinds of physically interesting solutions as solitons, vortices, dromions etc. We consider an integrable spin M-I equation (Myrzakulov and Vijayalakshmi in Phys. Lett. A 233:391–396, 1997). There is a corresponding Lax representation. And the equation allows an infinite number of integrals of motion. We construct a surface corresponding to soliton solution of the equation. Further, we investigate some geometrical features of the surface.

Keywords Surface · Soliton · Nonlinear equation

1 Introduction

We consider the connection between the surface and the soliton equation M-I which has the form [2],

$$\mathbf{S}_t = (\mathbf{S} \times \mathbf{S}_y + u\mathbf{S})_x, \tag{1.1}$$

$$u_x = -\big(\mathbf{S}, (\mathbf{S}_x \times \mathbf{S}_y)\big), \tag{1.2}$$

where $\mathbf{S}$ is spin vector, $S_1^2 + S_2^2 + S_3^2 = 1$, $\times$ is vector product, u is a scalar function. We identify the spin vector $\mathbf{S}$ and vector $\mathbf{r}_x$ according to [2]

$$\mathbf{S} \equiv \mathbf{r}_x. \tag{1.3}$$

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Then (1.1), (1.2) take the form

$$\mathbf{r}_{xt} = (\mathbf{r}_x \times \mathbf{r}_{xy} + u\mathbf{r}_x)_x, \tag{1.4}$$

$$u_x = -\big(\mathbf{r}_x, (\mathbf{r}_{xx} \times \mathbf{r}_{xy})\big). \tag{1.5}$$

If we integrate (1.4) by x, then it takes the form

$$\mathbf{r}_t = \mathbf{r}_x \times \mathbf{r}_{xy} + u\mathbf{r}_x. \tag{1.6}$$

Taking into account Gauss–Weingarten equation and $E = \mathbf{r}_x^2 = 1$ the system is defined as

$$\mathbf{r}_t = \left(u + \frac{MF}{\sqrt{\Lambda}}\right)\mathbf{r}_x - \frac{M}{\sqrt{\Lambda}}\mathbf{r}_y + \Gamma_{12}^2\sqrt{\Lambda}\mathbf{n}, \tag{1.7}$$

$$u_x = \sqrt{\Lambda}\big(L\Gamma_{12}^2 - M\Gamma_{11}^2\big), \tag{1.8}$$

where

$$\Gamma_{11}^2 = \frac{2EF_x - EE_t - FE_x}{2\Lambda}, \tag{1.9}$$

$$\Gamma_{12}^2 = \frac{EG_x - FE_t}{2\Lambda}, \tag{1.10}$$

$\Lambda = EG - F^2$. M-I equation is integrable equation and has soliton solutions.

2 Construction of Surface Corresponding to Soliton Solution

Here we present the one-soliton solution of (1.1), (1.2) [2],

$$S_3(x, y, t) = 1 - \frac{2\eta^2}{\eta^2 + \xi^2}\operatorname{sech}^2(\chi_{1R}), \tag{2.1}$$

$$S^+(x, y, t) = \frac{2\eta}{\eta^2 + \xi^2}\big[i\xi - \eta th(\chi_{1R})\big]\operatorname{sech}(\chi_{1R}), \tag{2.2}$$

$$\chi_1 = \chi_{1R} + i\chi_{1I}, \qquad \lambda_1 = \eta + i\xi, \tag{2.3}$$

$$m_1 = m_{1R}(\rho) + i m_{1I}(\rho), \quad m_j(y, t) = m_j(\rho), \tag{2.4}$$

$$\chi_{1R} = \eta x + m_{1R}(\rho) + c_{1R}, \quad \rho = y + i\lambda_j t, \tag{2.5}$$

$$\chi_{1I} = \xi x + m_{1I}(\rho) + c_{1I}, \quad c = \ln\big(2\eta/\lambda_1^*\big), \tag{2.6}$$

$$m_{1R}(\rho) = \operatorname{Re}\big[m_1(\rho)\big], \qquad m_{1I}(\rho) = \operatorname{Im}\big[m_1(\rho)\big], \tag{2.7}$$

which we use in the following theorem.

Theorem 2.1 (Main Theorem) *One-soliton solution* (2.1)–(2.7) *of the spin system M-I can be represented as components of the vector* $\mathbf{r}_x$, *where*

$$r_1 = \frac{2\eta}{(\eta^2+\xi^2)ch\chi_{1R}} + c_1, \tag{2.8}$$

$$r_2 = \frac{2\xi}{\eta^2+\xi^2}\operatorname{arctg}(sh\chi_{1R}) + c_2, \tag{2.9}$$

$$r_3 = x - \frac{2\eta}{\eta^2+\xi^2}th\chi_{1R} + c_3, \tag{2.10}$$

c_1, c_2, c_3 *are constants. Solution of the form* (2.8)–(2.10) *corresponds to the surface with the following coefficients of the first and second fundamental forms*

$$E = 1, \qquad G = \frac{4m_{1Ry}^2}{(\eta^2+\xi^2)ch^2\chi_{1R}}, \tag{2.11}$$

$$F = \frac{2\eta m_{1Ry}}{(\eta^2+\xi^2)ch^2\chi_{1R}}, \qquad L = \frac{4\eta^3\xi m_{1Ry}}{\sqrt{g}(\eta^2+\xi^2)^2ch^4\chi_{1R}}, \tag{2.12}$$

$$M = \frac{4\eta^2\xi m_{1Ry}^2}{\sqrt{g}(\eta^2+\xi^2)^2ch^4\chi_{1R}}, \qquad N = \frac{4\eta\xi m_{1Ry}^3}{\sqrt{g}(\eta^2+\xi^2)^2ch^4\chi_{1R}}. \tag{2.13}$$

Proof From (1.3) we have

$$(S_1, S_2, S_3) = (r_{1x}, r_{2x}, r_{3x}), \tag{2.14}$$

i.e.

$$r_{1x} = S_1, \qquad r_{2x} = S_2, \qquad r_{3x} = S_3. \tag{2.15}$$

Hence

$$r_1 = \int S_1 dx + c_1, \tag{2.16}$$

$$r_2 = \int S_2 dx + c_2, \tag{2.17}$$

$$r_3 = \int S_3 dx + c_3, \tag{2.18}$$

where c_1, c_2, c_3 are constants of integration. Note

$$S^+ = S_1 + iS_2 = r_x^+, \tag{2.19}$$

then

$$r^+ = r_1 + ir_2 = \int S^+ dx + c^+, \tag{2.20}$$

where c^+ is constant of integration. Substituting (2.1) in (2.18), we have

$$r_3 = \int S_3 dx + c_3 = \int \left[1 - \frac{2\eta^2}{\eta^2+\xi^2}\operatorname{sech}^2(\chi_{1R})\right]dx + c_3$$
$$= x - \frac{2\eta}{(\eta^2+\xi^2)} th(\chi_{1R}) + c_3^*, \tag{2.21}$$

where $c_3^* = c_3 + c_3'$. $c_3 \equiv c_3^*$, then

$$r_3 = x - \frac{2\eta}{(\eta^2+xi^2)} th(\chi_{1R}) + c_3. \tag{2.22}$$

Substituting (2.2) into (2.20) we have

$$r^+ = r_1 + ir_2 = \int S^+ dx + c^+$$
$$= \int \frac{2\eta}{\eta^2+\xi^2}\left[i\xi - \eta th(\chi_{1R})\right]\operatorname{sech}(\chi_{1R})dx + c^+$$
$$= \frac{2i\xi}{\eta^2+\xi^2}\operatorname{arctg}(sh\chi_{1R}) + \frac{2\eta}{\eta^2+\xi^2}\frac{1}{ch\chi_{1R}} + c'' + c^+ + c'''. \tag{2.23}$$

We denote $c_1 = c''$, $c_2 = c^+ + c'''$, then

$$r^+ = \frac{2\eta}{(\eta^2+\xi^2)ch\chi_{1R}} + c_1 + i\left(\frac{2\xi}{\eta^2+\xi^2}\operatorname{arctg}(sh\chi_{1R}) + c_2\right), \tag{2.24}$$

i.e. we have obtained

$$r_1 = \frac{2\eta}{(\eta^2+\xi^2)ch\chi_{1R}} + c_1, \qquad r_2 = \frac{2\xi}{\eta^2+\xi^2}\operatorname{arctg}(sh\chi_{1R}) + c_2. \tag{2.25}$$

Thus, (2.22), (2.25) give us (2.8)–(2.10).

We proceed to prove the second part of the theorem. From (2.22) and (2.25) we have

$$r_{1x} = -\frac{2\eta^2 sh\chi_{1R}}{(\eta^2+\xi^2)ch^2\chi_{1R}}, \qquad r_{2x} = \frac{2\eta\xi}{(\eta^2+\xi^2)ch\chi_{1R}}, \tag{2.26}$$

$$r_{3x} = 1 - \frac{2\eta^2}{(\eta^2+\xi^2)ch^2\chi_{1R}}, \qquad r_{1y} = -\frac{2\eta sh\chi_{1R} m_{1Ry}}{(\eta^2+\xi^2)ch^2\chi_{1R}}, \tag{2.27}$$

$$r_{2y} = \frac{2\xi m_{1Ry}}{(\eta^2+\xi^2)ch\chi_{1Ry}}, \qquad r_{3y} = -\frac{2\eta m_{1Ry}}{(\eta^2+\xi^2)ch^2\chi_{1Ry}}. \tag{2.28}$$

Then we can calculate

$$\begin{aligned} E = \mathbf{r}_x^2 = r_{1x}^2 + r_{2x}^2 + r_{3x}^2 \\ = \frac{4\eta^4 sh^2\chi_{1R}}{(\eta^2+\xi^2)^2 ch^4\chi_{1R}} + \frac{4\eta^2\xi^2}{(\eta^2+\xi^2)^2 ch^2\chi_{1R}} \\ + 1 - \frac{4\eta^2}{(\eta^2+\xi^2) ch^2\chi_{1R}} + \frac{4\eta^4}{(\eta^2+\xi^2)^2 ch^4\chi_{1R}} \equiv 1. \end{aligned} \tag{2.29}$$

Similarly, using (2.25) and (2.28) we obtain

$$G = \mathbf{r}_y^2 = r_{1y}^2 + r_{2y}^2 + r_{3y}^2 = \frac{4m_{1Ry}^2}{(\eta^2+\xi^2) ch^2\chi_{1R}} \equiv 1, \tag{2.30}$$

$$F = (\mathbf{r}_x, \mathbf{r}_y) = r_{1x}r_{1y} + r_{2x}r_{2y} + r_{3x}r_{3y} = \frac{2\eta m_{1Ry}}{(\eta^2+\xi^2) ch^2\chi_{1R}}. \tag{2.31}$$

Formulas (2.29)–(2.31) give us the first three equations (2.11)–(2.13). Using (2.29)–(2.31) we compute

$$\Lambda = EG - F^2 = \frac{4m_{1Ry}^2(\eta^2 sh^2\chi_{1R} + \xi^2 ch^2\chi_{1R})}{(\eta^2+\xi^2)^2 ch^4\chi_{1R}}. \tag{2.32}$$

We calculate the components of the vector $\mathbf{n}$

$$\mathbf{n} = \frac{\mathbf{r}_x \times \mathbf{r}_y}{|\mathbf{r}_x \times \mathbf{r}_y|} = \frac{\mathbf{r}_x \times \mathbf{r}_y}{\sqrt{\Lambda}} = \frac{1}{\sqrt{\Lambda}}(n_1, n_2, n_3), \tag{2.33}$$

$$n_1 = \frac{1}{\sqrt{\Lambda}}(r_{2x}r_{3y} - r_{3x}r_{2y}) = -\frac{2\xi m_{1Ry}}{\sqrt{\Lambda}(\eta^2+\xi^2) ch\chi_{1R}}. \tag{2.34}$$

Similarly, for the components

$$n_2 = \frac{1}{\sqrt{\Lambda}}(r_{3x}r_{1y} - r_{1x}r_{3y}) = -\frac{2\eta sh\chi_{1R} m_{1Ry}}{\sqrt{\Lambda}(\eta^2+\xi^2) ch^2\chi_{1R}}, \tag{2.35}$$

$$n_3 = \frac{1}{\sqrt{\Lambda}}(r_{1x}r_{2y} - r_{2x}r_{1y}) = 0. \tag{2.36}$$

Now, from (2.26), (2.28) we have

$$r_{1xx} = -\frac{2\eta^3 ch\chi_{1R}(ch^2\chi_{1R} - 2sh^2\chi_{1R})}{(\eta^2+\xi^2) ch^4\chi_{1R}} = -\frac{2\eta^3(1 - sh^2\chi_{1R})}{(\eta^2+\xi^2) ch^3\chi_{1R}}, \tag{2.37}$$

$$r_{2xx} = -\frac{2\eta^2\xi sh\chi_{1R}}{(\eta^2+\xi^2) ch^2\chi_{1R}}, \tag{2.38}$$

$$r_{3xx} = \frac{4\eta^3 sh\chi_{1R}}{(\eta^2+\xi^2) ch^3\chi_{1R}}. \tag{2.39}$$

Thus, using (2.34)–(2.39) we can compute

$$r_{3xx} = \frac{4\eta^3 sh\chi_{1R}}{(\eta^2+\xi^2)ch^3\chi_{1R}}. \tag{2.40}$$

Taking into account, that $n_3 = 0$,

$$L = \frac{4\eta^3\xi m_{1Ry}}{\sqrt{\Lambda}(\eta^2+\xi^2)^2 ch^4\chi_{1R}}. \tag{2.41}$$

Similarly, we calculate

$$M = \frac{4\eta^2\xi m_{1Ry}^2}{\sqrt{\Lambda}(\eta^2+\xi^2)^2 ch^4\chi_{1R}}, \tag{2.42}$$

$$N = \frac{4\eta\xi m_{1Ry}^3}{\sqrt{\Lambda}(\eta^2+\xi^2)^2 ch^4\chi_{1R}}. \tag{2.43}$$

The formulas (2.41)–(2.43) give us the last three equations (2.11)–(2.13). Using (2.11)–(2.13), for example, the Gaussian curvature can be calculated

$$\begin{aligned} K &= \frac{LN - M^2}{\Lambda} \\ &= \frac{1}{\Lambda}\left(\frac{4\eta^3\xi m_{1Ry}}{\sqrt{\Lambda}(\eta^2+\xi^2)^2 ch^4\chi_{1R}} \frac{4\eta\xi m_{1Ry}^3}{\sqrt{\Lambda}(\eta^2+\xi^2)^2 ch^4\chi_{1R}} - \frac{16\eta^4\xi^2 m_{1Ry}^4}{\Lambda(\eta^2+\xi^2)^4 ch^8\chi_{1R}}\right) \\ &= \frac{1}{\Lambda}\left(\frac{16\eta^4\xi^2 m_{1Ry}^4}{\Lambda(\eta^2+\xi^2)^4 ch^8\chi_{1R}} - \frac{16\eta^4\xi^2 m_{1Ry}^4}{\Lambda(\eta^2+\xi^2)^4 ch^8\chi_{1R}}\right) \equiv 0. \end{aligned} \tag{2.44}$$

We see that for the surface Gaussian curvature is equal to zero. Theorem is proved. □

3 Conclusion

Based on the results of work [2], where Gauss–Codazzi–Mainardi equation considered in multidimensional space, we have studied equation M-I and built the surface corresponding to soliton solution. Thus, this work fully reveals the meaning of the geometric approach [2] in $(2+1)$-dimensions.

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Part XII
M-Frame Constructions

Organizers: Krzysztof Rudol, Anna Grybos, Darian Onchis

Approximate Dual M-Frames Constructions: The Gabor Case

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Abstract The aim of this work is to provide an efficient method to realize constructively approximate duals of Gabor frames with multivariate atoms. The proposed method is independent of the number of atoms needed and it is applicable also in the case of non-separable atoms. Due to the small number of atoms used in the construction the method is computationally inexpensive.

Keywords Frames · Approximate dual constructions · Multivariate atoms · Gabor frames

Mathematics Subject Classification (2010) Primary 42C15 · Secondary 65D99

1 Introduction

Gabor analysis is concerned with the representation of signals using series of time-frequency shifted copies of a given atom. If the set of shifts used to build the Gabor family is a lattice Λ, one can use the so-called canonical dual Gabor atom (with respect to the same lattice Λ) in order to find the coefficients for such non-orthogonal expansions. Classically mostly separable Gabor lattices are used, which are described by a time shift a and a frequency shift b (resp. the number of channels). In contrast to this classical setting the general non-separable case has found only little attention in the literature.

In the paper [4], the authors have proposed a method to compute good approximations to the dual Gabor atom, using only a few neighbours from an adjoint lattice Λ°. In the current contribution we are extending that method to M-frames constructed using multivariate atoms.

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Existing work on approximating the inverse frame operator can be found in [2]. The frame operator has in general no bounded inverse. It is however possible to find a subspace, such that the projection is boundedly invertible.

The proposed computational procedure is intended to be used in applications when the signal must be reconstructed from the coefficients.

Frames $(g_i)_{i\in I}$ generalize the idea of a basis in a Hilbert space $\boldsymbol{H}$ and consist of the indexed families such that the so-called **frame operator** S

$$Sf = \sum_{i\in I} \langle f, g_i\rangle g_i \tag{1}$$

is invertible. Hence every element $f \in \boldsymbol{H}$ has an expansion of the form [2]:

$$f = SS^{-1}f = \sum_{i\in I} \langle S^{-1}f, g_i\rangle g_i = \sum_{i\in I} \langle f, S^{-1}g_i\rangle g_i. \tag{2}$$

The family $(\widetilde{g}_i)_{i\in I} = (S^{-1}g_i)_{i\in I}$ is again a frame and is called the **canonical dual frame**. It is not the only dual frame unless the frame is a basis. In general, any frame $(\gamma_i)_{i\in I}$ that allows the expansion of any $f \in \boldsymbol{H}$ as follow:

$$f = \sum_{i\in I} \langle f, g_i\rangle \gamma_i = \sum_{i\in I} \langle f, \gamma_i\rangle g_i \tag{3}$$

is called a **dual frame**. However, the canonical dual $(\widetilde{g}_i)_{i\in I}$ provides the coefficients $(\langle f, \widetilde{g}_i\rangle)_{i\in I}$ of minimal l^2-norm.

The main tool for time-frequency analysis is the **Short-Time Fourier Transform**, defined for functions $f, g \in \boldsymbol{L}^2(\mathbb{R}^d)$ at $\lambda = (\alpha, \beta) \in \mathbb{R}^{2d}$ by

$$V_g f(\lambda) = V_g f(\alpha, \beta) = \langle f, M_\beta T_\alpha g\rangle = \langle f, \pi(\lambda)g\rangle \tag{4}$$

where $T_\alpha f(t) = f(t-\alpha)$ is the translation (time shift) and $M_\beta f(t) = e^{2\pi i\beta\cdot t} f(t)$ is the modulation (frequency shift). The operators $\pi(\lambda) := M_\beta T_\alpha$ are called **time-frequency shifts** and the set $\Lambda = \{\lambda; \lambda = (\alpha, \beta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d\}$ is a **lattice**.

A sequence $(h_i)_{i\in I}$ in a separable Hilbert space $\boldsymbol{H}$ is a **Riesz basic sequence (RBS)** resp. **Riesz basis for its closed linear span** if there are two constants $0 < C \le D < \infty$ with

$$C\|c\|_{l^2}^2 \le \left\| \sum_i c_i h_i \right\|_{\boldsymbol{H}}^2 \le D\|c\|_{l^2}^2, \quad \forall c \in l^2(I). \tag{5}$$

The **Gabor system** $\mathcal{G}(g, \Lambda) = \{\pi(\lambda)g; \lambda \in \Lambda\}$ over the lattice Λ consisting of the translated and modulated versions of one atom g, is a **frame** for the space $L^2(\mathbb{R}^d)$, if and only if there exist $0 < A \le B < \infty$ (**frame bounds**) with

$$A\|f\|^2 \le \sum_{\lambda\in\Lambda} \left|\langle f, \pi(\lambda)g\rangle\right|^2 \le B\|f\|^2 \quad \text{for every } f \in L^2(\mathbb{R}^d). \tag{6}$$

Theorem 1.1 (Dual Gabor Frames [5]) *If $\mathcal{G}(g, \Lambda)$ is a Gabor frame for $\boldsymbol{L}^2(\mathbb{R}^d)$, for some lattice $\Lambda \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, then the canonical dual frame takes the form $\mathcal{G}(\gamma, \Lambda)$ for $\gamma = S^{-1}g$.*

The frame $\mathcal{G}(\widetilde{g}, \Lambda)$ for $\widetilde{g} = S^{-1}g$ is again a Gabor frame with respect to the same lattice.

For two Gabor systems we can write the frame operator (1) in the generalized form:

$$S_{g,\gamma,\Lambda} f := \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda) g \rangle \pi(\lambda)\gamma. \tag{7}$$

In particular, if $\mathcal{G}(g, \Lambda)$ is a frame and $\mathcal{G}(\widetilde{g}, \Lambda)$ is its canonical dual frame, then $S_{g,\widetilde{g},\Lambda}$ is an identity operator. Note that then $S = S_{g,g,\Lambda}$ and $S^{-1} = S_{\widetilde{g},\widetilde{g},\Lambda}$.

2 Multivariate M-Frames

We recall that Lyubarskii [6] and Seip and Wallstén [8] have proven that:

Proposition 2.1 *Let $g_1(x) = 2^{1/4} e^{-\pi x^2}$, $x \in \mathbb{R}$ and $\Lambda_{ab} = a\mathbb{Z} \times b\mathbb{Z}$. Then the Gabor system $\mathcal{G}(g_1, \Lambda_{ab})$ is a frame for $L^2(\mathbb{R})$ if and only if $ab < 1$.*

In order to construct M-frames, we need the following extension of Proposition 2.1, due to Bourouihiya [1]:

Lemma 2.2 *Let $g = g_1 \otimes \cdots \otimes g_1$ (d factors) and $\Lambda_{ab} = (a_1\mathbb{Z} \times \cdots \times a_d\mathbb{Z}) \times (b_1\mathbb{Z} \times \cdots \times b_d\mathbb{Z})$. Then $\mathcal{G}(g, \Lambda_{ab})$ is a frame if and only if $a_j b_j < 1$ for $1 \le j \le d$.*

The proof of this extension is straightforward using tensor products of Gabor systems.

Based on this result, another extension to non-separable lattices of the form $N\Lambda_{a,b}$ is possible [7], which besides a multi-variate sampling in each dimension, gives us a better packing of the time-frequency plane by using a (non-separable) hexagonal lattice that matches with the circular contour lines of the Gaussian. The representation of the non-separable lattice is based on the rectangular lattice via a shear operation denoted by N.

To construct multivariate M-frames, we will use generalized Gaussians.

Let $M = X + iY$ be a complex symmetric matrix of size d, we assume that $X = \operatorname{Re} M$ is positive-definite (hence M is invertible). We define the generalized Gaussian $g_M^\gamma = g_{X,Y}^\gamma$ by

$$g_M^\gamma(x) = 2^{d/4} e^\gamma (\det X)^{1/4} e^{-\pi \langle Mx, x \rangle}; \tag{8}$$

where γ is an arbitrary (real) phase. The function g_M^γ is normalized: $\|g_M^\gamma\|_{L^2} = 1$. When M is the identity matrix and $\gamma = 0$ one gets the standard (normalized) Gaussian

$$g_I^0(x) = 2^{d/4} e^{-\pi|x|^2}. \tag{9}$$

The theoretical results open the way for numerical constructions far beyond the standard Gabor processing on regular lattices. They gives us the freedom of generalizing both the lattice and the Gaussian window in a multi-variate way. The added flexibility will allow to find the most appropriate representation in the Gabor domain for different features identifications (e.g. 3D plane waves).

The numerical construction of multivariate Gaussian Gabor systems for multiple dimensions is possible as an extension of the 1D case. Using tensor products, it is well understood the potential of Gabor frames for analyzing nD-objects but also the numerical hurdles coming with their numerical implementation made it less feasible in practice.

In order to overcome these numerical hurdles, we will use the local construction of the dual system and we will extend it to the multidimensional setting. Basically, we apply the computationally inexpensive procedure in each dimension and then as a tensor product it gives the final approximate dual.

In these conditions it is feasible to perform a Gabor analysis followed by a Gabor synthesis at level of the constituting vectors, and therefore reducing the case of applying nD matrices to the fast case of applying the transform over each dimension. In this way, once we have the transform coefficients, we can approximate the nD object via the dual Gabor frame recovery.

3 Approximate Dual Gabor Frames

For the non-separable case we could still use the Janssen representation of the frame operator [5]:

$$S_{g,\Lambda} = C_\Lambda \sum_{\lambda^\circ \in \Lambda^\circ} V_g g(\lambda^\circ) \pi(\lambda^\circ),$$

where Λ° is the adjoint group (a symplectic variant of the orthogonal group).

As a consequence, for the M-frames constructions proposed in this contribution, we could use a sufficient condition given in [3]:

Theorem 3.1 *A pair* (g, Λ) *with* $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ *or* $g \in \boldsymbol{L}^2(\mathbb{R}^d)$, $g \neq 0$ *is a frame if the following Janssen-like condition is satisfied*:

$$s_2 := \|g\|_2{}^{-1} \sum_{\lambda^\circ \in \Lambda^\circ} \left| V_g g(\lambda^0) \right| \le 2 - \delta, \quad \delta > 0. \tag{10}$$

The state of the art for the calculation of the dual Gabor frames is that there exist many efficient algorithms for dual Gabor atoms in the separable case, already implemented and available in toolboxes. However, in the non-separable case the existing solutions are very time-consuming, hence an extension to multivariate atoms is necessary. The local approximation of the canonical dual atom for the Gabor frame relies on the following theorem and its consequences:

Theorem 3.2 (Wexler–Raz Identity) *Let Λ be a lattice in $\mathbb{R} \times \widehat{\mathbb{R}}$ with adjoint lattice Λ° and a pair (g, γ) in $\boldsymbol{L}^2 \times \boldsymbol{L}^2(\mathbb{R}^d)$. Assume that the synthesis operator $c \mapsto \sum_{\lambda\in\Lambda} c_\lambda \pi(\lambda) g$ and $c \mapsto \sum_{\lambda\in\Lambda} c_\lambda \pi(\lambda)\gamma$ are bounded on $\boldsymbol{\ell}^2(\Lambda)$. Then the following conditions are equivalent*:

(i)

$$S_{g,\gamma,\Lambda} f = S_{\gamma,g,\Lambda} = Id \quad on\ \boldsymbol{L}^2\big(\mathbb{R}^d\big).$$

(ii)

$$\big\langle \gamma, \pi\big(\lambda^\circ\big) g\big\rangle = red(\Lambda)\delta_{\lambda^\circ,0} \quad for\ \lambda^\circ \in \Lambda^\circ.$$

We introduce the following notation [4]: let $(h_i)_{i\in I}$ be a Riesz basic sequence in a Hilbert space $\boldsymbol{H}$ with the biorthogonal family $(\tilde{h}_i)_{i\in I}$. For $J \subseteq I$, we write $(h_i^J)_{i\in J}$ for the **local Riesz basis** which spans the closed subspace $\boldsymbol{H}^J$. We denote by $(\tilde{h}_i^J)_{i\in J}$ the **local biorthogonal family** to the local Riesz basis $(h_i^J)_{i\in J}$. It is obtained by forming linear combinations of the vectors in $(h_i)_{i\in J}$, using as coefficients the entries of the i-th column of the inverse **local Gramian matrix** $G_J = (\langle h_j, h_{j'}\rangle)_{j,j'\in J}$.

Theorem 3.3 ([4]) *Assume that a symmetric resp. Hermitian matrix $G = (g_{i,j})_{i,j\in I}$ satisfying $g_{i,i} = 1$ for all $i \in I$ is diagonal dominant, i.e.*:

$$\sup_{i\in I} \sum_{j\neq i} |g_{i,j}| \le s_1 < 1. \tag{11}$$

Additionally, for any subset $J \subseteq I$ let V_J denotes the pseudo-inverse of the matrix obtained from G by setting all the rows and columns with indices from $I \setminus J$ to zero, or equivalently by inverting the submatrix $G_J = (g_{j,j'})_{j,j'\in J}$.

Then G is invertible and the inverse can be approximated by local inverse matrices V_J as $J \nearrow I$ in the sense that for each fixed index $i \in I$ the columns of V_J converge to the corresponding column of $V_I = G^{-1}$ in the norm of $\ell^1(I)$.

The simulations presented in [4] show a good rate of approximation of the canonical dual atom by only a few elements of the adjoint Riesz basis. In order to define the subregions $J \subseteq I$ in the lattice Λ° the masks of the square block shape were considered in the tests. In the examples (Figs. 1 and 2) the size of the block changed from 3×3, 5×5 to 7×7, hence the number of used elements of the frame G increased from 9, 25, to 49 (see Fig. 1). In the case of a Gabor frame generated by a Gaussian window g of length $n = 240$ and time-frequency parameters $(a, b) = (12, 12)$, it constituted 2.25 %, 6.25 %, 12.25 % of all frame elements.

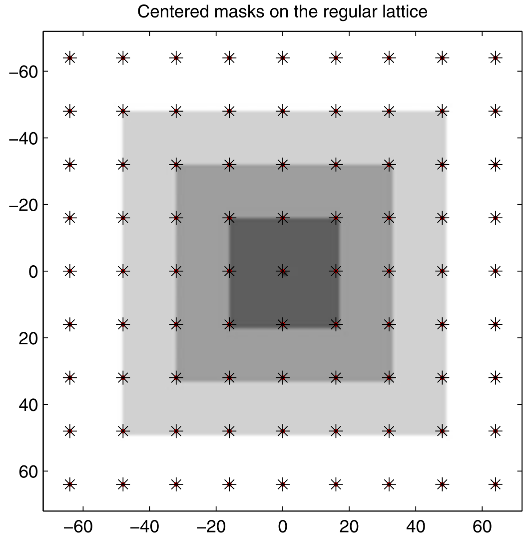

Fig. 1 The masks applied to the regular lattice give the areas of 3×3, 5×5 and 7×7 elements around the center

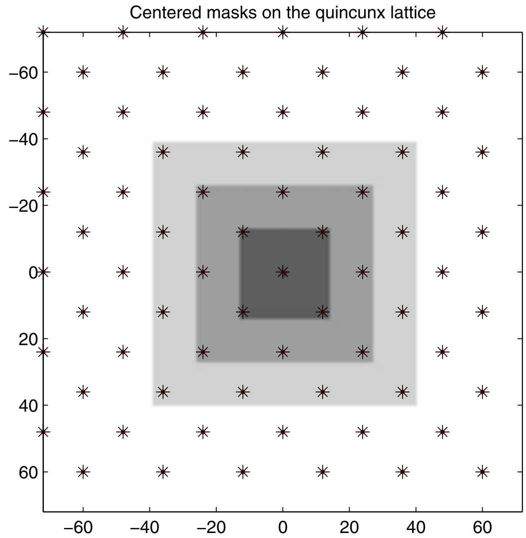

Fig. 2 The masks applied to the quincunx lattice give the areas of 3×3, 5×5 and 7×7 elements around the center

4 Conclusions

The computational efficiency of the new approach is based on the Wexler–Raz identity transferring the problem to the adjoint lattice and the convergence of Neumann series corresponding to the small Gramians G_J. Thus, the proposed procedure uses only a reduced number of time-frequency shifted atoms from the adjoint lattice to approximate the dual atom.

The proposed method is independent of the number of atoms and works in the multivariate case. It is applicable in the non-separable case (e.g. quincunx lattice) and it is computationally inexpensive due to the small number of atoms used. For the further developments the idea of intrinsic localization of the frame is necessary.

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Matrices of Operators on Some Function Spaces

Wojciech Mikołajczyk and Krzysztof Rudol

Abstract Some matrix representations for Hilbert space operators are considered. The corresponding matrices are related to frames and appear in a quite natural way especially in the case of reproducing kernel spaces. The membership in Schatten classes is discussed in terms of a discrete set of points, where the corresponding symbols are evaluated.

Keywords Schatten classes · Frames · Reproducing kernel · Berezin symbol

Mathematics Subject Classification (2010) Primary 47A05 · Secondary 47B10 · 42C15

1 Matrices with Respect to Frames

In order to study properties of frames in a Hilbert space H one needs to apply some operator theory. Even the very definition can be phrased as a statement concerning two basic operators of analysis and synthesis. Namely, a sequence $\mathcal{G} = (g_j)_{j \in \mathbb{N}}$ in H defines the *analysis operator* $C = C_{\mathcal{G}}$ assigning to $x \in H$ the sequence $(\langle x, g_j \rangle)$. Its range space will be denoted by $\mathcal{R}(C)$.

Definition 1.1 We say that the above sequence is

- a *Bessel sequence* if C maps boundedly H into ℓ^2, i.e. $\|C\| < \infty$;
- a *frame* if $\|C\| < \infty$ and $C^{-1} : \mathcal{R}(C) \to H$ is bounded in ℓ^2-norm.

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The *synthesis operator* $D = D_{\mathcal{G}} := C_{\mathcal{G}}^* : \ell^2 \to H$ sends scalar, square summable sequences (λ_j) to $\sum_j \lambda_j g_j$. The constants $\kappa_2 = \|C\|^2$, $\kappa_1 = \|(C|_{\mathcal{R}(C)})^{-1}\|^{-2}$—are the upper (= Bessel), resp. lower *frame bounds* are optimal in the following frame condition, usually taken as the definition of a frame:

$$\kappa_1 \|x\|^2 \le \sum_{j=1}^{\infty} \left|\langle x, g_j\rangle\right|^2 \le \kappa_2 \|x\|^2, \quad x \in H. \tag{1.1}$$

The *canonical dual frame*, $\widetilde{\mathcal{G}} = (\widetilde{g_j})$ is defined by $\widetilde{g}_j = (DC)^{-1} g_j$ and it appears in the reconstruction formulae: $x = \sum_j \langle x, \widetilde{g}_j\rangle g_j$ and $x = \sum_j \langle x, g_j\rangle \widetilde{g}_j$.

Basic operator theory is also involved in several frame constructions. It was quite recently that a converse attitude has been taken to apply frames (instead of orthonormal bases) in matrix representation of operators.

In [2] Peter Balazs has considered a pair of frames $(\mathcal{G}, \mathcal{H})$ in Hilbert spaces H_1, H_2 and defined the matrix $\mathcal{M}^{\mathcal{G},\mathcal{H}}(T) = \text{Matr}^{\mathcal{G},\mathcal{H}}(T)$ corresponding in the following natural way to a bounded linear operator $T : H_1 \to H_2$:

Definition 1.2 Let $\mathcal{M}^{\mathcal{G},\mathcal{H}}(T) = C_{\mathcal{G}} \circ T \circ D_{\mathcal{H}}$—which as an operator on ℓ^2 is the matrix with entries (T_{mn}) given by the inner products:

$$T_{nm} = \langle T h_m, g_n\rangle \quad \text{if } \mathcal{G} = (g_m)_{m=1}^{\infty},\ \mathcal{H} = (h_n)_{n=1}^{\infty}.$$

Conversely, the operator $\mathcal{O}^{\mathcal{G},\mathcal{H}}(A) = D_{\mathcal{G}} \circ A \circ C_{\mathcal{H}}$ associated to an infinite matrix $A = (a_{jn})$ bounded as an operator on ℓ^2 is given by

$$\mathcal{O}^{\mathcal{G},\mathcal{H}}(A)x = \sum_{j=1}^{\infty}\sum_{n=1}^{\infty} a_{jn}\langle x, h_n\rangle g_j, \quad x \in H$$

with the sum absolutely convergent.

If $H_1 = H_2 = H$ and $\mathcal{H} = \widetilde{\mathcal{G}}$ is the canonical dual (its frame bounds being $\kappa_2^{-1}, \kappa_1^{-1}$, its dual $\widetilde{\widetilde{\mathcal{G}}} = \mathcal{G}$), some regularity is achieved for the functor $\mathcal{M}(\cdot) := \mathcal{M}^{G,\widetilde{\mathcal{G}}}(\cdot)$—it becomes a Banach algebra homomorphism: it depends on $T \in \mathcal{B}(H)$ in both linear and multiplicative manner. The operator norms satisfy $\|\mathcal{M}(T)\| \le (\kappa_1^{-1}\kappa_2)^{\frac{1}{2}}\|T\|$ and similarly, the norm of $\|\mathcal{O}^{\mathcal{G},\mathcal{H}}(A)\|$ is estimated by $\|A\|$ times the geometric mean of the respective Bessel bounds. Analogous results hold for the Schatten–von-Neumann classes $\mathfrak{S}_p$ and their norms. The Reconstruction Formula holds only in one direction—in the form $\mathcal{O}^{\mathcal{G},\mathcal{H}}(\mathcal{M}^{\widetilde{\mathcal{G}},\widetilde{\mathcal{H}}}(T)) = T$. But although $\mathcal{O}^{\mathcal{G},\widetilde{\mathcal{G}}}(I_{\ell^2}) = I_H$, $\mathcal{M}^{\mathcal{G},\widetilde{\mathcal{G}}}$ sends the identity operator I_H to the Gram matrix of $\mathcal{G}$, which is noninvertible for frames that fail to be Riesz bases. These results of [2] were extended in [1], where it was shown that $\mathcal{M}^{\mathcal{G},\widetilde{\mathcal{G}}}$ preserves the hermitian conjugation iff the frame G is tight, i.e. $\mathcal{M}^{\mathcal{G},\widetilde{\mathcal{G}}}(T^*)$ is the adjoint for $\mathcal{M}^{\mathcal{G},\widetilde{\mathcal{G}}}(T)$ for any

$T \in \mathcal{B}(H)$ iff $\kappa_1 = \kappa_2$. Analysis of the spectrum of $M := \mathcal{M}^{\mathcal{G},\widetilde{\mathcal{G}}}(T)$ (and its parts) in terms of the spectrum of T, carried in [1] resulted in the following

Theorem 1.3 *The nonzero parts of the spectra*: $\sigma(T) \setminus \{0\}$ *and* $\sigma(M) \setminus \{0\}$ *are equal. The same is true for the point spectra*: $\sigma_p(T) \setminus \{0\} = \sigma_p(M) \setminus \{0\}$.

For the continuous spectrum $\sigma_c(T) := \{\lambda \in \sigma(T) \setminus \sigma_p(T) : \bar{\lambda} \notin \sigma_p(T^*)\}$ *and for* $\lambda \neq 0$ *we have* $\lambda \in \sigma_c(T)$ iff

$$\lambda \notin \sigma_p(M), \quad \mathcal{R}(M - \lambda I) \not\supset \mathcal{R}(C) \quad \textit{and} \quad \overline{\mathcal{R}(M - \lambda I)} \supset \mathcal{R}(C)$$

the essential spectra can be described using the orthogonal differences of the ranges and nullspaces (*with* $\tilde{D}$ *denoting the adjoint of the analysis operator* $\tilde{C}$ *for* $\widetilde{\mathcal{G}}$) *as follows*: $\lambda \notin \sigma_{\text{ess}}(T) \cup \{0\}$ *iff the range of* $M - \lambda I$ *is closed and* $\dim(\ker(M - \lambda I) \ominus \ker(\widetilde{D})) < \infty$ *and* $\dim(\mathcal{R}(C) \ominus \mathcal{R}(M - \lambda I)) < \infty$.

2 Frames from Reproducing Kernels

One instance when frames appear in a quite natural way is the case of *reproducing kernel Hilbert spaces* (RKH spaces—for short). Then H is a function space over some domain Ω, where for any $z \in \Omega$ there exists $K_z \in H$ representing the functional of evaluation at the point z so that $f(z) = \langle f, K_z \rangle$ for any $f \in H$. Then $K(z, w) := K_z(w)$ is called the kernel function and $k_z := K_z K(z, z)^{-\frac{1}{2}}$ is the *normalised reproducing kernel* at z, since $\|k_z\| = 1$. These normalised kernels span a dense subspace in H and in most RKH spaces there are natural conditions (in terms of the Bergman metrics) for lattices $\Gamma = \{\gamma_n : n \in \mathbb{N}\}$ of points in Ω, under which the sequence $\mathcal{G} = \{k_{\gamma_n} : n \in \mathbb{N}\}$ of normalised kernels forms a frame in H. We call such sequences *sampling sequences* for H. In [6] a general atomic decomposition is given in Theorem 8.2 (not only for Hilbert, but for the associated scale of Banach spaces, which for $p = 2$ give frames). In the case of Bergman spaces Chap. 5 of [5] seems the best source. The Segal–Bargmann space $\mathcal{F}_\alpha$ of entire functions on $\mathbb{C}^n$ square—integrable w.r. to the Gaussian measure $d\mu_\alpha(z) = (\frac{\alpha}{\pi})^n \exp(-\alpha|z|^2) dm(z)$, where dm is the Lebesgue measure is also called the Fock space. Here frames were constructed by Grossman with Daubechies, Lyubarski and Seip, using the normalised kernels $k_z(w) = \exp(\alpha \bar{z} w - \frac{\alpha}{2}|z|^2)$. In particular, the Euclidean metric lattices of sufficient density work. A recent treatment can be found in the book [8] by Kehe Zhu in the chapter on sampling sequences.

Definition 2.1 The Berezin symbol for a bounded linear operator $T : H \to H$ on a RKHS H is the function

$$\widetilde{T}(z, w) = \langle T\kappa_z, \kappa_w \rangle, \quad z, w \in \Omega$$

Its diagonal restriction is defined as $\widetilde{T}(z) = \widetilde{T}(z, z)$.

In [7] the following corollary is obtained in the case when $p = 2$.

Proposition 2.2 *The matrix of T w.r. to the frame κ_{z_n} has entries equal to Berezin symbol's values at the lattice points*: $T_{n,j} = \widetilde{T}(z_j, z_n)$. *Moreover, $T \in \mathfrak{S}_p(H)$ iff this infinite matrix as an operator on ℓ^2 belongs to the Schatten class $\mathfrak{S}_p(\ell^2)$. If $0 < p \le 2$ the latter holds if $\sum_{n,j} |\widetilde{T}(z_j, z_n)|^p < \infty$.*

Proof The first statement comes as a direct consequence of the definitions. The equivalence for the $\mathfrak{S}_p$-condition for an operator and its matrix in any frame is established in [1], Corollary 4.1. Its proof can actually avoid the assumption that $\mathcal{G}$ be a tight frame (in which case the matrix functor $\mathcal{M}$ satisfies $\mathcal{M}(|T|^{\frac{p}{2}}) = |\mathcal{M}(T)|^{\frac{p}{2}}$). Indeed, it follows directly from the ideal property of $\mathfrak{S}_p$, since $\mathcal{M}(T) = C_{\widetilde{\mathcal{G}}} \circ T \circ D_{\mathcal{G}}$. It is in some contrast with the results of [3], showing the dependence of the summability of $\|T f_n\|^p$ (and the related quantities) on the choice of the frame sequence (f_n in H). The last claim follows from Lemma 11 and Theorem 17 in [3]. □

Remark 2.3 Note that for nonnegative selfadjoint operators the latter condition is satisfied if $\sum_n |\widetilde{T}(z_n)|^p < \infty$. Indeed, then the Berezin symbol is a positive definite kernel and Schwarz inequality applies.

For $p > 2$ a similar sufficient condition requires the summability of p-th powers—but with respect to all orthonormal bases, or all frames, not just the considered ones. A partial remedy is provided by the following result in [4].

Let $\partial_H \Omega$ be the set of these boundary points $z_0 \in \partial\Omega$ for which the normalised kernels κ_z weakly tend to zero as $z \to z_0$, $z \in \Omega$.

Theorem 2.4 *Assume that $\partial_H \Omega$ is nonempty.* (i) *For a bounded linear $T : H \to H$ its compactness is then equivalent to the Berezin transform $\widetilde{U^{-1}TU}(z)$ tending to zero as z tends to an arbitrary (equivalently, for some) point $z_0 \in \partial_H \Omega$ for any unitary operator U on H.*

(ii) *If $p \ge 1$ and T is a compact operator on a RKHS H, then it belongs to the ideal $\mathfrak{S}_p$ iff for some point $z_0 \in \partial_H \Omega$ there exists a sequence (z_n) in Ω converging to z_0 and such that for all unitary operators $U : H \to H$ the sequence of values $\widetilde{U^{-1}TU}(z_n)$ is in ℓ^p. The supremum of the ℓ^p-norms taken over all unitaries U is equivalent to the $\mathfrak{S}_p$ norm of T.*

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Rank-M Frame Multipliers and Optimality Criterions for Density Operators of Rank M

Daniel Lantzberg

Abstract Ever since the introduction of frames in Duffin and Schaeffer (Trans. Am. Math. Soc. 72:341–366, 1952), the connection between frame theory and decompositions of certain operators, particularly the identity operator, into rank-ones began to be elaborated. Abandoning the idea of restricting to tight frame-like expansions, with respect to systems arising from a single template function, one is led to the concept of resolutions of the identity, with respect to more general systems than the usual rank-one expansions of the identity.

In this study, we will investigate various notions of possible generalizations of optimality criterions for rank-M frames and corresponding multipliers. Explicitly, we will lay stress on continuous M-frames, arising from irreducible group representations of locally compact groups, have a look at its connection to time-frequency analysis and comment on adequate notions of optimality.

Keywords Frames · Multipliers · Operator approximation · Locally compact groups

Mathematics Subject Classification (2010) Primary 47A05 · Secondary 42C15 · 47A58 · 43A22

1 Introduction

When considering rank-one expansions of the identity, one usually considers frame-like sets of vectors as well as corresponding duals in order to decompose the identity operator into (usually non-orthogonal) projections onto one-dimensional subspaces. By replacing the arising rank-ones, indexed by some measure space (X, μ), with

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suitable positive rank-M operators over the Hilbert space $\mathcal{H}$ of the form

$$P_x := \sum_{i=1}^{M} \lambda_i \langle \bullet, \varphi_x^i \rangle \varphi_x^i, \quad \lambda \in \mathbb{R}^M,\ x \in X \text{ and } \varphi_x^i \in \mathcal{H},\ \forall i, x,$$

which still retains the (weak) decomposition property of the identity, $\int_X P_x d\mu(x) = \mathbf{1}_{\mathcal{H}}$, we gain more flexibility regarding operator decompositions, without loosing too much of the well-established theory.

We will investigate the connection of frame-like families of rank-M operators to multipliers of the form

$$f \longmapsto \int_X m(x) P_x f d\mu(x), \quad \text{with } P_x f = \sum_{i=1}^{M} \lambda_i \langle f, \varphi_x^i \rangle \varphi_x^i \text{ as above,}$$

where we will concentrate on measure spaces (X, μ), arising from locally compact groups.

At first, we will sum up some mathematical preliminaries needed, in order to follow our arguments below. Afterwards, we'll introduce the not-so-standard topic of group multipliers before moving to our definition of a "Q-multiplier". In order to keep things on a fairly concrete level, we will conclude with a concretization to time-frequency analysis and make a brief comment on optimality criterions.

2 Notation and Preliminaries

Operators, Tensors and Its Relatives We'll denote the tensor product $(f, g) \in V \times W$ by $f \otimes g \in V \otimes W$ and represent the—unique—dual element of $f \in \mathcal{H}$, via Riesz' representation theorem, by

$$f^* : g \mapsto f^*(g) := \langle g, f \rangle, \quad f, g \in \mathcal{H}.$$

Consequently, rank-one operators $\mathcal{H}_1 \to \mathcal{H}_2$ are written as $f \otimes g^* \in \mathcal{H}_2 \otimes \mathcal{H}_1^*$, where $f \otimes g^* : h \mapsto f\langle h, g \rangle$. On the operator level, $S \otimes T^*$ is an action on the left resp. right factor, i.e., $S \otimes T^*(f \otimes g^*) = (Sf) \otimes (Tg)^*$. For compact operators, this amounts to a composition from the left and right, i.e., $(S \otimes T^*)O := S \circ O \circ T^*$, with T^* denoting the adjoint to T.

$\mathcal{HS}(V)$ will be the Hilbert–Schmidt operators over V and $\mathrm{tr}[\cdot]$ will denote the trace functional. Moreover, $\mathbf{1}_X$, δ_G and χ_E will be the identity on X, the Dirac distribution on G and the indicator function of the set E, respectively. $\mathcal{H}$ will always denote a Hilbert space and $\langle \cdot, \cdot \rangle$ an inner product.

Groups, Representations and Co. Let us be given a—not necessarily unimodular—locally compact group (LCG) (G, μ), with left Haar measure μ, modular function Δ_G, left resp. right regular representation

$$\big(\lambda(x) f\big)(y) = f\big(x^{-1} y\big) \quad \text{and} \quad \big(\rho(x) f\big)(y) = \Delta_G^{1/2}(x) f(yx),$$
$$x, y \in G,\ f \in L^2(G, \mu),$$

convolution being indicated by $*$ and its unitary dual $\hat{G}$, consisting of (equivalence classes of) unitary and *irreducible* representations (UIR). We will not make use of this unitary dual in general, we rather restrict ourselfs to a single equivalence class and work with some arbitrary representative of it. Denoting this representative by π, acting on $\mathcal{H}_\pi$, we may employ the (strongly) continuous map $G \ni x \mapsto \pi(x)\varphi \in \mathcal{H}_\pi$, for some arbitrary *cyclic* vector $\varphi \in \mathcal{H}_\pi$, to assemble *frames* for $\mathcal{H}_\pi$. Recall that

Recall 2.1 (Cyclicity [8]) *A vector $\varphi \in \mathcal{H}_\pi$ is* cyclic *for $(\pi, \mathcal{H}_\pi)$ iff the closure of the linear span of its orbit under the group action is all of $\mathcal{H}_\pi$, i.e., if*

$$\operatorname{span}\{\pi(x)\varphi \mid x \in G\} \quad \textit{is dense in } \mathcal{H}_\pi.$$

Recall 2.2 (Frame [1]) *With $0 < A \le B < \infty$, the family of vectors $\{\varphi_i \mid i \in I\} \subseteq V$ is called a* frame *for the vector space V iff*

$$A\|f\| \le \sum_{i \in I} \left|\langle f, \varphi_i\rangle\right|^2 \le B\|f\|, \quad \textit{for all } f \in V.$$

And moreover that

Recall 2.3 (Irreducibility [8]) *A representation $(\pi, \mathcal{H}_\pi)$ is* irreducible, *iff the only non-trivial subspace that is invariant under π is $\mathcal{H}_\pi$.*

Concerning UIRs, there's a neverending list of theorems and lemmata, one of which is the following, stated in a way that fits our needs the best.

Lemma 2.4 (Schur's Lemma) *Let $(\pi, \mathcal{H}_\pi)$ be an UIR of the LCG (G, μ), then its commutant is trivial, i.e., consists only of multiples of the identity*

$$\pi(G)' := \{c \cdot \mathbf{1}_{\mathcal{H}_\pi} \mid c \in \mathbb{C}\}.$$

We'll further denote (the inverse square-root of) the Duflo–Moore operator [5, 9] on $\mathcal{H}_\pi$, which is unbounded, but densely defined and has densely defined inverses, by K_π and make *excessive* use of the shorthand $\varphi_x := \pi(x)K_\pi^{-1}\varphi$. As the family $\{\varphi_x \mid x \in G\}$ is a proper subset of $\mathcal{H}_\pi$, we may define the inner product $\langle f, \varphi_x\rangle$, which is well defined for arbitrary $f \in \mathcal{H}_\pi$ and $x \in G$. Consequently, the map

$$x \mapsto \langle f, \varphi_x\rangle =: (V_\varphi f)(x), \quad x \in G, \; f \in \mathcal{H}_\pi,$$

is a bounded function on the group.

Definition 2.5 With $(\pi, \mathcal{H}_\pi)$, (G, μ), K_π as before, the map $V_\varphi : f \longmapsto \langle f, \varphi_\bullet\rangle$, $f \in \mathcal{H}_\pi$, is the $\|\varphi\|_{\mathcal{H}_\pi}$-multiple of an isometry and can therefore be inverted on its range by its adjoint

$$V_\varphi^* : F \longmapsto \int_G F(x)\varphi_x d\mu(x), \quad F \in L^2(G, \mu).$$

The *generalized wavelet transform* above—as well as its inverse—may be lifted to the operator level.

Definition 2.6 Let $(\pi, \mathcal{H}_\pi)$, (G, μ) and K_π be as before, then the *integrated representation*

$$F \longmapsto \int_G F(x)\pi(x)K_\pi^{-1} d\mu(x) =: \Pi(F), \quad F \in L^1(G, \mu), \tag{2.1}$$

is an *-algebra representation of $L^1(G, \mu)$, may be extended to an *isometry* $L^2(G, \mu) \to \mathcal{HS}(\mathcal{H}_\pi)$ by continuity and thus its inverse, *a.k.a. the spreading map* η, is given by its adjoint and maps $\mathcal{HS}(\mathcal{H}_\pi)$ to $L^2(G, \mu)$.

Remark 2.7 Note that from this it follows that (i) $V_\varphi^* F = \Pi(F)\varphi$ and $V_\varphi f = \eta(f \otimes \varphi^*)$, (ii) $\eta(T) = \mathfrak{tr}[T \ \pi^*(\cdot)]$ and (iii) composition of UIRs from the left resp. right are intertwined with left resp. right regular representations on G and vice versa, i.e.,

$$\Pi \circ (\lambda \times \rho) \circ \eta = (\pi \otimes \pi^*).$$

Let now all $\varphi_\bullet$ be normalized, i.e., $\|\varphi_x\| = 1$, for all $x \in G$, then, the family of rank-one projectors $\{\varphi_x \otimes \varphi_x^* \mid x \in G\}$ induce an integral operator of the very familiar form

$$Sf = \int_G \langle f, \varphi_x\rangle \varphi_x d\mu(x).$$

Unfortunately, by now, it is not even clear whether this integral converges with respect to any interesting topology and may thus not by defined at all. However, since $\pi(y)S = S\pi(y)$, it follows—by Schur's lemma above—that S is a multiple of the identity, S may—using the definitions above—be factorized into $S := V_\varphi^* V_\varphi$ and, with $A \le \|\varphi\|^2_{\mathcal{H}_\pi} \le B$, we find that

$$A\|f\|^2 \le \langle Sf, f\rangle = \int_G \bigl|\langle f, \varphi_x\rangle\bigr|^2 d\mu(x) \le B\|f\|^2, \quad \text{for all } f \in \mathcal{H}_\pi.$$

Thus, the family $\{\varphi_x \mid x \in G\}$ can be interpreted as a "continuous frame" [10], indexed by the measure space (G, μ). We take this one step further and define

Definition 2.8 With $(\pi, \mathcal{H}_\pi)$, (G, μ), A and B as before, a family of normalized and cyclic vectors $\{\varphi^i \mid i \in I\}$ and corresponding weights $\{\lambda_i \ge 0 \mid i \in I\}$, with $|I| := M$ and $\sum_{i \in I} \lambda_i = 1$, we *define* a (continuous and normalized) *rank M-frame* $\{(\lambda_i, \varphi_x^i) \mid i \in I, \ x \in G\}$ by

$$A\|f\|^2 \le \int_G \sum_{i \in I} \lambda_i \bigl|\langle f, \varphi_x^i\rangle\bigr|^2 d\mu(x) \le B\|f\|^2, \quad \text{for all } f \in \mathcal{H}_\pi. \tag{2.2}$$

Writing $Q := \sum_{i\in I} \lambda_i(\varphi^i \otimes (\varphi^i)^*)$ and $Q_x = (\pi \otimes \pi^*)(x)Q$, the above may be rephrased as

$$A\|f\|^2 \le \int_G \langle Q_x f, f\rangle d\mu(x) \le B\|f\|^2, \quad \text{for all } f \in \mathcal{H}_\pi,$$

which is a notation, we will frequently make use of.

3 Group Multipliers of Rank M

Having the above in mind, one is confronted with the possibility to put a "weight" on the generalized wavelet transform, before resynthesizing the transform, in complete analogy to the well-known Fourier multipliers, i.e., the bounded operators on $L^1(\mathbb{R}^n)$ that are diagonalized by the Fourier transform.

Theorem 3.1 *With* $(\pi, \mathcal{H}_\pi)$ *and* (G, μ) *as before and the multiplication operator* $M_{\mathfrak{m}} : F \mapsto \mathfrak{m} \cdot F$, *the* group multiplier $\mathcal{M}_{\mathfrak{m},\varphi}$, *defined by the integral*

$$\mathcal{M}_{\mathfrak{m},\varphi} f := \left(V_\varphi^* M_{\mathfrak{m}} V_\varphi\right) f = \int_G \mathfrak{m}(x)\langle f, \varphi_x\rangle \varphi_x d\mu(x), \quad f \in \mathcal{H}_\pi,$$

converges

(i) *w.r.t.* norm topology, *with* $\|\mathcal{M}_{m,\varphi}\|_{OP} \le \|\mathfrak{m}\|_{L^1(G,\mu)}\|\varphi\|^2_{\mathcal{H}_\pi}$, *iff* $\mathfrak{m} \in L^1(G, \mu)$,
(ii) *only* weakly, *whenever* $\mathfrak{m} \in L^\infty(G, \mu)$.

Moreover, it is trace-class *if* $\mathfrak{m} \in L^1(G, \mu)$.

Proof (i) Dividing $\|\mathcal{M}_{\mathfrak{m},\varphi} f\| \le \int_G |\mathfrak{m}(x)|\|f\|\|\varphi_x\|^2 d\mu$ by $\|f\|$, $f \in \mathcal{H}_\pi$, taking supremum, using the unitarity of π and the claim follows. The proof of (ii) follows from $|\langle \mathcal{M}_{\mathfrak{m},\varphi} f, g\rangle| \le \|\mathfrak{m}\|_{L^\infty} \int_G |V_\varphi f \overline{V_\varphi g}| d\mu$, $f, g \in \mathcal{H}_\pi$, the isometrical property of V_φ and Cauchy–Schwarz's inequality. And lastly, $\mathcal{M}_{\mathfrak{m},\varphi}$ is trace-class, because $|\mathrm{tr}[\mathcal{M}_{\mathfrak{m},\varphi}]| \le \|\mathfrak{m}\|_{L^1(G,\mu)}\|\varphi\|^2_{\mathcal{H}_\pi}$. □

Corollary 3.2 *From* Theorem 3.1, *and in the notation thereof, we find that for the* Q-Multiplier

$$\mathcal{M}_{\mathfrak{m},Q} f := \int_G \mathfrak{m}(x) Q_x f d\mu(x), \quad f \in \mathcal{H}_\pi,$$

the obvious analogues of (i) *and* (ii) *of Theorem* 3.1, *as well as its traceability, hold, too*.

Proof The proofs carry over almost unaltered, as soon as one realizes that linearity supplies us with

$$\mathcal{M}_{\mathfrak{m},Q} = \sum_{i\in I} \lambda_i \mathcal{M}_{\mathfrak{m},\varphi^i}.$$

(i) From $\|\mathcal{M}_{\mathfrak{m},Q}\|_{OP} \leq \sum_{i\in I} \lambda_i \|\mathcal{M}_{\mathfrak{m},\varphi^i}\|_{OP}$, the arguments above and the inequality $\sum_i \lambda_i \|\varphi^i\| \leq \sup_{i\in I} \|\varphi^i\|^2_{\mathcal{H}_\pi}$, the proof follows. (ii) The proof of *Theorem* 3.1 and the arguments of i show that $|\langle \mathcal{M}_{\mathfrak{m},Q} f, g\rangle| \leq \|\mathfrak{m}\|_{L^\infty} \|f\| \|g\| \times \sup_{i\in I} \|\varphi^i\|^2_{\mathcal{H}_\pi}$. Finally, it is traceable since $|\mathfrak{tr}[\mathcal{M}_{\mathfrak{m},\varphi}]| \leq \|\mathfrak{m}\|_{L^1(G,\mu)} \times \sup_{i\in I} \|\varphi^i\|^2_{\mathcal{H}_\pi}$. □

We will henceforth concentrate on Q-Multipliers only, since in the very special case of "classical" rank-one multiplier, that is, $Q = \varphi \otimes \varphi^*$, the results carry over immediately.

Theorem 3.3 *In the notation above, whenever we can find a—not necessarily bounded or compact—operator* $\mathcal{P}$, *such that* $\mathfrak{tr}[\mathcal{P}_x \mathcal{P}_y^*] = \delta_G(x^{-1}y)$ (*to be interpreted in the sense of distributions*), *where* $\mathcal{P}_x = (\pi \otimes \pi^*)(x)\mathcal{P}$, *we may introduce the invertible map*,

$$\sigma : T \mapsto \mathfrak{tr}\big[T\mathcal{P}_\bullet^*\big] =: \sigma(T), \tag{3.1}$$

which assigns to each operator T *its* symbol $\sigma(T)$. *The inverse map*, Φ, *is defined by the* weakly converging *integral*

$$\Phi : \mathfrak{s} \longmapsto \int_G \mathfrak{s}(x)\mathcal{P}_x d\mu_G(x). \tag{3.2}$$

Corollary 3.4 *With all notations as before, we conclude this section with the following implications.*

(i) $\sigma(\mathcal{M}_{\mathfrak{m},Q})(y) = (\mathfrak{m} * \sigma(Q))(y)$, $y \in G$, *and thus* $\mathcal{M}_{\mathfrak{m},Q} = \Phi(\mathfrak{m} * \sigma(Q))$.
(ii) *Combining* Definition 2.6 *and* Theorem 3.3 *we end up with a useful connection between a symbol and the spreading function of an operator, namely*, $(\eta \circ \Phi)$, *formally given by*

$$(\eta \circ \Phi)(\mathfrak{s}) = \int_G \mathfrak{s}(x)(\lambda \times \rho)(x)\eta(P) d\mu_G(x),$$

and mapping an operator's symbol to its spreading function, as well as its inverse, $(\sigma \circ \Pi)$, *defined by the integral operator*

$$(\sigma \circ \Pi)(F) = \int_G F(x)(\lambda \times \rho)(x)\overline{\eta(P)} d\mu_G(x).$$

4 Concrete Examples of Group Multipliers of Rank M

Although examples of rank M group multipliers of locally compact groups other than the (reduced) Heisenberg group exist, we will stick to the time-frequency case. More general examples, e.g. for the affine "$ax + b$"-group, will be presented elsewhere.

The Reduced Heisenberg Group Introducing the *symmetric* time-frequency shift operator

$$\big(\widetilde{\pi}_s(t,\omega)\varphi\big)(x) := e^{i\pi\langle\omega,t\rangle}e^{i2\pi\langle\omega,x\rangle}\varphi(x-t), \quad x\in\mathbb{R}^n,\ (t,\omega)\in\mathbb{R}^{2n},\ n\in\mathbb{N},$$

as well as the *asymmetric* one, $\widetilde{\pi}_a(t,\omega) := e^{-i\pi\langle\omega,t\rangle}\widetilde{\pi}_s(t,\omega)$, we find that the composition of two such time-frequency shifts lead to another one, along with a certain phaseshift. Thus, one needs to introduce another *toral* parameter, say, $\tau\in\mathbb{T}$, in order for the shifts to constitute a group (representation). The underlying group structures of the emerging representations, $\pi_s(t,\omega,\tau) := \tau\cdot\widetilde{\pi}_s(t,\omega)$ and $\pi_a(t,\omega,\tau) := \tau\cdot\widetilde{\pi}_a(t,\omega)$, are usually called the *reduced* resp. *reduced and polarized* Heisenberg group, which will be our starting point for concrete realizations of (3.1) and (3.2).

Remark 4.1 The reader should be aware of the fact that—at least for the symbol-to-operator correspondences, we are considering—the Heisenberg group constitutes a very special case. Its unique role arises from one very simple but consequential equality, namely

$$\begin{aligned}(\lambda\times\rho)(t,\omega,\tau)F(x,y) &= e^{i2\pi(\langle\omega,x\rangle-\langle t,y\rangle)}F(x,y),\\ (x,y)\in\mathbb{R}^{2n},\ &(t,\omega,\tau)\in\mathbb{R}^{2n+1},\end{aligned} \tag{4.1}$$

that is, conjugation, with both the reduced as well as the reduced and polarized group, boils down to a phase-shift (a "symplectic homomorphism", $\mathbb{R}^{2n}\times\mathbb{R}^{2n}\to\mathbb{T}$, to be more precise)—note the absence of τ on the RHS!—and thus the conjugation action on functions, $(\lambda\times\rho)$, as well as operators, $(\pi\otimes\pi^*)$, may both be interpreted as a (unitary and irreducible) representation of the locally compact and *abelian* group $\mathbb{R}^{2n}$ (and as such we will henceforth refrain from referring to the toral parameter τ).

Definition 4.2 There are two well-known symbol to operator correspondences, both related to time-frequency analysis as well as pseudodifferential theory ($\mathcal{F}_s$ denotes the *symplectic* Fourier transform).

- The Wigner–Weyl correspondence, Φ_w, weakly defined by

$$\big\langle\Phi_w(\mathfrak{s})f,f\big\rangle := \iint_{\mathbb{R}^2}\mathfrak{s}(t,\omega)\overline{\mathfrak{W}_f(t,\omega)}\,d\omega dt, \quad \text{for all } f\in\mathcal{H}_\pi,$$

  with $\mathfrak{W}_f(t,\omega) := \langle f, P^w_{(t,\omega)}f\rangle = \mathcal{F}_s(\langle f,\widetilde{\pi}_s(t,\omega)f\rangle)$ denoting Wigner's distribution.
- The Kohn–Nirenberg correspondence, Φ_{kn}, weakly defined by

$$\big\langle\Phi_{kn}(\mathfrak{s})f,f\big\rangle := \iint_{\mathbb{R}^2}\mathfrak{s}(t,\omega)\overline{\mathfrak{R}_f(t,\omega)}\,d\omega dt, \quad \text{for all } f\in\mathcal{H}_\pi,$$

  with $\mathfrak{R}_f(t,\omega) := \langle f, P^{kn}_{(t,\omega)}f\rangle = \mathcal{F}_s(\langle f,\widetilde{\pi}_a(t,\omega)f\rangle)$ being Rihaczek's distribution.

Lemma 4.3 *With all notations as above, we summarize that, as far as time-frequency analysis is concerned, we have* (i) *$(\eta \circ \Phi)$ as well as $(\sigma \circ \Pi)$ each boil down to a* symplectic *Fourier transform, that is,*

$$(\eta \circ \Phi)(\mathfrak{s}) = \iint_{\mathbb{R}^2} \mathfrak{s}(t,\omega)\ (\lambda \times \rho)(t,\omega) d\omega dt = \widehat{\mathfrak{s}} \quad \textit{and} \quad (\sigma \circ \Pi)(F) := \widehat{F}, \tag{4.2}$$

and (ii) *a rank M time-frequency multiplier $\mathcal{M}_{\mathfrak{m},Q}$ is best described via its spreading representation*

$$\mathcal{M}_{\mathfrak{m},Q} = \iint_{\mathbb{R}^2} \underbrace{\widehat{\mathfrak{m}}(t,\omega)\eta(Q)(t,\omega)}_{\eta(\mathcal{M}_{\mathfrak{m},Q})(t,\omega)} \widetilde{\pi}_a(t,\omega) d\omega dt \tag{4.3}$$

$$= \iint_{\mathbb{R}^2} \widehat{\mathfrak{m}}(t,\omega)\Big(\sum_i \lambda_i V_{\varphi^i}\varphi^i(t,\omega)\Big)\widetilde{\pi}_a(t,\omega) d\omega dt \tag{4.4}$$

Proof (i) follows from (3.2) and (4.1) (see also Remark 2.7), and (ii) from utilizing (4.2), Corollary 3.4 and the fact that (abelian) convolution is diagonalized by the (symplectic) Fourier transform. □

Theorem 4.4 *An operator, T, can be represented* properly *as a Q-Multiplier, whenever there exists a Q with* $\operatorname{supp}(\eta(T)) \subset \operatorname{supp}(\eta(Q))$ *and may only be* approximated, *if* $\operatorname{supp}(\eta(T)) \not\subset \operatorname{supp}(\eta(Q))$.

Proof We will indicate the support-sets of spreading functions by $|\eta(T)|$ and $|\eta(Q)|$ and $F|_E$ is the restriction of a function to a subset $E \subset Dom(F)$. If $|\eta(T)| \subset |\eta(Q)|$, then

$$T = \iint_{\mathbb{R}^2} \eta(T)(t,\omega)\chi_{|\eta(T)|}(t,\omega)\chi_{|\eta(Q)|}(t,\omega)\pi(t,\omega) d\omega dt \tag{4.5}$$

$$= \iint_{\mathbb{R}^2} \eta(T)(t,\omega)\chi_{|\eta(T)|}(t,\omega)\frac{\eta(Q)(t,\omega)}{\eta(Q)(t,\omega)}\bigg|_{|\eta(Q)|}\pi(t,\omega) d\omega dt$$

$$= \iint_{\mathbb{R}^2} \frac{\eta(T)(t,\omega)}{\eta(Q)(t,\omega)}\bigg|_{|\eta(T)|}\chi_{|\eta(T)|}(t,\omega)\eta(Q)(t,\omega)\pi(t,\omega) d\omega dt \tag{4.6}$$

$$= \iint_{\mathbb{R}^2} \mathfrak{m}\big(t',\omega'\big) Q_{(t',\omega')} d\omega' dt', \tag{4.7}$$

where $\widehat{\mathfrak{m}} = \frac{\eta(T)}{\eta(Q)} \cdot \chi_{|\eta(T)|}$. The above calculation is valid, since (4.5) follows from $\eta(T) = \eta(T) \cdot \chi_{|\eta(T)|}$ and $|\eta(T)| \subset |\eta(Q)|$, (4.6) holds as $\eta(Q)$ is non-zero on $|\eta(T)|$ and finally (4.7) is due to Lemma 4.3.

If $|\eta(T)| \not\subset |\eta(Q)|$ the conditions are not as convenient as before, since one is constrained to restrict the support of $\eta(T)$ to $|\eta(Q)|$, that is,

$$\widehat{\mathfrak{m}}_{appr} = \frac{\eta(T)}{\eta(Q)} \cdot \chi_{|\eta(Q)|},$$

which clearly purges information and thus entails an operator, $\mathcal{M}_{\mathfrak{m}_{appr},Q}$, approximating T only up to a certain degree. □

Remark 4.5 It is worth noting that, in the special case of time-frequency analysis, we have

(i) whenever T can only be *approximated*, that is, $\operatorname{supp}(\eta(Q)) \not\subset \operatorname{supp}(\eta(T))$, this is related to (multi-window) spline-type spaces [7], via finding Q and its "dual", and in general to Wiener's Tauberian Theorem, a.k.a. the invertibility problem;
(ii) although in the time-frequency case, this in essence overlaps with the theory of coadjoint orbits [11], this does not hold in general, i.e., is not an alternative approach to Kirillov's orbit method;
(iii) this is connected to the approach of operator approximation by multiple Gabor multiplier [3] and related approaches to "C-frames of subspaces"—with different emphasis—is taken in [6];
(iv) another approach to find $\mathfrak{m}_{appr}$ for the operator T would be to minimize

$$\arg\min_{\mathfrak{m}_{appr}} \|T - \mathcal{M}_{\mathfrak{m}_{appr},Q}\|_{L^p}$$

subject to some adequate constraint;
(v) the symbol-to-operator correspondences referred to in Definition 4.2 are two very special cases, since, in fact, there is a continuously $\mathbb{R}$-indexed family of correspondences. The majority of scientific investigations, however, only deals with the ones stated above and so did we.

Comment 4.6 (Optimality Criterions) *It is clear that there is no* definite optimum, *considering the various areas of application. One needs to find the most adequate one, when fixing the application. Nonetheless, there are a few basic criterions which in the majority of cases should lead to nice results.*

- *The more* localized *the symbol of* Q, *the least* "smeared", *the information about* T *becomes, since* $\sigma(Q)$ *is the convolution kernel, mapping the multiplier symbol* $\mathfrak{m}$ *to* $\sigma(T)$. *Localization could be measured via* L^1 norm, $\|\sigma(Q)\|_1$, *via a* second-moment, "variance-like" uncertainty *of the form*

$$\iint_{\mathbb{R}^{2n}} \left(t^2+\omega^2\right)\sigma(Q)(t,\omega)d\omega = \sum_{i\in I} \lambda_i \left(\|xg_i\|_2^2 + \|x\widehat{g_i}\|_2^2\right)$$

or minimal essential support *of* $\sigma(Q)$ *in the sense of Donoho and Stark* [2], *among others.*
- *Or, optimality in the sense of a minimal number of non-zero* λ_i, *while still representing* T *properly, since this will most likely lead to faster algorithms (as well as easier calculations).*

5 Summary

With the notion of a rank-M group multiplier, we have introduced a generalization of multiple Gabor multiplier to other locally compact groups, without the need to combine several individual Gabor multiplier into a MGM—when doing time-frequency analysis—but with the convenient feature of having to deal with only one operator Q, and as such constructing an MGM-like operator from a rank-M frame, considered as a collective of operators $\{Q_x \mid x \in G\}$.

For numerical reasons, the connection to the individual frames must not be lost and for that very reason, a decomposition into a set of (λ_i-weighted) "classical" multiplier is available at any time. This is worth striving for, especially when fast numerical algorithms are needed and are already available for the individual frames, making up the rank-M frame as a whole.

And finally, for the sake of classification, we have listed connections, parallels and differences to other publications as well as pointed towards desirable criterions, in order to assist the fellow researches in its quest for optimality.

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Audio Inpainting Using M-Frames

Florian Lieb

Abstract Classical short-time Fourier constructions lead to a signal decomposition with a fixed time-frequency resolution. However, having signals with varying features, such time-frequency decompositions are very restrictive. A more flexible and adaptive sampling of the time-frequency plane is achieved by the nonstationary Gabor transform. Here, the resolution can evolve over time or frequency, respectively, by using different windows for the different sampling positions in the time or frequency domain (Multiwindow-frames). This adaptivity in the time-frequency plane leads to a sparser signal representation.

In terms of audio inpainting, i.e., filling in blanks of a depleted audio signal, sparsity in some representation space profoundly influences the quality of the reconstructed signal. We will compare this quality using different nonstationary Gabor transforms and the regular Gabor transform with different types of audio signals.

Keywords Audio inpainting · Convex optimization · Nonstationary Gabor frames · ERBlet Transform · Constant-Q transform

Mathematics Subject Classification (2010) Primary 42C15 · Secondary 42C40 · 65K10 · 94A12

1 Introduction

1.1 Audio Inpainting

Let $x \in \mathbb{R}^N$ be an audio signal, which is depleted by a degradation matrix $A \in \mathbb{R}^{N \times N}$, randomly setting values of x to zero (e.g. 66 %), such that the resulting depleted audio signal is $b = Ax$. Audio inpainting can then be considered as the

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following optimization problem

$$\arg\min_x \|Ax - b\|_2^2 + \tau \left\|\Psi(x)\right\|_1, \tag{1.1}$$

where $\tau \in \mathbb{R}^+$ is a regularization parameter for the L1-norm of some sparse signal representation of x and Ψ a linear operator, transforming x to such a sparse representation.

This sparse representation is crucial to the performance of the optimization problem. The recently introduced concept of nonstationary Gabor frames provide an even sparser time frequency representation than regular Gabor frames. Hence, the question arises, if these nonstationary Gabor frames have an advantage over regular Gabor frames when doing audio inpainting. In the following we will consider the operator Ψ to be either a regular Gabor transform or a nonstationary Gabor transform (Wavelet transform, ERBlet Transform or Constant-Q transform).

The optimization problem is then solved using the Douglas–Rachford algorithm from [3], which is implemented in the UNLocBox [1].

1.2 Nonstationary Gabor Frames

Given a unique window function g, the regular Gabor transform of a signal f is given by the inner product $\langle f, g_{\tau,\omega}\rangle$ of f with time-frequency atoms $g_{\tau,\omega}(t) = g(t-\tau)e^{2\pi i t\omega}$, yielding a fixed time-frequency resolution. In a discrete setting, the set of time frequency shifts of g, i.e.,

$$g_{m,n}(t) = g(t - mb)e^{2\pi i t a n} \quad m, n \in \mathbb{Z}$$

and parameters $a, b > 0$, is called a Gabor frame if it satisfies the frame condition, i.e., the lower and upper frame bounds $A, B > 0$ exist

$$A\|f\|^2 \le \sum_{m,n\in\mathbb{Z}} \left|\langle f, g_{m,n}\rangle\right|^2 \le B\|f\|^2 \quad \forall f \in L^2(\mathbb{R}). \tag{1.2}$$

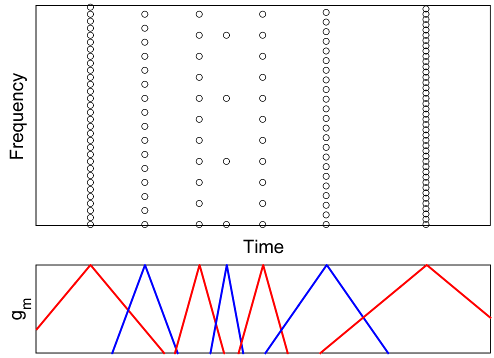
The following definitions for nonstationary frames are extracted from [2].

Definition 1.1 Let $\{g_m\}_{m\in\mathbb{Z}}$ be a set of window functions in $L^2(\mathbb{R})$, which are well localized and centered around time points b_m. With $a_m \in \mathbb{R}$ the frequency sampling step for the corresponding window g_m, the frame elements of a nonstationary Gabor frame are defined as

$$g_{m,n}(t) = g_m(t)e^{2\pi i n a_m t}, \tag{1.3}$$

for $(m, n) \in \mathbb{Z}^2$. Then, the analysis coefficients may then be written as $c_{m,n} = \langle f, g_{m,n}\rangle$.

Fig. 1 Irregular sampling of the time-frequency plane



This allows to vary the window functions at each time sampling point mb. An example of such an irregular sampling grid can be seen in Fig. 1, where the analysis window, and hence, the resolution, changes over time.

Definition 1.2 Analogously, one can define a nonstationary Gabor frame where the resolution changes over frequency, i.e.,

$$h_{m,n}(t) = h_n(t - mb_n), \tag{1.4}$$

for a set of functions $\{h_n\}_{n\in\mathbb{Z}}$ in $L^2(\mathbb{R})$ (with center frequencies a_n) and for $(m,n) \in \mathbb{Z}^2$.

This construction is a generalization of regular Gabor frames and if certain conditions of $g_{m,n}(t)$ or $h_{m,n}(t)$ are satisfied, perfect and efficient reconstruction is possible:

Definition 1.3 The frame operator

$$Sf = \sum_m \sum_n \langle f, g_{m,n}\rangle g_{m,n}, \tag{1.5}$$

is bounded and invertible if the following conditions are satisfied (cf. [2]):

- compactly supported windows g_m,
- sufficiently dense frequency sampling points for each g_m,
- controlled overlap of adjacent windows.

The equivalent holds for $h_{m,n}$.

The concept of nonstationary Gabor frames allows implementations of wavelet, constant-Q, ERBlet as well as regular Gabor transforms.

1.2.1 Wavelet Transform

Frame elements for a wavelet transform in a discrete subset of the upper half-plane are defined as

$$\psi_{m,n}(t) = \alpha^{-n/2}\psi\big(\alpha^{-n}(t - m\beta)\big) \quad m, n \in \mathbb{Z}, \tag{1.6}$$

with some parameters $\alpha, \beta > 0$. To obtain a nonstationary Gabor frame with resolution changing over frequency as in (1.4) we set $h_n(t) = \alpha^{-n/2}\psi(\alpha^{-n}t)$ and $b_n = \alpha^{-n}\beta$.

Considering a logarithmic frequency scaling, the dilates of the mother wavelet, which is in our case a exponentially warped Hanning window, will turn into translates in the frequency domain. Hence, the controlled overlap of wavelets can be controlled by the dilation parameter. We have chosen the mother wavelet because it is closely related to the uncertainty minimizer for the affine group mentioned in [4], but still has compact support.

1.2.2 Constant-Q Transform

The constant-Q transform provides a frequency resolution that depends on the center frequencies of the analysis windows. In particular, the ratio of center frequency to bandwidth of each analysis window is constant, hence the name constant-Q transform. This will lead to a finer frequency resolution at low frequencies and a better time resolution at higher frequencies. In [6] Dörfler et al. developed an algorithm for an invertible constant-Q transform based on nonstationary Gabor frames. The window function we used for the signal decomposition is a plain Hanning window.

1.2.3 ERBlet Transform

The ERB (equivalent rectangular bandwidth) frequency scale is adapted to human auditory perception, which is somehow linear in low frequencies and logarithmic in high frequencies (cf. [5]). The bandwidth of the filter centered at frequency F satisfies

$$\mathrm{ERB}(F) = 24.7 + \frac{F}{9.265}. \tag{1.7}$$

If the center frequencies of the analysis windows (again, we used a Hanning window) are well chosen, they constitute a nonstationary Gabor frame, with resolution changing over frequency.

2 Problem

We will consider several different audio signals x (noisy, synthetic, speech,...) and randomly set 66 % of their coefficients to zero. Using (1.1) to reconstruct the original signal from the depleted versions, we will compare the signal-to-noise ratio

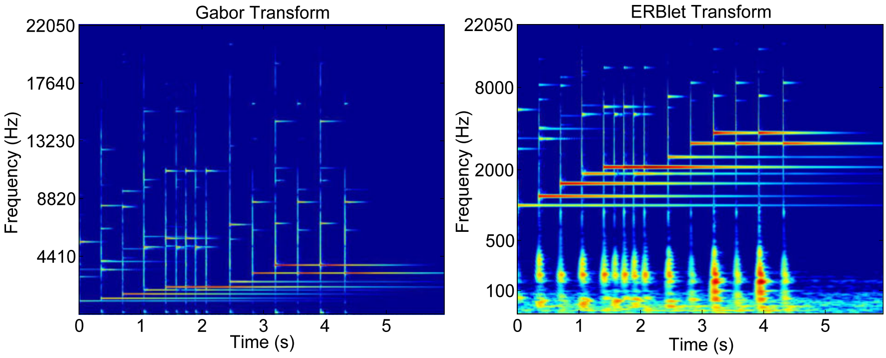
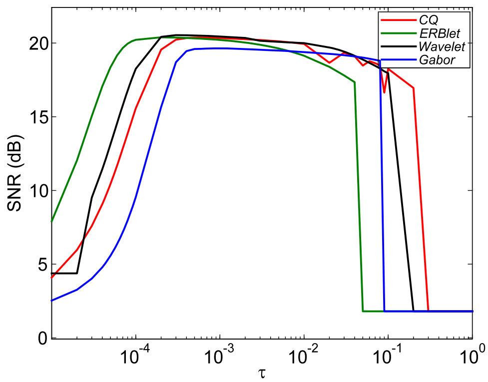


Fig. 2 Gabor and ERBlet transform of the glockenspiel signal

Fig. 3 Signal-to-noise ratios of the glockenspiel signal



(SNR) when using the different time-frequency representations mentioned above. The SNR is defined as

$$\mathrm{SNR}_{dB} = 20 \times \log_{10} \frac{\sigma(x)}{\sigma(x_{rec} - x)}, \tag{2.1}$$

where σ is the standard deviation and x_{rec} the reconstructed audio signal. For each of the transforms the SNR is plotted against the constraint parameter τ in (1.1), ranging from 10^{-5} to 1.

Remark 2.1 Steadily increasing the parameter τ in (1.1) will eventually result in a point where the Douglas–Rachford algorithm simply does not converge anymore. In this case the output of the algorithm is equal to its input, which can be observed in some cases in the following results where the SNR suddenly drops to a low value.

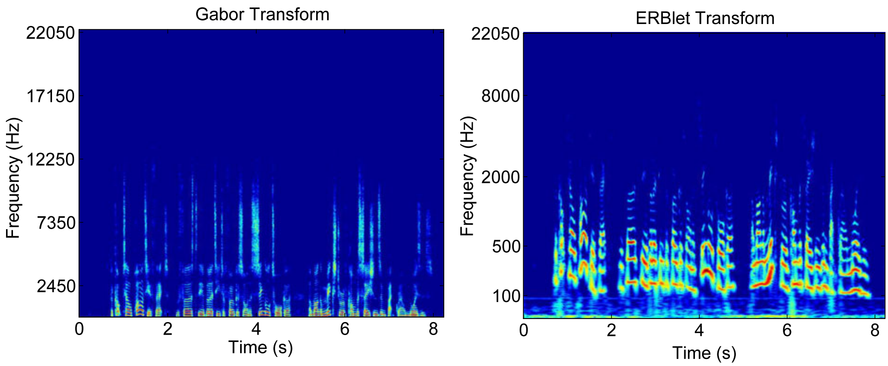
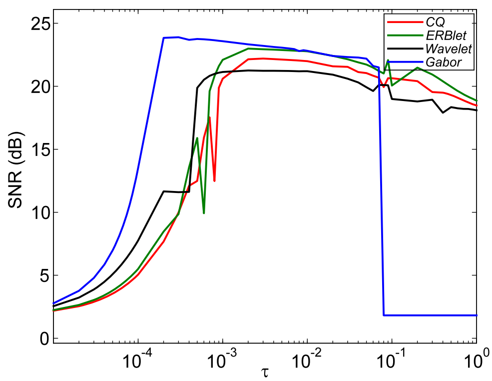


Fig. 4 Gabor and ERBlet transform of the speech signal

Fig. 5 Signal-to-noise ratios of the speech signal



3 Results

3.1 Audio Signal

The first audio signal is the famous glockenspiel, which has a wide range of transient frequencies, but no prominent features in low frequencies as can be seen by its Gabor transform and its ERBlet transform in Fig. 2. The noise level is relatively low.

Solving the inpainting problem will result in the signal-to-noise ratios plotted in Fig. 3. It can be seen, that all of the nonstationary frame implementations yield slightly better results than the regular Gabor transform. Of course, in the range of τ, where the algorithm converges.

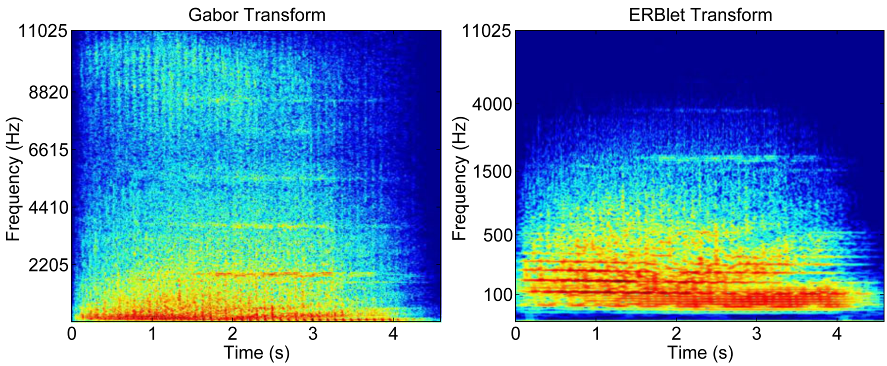
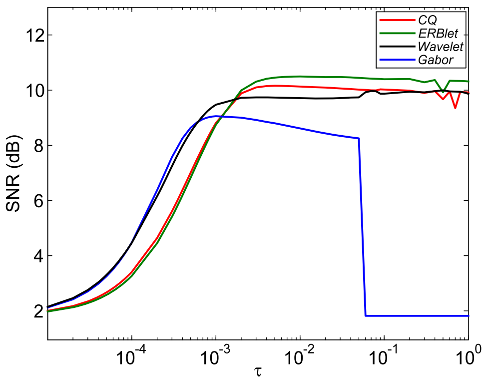


Fig. 6 Gabor and ERBlet transform of the noisy helicopter signal

Fig. 7 Signal-to-noise ratios of the helicopter signal



3.2 Speech Signal

The speech signal is a native British speaker saying: *"The cocktail party effect refers to the ability to focus on a single talker among a mixture of conversations in background noises"*, which was recorded at an anechoic environment and is therefore almost free of noise. Figure 4 shows its Gabor and ERBlet transform.

Nevertheless, when doing inpainting the regular Gabor transform yields the best SNR, as can be seen in Fig. 5.

3.3 Helicopter Sound with Noise

The following signal is a recording of a helicopter with a relative high level of background noise, which can easily be verified from its time-frequency representations in Fig. 6.

Due to the high noise level, the SNR values are relatively low compared to the rest of the test signals (cf. Figure 7). All of the nonstationary frame approaches yield

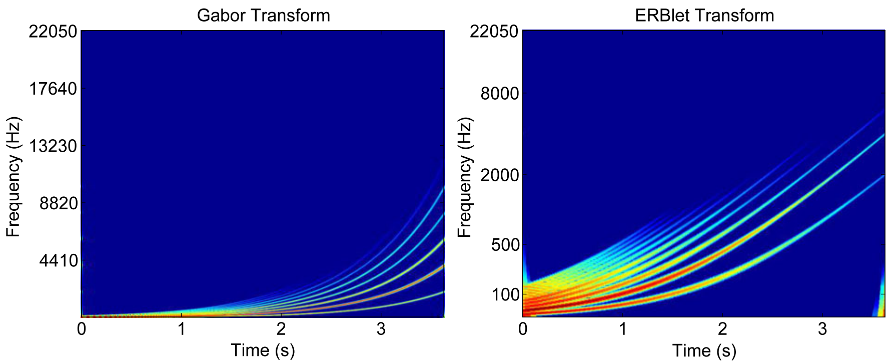
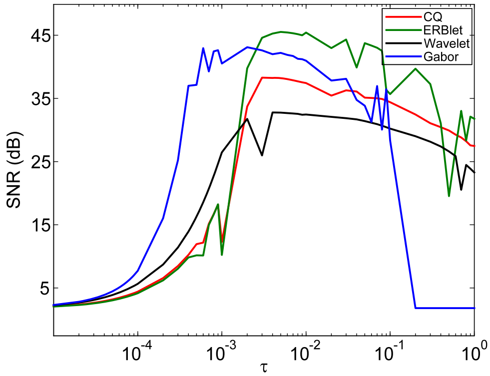


Fig. 8 Gabor and ERBlet transform of a synthetically generated RPM raise

Fig. 9 Signal-to-noise ratios of the synthetic signal



better reconstructed results than doing the inpainting with a fixed time-frequency resolution.

3.4 Synthetic Signal

The last test audio signal consists of a synthetic generated signal, inspired by an exponential RPM raise of an engine with 10 harmonics.

The ERBlet transform also gives the highest SNR directly followed by the regular Gabor transform, whereas the other two nonstationary approaches are significantly worse (Figs. 8–9).

4 Summary

Apart from the recorded voice signal, the nonstationary Gabor frame approaches yield the better reconstruction results, and hence, a better (sparser) time-frequency representation of the above test signals. For the speech signal it seems, that the old-fashioned time-frequency representation is still the best choice in terms of the inpainting problem. A generalization to other speech signals can, of course, not be made without further research.

Furthermore, it seems nonstationary Gabor frames bring out the important features better, even if the signal has a high level of noise (see helicopter example). For the synthetic RPM Raise the nonstationary frames give even a better time-frequency resolution at low frequencies during the first seconds.

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Wavelet Frames to Optimally Learn Functions on Diffusion Measure Spaces

Martin Ehler and Frank Filbir

Abstract Based on the theory of wavelets on data defined manifolds we study the Kolmogorov metric entropy and related measures of complexity of certain function spaces. We also develop constructive algorithms to represent those functions within a prescribed accuracy that is asymptotically optimal up to a logarithmic factor.

Keywords Data-defined manifold · Diffusion measure space

Mathematics Subject Classification (2010) Primary 41A46 · Secondary 42C40

1 Introduction

A common problem in computational mathematics is the approximation of functions from few sample values under certain smoothness assumptions. In other words, we aim to learn a function from few observations. Accuracy does not only depend on the sampling set in general but also on the complexity of the underlying smoothness space. In the present note we shall study the metric entropy [5, 9] as a measure of complexity of function spaces on manifolds. We shall also consider related measures of complexity, so-called n-widths [4], which were studied for some classical function spaces in [7, 8]. Both concepts, metric entropy and n-widths, are important complexity measures for the analysis of functions on high-dimensional datasets occurring in biology, medicine, and related areas.

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The metric entropy and n-widths can be considered as optimality criteria for approximation schemes. We will use wavelet frame expansions for approximation and observe that our scheme is asymptotically optimal up to a logarithmic factor in the sense of the metric entropy. The approximation error is measured in the L_p norm, $1 < p \le \infty$, so that the presented results generalize findings in [3], where $p = \infty$ was considered exclusively. We refer to [2] for an extended version of the present manuscript including proofs and further results.

2 Sobolev Spaces and Their Metric Entropy and *n*-Widths

2.1 Diffusion Measure Space

Let $(\mathbb{X}, \rho)$ be a quasi-metric space endowed with a Borel probability measure μ. The system $\{\varphi_k\}_{k=0}^{\infty} \subset L_2(\mathbb{X}, \mu)$ is supposed to be an orthonormal basis of continuous functions with $\varphi_0 \equiv 1$ and our results also involve a sequence of nondecreasing real numbers $\{\lambda_k\}_{k=0}^{\infty}$ such that $\lambda_0 = 0$ and $\lambda_k \to \infty$ as $k \to \infty$. Let N be a positive integer and we shall restrict us to $N = 2^n$, where n is some nonnegative integer. The space of *diffusion polynomials* up to degree N is $\Pi_N := \operatorname{span}\{\varphi_k : \lambda_k \le N\}$. We assume that the *strong product assumption* holds, i.e., there is a constant $a > 0$ such that $P \cdot Q \in \Pi_{aN}$ for all $P, Q \in \Pi_N$. Moreover, we make use of the *generalized heat kernel*

$$G_t(x, y) = \sum_{k=0}^{\infty} \exp\bigl(-\lambda_k^2 t\bigr)\varphi_k(x)\varphi_k(y), \quad t > 0. \tag{2.1}$$

Let us write $\lesssim$ if the left-hand-side is bounded by a generic constant times the right-hand-side, and if both hold, $\lesssim$ and $\gtrsim$, then we write $\asymp$. Recall that a quasi-metric satisfies the standard requirements of a metric but the triangle inequality only needs to hold up to a constant factor. We also summarize the technical assumptions that are related to so-called upper and lower Gaussian bounds on the generalized heat kernel:

Definition 2.1 ([1]) Under the above notation, a quasi-metric space $\mathbb{X}$ is called a *diffusion measure space* if each of the following properties is satisfied:

(i) For each $x \in \mathbb{X}$ and $t > 0$, the closed ball $B_t(x)$ of radius t at x is compact, and there is $\alpha > 0$ such that

$$\mu\bigl(B_t(x)\bigr) \lesssim t^{\alpha}, \quad x \in \mathbb{X},\ t > 0.$$

(ii) There is $c > 0$ such that

$$\bigl|G_t(x, y)\bigr| \lesssim t^{-\alpha/2} \exp\left(-c\frac{\rho(x, y)^2}{t}\right), \quad x, y \in \mathbb{X},\ 0 < t \le 1.$$

(iii) We have $t^{-\alpha/2} \lesssim G_t(x, x)$, $x \in \mathbb{X}$, $0 < t < 1$.

From here on, we suppose that $\mathbb{X}$ is a diffusion measure space throughout the present paper. Given an arbitrary normed space X and a subset $Y \subset X$, we define, for $f \in X$,

$$E(f, Y, X) := \inf_{g \in Y} \|f - g\|_X. \tag{2.2}$$

We can now introduce some function spaces. For a nontrivial ball $B \subset \mathbb{X}$ and $1 \leq p \leq \infty$, the *Sobolev space* of order $s > 0$ is

$$W^s\big(L_p(B)\big) = \big\{f \in L_p(B) : \|f\|_{W^s(L_p(B))} < \infty\big\}, \tag{2.3}$$

where the Sobolev norm is given by

$$\|f\|_{W^s(L_p(B))} := \|f\|_{L_p(B)} + \sup_{N \geq 1} N^s E\big(f, \Pi_N, L_p(B)\big).$$

The ball of radius $r > 0$ in $W^s(L_p(B))$ is denoted by

$$\overline{W_r^s}\big(L_p(B)\big) := \big\{f \in L_p(B) : \|f\|_{W^s(L_p(B))} \leq r\big\}. \tag{2.4}$$

2.2 Kolmogorov Metric Entropy and n-Widths

Let Y be a compact subset of a metric space (X, ϱ) and, for $\varepsilon > 0$, let $N_\varepsilon(Y)$ be the ε-covering number of Y in X. Then

$$H_\varepsilon(Y, X) := \log_2\big(N_\varepsilon(Y)\big) \tag{2.5}$$

is called the *metric entropy* of Y in X and is the minimal number of bits necessary to represent any f with precision ε, cf. [5]. Let us also introduce some alternative notions of complexity. Let Y be a subset of a linear normed space $(X, \|\cdot\|)$ and let $n \geq 1$ be an integer.

(i) The *Kolmogorov n-width of Y in X* is

$$\mathscr{K}_n(Y, X) := \inf_{L_n} \sup_{y \in Y} \inf_{x \in L_n} \|x - y\|,$$

where the infimum is taken over all n-dimensional linear subspaces L_n in X.

(ii) The *linear n-width of Y in X* is

$$\mathscr{L}_n(Y, X) := \inf_{F_n} \sup_{x \in Y} \big\|x - F_n(x)\big\|,$$

where the infimum is taken over all bounded linear operators F_n on X whose range is of dimension at most n.

(iii) The *Gelfand n-width of Y in X* is

$$\mathscr{G}_n(Y, X) := \inf_{\mathcal{L}_n} \sup_{x \in Y \cap \mathcal{L}_n} \|x\|,$$

where the infimum is taken over all closed subspaces $\mathcal{L}_n$ of X of codimension at most n.

(iv) The *Bernstein n-width of Y in X* is

$$\mathscr{B}_n(Y, X) := \sup_{X_{n+1}} \sup\{\lambda \geq 0 : \lambda \overline{X_{n+1}} \subset Y\}$$

where the supremum is taken over all subspaces X_{n+1} of X of dimension at least $n+1$ and $\overline{X_{n+1}}$ denotes the unit ball in X_{n+1}.

The following result extends findings in [6] from the sphere to balls in diffusion measure spaces:

Theorem 2.2 *If $s > 0$ is fixed, B is a nontrivial ball in $\mathbb{X}$, and $0 < \varepsilon \leq r$, then*

$$H_\varepsilon\big(\overline{W_r^s}\big(L_p(B)\big), L_p(B)\big) \asymp (r/\varepsilon)^{\alpha/s} \tag{2.6}$$

holds, where the generic constants neither depend on ε nor on r, and α is the constant in Definition 2.1.

We also determine the n-widths for the global Sobolev space:

Theorem 2.3 *The n-widths of $\overline{W_r^s}(L_p(\mathbb{X}))$ in $L_p(\mathbb{X})$ satisfy*

$$rn^{-s/\alpha} \asymp \mathscr{K}_n \asymp \mathscr{L}_n \asymp \mathscr{G}_n \asymp \mathscr{B}_n.$$

3 Approximation Schemes Using Scattered Data

This section is dedicated to introduce our approximation scheme. A signed Borel measure ν on $\mathbb{X}$ is called a *quadrature measure* of order N if

$$\int_{\mathbb{X}} P(x)d\mu(x) = \int_{\mathbb{X}} P(x)d\nu(x), \quad \text{for all } P \in \Pi_{aN}.$$

For fixed $1 \leq p \leq \infty$, a signed Borel measure ν on $\mathbb{X}$ is called a *Marcinkiewicz–Zygmund measure* of order N if the L_p-norm $\|P\|_{|\nu|, L_p(\mathbb{X})}$ of P with respect to $|\nu|$ satisfies

$$\|P\|_{|\nu|, L_p(\mathbb{X})} \asymp \|P\|_{L_p(\mathbb{X})}, \quad \text{for all } P \in \Pi_{aN}, \tag{3.1}$$

and $|\nu|$ denotes the total variation measure of ν. A signed Borel measure is called a *Marcinkiewicz–Zygmund quadrature measure* of order N if it is both, a quadrature and a Marcinkiewicz–Zygmund measure of order N.

For fixed $1 \le p \le \infty$, a family $(\nu_N)_{N=1}^{\infty}$ of Marcinkiewicz–Zygmund (quadrature) measures, each of order N, respectively, is called *uniform* if the generic constants in (3.1) can be chosen independently of N.

Definition 3.1 We call an infinitely often differentiable function $H : \mathbb{R}_{\ge 0} \to \mathbb{R}$ a *low-pass filter* if it is non-increasing and $H(t) = 1$ for $t \le 1/2$ and $H(t) = 0$ for $t \ge 1$.

For some signed Borel measure ν on $\mathbb{X}$ and $f \in L_1(\mathbb{X}, |\nu|)$, we can define, for $N = 2^n$, $n = 0, 1, 2, \ldots$,

$$\sigma_N(f, \nu) := \sum_{k=0}^{\infty} H\left(\frac{\lambda_k}{N}\right) \int_{\mathbb{X}} f(y)\varphi_k^*(y)d\nu(y)\varphi_k. \tag{3.2}$$

4 Bit Representation in Global Sobolev Spaces

This section is dedicated to verify that linear quantization of the approximation scheme $\sigma_N(f, \mu_N)$ enables bit representations matching the optimality bounds derived in Theorem 2.2 up to a logarithmic factor. First, we recall the formula (3.2),

$$\sigma_N(f, \mu_N) = \sum_{k=0}^{\infty} H\left(\frac{\lambda_k}{N}\right) \int_{\mathbb{X}} f(y)\varphi_k^*(y)d\mu_N(y)\varphi_k,$$

where $(\mu_N)_{N=1}^{\infty}$ is a uniform family of Marcinkiewicz–Zygmund quadrature measures, each of order N, respectively, provided that $p = \infty$. Again, if $1 \le p < \infty$, then we choose $\mu_N = \mu$, $N = 1, 2, 4, \ldots$. Since $H(t) = 1$, for $t \in [0, 1/2]$ and $H(t) = 0$, for $t > 1$, we observe that $H(\frac{\lambda_k}{N})H(\frac{\lambda_k}{2N}) = H(\frac{\lambda_k}{N})$. If $(\nu_N)_{N=1}^{\infty}$ is a family of quadrature measures of order N, respectively, then a straight-forward calculation using the strong product assumption yields

$$\sigma_N(f, \mu_N) = \int_{\mathbb{X}} \sigma_N(f, \mu_N, y) \sum_{k=0}^{\infty} H\left(\frac{\lambda_k}{2N}\right)\varphi_k^*(y)d\nu_N(y)\varphi_k. \tag{4.1}$$

Hence, the scheme (4.1) involves the quadrature measure ν_N and the Marcinkiewicz–Zygmund quadrature measure μ_N. To design the final approximation scheme, we fix some $S > 1$ and apply the quantization

$$I_N(f, \mu_N, y) = \lfloor N^S \sigma_N(f, \mu_N, y) \rfloor, \tag{4.2}$$

and define the actual approximation by

$$\sigma_N^\circ(f, \mu_N, \nu_N) := N^{-S} \int_{\mathbb{X}} I_N(f, \mu_N, y) \sum_{k=0}^{\infty} H\left(\frac{\lambda_k}{2N}\right)\varphi_k^*(y)d\nu_N(y)\varphi_k. \tag{4.3}$$

In other words, we replace $\sigma_N(f, \mu_N, y)$ in (4.1) with a number on the grid $\frac{1}{N^S}\mathbb{Z}$.

We have the following result for the ball $\overline{W_r^s}(L_p(\mathbb{X}))$ of radius r of the global Sobolev space given by (2.3). It extends results in [3] from compact Riemannian manifolds and $p = \infty$ to diffusion measure spaces and to the entire range $1 \le p \le \infty$:

Theorem 4.1 *Suppose that* $(\mu_N)_{N=1}^{\infty}$ *is a uniform family of Marcinkiewicz–Zygmund quadrature measures, each of order* N*, respectively, provided that* $p = \infty$*. For* $1 \le p < \infty$ *we choose* $\mu_N = \mu$, $N = 1, 2, 4, \ldots$*. Assume further that* H *is a low-pass filter as defined above. We also suppose that* $(\nu_N)_{N=1}^{\infty}$ *are Marcinkiewicz–Zygmund quadrature measures with* $\#\operatorname{supp}(\nu_N) \lesssim N^{\alpha}$*. For fixed* $s > 0$ *and* $S > \max(1, s)$, *we apply the discretizations* (4.2) *and* (4.3)*. Then there is a constant* $c > 0$ *such that, for all* $f \in \overline{W_r^s}(L_p, \mathbb{X})$,

$$\left\| f - \sigma_N^{\circ}(f, \mu_N, \nu_N) \right\|_{L_p(\mathbb{X})} \le c r N^{-s} \tag{4.4}$$

holds. For $c r N^{-s} = \varepsilon \le 1$ *and* $\varepsilon \le r$, *the number of bits needed to represent all integers* $\{I_N(f, \mu_N, y) : y \in \operatorname{supp}(\nu_N)\}$ *does not exceed a positive constant* (*independent of* ε *and* r) *times*

$$(r/\varepsilon)^{\alpha/s}\big(1 + \log_2(r/\varepsilon)\big). \tag{4.5}$$

For proofs and an extended version of the present paper, we refer to [2], where also approximation schemes for local smoothness spaces are studied.

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Part XIII
Applications of Queueing Theory in Modelling and Performance Evaluation of Computer Networks

Organizer: Krzysztof Grochla

A Study on Ateb Transform as a Generalization of Fourier Transform

Ivanna Dronjuk and Maria Nazarkevich

Abstract The aim of this work is to construct the Ateb transforms based on Ateb-functions as a generalization of orthogonal Fourier transform. It was proved that these transforms satisfy the properties of linearity, symmetry and similarity. The Hartley transform is a real linear operator, and symmetric and self-inverse properties for Hartley Ateb-transform were proved. The one-dimensional discrete and two-dimensional discrete Ateb transforms were represented. Discrete transforms were used for construction digital watermark for the information security aim in the computer networks.

Keywords Ateb-transform · Fourier transform · Ateb-function · Digital watermark

Mathematics Subject Classification (2010) Primary 42B10 · Secondary 94A08

1 Introduction

Methods, based on mathematical apparatus of orthogonal trigonometric transforms, are widely used while designing and developing the digital signal processing and the information protection systems [1]. On the other hand, methods of wavelet transforms as modern and advanced methods of data processing also are widely used [2]. Unlike conventional spectral transform, wavelet analysis provides approximation with the same accuracy both for smooth functions, as well as for functions of the rapid changes in slope. Different types of wavelets are considered. Among them wavelets, which are described analytically, comprise a short list. However, most types of wavelets, which are used in data processing problems, have no analytical description in the form of a formula, and are described like iterative expressions,

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which are easily calculated by computers. Daubechies wavelets are examples of such functions, one of which (db4) is built into Mathcad [3].

The method of orthogonal transforms based on periodic Ateb-functions was proposed. Wel call it orthogonal Ateb-transform (OAT). The ability to build an OAT is based on the following provisions. First, in [4] it was shown that the Ateb-functions are a generalized case of the ordinary trigonometric functions. Second, in the work [5] orthonormality of system of periodic Ateb-functions is proved. In the work [6] methods and algorithms for computing Ateb-functions, depending on the parameter, that allows to successful using the proposed OAT method, similar to the functions of Daubechies, were developed.

2 Trigonometric Orthogonal Transforms

Denote $i = \sqrt{-1}$ and suppose that $x(t)$ is a real function. Then its Fourier transform is written as

$$V(\omega) = A(\omega) - iB(\omega), \tag{2.1}$$

(see [1]), where

$$A(\omega) = \int_{-\infty}^{\infty} x(t)\cos(\omega t)dt, \tag{2.2}$$

$$B(\omega) = \int_{-\infty}^{\infty} x(t)\sin(\omega t)dt. \tag{2.3}$$

Consider the function of the form

$$cas(t) = \cos(t) + \sin(t), \tag{2.4}$$

then the Hartley transform is given by

$$H(\omega) = \int_{-\infty}^{\infty} x(t)cas(\omega t)dt, \tag{2.5}$$

(see [1]). It is known [4] that the Ateb-functions are generalizations of the usual trigonometric functions. Therefore, similarly to the formulas (2.1)–(2.5), we introduce construction of the transform, based on the periodic Ateb-functions.

3 Construction of the Orthogonal Ateb-Transforms

3.1 OAT with a Single Parameter

We introduce the Ateb-sine and cosine functions [4] depending on the one parameter and having the form $sa(n, 1, t)$ and $ca(1, n, t)$. Suppose that $x(t)$ is a real function, then its Ateb-transform is written as

$$X(n,\omega) = A(n,\omega) - iB(n,\omega), \tag{3.1}$$

where

$$A(n,\omega) = \int_{-\infty}^{\infty} x(t)ca(1,n,\omega t)dt, \tag{3.2}$$

$$B(n,\omega) = \int_{-\infty}^{\infty} x(t)sa^{n}(n,1,\omega t)dt. \tag{3.3}$$

Since the Ateb-sine and Ateb-cosine are odd even functions, respectively, we write the inverse Ateb-transform

$$x(t) = \frac{1}{\Pi}\int_{-\infty}^{\infty}\big(A(n,\omega)ca(1,n,\omega t) - B(n,\omega)sa(n,1,\omega t)\big)d\omega, \tag{3.4}$$

where $\Pi(1,n)$ is the period of the Ateb-functions. The right side of formula (3.4) depends on the parameter n. For each value of n the graph of the function $x(t)$ will be different. Nature, i.e. the steepness, of the period of Ateb-functions $ca(1,n,t)$ and $sa(n,1,t)$ will vary from n. Dependence of Ateb-function on the parameter n makes it possible to choose the appropriate $x(t)$ form of $ca(1,n,t)$ and $sa(n,1,t)$, that fits the computation algorithms.

We introduce the function $casa(1,n,t)$ as follows

$$casa(1,n,t) = ca(1,n,t) + sa^{n}(n,1,t). \tag{3.5}$$

We introduce the direct and inverse Hartley Ateb-transforms using formulas

$$H(n,\omega) = \int_{-\infty}^{\infty} x(t)casa(1,n,\omega t)dt, \tag{3.6}$$

$$x(t) = \frac{1}{\Pi}\int_{-\infty}^{\infty} H(n,\omega)casa(1,n,\omega t)d\omega. \tag{3.7}$$

When $n = 1$ Ateb-transforms introduced by formulas (3.1)–(3.4), (3.6), (3.7) will be known like the orthogonal Fourier and Hartley transforms. For the existence of Ateb-transform of function $x(t)$, it is sufficient to perform the same conditions which are sufficient for the existence of orthogonal Fourier transform.

3.2 OAT with Two Parameters

Let Ateb-functions depend on two parameters. Suppose that $x(t)$ is a real function, then the analog of the Fourier transform [1], an Ateb-transform, is written as

$$X(m,n,\omega) = A(m,n,\omega) - iB(n,m,\omega) \tag{3.8}$$

where

$$A(m, n, \omega) = \int_{-\infty}^{\infty} x(t) ca^m(m, n, \omega t) dt, \quad (3.9)$$

$$B(n, m, \omega) = \int_{-\infty}^{\infty} x(t) sa^n(n, m, \omega t) dt \quad (3.10)$$

where $ca(m, n, \omega)$ is the Ateb-cosine function, $sa(n, m, \omega)$ the Ateb-sine function. Taking into account the identity $ca^{m+1}(m, n, \omega) + sa^{n+1}(n, m, \omega) = 1$ [4], we obtain an expression for the inverse transform:

$$x(m, n, t) = \frac{1}{\Pi} \int_{-\infty}^{\infty} \big(A(m, n, \omega) ca(m, n, \omega t) + B(n, m, \omega) sa(n, m, \omega t) \big) d\omega, \quad (3.11)$$

where $\Pi(m, n)$ is the period of the Ateb-functions. Let us introduce

$$casa(m, n, t) = ca^m(m, n, t) + sa^n(n, m, t). \quad (3.12)$$

Then, according to formula (2.5), the direct Hartley Ateb-transform can be written as

$$H(m, n, \omega) = \int_{-\infty}^{\infty} x(t) casa(m, n, \omega t) dt. \quad (3.13)$$

When $n = 1, m = 1$, Ateb-transforms introduced by formulas (3.8)–(3.11), (3.13) are known as the orthogonal Fourier and Hartley transforms [1]. The validity of linearity, symmetry and similarity are proved: which are similar to the properties of the trigonometric Fourier and Hartley transforms.

3.3 Properties of Orthogonal Ateb-Transforms

Similarly to the properties presented for Fourier and Hartley transforms we derive some properties of orthogonal Ateb-transforms.

1. Linearity. Let a function $x(t)$ be a linear combination of any two functions $x(t) = ax_1(t) + bx_2(t)$. Then

$$X(m, n, \omega) = aX_1(m, n, \omega) + bX_2(m, n, \omega) \quad (3.14)$$

   where $X(m, n, \omega)$ is the image of this function $x(t)$ and $X_1(m, n, \omega)$, $X_2(m, n, \omega)$—are the images of the functions $x_1(t)$ and $x_2(t)$, respectively constructed according to formula (3.8). The proof follows directly from the linearity of the integral.

   A similar property holds for the Hartley Ateb-transform

$$H(m, n, \omega) = aH_1(m, n, \omega) + bH_2(m, n, \omega), \quad (3.15)$$

where $H(m, n, \omega)$ is the image of the function $x(t)$, and $H_1(m, n, \omega)$, $H_2(m, n, \omega)$ are the images of the functions $x_1(t)$, $x_2(t)$, respectively, for the Hartley Ateb-transform, constructed according to formula (3.13).

2. Symmetry. The image of the function $x(-t)$ is $X(n, -\omega)$ and $H(n, -\omega)$, respectively.

   The proof follows from the fact that the Ateb-sine and Ateb-cosine are odd even functions, respectively [4].

3. Similarity. Consider the function $x(\frac{t}{T})$. Then the image of this function is $|T| \cdot H(m, n, T\omega)$.

3.4 Orthogonal Trigonometric Transform for Ateb-Functions

For identification and reproduction of information mathematical apparatus of orthogonal trigonometric transforms, including Fourier transform is widely used. To solve the problem of identification information, which is protected with Ateb-functions, we use orthogonal trigonometric Fourier transform.

Considering the fact that the Ateb-sine $sa(n, m, \omega)$ is the odd function, it can be represented as the direct Fourier sine-transform $B(n, m, x)$ [2]

$$B(n, m, x) = \int_{-\infty}^{\infty} sa(n, m, \omega t) \sin(x, \omega) d\omega. \tag{3.16}$$

Then Ateb-sine is represented like the inverse Fourier sine-transform by the formula

$$sa(n, m, \omega) = \frac{1}{\Pi} \int_{0}^{\infty} B(n, m, x) \sin(x\omega) dx. \tag{3.17}$$

Using the fact that Ateb-cosine $ca(m, n, \omega)$ is even function, we represent it as a direct cosine Fourier transform $A(m, n, x)$

$$A(m, n, x) = \int_{-\infty}^{\infty} ca(m, n, \omega) \cos(x\omega) d\omega. \tag{3.18}$$

Then the Ateb-cosine is represented like the inverse Fourier cosine-transform by the formula

$$ca(m, n, \omega) = \frac{1}{\Pi} \int_{0}^{\infty} A(m, n, x) \cos(x\omega) dx. \tag{3.19}$$

The cosine and sine Fourier transforms are used for continuous functions. Formulas (3.16), (3.18) are used to construct a continuous spectrum of Fourier-images of functions. However, for problems which are associated with information technology more expedient is the usage of discrete functions and transforms. In this case, the discrete Fourier transform is applied.

4 Constructing of Discrete Ateb-Transform

4.1 One-Dimensional DAT

Introduce the discrete Ateb-transform (DAT). Let a signal be given in the form of a discrete sequence $S(p)$. We introduce the functions $A(m, n, k)$ and $B(n, m, k)$ by the formulas

$$A(m,n,k) = \sum_{p=1}^{N-1} S(p) ca^m \left(m, n, -i \frac{2\Pi pk}{N} \right), \quad k = 1, \ldots, N, \tag{4.1}$$

$$B(n,m,k) = \sum_{p=1}^{N-1} S(p) sa^n \left(n, m, -i \frac{2\Pi pk}{N} \right), \quad k = 1, \ldots, N, \tag{4.2}$$

where p is the number of harmonics, N the sample size, $ca(m, n, \omega)$ the Ateb-cosine function, $sa(n, m, \omega)$ the Ateb-sine function.

Then direct DAT is defined by the formula

$$X(m,n,k) = A(m,n,k) - i B(n,m,k). \tag{4.3}$$

Hence we obtain an expression for the inverse transform as

$$S(m,n,p) = \frac{1}{N} \sum_{k=1}^{N-1} \left\{ A(m,n,k) ca^m \left(m, n, -i \frac{2\Pi pk}{N} \right) \right.$$
$$\left. + B(n,m,k) sa^n \left(n, m, -i \frac{2\Pi pk}{N} \right) \right\}, \quad k = 1, \ldots, N. \tag{4.4}$$

The input signal $S(p)$ is formally under the action of the direct and inverse DAT transformed into the signal $S(m, n, p)$. However, for fixed values of the parameters m, n the value of the signal $S(p)$ can be reproduced. This representation allows for using the proposed transforms for creating a digital watermark (DWM) and personalized security in electronic documents.

4.2 Two-Dimensional DAT

Introduce the two-dimensional discrete Ateb-transform. Consider the document set as a two-dimensional discrete sequence $S(p, q)$ where (p, q) is the running pixel image of size $N \times N$. We introduce the functions $A(m, n, k, g)$ and $B(m, n, k, g)$ by the formulas

$$A(m,n,k,g) = \sum_{p=1}^{N-1} \sum_{q=1}^{N-1} S(p,q) ca^m \left(n, m, -i \frac{2\Pi pk}{N}, -i \frac{2\Pi gq}{N} \right),$$
$$k, g = 1, \ldots, N, \tag{4.5}$$

$$B(n,m,k,g)=\sum_{p=1}^{N-1}\sum_{q=1}^{N-1}S(p,q)sa^{n}\left(m,n,-i\frac{2\Pi pk}{N},-i\frac{2\Pi gq}{N}\right),$$
$$k,g=1,\ldots,N, \tag{4.6}$$

where p,q are the numbers of harmonics, $ca()$ the Ateb-cosine function, $sa()$ the Ateb-sine function, $\Pi()$ the period of the Ateb-function.

Direct DAT can be defined by the formula

$$X(m,n,k,g)=A(m,n,k,g)-iB(n,m,k,g). \tag{4.7}$$

We will obtain an expression for the inverse transform as

$$\begin{aligned}S(m,n,p,q)&\\&=\frac{1}{N}\sum_{k=1}^{N-1}\sum_{g=1}^{N-1}\left\{A(m,n,k,g)ca^{m}\left(m,n,-i\frac{2\Pi pk}{N},-i\frac{2\Pi gq}{N}\right)\right.\\&\quad\left.+B(n,m,k,g)sa^{n}\left(n,m,-i\frac{2\Pi pk}{N},-i\frac{2\Pi gq}{N}\right)\right\},\quad p,q=1,\ldots,N.\end{aligned} \tag{4.8}$$

The input image of the document in the form of a matrix $S(p,q)$ formally under the action of direct and inverse DAT is transformed into a matrix $S(m,n,p,q)$. The parameters m,n of the Ateb-functions can be used to personalize documents. For fixed values of the parameters m,n a pixel value of image $S(p,q)$ can be reproduced. This representation allows for using the proposed transforms for creation of embedded hidden messages and personalized protection of the documents.

5 The Digital Watermark Embedding

Consider the digital image Z. We transform this image with formula (4.8) to Z^p. For the digital watermark embedding we used the formula (5.1) for r maximum values in Z^p

$$Z^{wp}=Z^{p}+\alpha w, \tag{5.1}$$

where Z^{wp} is the image transformed with watermark, Z^p is the transformed image, α is the embedding coefficient, w is the watermark of size r.

Figure 1 represents embedding watermark algorithm. It is impossible to detect watermark visually as images with a watermark and without the one look identical. For detecting a watermark specific methods of image processing have to be used. In particular we use a correlation criterion K, calculated according to formula

$$K=\frac{1}{r-1}\sum_{i=1}^{r}\frac{(c_i^w-\bar{c}_i^w)(w_i-\bar{w})}{\sigma_c\sigma_w} \tag{5.2}$$

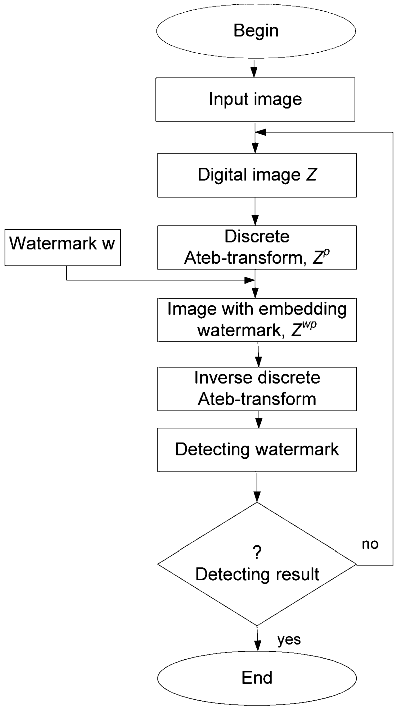

Fig. 1 Embedding and detecting watermark algorithm

where c_i^w is an image element number i, w_i is a watermark element number i, $\bar{c}_i^w$ is an image element average value, $\bar{w}$ is a watermark average value, σ_c is an image with watermark standard deviation, σ_w is a watermark standard deviation. If calculated criterion K is greater than given critical criterial value K_{kr}, we decided that watermark is detected.

6 Conclusions

The method of generalized trigonometric transforms based on periodic Ateb-functions was proposed. Properties of orthogonal Ateb-transforms Fourier and Hartley were introduced and proved. Properties of linearity, symmetry and similarity of the introduced orthogonal Ateb-transform were proved. The one-dimensional and two-dimensional discret Ateb-transforms are constructed. The algorithm of embedding a digital watermark into the image was implemented. Formulas of discrete Ateb transforms shall be applied to the image. For the transformed image, the well-known additive algorithm of embedding the digital watermark in the frequency domain shall be applied.

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Queue-Size Distribution in Energy-Saving Model Based on Multiple Vacation Policy

Wojciech M. Kempa

Abstract An energy-saving model based on the $M/G/1/N$-type finite-buffer queue with independent and generally distributed repeated vacations is considered. Using the formula of total probability and the idea of embedded Markov chain, a system of integral equations for conditional transient queue-size distributions is found. A closed-form representation for the solution of the corresponding system built for Laplace transforms is obtained. Numerical example is attached as well.

Keywords Energy saving · Finite-buffer queue · Multiple vacation policy · Queue-size · Transient distribution

Mathematics Subject Classification (2010) Primary 60K25 · Secondary 90B22

1 Introduction

The problems of power saving and minimization of energy consumption are fundamental ones in wireless sensor networks (WSNs) or mobile stations of WiMAX. Typically, majority of sensor nodes are equipped with non-rechargeable batteries. So, in practice, for elongation the lifetime of the battery, a mechanism based on cyclic succession of sleep and listening modes of the node radio adapter/receiver is being usually implemented. The IEEE 802.16e standard of mobile WiMAX defines three different classes of power-saving mechanisms (see e.g. [3, 12]). Since, during the sleep-mode operation, the energy consumption is much smaller than in the listening mode, the key problem is the efficient reduction the wake mode duration. In the literature different solutions are being proposed for this issue. In [4] a threshold-type discipline is considered in which the accumulation of N packets in the buffer queue is necessary to activate the radio (server) after the idle period (N-policy). An infinite-buffer $M/G/1$-type queue with repeated (multiple) server vacations is

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proposed in [11] as a model of Type I power-saving mode in IEEE 802.16e standard, and some performance measures are derived there.

In the article we investigate the transient queue-size distribution in the model of power saving mechanism based on the $M/G/1/N$-type finite-buffer queue with multiple vacation policy. After each busy period (listening mode) of the system (every time when the system empties) the server takes successive (repeated) independent and generally distributed vacations until, at the end of one of them, the buffer queue contains at least one packet waiting for service (sleep mode). Then, after completion of this vacation, a new busy period (listening mode) begins immediately, during which the queue empties and so on. In [5] and [6] a similar model but with infinite buffer capacity is being considered, and the formulae for transforms of the queue-size distribution and departure process are obtained there, respectively.

In the next Sect. 2, after the mathematical description of the model, for the transient queue-size distributions conditioned by the numbers of packets present in the buffer queue initially (at the opening of the system), we build a system of integral equations, by virtue of the idea of embedded Markov chain and the formula of total probability. Using the approach introduced in [10] (applied also e.g. in [7–9]), we obtain in Sect. 3 the general solution of the corresponding system written for Laplace transforms in a compact form. The last Sect. 4 contains numerical example.

2 System of Equations for Conditional Queue-Size Distributions

Consider the $M/G/1/N$-type queuing model in which packets arrive according to a Poisson process with intensity λ and are being served with a distribution function (d.f.) $F(\cdot)$. The maximal system capacity equals N i.e. we have $N-1$ places in the buffer queue and one place for service. Every time when the system becomes empty the server begins a multiple vacation period (MVP), consisting of a number of independent single vacations, generally distributed with a d.f. $G(\cdot)$.

Denote by $X(t)$ the number of packets present in the system at time t, and introduce the conditional transient queue-size distribution as follows:

$$P_n(t,m) = \mathbf{P}\big\{X(t) = m \,\big|\, X(0) = n\big\}, \quad t > 0,\ 0 \le m,n \le N. \tag{2.1}$$

Assume firstly that the system is empty before the opening and starts its operation at time $t = 0$ with a MVP. The formula of total probability gives

$$\begin{aligned} P_0(t,m) = {} & \sum_{i=0}^{\infty} \int_{u=0}^{t} dG^{i*}(u) \int_{y=u}^{t} \lambda e^{-\lambda y} dy \\ & \times \Bigg\{ \int_{v=y-u}^{t-u} \Bigg[\sum_{k=0}^{N-2} \frac{[\lambda(u+v-y)]^k}{k!} e^{-\lambda(u+v-y)} P_{k+1}(t-u-v,m) \\ & + P_N(t-u-v,m) \sum_{k=N-1}^{\infty} \frac{[\lambda(u+v-y)]^k}{k!} e^{-\lambda(u+v-y)} \Bigg] dG(v) \end{aligned}$$

$$+ I\{1 \le m \le N-1\} \frac{[\lambda(t-y)]^{m-1}}{(m-1)!} e^{-\lambda(t-y)} \overline{G}(t-u)$$

$$+ \delta_{m,N} \sum_{k=N-1}^{\infty} \frac{[\lambda(t-y)]^k}{k!} e^{-\lambda(t-y)} \overline{G}(t-u) \Bigg\} + \delta_{m,0} e^{-\lambda t}, \tag{2.2}$$

where $\delta_{i,j}$ is the Kronecker delta function and the notation $\overline{H}(\cdot)$ stands for the tail of arbitrary d.f. $H(\cdot)$. Indeed, if the MVP ends before t and during it the buffer does not become saturated (1st summand on the right side of (2.2)), then at the completion epoch of the MVP (at the time $u + v$) the system starts the operation with the number of packets which have arrived during the server vacation. If the buffer becomes saturated before the MVP completion epoch (2nd summand), the service begins with N packets present, exactly. If the MVP ends after t and at least one arrival occurs before t, then $X(t) = m$ if and only if the number of arrivals till t equals m, if only $m \le N - 1$ (3rd summand on the right side of (2.2)), or equals at least N if $m = N$ (4th summand). If the first packet arrives after time t, then the random event $\{X(t) = m\}$ is equivalent to $\{m = 0\}$ (5th summand).

As it is well known, service completion epochs in the $M/G/1$-type queue are Markov moments (see e.g. [2]). If $X(0) = n$, where $1 \le n \le N$, then, applying the formula of total probability with respect to the first service completion epoch after $t = 0$, we obtain

$$P_n(t,m) = \int_0^t \Bigg[\sum_{k=0}^{N-n-1} \frac{(\lambda y)^k}{k!} e^{-\lambda y} P_{n+k-1}(t-y,m)$$

$$+ P_{N-1}(t-y,m) \sum_{k=N-n}^{\infty} \frac{(\lambda y)^k}{k!} e^{-\lambda y} \Bigg] dF(y)$$

$$+ \Bigg(I\{n \le m \le N-1\} \frac{(\lambda t)^{m-n}}{(m-n)!} + \delta_{m,N} \sum_{k=N-n}^{\infty} \frac{(\lambda t)^k}{k!} \Bigg) e^{-\lambda t} \overline{F}(t), \tag{2.3}$$

where $I\{\mathbb{A}\}$ is the indicator of the random event $\mathbb{A}$. The interpretation of the right side of (2.3) is similar to that in (2.2).

Introduce the Laplace transform of $P_n(t,m)$ as

$$\widehat{P}_n(s,m) = \int_0^\infty e^{-st} P_n(t,m)\,dt, \quad \mathrm{Re}(s) > 0, \tag{2.4}$$

and define the following functions:

$$a_k(s) = \int_0^\infty e^{-(s+\lambda)y} \frac{(\lambda y)^k}{k!} dF(y), \tag{2.5}$$

$$b_k(s) = \big(1 - g(s+\lambda)\big)^{-1} \int_0^\infty e^{-(s+\lambda)y} \frac{(\lambda y)^k}{k!} dG(y), \tag{2.6}$$

$$d(s,m) = \big(1 - g(s+\lambda)\big)^{-1}\Bigg(I\{1 \le m \le N-1\}\varphi_{G,m}(s) + \delta_{m,N}\sum_{k=N-1}^{\infty}\varphi_{G,k+1}\Bigg)(s), \tag{2.7}$$

$$h_k(s,m) = I\{k \le m \le N-1\}\varphi_{F,m-k}(s) + \delta_{m,N}\sum_{i=N-k}^{\infty}\varphi_{F,i}(s), \tag{2.8}$$

where

$$\varphi_{H,k}(s) = \int_0^{\infty} e^{-(s+\lambda)t}\frac{(\lambda t)^k}{k!}\overline{H}(t)dt \tag{2.9}$$

for any distribution function $H(\cdot)$ and $g(\cdot)$ stands for the Laplace–Stieltjes transform of the d.f. $G(\cdot)$.

Now, the system (2.2)–(2.3) can be rewritten in the following form:

$$\widehat{P}_0(s,m) = \sum_{k=1}^{N-1} b_k(s)\widehat{P}_k(s,m) + \widehat{P}_N(s,m)\sum_{k=N}^{\infty} b_k(s) + d(s,m) + \frac{\delta_{m,0}}{s+\lambda} \tag{2.10}$$

and

$$\widehat{P}_n(s,m) = \sum_{k=0}^{N-n-1} a_k(s)\widehat{P}_{n+k-1}(s,m) + \widehat{P}_{N-1}(s,m)\sum_{k=N-n}^{\infty} a_k(s) + h_n(s,m), \tag{2.11}$$

where $1 \le n \le N$.

Applying, additionally, to (2.10)–(2.11) the following substitution:

$$\widehat{Q}_n(s,m) = \widehat{P}_{N-n}(s,m), \quad 0 \le n \le N, \tag{2.12}$$

we obtain

$$\sum_{k=-1}^{n} a_{k+1}(s)\widehat{Q}_{n-k}(s,m) - \widehat{Q}_n(s,m) = \psi_n(s,m), \quad 0 \le n \le N-1, \tag{2.13}$$

and

$$\widehat{Q}_N(s,m) = \sum_{k=1}^{N-1} b_k(s)\widehat{Q}_{N-k}(s,m) + \widehat{Q}_0(s,m)\sum_{k=N}^{\infty} b_k(s) + d(s,m) + \frac{\delta_{m,0}}{s+\lambda}, \tag{2.14}$$

where

$$\psi_n(s,m) = a_{n+1}(s)\widehat{Q}_0(s,m) - \widehat{Q}_1(s,m)\sum_{k=n+1}^{\infty} a_k(s) - h_{N-n}(s,m). \qquad (2.15)$$

3 Main Result

In this section we prove the following main theorem:

Theorem 3.1 *In the $M/G/1/N$-type system with multiple vacation policy the representation for the Laplace transform $\widehat{P}_n(s,m)$ of the conditional transient queue-size distribution is following*:

$$\begin{aligned}\widehat{P}_n(s,m) &= \Phi_{N-n}(s,m) \\ &\quad + \frac{\sum_{k=1}^{N-1} b_{N-k}(s)\Phi_k(s,m) + d(s,m) + \delta_{m,0}(s+\lambda)^{-1} - \Phi_N(s,m)}{\Theta_N(s) - \sum_{k=1}^{N-1} b_{N-k}(s)\Theta_k(s) - \sum_{k=N}^{\infty} b_k(s)}\Theta_{N-n}(s),\end{aligned} \qquad (3.1)$$

where

$$\Theta_n(s) = a_0(s)R_{n+1}(s) + \sum_{k=0}^{n} R_{n-k}(s)\left(a_{k+1}(s) - f^{-1}(s)\sum_{i=k+1}^{\infty} a_i(s)\right), \qquad (3.2)$$

$$\Phi_n(s,m) = \sum_{k=0}^{n} R_{n-k}(s)\left(h_N(s,m)f^{-1}(s)\sum_{i=k+1}^{\infty} a_i(s) - h_{N-k}(s,m)\right), \qquad (3.3)$$

the formulae for $a_k(s)$, $b_k(s)$, $d(s,m)$, $h_k(s,m)$ are given in (2.5), (2.6), (2.7), (2.8), *respectively, and the sequence $(R_k(s))$ is defined recursively as follows*:

$$\begin{aligned} &R_0(s) = 0, \qquad R_1(s) = \frac{1}{a_0(s)}, \\ &R_{k+1}(s) = R_1(s)\left(R_k(s) - \sum_{i=0}^{k} a_{i+1}(s)R_{k-i}(s)\right), \end{aligned} \qquad (3.4)$$

where $k \ge 1$.

Proof There is proved in [10] that each solution of the infinite-sized system of type (2.13), written for $n \ge 0$, can be stated as

$$\widehat{Q}_n(s,m) = C(s,m)R_{n+1}(s) + \sum_{k=0}^{n} R_{n-k}(s)\psi_k(s,m), \quad n \ge 0, \qquad (3.5)$$

where $C(s,m)$ is independent on n. Since (2.13) has finite number of equations, then the formula (2.14) can be treated as a boundary condition which allows for finding $C(s,m)$ explicitly. Indeed, substituting $n=0$ into (3.5) we get

$$C(s,m)=a_0(s)\widehat{Q}_0(s,m). \tag{3.6}$$

Similarly, substituting $n=0$ into (2.13), we obtain

$$a_0(s)\widehat{Q}_1(s,m)+a_1(s)\widehat{Q}_0(s,m)-\widehat{Q}_0(s,m)=\psi_0(s,m). \tag{3.7}$$

Hence, since $\sum_{k=0}^{\infty}a_k(s)=f(s)$, we have

$$\widehat{Q}_1(s,m)=\frac{\widehat{Q}_0(s,m)-h_N(s,m)}{f(s)}. \tag{3.8}$$

Substituting now (3.6) and (3.7) into (3.5) we obtain

$$\begin{aligned}\widehat{Q}_n(s,m)&\\ &=a_0(s)R_{n+1}(s)\widehat{Q}_0(s,m)\\ &\quad+\sum_{k=0}^{n}R_{n-k}(s)\Bigg[a_{k+1}(s)\widehat{Q}_0(s,m)-\widehat{Q}_1(s,m)\sum_{i=k+1}^{\infty}a_i(s)-h_{N-k}(s,m)\Bigg]\\ &=\widehat{Q}_0(s,m)\Theta_n(s)+\Phi_n(s,m),\end{aligned} \tag{3.9}$$

where $\Theta_n(s)$ and $\Phi_n(s,m)$ are defined in (3.2) and (3.3), respectively.

Now we must find the representation for $\widehat{Q}_0(s,m)$ to derive $\widehat{Q}_1(s,m)$ and $C(s,m)$.

After substituting (3.9) into (2.14), we eliminate $\widehat{Q}_0(s,m)$ as follows:

$$\widehat{Q}_0(s,m)=\frac{\sum_{k=1}^{N-1}b_{n-k}(s)\Phi_k(s)+d(s,m)+\delta_{m,0}(s+\lambda)^{-1}-\Phi_N(s,m)}{\Theta_N(s)-\sum_{k=1}^{N-1}b_{N-k}(s)\Theta_k(s)-\sum_{k=N}^{\infty}b_k(s)}. \tag{3.10}$$

Now, (3.9) and (3.10), after taking into consideration (2.12), lead to (3.1). □

4 Numerical Example

Let packets of average sizes 100 B arrive at the node of the WSN with intensity 400 kb/s, and let the throughput of the output link equals 480 kb/s, so the link utilization is $\rho=0.833$. Assuming that the arrival stream is described by the Poisson process, we have $\lambda=500$ packets/s. Similarly, comparing to the given transmission rate, the service time can be described by the following 2-order hyperexponential distribution function:

$$F(t)=0.5\big(2-e^{-500t}-e^{-750t}\big),\quad t>0.$$

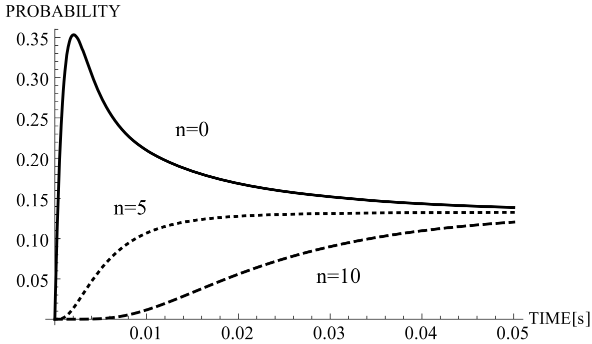

Fig. 1 Probabilities $\mathbf{P}\{X(t) = 1 \mid X(0) = n\}$ for $\rho = 0.833$

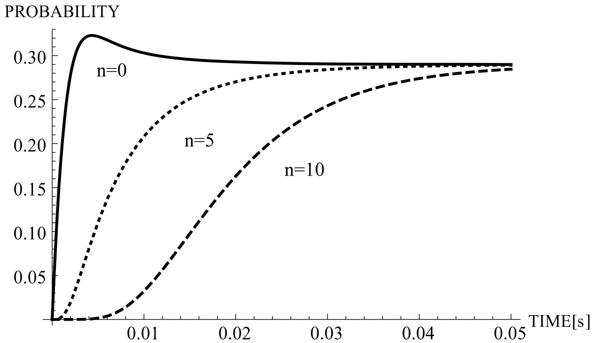

Fig. 2 Probabilities $\mathbf{P}\{X(t) = 1 \mid X(0) = n\}$ for $\rho = 0.417$

Moreover, assume that the energy-saving mechanism of the node is described by the multiple vacation policy in which successive single vacations have exponential distributions with means 0.002 [s]. Let us investigate the behavior of transient conditional distributions $\mathbf{P}\{X(t) = 1 \mid X(0) = n\}$ in dependence on the initial buffer state for $n = 0, 5$ and 10, where the maximal system capacity equals $N = 10$. To obtain the results we use directly the formula (3.1) for $m = 1$ and $n = 0$, 5 and 10 to obtain explicit representations for Laplace transforms of conditional queue-size distributions. Next we use the algorithm of numerical Laplace transform inversion, based on the Bromwich integral and Euler's summation formula, proposed by Abate et al. in [1]. The results are visualized in Fig. 1. As one can observe the relaxation time (time of reaching the stationary state) equals approximately 0.05 [s]. In Fig. 2 we present results for the low input stream intensity, namely for the flow of packets arriving with speed 200 kb/s (that gives $\lambda = 250$ packets/s and $\rho = 0.417$) and the remaining traffic characteristic being the same.

5 Conclusion

In the paper an energy-saving model based on the $M/G/1/N$-type finite-buffer queue with multiple vacation policy is investigated. The explicit representation for the Laplace transform of the conditional queue-size distribution is obtained via the approach using the formula of total probability, the idea of embedded Markov chain and the solution of a specific-type system of linear equations. Applicability of theoretical formulae is illustrated by numerical examples motivated by real-life IP traffic in wireless sensor networks.

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Automobile System Safety Based on the Model for Stochastic Networks with Dependent Service Times

Vladimir Vishnevsky and Vladimir Rykov

Abstract Broadband wireless data transmission network for providing of automobile transport system safety is considered. The network operates under IEEE802.11n-2012 protocol that guarantees high-speed transmission of multimedia information from stationary and mobile automatic systems of traffic control. The model of stochastic network with dependent service time and processor sharing discipline for the problem solution is used. Product-form representation for the model steady-state probabilities is presented.

1 Introduction and Motivation

The Stochastic Networks (SN) have a wide spectrum of applications, including computer, data transmission and telecommunication network. Nowadays telecommunication technologies give extremely wide possibilities for information interchange. However exponentially growth of the number of Internet users and local and corporative networks create the problems of needed quality of service (QoS) providing. To guarantee it in the networks it is necessary to rationalize the using of the network resources, which need to invoke the mathematical models and methods.

Most of really applicable network characteristics are macro-state characteristics such as queue length in buffers, mean time message transmission etc. are usually represented in terms of its steady-state probabilities (SSP). Therefore their calculation is one of very important problems in this topic. Network decomposition and the product form representation of the SSP is a real way for the problem solution.

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There are vast bibliography devoted to both precision and approximate methods of decomposition.

The first strong result that have been proposed by J. Jackson deals with open exponential networks. It was shown in the papers [1, 2] that the equilibrium probabilities for these kind of networks have a product form. For the closed networks analogous result have been obtained by Gordon and Newell [3]. Baskett et al. [4] spread out Jackson's results to the network, which service times cumulative distribution functions (c.d.f.) have fractional rational Laplace–Stiltjes transform. Kelly [5, 6] introduce new principle of routing and propose the general concept for product form representation of the network steady state probability distributions. Latest generalizations and other aspects of queueing networks analysis one can find in reviews [7, 8] and special monographs of R. Boucherie [9] and N. van Dijk [10]. The results of these investigations have been summarized in an excellent book of R. Serfozo [11]. For applications of SN to data transmission systems see [12].

However in most of all these publications it is assumed that the service times in different nodes are independent random variables (r.v.). But, this assumption is not adequate for real data transmission systems such as tele- and computer-communication networks. Indeed, the same message (call) during its transmission through the network has the same size (work requirement, workload), but may be transmitted with different rates by different channels. It is necessary to take into account that in data transmission and telecommunication networks the role of nodes (service stations) usually play the links, while the stations (terminals) are usually considered as buffers. That is, the service times in different nodes of the message route must be dependent.

The models with invariable service times along the route was considered in [13, 14], and the models with generally dependent service times but for the special case of infinity servers nodes have been presented in [15]. Product form of steady state probabilities for network with regenerative service mechanism and Kelly's service disciplines, which are determined by some collection of service rates in nodes was obtained in [16]. Product form for equilibrium probabilities of the open hierarchical networks with dependent service times was obtained in [17]. In [18] another approach to decomposition of the complex hierarchical stochastic networks was proposed. In the paper [19] it was shown that the macro-state stationary probabilities for the network with Poisson input, infinite-servers nodes and processor sharing discipline have a product form. These results has been generalized for open and closed queueing networks with dependent service times in [20–22] that includes and generalize almost all previously considered models where waist bibliography has been also presented.

In this paper this approach will be used for investigation of the broadband wireless data transmission network aimed automobile transport system safety. The paper is organized as follows. In the next section a special model of the automobile transport system safety will be proposed. In the third section an appropriate mathematical model will be propose and analyzed. The paper closes with conclusion and the bibliography.

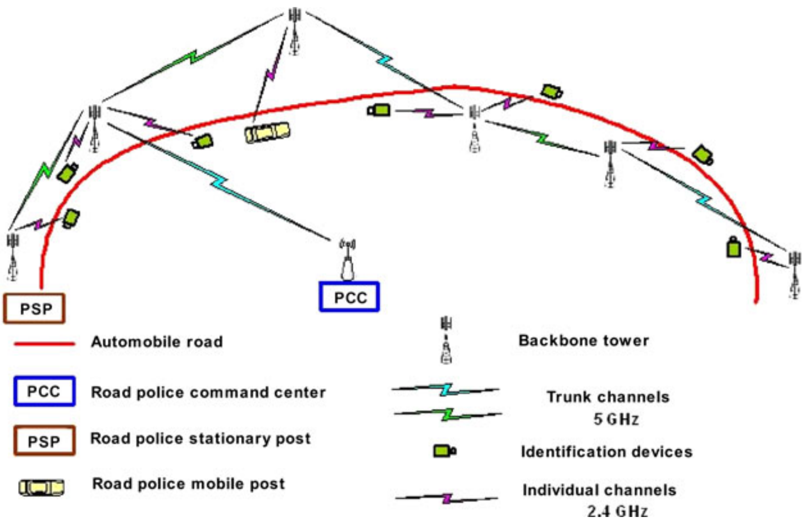

Fig. 1 A system of automatic RRV fixing and transmission

2 The Problem Setting

Accidents on roads is one of serious problems over all the world. Accordingly to the International Public Health Service Organization about 1.2 mln peoples died at the roads over all the world (including 27000 in Russia). It looks like a war in the piece time.

As a fighting tools against accidents at the roads the Automobile System Safety (ASS) is usually used (see, for example, [23]). It represents a system of automatic fixing the Road Rules Violation (RRV) and the information transmission to the appropriate Police Center as it is shown in Fig. 1.

It consists of fixing system located on Stationary or/and Mobile Modules an appropriate transmission system. The last one is a broadband wireless data transmission network along the road. The system usually uses a radar tool for the velocity measuring and optic camera for the vehicle sign fixing. However, this system has two defects:

- Impossibility of recognition of too dirty signs, and
- Too long transmission of the information to the Central Police Office (CPO)

that decrease the system efficiency up to 40 %. In the another approach the RFID-technology that uses special Radio Fixing Identification Marc at the vehicle sign is applied.

3 The Model

The system is modelled as a tandem-type network with r nodes of infinity-size buffers and processor sharing discipline as it is presented in Fig. 2. This means that

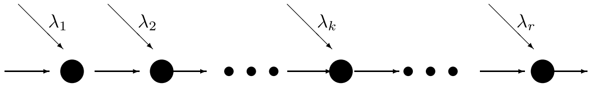

Fig. 2 The model of automobile safety system

all calls are served simultaneously and transferred from k-th node to the $k+1$-th one, and from the last node they go out of the system.

Over all the paper the following assumption and notations will be used:

$\mathbf{n} = (n_1, \dots, n_k, \dots, n_r)$ is the system state vectors, where n_k is number of calls at the k-th node;
λ_k is the intensity of input Poisson flow to the k-th node from the outside and therefore the summary flow to k-th node is

$$\Lambda_k = \lambda_1 + \cdots + \lambda_k;$$

The sizes of all calls are independent r.vs. Y with the same c.d.f.

$$G(y) = \mathbf{P}\{Y \le y\}$$

and they are the same along all the system;
c_k is the capacity of k-th channel, and the service discipline is supposed to be processor sharing discipline, which allows to be served all calls simultaneously with the rate $\frac{c_k}{n_k}$ for k-th node.

Therefore the conditional c.d.fs. of service times on each stage of service given call volume y are

$$B_k(x|y) = \mathbf{P}\{X_k \le x \mid Y = y\} = \Theta\left(x - \frac{yn_k}{c_k}\right) \quad (k = \overline{1,r}).$$

This means that the unconditional service time c.d.f. is

$$B_k(x) = \mathbf{P}\{X_k \le x\} = \int_0^\infty \Theta\left(x - \frac{yn_k}{c_k}\right) dG(y)$$
$$= \int_0^\infty \Theta\left(\frac{c_k x}{n_k} - y\right) dG(y) = G\left(\frac{c_k x}{n_k}\right) \quad (k = \overline{1,r}), \tag{3.1}$$

and therefore the joint service times c.p.d. for the calls are

$$B(\mathbf{x}) = \mathbf{P}\{X_k \le x_k,\ k = \overline{1,r}\} = \prod_{1 \le k \le r} G\left(\frac{c_k x}{n_k}\right).$$

4 The Model Investigation

Consider the stochastic process

$$\mathbf{Z}(t) = \left\{\mathbf{N}(t),\ \mathbf{X}_k(t),\ \mathbf{Y}_k(t);\ k = \overline{1,r}\right\}$$

under the set space for $t \geq 0$

$$\mathcal{E} = \left\{0 \leq x_{k,i_k} \leq y_{k,i_k} < \infty \ (i_k = \overline{1, n_k}, k = \overline{1, r})\right\},$$

where

$\mathbf{N}(t) = (N_1, \ldots, N_k(t), \ldots, N_r(t))$ is the random vector of calls at each node;
$\mathbf{X}_k(t) = (X_{k,1}, \ldots, X_{k,n_k})$ is the vector of remained service times of calls at the k-th node ($k = \overline{1, r}$);
$\mathbf{Y}_k(t) = (Y_{k,1}, \ldots, Y_{k,n_k})$ is the vector of initial service times of calls at the k-th node ($k = \overline{1, r-1}$) arranged in the same order as vector $\mathbf{X}_k(t)$.

Denote by

$\pi(t; \mathbf{z}) = \pi(t; \mathbf{n}, \mathbf{x}, \mathbf{y})$ the probability density function (p.d.f.) of the process $\mathbf{Z}(t)$,
$\mathbf{e}_k = (0, \ldots, 0, 1, 0, \ldots, 0)$ the vector, which k-th component equals 1, while others equal 0,
$\mathbf{x}^-_{k,i} = (x_{k,1}, \ldots, x_{k,i-1}, x_{k,i+1}, \ldots, x_{k,n})$ the vector, in which i-th component is eliminated,
$\mathbf{x}_i(k, y) = (x_{k,1}, \ldots, x_{k,i-1}, y, x_{k,,i+1}, \ldots, x_{k,n})$ the vector, to which additional value y is added at i-th place, while others move at the next positions,

and introduce operators $A_{(k,i)}$, $T_{(k,i,j)}$, $D_{(i)}$ in the system state space corresponding to

arrival of call at i-th position of the k-th node,
service completions of call, being in i-th position at the k-th node and its transmission to the j-th position of the next one, and
service completions of call from i-th position of the last node and its leaving the system.

Formally these operators can be represented by the following relations:

$$A_{(k,i)}\mathbf{z}(y) = \big(\mathbf{n} + \mathbf{e}_k, \mathbf{x}_{k,i}(y), \mathbf{y}_{k,i}(y)\big);$$

$$T_{(k,i,j)}\mathbf{z} = \big(\mathbf{n} - \mathbf{e}_k + \mathbf{e}_{k+1}, \mathbf{x}^-_{k,i}, \mathbf{x}_{k+1,j}(y_{k,i}), \mathbf{y}^-_{k+1,i}(y_{k,i})\big);$$

$$D_{(i)}\mathbf{z} = \big(\mathbf{n} - \mathbf{e}_r, \mathbf{x}^-_{r,i}\big).$$

Theorem 1 *If the p.d.fs.* $\pi(t; \mathbf{z}) = \pi(t; \mathbf{n}, \mathbf{x}, \mathbf{y})$ *of the process* $\mathbf{Z}(t)$ *are differentiable with respect to all its arguments in the set* $\mathbf{z} \in \mathcal{E}$ *for* $t \geq 0$, *then they satisfies to the following Kolmogorov system of differential equations*

$$\frac{\partial \pi(t, \mathbf{z})}{\partial t} - \sum_{\substack{1 \leq k \leq r,\\ 1 \leq i_k \leq n_k}} \frac{1}{n_k} \frac{\partial \pi(t, \mathbf{z})}{\partial x_{k,i_k}} + \Lambda \pi(t, \mathbf{z})$$

$$= \sum_{\substack{1 \leq k \leq r,\\ 1 \leq i_k \leq n_k}} \frac{\Lambda_k}{n_k} \pi\big(t; A^{-1}_{(k,i_k)}\mathbf{z}\big) + \sum_{\substack{1 \leq k \leq r-1,\\ 1 \leq i_k \leq n_k,\\ 1 \leq i_{k+1} \leq n_{k+1}+1}} \frac{1}{n_k(n_{k+1}+1)} \pi\big(t, T^{-1}_{(k,i_k,i_{k+1})}\mathbf{z}\big)$$

$$+\frac{1}{n_r+1}\sum_{1\le i_r\le n_r}\pi\left(t, D^{-1}_{(r,i)}\mathbf{z}\right) \tag{4.1}$$

while the initial conditions in terms of Dirac δ-function are

$$\pi(0;\mathbf{z})\equiv\pi\left(0;(\mathbf{n},\mathbf{x},\mathbf{y})\right)=\delta_{(\mathbf{n},\mathbf{0})}(t)\delta(\mathbf{x})\delta(\mathbf{y})\quad(\mathbf{z}\in\mathcal{E}). \tag{4.2}$$

Proof The proof uses the usual argumentations based on complete probability formulae, Markov property of the process and connection of the process p.d.f. in times t and $t+h$ for small h, when it tends to zero. □

For the Harris irreducible process [24] the following corollary holds.

Corollary 2 *If the process $\mathbf{Z}(t)$ is Harris irreducible one, then these equations have a unique solution that with $t\to\infty$ converges to the solution of stationary regime, which p.d.fs. for $\mathbf{z}\in\mathcal{E}$ satisfy the following system of equations*

$$\begin{aligned}
&\Lambda\pi(\mathbf{z})-\frac{1}{n_r+1}\sum_{1\le i_r\le n_r}\pi\left(D^{-1}_{(r,i)}\mathbf{z}\right)\\
&=\sum_{\substack{1\le k\le r,\\1\le i_k\le n_k}}\frac{1}{n_k}\frac{\partial\pi(\mathbf{z})}{\partial x_{k,i_k}}+\sum_{1\le k\le r}\frac{\Lambda_k}{n_k}\sum_{1\le i_k\le n_k}\pi\left(A^{-1}_{(k,i_k)}\mathbf{z}\right)\\
&\quad+\sum_{\substack{1\le k\le r-1,\\1\le i_k\le n_k,\\1\le i_{k+1}\le n_{k+1}+1}}\frac{1}{n_k(n_{k+1}+1)}\pi\left(T^{-1}_{(k,i_k,i_{k+1})}\mathbf{z}\right).
\end{aligned} \tag{4.3}$$

By the simple substitution the following theorem is proved

Theorem 3 *The functions*

$$\pi(\mathbf{z})=C\prod_{1\le k\le r-1}\Lambda_k^{n_k}\prod_{1\le i\le n_k}\Theta(y_{k,i}-x_{k,i})b_k(y_{k,i})\times\lambda_r^{n_r}\prod_{1\le i\le n_r}\left(1-B_r(x_{r,i})\right) \tag{4.4}$$

with some constant C are the solution of Eq. (4.3) and therefore represent an invariant measure of the process $\mathbf{Z}(t)$.

At last integration with respect to all admissible continuous variables and calculation of the constant C gives in terms of calls size the final macro-states distribution.

Corollary 4 *The macro-states distribution of the process has the product form*

$$\pi(\mathbf{n})=\prod_{1\le k\le r}(1-\rho_k)\rho_k^{n_k} \tag{4.5}$$

with $\rho_k=\frac{\Lambda_k m}{rc_k}$ and $m=\mathbf{E}V=\int(1-G(x))dx$.

5 Proof the Theorem

In order to omit the long and cumbersome calculation we prove the theorem only for the case of simple tandem system for $r = 2$. In this case the system is described by the stochastic process $\mathbf{Z}(t) = \{\mathbf{N}(t), \mathbf{X}_k(t), \mathbf{Y}_k(t); k = 1, 2\}$ for $t \geq 0$ under the set space

$$\mathcal{E} = \left\{0 \leq x_{1,i_1} \leq y_{i_1} < \infty \ (i_1 = \overline{1, n_1}), 0 \leq x_{2,i_1} < \infty \ (i_2 = \overline{1, n_2})\right\}$$

and Eqs. (4.1) take the form

$$\begin{aligned}
&\frac{\partial \pi(t; \mathbf{z})}{\partial t} - \frac{1}{n_1} \sum_{1 \leq i_1 \leq n_1} \frac{\partial \pi(t; \mathbf{z})}{\partial x_{1,i_1}} - \frac{1}{n_2} \sum_{1 \leq i_2 \leq n_2} \frac{\partial \pi(t; \mathbf{z})}{\partial x_{2,i_2}} + \lambda \pi(\mathbf{z}) \\
&= \frac{1}{n_1} \sum_{1 \leq i_1 \leq n_1} \lambda \pi\left(t; A^{-1}_{(i_1)} \mathbf{z}(y_{i_1})\right) + \frac{1}{n_2(n_1 + 1)} \sum_{\substack{1 \leq i_1 \leq n_1, \\ 1 \leq i_2 \leq n_2}} \pi\left(t; T^{-1}_{(i_1,i_2)} \mathbf{z}\right) \\
&+ \frac{1}{n_2 + 1} \sum_{1 \leq i_2 \leq n_2} \pi\left(D^{-1}_{(i_2)} t; \mathbf{z}\right)
\end{aligned} \tag{5.1}$$

while Eqs. (4.3) for the stationary regime are

$$\begin{aligned}
&-\frac{1}{n_1} \sum_{1 \leq i_1 \leq n_1} \frac{\partial \pi(\mathbf{z})}{\partial x_{1,i_1}} - \frac{1}{n_2} \sum_{1 \leq i_2 \leq n_2} \frac{\partial \pi(\mathbf{z})}{\partial x_{2,i_2}} + \lambda \pi(\mathbf{z}) \\
&= \frac{1}{n_1} \sum_{1 \leq i_1 \leq n_1} \lambda \pi\left(t; A^{-1}_{(i_1)} \mathbf{z}(y_{i_1})\right) + \frac{1}{n_2(n_1 + 1)} \sum_{\substack{1 \leq i_1 \leq n_1 + 1, \\ 1 \leq i_2 \leq n_2}} \pi\left(t; T^{-1}_{(i_1,i_2)} \mathbf{z}\right) \\
&+ \frac{1}{n_2 + 1} \sum_{1 \leq i_2 \leq n_2} \pi\left(D^{-1}_{(i_2)} t; \mathbf{z}\right).
\end{aligned} \tag{5.2}$$

By the simple substitution the following theorem is proved.

Theorem 5 *The functions*

$$\pi(\mathbf{z}) = C \lambda^n \prod_{\substack{1 \leq i_1 \leq n_1, \\ 1 \leq i_2 \leq n_2}} \Theta(y_{i_1} - x_{1,i_1}) b_1(y_{i_1}) \left(1 - B_2(x_{2,i_2})\right) \tag{5.3}$$

with some constant C are the solution of Eq. (5.2) *and therefore represent an invariant measure of the process* $\mathbf{Z}(t)$.

Proof In order to simplify the proof rewrite the last equation in the form

$$\lambda\pi(\mathbf{z}) - \frac{1}{n_2+1}\sum_{1\le i_2\le n_2+1}\pi\left(D_{(i_2)}^{-1}\mathbf{z}\right)$$
$$= \frac{1}{n_1}\sum_{1\le i_1\le n_1}\left[\frac{\partial\pi(\mathbf{z})}{\partial x_{1,i_1}} + \lambda\pi\left(A_{(i_1)}^{-1}\mathbf{z}(y_{i_1})\right)\right]$$
$$+ \frac{1}{n_2}\sum_{1\le i_2\le n_2}\left[\frac{\partial\pi(\mathbf{z})}{\partial x_{2,i_2}} + \frac{1}{n_1+1}\sum_{\substack{1\le i_1\le n_1+1,\\ 1\le i_2\le n_2}}\pi\left(T_{(i_1,i_2)}^{-1}\mathbf{z}\right)\right]. \tag{5.4}$$

Now each part of this equation equals zero, that shows that for the system some kind of partial balance takes place.

Indeed, the substitution of the functions (5.3) to the left side of Eq. (5.4) gives

$$\lambda\pi(\mathbf{z}) - \frac{C\lambda^{n+1}}{n_2+1}\sum_{1\le i_2\le n_2+1}\prod_{\substack{1\le i_1\le n_1,\\ 1\le j\ne i_2\le n_2+1}}\Theta(y_{i_1}-x_{1,i_1})b_1(y_{i_1})\big(1-B_2(x_{2,j})\big)$$
$$= \lambda\pi(\mathbf{z}) - \lambda\pi(\mathbf{z})\frac{1}{n_2+1}\sum_{1\le i_2\le n_2+1}1 = 0.$$

For first summand on the right side of Eq. (5.4) consider expression in brackets

$$\frac{\partial\pi(\mathbf{z})}{\partial x_{1,i_1}} + \lambda\pi\big(t;\, A_{(i_1)}^{-1}\mathbf{z}\big)b_1(y_{i_1})$$
$$= -C\lambda^n\delta(y_{i_1}-x_{1,i_1})b_1(y_{i_1})\prod_{\substack{1\le j_1\ne i_1\le n_1-1,\\ 1\le j_2\le n_2}}\Theta(y_{j_1}-x_{1,j_1})\big(1-B_2(x_{2,i_2})\big)$$
$$+ \lambda C\lambda^{n-1}\prod_{\substack{1\le j_1\ne i_1\le n_1-1\\ 1\le j_2\le n_2}}\Theta(y_{j_1}-x_{1,j_1})b_1(y_{j_1})\delta(y_{i_1}-x_{1,i_1})\big(1-B_2(x_{2,i_2})\big)$$
$$= 0.$$

At last for the last expression it is true

$$\frac{\partial\pi(\mathbf{z})}{\partial x_{2,i_2}} + \frac{1}{n_1+1}\sum_{1\le i_1\le n_1+1}\pi\big(T_{(i_1,i_2)}^{-1}\mathbf{z}\big)$$
$$= -C\lambda^n\frac{b_2(x_{2,i_2})}{1-B_2(x_{2,i_2})}\prod_{\substack{1\le j_1\le n_1,\\ 1\le j_2\ne i_2\le n_2}}\Theta(y_{j_1}-x_{1,j_1})b_1(y_{j_1})\big(1-B_2(x_{2,i_2})\big)$$
$$+ \frac{1}{n_1+1}\sum_{1\le i_1\le n_1+1}C\lambda^n\frac{b_2(x_{2,i_2})}{1-B_2(x_{2,i_2})}\Theta(y_{j_1}-x_{1,j_1})$$

$$\times \prod_{\substack{1\le j_1\le n_1,\\ 1\le j_2\le n_2}} \Theta(y_{j_1}-x_{1,j_1})b_1(y_{j_1})\big(1-B_2(x_{2,i_2})\big)$$

$$= -\beta_2(x_{2,i_2})\pi(\mathbf{z}) + \Theta(y_{j_1}-x_{1,j_1})\beta_2(y_{i_2})\frac{\pi(\mathbf{z})}{n_1+1}\sum_{1\le i_1\le n_1+1} 1 = 0. \qquad \square$$

Corollary 6 *Substitution to the formulae* (5.3) *the expression* (3.1) *of calls service time distribution in terms of their size gives*

$$\pi(\mathbf{z}) = C\lambda^n \prod_{\substack{1\le i_1\le n_1,\\ 1\le i_2\le n_2}} \Theta(y_{i_1}-x_{1,i_1})b_1(y_{i_1})\frac{n_1}{c_1}\left(1-G_2\left(\frac{n_2x_{2,i_2}}{c_2}\right)\right)g_1\left(\frac{n_1y_{i_1}}{c_1}\right). \tag{5.5}$$

Corollary 7 *Integration of the previous expression over all admissible values of continuous variables and the constant C calculation gives the macro-state probability distribution*

$$\pi(\mathbf{n}) = (1-\rho_1)(1-\rho_2)\rho_1^{n_1}\rho_2^{n_2} \tag{5.6}$$

with

$$\rho_i = \frac{\lambda m}{c_i}\big(i=\{1,2\}\big) \quad \text{and} \quad m = \mathbf{E}V = \int \big(1-G(x)\big)dx.$$

6 Conclusion

As a model for investigation of the broadband wireless data transmission network aimed automobile transport system safety a tandem-type network with nodes of infinity buffers and processor sharing discipline is proposed. The product-form representation for the steady-state probabilities of the model is found.

The algorithms for the numerical investigations of the model are needed, their computer realization should be done and applied to different concrete situations.

The authors are very grateful to anonymous reviewers, whose remarks was very helpful and promote for the text improvement.

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Part XIV
Applied Mathematics

Organizer: Siarhei Bosiakov

Damage Prediction of the Femur with Postresection Defect

S. Bosiakov, D. Alekseev, and I. Shpileuski

Abstract The aim of this study is to develop an approach to assessing the strength of the femur after sectoral resection in cases of benign bone tumors, tumor-like and metastatic lesions. The proposed approach is based on the finite element calculation of dangerous volumes in the area of bone defect. Load is static and equivalent to average human weight. Model of the femur is based on tomographic data. Postresection defect is localized in the middle third of the lateral side of the femur. As a conditions for the selection of dangerous volume fracture criterion Coulomb–Mohr is used. The analysis of damage near the concentrators of the bone defect is carried out for different loads. The domain of the bone defect with the largest damage is determined. For concentrators of the postresection hole the emergence and growth of crack is considered as a change of the dangerous volume with taking account the removal of the damaged finite elements. The ranges of the load corresponding to the various cases of damage development are determined. Three options to compensate for bone strength and the prevention the pathological bone fracture after sectoral resection are suggested.

Keywords Femur · Benign bone tumors · Tumor-like lesions · Metastatic lesion · Sectoral resection · Failure criterion · Damage prediction · Finite element modeling

Mathematics Subject Classification (2010) Primary 99Z99 · Secondary 00A00

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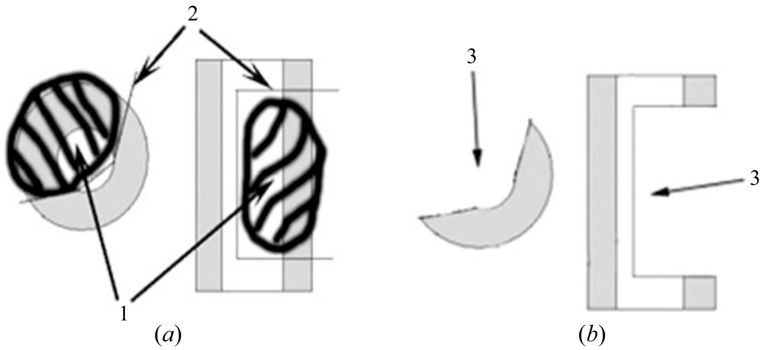
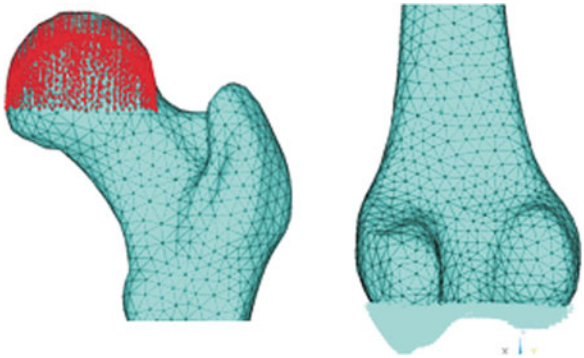


Fig. 1 Scheme of sectoral resection: (**a**) is the fragment of cortical bone before resection (1 is the lesion, 2 is the line of excision of the bone); (**b**) is the fragment of cortical bone after resection (3 is the postresection defect)

1 Introduction

The leading method of treatment of the benign bone tumors, tumor-like and metastatic lesions in tubular bones is the surgical removal of the pathological focus within intact bone. Thus there is formed a sectoral bone defect that leads to a reduction its strength. Figure 1 shows a schematic diagram of sectoral bone resection.

Functional capabilities of the limb and quality of life after the operation are reduced significantly because there is a risk of pathologic bone fracture on the level of the resection. A review of research on the development of criteria for predicting of fractures of the bone with defects and of intact bone, shows that for this purpose finite element modeling, X-ray computed tomography or routine radiography are used. In particular, on basis a retrospective analysis of radiographs of patients with a high probability of pathological fracture and with actually fracture an approach for predicting fractures in trochanteric region of femur with metastatic lesion is suggested [1]. In a study [2] on the basis of subject-specific finite element analysis the torsional stiffness and strain energy density for femora the ratio for the torque at which the fracture occurs in the vicinity of the defect or post-resection of metastatic lesions is determined. In [3] it was proposed to predict fractures in the neighborhood of metastatic lesions on the basis of plastic strains. The method of predicting of pathologic fracture after sectoral resection based on the direct use of quantitative computed tomography is presented in study [4]. When predicting the pathologic fractures of long bones with post-resection defects and other lesions much attention is paid to the principal and the equivalent (von Mises) stresses. Results obtained in this direction are presented in [5, 6]. In [7] the use the factors of safety is proposed. They are defined as the ratio of the allowable to the computed stress or strain according to a particular failure theory, were computed for each element using several failure theories. Values of safety factors are determined for each of the finite elements. The coefficient of less than one indicates the damage of the corresponding



(*a*) (*b*)

Fig. 2 Area of application of the load to the femur head and its direction of action (**a**), as well as rigid fixing of the femoral condyles (**b**)

element and allows to predict the location and nature of the fracture. Direct measurement of the volume of damaged finite elements for assessing the strength of the distal radius and proximal femur is proposed in [8–10]. In these studies, the element is considered damaged if certain strength criteria are not carried out. The advantage of this approach is not only the ability to predict occurrence of cracks or fracture, but also the possibility of determining the location and nature of the fracture. This study develops this topical direction. The aim is to research on the basis the finite element method the damage femur with postresection defect localized in the middle third.

2 Modeling

Computed tomography of the femur is performed on the spiral X-ray CT Siemens Somatom Emotion 16, step of slice is 2 mm. Development of three-dimensional solid model of the femur carried out using a computer system processing of medical images ScanIP (Simpleware Ltd., UK). STL-model converted into a solid model using CAD-package CATIA V5 (Dassault Systémes, France). Postresection defect is constructed after importing model in ANSYS 14.0 (ANSYS Inc., USA). Hole is localized in the middle third of the femur and is positioned on its medial surface. Defect length along the axis of the bone is $2d$ (d is a diameter of the median surface of the bone at the level of the defect). The angular size of the defect is equals to 180°.

Static load is directed along the biomechanical axis if the femur extending from the upper pole of the femoral head to the middle of the segment between the extreme points of the articular surfaces of the femoral condyles. Load zone corresponds to a contact region of the femoral head with the roof of the acetabulum. The lower regions of the femoral condyles (contact areas with the condyles of the tibia and meniscus) are rigidly fixed. The application of the load, its direction and embedded region of the femur condyles are shown in Fig. 2.

Bone tissue modeled by homogeneous isotropic medium. The elastic modulus is 13.7 GPa, Poisson's ratio is 0.3 (the constants of elasticity correspond to compact bone [11]).

3 Definition of Damage

According to the model of the body with a dangerous volume the damage is determined by the volume of the material with a critical level of stresses therein [12]. As a criterion for limiting the dangerous volume in region of postresection hole Coulomb–Mohr failure theory has been used [10, 13]:

$$\frac{\sigma_1 - \sigma_3}{\sigma_{yc}} \geq 1,$$

where $\sigma_1 \geq \sigma_2 \geq \sigma_3$ are principal stresses; $\sigma_{yc} = E\varepsilon_{yc}$ is limit of strength for compressed compact bone tissue; ε_{yc} is the limit compressive deformation (for compact bone $\varepsilon_{yc} = 0.0154$ [10]).

4 Assessment of Damage

Analysis of the damage in the region of bone defect is carried out under a load of 0.55 kN to 1.0 kN. Finite element calculation of the stress-strain state was performed in several stages, to trace the development of dangerous volumes. At the first stage determined the $\psi = \frac{(\sigma_1 - \sigma_3)}{\sigma_{yc}}$ value for the four concentrators of postresection hole. The next step of calculating is removing the finite elements, for which inequality $\psi \geq 1$ is performed. After that the finite element calculation of model is carried out under the same boundary conditions, and dangerous volumes are determined in the vicinity of each concentrator. Dangerous volumes are removed again, and finite element calculation is repeated. Discrete model of femur with designations of concentrators, as well as the visualization of dangerous volumes at various steps of the calculation are shown in Fig. 3.

We note that the finite-element partition has been carried out in semi-automatic mode. The finite element mesh is tetrahedral, the maximum size of finite element edge is 5 mm (see Fig. 3a). The number of elements in the model is 101246, number of nodes is 146290, the element type is Solid187. In the neighborhood of concentrators 1–4 maximum of the edge finite element is 0.5 mm (see Fig. 3a). Figure 4 shows the change of the dangerous volumes under various loads for five consistent stages of the finite element calculation.

Figure 4 also shows that at the first stage of the calculation the largest dangerous volumes are observed in the vicinity of the first, second and fourth hubs. At subsequent stages of calculation the dangerous volume to a greater extent increases near of the concentrator 4 as compared with the concentrators 1, 2 and 3. This allows to conclude that the fracture in the femur, primarily occurs in the concentrator 4 of

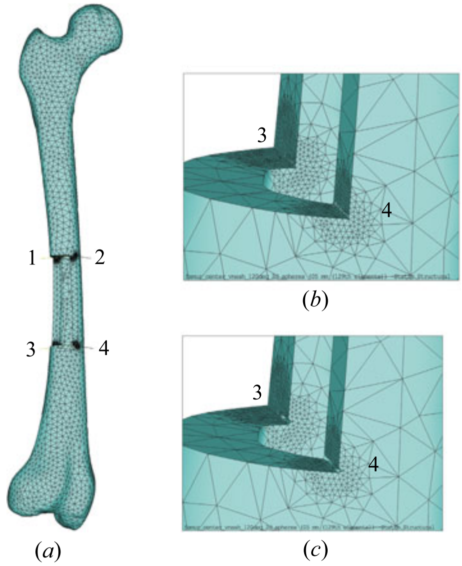


Fig. 3 Finite element model of the femur with notations of concentrators of the bone defect (**a**); dangerous volumes in the neighborhood of the fourth concentrator on the first (**b**) and on the fifth (**c**) step of the calculation

the bone defect. In descending order of magnitudes of dangerous volumes after concentrator 4 follows concentrators 1 or 2 (depending on the step of the finite element calculation). Smallest dangerous volume at any load observed near the third concentrator. Taking this into account it can be concluded that the fracture will propagate in a direction away from the concentrator 4 to the concentrator 1.

4.1 Discussion and Conclusions

At various stages of the finite element calculation the dangerous volumes are changed differently depending on the load (see, for example, Figs. 3a and 3d). Figure 3a shows that the dangerous volume decreases with sequential removal of elements for which an inequality $\psi \geq 1$ is carried out. A similar behavior of dangerous volume is observed at the load of 600 N and 650 N. At a load of 550 N the dangerous volumes are not occurs. With increase of the load (750 N to 1000 N) the dangerous volumes increase for all concentrators at each stage of the finite element calculation.

Nature of the change dangerous volumes at various stages of the finite element calculation suggests three variants of behavior femur with postresection defect and three variants of recommendations for the patient. If in the neighborhood concentrators do not appear the dangerous volumes from routine human activity in region of the defect don't occur—the fracture will not happen. In this case, unloading regime should be the recommended for the patient. Else, if defect arise near the concentrators of bone the dangerous volumes, but the magnitudes of the volume are decreased at following stages of the calculation with the removal of elements with

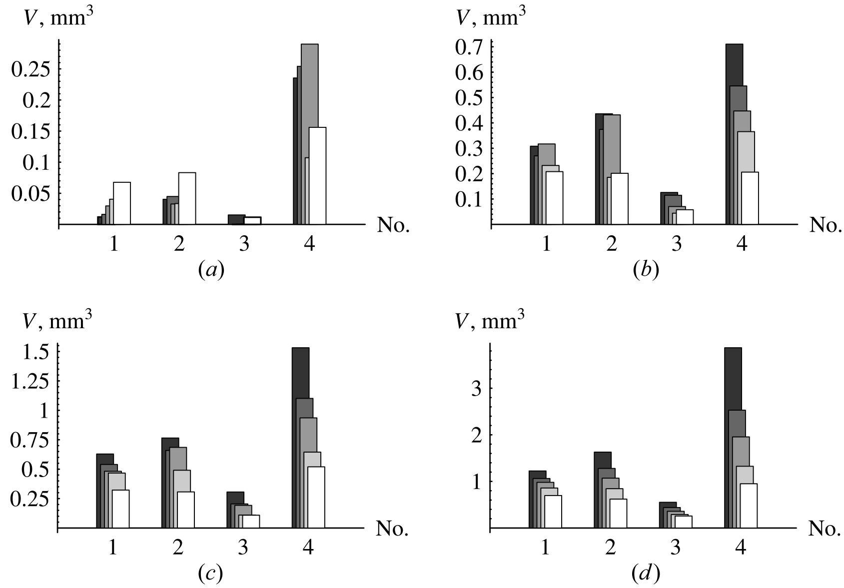


Fig. 4 Dangerous volumes at various stages of the finite-element calculation at the area of bone defect by the action of a static load: (**a**) for 700 N; (**b**) for 800 N; (**c**) for 900 N; (**d**) for 1000 N. The number of concentrator defect is indicated on the *horizontal axis*. The bar chart corresponding to the first step of the calculation is shown on the foreground; the bar chart for the fifth step of the calculation is shown on the background

critical values of ψ, there is a damage of the femur in the neighborhood of concentrators. Development of damage can be stopped if certain preventive measures are undertake. In this case, the use of means of external immobilization (plaster cast) should be recommended the patient. With increasing of dangerous volumes in the successive stages of the calculation, we assume that the damage will develop even in static position and fracture will arise. In this case, the mandatory bone reinforcement ("preventive osteosynthesis") should be applied to the bone.

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Representative Elements for Polydispersed Composites

Natalia Rylko

Abstract Effective properties of random 2D composites are discussed in the framework of the representative volume element (RVE) theory proposed by Mityushev (Complex Var. Elliptic Equ. 51:1033–1045 2006). This theory is extended to 2D fiber composites with sections perpendicular to fibers of different radii. The RVE theory is applied to the mixture problem arisen in technological processes.

Keywords Representative volume element · Random composite

Mathematics Subject Classification (2010) 74Q99

1 Introduction

The physical properties of composites can be determined by measurement of macroscopic properties of testing specimen. Analogous to experimental investigations computational methods are used in theoretical study of the specimen which represent the entire material. If inclusions or pores are distributed statistically homogeneous in the bulk material, the effective properties are described by constant tensors [1]. The macroscopic tensors do not depend on the size, shape of the chosen specimen or on boundary conditions [2, 9]. These tensors can be determined via solution of the periodic problem when the periodicity cell represents the material under consideration. This concerns also statistically homogeneous media when a cell represents the macroscopic properties of the random media not necessary periodic [7, 9, 18]. Such media constitute a subclass of heterogeneous fields discussed in [11, 15, 17] and functionally gradient materials [8].

Statistically homogeneous media defined in [7, 9, 18] can be represented by a cell which is called by the representative volume element (RVE). Statistical methods to construct RVEs have been described in details in [10] and have been developing in many recent papers, see for instance [16] and works cited therein. The statistical

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methods are based on the overall testing process. For dispersed two phase composites with identical inclusions, the number of particles contained in a sample is increased and the effective constants can be computed by numerically [19]. The process of increasing is stopped when the fluctuations of the effective constants become sufficiently small. The number of particles 64 per cell frequently arises in literature [19] as a sufficiently large number for the non-overlapping uniform distribution of inclusions. Analytical and numerical results [4, 5] rigorously confirm this fact.

Numerical methods [19] are restricted by special distributions of inclusions. The properties of constitutes are also given numerically. These difficulties were overcame in [13] where a rigorous and constructive theory of the RVE for plane composites with identical circular inclusions was described. The RVE theory for identical disks [13] is based on the representation of the effective conductivity tensor [3] and [12] in the form of a double series on the concentration of inclusions and on "basic elements" which depend only on the locations of the inclusions. These basic elements are written in terms of the Eisenstein series. Coefficients in the double series depend on the physical properties of constitutes. Two composites were defined as equivalent if they have the same basic elements. Therefore, the set of the composites with circular identical inclusions into classes of equivalence determined only by geometrical structure of the composite. In each class of equivalence a composite having the minimal size cell is chosen. This cell is called the representative cell [13] of equivalent composite materials. A constructive algorithm to determine the representative cell for any distribution of inclusions using only pure geometrical parameters was described in [13]. Examples presented in [13] yield quick transformations from cells to the representative cell of small size. This can be used in optimal computations of the macroscopic properties by applications of the numerical and analytical methods.

In the present paper, we extend the RVE theory [13] to 2D polydispersed composites with circular inclusions. The results can be applied to the mixture problem which can be outlined as follows. Take many particles, put them in a vessel and stir. There are the original and final locations of particles in the vessel. The mixture problem consists in the process to reach the required mixture. Here, we arrive at the questions how to measure the macroscopic thermal and mechanical properties of the mixture during the process. It can be done by expensive experiments. The RVE theory yields a simple method to solve this mixture problem.

2 2D RVE Theory

Following [13] we develop the RVE theory to disks of different radii. Consider 2D two-component composite made from a collection of non-overlapping disks embedded in an otherwise matrix. It is assumed that the distribution $\mathcal{R}$ of the radii of the disks r_k is given and it does not depend on the locations of the disks. The centers of disks satisfies a distribution $\mathcal{A}$ corresponding to a non-overlapping disks distribution on the plane. The distribution $\mathcal{A}$ formally does not depend on the radii

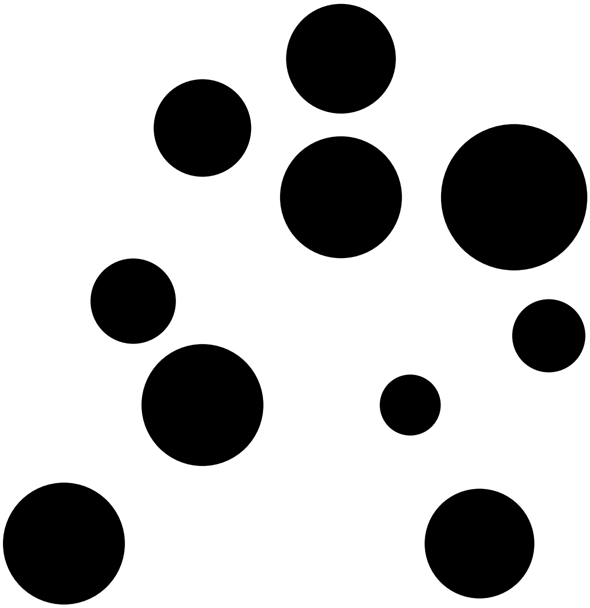

Fig. 1 Section of the polydispersed fibrous composite

distribution $\mathcal{R}$. But the choice of the distribution $\mathcal{A}$ is restricted by $\mathcal{R}$. This situation can be demonstrated by the following observation. For identical disks of radius r the distance between any two centers must exceed $2r$.

It is assumed that the distribution $\mathcal{A}$ generates a random homogeneous field [7, 9, 18] for which the macroscopic properties are correctly defined. One of the most important distribution $\mathcal{A}$ is the non-overlapping uniform distribution $\mathcal{U}$ which corresponds to the perfect mixture of inclusions. The distribution $\mathcal{U}$ can be realized by the sequence location method or by random walks described in [4]. Other distributions are described in the book [17] in terms of the correlation functions. In the present paper, we do not discuss the question of the statistical generation of the theoretical distributions and assume that realizations of $\mathcal{A}$ are given in the form of the pares $(\mathbf{a}_k, r_k)$ where $\mathbf{a}_k = (x_k, y_k)$ denotes the center and r_k the radius of the kth inclusion. Further, it will be convenient to identify $\mathbf{a}_k$ to the complex number $a_k = x_k + iy_k$ (see Fig. 1).

According to the homogenization theory there exist a periodicity cell with a finite number of inclusions representing the composite. First, we describe parameters of this cell. Consider a lattice $\mathcal{Q}$ on the complex plane $\mathbb{C}$ which is defined by two fundamental translation vectors ω_1 and ω_2. Without loss of generality we assume that $\omega_1 > 0$ and $\operatorname{Im}\omega_2 > 0$ where Im stands for the imaginary part. Introduce the zero-th cell $Q_0 := \{z \in \mathbb{C} : z = t_1\omega_1 + t_2\omega_2 - 1/2 < t_{1,2} < 1/2\}$. The lattice $\mathcal{Q}$ is generated by the cells $Q_m := \{z \in \mathbb{C} : z - m_1\omega_1 - m_2\omega_2 \in Q_0\}$ where $m = m_1 + im_2$ denote complex numbers with integers m_1 and m_2.

Let $\mathcal{C}_N$ denote the set of the elements (a_k, r_k), $k = 1, 2, \ldots, N$, where the radii r_k satisfies the distribution $\mathcal{R}$ and the centers a_k correspond to the non-overlapping uniformly distributed disks in a cell Q_0. Introduce the set $\mathcal{C} = \bigcup_{N=N_0}^{\infty} \mathcal{C}_N$ with sufficiently large N_0. Actually, the number N_0 gives the size of the minimal representative set $\mathcal{C}_{N_0}$. The set $\mathcal{C}$ consists of all the configurations of mutually disjoint disks uniformly distributed on the plane whose radii satisfy the distribution $\mathcal{R}$. It is worth noting that $\mathcal{C}$ describes random locations of disks on the plane. In practical measurements, we observe finite fragments of $\mathcal{C}$. If these fragments represent the considered material, it is possible to statistically recover the distributions $\mathcal{R}$ and $\mathcal{A}$. The radii distribution $\mathcal{R}$ can be easily constructed since it describes a 1D random variable. The 2D distribution $\mathcal{A}$ is theoretically described by correlation functions [17]. But we do not follow [17] and consider $\mathcal{A}$ as a set of the given center coordinates a_k (measured and statistically presented). In particular, the 2D concentration of inclusions ϕ_2 can be measured. Theoretically, the 2D concentration ϕ_2 can be considered as the mean value

$$\phi_2 = \frac{1}{|Q|}\sum_{k=1}^{N} \pi r_k^2, \tag{2.1}$$

where $|Q|$ stands for the area of the domain Q.

According to the theory [13] we have to compare two different representative elements of $\mathcal{A}$. Consider a large fundamental region Q' constructed by the fundamental translation vectors ω_1' and ω_2'. Let Q' contains N' non-overlapping circular disks D_k' of radius r_k' with the centers $a_k' \in Q'$ $(k = 1, 2, \ldots, N')$ representing the distributions $\mathcal{R}$ and $\mathcal{A}$. Let $\widehat{\Lambda}'$ be the effective tensor of the composite represented by the region Q' with inclusions D_k'. Let the cell Q' corresponds to another small cell Q which contains inclusions $D_k = \{z \in \mathbb{C} : |z - a_k| < r_k\}$ $(k = 1, 2, \ldots, N)$ also representing the distributions $\mathcal{R}$ and $\mathcal{A}$ such that an effective tensor $\widehat{\Lambda}$ close to $\widehat{\Lambda}'$. Closeness is defined by the concentration accuracy $O(\phi_2^{L+1})$ for the difference $\Delta\widehat{\Lambda} = \widehat{\Lambda} - \widehat{\Lambda}'$ with prescribed L. According to [13] the cell Q is a representative cell for the region Q' with the accuracy $O(\phi_2^{L+1})$ if

$$\Delta\widehat{\Lambda} = O\left(\phi_2^{L+1}\right). \tag{2.2}$$

Let a representative cell Q has the minimal possible area from all the representative cells equivalent to Q'. This cell is called the RVE [13]. The existence of the RVE is evident since in the worst case one can take $Q = Q'$. The numerical statistical methods [10] are also based on the relation (2.2). Since ϕ_2 is fixed in numerical computations, (2.2) becomes $\Delta\widehat{\Lambda} \approx 0$.

Instead of (2.2) Mityushev [13] proposed to compare the basic elements of the expansion of the effective tensor. These basic elements are introduced as follows. Let a cell Q contains N inclusions with centers a_k. Let $E_m(z)$ denote the Eisenstein function of order m (see for instance Appendix A to [4]). Let $\mathbf{C}$ denote the operator of complex conjugation which is identical for even q, i.e., $\mathbf{C}^q z = z$ and $\mathbf{C}^q z = \mathbf{C}z = \overline{z}$ for odd q.

The following sum of multi-order $(m_1, \ldots, m_q)$ were introduced by Mityushev [13]

$$e_{m_1 \ldots m_q} := \left(\frac{\pi}{\phi_2}\right)^{[1+\frac{1}{2}(m_1+\cdots+m_q)]} \sum_{k_0 k_1 \ldots k_q} r_{k_0}^2 r_{k_1}^{2t_1} \ldots r_{k_q}^{2t_q} E_{m_1}(a_{k_0} - a_{k_1}) \times \overline{E_{m_2}(a_{k_1} - a_{k_2})} \ldots \mathbf{C}^q E_{m_q}(a_{k_{q-1}} - a_{k_q}), \tag{2.3}$$

where $k_s = 1, 2, \ldots, N$ $(s = 0, 1, \ldots, q)$, $t_0 = 1$ and $t_s = m_s - t_{s-1}$. We call[1] (2.3) by M-sum of order $(m_1, \ldots, m_q)$. For instance, the M-sum of order (2, 2) has the form

$$e_{22} := \left(\frac{\pi}{\phi_2}\right)^3 \sum_{k_0,k_1,k_2} r_{k_0}^2 r_{k_1}^2 r_{k_2}^2 E_2(a_{k_0} - a_{k_1}) \overline{E_2(a_{k_1} - a_{k_2})}. \tag{2.4}$$

It is justified in [13] that the effective conductivity tensor for 2D composites can be presented in the form of the power series on the total concentration ϕ_2 with coefficients linearly depending on $e_{m_1 \ldots m_q}$. In order to obtain the effective elastic tensor one has to add to the M-sums analogous sums when the Eisenstein functions are replaced by quasi-elliptic functions introduced in Appendix 2 of the book [6]. However, the quasi-elliptic functions are expressed via the Eisenstein functions by algebraic equations [6]. Therefore, it is sufficient to consider only the M-sums (2.3) for elastic media. Not all the M-sums participate in the effective tensor. For instance, the effective conductivity up to $O(\phi_2^5)$ contains eight M-sums: e_2, e_{22}, e_{33}, e_{222}, e_{44}, e_{322}, e_{223}, e_{2222}. For macroscopically isotropic composites $e_2 = \pi$ and many other M-sums are dependent [14]. This reduces the number of the basic elements to achieve the accuracy $O(\phi_2^4)$ to the following four M-sums:

$$e_{22}, \quad e_{33}, \quad e_{2222}, \quad e_{44}. \tag{2.5}$$

The M-sum (2, 2) can be calculated by (2.4). Explicit form of other M-sums (2.5) is given by the following formulae

$$e_{33} = \left(\frac{\pi}{\phi_2}\right)^4 \sum_{k_0,k_1,k_2} r_{k_0}^2 r_{k_1}^4 r_{k_2}^2 E_3(a_{k_0} - a_{k_1}) \overline{E_3(a_{k_1} - a_{k_2})}, \tag{2.6}$$

$$e_{2222} = \left(\frac{\pi}{\phi_2}\right)^5 \sum_{k_0,k_1,k_2,k_3,k_4} r_{k_0}^2 r_{k_1}^2 r_{k_2}^2 r_{k_3}^2 r_{k_4}^2 \times E_2(a_{k_0} - a_{k_1}) \overline{E_2(a_{k_1} - a_{k_2})} E_2(a_{k_2} - a_{k_3}) \overline{E_2(a_{k_3} - a_{k_4})}, \tag{2.7}$$

$$e_{44} = \left(\frac{\pi}{\phi_2}\right)^5 \sum_{k_0,k_1,k_2} r_{k_0}^2 r_{k_1}^6 r_{k_2}^2 E_4(a_{k_0} - a_{k_1}) \overline{E_4(a_{k_1} - a_{k_2})}. \tag{2.8}$$

[1] M-sum is short for Mityushev's sum.

Remark The M-sums (2.3), in particular (2.5), can be considered as the moments of the correlation functions [17]. Hence, the RVE theory [13] implicitly uses the correlation functions and do not requires their explicit computations.

3 Conclusion

In the present paper, we extend the RVE theory [13] to 2D composites with different circular inclusions. This theory can be applied to solution of the mixture problem. Namely, one can check, whether the M-sums (2.5) coincide with the theoretical ones computed in [4, 5]. If they do coincide with an appropriate accuracy, one can say that the inclusions are well stirred in the host.

One can investigate by the same method composites with inclusions having other shapes and compare the obtained M-sums with the corresponding theoretical M-sums which can be computed by methods presented in [4, 5]. This idea was presented and justified in [13].

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Random Non-overlapping Walks of Disks on the Plane

Wojciech Nawalaniec

Abstract The initial locations of disks on the plane form regular lattice. Random non-overlapping walks of disks approach to the uniform non-overlapping distribution of disks on the plane. The basic e-sums are computed in time to describe such a dynamic process within random walks.

Keywords e-Sums · Representative volume element · Composite material

1 Introduction

Consider two-dimensional two-component periodic composite made from a collection of non-overlapping, identical, circular disks, embedded in a matrix. In accordance with a theory of the representative cells (representative volume elements, RVE), the effective conductivity of disks is expressed in terms of the e-sums (see below formula (2.8)). It was established in [4] that the effective conductivity tensor Λ_e of the considered composites has the form of double series on the concentration of inclusions and on "basic elements" which depend only on locations of the inclusions. These basic elements are written in terms of the Eisenstein series. Coefficients in the double series depend on conductivity of constitutes. Two composites are equivalent if expansions of their Λ_e have the same basic elements. Therefore, the set of the composites with circular identical inclusions is divided onto classes of equivalence determined only by geometrical structure of the composite. Each composite is represented by a periodicity cell. In each class of equivalence a composite having the minimal size cell is chosen. This cell is called the representative cell of the considered class of equivalent composites.

This approach was used in [2] to simulate representative volume elements for random 2D composites with circular non-overlapping inclusions. A method of random walks to simulate random locations of inclusions with high concentrations was applied. The initial locations of disks were fixed in various periodical nodes: square,

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hexagonal and rectangular. The results showed that lower order e-sums which essentially impact on the effective conductivity tensor do not depend on the choice of the original location. But some higher order e-sums "remember" the initial location. This yield the hypothesis that the original regular location of inclusions can be restored for dispersed composites produced by stirring. This observation can be explained by the different scales of convergence of the e-sums when time of random walks tends to infinity. In the present paper, we consider particular e-sums as functions depending on *time* of random walk of disks in plane. Our considerations show the strong relation of e-sums to the geometrical structure of composites.

2 General Theory of *e*-Sums

Following [3, 4] we present constructive formulae for the Eisenstein–Rayleigh sums S_m and the Eisenstein functions $E_m(z)$ corresponding to lattice $\mathcal{Q}$. Let ω_1 and ω_2 be the fundamental pair of periods on the complex plane $\mathbb{C}$ such that $\operatorname{Im}\frac{\omega_2}{\omega_1} > 0$. The fundamental parallelogram Q is defined by its vertices $\pm\frac{\omega_1}{2}$ and $\pm\frac{\omega_2}{2}$. Without loss of generality the area of Q can be normalized to one. The points $m_1\omega_1 + m_2\omega_2$ $(m_1, m_2 \in \mathbb{Z})$ generates a doubly periodic lattice $\mathcal{Q}$. Here, $\mathbb{Z}$ stands for the set of integer numbers.

The Eisenstein–Rayleigh lattice sums S_m can be easily calculated through the rapidly convergent series

$$S_2 = \left(\frac{\pi}{\omega_1}\right)^2 \left(\frac{1}{3} - 8\sum_{m=1}^{\infty} \frac{mq^{2m}}{1-q^{2m}}\right), \quad \text{where } q = \exp\left(\pi i \frac{\omega_2}{\omega_1}\right), \tag{2.1}$$

$$S_4 = 60\left(\frac{\pi}{\omega_1}\right)^4 \left(\frac{4}{3} + 320\sum_{m=1}^{\infty} \frac{m^3 q^{2m}}{1-q^{2m}}\right), \tag{2.2}$$

$$S_6 = 1400\left(\frac{\pi}{\omega_1}\right)^6 \left(\frac{8}{27} - \frac{448}{3}\sum_{m=1}^{\infty} \frac{m^5 q^{2m}}{1-q^{2m}}\right). \tag{2.3}$$

S_{2n} $(n \geq 4)$ can be calculated by the recurrent formula

$$S_{2n} = \frac{3}{(2n+1)(2n-1)(n-3)} \sum_{m=2}^{n-2} (2m-1)(2n-2m-1) S_{2m} S_{2(n-m)}. \tag{2.4}$$

The rest sums vanish.

The Eisenstein functions [5] are related to the Weierstrass function $\wp(z)$ [1] by the identities

$$E_2(z) = \wp(z) + S_2, \qquad E_m(z) = \frac{(-1)^m}{(m-1)!} \frac{d^{m-2}\wp(z)}{dz^{m-2}}, \quad m = 3, 4, \ldots. \tag{2.5}$$

Every function (2.5) is doubly periodic and has a pole of order m at $z = 0$. The Eisenstein functions of the even order $E_{2m}(z)$ can be presented in the form of the series [5]

$$E_{2m}(z) = \frac{1}{z^{2m}} + \sum_{k=1}^{\infty} \sigma_k^{(m)} z^{2(k-1)}, \tag{2.6}$$

where

$$\sigma_k^{(m)} = \frac{(2m+2k-3)!}{(2m-1)!(2k-2)!} S_{2(m+k-1)}. \tag{2.7}$$

We follow [4] to introduce e-sums. Let a_k $(k = 1, 2, \dots, N)$ be a set of points. Let q be a positive integer; k_t runs over 1 to N; $m_j = 2, 3, \dots$. Let $\mathbf{C}$ be the operator of complex conjugation. Introduce the following sum of multi-index $(m_1, \dots, m_q)$

$$e_{m_1 \dots m_q} := N^{-[1+\frac{1}{2}(m_1+\dots+m_q)]} \sum_{k_0 k_1 \dots k_q} E_{m_1}(a_{k_0} - a_{k_1}) \times \overline{E_{m_2}(a_{k_1} - a_{k_2})} \dots \mathbf{C}^q E_{m_q}(a_{k_{q-1}} - a_{k_q}). \tag{2.8}$$

Here, it is assumed for convenience that

$$E_m(0) := S_m. \tag{2.9}$$

According to (2.8)–(2.9), e_m becomes the classical e-sum S_m in the case $N = 1$. The sums (2.8) constitute the basic elements to calculate the effective conductivity [2, 3] dependent only on the locations of inclusion.

3 Random Walk Model

Consider N non-overlapping circular disks D_k of radius r with the centres $a_k \in Q$ (see Fig. 1). Let D_0 be the complement of all closure disks $|z - a_k| \le r$ to the domain Q. We study conductivity of the doubly periodic composite when the host $\bigcup_{m_1,m_2}(D_0 + m_1\omega_1 + m_2\omega_2)$ and the inclusions $D_k + m_1\omega_1 + m_2\omega_2$ are occupied by conducting materials. It is assumed that inclusions are occupied by a perfect conductor.

The concentration of the inclusions has the form $\nu = N\pi r^2$. The centres a_k are considered as random variables distributed in such a way that the disks $D_k = \{z \in \mathbb{C} : |z - a_k| < r\}$ generate a set of uniformly distributed non-overlapping disks. Theoretically this distribution denoted below as $\mathcal{U}$ can be introduced as the distribution of the variable $\mathbf{a} = (a_1, a_2, \dots, a_N) \in Q^N$ with the restrictions $|a_m - a_k| > 2r$ for $m \ne k$ $(m, k = 1, 2, \dots, N)$. According to [2], $0 \le \nu \le \frac{\pi}{\sqrt{12}}$ where $\frac{\pi}{\sqrt{12}}$ is the maximal concentration attained for the hexagonal array. It is worth noting that the disks D_k belong to Q in the torus topology when the opposite sides of Q are identified.

A constructive description of the distribution $\mathcal{U}$ for high concentrations is based on random walks. Put the centres a_k onto the nodes of a regular array. Take a positive

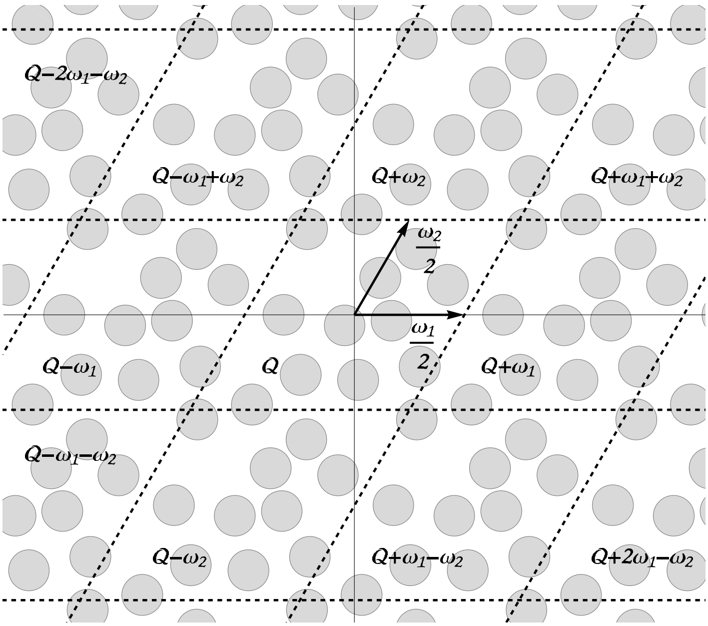


Fig. 1 Doubly periodic composite with inclusions $D_k + m_1\omega_1 + m_2\omega_2$ where $m_1, m_2 \in \mathbb{Z}$

number d less than $\min_{k\neq m} |a_k - a_m| - 2r$. Let each a_k moves in a randomly chosen direction $\phi_k \in [0, 2\pi)$ (i.e. program calculates range of possible move to avoid collisions) in the torus topology of Q. Then, each center obtain new complex coordinate $a'_k = a_k + de^{i\phi_k}$.

After sufficiently large number of the walks the obtained location of the centres can be considered as a statistical realization of the distribution $\mathcal{U}$. This method can be applied for arbitrary concentrations satisfying $0 \leq \nu \leq \frac{\pi}{\sqrt{12}}$.

If every center a_k move to a'_k with a randomly chosen direction, for all $k = 1, 2, \ldots, N$, we say that a cycle is performed. As a result, we have one ultimate location $\mathbf{a} = (a_1, a_2, \ldots, a_N)$. The time T of random walk is measured in number of cycles performed from the start.

Particular e-sums are considered as functions on T (i.e. numerical values of e-sums are calculated for every performed cycle). All computations are based on the fast algorithm presented in [2]. Below, the numerical computations are performed for the *square* and *hexagonal-like* lattices in Fig. 2, both in the square cell (i.e. $\omega_1 = 1$ and $\omega_2 = i$, where i denote the imaginary unit). The positions of inclusions at time $T = 1500$ of random walk are presented in Fig. 3.

4 Computation of *e*-Sums and Examples of Their Convergence

The *hexagonal-like* and square lattices are considered (see Fig. 2) with the concentration $\nu = 0.7$ as initial locations of the points a_k and further random walks

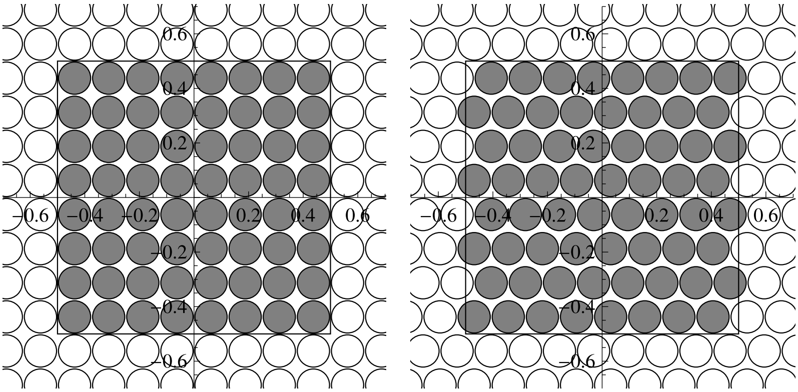


Fig. 2 Considered lattices for $N = 64$. The hexagonal-like lattice is obtained from square lattice by translating disks of every second row by vector $\tilde{v} = [\frac{\omega_1}{2\sqrt{N}}, 0]$

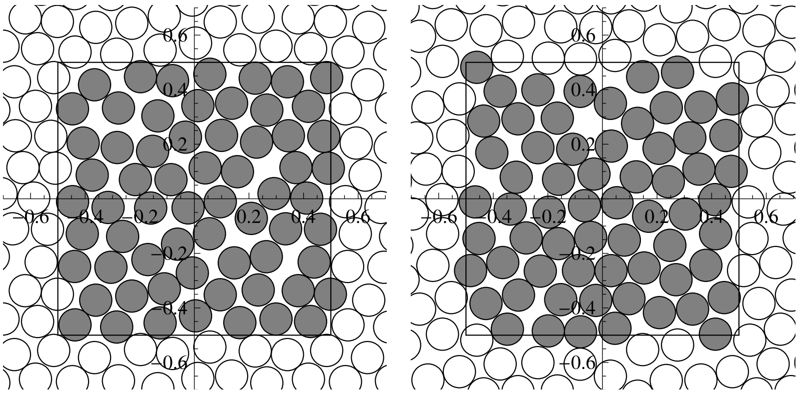


Fig. 3 Considered lattices for $N = 64$ and $T = 1500$. Square lattice is on the left; hexagonal-like lattice is on the right

described in Sect. 3. Both lattices contains $N = 64$ inclusions and are placed within the square cell. The values of the considered e-sums are calculated for positions of disks at each time T ($T = 0, 1, 2, \ldots, 1500$). The results are presented in Fig. 4. It is clear that beginning from the special value $T_0 = 100$, the considered e-sums are oscillating near the same value for both lattices. One can see (Figs. 5 and 6) that their values are different for T less about 100.

5 Discussion and Conclusion

One can consider the set $\{e_{m_1 \ldots m_q}, m_j = 2, 3, \ldots\}$ as a basis in the space of the deterministic or random locations of inclusions. This observation was used in [4] to

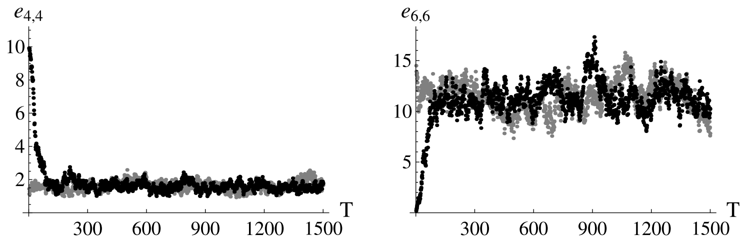


Fig. 4 Sums $e_{4,4}$ and $e_{6,6}$ as the functions of time of random walk T. *Black dots*—square lattice; *grey dots*—hexagonal-like lattice

Fig. 5 Sum $e_{4,4}$ as a function of time of random walk T. *Black dots*—square lattice; *grey dots*—hexagonal-like lattice

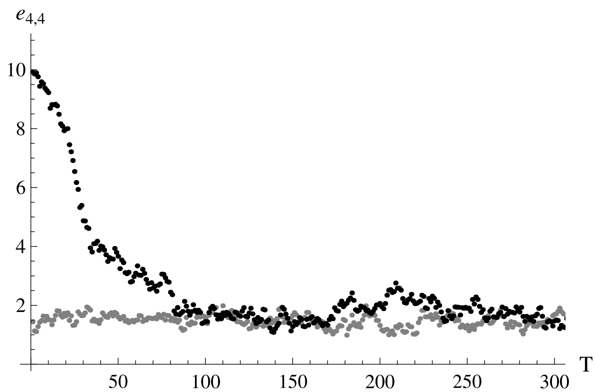
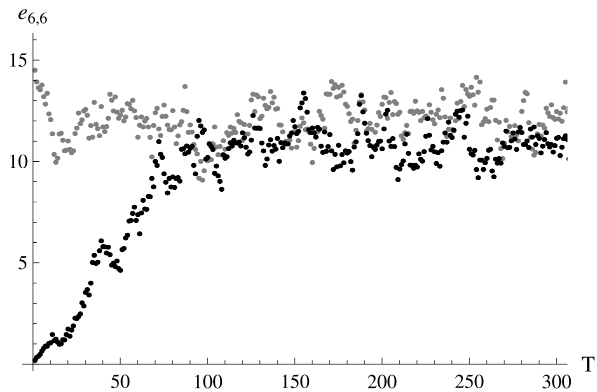


Fig. 6 Sum $e_{6,6}$ as a function of time of random walk T. *Black dots*—square lattice; *grey dots*—hexagonal-like lattice



create a constructive theory of RVEs and in [2] to simulate the RVEs for random 2D composites with circular non-overlapping inclusions. The convergence of e-sums reflects the *geometrical convergence* of cells to the one class of equivalent composites.

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Symbolic Computation of Conformal Mappings onto Slit Domains

Roman Czapla

Abstract A present paper, the symbolic implementation of conformal mappings of arbitrary circular multiply connected domains onto the complex plane with slits of prescribed inclinations is applied to examine of distribution of lengths of slits.

Keywords Riemann–Hilbert problem · Multiply connected domain · Complex plane with slits

1 Introduction

The study of the conformal mappings between multiply connected domains becomes more convenient by introducing the canonical domains and to study conformal mappings of arbitrary domains onto these canonical domains. Multiply connected domains on the extended complex plane whose boundaries consist of mutually disjoint circles form one of the most important class of the canonical domains. Domains bounded by mutually disjoint arbitrarily oriented slits are important in fracture mechanics. The method of Riemann–Hilbert problems [1] was used to unify and to simplify construction of conformal mappings of multiply connected domains. Authors derived the conformal mappings of arbitrary circular multiply connected domains onto the complex plane with slits of prescribed inclinations in terms of uniformly convergent Poincaré series (see below (3.7)). The method of successive approximations was applied to obtain the iterative formula (3.8). In the present paper, the symbolic implementation of (3.8) is presented. As an application, we examine the distribution of lengths of slits obtained in particular example.

2 Riemann–Hilbert and $\mathbb{R}$-Linear Problems

Let $z = x + iy$ denote a complex variable on the complex plane $\mathbb{C}$. Consider non-overlapping disks $\mathbb{D}_k = \{z \in \mathbb{C} : |z - a_k| < r_k\}$, $k = 1, 2, \ldots, n$. Let the boundary

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of $\mathbb{D}_k$, the circle $\partial\mathbb{D}_k$, is oriented in counterclockwise direction and D denote the complement of the closed disks $|z-a_k|\le r_k$ in the extended complex plane $\widehat{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$. Consider the second complex variable $\zeta=u+iv$ on the complex plane with slits Γ_k lying on the lines

$$-\sin\alpha_k u+\cos\alpha_k v=c_k, \tag{2.1}$$

where c_k are real constants. Let D' denote the complement of all the segments Γ_k to $\widehat{\mathbb{C}}$. Let $\zeta=\varphi(z)$ be a conformal mapping of the circular multiply connected domain D onto D', which transforms the circle $|z-a_k|=r_k$ to the slit Γ_k. For definiteness, it is assumed that $\varphi(z)$ satisfies the hydrodynamic normalisation at infinity

$$\varphi(z)=z+\varphi_0+\frac{\varphi_1}{z}+\frac{\varphi_2}{z^2}+\cdots. \tag{2.2}$$

Such a conformal mapping always exists and unique up to an arbitrary additive constant for the given inclinations α_k [3]. It follows from (2.1) that $\varphi(z)$ satisfies the following Riemann–Hilbert problem [4]

$$\operatorname{Im}\big[e^{-i\alpha_k}\varphi(t)\big]=c_k,\quad |t-a_k|=r_k,\ k=1,2,\dots,n, \tag{2.3}$$

where c_k are undetermined constants, Im stands for the imaginary part. The problem (2.3) with $c_k=0$ in classes of meromorphic functions were investigated in [6].

One can prove the following lemma [1].

Lemma 2.1 *The problem* (2.2)–(2.3) *has a unique solution up to an arbitrary additive constant.*

Remark 2.2 One can see that Lemma 2.1 is valid for an arbitrary multiply connected D with smooth boundary.

The problem (2.3) can be reduced to the $\mathbb{R}$-linear problem [5]

$$\varphi(t)=\varphi_k(t)+e^{2i\alpha_k}\overline{\varphi_k(t)}+ie^{i\alpha_k}c_k,\quad |t-a_k|=r_k,\ k=1,2,\dots,n, \tag{2.4}$$

where $\varphi_k(z)$ is analytic in $|z-a_k|<r_k$ and continuously differentiable in $|z-a_k|\le r_k$. Transforming (2.4) we obtain

$$\psi(t)=\psi_k(t)-e^{2i\alpha_k}\left(\frac{r_k}{t-a_k}\right)^2\overline{\psi_k(t)},\quad |t-a_k|=r_k,\ k=1,2,\dots,n, \tag{2.5}$$

where $\psi(z)=\varphi'(z)$ and $\psi_k(z)=\varphi_k'(z)$.

3 Functional Equations and Method of Successive Approximations

The $\mathbb{R}$-linear problem (2.5) can be reduced to functional equations. Following [4, 5] introduce the function

$$\Phi(z) := \begin{cases} \psi_k(z) + \sum_{m\neq k} e^{2i\alpha_m}(\frac{r_m}{z-a_m})^2\overline{\psi_m(z^*_{(m)})}, & |z-a_k|\leq r_k, \\ & k=1,2,\dots,n, \\ \psi(z) + \sum_{m=1}^{n} e^{2i\alpha_m}(\frac{r_m}{z-a_m})^2\overline{\psi_m(z^*_{(m)})}, & z\in D, \end{cases}$$

analytic in the domains $\mathbb{D}_k$ $(k=1,2,\dots,n)$ and D, where $z^*_{(m)} = \dfrac{r_m^2}{\overline{z-a_m}} + a_m$ denote the inversion with respect to the circle $|t-a_m| = r_m$.

Calculate the jump across the circle $|t-a_k| = r_k$

$$\Delta_k := \Phi^+(t) - \Phi^-(t), \quad |t-a_k| = r_k,$$

where $\Phi^+(t) := \lim_{z\to t z\in D}\Phi(z)$, $\Phi^-(t) := \lim_{z\to t z\in D_k}\Phi(z)$. Using (2.5) we get $\Delta_k = 0$. It follows from the principle of analytic continuation that $\Phi(z)$ is analytic in the extended complex plane. Moreover, $\psi(\infty) = \varphi'(\infty) = 1$ yields $\Phi(\infty) = 1$. Then Liouville's theorem implies that $\Phi(z)\equiv 1$. The definition of $\Phi(z)$ in $|z-a_k|\leq r_k$ yields the following system of functional equations

$$\psi_k(z) = -\sum_{m\neq k} e^{2i\alpha_m}\left(\frac{r_m}{z-a_m}\right)^2 \overline{\psi_m\left(z^*_{(m)}\right)} + 1,$$
$$|z-a_k|\leq r_k,\ k=1,2,\dots,n. \tag{3.1}$$

Let $\psi_k(z)$ $(k=1,2,\dots,n)$ be a solution of (3.1). Then the function $\psi(z)$ can be found from the definition of $\Phi(z)$ in D

$$\psi(z) = -\sum_{m=1}^{n} e^{2i\alpha_m}\left(\frac{r_m}{z-a_m}\right)^2 \overline{\psi_m\left(z^*_{(m)}\right)} + 1, \quad z\in D\cup\partial D. \tag{3.2}$$

Was proved in [1] that there is convergence of the method of successive approximations applied to the system (3.1).

Let $\psi_k(z)$ be a solution to the system of functional equations (3.1). Let $w\in D$ be a fixed point not equal to infinity. Introduce the functions

$$\phi_m(z) = \int_{w^*_{(m)}}^{z} \psi_m(t)dt + \phi_m\left(w^*_{(m)}\right), \quad |z-a_m|\leq r_m,\ m=1,2,\dots,n, \tag{3.3}$$

and

$$\omega(z) = \sum_{m=1}^{n} e^{2i\alpha_m}\left[\overline{\phi_m\left(z^*_{(m)}\right)} - \overline{\phi_m\left(w^*_{(m)}\right)}\right]. \tag{3.4}$$

One can prove, inter alia by Lemma 2.1, that the required conformal mapping has the form

$$\varphi(z) = z + \omega(z) + constant, \tag{3.5}$$

where $\omega(z)$ is calculated by (3.4). Application of the method of successive approximations to (3.1) and integration terms by terms of the obtained uniformly convergent series yields the exact formula

$$\begin{aligned} \varphi_k(z) = q_k + z &+ \sum_{k_1 \neq k} e^{2i\alpha_{k_1}} \overline{\left(z^*_{(k_1)} - w^*_{(k_1)}\right)} \\ &+ \sum_{k_1 \neq k} \sum_{k_2 \neq k_1} e^{2i(\alpha_{k_1} - \alpha_{k_2})} \left(z^*_{(k_2 k_1)} - w^*_{(k_2 k_1)}\right) \\ &+ \sum_{k_1 \neq k} \sum_{k_2 \neq k_1} \sum_{k_3 \neq k_2} e^{2i(\alpha_{k_1} - \alpha_{k_2} + \alpha_{k_3})} \overline{\left(z^*_{(k_3 k_2 k_1)} - w^*_{(k_3 k_2 k_1)}\right)} + \cdots, \\ |z - a_k| \leq r_k. \end{aligned} \tag{3.6}$$

Using (3.4) and (3.6) we write the function (3.5) up to an arbitrary additive constant in the form

$$\begin{aligned} \varphi(z) = z &+ \sum_{k=1}^{n} e^{2i\alpha_k} \overline{\left(z^*_{(k)} - w^*_{(k)}\right)} \\ &+ \sum_{k=1}^{n} \sum_{k_1 \neq k} e^{2i(\alpha_k - \alpha_{k_1})} \left(z^*_{(k_1 k)} - w^*_{(k_1 k)}\right) \\ &+ \sum_{k=1}^{n} \sum_{k_1 \neq k} \sum_{k_2 \neq k_1} e^{2i(\alpha_k - \alpha_{k_1} + \alpha_{k_2})} \overline{\left(z^*_{(k_2 k_1 k)} - w^*_{(k_2 k_1 k)}\right)} + \cdots. \end{aligned} \tag{3.7}$$

We use another implementation based on the functional equations (3.1). The method of successive approximations is applied to (3.1). We start with the initial guess, $\psi_k^{(0)} = 1$, $k = 1, 2, \ldots, n$. The iteration is then given by

$$\psi_k^{(\text{it}+1)}(z) = -\sum_{m \neq k} e^{2i\alpha_m} \left(\frac{r_m}{z - a_m}\right)^2 \overline{\psi_m^{(\text{it})}\left(z^*_{(m)}\right)} + 1, \tag{3.8}$$

for $|z - a_k| \leq r_k$, $k = 1, 2, \ldots, n$. The approximations converge uniformly for any location of non-overlapping disks. Further, the function $\psi(z)$ is constructed by (3.2). The conformal mapping $\varphi(z)$ is constructed by integration of $\psi(z)$.

Table 1 Considered random centers an inclinations

a_k	α_k
$-0.1435-0.0946i$	-0.8843
$0.1519+0.2254i$	1.1627
$0.2586-0.2910i$	-1.3488
$-0.3058+0.2416i$	0.3787

Now, we implement the iterative formula (3.8) in *Mathematica*®:

```
ψ[0, k_Integer, R_, z_, α_, A_] := (ψ[0, k, R, z, α, A] = 1) /; 0 < k ≤ n

Inv[m_Integer, R_, z_, A_] := (Inv[m, z] = R[[m]]^2/(z - A[[m]])* + A[[m]]) /; 0 < m ≤ n

MT[z_, R_, A_] := Min[Table[Abs[z - A[[k]] - R[[k]]], {k, 1, n}]]

ψ[Step_Integer?Positive, k_Integer, R_, z_, α_, A_] := (ψ[Step, k, R, z, α, A] =
    1 - Sum[{ 0                                                                       m == k
              e^(2 i α[[m]]) (R[[m]]/(z-A[[m]]))^2 ψ[Step - 1, m, R, Inv[m, R, z, A], α, A]*   True
            }, {m, 1, n}]) /; 0 < k ≤ n

Ψ1[Step_, R_, z_, α_, A_] := Ψ1[Step, R, z, α, A] =
    -Sum[e^(2 i α[[m]]) (R[[m]]/(z - A[[m]]))^2 ψ[Step, m, R, Inv[m, R, z, A], α, A]*, {m, 1, n}] + 1;

Ψ[Step_, R_, z_, α_, A_] := Ψ[Step, R, z, α, A] = If[MT[z, R, A] < 0, 0, Ψ1[Step, R, z, α, A]]
ΨA[Step_, R_, z_, α_, A_] := ΨA[Step, R, z, α, A] = Apart[Ψ1[Step, R, z, α, A] // N] // Chop

IΨA[Step_, R_, w_, α_, A_] := IΨA[Step, R, z, α, A] = ∫ΨA[Step, R, z, α, A] dz /. z → w;

def[rN_, αN_, B_, {it_}] := Module[{}, Clear[f]; f[it][z_] = IΨA[it, rN, z, αN, B];]
```

For example, let a_k and α_k be defined by Table 1, formula (3.8) takes the following form (3.9).

$$\begin{aligned}\varphi(z) = z &- \frac{0.0287442+0.0136743i}{1.z-(0.258645-0.291009i)} - \frac{0.00035354-0.003626i}{z-(0.246426-0.2319i)}\\ &- \frac{0.000424771+0.00162785i}{z-(0.228812-0.26286i)} + \frac{0.00133436+0.00487988i}{z-(0.194735-0.259791i)}\\ &+ \frac{0.00239752-0.00274351i}{z-(0.164111+0.166303i)} - \frac{0.0218029-0.0231915i}{z-(0.151891+0.225414i)}\\ &- \frac{0.00458698-0.00273962i}{z-(0.102315+0.171702i)} - \frac{0.000354363+0.00481762i}{z-(0.0824317+0.227868i)}\\ &- \frac{0.00505799+0.0001021i}{z+(0.0795605+0.125805i)} + \frac{0.00409164+0.0034358i}{z+(0.0938938+0.040876i)}\\ &- \frac{0.00625544+0.0312103i}{z+(0.14347+0.0945875i)} - \frac{0.000402701+0.00725921i}{z+(0.180548+0.0178048i)}\\ &+ \frac{0.000327288+0.00481953i}{z+(0.236346-0.239127i)} - \frac{0.00697891+0.00203792i}{z+(0.268727-0.164798i)}\\ &+ \frac{0.00061054-0.0015677i}{z+(0.275972-0.213432i)} + \frac{0.0231302+0.0218679i}{z+(0.305805-0.241581i)} \end{aligned} \tag{3.9}$$

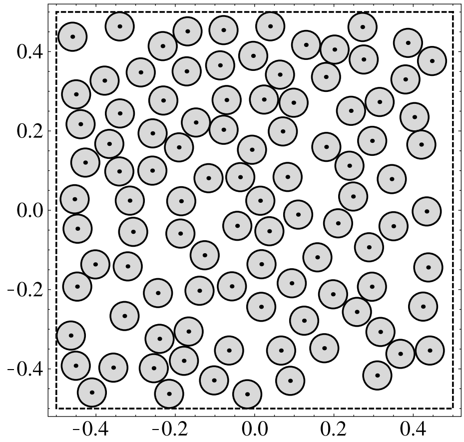

Fig. 1 Unit square with randomly distributed inclusions

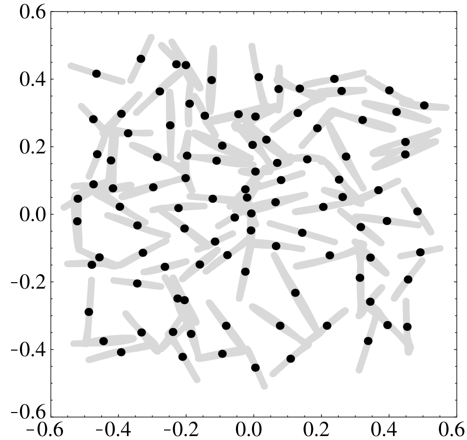

Fig. 2 Plane with slits of the inclinations randomly chosen on $(-\frac{\pi}{2}, \frac{\pi}{2})$

4 Numerical Example

Let Q be the unit square (Fig. 1). Consider 100 non-overlapping circular disks D_k of radius $r = 0.0356825$ randomly distributed in Q (see the Method I described in Chap. 2 of [2]).

Conformal mapping of the exterior of considered disks onto the plane with slits of the inclinations randomly chosen on $(-\frac{\pi}{2}, \frac{\pi}{2})$ is presented in Fig. 2.

The histogram in Fig. 3 presents the distribution of lengths of obtained slits. The values seem to be normally distributed. Note that the lengths of the slits are greater than $2r$.

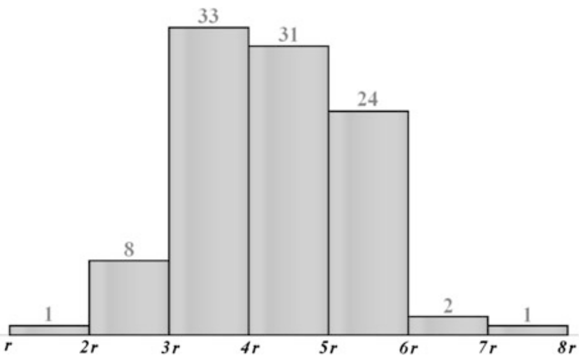

Fig. 3 Distrubutions lengths of the slits ($r = 0.0356825$)

5 Conclusion

In this symbolic implementation (3.8), the cost of each iteration increases rapidly as the number of iterations increases. A preliminary numerical method, similar to [7, Sect. 12.4], has been implemented in MATLAB. It is much faster than the symbolic calculation, but does not yield analytic formulas like (3.9).

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On One Approach to the Simulation of the Periodontal Ligament Takes into Account Its Viscoelastic Properties

K. Yurkevich and S. Bosiakov

Abstract The mathematical model of the viscoelastic periodontal ligament is presented. The relaxation kernel corresponds to the Maxwell model. Model describes the viscoelastic deformations of the periodontal membrane and the tooth movements. Analysis of premolar root movements in the form of an elliptical hyperboloid under the vertical load is performed.

Keywords Viscoelastic periodontal ligament · Elliptical hyperboloid · Tooth root displacements · Maxwell's kernel

1 Introduction

Periodontal ligament is a thin membrane that holds the tooth root in the alveolar bone, absorbs and distributes the occlusal forces on the tooth by the collagen fibers. Under normal conditions the contact between the tooth root and the bone tissue is absent. The load acting on the crown of the tooth is transferred to the alveolar bone through the periodontal ligament. Short-term (initial) and long-term (orthodontic) tooth displacements are regulated by strains and stresses of the periodontal ligament, because teeth and alveolar bone are considered almost completely rigid [1–5].

Adequately describe the function of the periodontal tissues without using of simple or too complex mathematical models allow the viscoelastic equations [6, 7]. In particular, the viscoelastic model allows to avoid differences between the physiological and calculated stresses in periodontal tissues. Also this model allows to explain the dependence of the physiological response of the periodontal tissue to the action of the load on the time and to combine movement of unsteady viscous liquid phase with deformation of rigid body [8, 9].

Known existing viscoelastic model are based on the use of single Maxwell's element [10], Kelvin–Voight model (parallel connected springs and shock absorber)

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[11, 12], nonlinear springs with three parameters [13]. Attempts to run the simulation of the periodontal ligament using a linear viscoelastic law are undertaken in the studies [14, 15]. Nevertheless, in [16] it is shown that the simulation of the periodontium nonlinear properties provides a more accurate and reliable calculation of stresses and strains in a wide range of the tooth movements. Some authors demonstrated the viscoelastic behavior of human and primate periodontal ligaments, but not offered a quantitative description [17–20]. The enhanced approach to the study of the mechanical behavior of the periodontium based on the quasi-linear viscoelastic phenomenological model is proposed in [21]. However the results obtained here been challenged since the nonlinear behavior of periodontium may not be well described by a quasi-linear viscoelastic theory which is usually used to describe the biomechanics of tissue [22]. The most important results related to the finite element calculation of viscoelastic models of the "tooth root—periodontal ligament" are represented in [23–26].

A review of the results concerning the use of viscoelastic model for describing the periodontal ligament's behavior shows that precise information about relationships between the viscoelastic response and periodontium structure, as well as uniform approach to the description of the periodontal ligament properties are absent. A lot of studies on the properties and behavior of the periodontal membrane still interprets its as linear elastic material. The aim of this work is the formulation of the motion equations of the viscoelastic periodontal ligament in the shape of elliptic hyperboloid.

2 Material and Methods

Let us assume that external and internal periodontal surfaces are defined by equations of the elliptical hyperboloids $F_0(x_1, x_2, x_3)$ and $F(x_1, x_2, x_3)$, respectively:

$$F_0(x_1, x_2, x_3) = F(x_1, x_2, x_3) + h_0 = 0, \tag{2.1}$$

$$F(x_1, x_2, x_3) = x_3 - \frac{H}{\sqrt{1+p^2} - p}\left(\sqrt{\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + p^2} - p\right) = 0, \tag{2.2}$$

where H is the height of the root; p is the parameter characterizing the rounding of the root; a and b are semiaxes of the ellipse in the cross-section of the tooth root near the alveolar crest; h_0 is the parameter characterizing the thickness of periodontal ligament; the surface $F_0(x_1, x_2, x_3)$ restricts the periodontal surface from the alveolar bone; the surface $F(x_1, x_2, x_3)$ limits the surface of the periodontal ligament from the side of tooth root (see Fig. 1).

Any displacement of the root (and points on the internal surface of the periodontal ligament) can be described as a combination of a translation of the apex and a rotation around the apex of the root. The displacements of points on the external surface of the periodontal ligament are equal to zero, because the periodontium

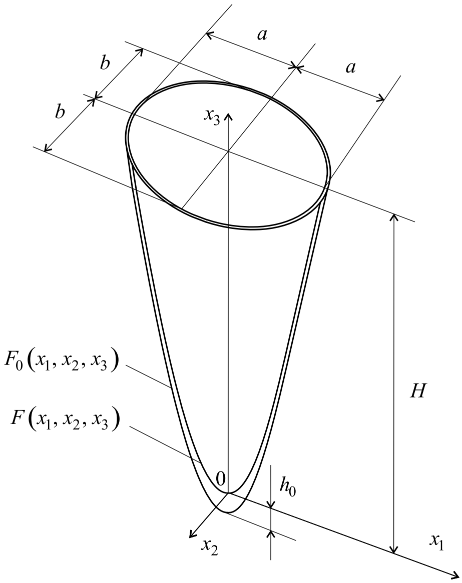

Fig. 1 The geometrical parameters of the tooth root

is rigidly fixed in alveolar bone. Taking this into account the displacements of the periodontal ligament are represented as follows:

$$\mathbf{u}(\mathbf{r}, t) = \frac{1}{h_0}\big(F(x_1, x_2, x_3) + h_0\big)\big(\mathbf{u}^{(0)}(t) + \boldsymbol{\varphi}(t) \times \mathbf{r}\big). \tag{2.3}$$

Here $\mathbf{u}^{(0)}(t)$ is the translational displacement vector of tooth root; $\boldsymbol{\varphi}(t)$ is the rotational angle vector of tooth root; $\mathbf{r} = (x_1, x_2, x_3)$ is the radius-vector. Geometric parameters of tooth root and the coordinate system are shown in Fig. 1.

The relationship between stresses and strains for the periodontal membrane takes into account their viscoelastic properties written as follows:

$$\sigma_{ij}(\mathbf{r}, t) = 2G\Bigg(e_{ij}(\mathbf{r}, t) - \int_0^t K(t-\tau)e_{ij}(\mathbf{r}, \tau)d\tau + \frac{\nu\delta_{ij}}{1-2\nu}\Bigg(\sum_{k=1}^{3} e_{kk}(\mathbf{r}, t) - \int_0^t K(t-\tau)\sum_{k=1}^{3} e_{kk}(\mathbf{r}, \tau)d\tau\Bigg)\Bigg), \quad i, j = \overline{1, 3}, \tag{2.4}$$

where G is the shear modulus; ν is Poisson's ratio; $K(t)$ is the relaxation kernel for volume and shear stresses; δ_{ij} is Kronecker delta.

The components of the strain tensor have the following form:

$$e_{ij}(\mathbf{r}, t) = \frac{1}{2}\left(\frac{\partial u_i(\mathbf{r}, t)}{\partial x_j} + \frac{\partial u_j(\mathbf{r}, t)}{\partial x_i}\right), \quad i, j = \overline{1, 3}. \tag{2.5}$$

Let's substitute the relations (2.5) to (2.4), and then insert the resulting expressions in the equations of motion:

$$\iint_F \big(\mathbf{n}\cdot\sigma(\mathbf{r},t)\big)dF + M\frac{d^2\mathbf{u}^{(0)}(t)}{dt^2} - \mathbf{P} = 0,$$
$$\iint_F \mathbf{r}\times\big(\mathbf{n}\cdot\sigma(\mathbf{r},t)\big)dF + J\frac{d^2\boldsymbol{\varphi}(t)}{dt^2} - \mathbf{m} = 0, \quad (2.6)$$

where $\mathbf{m} = (m_1, m_2, m_3)$ is the principal moment of the external forces; $\mathbf{P} = (P_1, P_2, P_3)$ is the principal vector of the external forces; $\mathbf{n} = (n_1, n_2, n_3)$ is the unit normal vector to the surface $F(x_1, x_2, x_3)$; $\sigma(\mathbf{r}, t)$ is the stress tensor whose components defined by the expressions (2.4); M is the mass of the tooth; J is the axial moment of inertia of the tooth.

3 Tooth Translation (Extrusion)

Let's consider one of the clinical cases of orthodontic movements of tooth, when the tooth displacement is only vertical, in particular by extrusion. During translation along the vertical axis under the force $P = P_3$ from system (2.6) we have ($P_1 = P_2 = 0$, the translational displacements $u_1^{(0)}$ and $u_2^{(0)}$, as well as all rotation angles are equal to zero):

$$c_3\left(u_3^{(0)}(t) - \int_0^t K(t-\tau)u_3^{(0)}(\tau)d\tau\right) + M\frac{d^2u_3^{(0)}(t)}{dt^2} = P,$$
$$c_3 = \frac{2aG\pi(b^2H^2 + a^2(H^2 + 2\gamma r_1 b^2) - H^2p^2r_2(a^2+b^2))}{4a^2bh_0r_1},$$
$$r_1 = \big(\sqrt{1+p^2} - p\big)^2,\ r_2 = \ln\left(\frac{1}{p^2}+1\right),\ \gamma = \frac{2(1-\nu)}{1-2\nu}, \quad (3.1)$$

where c_3 is the stiffness of the periodontal ligament.

Let's find the solution of (3.1) for Maxwell's kernel $K(t) = A\exp^{-bt}$, $(A > 0, b > 0)$ with the initial conditions corresponding to the absence of initial velocity. The initial displacement was determined in accordance with the previously developed linear elastic model of the periodontal ligament [27]. As a result one obtains:

$$u_3^{(0)}(t) = \frac{Pb}{c_3(b-A)} + \frac{PA}{c_3m(A-b)}\left(\frac{\exp(p_1t)(c_z + Mp_1(b+p_1))}{(p_1-p_2)(p_1-p_3)}\right.$$
$$\left. + \frac{\exp(p_2t)(c_3 + Mp_2(b+p_2))}{(p_2-p_1)(p_2-p_3)} + \frac{\exp(p_3t)(c_3 + Mp_3(b+p_3))}{(p_3-p_1)(p_3-p_2)}\right),$$
$$p_1 = -\frac{b}{3} + \alpha + \beta,\ p_2 = -\frac{b}{3} - \frac{1}{2}(\alpha+\beta) + \frac{i\sqrt{3}}{2}(\alpha-\beta),$$

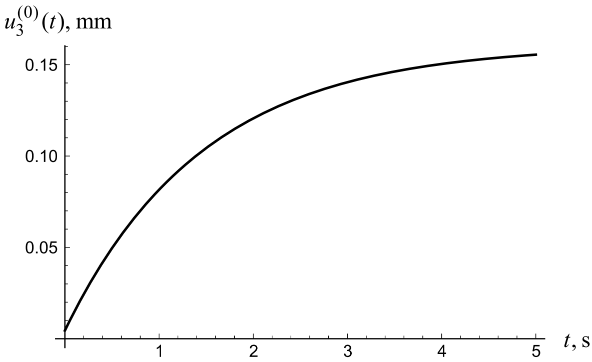

Fig. 2 The dependence of the tooth root translation in the viscoelastic periodontal ligament in the vertical axis direction

$$p_3 = -\frac{b}{3} - \frac{1}{2}(\alpha + \beta) - \frac{i\sqrt{3}}{2}(\alpha - \beta),\ \alpha = \sqrt[3]{-\frac{q_2}{2} + \sqrt{\left(\frac{q_2}{2}\right)^2 + \left(\frac{q_1}{2}\right)^3}},$$

$$\beta = -\frac{q_1}{3\alpha},\ q_1 = \frac{c_3}{M} - \frac{b^2}{3},\ q_2 = \frac{2b^3}{27} - \frac{Ac_3}{M} + \frac{2bc_3}{3M}. \tag{3.2}$$

Relaxation kernel parameters are defined based on results of the clinical observations of the periodontal membrane deformations. It is assumed that the absolute maximum deformation of the periodontal ligament under constant force is $h_1 \approx 2h_0/3$ (h_0 is the thickness of the periodontal ligament at the apex). If the thickness h_0 is 0.25 mm [28] then displacement h_1 of the tooth root approximately is 0.17 mm. The calculation of the kernel parameters based on the solution of (3.2) for a premolar ($a = 5$ mm, $b = 3.5$ mm, $H = 14.3$ mm, $p = 0.4$) under vertical load $P = 100$ N shows that $A = 23.80$ 1/s, $b = 24.48$ 1/s. Elasticity modulus of the periodontal ligament is 10 MPa, Poisson's ratio is 0.45 [29]. The stiffness c_3 is 22.4 MN/m and the mass of premolar is 1 gram. Figure 2 shows the dependence of the premolar's root movement in the periodontal ligament from time for given geometrical, physical and mechanical properties, the time variable varies from 0 to 5 seconds.

As shown in Fig. 2, with further change of time translational movement takes almost constant value approximately is 0.16 mm. Using found values for the parameters of relaxation kernel the dependence of stresses from time can be determined.

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Biomechanical Effects of Maxillary Expansion in Cross-Bite Patients During Orthodontic Treatment with Hyrax Screw

S. Bosiakov, A. Vinokurova, and A. Dosta

Abstract The aim of this study was finite element analysis of stress-strain state of the human maxillary complex with and without cleft palate. Loading the skull is carried out by activating orthodontic device HYRAX. Model of the skull and supporting teeth of upper jaw obtained on the basis of tomographic data for dry intact skull of an adult. Design of orthodontic device differ position of screws and rods relative to the palate. Equivalent stresses in the bones of the craniofacial complex are assessed. It is shown that large stresses occur in the maxillary complex, if the screw and rods of orthodontic devices are located in a horizontal plane for skull with and without cleft. Also in the intact skull big stresses appear in the bone of the upper jaw with location of the screw and rods of orthodontic device in a horizontal plane. In the rest of the skull bones stresses are insignificant. By moving the device screw to the palate the values of maximum stresses are reduced, but the region of big stresses displaced to the pterygoid plate and pharyngeal tubercle. In the skull with cleft for different positions of screws and rods orthodontic device the upper jaw is loaded fragmentary. High stresses are observed in the region of the maxilla near the zygomatic arches and along the edges of eye-sockets. When placing screw of orthodontic device close to palate the stresses decreases, but are observed in most part of the zygomatic arches.

Keywords Maxillary expansion · Orthodontic device Hyrax · Craniofacial complex · Cross-bite · Palate cleft · Finite-element modeling

Mathematics Subject Classification (2010) Primary 99Z99 · Secondary 00A00

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1 Introduction

Correction of cross-bite, usually performed using transverse forces that create a fixed or removable orthodontic appliances [1]. It allows to increase the maxillary transverse dimension, but is sufficiently a complicated procedure that causes significant side effects, in particular dislocation of the supporting teeth, fenestration of the cortical plate, root resorption and gingival recession [2]. The maxillary expansion may be associated with a pressure in the various areas of its joint, for example in the zygomatic bones and bridge of the nose, and may also be accompanied by complications [3]. To better understand the effects of the rapid expansion of the maxillary complex was carried out a lot of finite-element studies [4–14]. At the same time, in most of these studies don't considered the real design of the orthodontic appliance. Loading is performed through the application of the transversal concentrated forces to the teeth in particular the molars and premolars (from 1 N to 300 N) [4, 5, 9, 10, 12], or by displacement of teeth by a distance corresponding some number of revolutions of orthodontic screw (from 0.2 mm to 5.0 mm) [7, 10, 11, 13, 14]. Such simplified approaches do not represent real situation and give inaccurate results in the simulation of the maxillary expansion [6]. The aim of this paper is to analyze the influence of design of the orthodontic appliance HYRAX on the stress-strain state of the intact human skull and skull with unilateral cleft, which occurs after the activation of the device.

2 Model the Skull and Orthodontic Device

Stereolithography model format of the skull was obtained using the program for medical imaging MIMICS 14.12 (Materialise's Interactive Medical Image Control Systems, Materialise BV, Leuven, Belgium) on the basis of 210 tomographic images of the dried cadaveric intact skull of an adult. Note that for the development of the skull model enough tomography data obtained from of dry human skull [9–11]. In addition, under the action of orthodontic forces the initial reaction of dry skull and skull in vivo are similar [15]. Step tomographic slices is equal to 1 mm. Finite element model is obtained after processing model in MIMICS 3-matic 6.1. Teeth of the maxilla (first and second premolars, permanent molars), on which established orthodontic device were removed when generating stereolithographic format. Discrete model of the skull contains 26445 nodes and 91731 elements like Solid72. Finite element partition of is performed automatically. Simulation of the periodontal ligament was not carried out, as it has virtually no effect on the stress distribution in the bones of the maxillofacial complex under the action of loads of different types [16]. Sutures in the craniofacial complex also not taken into account in the finite element model. This is because in the adult the skull sutures partially or fully are ossified [7].

Solid models premolar and first molar of the upper dental row also obtained on the basis of tomographic data of the human skull. For this SolidWorks 2010

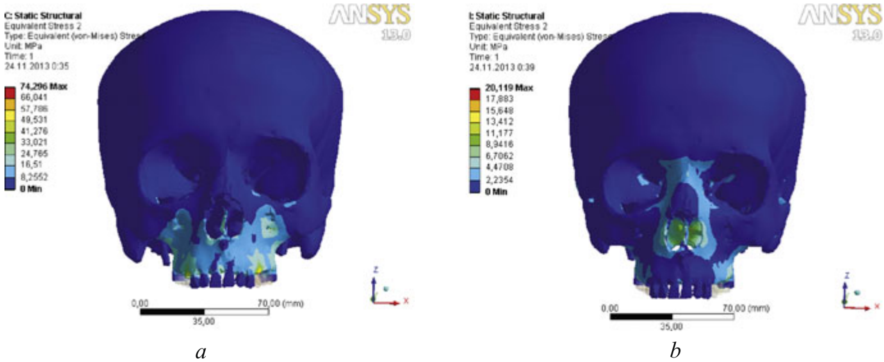


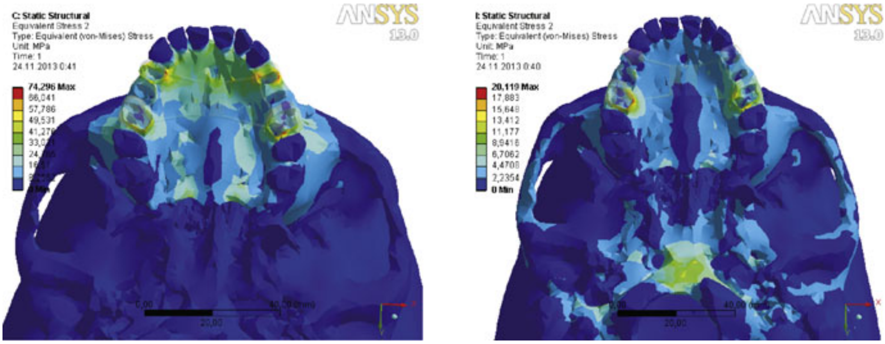
Fig. 1 Distribution of equivalent stresses in the intact skull (frontal view): (**a**) is skull with horizontal position of the orthodontic device; (**b**) is skull with screw higher on 8 mm of the horizontal position

(SolidWorks Corporation, USA) is used. Model orthodontic device HYRAX using graphical primitives of this package is constructed. Crowns device are installed on the first permanent molars and premolars. The length and width of the plates in a model orthodontic device is 4.0 mm and 10.0 mm respectively, the radius of the cross section is 1.0 mm rods, thickness of crown is 0.2 mm.

Boundary conditions correspond to anchorage of nodes near the foramen magnum [4, 6]. Displacement of each plate is 0.25 mm (corresponding to the activation screw of orthodontic device a half turn [5, 12]) and transversally is directed (in the direction of the axis $0x$). Modulus of elasticity material plates and rods of the orthodontic device is 200 GPa, Poisson's ratio is 0.3. The elastic modulus of cortical bone and teeth equal to 15 GPa and 20 GPa, Poisson's ratio for the cortical bone and teeth equal to 0.3. Finite element analysis of the stress-strain state of a skull with orthodontic appliances is carried out for different locations of the screw and rods. In one design, the screw and rod of the device were located in a horizontal plane. In other configurations, the screw of device was located on the 0.5, 1, 2 and 8 mm above relative to the horizontal construction of the orthodontic appliance. Geometric dimensions of orthodontic appliances, except for the lengths of rods, are not changed. Rods lengths between the plates and crowns on the first premolars is vary from 8.15 mm to 12.20 mm, length of rods between the plates and the molars is changes from 11.05 mm to 16.45 mm.

3 Stress-Strain State of the Intact Skull

Figures 1 and 2 show the distribution of the equivalent stresses in the front part and in the base of the skull respectively for two different designs of orthodontic device. Here and further, the case a corresponds to the design of the device with horizontal position of screw and rods; case b meets the construction in which the screw is shifted by 8 mm closer to the palate relative to first case.



a *b*

Fig. 2 Distribution of equivalent stresses in the intact skull (bottom view): (**a**) is skull with horizontal position of the orthodontic device; (**b**) is skull with screw higher on 8 mm of the horizontal position

From distributions of stress is seen (see Figs. 1a and 2a) that significant stresses occur mainly in the maxilla, the maximum equivalent stress is equal ≈74.3 MPa. Also high stresses are observed in the middle and lower part of nasal concha. Stresses are present at the bottom of the left orbit. Figure 1b shows that when installing the screw device in the palate, the maxilla is loaded fragmentary, and most equivalent stress is ≈20.12 MPa. In particular, the stresses are observed in the frontal area and the alveolar processes region of the maxilla and also in the frontonasal suture. Insignificant stresses arise in the sphenoid and occipital bones.

As Fig. 2b shows, the palatal suture loaded slightly. Thus for rapid expansion of the palatal suture the case a is more preferable than case b. Note that the maximum stresses in the case of a and b occur in the bone tissue around teeth on which the orthodontic device is installed. This is confirmed by the distribution of equivalent stress at the base of the skull (see Figs. 2a, b). At the same time, the skull bone differently are loaded in cases a and b. In case a, large stresses occur in incisal bone (33.02 MPa), the maxilla and the palatal bone (8.26–33.02 MPa). In case b, stress in the maxilla (6.71 MPa) is less than stress in the occipital bone, particularly in the region of the foramen magnum (11.18 MPa). Besides, in the case b the non-zero stress in the zygomatic arcs is observed, as well as in the pterygoid plate.

4 Stress-Strain State of Maxilla with Cleft Palate

Figures 3 and 4 shows the distribution of equivalent stresses in the front part (see Figs. 3a and 4a) and in the base (see Figs. 3b and 4b) of the skull with a cleft for two different designs orthodontic device.

Figures 3 and 4 show that the maximum stresses in the skull with palate cleft are occur when placing of the screw and rod in a horizontal plane. At the same time, maximum stresses in the skull with a cleft (≈69.1 MPa, see Figs. 3, 4a) is less than

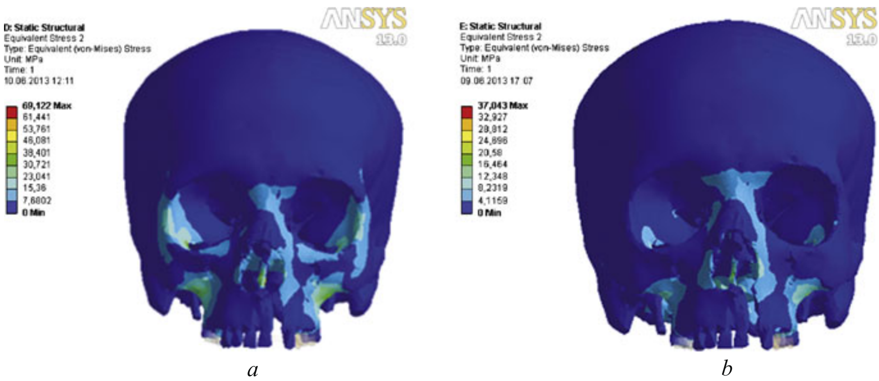


Fig. 3 Distribution of equivalent stresses in the skull with cleft palate (frontal view): (**a**) is skull with horizontal position of the orthodontic device; (**b**) is skull with screw higher on 8 mm of the horizontal position

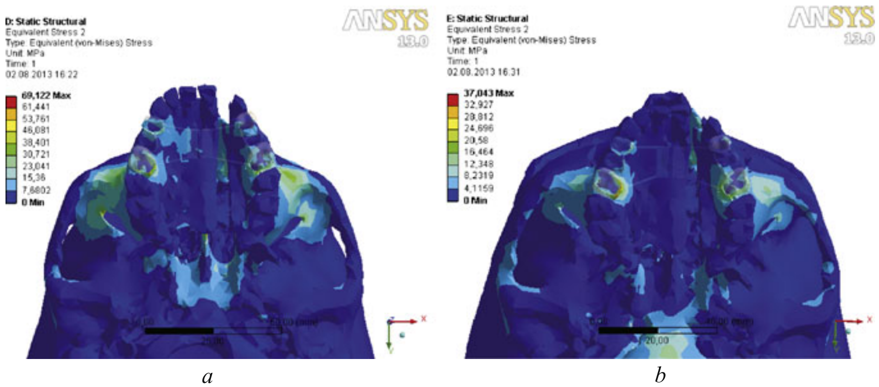


Fig. 4 Distribution of equivalent stresses in the skull with cleft palate (frontal view): (**a**) is skull with horizontal position of the orthodontic device; (**b**) is skull with screw higher on 8 mm of the horizontal position

the maximum stress in the intact skull (≈74.3 MPa, see Figs. 1, 2a). Conversely, in the case b for skull with cleft the maximum stress increases (≈37.0 MPa compared to ≈20.1 MPa, see Figs. 1–4b). Like for the intact skull, the maximum stresses in the skull with palate cleft are observed in the bone tissue of maxilla adjacent to the supporting teeth. Along with the change of stresses can observe significant qualitative differences in the distribution of stresses in cases a and b for the intact skull and skull with cleft. In particular, in the skull with a cleft palate the region of the median palatal suture practically is not loaded during activating the orthodontic device (see Figs. 4a, b). In the case a (see Fig. 4a) can be observed insignificant stresses (less than 15.4 MPa) in pterygoid plate. More great stresses appear in the eye-sockets and in the zygomatic arches (less than 46.1 MPa). In case b the region of stress in the skull base shifts from pterygoid plate aside large pharyngeal tubercle (stresses reach ≈24.7 MPa). Stresses (less than ≈23.0 MPa) are observed along the entire length of the zygomatic arches (see Figs. 3b and 4b).

5 Conclusions

- When installing screws and rods of the orthodontic device in the horizontal plane significant stresses appear in the upper jaw and the lower part of the nasal cavity of a human skull. By moving the screw of orthodontic device to the palate the stresses redistributed and maxilla slightly is loading. The stress state is observed around the nasal cavity, in the region of fronto-nasal suture, as well as in the pterygoid plate and pharyngeal tubercle. Therefore, in the case a of the installation of orthodontic device on an intact skull expediently carry out osteotomy median palatal suture. In case b, we can recommend an osteotomy, which will prevent the loading of the skull base.
- During expansion upper jaw with a cleft palate advisable to carry out an osteotomy to separate the maxilla and pterygoid plate in region of pterygoid process regardless of orthodontic device design. When installing screw of orthodontic device near the palate in the skull with cleft also advisable to carry out an osteotomy of the zygomatic arches.

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Part XV
Others

Organizers: Vladimir Mityushev, Michael Ruzhansky, Erdal Karapınar, Michael Oberguggenberger, Stevan Pilipovic

On the Solvability of a Nonlinear Optimal Control Problem for the Thermal Processes Described by Fredholm Integro-Differential Equations

Akylbek Kerimbekov

Abstract The problem of nonlinear optimal control of the thermal process described by Fredholm integro-differential equation was investigated. The concept of a weak generalized solution of the boundary problem was introduced and the algorithm for its construction was indicated. It was established that the optimal control is defined as a solution of a nonlinear integral equation satisfying the additional condition in the form of inequality. Sufficient conditions for unique solvability of nonlinear optimization were found and the algorithm for constructing approximate solutions was developed. The convergence of approximate solutions with respect to control, optimal process and functional was investigated.

Keywords Boundary value problem · Weak generalized solution · Functional · The maximum principle · The optimality condition · Integral equation · Approximate solution · Convergence

1 Introduction

Many applied problems are described by integro-differential equations [5, 7]. As it was noticed in [3, Introduction] in many applications mathematical models which contain integro-differentional operators haven't been studied or have been studied not enough because of controlled system's difficulty. Problems of control processes described by integro-differential equations, in the case in which control functions enter the equations non-linearly, were almost not studied. In this paper, the solvability of control problem with the quadratic quality criterion was investigated. Using the maximum principle in the case in which the controlled process is described by a Fredholm integro-differential equation, the optimality condition was obtained in the form of a nonlinear integral equation and differential inequality, i.e. optimal control is defined as a solution of the specific problem that is new in the theory of integral equations. By applying the method of [2] sufficient conditions were found for unique solvability of this problem and the algorithm was indicated for constructing

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solutions of nonlinear optimization problems with arbitrary precision in the form of a triplet $(u^0(t), v^0(t,x), J[u^0(t)])$, where $u^0(t)$ is the optimal control, $v^0(t,x)$ is the optimal process, $J[u^0(t)]$ is the minimal value of the functional.

2 Boundary Value Problem of the Controlled Process

Let the state of a thermal process be described by a scalar function $v(t,x)$, which in the region $Q_T = Q \times (0,T]$, where Q is a region of the space R^n bounded by a piecewise smooth curve γ, satisfies the integral-differential equation [5, 7]

$$v_t - Av = \lambda \int_0^T K(t,\tau)v(\tau,x)d\tau + g(t,x)f\big[t,u(t)\big],$$
$$x \in Q \subset R_n,\ 0 < t \le T \tag{2.1}$$

and on the boundary of Q satisfies the initial condition

$$v(0,x) = \psi(x), \quad x \in Q \tag{2.2}$$

and the boundary condition

$$\Gamma v(t,x) \equiv \sum_{i,j=1}^{n} a_{ij}(x)v_{x_j}(t,x)\cos(\delta, x_i) + a(x)v(t,x) = 0,$$
$$x \in q,\ 0 < t \le T. \tag{2.3}$$

Here A is the elliptic operator defined by the formula:

$$Av(t,x) \equiv \sum_{i,j=1}^{n} \big(a_{ij}(x)v_{x_j}(t,x)\big)_{x_i} - c(x)v(t,x),$$

$$a_{ij}(x) = a_{ji}(x), \quad \sum_{i,j=1}^{n} a_{ij}(x)a_i a_j \ge a_0 \sum_{i=1}^{n} a_i^2, \quad a_0 > 0;$$

δ is a normal vector, outgoing from the point $x \in q$; T is a fixed moment of time, $K(t,\tau)$ is a given function defined in the region $D = (0 \le t \le 1, 0 \le \tau \le 1)$ and satisfying the condition

$$\int_0^T \int_0^T K^2(t,\tau)d\tau dt = K_0 < \infty, \tag{2.4}$$

i.e. $K(t,\tau)$ is an element of the Hilbert space $H(D) \equiv L_2(D)$;

$$g(t,x) \in H(Q), \qquad \psi(x) \in H(0,1), \qquad f\big[t,u(t)\big] \in H(0,T),$$
$$f_u\big[t,u(t)\big] \ne 0, \quad \forall t \in (0,T), \tag{2.5}$$

are given functions; $a(x) \geq 0$, $c(x) \geq 0$ are known measurable functions; $u(t) \in H(0,T)$ is a control function, λ is a parameter and $\alpha > 0$ is a constant.

As is known, under conditions (2.5) problem (2.1)–(2.3) has no classical solutions. Therefore, we will use the notion of a weak generalized solution of problem (2.1)–(2.3).

The solution of problem (2.1)–(2.3) we will seek in the form:

$$\begin{gathered} v(t,x) = \sum_{n=1}^{\infty} v_n(t) z_n(x), \\ v_n(t) = \langle v(t,x), z_n(x) \rangle = \int_Q v(t,x) z_n(x) dx, \end{gathered} \tag{2.6}$$

where $z_n(x)$, $n = 1, 2, 3, \ldots$ are eigenfunction function of the boundary value problem

$$Az(x) = -\lambda^2 z(x), \quad x \in Q,$$
$$\Gamma z(x) = 0, \quad x \in q,$$

which form a complete orthonormal system in the Hilbert space $H(Q)$, and the corresponding eigenvalues λ_n satisfy the following conditions

$$\lambda_n \leq \lambda_{n+1} \leq \cdots, \qquad \lim_{n\to\infty} \lambda_n = \infty$$

Definition 2.1 A weak generalized solution of problem (2.1)–(2.3) is a function $v(t,x) \in H(Q)$ that satisfies the initial condition in a weak sense, i.e. for any function $\phi_0(x) \in H(Q)$ we have the equality:

$$\lim_{t\to+0} \int_Q v(t,x)\phi_0(x)dx = \int_Q \psi(x)\phi_0(x)dx,$$

and the Fourier coefficients $v_n(t)$ satisfy the linear Fredholm integral equation of the second type

$$\begin{aligned} v_n(t) = &\int_0^t e^{-\lambda_n^2(t-\tau)}\left(\lambda \int_0^T K(\tau,s)v_n(s)ds + g_n(\tau) f\left[\tau, u(\tau)\right]\right)d\tau \\ &+ e^{-\lambda_n^2 t}\psi_n, \end{aligned} \tag{2.7}$$

where ψ_n and $g_n(t)$ are the Fourier coefficients of the functions $\psi(x)$, $g(t,x)$ respectively.

To determine the Fourier coefficients $v_n(t)$ (2.7) can be rewritten as

$$v_n(t) = \lambda \int_0^T K_n(t,s)v_n(s)ds + \alpha_n(t), \tag{2.8}$$

where

$$K_n(t,s) = \int_0^t e^{-\lambda_n^2(t-\tau)} K(\tau,s)d\tau, \tag{2.9}$$

$$\alpha_n(t) = e^{-\lambda_n^2 t}\psi_n + \int_0^t e^{-\lambda_n^2(t-\tau)} g_n(\tau) f\big[\tau, u(\tau)\big]d\tau \tag{2.10}$$

The solution of integral equation (2.8) we find by the formula [1]:

$$v_n(t) = \lambda \int_0^T R_n(t,s,\lambda)\alpha_n(s)ds + \alpha_n(t), \tag{2.11}$$

where

$$R_n(t,s,\lambda) = \sum_{n=1}^{\infty} \lambda^{i-1} K_{n,i}(t,s), \quad n = 1,2,3,\ldots, \tag{2.12}$$

is the resolvent $K_{n,1}(t,s) \equiv K_n(t,s)$, and the iterated kernels $K_{n,i}(t,s)$ are defined by the formula [1]

$$K_{n,i+1}(t,s) = \int_0^T K_n(t,\eta)K_{n,i}(\eta,s)d\eta, \quad i = 1,2,3,\ldots, \tag{2.13}$$

for each $n = 1,2,3,\ldots$. We investigate the convergence of Neumann series (2.12). According to (2.9) and (2.13) by direct calculation the following estimates are established

$$\big|K_{n,i}(t,s)\big|^2 \le \frac{(K_0T)^{i-1}}{(2\lambda_n^2)^i} \int_0^T K^2(\eta,s)d\eta, \quad i = 1,2,3,\ldots. \tag{2.14}$$

Neumann series (2.12) is dominated by the numerical series

$$\begin{aligned}\sum_{i=1}^{\infty} \lambda^{i-1} K_{n,i}(t,s) &\le \sum_{i=1}^{\infty} |\lambda|^{i-1} \big|K_{n,i}(t,s)\big| \\ &\le \left(\int_0^T K^2(\eta,s)d\eta\right)^{1/2} \frac{1}{\sqrt{2\lambda_n^2}} \sum_{i=1}^{\infty} \left(|\lambda| \frac{\sqrt{K_0T}}{\sqrt{2\lambda_n^2}}\right)^{i-1},\end{aligned}$$

which converges for every $n = 1,2,3,\ldots$ for the values of the parameter λ that satisfy the inequality

$$|\lambda| \frac{\sqrt{K_0T}}{\sqrt{2\lambda_n^2}} < 1.$$

Note that

$$|\lambda| < \frac{\sqrt{2}}{\sqrt{K_0T}} \lambda_n \xrightarrow[n\to\infty]{} \infty,$$

i.e. the radius of convergence increases when n is growing. However, the Neumann series, for the parameter values λ that satisfy the condition

$$|\lambda| < \frac{\sqrt{2}}{\sqrt{K_0 T}}\lambda_1, \quad \lambda \neq 0 \tag{2.15}$$

converges absolutely for any $n = 1, 2, 3, \ldots$. In this case the resolvent as the sum of an absolutely convergent series is a continuous function and satisfies the following estimates

$$\begin{aligned}
\left|R_n(t, s, \lambda)\right| &\leq \left(\int_0^T K^2(\eta, s)d\eta\right)^{1/2} \frac{1}{\sqrt{2\lambda_n^2}} \sum_{i=1}^{\infty}\left(|\lambda|\frac{\sqrt{K_0 T}}{\sqrt{2\lambda_n^2}}\right)^{i-1} \\
&= \frac{1}{\sqrt{2\lambda_1^2} - |\lambda|\sqrt{K_0 T}}\left(\int_0^T K^2(\eta, s)d\eta\right)^{1/2}, \\
\int_0^T R_n^2(t, s, \lambda)ds &\leq \frac{1}{(\sqrt{2\lambda_1^2} - |\lambda|\sqrt{K_0 T})^2} \int_0^T \int_0^T K^2(\eta, s)d\eta ds \\
&= \frac{K_0}{(\sqrt{2\lambda_1^2} - |\lambda|\sqrt{K_0 T})^2}.
\end{aligned} \tag{2.16}$$

Thus, the solution of problem (2.1)–(2.3) we find by (2.6), where $v_n(t)$ is defined by formula (2.11) as the unique solution of integral equation (2.8). It is easy to verify that this solution satisfies initial condition (2.2).

Now we show that this solution is an element of the space $H(Q_T)$. Taking into account (2.9) and (2.10) by direct calculation it is easy to show that the following inequality holds

$$\begin{aligned}
&\int_0^T \int_Q v^2(t, x)dxdt \\
&\leq \int_0^T \int_Q \left(\sum_{n=1}^{\infty} v(t)z_n(x)\right)^2 dxdt = \int_0^T \sum_{n=1}^{\infty} v_n^2(t)dt \\
&\leq \int_0^T \sum_{n=1}^{\infty}\left(\lambda \int_0^T R_n(t, s, \lambda)\alpha_n(s)ds + \alpha_n(t)\right)^2 dt \\
&\leq 2\int_0^T \sum_{n=1}^{\infty}\left(\lambda^2 \int_0^T R_n^2(t, s, \lambda)ds \int_0^T \alpha_n^2(s)ds + \alpha_n^2(t)\right)dt \\
&\leq 2\left(\frac{\lambda^2 K_0 T}{(\sqrt{2\lambda_1^2} - |\lambda|\sqrt{K_0 T})^2} \sum_{n=1}^{\infty} \int_0^T \alpha_n^2(s)ds + \int_0^T \sum_{n=1}^{\infty} \alpha_n^2(t)dt\right)
\end{aligned}$$

$$
\le 2\Big(\frac{2\lambda^2 K_0 T}{(\sqrt{2\lambda_1^2} - |\lambda|\sqrt{K_0 T})^2} + 1\Big)
$$

$$
\times 2T\left(\sum_{n=1}^{\infty}\psi_n^2 + \sum_{n=1}^{\infty}\int_0^T g_n^2(\tau)d\tau\int_0^T f^2\big[\tau, u(\tau)\big]d\tau\right)
$$

$$
= 4T\Big(\frac{\lambda^2 K_0 T}{(\sqrt{2\lambda_1^2} - |\lambda|\sqrt{K_0 T})^2} + 1\Big)
$$

$$
\times \big\{\|\psi(x)\|_H^2 + \big(\|g(t,x)\|_H^2\|f\big[t, u(t)\big]\|_H^2\big)\big\}.
$$

From this inequality it follows that $v(t,x) \in H(Q_T)$. When the functions $v_n(t)$, $n = 1, 2, 3, \dots$, are determined by formulas (2.11)–(2.12), it is not always possible to find the exact resolvent $R_n(t,s,\lambda)$. In practice, the approximations of the resolvent are considered most often. The truncated series of the form

$$
R_n^m(t,s,\lambda) = \sum_{i=1}^{m}\lambda^{i-1}K_{n,i}(t,s), \quad n = 1, 2, 3, \dots, \tag{2.17}
$$

is called mth approximation of the resolvent $R_n(t,s,\lambda)$ for each fixed $n = 1, 2, 3, \dots$.

The function $v_n^m(t)$ defined by the formula

$$
v_n^m(t) = \lambda\int_0^T R_n^m(t,s,\lambda)\alpha_n(s)ds + \alpha_n(t), \quad n = 1, 2, 3, \dots, \tag{2.18}
$$

is called the mth approximation of the function $v_n(t)$ for each fixed $n = 1, 2, 3, \dots$.

According to the formula (2.6), the mth approximation of the solution $v(t,x)$ of boundary value problem (2.1)–(2.3) we find from the formula

$$
v^{(m)}(t,x) = \sum_{n=1}^{\infty} v_n^m(t) z_n(x), \tag{2.19}
$$

where $v_n^m(t)$ have the form (2.18). We show that the approximate solution $v_n^m(t,x)$ of boundary value problem (2.1)–(2.3) converges to the exact solution $v(t,x)$ with respect to the norm of the space $H(Q_T)$. Taking into account (2.12), (2.14), (2.15), (2.17), (2.18) and the inequality

$$
\sum_{i=m+1}^{\infty}\alpha^i \le \alpha^{m+1} + \int_{m+1}^{\infty}\alpha^x dx = \alpha^{m+1} + \frac{1}{\ln\alpha}\alpha^x\Big|_{m+1}^{\infty} = \alpha^{m+1}\Big(1 - \frac{1}{\ln\alpha}\Big),
$$

$$
0 < \alpha < 1,
$$

by direct computation we find that

$$\begin{aligned}
\left[v_n(t)-v_n^m(t)\right]^2 &= \left(\lambda\int_0^T\left[R_n(t,s,\lambda)-R_n^m(t,s,\lambda)\right]\alpha_n(s)ds\right)^2\\
&\le \lambda^2\int_0^T\left[R_n(t,s,\lambda)-R_n^m(t,s,\lambda)\right]^2 ds\int_0^T\alpha_n^2(s)ds\\
&\le \lambda^2\int_0^T\left(\sum_{i=m+1}^{\infty}|\lambda|^{i-1}\left|K_{n,i}(t,s)\right|\right)^2 ds\int_0^T\alpha_n^2(s)ds\\
&\le \lambda^2\frac{K_0}{2\lambda_n^2}\left(\sum_{i=m+1}^{\infty}\left(|\lambda|\sqrt{\frac{K_0T}{2\lambda_1^2}}\right)^{i-1}\right)^2\int_0^T\alpha_n^2(s)ds\\
&\le \frac{\lambda^2K_0}{2\lambda_n^2}\left(|\lambda|\sqrt{\frac{K_0T}{2\lambda_1^2}}\right)^{2m}\left(1-\frac{1}{\ln|\lambda|\sqrt{\frac{K_0T}{2\lambda_1^2}}}\right)^2\\
&\quad\times\int_0^T\alpha_n^2(s)ds\le C_n(\lambda)\left(|\lambda|\sqrt{\frac{K_0T}{2\lambda_1^2}}\right)^{2m},
\end{aligned}\tag{2.20}$$

where

$$\begin{aligned}
C_n(\lambda) &= \frac{\lambda^2K_0}{2}\left(1-\frac{1}{\ln|\lambda|\sqrt{\frac{K_0T}{2\lambda_1^2}}}\right)^2\\
&\quad\times\left(\psi_n^2(x)+\int_0^T g_n^2(\tau)d\tau\left\|f\left[t,u(t)\right]\right\|_H^2\right).
\end{aligned}\tag{2.21}$$

Note that, because the parameter λ satisfies (2.15), we have the inequality

$$0<1-\frac{1}{\ln|\lambda|\sqrt{\frac{K_0T}{2\lambda_1^2}}}<\infty. \tag{2.22}$$

The convergence of the approximate solutions of boundary-value problem follows from

$$\begin{aligned}
&\left\|v(t,x)-v^m(t,x)\right\|_H^2\\
&=\int_0^T\int_Q\left(\sum_{n=1}^{\infty}\left[v_n(t)-v_n^m(t)\right]z_n(x)\right)^2dxdt\\
&=\int_0^T\sum_{n=1}^{\infty}\left[v_n(t)-v_n^m(t)\right]^2dt\le\int_0^T\sum_{n=1}^{\infty}C_n(\lambda)\left(|\lambda|\sqrt{\frac{K_0T}{2\lambda_1^2}}\right)^{2m}dt
\end{aligned}$$

$$\leq \int_0^T \sum_{n=1}^{\infty} \frac{\lambda^2 K_0}{\lambda_1^2}\left(1-\frac{1}{\ln|\lambda|\sqrt{\frac{K_0T}{2\lambda_1^2}}}\right)^2$$

$$\times\left(\psi_n^2+\int_0^T g_n^2(\tau)d\tau\left\|f\left[t,u(t)\right]\right\|_n^2\right)\left(|\lambda|\sqrt{\frac{K_0T}{2\lambda_1^2}}\right)^{2m}dt$$

$$\leq \frac{\lambda^2 K_0 T}{\lambda_1^2}\left(1-\frac{1}{\ln|\lambda|\sqrt{\frac{K_0T}{2\lambda_1^2}}}\right)^2$$

$$\times\left(\left\|\psi(x)\right\|_H^2+\left\|g(t,x)\right\|_H^2\left\|f\left[t,u(t)\right]\right\|_H^2\right)\left(|\lambda|\sqrt{\frac{K_0T}{2\lambda_1^2}}\right)^{2m}\xrightarrow[m\to\infty]{} 0.$$

3 Formulation of Optimal Control Problem and Conditions of Optimality

Consider the optimization problem in which it is required to minimize the integral functional

$$J\left[u(t)\right]=\int_Q\left[v(T,x)-\xi(x)\right]^2dx+2\beta\int_0^T M\left[t,u(t)\right]dt,\quad \beta>0, \tag{3.1}$$

where $\xi(x)\in H(Q)$, $M_u[t,u(t)]\in H(0,T)$—are given functions on the set of solutions of problem (2.1)–(2.3), i.e. we need to find the control $u^0(t)\in H(0,T)$ which together with the corresponding solution $v^0(t,x)$ of boundary value problem (2.1)–(2.3) gives the smallest possible value of functional (3.1). In this case $u^0(t)$ is called the optimal control, and $v^0(t,x)$ the optimal process.

Since by condition (2.5) each control $u(t)$ uniquely defines the controlled process $v(t,x)$, the solution of boundary value problem (2.1)–(2.3) of the form $v(t,x)+\Delta v(t,x)$ corresponds to the control $u(t)+\Delta u(t)$, where $\Delta v(t,x)$ is the increment corresponding to the increment $\Delta u(t)$. According to the procedure of application of the maximum principle [3, 4, 6], the increment of the functional (3.1) can be written as

$$\begin{aligned}\Delta J[u]&=J[u+\Delta u]-J[u]\\&=-\int_0^T\Delta\Pi\left[t,v(t,x),\omega(t,x),u(t)\right]dt+\int_Q\Delta v^2(T,x)dx,\end{aligned}\tag{3.2}$$

where

$$\Delta\Pi(t,v,\omega,u) = \Pi\big(t, v(t,x), \omega(t,x), u(t) + \Delta u(t)\big) - \Pi\big(t, v(t,x), \omega(t,x), u(t)\big),$$

$$\Pi\big(t, v(t,x), \omega(t,x), u(t)\big) = \int_Q g(t,x)\omega(t,x)dx f\big[t,u(t)\big] - 2\beta M\big[t,u(t)\big], \tag{3.3}$$

and the function $\omega(t,x)$ is a solution of the adjoint boundary value problem

$$\begin{aligned} \omega_t + A\omega + \int_0^T K(\tau,t)\omega(\tau,x)d\tau = 0, &\quad x \in Q,\ 0 \le t < T, \\ \omega(T,x) + 2\big[v(T,x) - \xi(x)\big] = 0, &\quad x \in Q, \\ \Gamma\omega(t,x) = 0, &\quad x \in q. \end{aligned} \tag{3.4}$$

According to the maximum principle for systems with distributed parameters [6], the optimal control is determined by the relations

$$2\beta M_u\big[t,u(t)\big] f_u^{-1}\big[t,u(t)\big] = \int_Q g(t,x)\omega(t,x)dx, \tag{3.5}$$

$$f_u\big[t,u(t)\big]\left(\frac{M_u[t,u(t)]}{f_u[t,u(t)]}\right)_u > 0, \tag{3.6}$$

which are called the optimality conditions.

4 Solution of the Adjoint Boundary-Value Problem

We are looking for solution of boundary value problem (3.4) in the form of the series

$$\omega(t,x) = \sum_{i=1}^{\infty} \omega_n(t) z_n(x). \tag{4.1}$$

It is easy to verify that the Fourier coefficients $\omega(t,x)$ for each fixed $n = 1, 2, 3, \ldots,$ satisfy the conditions

$$\omega_n'(t) - \lambda_n^2\omega_n(t) = -\lambda \int_0^T K(\tau,t)\omega_n(\tau)d\tau,$$

$$\omega_n(T) + 2\big[v_n(T) - \xi_n\big] = 0,$$

which can be converted to the linear non-homogeneous Fredholm integral equation of the second type

$$\omega_n(t) = \lambda \int_0^T B_n(s,t)\omega_n(s)ds - 2e^{-\lambda_n^2(T-t)}\big[v_n(T) - \xi_n\big], \tag{4.2}$$

where the kernel

$$B_n(s,t) = \int_t^T e^{-\lambda_n^2(T-t)} K(s,\tau)d\tau \quad \text{and} \quad B_n(s,T) = 0. \tag{4.3}$$

The solution of (4.2) we find by the formula [1]

$$\omega_n(t) = -2\big[v_n(T) - \xi_n\big]\bigg(e^{-\lambda_n^2(T-t)} + \lambda \int_0^T P_n(s,t,\lambda)e^{-\lambda_n^2(T-s)}ds\bigg), \tag{4.4}$$

where the resolvent $P_n(s,t,\lambda)$ of the kernel $B_n(s,t)$ is given by

$$P_n(s,t,\lambda) = \sum_{i=1}^{\infty} \lambda^{i-1} B_{n,i}(s,t),$$

$$B_{n,i+1}(s,t) = \int_0^T B_n(\eta,t)B_{n,i}(s,\eta)d\eta, \quad i = 1,2,3,\ldots,$$

and by the condition (2.14) it is a continuous function, and satisfies the inequality

$$\big|P_n(s,t,\lambda)\big| \le \frac{1}{\sqrt{2\lambda_1^2} - |\lambda|\sqrt{K_0T}} \bigg(\int_0^T K^2(\eta,s)d\eta\bigg)^{1/2}. \tag{4.5}$$

It is easy to verify that $\omega(t,x)$ is an element of the space $H(Q)$.
This follows from the inequality

$$\begin{aligned}
&\int_0^T \int_Q \omega^2(t,x)dxdt \\
&= \int_0^T \int_Q \bigg(\sum_{n=1}^{\infty} \omega_n(t)z_n(x)\bigg)^2 dxdt = \int_0^T \sum_{n=1}^{\infty} \omega_n^2(t)dt \\
&\le 8\int_0^T \sum_{n=1}^{\infty} \big[v_n(T) - \xi_n\big]^2 \\
&\quad \times \bigg(e^{-2\lambda_n^2(T-t)} + \lambda^2 \int_0^T P_n^2(s,t,\lambda)ds \int_0^T e^{-2\lambda_n^2(T-s)}ds\bigg)dt \\
&\le 8\int_0^T \sum_{n=1}^{\infty} \big[v_n(T) - \xi_n\big]^2 \\
&\quad \times \bigg(1 + \lambda^2 \frac{1}{(\sqrt{2\lambda_1^2} - |\lambda|\sqrt{K_0T})^2} \int_0^T \int_0^T K^2(s,\eta)d\eta ds \frac{1}{2\lambda_1^n}\bigg)dt
\end{aligned}$$

$$\leq 16T\left(1+\frac{\lambda^2 K_0}{(2\lambda_1^2\sqrt{2\lambda_1^2}-|\lambda|\sqrt{K_0 T})^2}\right)\sum_{n=1}^{\infty}\left(v_n^2(T)-\xi_n^2\right)<\infty,$$

which holds by the following relations

$$\sum_{n=1}^{\infty} v_n^2(T)<\infty \quad \text{and} \quad \sum_{n=1}^{\infty}\xi_n^2=\left\|\xi(x)\right\|_H^2.$$

5 Nonlinear Integral Equation of Optimal Control

We find the optimal control according to optimality conditions (3.5) and (3.6). We substitute in (3.5) the solution of adjoint boundary value problem (3.4) defined by (4.1) and (4.4). First, we calculate the integral

$$\int_Q g(t,x)\omega(t,x)dx=\int_Q \sum_{n=1}^{\infty} g_n(t)z_n(x)\sum_{k=1}^{\infty}\omega_k(t)z_k(x)dx=\sum_{n=1}^{\infty} g_n(t)\omega_n(t)$$

and rewrite equality (3.5) in the form

$$\beta M_u\big[t,u(t)\big]f_u^{-1}\big[t,u(t)\big]$$
$$=-\sum_{n=1}^{\infty} g_n(t)\big[v_n(T)-\xi_n\big]\left(e^{-\lambda_n^2(T-t)}+\lambda\int_0^T P_n(s,t,\lambda)e^{-\lambda_n^2(T-s)}ds\right).$$

According to (2.6), we reduce this equality to the form

$$\beta M_u\big[t,u(t)\big]f_u^{-1}\big[t,u(t)\big]+\sum_{n=1}^{\infty} L_n(t,\lambda)\int_0^T G_n(s,\lambda)f\big[s,u(s)\big]ds$$
$$=\sum_{n=1}^{\infty} L_n(t,\lambda)h_n, \tag{5.1}$$

where

$$L_n(t,\lambda)=g_n(t)\left[e^{-\lambda_n^2(T-t)}+\lambda\int_0^T P_n(\tau,t,\lambda)e^{-\lambda_n^2(T-\tau)}d\tau\right], \tag{5.2}$$

$$G_n(t,\lambda)=g_n(t)\left[e^{-\lambda_n^2(T-t)}+\lambda\int_t^T R_n(T,\tau,\lambda)e^{-\lambda_n^2(\tau-t)}d\tau\right], \tag{5.3}$$

$$h_n=\xi_n-\psi_n\left[e^{-\lambda_n^2 T}+\lambda\int_0^T R_n(T,\tau,\lambda)e^{-\lambda_n^2\tau}d\tau\right]. \tag{5.4}$$

Thus, the optimal control is defined as the solution of nonlinear integral equation (5.1), and here we must have condition (3.6). Condition (3.6) restricts the class of functions of external actions $f[t, u(t)]$. Therefore, we assume that the function $f[t, u(t)]$ satisfies (3.6) for any control $u(t) \in H(0, T)$.

Nonlinear integral equation (5.1) is solved according to the procedure of work [7]. We set

$$\beta M_u\big[t, u(t)\big] f_u^{-1}\big[t, u(t)\big] = p(t). \tag{5.5}$$

Lemma 5.1 *The function $p(t)$ is an element of space $H(0, T)$.*

Proof By (2.5), we have the estimate

$$\big|f_u^{-1}\big[t, u(t)\big]\big| \le M_0, \quad \forall t \in [0, T]. \qquad \square$$

Since $u(t) \in H(0, T)$, the statement of the lemma follows by the inequality

$$\int_0^T p^2(t)dt \le \beta^2 \int_0^T \big|f_u^{-1}\big[t, u(t)\big]\big|^2 \big|M_u\big[t, u(t)\big]\big|^2 dt$$
$$\le \beta^2 M_0^2 \int_0^T M_u^2\big[t, u(t)\big](t)dt < \infty.$$

According to (3.6), the control $u(t)$ is uniquely determined by equality (5.5), i.e. there is a function φ such that

$$u(t) = \varphi\big(t, p(t), \beta\big). \tag{5.6}$$

By (5.5) and (5.6) we rewrite (5.1) in the form

$$p(t) + \sum_{n=1}^{\infty} L_n(t, \lambda) \int_0^T G_n(s, \lambda) f\big[s, \varphi\big(s, p(s), \beta\big)\big]ds = \sum_{n=1}^{\infty} L_n(t, \lambda) h_n, \tag{5.7}$$

or in the operator form

$$p(t) = G\big[p(t)\big], \tag{5.8}$$

where

$$G\big[p(t)\big] = \sum_{n=1}^{\infty} L_n(t, \lambda)\Big[h_n - \int_0^T G_n(s, \lambda) f\big[s, \varphi\big(s, p(s), \beta\big)\big]ds\Big]. \tag{5.9}$$

Now we turn to the problem of unique solvability of operator equation (5.8).

Lemma 5.2 *The operator G maps the space $H(0, T)$ into itself, i.e. $G[p(t)]$ is an element of the space $H(0, T)$.*

Proof By direct calculation we have the inequality

$$\int_0^T G^2\big[p(t)\big]dt$$

$$= \int_0^T \left(\sum_{n=1}^{\infty} L_n(t,\lambda) \left[h_n - \int_0^T G_n(s,\lambda) f\big[s, \varphi\big(s, p(s), \beta\big)\big]ds \right] \right)^2 dt$$

$$\leq 2\int_0^T \sum_{n=1}^{\infty} L_n^2(t,\lambda) \sum_{n=1}^{\infty} \left[h_n^2 + \int_0^T G_n^2(s,\lambda)ds \int_0^T f^2\big[s, \varphi\big(s, p(s), \beta\big)\big]ds \right] dt$$

$$\leq 2\int_0^T \sum_{n=1}^{\infty} 2g_n^2(t) \Big[e^{-2\lambda_n^2(T-t)}$$

$$+ \lambda^2 \int_0^T P_n^2(\tau, t, \lambda)d\tau \int_0^T e^{-2\lambda_n^2(T-\tau)}d\tau \Big] \Big\{ 2\Big[\|\xi(x)\|_H^2$$

$$+ 2\left(1 + \frac{\lambda^2 K_0}{(\sqrt{2\lambda_1^2} - |\lambda|\sqrt{K_0 T})^2} \frac{1}{2\lambda_1^2}\right) \|\psi(x)\|_H^2 \Big]$$

$$+ \sum_{n=1}^{\infty} 2g_n^2(t) \left(1 + \frac{\lambda^2 K_0 T}{(\sqrt{2\lambda_1^2} - |\lambda|\sqrt{K_0 T})^2}\right) \big\| f\big[s, \varphi\big(s, p(s), \beta\big)\big]\big\|_H^2 \Big\} dt$$

$$\leq C \int_0^T \sum_{n=1}^{\infty} g_n^2(t)dt < \infty,$$

from which the statement of the lemma follows. □

Lemma 5.3 *Suppose conditions*

$$\big\| f\big[t, u(t)\big] - f\big[t, \overline{u}(t)\big]\big\|_H \leq f_0 \big\|u(t) - \overline{u}(t)\big\|_H, \quad f_0 > 0 \tag{5.10}$$

and

$$\big\|\varphi\big[t, p(t), \beta\big] - \varphi\big[t, \overline{p}(t), \beta\big]\big\|_H \leq \varphi_0(\beta)\big\|p(t) - \overline{p}(t)\big\|_H, \quad \varphi_0(\beta) > 0 \tag{5.11}$$

are satisfied. Then if the condition

$$\gamma = 2\|g(t,x)\|_H^2 \left(1 + \frac{a_0^2 K_0}{(\sqrt{2\lambda_1^2} - a_0\sqrt{K_0 T})^2}\right) f_0 \varphi_0(\beta) < 1, \tag{5.12}$$

is met, where α_0 *is a positive constant satisfying the inequality*

$$|\lambda| \leq a_0 < \frac{\sqrt{2}\lambda_1}{\sqrt{K_0 T}}, \tag{5.13}$$

then the operator G is contractive.

Proof By direct calculations, we have the inequality

$$\int_0^T \left|G[p] - G[\overline{p}]\right|^2 dt$$

$$= \int_0^T \left(\sum_{n=1}^{\infty} L_n(t,\lambda) \int_0^T G_n(s,\lambda)\right.$$

$$\left. \times \big(f\big[s, \varphi\big(s, p(s), \beta\big)\big] - f\big[s, \varphi\big(s, \overline{p}(s), \beta\big)\big]\big) ds\right)^2 dt$$

$$\leq \int_0^T \sum_{n=1}^{\infty} L_n^2(t,\lambda) \sum_{n=1}^{\infty} \int_0^T G_n^2(s,\lambda) ds \int_0^T \big(f\big[s, \varphi\big(s, p(s), \beta\big)\big]$$

$$- f\big[s, \varphi\big(s, \overline{p}(s), \beta\big)\big]\big)^2 ds dt$$

$$\leq \left[2\left\|g(t,x)\right\|_H^2 \left(1 + \frac{\lambda^2 K_0}{(\sqrt{2\lambda_1^2} - |\lambda|\sqrt{K_0 T})^2}\right) f_0 \varphi_0(\beta) \left\|p(s) - \overline{p}(s)\right\|_H\right]^2,$$

from which we find that

$$\left\|G[p] - G[\overline{p}]\right\|_H$$

$$\leq 2\left\|g(t,x)\right\|_H^2 \left(1 + \frac{\lambda^2 K_0}{(\sqrt{2\lambda_1^2} - |\lambda|\sqrt{K_0 T})^2}\right) f_0 \varphi_0(\beta) \left\|p(t) - \overline{p}(t)\right\|_H.$$

□

Theorem 5.4 *Suppose that conditions* (2.4)–(2.5), (3.6), (4.5), (5.10)–(5.13) *are satisfied. Then operator equation* (5.8) *has a unique solution in the space* $H(0,T)$.

Proof According to Lemmas 5.1 and 5.2, operator equation (5.8) can be considered in the space $H(0,T)$. According to Lemma 5.3 operator G is contractive. Since the Hilbert space $H(0,T)$ is a complete metric space, by the theorem on contraction mappings the operator G has a unique fixed point, i.e. operator equation (5.8) has a unique solution. □

The solution of operator equation (5.8) can be found by the method of successive approximations, i.e. nth approximation of the solution is found by the formula

$$p_n(t) = G\big[p_{n-1}(t)\big], \quad n = 1, 2, 3, \ldots,$$

where $p_0(t)$ is an arbitary element of the space $H(0,T)$, and we have the estimate

$$\left\|\overline{p}(t) - p_n(t)\right\| \leq \frac{\gamma^n}{1-\gamma} \left\|G\big[p_0(t)\big] - p_0(t)\right\|_H, \tag{5.14}$$

where $0 < \gamma < 1$ is the construction constant. The exact solution can be found as the limit of the approximate solutions, i.e.

$$\overline{p}(t) = \lim_{n\to\infty} p_n(t).$$

Substituting this solution in (5.6) we find the required optimal control

$$u^0(t) = \varphi\big[t, \overline{p}(t), \beta\big]. \tag{5.15}$$

The optimal process $v^0(t,x)$, i.e. the solution of boundary value problem (2.1)–(2.5) corresponding to the optimal control $u^0(t)$, according to (2.6) and (2.7) we find from the formula

$$\begin{aligned} v^0(t,x) &= \sum_{n=1}^{\infty}\left(\lambda\int_0^T R_n(t,s,\lambda)a_n(s)ds - a_n(t)\right)z_n(x) \\ &= \sum_{n=1}^{\infty}\Bigg[\psi_n\left(e^{-\lambda_n^2 t} + \lambda\int_0^T R_n(t,s,\lambda)e^{-\lambda_n^2 s}\right)ds \\ &\quad + \int_0^T e^{-\lambda_n^2(t-\tau)}g_n(\tau)f\big[\tau,u^0(\tau)\big]d\tau \\ &\quad + \lambda\int_0^T R_n(t,s,\lambda)\int_0^s e^{-\lambda_n^2(s-\eta)}g_n(\eta)f\big[\eta,u^0(\eta)\big]d\eta ds\Bigg]z_n(x). \end{aligned} \tag{5.16}$$

The minimum value of the functional (3.2) is calculated by the formula

$$J\big[u^0(t)\big] = \int_0^1\big[v^0(T,x) - \xi(x)\big]^2dx + \beta\int_0^T M\big[t,u^0(t)\big]dt. \tag{5.17}$$

The found triple $(u^0(t), v^0(t,x), J[u^0(t)])$ is a solution of the nonlinear optimization problem.

6 An Approximate Solution of the Optimization Problem

In practice, it is not always possible to find the exact solution of (5.8), i.e. the limit function $\overline{p}(t)$. Therefore, in most of the cases only approximate solutions $p_k(t)$ of (5.8) are looked for, where the number k is determined by the inequality

$$\big\|\overline{p}(t) - p_k(t)\big\|_H \le \frac{\gamma^k}{1-\gamma}\big\|G\big[p_0(t)\big] - p_0(t)\big\|_H < \varepsilon \tag{6.1}$$

for given $\varepsilon > 0$. By substituting the approximate solution $p_k(t)$ in (5.6) we find the kth approximation of optimal control

$$u_k(t) = \varphi\big[t, p_k(t), \beta\big]. \tag{6.2}$$

Lemma 6.1 *Let the function $\varphi[t, \vartheta(t), \beta]$ satisfy the Lipschitz condition with respect to the functional variable $\vartheta(t)$, i.e.*

$$\begin{aligned}&\big\|\varphi\big[t, \vartheta_1(t), \beta\big] - \varphi\big[t, \vartheta_2(t), \beta\big]\big\|_H \le \varphi_0(\beta)\big\|\big(\vartheta_1(t) - \vartheta_2(t)\big\|\big)_H,\\ &\varphi_0(\beta) > 0.\end{aligned} \tag{6.3}$$

Then the kth approximate controls converge to the optimal control $u^0(t)$ in the norm of the Hilbert space $H(Q)$ as $k \to \infty$.

Proof Lemma's assertion follows from the inequality

$$\begin{aligned}\big\|u^0(t) - u_k(t)\big\|_H &= \big\|\varphi\big[t, p^0(t), \beta\big] - \varphi\big[t, p_k(t), \beta\big]\big\|_H\\ &\le \varphi_0(\beta)\big\|p^0(t) - p_k(t)\big\|_H\\ &\le \varphi_0(\beta)\frac{\gamma^k}{1-\gamma}\big\|G\big[p_0(t)\big] - p_0(t)\big\|_H \xrightarrow[k\to\infty]{} 0.\end{aligned} \tag{6.4}$$

□

Lemma 6.2 *Let the function $f[t, u(t)]$ satisfy the Lipschitz condition with respect to the functional variable $u(t)$, i.e.*

$$\big\|f\big[t, u_1(t)\big] - f\big[t, u_2(t)\big]\big\|_H \le f_0\big\|u_1(t) - u_2(t)\big\|_H \tag{6.5}$$

and we have (6.3). Then m, kth approximations of the solution $v_k^m(t, x)$ of boundary value problem (2.1)–(2.3) converge to the exact solution $v(t, x)$ in the norm of the Hilbert space $H(Q)$ as $m, k \to \infty$.

Proof Approximations of the optimal process $v^0(t, x)$ are determined by two indices k and m and have the form

$$\begin{aligned}v_k^m(t, x) = \sum_{n=1}^{\infty}\bigg\{&\psi_n\bigg(e^{-\lambda_n^2 t} + \lambda\int_0^T R_n^m(t, s, \lambda)e^{-\lambda_n^2 s}ds\bigg)\\ &+ \int_0^t e^{-\lambda_n^2(t-\tau)}g_n(\tau)f\big[\tau, u_k(\tau)\big]d\tau\\ &+ \lambda\int_0^T g_n(\tau)\int_0^T R_n^m(t, s, \lambda)e^{-\lambda_n^2(s-\tau)}ds f\big[\tau, u_k(\tau)\big]d\tau\bigg\}z_n(x).\end{aligned} \tag{6.6}$$

Since

$$\begin{aligned}&v^0(t, x) - v_k^m(t, x)\\ &\quad = \sum_{n=1}^{\infty}\bigg\{\psi_n\lambda\int_0^T\big[R_n(t, s, \lambda) - R_n^m(t, s, \lambda)\big]e^{-\lambda_n^2 s}\bigg\}ds\end{aligned}$$

$$+\int_0^T e^{-\lambda_n^2(t-\tau)} g_n(\tau)\big[f\big(\tau, u^0(\tau)\big) - f\big(\tau, u_k(\tau)\big)\big]d\tau$$
$$+\lambda \int_0^T g_n(\tau) \int_\tau^T \big[R_n(t,s,\lambda) - R_n^m(t,s,\lambda)\big]e^{-\lambda_n^2(s-\tau)} ds f\big[\tau, u^0(\tau)\big]d\tau$$
$$+\lambda \int_0^T g_n(\tau) \int_\tau^T R_n^m(t,s,\lambda) e^{-\lambda_n^2(s-\tau)} ds$$
$$\times \big[f\big(\tau, u^0(\tau)\big) - f\big(\tau, u_k(\tau)\big)\big]d\tau z_n(x),$$

by calculations, used to prove the convergence of the approximate solutions of boundary value problem (2.1)–(2.3), we get the relation

$$\big\|v^0(t,x) - v_k^m(t,x)\big\|_H^2 \le C_1(\lambda)\left(|\lambda|\sqrt{\frac{K_0 T}{2\lambda_1^2}}\right)^{2m} + C_2(\lambda)\left(\frac{\gamma^k}{1-\gamma}\right)^2 \xrightarrow[m,k\to\infty]{} 0,$$

where

$$C_1(\lambda) = \big(\|\psi(x)\|_H^2 + \big\|f\big[t,u(t)\big]\big\|_H^2 \cdot \|g(t,x)\|_H^2\big)\frac{\lambda^2 K_0 T}{4}\left(1 - \frac{1}{\ln|\lambda|\sqrt{\frac{K_0 T}{2\lambda_1^2}}}\right),$$

$$C_2(\lambda) = 4\left(1 + \frac{\lambda^2 K_0}{2\lambda_1^2(\sqrt{2\lambda_1^2} - |\lambda|\sqrt{K_0 T})^2}\right)$$
$$\times T\|g(t,x)\|_H^2 f_0^2 \varphi_0^2(\beta)\big\|G\big[p_0(t)\big] - p_0(t)\big\|_H^2,$$

by which assertion of the lemma follows. □

Lemma 6.3 *The m,kth approximations $J_m[u_k(t)]$ of the minimum value of the functional $J[u^0(t)]$ converges to the exact value as $m,k \to \infty$.*

Proof Since

$$J\big[u^0(t)\big] = \int_Q \big[v^0(T,x) - \xi(x)\big]^2 dx + \beta\int_0^T M\big[t, u^0(t)\big]dt,$$
$$J_m\big[u_k(t)\big] = \int_Q \big[v_k^m(T,x) - \xi(x)\big]^2 dx + \beta\int_0^T M\big[t, u_k(t)\big]dt,$$

it is not difficult to obtain the inequality

$$\big|J\big[u^0(t)\big] - J_m\big[u_k(t)\big]\big|$$
$$= \int_Q \big\{\big[v^0(T,x) - \xi(x)\big]^2 - \big[v_k^m(T,x) - \xi(x)\big]^2\big\}dx$$
$$+ 2\beta\int_0^T \big(M\big[t, u^0(t)\big] - M\big[t, u_k(t)\big]\big)dt$$

$$\le \left\| v^0(T,x) + v_k^m(T,x) - 2\xi(x) \right\|_H \left\| v^0(T,x) - v_k^m(T,x) \right\|_H + 2\beta\sqrt{T} \left\| M\left[t, u^0(t)\right] - M\left[t, u_k(t)\right] \right\|_H$$

$$\le \left\| v^0(T,x) + v_k^m(T,x) - 2\xi(x) \right\|_H \left\| v^0(T,x) - v_k^m(T,x) \right\|_H + 2\beta\sqrt{T} m_0 \left\| u^0(t) - u_k(t) \right\|_H.$$

By Lemmas 6.1 and 6.2, and in view of the fact that

$$u^0(t) \in H(0,T), \qquad v(T,x) \in H(0,1),$$
$$\xi(x) \in H(0,1), \qquad f\left[t, u(t)\right] \in H(0,T),$$

we obtain the relation

$$\left| J\left[u^0(t)\right] - J_m\left[u_k(t)\right] \right| \le C_0(\lambda) \left[C_1(\lambda) \left(|\lambda| \sqrt{\frac{K_0 T}{2\lambda_1^2}} \right)^{2m} + C_2(\lambda) \left(\frac{\gamma^k}{1-\gamma} \right)^2 \right]^{1/2} + 2\beta\sqrt{T} \varphi_0(\beta) \left\| G\left[p_0(t)\right] - p_0(t) \right\|_H \frac{\gamma^k}{1-\gamma} \xrightarrow[m,k\to\infty]{} 0,$$

where

$$C_0(\lambda) \ge \left\| v^0(T,x) + v_m^k(T,x) - 2\xi(x) \right\|_H,$$

by which the statement of the lemma follows. □

Thus, by Lemmas 6.1–6.3 it is proved that the approximate solutions $(u^k(t), v_m^k(t,x), J[u^k(t)])$ of the problem of the nonlinear optimization converge to the exact solution $(u^0(t), v^0(t,x), J[u^0(t)])$ with respect to control, optimal process and functional.

7 Conclusion

The obtained results are theoretical and can be used to develop methods for studying optimal control of systems with nonlinear by distributed parameter and constructive methods for solving them. They can be applied to solving applied problems.

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Exact Null-Controllability of Evolution Equations by Smooth Controls and Applications to Controllability of Interconnected Systems

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Abstract The aim of this work is to establish exact null-controllability conditions for a linear evolution equation in the class of smooth controls. Applications to the controllability of system consisting of two serially connected abstract control systems are considered.

Keywords Controllability by smooth controls · Interconnected evolution equations · Strongly minimal families

Mathematics Subject Classification (2010) Primary 93B05 · Secondary 93B28

1 Introduction and Problem Statement

Let X be a separable Hilbert space, and let A be an infinitesimal generator of a strongly continuous semigroup $S(t)$ in X of the class C_0 [5].

Consider the control evolution equation [5] with scalar control[1]

$$\dot{x}(t) = Ax(t) + bv(t), \quad x(0) = x_0, \tag{1.1}$$

where $x(t), x_0, b \in X$, $v(t) \in \mathbb{R}$.

2 Preliminaries

We assume that the operator A has the following properties.

1. The operator A has the purely point spectrums σ with no finite limit points. Since we use scalar controls we assume the geometrical multiplicity[2] of eigenvalues of the operator A to be equal to 1.

[1]Scalar controls will be considered only for the sake of the simplicity.

[2]The geometric multiplicity is the number of Jordan blocks corresponding to $\lambda_j \in \sigma_1$. Throughout in the paper it is equal to 1.

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2. The sequence of all eigenvectors of the operators A produces a Riesz basis in their linear span.

Let the eigenvalues $\lambda_j \in \sigma$, $j = 1, 2, \ldots$, be enumerated in the order of non-decreasing absolute values, let α_j be the algebraic multiplicities of $\lambda_j \in \sigma$ correspondingly, let φ_{jk}, $j \in \mathbb{N}$, $k = 1, 2, \ldots, \alpha_j$ be the generalized eigenvectors of the operator A, $A\varphi_{j\alpha_j} = \lambda_j \varphi_{j\alpha_j}$, $j \in \mathbb{N}$, and let ψ_{jk}, $j \in \mathbb{N}$, $k = 1, 2, \ldots, \alpha_j$, be the generalized eigenvectors of the adjoint operator A^*, $A^*\psi_{j\alpha_j} = \bar{\lambda}_j \psi_{j\alpha_j}$, $j \in \mathbb{N}$, chosen such that $(\varphi_{s\alpha_s - l+1}, \psi_{jk}) = \delta_{sj}\delta_{lk}$, $s, j \in \mathbb{N}$, $l = 1, \ldots, \alpha_s$, $k = 1, \ldots, \alpha_j$.

We use the following notations:[3] $x(t, x^0, v(\cdot))$ is a mild solution of (1.1) with initial condition $x(0) = x^0$, generated by the control $v(t)$, $x_{jk}(t) = (x(t), \psi_{jk})$, $x^0_{jk} = (x_0, \psi_{jk})$, $b_{jk} = (b, \psi_{jk})$, $j \in \mathbb{N}$, $k = 1, 2, \ldots, \alpha_j$,

$$g_{jk}(-t) = e^{-\lambda_j t} \sum_{l=0}^{\alpha_j - k} b_{jk+l} \frac{(-t)^l}{l!}, \quad t \in [0, t_1], \ j \in \mathbb{N}, \ k = 1, 2, \ldots, \alpha_j. \tag{2.1}$$

For the simplicity of the exposition we assume below that all the eigenvalues of the operator A are simple. In this case the eigenvector of the operator A, corresponding to the eigenvalue λ_j, can be denoted by ψ_j, $b_j = (b, \psi_j)$, $j = 1, 2, \ldots$, and the family of generalized exponents (2.1) can be written by exponents $\{g_j(-t) = b_j e^{-\lambda_j t}, j = 1, 2, \ldots\}$. If $0 \in \sigma$, then according to our assumption $\lambda_0 = 0$ is a simple eigenvalue, and $\sigma = \{\lambda_j, j = 0, 1, 2, \ldots,\}$. Otherwise $\sigma = \{\lambda_j, j = 1, 2, \ldots\}$. In both cases $\lambda_j \neq 0$, $j = 1, 2, \ldots$, and $\{g_j(-t) = b_j e^{-\lambda_j t}, j = 0, 1, 2, \ldots\}$.

The following property of sequences $\{x_j \in X, j = 1, 2, \ldots\}$ is very significant throughout in this paper.

Definition 2.1 The sequence $\{x_j \in X, j = 1, 2, \ldots\}$ is said to be strongly minimal, if there exists a positive number $\gamma > 0$ such that $\gamma \sum_{k=1}^n |c_k|^2 \leq \| \sum_{k=1}^n c_k x_k \|^2$, $n = 1, 2, \ldots$, where $\gamma = \lim_{n\to\infty} \min_{\substack{c_1,\ldots,c_n: \\ \sum_{k=1}^n |c_k|^2 = 1}} \| \sum_{k=1}^n c_k x_k \|^2$.

3 Controllability of (1.1) by Smooth Controls

Various types of controllability by **square integrable** and **pointwise** controls, investigated by the method of moments, and solvability conditions for the moments problem, are widely investigated in the literature (see, for example, the books of [1, 8] and references therein). Below exact null-controllability conditions for (1.1) in the class of **smooth** controls are presented. These results are also obtained by the method of moments.

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[3] If $0 \in \sigma$, we denote $\lambda_0 = 0$ and we will use $j \in 0 \cup \mathbb{N}$.

Denote: $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{m-1})$, $\beta = (\beta_0, \beta_1, \dots, \beta_{m-1}) \in \mathbb{R}^m$, $m \in \mathbb{N}$,

$$H^m_{\alpha\beta}[0, t_1] = \left\{ \begin{array}{c} v^{(k)}(\cdot) \in C^{(m-k-1)}[0, t_1], k = 0, 1, \dots, m-2, m \geq 2; \\ v^{(m-1)}(\cdot) \in AC[0, t_1], v^{(m)}(\cdot) \in L_2[0, t_1], \\ v^{(k)}(0) = a_k, v^{(k)}(t_1) = \beta_k, k = 0, 1, \dots, m-1. \end{array} \right\}.$$

Definition 3.1 Equation (1.1) is said to be exact null-controllable on $[0, t_1]$ by m-smooth controls $v(\cdot) \in H^m_{a0}[0, t_1]$, if for any $x_0 \in X$ and $\alpha \in \mathbb{R}^m$ there exists a control $v(\cdot) \in H^m_{\alpha 0}[0, t_1]$, such that $x(t_1, x_0, v(\cdot)) = 0$.

Theorem 3.2 *Equation* (1.1) *is exact null-controllable on* $[0, t_1]$ *by* m*-smooth controls* $v(\cdot) \in H^m_{a0}[0, t_1]$, *if and only if either* $0 \notin \sigma$ *or* $0 \in \sigma$ *and* $b_0 \neq 0$, *and the family*

$$\left\{1, t, \dots, t^{m-1}, \frac{b_j}{\lambda_j^m} e^{-\lambda_{1j} t}, j = 1, 2, \dots \right\} \tag{3.1}$$

us strongly minimal.

The theorem is proved by reducing the original system to the composite system with respect to the pair $(x(t), w(t))$, where $x(t)$ is a solution of equation $w(t) = (v(t), v'(t), \dots, v^{(m-1)}(t))$. Then eigenvalues and eigenvectors of the obtained system are investigated. The proof is finished by using solvability conditions for moments problem [3, 9], which are exactly equivalent to the strong minimality of the family (3.1) of generalized exponentials.

4 Applications. Controllability Conditions of Interconnected Equations by Distributed Controls

The results obtained in the previous sections may be applied to the controllability of interconnected systems.

Let X_1, X_2 be separable Hilbert spaces. Consider the control evolution equation [5] with scalar control

$$\dot{x}_1(t) = A_1 x_1(t) + b_1 v(t), \quad x_1(0) = x_1^0, \tag{4.1}$$

$$v(t) = \big(c, x_2(t)\big), \quad 0 \leq t < +\infty, \tag{4.2}$$

where $x_1(t)$ is a mild solution of (4.1) with initial condition $x_1(0) = x_1^0$, and $x_2(t)$ is a mild solution of the another control equation of the form

$$\dot{x}_2(t) = A_2 x_2(t) + b_2 u(t), \quad 0 \leq t < +\infty, \ x_2(0) = x_2^0. \tag{4.3}$$

Here $x_1(t)$, x_1^0, $b_1 \in X_1$, where X_1 is the state space of (4.1), $v(t) \in \mathbb{R}$, $x_2(t)$, x_2^0, c, $b_2 \in X_2$, where X_2 is the state space of (4.3), $u(t) \in \mathbb{R}$, and the linear operators A_1

and A_2 generate strongly continuous C_0-semigroups $S_1(t)$ in X_1 and $S_2(t)$ in X_2 correspondingly [5].

Equation (4.1) is governed by control $u(t)$ of (4.3) via the output (4.2) of (4.3).

Definition 4.1 Interconnected system (4.1)–(4.3) is said to be exact null-controllable on $[0, t_1]$ if for any $x_1^0 \in X_1$ there exists a scalar control $u(\cdot) \in L_2[0, t_1]$, such that a mild solution $x_1(t, x_1^0, v(\cdot))$ of (1.1) with a control $v(t)$ defined by (4.2) satisfies the condition $x_1(t, x_1^0, v(\cdot)) = 0$.

We know nothing about differential properties of a generalized solution $x_2(t)$, generated by control $u(\cdot) \in L_2[0, t_1]$, but in accordance with the definition of a generalized solution of (4.3) the function $v(t) = (c, x_2(t))$ defined by (4.2) is absolutely continuous for any $c \in D(A_2^*)$ [2]. In order to keep the control object in the equilibrium state, we will turn off the control $v(t)$ at the end of the control process, i.e. $v(t) \equiv 0, t \geq t_1$.

The equation

$$v(t) - \big(c, x_2^0\big) = \big(c, x_2(t)\big) = \int_0^t \big(c, S_2(t-\tau)b_2\big)u(\tau)d\tau \tag{4.4}$$

is the Volterra integral equation of the first kind with the kernel $K(t, \tau) = (c, S_2(t - \tau)b_2)$. So the exact controllability conditions of interconnected system (4.1)–(4.3) is obtained by joining of exact null controllability conditions established in Theorem 3.2 for $m = 1$ and the solvability conditions of Volterra integral equation (4.4) of the first kind with absolutely continuous function $v(\cdot)$ in the space of square integrable controls $u(\cdot)$.

4.1 Regular Case

Theorem 4.2 *If*

- *the family of exponents* $\{1, \frac{b_j}{\lambda_j} e^{-\lambda_{1j} t}, j = 1, 2, \ldots\}$ *is strongly minimal,*
- $c \in D(A_2^*)$ *and* $(c, b_2) \neq 0$,

then interconnected equation (4.1)–(4.3) *is exact null-controllable on* $[0, t_1]$.

4.2 Singular Case

If $(c, b_2) = 0$, then it turns out that the proof of theorem can be used, if everywhere in the proof to replace the vector $c \in D(A_2^*)$ by the vector $A_2^* c \in D(A_2^*)$, provided that the function $v(t)$ is continuously differentiable, $\dot{v}(\cdot)$ is absolutely continuous and $\ddot{v}(\cdot) \in L_2[0, t_1]$. So the exact controllability conditions of interconnected system

(4.1)–(4.3) is obtained by joining of exact null controllability conditions established in Theorem 3.2 for $m = 2$ and the solvability conditions of Volterra integral equation $\dot{v}(t) = \int_0^t (c, A_2 S_2(t-\tau) b_2) u(\tau) d\tau$ of the first kind with absolutely continuous function $\dot{v}(t)$ in the space of square integrable controls $u(\cdot)$.

Theorem 4.3 *If*

- *the family of exponents* $\{1, t, \frac{b_{1j}}{\lambda_j^2} e^{-\lambda_{1j} t}, j = 1, 2, \ldots\}$ *is strongly minimal,*
- $c \in D(A_2^{*^2})$, $(c, b_2) = 0$ *and* $(A_2^* c, b_2) \neq 0$,

then interconnected equation (4.1)–(4.3) *is exact null-controllable on* $[0, t_1]$.

Remark 4.4 The same approach can be used, if $c \in D(A_2^{*^m})$, $(A_2^{*^k} c, b_2) = 0$, $k = 0, 1, \ldots, m-2$, $(A_2^{*^{m-1}} c, b_2) \neq 0$ for some $m \in \mathbb{N}$, $m \geq 2$.

4.3 Strong Minimality of Real Exponentials

A direct proof of the strong minimality for a given sequence of exponents sometimes can be tough. Below we present two lemmas which substantially facilitate the establishment of the strong minimality for real exponential families.

Lemma 4.5 *If* (i) $0 < \mu_1 < \mu_2 < \cdots$, (ii) $\inf\{\mu_n - \mu_{n-1}\} > 0$, $n = 1, 2, \ldots$, (iii) *the series* $\sum_{n=1}^{\infty} \frac{1}{\mu_n}$ *converges*, (iv) $\beta_n \neq 0$, $n = 1, 2, \ldots$, *and the Dirichlet series* $\sum_{n=1}^{\infty} \frac{e^{-\mu_n \alpha}}{\beta_n}$ *converges for some* $\alpha > 0$, *then the family of real exponentials* $\{\beta_n e^{\mu_n t}, n = 1, 2, \ldots, t \in [0, t_1], \forall t_1 > 0\}$ *is strongly minimal.*

Lemma 4.6 *If conditions of Lemma* 4.5 *hold, then the sequence* $\{1, \beta_n e^{\mu_n t}, n = 1, 2, \ldots, t \in [0, t_1], \forall t_1 > 0\}$ *is strongly minimal.*

The proofs of both lemmas are based on results of [4] and [7].

5 Examples. Exact Null Controllability of Interconnected Heat Equation and Wave Equation by Distributed Control

Let $H^2[0, \pi]$, $H_0^1[0, \pi]$ be Sobolev spaces (see [6] for definitions of the spaces $H^m[a, b]$, $H_0^m[a, b]$, $a, b \in \mathbb{R}$).

We consider the heat equation with distributed control

$$y'_t = y''_{xx} + b_1(x) v(t), \quad 0 \leq t \leq t_1, \ 0 \leq x \leq \pi, \tag{5.1}$$

$$y(0, t) = y(\pi, t) = 0, \quad 0 \leq t \leq t_1, \tag{5.2}$$

$$y(x, 0) = \varphi(x), \quad 0 \leq x \leq \pi, \tag{5.3}$$

governed by a control $u(\cdot)$ of the wave equation

$$z''_{tt} - z''_{xx} = b_2(x)u(t), \quad 0 \le t \le t_1,\ 0 \le x \le \pi, \tag{5.4}$$

$$z(0,t) = z(\pi,t) = 0, \quad 0 \le t \le t_1, \tag{5.5}$$

$$z(x,0) = \psi_0(x), \quad z'_t(x,0) = \psi_1(x),\ 0 \le x \le \pi, \tag{5.6}$$

via the observation

$$v(t) = \int_0^{\pi} \left(c'_1(x) z'_x(x,t) + c_2(x) z'_t(x,t)\right) dx,$$

$$0 \le t \le t_1\pi,\ c_2(\cdot) \in L_2[0,\pi] \tag{5.7}$$

of wave equation (5.4)–(5.6), where $c(\cdot) = (c_1(\cdot), c_2(\cdot)) \in H_0^1[0,\pi] \times L_2[0,\pi]$ and $\varphi, \psi_0, \psi_1, b_1, b_2 \in L_2[0,\pi]$.

5.1 Controllability Conditions

Denote $b_{1n} = \int_0^{\pi} b_1(x) \sin nx$, $n = 1, 2, \dots$.

Theorem 5.1 *If the Dirichlet series* $\sum_{n=1}^{\infty} \frac{n^2}{b_{1n}} e^{-n^2\alpha}$ *converges for some* $\alpha > 0$, *then* (5.1)–(5.3) *is exact null-controllable on* $[0, t_1]$, $\forall t_1 > 0$, *by smooth controls.*

Theorem 5.2 *If*

1. *the Dirichlet series* $\sum_{n=1}^{\infty} \frac{n^2}{b_{1n}} e^{-n^2\alpha}$ *converges for some* $\alpha > 0$,
2. $b_2(\cdot), c_2(\cdot) \in L_2[0,\pi]$,
3. $\int_0^{\pi} c_2(x) b_2(x) dx \neq 0$,

then system equation (5.1)–(5.3), (5.4)–(5.6), *interconnected by* (5.7) *is exact null-controllable on* $[0, t_1]$, $\forall t_1 > 0$.

For example, conditions of Theorem 5.2 hold true for $b_1(x) = b_2(x) = x$, $c_2(x) = 1$, $x \in [0,\pi]$.

6 Conclusion

Exact null-controllability conditions for abstract control equation (1.1) in the class of smooth controls is established.

One of applications of these results is the exact null-controllability conditions for two interconnected abstract control equations (4.1)–(4.3) governed by a scalar distributed control $u(t)$ of (4.3).

Of course, these results can be extended for series of a number interconnected equations, governed by a control of the last one.

The mutual independence of two exact null-controllability conditions for interconnected systems allow us to use the abstract approach developed in the paper for investigation of various control problems for interconnected systems contained equations of a different nature. For example, (4.1) may be a parabolic control equation, governed by distributed control, and (4.3) may be a linear differential control system with delays. The singular case does not seem to be essential (in our opinion), because there are a lot of practical situations, for which (1.1) and (4.3) are given, and need to decide, how to connect them. It means, that if the case $(c, b_2) = 0$ occurs, one can always choose other vector c, slightly different from the first one, such that $(c, b_2) \neq 0$.

The exact null-controllability for interconnected heat-wave equations is considered as illustrative example only.

The unifying abstract approach for the controllability problem of abstract evolution equations by smooth boundary controls will be considered later.

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On the Sum of Contractive Type of Mappings I: Maps on the Same Class

J.R. Morales and E.M. Rojas

Abstract In this paper, we will show that under some conditions the sum of two mappings belonging to a contractive class of maps is a mapping on the same class (but with different contractive parameters).

Keywords Banach space · Fixed point · Contractive mapping

Mathematics Subject Classification (2010) Primary 47H09 · 47H10 · Secondary 54E50

1 Motivation and Preliminaries

Many real world applications are modeled by equations whose corresponding operators can be decompose as the sum of two or more well known operators. For instance, many concrete problems have recently appeared in mathematical physics such as *boundary-transmission problems for the Helmholtz equation* which arises within the context of the analysis of problems of wave diffraction by wedges, that can be reduced to equations characterized by Wiener–Hopf plus Hankel operators [3–6, 12]. This is also the case of some problems in *mechanics* and *control theory, electro-magnetic fluid dynamic* and *reformulation of boundary value problems with a nonlinear boundary condition* which are modeled by Volterra–Fredholm–Hammerstein integral equations [1, 2] and *neural networks* and *spread of diseases* studied by integro-differential equations [16, 21].

On the other hand, the metric theory of fixed points is a very important tool in the study of solvability of equations, giving conditions under which maps have solutions. However, despite that if we consider equations whose related operators have

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well-developed fixed point's existence conditions, e.g. Volterra or Fredholm second kind integral operators and, in general, mapping of contractive type, the conditions for the existence and uniqueness of fixed points for linear combinations of these operators is not clear (and therefore the solvability conditions of the associated equation).

Thus, this paper is devoted to study conditions under which the sum of two mappings with fixed point also has a fixed point. For this aim, we will consider some classical classes of contractive type mappings, for which we will prove that under some conditions the sum of two mappings on each one of these classes belongs to the same class but with different contractive parameters. To attain our goals, we are going to assume that a Diaz–Metcalf type inequality is hold.

Proposition 1.1 (Diaz and Metcalf, 1968, [10]) *If $F : X \longrightarrow \mathbb{R}$ is a linear functional of a unit norm defined on the normed linear space X endowed with the norm $\|\cdot\|$ and the vectors $x_1, \ldots, x_n$ satisfy the condition*

$$0 \le r \le F(x_i) \quad i \in \{1, \ldots, n\}$$

then

$$r \sum_{i=1}^{n} \|x_i\| \le \left\| \sum_{i=1}^{n} x_i \right\|,$$

where equality holds if and only if both

$$F\left(\sum_{i=1}^{n} x_i\right) = r \sum_{i=1}^{n} \|x_i\| \quad \text{and} \quad F\left(\sum_{i=1}^{n} x_i\right) = \left\| \sum_{i=1}^{n} x_i \right\|.$$

In this paper we are going to consider mappings satisfying the next classical result.

Theorem 1.2 *Let (M, d) be a complete metric space and $T : M \longrightarrow M$ a map. Then T has a fixed point in M if it satisfies any of the following conditions*:

BC(α) (*Banach*, 1922, *see* [15]) *T is an α-contraction or Banach contraction, this is*:

$$d(Tx, Ty) \le \alpha d(x, y) \quad \forall x, y \in M,\ 0 \le \alpha < 1.$$

KA(α) (*Kannan*, 1969, 1971, [17, 18]) *T satisfies*: *there is $\alpha \in [0, \frac{1}{2})$ such that*

$$d(Tx, Ty) \le \alpha\big(d(x, Tx) + d(y, Ty)\big) \quad \forall x, y \in M.$$

CH(α) (*Chatterge*, 1972, [7]) *T satisfies the following condition*: *there is $\alpha \in [0, \frac{1}{2})$ such that*

$$d(Tx, Ty) \le \alpha\big(d(x, Ty) + d(y, Tx)\big) \quad \forall x, y \in M.$$

RE$(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ (*Reich*, 1971, [24, 25]) *T satisfies*:

$$d(Tx, Ty) \le a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty),$$

for all $x, y \in M$, *with* $0 \le a_1 + a_2 + a_3 < 1$.

RH$(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ (*B.E. Rhoades*, 1977, [26] *or see* [19]) *T satisfies*:

$$d(Tx, Ty) \le a_1 d(x, y) + a_2 d(x, Ty) + a_3 d(y, Tx),$$

for all $x, y \in M$, $0 \le a_1 + a_2 + a_3 < 1$.

HR$(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5)$ (*Hardy and Rogers*, 1973, [14] *or see* [13, 20] *for instance*) $\forall x, y \in M$, *T satisfies*: *there are* $a_i \ge 0$ *such that* $A = \sum_{i=1}^{5} a_i < 1$ *and*

$$d(Tx, Ty) \le a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty) + a_5 d(y, Tx).$$

D$(\mathbf{a}, \mathbf{b})$ (*L. Nova*, 1986, [23] *or see* [8, 9] *for instance*) $K \subset M$ *closed and* $T : K \longrightarrow K$ *an arbitrary operator that satisfies the following condition*, *for* $a, b \ge 0$ *and any* $x, y \in K$

$$d(Tx, Ty) \le ad(x, y) + b\big[d(x, Tx) + d(y, Ty)\big].$$

We shall say T belong or is of class $BC(\alpha)$ (respectively $KA(\alpha)$, $CH(\alpha)$, $RE(a_1, a_2, a_3)$, $RH(a_1, a_2, a_3)$, $HR(a_1, a_2, a_3, a_4, a_5)$, $D(a, b)$) when T satisfies the condition $BC(\alpha)$ (respectively $KA(\alpha)$, $CH(\alpha)$, $RE(a_1, a_2, a_3)$, $RH(a_1, a_2, a_3)$, $HR(a_1, a_2, a_3, a_4, a_5)$, $D(a, b)$) where α indicates the contraction's constant (the same indicate the parameters in each of the remainder classes).

Examples showing that the above conditions are independent among each other we can be found in [22]. Also, in [22] the following result was proved.

Theorem 1.3 *Let X be a strictly convex Banach space*, *and let* $S, T : B_X \longrightarrow B_X$, *where* B_X *is the open unit ball of X*. *If the following conditions hold*

(i) $S, T \in D(a, b)$.
(ii) $x - Tx = r(x - Sx)$ *for any scalar r and every* $x \in B_X$.

Then $S + T \in D(a, b)$ *for* $a + b$ *sufficiently small*.

In the next section we will show analogous results for the classes of mappings consider in the theorem above.

2 Main Results

As was established in the introduction, our principal objective is the existence of a fixed point for the map resulting of the sum of two contractive type of mappings. For this reason, we are going to consider the contractive parameters on each one of the classes considered sufficiently small such that the uniqueness of the fixed point can be guarantee for the mappings on each class (Theorem 1.2).

Let $(X, \|\cdot\|)$ be a Banach space and $T, S : X \longrightarrow X$ be two mappings. To establish our results we are going to assume that the Diaz–Metcalf's Theorem is satisfies for $(I - T)x$ and $(I - S)x$ for each $x \in X$. I.e.,

$$\begin{cases} 0 < r \leq F(x - Tx) \\ 0 < r \leq F(x - Sx), \end{cases} \qquad \text{for all } x \in X. \tag{1}$$

We would like to point out that the case when $0 = r = F(Tx - x) = F(Sx - x)$ corresponds to the case when x in the common fixed point for the pair (T, S) which or is unique nor exists. This fact justify that in our results we consider only the case $r > 0$.

First, notice that the closeness under sum for Banach contraction mappings can be assure without any extra assumption.

Proposition 2.1 *Let X be a Banach space, and $T, S : X \longrightarrow X$ be of classes $BC(\alpha_1)$, and $BC(\alpha_2)$ respectively, then $T + S \in BC(\alpha_1 + \alpha_2)$.*

Proof Let $x, y \in X$ and $T, S : X \longrightarrow X$

$$\begin{aligned} \big\|(T+S)x - (T+S)y\big\| &\leq \|Tx - Ty\| + \|Sx - Sy\| \\ &\leq \alpha_1\|x - y\| + \alpha_2\|x - y\| = (\alpha_1 + \alpha_2)\|x - y\|. \quad \square \end{aligned}$$

Theorem 2.2 *Let X be a Banach space, and let $T, S : B_X(r) \longrightarrow B_X(r)$, where $B_X(r)$ is the ball of X with radius $r \in (0, 1]$. If the following conditions hold*

(i) $T \in KA(\alpha_1)$, $S \in KA(\alpha_2)$.
(ii) *For each $x \in X$, the Diaz–Metcalf's condition (1) holds.*

Then $S + T \in KA(\mu^*)$, where $\mu^* = \max(\alpha_1/r, \alpha_2/r)$.

Proof Let $x, y \in B_X(r)$

$$\begin{aligned} &\big\|(T+S)x - (T+S)y\big\| \\ &\quad \leq \|Tx - Ty\| + \|Sx - Sy\| \\ &\quad \leq \alpha_1\big(\|x - Tx\| + \|y - Ty\|\big) + \alpha_2\big(\|x - Sx\| + \|y - Sy\|\big) \\ &\quad \leq \mu\big(\|x - Tx\| + \|y - Ty\| + \|x - Sx\| + \|y - Sy\|\big) \end{aligned}$$

where $\mu = \max(\alpha_1, \alpha_2)$.

Condition (ii) implies that $\|x - Tx\| + \|x - Sx\| \leq \frac{1}{r}\|2x - (T+S)x\|$, thus, we have

$$\begin{aligned}\big\|(T+S)x - (T+S)y\big\| &\leq \frac{\mu}{r}\big[\big\|2x - (T+S)x\big\| + \big\|2y - (T+S)y\big\|\big] \\ &\leq \frac{\mu}{r}\big[\|x\| + \big\|x - (T+S)x\big\| + \|y\| + \big\|y - (T+S)y\big\|\big] \\ &\leq \frac{\mu}{r}\big[\big\|x - (T+S)x\big\| + \big\|y - (T+S)y\big\|\big] + 2\mu. \qquad (2)\end{aligned}$$

Since we are considering contractive parameters μ sufficiently small, in particular we choose it such that the next inequality is fulfilled

$$\big\|(T+S)x - (T+S)y\big\| \leq \mu^*\big[\big\|x - (T+S)x\big\| + \big\|y - (T+S)y\big\|\big]. \qquad (3)$$

Hence $T + S \in KA(\mu^*)$, where $\mu^* = \max(\alpha_1/r, \alpha_2/r)$. □

Remark 1 We would like to point out that the value of μ^* is not unique. For instance, if we assume

$$\mu \leq \inf_{x,y \in B_X(r)} \left\{ \frac{1}{2\beta r}\big[\big\|x - (T+S)x\big\| + \big\|y - (T+S)y\big\|\big] \right\},$$

where $\beta > 0$, then from the inequality (2), the inequality (3) holds for $\mu^* = \frac{1}{r}(\mu + \frac{1}{\beta})$.

In a similar way, different assumptions on the contractive parameters of each one of the classes of mappings in consideration here, gives different values for the contractive parameters on the resulting class.

Theorem 2.3 *Let X be a Banach space, and let $T, S : B_X(r) \longrightarrow B_X(r)$. If the following conditions hold*

(i) $T \in CH(\alpha_1)$, $S \in CH(\alpha_2)$.
(ii) *For each $x \in X$, the Diaz–Metcalf's condition (1) holds.*

Then $S + T \in CH(\mu^*)$, where $\mu^* = \max(\alpha_1/r, \alpha_2/r)$.

Proof Let $x, y \in B_X(r)$

$$\begin{aligned}&\big\|(T+S)x - (T+S)y\big\| \\ &\quad \leq \|Tx - Ty\| + \|Sx - Sy\| \\ &\quad \leq \alpha_1\big(\|x - Ty\| + \|y - Tx\|\big) + \alpha_2\big(\|x - Sy\| + \|y - Sx\|\big) \\ &\quad \leq \mu\big(\|x - Ty\| + \|y - Tx\| + \|x - Sy\| + \|y - Sx\|\big)\end{aligned}$$

where $\mu = \max(\alpha_1, \alpha_2)$.

The Diaz–Metcalf's condition (1) implies that $\|x - Ty\| + \|x - Sy\| \leq \frac{1}{r}\|2x - (T + S)y\|$. Therefore

$$\begin{aligned}\big\|(T+S)x-(T+S)y\big\| &\leq \frac{\mu}{r}\big[\big\|2x-(T+S)y\big\|+\big\|2y-(T+S)x\big\|\big]\\ &\leq \frac{\mu}{r}\big[\|x\|+\big\|x-(T+S)y\big\|+\|y\|+\big\|y-(T+S)x\big\|\big]\\ &\leq \frac{\mu}{r}\big[\big\|x-(T+S)y\big\|+\big\|y-(T+S)x\big\|\big]+2\mu.\end{aligned}$$

As before, from here we get

$$\big\|(T+S)x-(T+S)y\big\| \leq \mu^*\big[\big\|x-(T+S)y\big\|+\big\|y-(T+S)x\big\|\big].$$

Hence $T + S \in KA(\mu^*)$, where $\mu^* = \max(\alpha_1/r, \alpha_2/r)$. □

Theorem 2.4 *Let X be a Banach space, and let $T, S : B_X(r) \longrightarrow B_X(r)$. If the following conditions hold*

(i) $T \in RE(a_1, a_2, a_3)$, $S \in RE(b_1, b_2, b_3)$.
(ii) *For each $x \in X$, the Diaz–Metcalf's condition* (1) *holds.*

Then $S + T \in RE(\mu_1, \mu_2^*, \mu_3^*)$, where $\mu_1 = a_1 + b_1$, $\mu_2^* = \max(a_2/r, b_2/r)$, $\mu_3^* = \max(a_3/r, b_3/r)$.

Proof Let $x, y \in B_X(r)$

$$\begin{aligned}\big\|(T+S)x-(T+S)y\big\| &\leq \|Tx-Ty\|+\|Sx-Sy\|\\ &\leq a_1\|x-y\|+a_2\|x-Tx\|+a_3\|y-Ty\|\\ &\quad+b_1|x-y\|+b_2\|x-Sx\|+b_3\|y-Sy\|\\ &= \mu_1\|x-y\|+a_2\|x-Tx\|+a_3\|y-Ty\|\\ &\quad+b_2\|x-Sx\|+b_3\|y-Sy\|\\ &\leq \mu_1\|x-y\|+\mu_2\big[\|x-Tx\|+\|x-Sx\|\big]\\ &\quad+\mu_3\big[\|y-Ty\|+\|y-Sy\|\big],\end{aligned}$$

where $\mu_1 = a_1 + a_2$, $\mu_2 = \max(a_2, b_2)$ and $\mu_3 = \max(a_3, b_3)$. Again condition (ii) implies that

$$\|x-Tx\|+\|x-Sx\| \leq \frac{1}{r}\big\|2x-(T+S)x\big\|,$$

then we obtain

$$\begin{aligned}\big\|(T+S)x-(T+S)y\big\| &\leq \mu_1\|x-y\|+\frac{\mu_2}{r}\big\|2x-(T+S)x\big\|\\ &\quad+\frac{\mu_3}{r}\big\|2y-(T+S)y\big\|\end{aligned}$$

$$\begin{aligned}&\le \mu_1\|x-y\| + \frac{\mu_2}{r}\big[\|x\| + \|x-(T+S)x\|\big]\\&\quad + \frac{\mu_3}{r}\big[\|y\| + \|y-(T+S)y\|\big]\\&\le \mu_1\|x-y\| + \frac{\mu_2}{r}\big[\|x-(T+S)x\|\big]\\&\quad + \frac{\mu_3}{r}\big[\|y-(T+S)y\|\big] + \mu_2 + \mu_3.\end{aligned}$$

Since, $\mu_2+\mu_3$ can be as small as we please, we have

$$\|(T+S)x-(T+S)y\| \le \mu_1\|x-y\| + \mu_2^*\|x-(T+S)x\| + \mu_3^*\|y-(T+S)y\|.$$

So, $T+S \in RE(\mu_1, \mu_2^*, \mu_3^*)$, where $\mu_1 = a_1 + b_1$, $\mu_2^* = \max(a_2/r, b_2/r)$, $\mu_3^* = \max(a_3/r, b_3/r)$. □

Theorem 2.5 *Let X be a Banach space, and let $T, S : B_X(r) \longrightarrow B_X(r)$. If the following conditions hold*

(i) $T \in RH(a_1, a_2, a_3)$, $S \in RH(b_1, b_2, b_3)$.
(ii) *For each $x \in X$, the Diaz–Metcalf's condition (1) holds.*

Then $S+T \in RH(\mu_1, \mu_2^*, \mu_3^*)$, where $\mu_1 = a_1 + b_1$, $\mu_2^* = \max(a_2/r, b_2/r)$, $\mu_3^* = \max(a_3/r, b_3/r)$.

Proof This theorem can be proved as the previous theorem. In this case the Diaz–Metcalf's condition (1) guarantee that the following inequality holds

$$\|x-Ty\| + \|x-Sy\| \le \frac{1}{r}\|2x-(T+S)y\|.$$

From here the proof runs analogously to Theorem 2.4. □

Theorem 2.6 *Let X be a Banach space, and let $T, S : B_X(r) \longrightarrow B_X(r)$. If the following conditions hold*

(i) $T \in HR(a_1, a_2, a_3, a_4, a_5)$, $S \in HR(b_1, b_2, b_3, b_4, b_5)$.
(ii) *For each $x \in X$, the Diaz–Metcalf's condition (1) holds.*

Then $S+T \in HR(\mu_1, \mu_2^*, \mu_3^*, \mu_4^*, \mu_5^*)$, where $\mu_1 = a_1+b_1$, $\mu_i^* = \max(a_i/r, b_i/r)$, $i = 2, \dots, 5$.

Proof Let $x, y \in B_X(r)$

$$\begin{aligned}&\|(T+S)x-(T+S)y\|\\&\quad\le \|Tx-Ty\| + \|Sx-Sy\|\\&\quad\le a_1\|x-y\| + a_2\|x-Tx\| + a_3\|y-Ty\| + a_4\|x-Ty\|\end{aligned}$$

$$
\begin{aligned}
&+a_5\|y-Tx\|+b_1|x-y\|+b_2\|x-Sx\|+b_3\|y-Sy\|\\
&+b_4\|x-Sy\|+b_5\|y-Sx\|\\
=&\ \mu_1\|x-y\|+a_2\|x-Tx\|+a_3\|y-Ty\|+a_4\|x-Ty\|\\
&+a_5\|y-Tx\|+b_2\|x-Sx\|+b_3\|y-Sy\|\\
&+b_4\|x-Sy\|+b_5\|y-Sx\|,
\end{aligned}
$$

where $\mu_1 = a_1 + b_1$, therefore we have

$$
\begin{aligned}
&\big\|(T+S)x-(T+S)y\big\|\\
&\quad\le \mu_1\|x-y\|+\mu_2\big[\|x-Tx\|+\|x-Sx\|\big]\\
&\qquad+\mu_3\big[\|y-Ty\|+\|y-Sy\|\big]+\mu_4\big[\|x-Ty\|\|x-Sy\|\big]\\
&\qquad+\mu_5\big[\|y-Tx\|+\|y-Sx\|\big]
\end{aligned}
$$

here, $\mu_i = \max(a_i, b_i)$, $i = 2, \ldots, 5$. From The Diaz–Metcalf's condition (1) we conclude that

$$
\|x-Ty\|+\|x-Sy\|\le \frac{1}{r}\big\|2x-(T+S)y\big\|.
$$

So, we obtain the following estimates

$$
\begin{aligned}
&\big\|(T+S)x-(T+S)y\big\|\\
&\quad\le \mu_1\|x-y\|+\frac{\mu_2}{r}\big\|2x-(T+S)x\big\|+\frac{\mu_3}{r}\big\|2y-(T+S)y\big\|\\
&\qquad+\frac{\mu_4}{r}\big\|2x-(T+S)y\big\|+\frac{\mu_5}{r}\big\|2y-(T+S)x\big\|.
\end{aligned}
$$

Let $\mu_i^* = \frac{\mu_i}{r}$, $i = 2, \ldots, 5$. Then

$$
\begin{aligned}
\big\|(T+S)x-(T+S)y\big\| \le&\ \mu_1\|x-y\|+\mu_2^*\big[\|x\|+\big\|x-(T+S)x\big\|\big]\\
&+\mu_3^*\big[\|y\|+\big\|y-(T+S)y\big\|\big]\\
&+\mu_4^*\big[\|x\|+\big\|x-(T+S)y\big\|\big]\\
&+\mu_5^*\big[\|y\|+\big\|y-(T+S)x\big\|\big]\\
\le&\ \mu_1\|x-y\|+\mu_2^*\big\|x-(T+S)x\big\|+\mu_3^*\big\|y-(T+S)y\big\|\\
&+\mu_4^*\big\|x-(T+S)y\big\|+\mu_5^*\big\|y-(T+S)x\big\|\\
&+\mu_2+\mu_3+\mu_4+\mu_5.
\end{aligned}
$$

Again, $\sum_{i=2}^{5} \mu_i$, can be as small as we please and just like Theorems before, we have

$$\big\|(T+S)x-(T+S)y\big\| \le \mu_1\|x-y\| + \mu_2^*\big\|x-(T+S)x\big\| + \mu_3^*\big\|y-(T+S)y\big\| + \mu_4^*\big\|x-(T+S)y\big\| + \mu_5^*\big\|y-(T+S)x\big\|.$$

Thus we conclude that, $T+S \in HR(\mu_1, \mu_2^*, \mu_3^*, \mu_4^*, \mu_5^*)$, where $\mu_1 = a_1 + b_1$, $\mu_i^* = \max(a_i/r, b_i/r)$, $i = 2, \dots, 5$. □

Theorem 2.7 *Let X be a Banach space, and let $S, T : B_X(r) \longrightarrow B_X(r)$. If the following conditions hold*

(i) $S, T \in D(a,b)$.
(ii) *For each $x \in X$, the Diaz–Metcalf's condition (1) holds.*

Then $S+T \in D(a, b/r)$ for $ar+b$ sufficiently small.

Proof Let $x, y \in B_X(r)$

$$\|Tx-Ty\| \le a\|x-y\| + b\big[\|x-Tx\| + \|y-Ty\|\big]$$
$$\|Sx-Sy\| \le a\|x-y\| + b\big[\|x-Sx\| + \|y-Sy\|\big].$$

Then,

$$\|Tx-Ty\| + \|Sx-Sy\| \le 2a\|x-y\| + b\big[\|x-Tx\| + \|y-Ty\| + \|x-Sx\| + \|y-Sy\|\big]$$
$$\big\|(S+T)x-(S+T)y\big\| \le 2a\|x-y\| + b\big[\|x-Tx\| + \|y-Ty\| + \|x-Sx\| + \|y-Sy\|\big].$$

The condition (ii) gives us the estimate $\|x-Tx\| + \|x-Sx\| \le \frac{1}{r}\|2x-(T+S)x\|$. From which we obtain

$$\begin{aligned}
&\big\|(S+T)x-(S+T)y\big\| \\
&\le 2a\|x-y\| + \frac{b}{r}\big[\|2x-Tx-Sx\| + \|2y-Ty-Sy\|\big] \\
&= 2a\|x-y\| + \frac{b}{r}\big[\big\|2x-(T+S)x\big\| + \big\|2y-(S+T)y\big\|\big] \\
&\le 2a\|x-y\| + \frac{b}{r}\big[\big\|x-(S+T)x\big\| + \|x\| + \big\|y-(S+T)y\big\| + \|y\|\big] \\
&= 2a\|x-y\| + \frac{b}{r}\big[\big\|x-(S+T)x\big\| + \big\|y-(S+T)y\big\|\big] + \frac{b}{r}\big(\|x\| + \|y\|\big) \\
&\le a\|x-y\| + \frac{b}{r}\big[\big\|x-(S+T)x\big\| + \big\|y-(S+T)y\big\|\big] + 2(ar+b).
\end{aligned}$$

Since $ar + b$ can be as small as we please, we have

$$\|(S+T)x - (S+T)y\| \leq a\|x-y\| + \frac{b}{r}\big[\|x-(S+T)x\| + \|y-(S+T)y\|\big].$$

Hence $S + T \in D(a, b/r)$. □

3 Remarks

Notice that in our results we consider the radius r of the ball $B_X(r)$ the same that the Diaz–Metcalf parameter in (1). However we can relax such dependence by considering the unit ball on X, $B_X(1)$, and taking the contractive parameters on each class of mappings sufficiently smaller than the Diaz–Metcalf parameter r. For instance, if in Theorem 2.2 we assume that the contractive parameters satisfy $0 \leq \alpha_1, \alpha_2 \ll r$, then the conclusion is maintained and the proof runs analogously with obvious changes.

Also, we would like to point out that there are several inequalities of the type Diaz–Metcalf that can replace the condition (ii) in our results. These alternatives include the consideration of more that one linear functional, as well as the explicit construction of such functional for the case Hilbert spaces, see [11].

If we consider mappings acting into $B_X(1)$, where X is a strictly convex Banach space, as in [22], we can replace the Diaz–Metcalf condition (condition (ii) in our results) for the more suitable one:

$$Tx - x = k(x - Sx) \quad \text{for any scalar } k \text{ and every } x \in B_X(1)$$

and the conclusions of our results are still valid.

Even more, if in Theorems 2.2, 2.3, 2.4, 2.5 and 2.6 we define the mappings on X instead of $B_X(1)$, then we have a similar result of the theorems above, only for the points of the form $x = \lambda y$, $x, y \in X$ and any scalar λ. On the other words, similar conclusions are hold for x out of $B_X(1)$ and for y any scalar multiple of x.

Let A and B be classes of mappings. By $A + B$ we will mean the sum of the mappings $T \in A$ and $S \in B$, and $A + B = C$ will mean that the mapping $T + S$ belongs to the class C.

Proposition 3.1 *Let X be a strictly convex Banach space, and $T, S : X \longrightarrow X$, suppose that $x - Ty = r(x - Sy)$ for any scalar r and every $x, y \in X$. If in addition, $x = \lambda y$. Then*

(i) $KA(\alpha) + KA(\beta) = D(\mu, \mu)$, $\mu = \max(\alpha, \beta)$.
(ii) $CH(\alpha) + CH(\beta) = RH(\mu, \mu, \mu)$, $\mu = \max(\alpha, \beta)$.
(iii) $RE(a_1, a_2, a_3) + RE(b_1, b_2, b_3) = RE(\mu_4, \mu_2, \mu_3)$, $\mu_4 = \mu_1 + \max(\mu_2, \mu_3)$, $\mu_1 = a_1 + b_1$, $\mu_2 = \max(a_2, b_2)$, $\mu_3 = \max(a_3, b_3)$.
(iv) $RH(a_1, a_2, a_3) + RH(b_1, b_2, b_3) = RH(\mu_4, \mu_2, \mu_3)$, $\mu_1 = a_1 + b_1$, $\mu_2 = \max(a_2, b_2)$, $\mu_3 = \max(a_3, b_3)$, $\mu_4 = \mu_1 + \max(\mu_2, \mu_3)$.

(v) $HR(a_1, a_2, a_3, a_4, a_5) + HR(b_1, b_2, b_3, b_4, b_5) = HR(\mu, \mu_2, \mu_3, \mu_4, \mu_5)$, $\mu_1 = a_1 + b_1$, $\mu_i = \max(a_i, b_i)$, $i = 2, \ldots, 5$, $\mu = \mu_1 + \max(\mu_2 + \mu_4, \mu_3 + \mu_5)$.

Proof The proof follows from Theorems 2.2, 2.3, 2.4, 2.5 and 2.6. Using the hypothesis $x = \lambda y$, we can guarantee that $\|x\| + \|y\| = \|x - y\|$, the rest is repeat each proof of Theorems mentioned above. □

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On (α, ψ) Contractions of Integral Type on Generalized Metric Spaces

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Abstract In this paper, we investigate the existence and uniqueness of fixed points of (α, ψ)-contractive mappings of integral type in complete generalized metric spaces, introduced by Branciari. Our results generalize and improve several results in literature.

Keywords Generalized metric spaces · α-ψ contractions of integral type

Mathematics Subject Classification (2010) Primary 46T99 · Secondary 47H10 · 54H25

1 Introduction and Preliminaries

The notion of a generalized metric, also known as rectangular metric, was introduced by Branciari [1] via replacing the triangle inequality with a quadrilateral inequality. It is evident that triangle inequality is more strong then quadrilateral inequality, and hence each metric space is generalized metric space, but, the converse of this statement is false [1]. In this initial paper, Branciari [1] successfully defined an open ball and hence a topology on generalized metric space. As it is expected, the topology of the generalized metric space could not provide some basic topological properties, such as,

(P1) generalized metric needs not to be continuous,
(P2) a convergent sequence in generalized metric space needs not to be Cauchy,
(P3) generalized metric space needs not to be Hausdorff, and hence the uniqueness of limits can not be guaranteed.

Dedicated to Prof. V.P. Zakharyuta.

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Despite the weakness of the topology of generalized metric space, Branciari [1] proved the analog of well-known Banach Contraction Principle without any further conditions. On the other hand, the proofs of Branciari [1] had gaps which was pointed first by Samet [2]. Later, it was understood that these gaps can be annihilated in some recent works see e.g. [3–5]. Another interesting result, the characterization of Caristi theorem in the context of generalized metric spaces, was obtained [4]. Getting fundamental theorems of fixed point theory in the setting of generalized metric space without additional conditions has attracted attention of several authors (see e.g. [2–19]).

In this paper, we investigate the existence and uniqueness of fixed point of α-ψ contraction mappings of integral type, in the setting of generalized metric spaces by regarding the problems (P1)–(P3) above.

We, first, recall some basic definitions, notations and fundamental results that will be used in the sequel.

Let Ψ be the family of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(i) ψ is nondecreasing;
(ii) there exist $k_0 \in \mathbb{N}$ and $a \in (0, 1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that

$$\psi^{k+1}(t) \leq a\psi^k(t) + v_k,$$

for $k \geq k_0$ and any $t \in \mathbb{R}^+$.

These functions are called as either Bianchini–Grandolfi gauge functions (see e.g. [20–22]) or (c)-comparison functions (see e.g. [23]).

Lemma 1 (See e.g. [23]) *If $\psi \in \Psi$, then the following hold*:

(i) $(\psi^n(t))_{n\in\mathbb{N}}$ *converges to* 0 *as* $n \to \infty$ *for all* $t \in \mathbb{R}^+$;
(ii) $\psi(t) < t$, *for any* $t \in \mathbb{R}^+$;
(iii) ψ *is continuous at* 0;
(iv) *the series* $\sum_{k=1}^{\infty} \psi^k(t)$ *converges for any* $t \in \mathbb{R}^+$.

In what follows, we recollect the notion of generalized metric spaces.

Definition 2 ([1]) Let X be a nonempty set and let $d : X \times X \longrightarrow [0, \infty]$ satisfy the following conditions for all $x, y \in X$ and all distinct $u, v \in X$ each of which is different from x and y

$$\begin{array}{ll} \text{(GMS1)} & d(x, y) = 0 \quad \text{if and only if } x = y \\ \text{(GMS2)} & d(x, y) = d(y, x) \\ \text{(GMS3)} & d(x, y) \leq d(x, u) + d(u, v) + d(v, y). \end{array} \tag{1.1}$$

Then the map d is called generalized metric and abbreviated as GMS. Here, the pair (X, d) is called generalized metric space.

In the above definition, if d satisfies only (GMS1) and (GMS2), then it is called semimetric (see e.g. [6]).

The basic topological properties, such as concepts of convergence, Cauchy sequence and completeness in a GMS are defined below.

Definition 3

1. A sequence $\{x_n\}$ in a GMS (X, d) is GMS convergent to a limit x if and only if $d(x_n, x) \to 0$ as $n \to \infty$.
2. A sequence $\{x_n\}$ in a GMS (X, d) is GMS Cauchy if and only if for every $\varepsilon > 0$ there exists positive integer $N(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$ for all $n > m > N(\varepsilon)$.
3. A GMS (X, d) is called complete if every GMS Cauchy sequence in X is GMS convergent.

Wilson [6] the following assumption to weakened the triangle inequality by preserving the benefits.

(W) For each pair of (distinct) points u, v there is a number $r_{u,v} > 0$ such that for every $z \in X$,

$$r_{u,v} < d(u, z) + d(z, v).$$

Proposition 4 ([4]) *In a semimetric space, the assumption* (W) *is equivalent to the assertion that limits are unique.*

Proposition 5 ([4]) *Suppose that* $\{x_n\}$ *is a Cauchy sequence in a GMS* (X, d) *with* $\lim_{n\to\infty} d(x_n, u) = 0$, *where* $u \in X$. *Then* $\lim_{n\to\infty} d(x_n, z) = d(u, z)$ *for all* $z \in X$. *In particular, the sequence* $\{x_n\}$ *does not converge to* z *if* $z \neq u$.

Definition 6 ([24]) For a nonempty set X, let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be mappings. We say that T is α-admissible if for all $x, y \in X$, we have

$$\alpha(x, y) \geq 1 \quad \Longrightarrow \quad \alpha(Tx, Ty) \geq 1. \tag{1.2}$$

We refer e.g. [24–29] for interesting examples of such mappings.

In what follows, the notion of α-ψ contractive mapping is defined.

Definition 7 ([24]) Let (X, d) be a metric space and $T : X \to X$ be a given mapping. We say that T is an α-ψ contractive mapping if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that

$$\alpha(x, y)d(Tx, Ty) \leq \psi\big(d(x, y)\big), \quad \text{for all } x, y \in X. \tag{1.3}$$

Note that, any contractive mapping, that is, a mapping satisfying Banach contraction, is an α-ψ contractive mapping with $\alpha(x, y) = 1$ for all $x, y \in X$ and $\psi(t) = kt$, $k \in (0, 1)$. The notion of transitivity of mapping $\alpha : X \times X \to [0, +\infty)$ was introduced in [30, 31] as follows:

Definition 8 (See [30, 31]) Let $N \in \mathbb{N}$. We say that α is N-transitive (on X) if

$$x_0, x_1, \ldots, x_{N+1} \in X : \alpha(x_i, x_{i+1}) \geq 1,$$

for all $i \in \{0, 1, \ldots, N\} \Rightarrow \alpha(x_0, x_{N+1}) \geq 1$.

In particular, we say that α is transitive if it is 1-transitive, i.e.,

$$x, y, z \in X : \alpha(x, y) \geq 1 \quad \text{and} \quad \alpha(y, z) \geq 1 \Rightarrow \alpha(x, z) \geq 1.$$

2 Main Results

We shall present our main results in this section. First, we define $\Phi = \{\varphi : \varphi : \mathbb{R}^+ \to \mathbb{R}\}$ such that φ is nonnegative, Lebesgue integrable, and satisfy

$$\int_0^{\epsilon} \varphi(t)dt > 0 \quad \text{for each } \epsilon > 0. \tag{2.1}$$

In what follows that we introduce the notion of α-ψ-contractive type mappings of integral type.

Definition 9 Let (X, d) be a generalized metric space and $T : X \to X$ be a given mapping. We say that T is an α-ψ-contractive mapping of integral type if there exist two functions $\alpha : X \times X \to [0, +\infty)$ and $\psi \in \Psi$ such that for each $x, y \in X$

$$\alpha(x, y) \int_0^{d(Tx,Ty)} \varphi(t)dt \leq \psi\left(\int_0^{d(x,y)} \varphi(t)dt\right), \tag{2.2}$$

where $\varphi \in \Phi$.

Now, we state the following fixed point theorem.

Theorem 10 *Let (X, d) be a complete generalized metric space and $T : X \to X$ be an α-ψ contractive mapping of integral type. Suppose that*

(i) *T is α-admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$;*
(iii) *T is continuous.*

Then there exists a $u \in X$ such that $Tu = u$.

Proof Let $x_0 \in X$ be an arbitrary point such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, Tx_0) \geq 1$. Notice that the existence of such a point guaranteed from assumption (ii) of theorem. We construct an iterative sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n = T^{n+1}x_0$ for all $n \geq 0$. If we have $x_{n_0} = x_{n_0+1}$ for some n_0, then $u = x_{n_0}$ is a fixed point of T. Hence, for the rest of the proof, we presume that

$$x_n \neq x_{n+1} \quad \text{for all } n. \tag{2.3}$$

Since T is α-admissible, we have

$$\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1 \quad \Rightarrow \quad \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1.$$

Recursively, we obtain that

$$\alpha(x_n, x_{n+1}) \geq 1, \quad \text{for all } n = 0, 1, \ldots. \tag{2.4}$$

Analogously, we derive that

$$\alpha(x_0, x_2) = \alpha\big(x_0, T^2x_0\big) \geq 1 \quad \Rightarrow \quad \alpha(Tx_0, Tx_2) = \alpha(x_1, x_3) \geq 1.$$

Iteratively, we get that

$$\alpha(x_n, x_{n+2}) \geq 1, \quad \text{for all } n = 0, 1, \ldots. \tag{2.5}$$

Regarding (2.2) and (2.4), we deduce that

$$\begin{aligned}\int_0^{d(x_{n+1},x_n)} \varphi(t)dt &= \int_0^{d(Tx_n,Tx_{n-1})} \varphi(t)dt \\ &\leq \alpha(x_n, x_{n-1}) \int_0^{d(Tx_n,Tx_{n-1})} \varphi(t)dt \\ &\leq \psi\left(\int_0^{d(x_n,x_{n-1})} \varphi(t)dt\right),\end{aligned} \tag{2.6}$$

for all $n \geq 1$.

Inductively, we find that

$$\int_0^{d(x_{n+1},x_n)} \varphi(t)dt \leq \psi^n\left(\int_0^{d(x_1,x_0)} \varphi(t)dt\right), \quad \text{for all } n \geq 1. \tag{2.7}$$

It is clear from Lemma 1 that

$$\lim_{n\to\infty} \int_0^{d(x_{n+1},x_n)} \varphi(t)dt = 0 \tag{2.8}$$

and hence

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{2.9}$$

On account of (2.2) and (2.5), we get that

$$\begin{aligned}\int_0^{d(x_{n+2},x_n)} \varphi(t)dt &= \int_0^{d(Tx_{n+1},Tx_{n-1})} \varphi(t)dt \\ &\leq \alpha(x_{n+1}, x_{n-1}) \int_0^{d(Tx_{n+1},Tx_{n-1})} \varphi(t)dt \\ &\leq \psi\left(\int_0^{d(x_{n+1},x_{n-1})} \varphi(t)dt\right),\end{aligned} \tag{2.10}$$

for all $n \geq 1$. By elementary calculations, the inequality (2.10) yields that

$$\int_0^{d(x_{n+2},x_n)} \varphi(t)dt \leq \psi^n\left(\int_0^{d(x_2,x_0)} \varphi(t)dt\right), \quad \text{for all } n \geq 1. \tag{2.11}$$

Again by Lemma 1, we find that

$$\lim_{n\to\infty} \int_0^{d(x_{n+2},x_n)} \varphi(t)dt = 0, \quad \text{and consequently} \quad \lim_{n\to\infty} d(x_{n+2},x_n) = 0. \tag{2.12}$$

Let $x_n = x_m$ for some $m, n \in \mathbb{N}$ with $m \neq n$. Without loss of generality, assume that $m > n$. Thus, $x_m = T^{m-n}(T^n x_0) = T^n x_0 = x_n$. Consider now,

$$\begin{aligned}\int_0^{d(x_{n+1},x_n)} \varphi(t)dt &= \int_0^{d(Tx_n,x_n)} \varphi(t)dt \\ &= \int_0^{d(Tx_m,Tx_{m-1})} \varphi(t)dt \\ &\leq \alpha(x_m,x_{m-1}) \int_0^{d(Tx_m,Tx_{m-1})} \varphi(t)dt \\ &\leq \psi^{m-n}\left(\int_0^{d(x_{n+1},x_n)} \varphi(t)dt\right). \end{aligned} \tag{2.13}$$

Due to (ii) of Lemma 1, the inequality (2.13) turns into

$$\int_0^{d(x_{n+1},x_n)} \varphi(t)dt \leq \psi^{m-n}\left(\int_0^{d(x_{n+1},x_n)} \varphi(t)dt\right) < \int_0^{d(x_{n+1},x_n)} \varphi(t)dt, \tag{2.14}$$

which is a contradiction. Now we shall show that the sequence $\{x_n\}$ is Cauchy. First observe that

$$\int_{d(x_n,x_{n-1})}^{d(x_{n+1},x_n)} \varphi(t)dt \leq \int_0^{d(x_{n+1},x_n)} \tag{2.15}$$

since $\{d(x_{n+1},x_n)\}$ is a non-negative sequence. For this aim, it is sufficient to investigate the following cases. Case (I): Suppose that $k > 2$ and k is odd. Let $k = 2m+1$, $k \geq 1$. Then, by using the quadrilateral inequality together with (2.11) and (2.15), we find

$$\begin{aligned}\int_0^{d(x_n,x_{n+k})} \varphi(t)dt &= \int_0^{d(x_n,x_{n+2m+1})} \varphi(t)dt \\ &\leq \int_0^{d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+\cdots+d(x_{n+2m},x_{n+2m+1})} \varphi(t)dt \\ &\leq \int_0^{d(x_n,x_{n+1})} \varphi(t)dt + \int_{d(x_n,x_{n+1})}^{d(x_{n+1},x_{n+2})} \varphi(t)dt\end{aligned}$$

$$
\begin{aligned}
&+\cdots+\int_{d(x_{n+2m-1},x_{n+2m})}^{d(x_{n+2m},x_{n+2m+1})}\varphi(t)dt \\
&\leq \int_0^{d(x_n,x_{n+1})}\varphi(t)dt+\int_0^{d(x_{n+1},x_{n+2})}\varphi(t)dt \\
&\quad +\cdots+\int_0^{d(x_{n+2m},x_{n+2m+1})}\varphi(t)dt \\
&\leq \sum_{p=n}^{n+k-1}\psi^n\left(\int_0^{d(x_1,x_0)}\varphi(t)dt\right) \\
&\leq \sum_{p=n}^{+\infty}\psi^n\left(\int_0^{d(x_1,x_0)}\varphi(t)dt\right)\to 0 \quad \text{as } n\to\infty. \qquad (2.16)
\end{aligned}
$$

Case (II): Let $k > 2$ and k is even. Let $k = 2m$, $k \geq 1$. Then, by applying the quadrilateral inequality together with (2.11), (2.12) and (2.15), we get

$$
\begin{aligned}
\int_0^{d(x_n,x_{n+k})}\varphi(t)dt &= \int_0^{d(x_n,x_{n+2m})}\varphi(t)dt \\
&\leq \int_0^{d(x_n,x_{n+2})}\varphi(t)dt+\int_0^{d(x_{n+2},x_{n+3})}\varphi(t)dt \\
&\quad +\cdots+\int_0^{d(x_{n+2m-1},x_{n+2m})}\varphi(t)dt \\
&\leq \sum_{p=n}^{n+k-1}\psi^n\left(\int_0^{d(x_2,x_0)}\varphi(t)dt\right) \\
&\leq \sum_{p=n}^{+\infty}\psi^n\left(\int_0^{d(x_2,x_0)}\varphi(t)dt\right)\to 0 \quad \text{as } n\to\infty. \quad (2.17)
\end{aligned}
$$

By (2.16) and (2.16) we observe that $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is complete, there exists $u \in X$ such that

$$\lim_{n\to\infty} d(x_n, u) = 0. \tag{2.18}$$

Since T is continuous, we obtain from (2.18) that

$$\lim_{n\to\infty} d(x_{n+1}, Tu) = \lim_{n\to\infty} d(Tx_n, Tu) = 0. \tag{2.19}$$

From (2.18) and (2.19) we get immediately that $\lim_{n\to\infty} Tx_n x_0 = \lim_{n\to\infty} Tx_n = Tu$. Taking Proposition 5 into account, we conclude that u is a fixed point of T, that is, $Tu = u$. □

Theorem 11 *Let (X, d) be a complete generalized metric space and $T : X \to X$ be an α-ψ contractive mapping of integral type. Suppose that*

(i) *T is α-admissible*;
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$*;
(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all n.*

Then there exists a $u \in X$ such that $Tu = u$.

Proof Following the proof of Theorem 10, we know that the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \geq 0$, converges for some $u \in X$. From (2.4) and condition (iii), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, u) \geq 1$ for all k. Applying (2.2), for all k, we get that

$$\begin{aligned}\int_0^{d(x_{n(k)+1},Tu)} \varphi(t)dt &= \int_0^{d(Tx_{n(k)},Tu)} \varphi(t)dt \\ &\leq \alpha(x_{n(k)}, u) \int_0^{d(Tx_{n(k)},Tu)} \varphi(t)dt \\ &\leq \psi\left(\int_0^{d(x_{n(k)},u)} \varphi(t)dt\right). \end{aligned} \tag{2.20}$$

Letting $k \to \infty$ in the above equality, we find that

$$\lim_{k\to\infty} \int_0^{d(x_{n(k)+1},Tu)} \varphi(t)dt = 0 \quad \text{and} \quad \lim_{k\to\infty} d(x_{n(k)+1}, Tu) = 0, \tag{2.21}$$

and hence $\lim_{n\to\infty} T^n x_0 = \lim_{n\to\infty} Tx_n = Tu$. By Proposition 5, we obtain that u is a fixed point of T, that is, $Tu = u$. □

Corollary 12 *Let (X, d) be a complete generalized metric space and $T : X \to X$ be an α-ψ contractive mapping of integral type. Suppose that*

(i) *T is α-admissible*;
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and α is transitive*;
(iii) *either*,
   (a) *T is continuous.*
   (b) *If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all n.*

Then there exists a $u \in X$ such that $Tu = u$.

Proof By assumption of (ii) theorem, there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Thus, we get $\alpha(Tx_0, T^2x_0) \geq 1$, by (i). Owing to the fact that α is transitive, we conclude that $\alpha(x_0, T^2x_0) \geq 1$. Consequently, all conditions of Theorem 10 (and, respectively Theorem 11) are satisfied. □

For the uniqueness, we need an additional condition:

(U) For all $x, y \in Fix(T)$, we have $\alpha(x, y) \geq 1$, where $Fix(T)$ denotes the set of fixed points of T.

Theorem 13 *Adding condition (U) to the hypotheses of Theorem 10 (resp. Theorem 11 and Theorem 12), we obtain that u is the unique fixed point of T.*

Proof In what follows we shall show that u is a unique fixed point of T. We shall use the *reductio ad absurdum*. Let v be another fixed point of T with $v \neq u$. It is evident that $\alpha(u, v) = \alpha(Tu, Tv)$.

Now, due to (2.2), we have

$$\begin{aligned}\int_0^{d(u,v)} \varphi(t)dt &\leq \alpha(u, v) \int_0^{d(u,v)} \varphi(t)dt \\ &\leq \alpha(Tu, Tv) \int_0^{d(Tu,Tv)} \varphi(t)dt \\ &\leq \psi\left(\int_0^{d(u,v)} \varphi(t)dt\right) < \int_0^{d(u,v)} \varphi(t)dt \end{aligned} \tag{2.22}$$

which is a contradiction. Hence, $u = v$. □

As an alternative condition for the uniqueness of a fixed point of a α-ψ contractive mapping, we shall consider the following hypothesis.

(H) For all $x, y \in Fix(T)$, there exists $z \in X$ such that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$.

Theorem 14 *Adding conditions (H) and (W) to the hypotheses of Theorem 10 (resp. Theorem 11), we obtain that u is the unique fixed point of T.*

Proof Suppose that v is another fixed point of T. From (H), there exists $z \in X$ such that

$$\alpha(u, z) \geq 1 \quad \text{and} \quad \alpha(v, z) \geq 1. \tag{2.23}$$

Since T is α-admissible, from (2.23), we have

$$\alpha\big(u, T^n z\big) \geq 1 \quad \text{and} \quad \alpha\big(v, T^n z\big) \geq 1, \quad \text{for all } n. \tag{2.24}$$

Define the sequence $\{z_n\}$ in X by $z_{n+1} = Tz_n$ for all $n \geq 0$ and $z_0 = z$. From (2.24), for all n, we have

$$\begin{aligned}\int_0^{d(u,z_{n+1})} \varphi(t)dt = \int_0^{d(Tu,Tz_n)} \varphi(t)dt &\leq \alpha(u, z_n) \int_0^{d(Tu,Tz_n)} \varphi(t)dt \\ &\leq \psi\left(\int_0^{d(u,z_n)} \varphi(t)dt\right) \end{aligned} \tag{2.25}$$

Iteratively, by using the inequality (2.25), we get that

$$\int_0^{d(u,z_{n+1})} \varphi(t)dt \leq \psi^n\left(\int_0^{d(u,z_0)} \varphi(t)dt\right), \tag{2.26}$$

for all n. Letting $n \to \infty$ in the above inequality, we obtain

$$\lim_{n\to\infty} d(z_n, u) = 0. \tag{2.27}$$

Similarly, one can show that

$$\lim_{n\to\infty} d(z_n, v) = 0. \tag{2.28}$$

Regarding (W) together with (2.27) and (2.28), it follows that $u = v$. Thus we proved that u is the unique fixed point of T. □

Corollary 15 *Adding condition (H) to the hypotheses of Theorem 10 (resp. Theorem 11 and Theorem 12) and assuming that (X, d) is Hausdorff, we obtain that u is the unique fixed point of T.*

The proof is clear and hence it is omitted. Indeed, Hausdorffness implies the uniqueness of the limit. Thus, the theorem above yields the conclusions.

3 Consequences

In this section, we shall state some existing results in the literature that can be inferred from the main results.

Corollary 16 *Let (X, d) be a complete generalized metric space and $T : X \to X$ be a given mapping. Suppose that there exists a function $\psi \in \Psi$ such that*

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \leq \psi\left(\int_0^{d(x,y)} \varphi(t)\right)dt,$$

for all $x, y \in X$. Then T has a unique fixed point.

Proof Let $\alpha : X \times X \to [0, \infty)$ be the mapping defined by $\alpha(x, y) = 1$, for all $x, y \in X$. Then T is an α-ψ-contraction mapping. It is evident that all conditions of Theorem 10 are satisfied. Hence, T has a unique fixed point. □

The following fixed point theorems follow immediately from Corollary 16 by taking $\psi(t) = \lambda t$, where $\lambda \in (0, 1)$.

Corollary 17 (Samet [32]) *Let (X, d) be a complete generalized metric space and $T : X \to X$ be a given mapping. Suppose that there exists a constant $\lambda \in (0, 1)$ such that*

$$\int_0^{d(Tx,Ty)} \varphi(t))dt \le \lambda \int_0^{d(x,y)} \varphi(t))dt,$$

for all $x, y \in X$. Then T has a unique fixed point.

Corollary 18 *Let (X, d) be a complete generalized metric space and $T : X \to X$ be a given mapping. Suppose that there exists a function $\psi \in \Psi$ such that*

$$d(Tx, Ty) \le \psi\big(d(x, y)\big),$$

for all $x, y \in X$. Then T has a unique fixed point.

Proof Let $\alpha : X \times X \to [0, \infty)$ be the mapping defined by $\alpha(x, y) = 1$, for all $x, y \in X$. Then T is an α-ψ-contraction mapping. It is evident that all conditions of Theorem 10 are satisfied. Hence, T has a unique fixed point. □

The following fixed point theorems follow immediately from Corollary 18 by taking $\psi(t) = \lambda t$, where $\lambda \in (0, 1)$.

Corollary 19 (Branciari [1]) *Let (X, d) be a complete generalized metric space and $T : X \to X$ be a given mapping. Suppose that there exists a constant $\lambda \in (0, 1)$ such that*

$$d(Tx, Ty) \le \lambda d(x, y),$$

for all $x, y \in X$. Then T has a unique fixed point.

Remark 20 These result improve the results of Samet [32] and Branciari [1]. Notice that in the literature, to prove fixed point theorem in generalized metric spaces, some superfluous conditions have been assumed such as Hausdorffness, continuity and so on. By following the interesting results [3, 4] we prove our results without any further condition.

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Infinite Dimensional Stochastic Cauchy Problems in Ito and Differential Forms: Comparison of Solutions

Irina V. Melnikova and Olga Starkova

Abstract We consider three types of solutions to the infinite dimensional stochastic Cauchy problem $X'(t) = AX(t) + B\mathbb{W}(t)$, $t \geq 0$, $X(0) = \zeta$, with A being the generator of a regularized semigroup in a Hilbert space and a white noise $\mathbb{W}$ in another Hilbert space: weak, generalized in t, and generalized in a random variable. It is proved coincidence of the solutions under the conditions they exist.

Keywords Distribution · Semigroup of operators · White noise · Wiener process · Generalized · Weak · Regularized solutions

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Models of various evolution processes considered with regard for random perturbations lead to the Cauchy problem for equations with an inhomogeneity in the form of white noise in infinite dimensional spaces. Among them, important for applications is the first-order Cauchy problem:

$$X'(t) = AX(t) + B\mathbb{W}(t), \quad t \geq 0,\ X(0) = \zeta;\ B \in \mathcal{L}(\mathbb{H}, H), \tag{0.1}$$

with A, the generator of a generally regularized semigroup in a Hilbert space H and a white noise process $\mathbb{W} = \{\mathbb{W}(t), t \geq 0\}$ in another separable Hilbert space $\mathbb{H}$. Due to the singularities of $\mathbb{W}$ which is not a process in the usual sense, the problem (0.1), similarly to the finite dimensional case, is considered in the integral form:

$$X(t) = \zeta + \int_0^t AX(s)ds + \int_0^t BdW(t), \quad t \geq 0, \tag{0.2}$$

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where $\int_0^t BdW(t)$ is the Ito integral with respect to an $\mathbb{H}$-valued Wiener process $\{W(t), t \geq 0\}$, a "primitive" of $\mathbb{W}$ (see, e.g. [2, 3, 5, 7, 8]). The problem (0.2) is usually written in the short form: $dX(t) = AX(t)dt + BdW(t)$, $t \geq 0$, $X(0) = \zeta$.

In addition to (0.2), we study the problem in the differential form (0.1) in spaces of distributions using the known technique of abstract distribution L. Sczwartz and a new one generalizing the white noise theory to the infinite dimensional case (see, e.g. [1, 6, 9] and in the finite dimensional case [5, 10]).

The present paper is devoted to the important problem naturally arising in this situation—to comparison of solutions for the Cauchy problem considered in different forms, more specifically, to comparison of weak solutions, generalized in t-variable solutions, and generalized in ω-variable solutions. We show that under conditions when each of them exists, these solutions, obtained in absolutely different techniques, coincide.

1 Solutions of the Stochastic Cauchy Problem in Integral and Differential Forms

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space with normal filtration $\{\mathcal{F}_t, t \geq 0\}$; H and $\mathbb{H}$ be separable Hilbert spaces.

At the beginning we consider the abstract stochastic Cauchy problem in the form (0.2), where ζ is an $\mathcal{F}_0$-measurable H-valued random variable, A is the generator of a semigroup $S = \{S(t), t \geq 0\}$, generally regularized, $B \in \mathcal{L}(\mathbb{H}, H)$, and $W = \{W(t), t \geq 0\}$, $W(t) = W(\omega, t)$, $\omega \in \Omega$, is an $\mathbb{H}$-valued Q-Wiener process.

An H-valued process $X = \{X(t), t \geq 0\}$ is said to be *a weak solution* for problem (0.2) if $\int_0^t \|X(s)\|_H ds < \infty$ and for each $y \in \operatorname{dom} A^*$

$$\langle X(t), y\rangle = \langle \zeta, y\rangle + \int_0^t \langle X(s), A^* y\rangle ds + \langle BW(t), y\rangle, \quad \mathbf{P}_{a.s.}$$

Similarly to the classical case, we look for a solution in the following form

$$X(t) = S(t)\zeta + \int_0^t S(t-s)BdW(s) =: S(t)\zeta + W_A(t), \quad t \geq 0, \tag{1.1}$$

where $S(t)$ are solution operators to the corresponding homogeneous problem.

To prove the existence and uniqueness of a weak solution for the Cauchy problem we need the stochastic convolution W_A, the important component of a solution, to be well defined. That is $\Psi(s) = S(t-s)B$ for $t \geq 0$ must satisfy the condition for existence of the Ito integral $\int_0^t \Psi(s)dW(s)$:

$$\int_0^t \|\Psi(s)\|^2_{\mathrm{HS}} ds < \infty, \quad \|\Psi(s)\|^2_{\mathrm{HS}} := \sum_{j=1}^{\infty} \|\Psi(s)Q^{\frac{1}{2}} e_j\|^2 = \operatorname{Tr} \Psi(s) Q^{\frac{1}{2}} Q^{*\frac{1}{2}} \Psi^*(s). \tag{1.2}$$

In the case of a semigroup of class C_0 the following result holds [1, 2].

Theorem 1.1 *Let A be the generator of a C_0-class semigroup S and W be a Q-Wiener process. If $\Psi(s) = S(t-s)B$, $t \geq 0$, satisfies (1.2), then for each $\mathcal{F}_0$-measurable $\zeta \in H$ the random process $\{X(t) = S(t)\zeta + W_A(t), t \geq 0\}$ exists and is a unique weak solution.*

In [1] it is shown that if A generates a K-convoluted semigroup $\{S(t), t \geq 0\}$, a *weak K-convoluted solution* for the problem (0.2) can be constructed in the form (1.1). An H-valued process X defined by (1.1) is a weak K-convoluted solution of (0.2) if for each $y \in \operatorname{dom} A^*$

$$\langle X(t), y\rangle = \left\langle \int_0^t K(s)\zeta ds, y\right\rangle + \left\langle \int_0^t X(s)ds, A^* y\right\rangle + \left\langle \int_0^t \int_0^s K(s-r)BdW(r)ds, y\right\rangle.$$

In particular, X is a *weak n-times integrated solution* if

$$\langle X(t), y\rangle = \left\langle \frac{t^n}{n!}\zeta, y\right\rangle + \int_0^t \langle X(s), A^* y\rangle ds + \left\langle \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} BW(s)ds, y\right\rangle.$$

Now we consider the Cauchy problem (0.1) in spaces of abstract distributions. Let H be a Hilbert space. By $\mathcal{D}'(H)$ we denote a space of H-valued distributions over the space of L. Schwartz test functions $\mathcal{D}$ and by $\mathcal{D}'_0(H)$ the space of H-valued distributions with supports on $[0, \infty)$.

In the space of distributions $\mathcal{D}'(\mathbb{H})$ a Q-white noise is well defined as the generalized derivative of a Q-Wiener process continued by zero for $t \leq 0$:

$$\langle \varphi, \mathbb{W}\rangle := -\int_0^\infty W(t)\varphi'(t)dt, \quad \varphi \in \mathcal{D},\ \mathbf{P}_{a.s.};\ \mathbb{W} \in \mathcal{D}'_0\big(L_2(\Omega, \mathbb{H})\big). \tag{1.3}$$

Using [1, 3, 7], the Cauchy problem (0.1) in spaces of abstract distributions can be written as follows:

$$P * X = \delta \otimes \zeta + B\mathbb{W}, \tag{1.4}$$

where $P := \delta' \otimes I - \delta \otimes A \in \mathcal{D}'_0(\mathcal{L}([\operatorname{dom} A], H))$ and $[\operatorname{dom} A]$ is the domain of A with the graph-norm $\|x\|_{[\operatorname{dom} A]} = \|x\| + \|Ax\|$.

A distribution $G \in \mathcal{D}'_0(\mathcal{L}(H, [\operatorname{dom} A]))$ is called *inverse with respect to convolution* with $P \in \mathcal{D}'_0(\mathcal{L}([\operatorname{dom} A], H))$ if $G * P = \delta \otimes I_{[\operatorname{dom} A]}$, $P * G = \delta \otimes I_H$, where $I_{[\operatorname{dom} A]}$ and I_H are identity operators in $[\operatorname{dom} A]$ and H respectively. By the use the properties of inverse distribution G, it is proved that a unique solution of a Cauchy problem $P * X = \delta \otimes \zeta + F$ with $F \in \mathcal{D}'_0(H)$ is defined as $X = G * \delta\zeta + G * F$. It follows that if A generates a C_0-class semigroup or an n-times integrated semigroup, the solution

$$X = G * \delta\zeta + G * B\mathbb{W}, \quad X \in \mathcal{D}'_0\big([\operatorname{dom} A]\big) \cap \mathcal{D}'_0\big(L_2\big(\Omega, [\operatorname{dom} A]\big)\big) \tag{1.5}$$

exists and is a unique solution of the problem (1.4). Here we also have a stochastic convolution $W_A = G * B\mathbb{W}$, but now in a generalized sense.

If A is the generator a semigroup $S = \{S(t), t \geq 0\}$ of class C_0, then by the definition of a convolution in spaces of distributions and distribution G inverse to P (in the case of C_0-class semigroup G coincides with distribution $\mathbf{S}$ defined as a semigroup S, extended by zero as $t < 0$), we get a solution of (1.4) in the form (1.5) with $G = \mathbf{S}$ and $\mathbb{W}$ defined by (1.3).

If A generates a n-times integrated semigroup $\{S(t), t \geq 0\}$, then we obtain the solution in the form (1.5) with $G = \mathbf{S}^{(n)}$.

For a solution of (1.4) with the generator of a K-convoluted semigroup S we need a larger space of distributions, namely ultra-distributions, since we have to find an operator inverse to convolution with function K, which is an infinite differentiation operator $P_{ult}(\frac{d}{dt})$ defined in the spaces. In this case the solution of (1.4) is presented as follows [1]:

$$\langle \varphi, X\rangle = \left\langle P^*_{ult}\left(\frac{d}{dt}\right)\varphi, \mathbf{S}\zeta\right\rangle - \int_0^\infty S(t)\left\langle P^*_{ult}\left(\frac{d}{dt}\right)\varphi'(t+s), BW(s)\right\rangle dt$$
$$= \int_0^\infty P^*_{ult}\left(\frac{d}{dt}\right)\varphi(t)S(t)\zeta\, dt + \int_0^\infty P^*_{ult}\left(\frac{d}{dt}\right)\varphi(t)dt\int_0^t S(t-s)B\,dW(s). \tag{1.6}$$

At last, we consider the problem (0.1) in the spaces of abstract stochastic distributions $(\mathcal{S})^*(H)$ and a solution generalized in random variable ω.

The series of inclusions

$$(\mathcal{S})(H) \subset \cdots \subset (\mathcal{S}_p)(H) \subset \cdots \subset (L_2)(H) \subset \cdots \subset (\mathcal{S}_{-p})(H) \subset \cdots \subset (\mathcal{S})^*(H)$$

is built for a Hilbert space H (see, e.g. [1]) by analogy with the Gelfand triple $\mathcal{S} \subset L_2(\mathbb{R}) \subset \mathcal{S}^*$ [9], where elements of the spaces $(\mathcal{S}_p)(H)$ and $(\mathcal{S}_{-p})(H)$ are defined in terms of the behavior (decrease or increase, respectively) of the Fourier coefficients with respect to the basis in $(L_2)(\mathbb{R})$ consisting of stochastic Hermite polynomials defined by

$$\mathbf{h}_\alpha(\omega) := \prod_{i=1}^\infty h_{\alpha_i}\big(\langle \xi_i, \omega\rangle\big), \quad \omega \in \mathcal{S}',\ \alpha = (\alpha_1, \alpha_2, \ldots) \in \mathcal{T},$$

where $\xi_i(x) = \pi^{-\frac{1}{4}}((i-1)!)^{-\frac{1}{2}}e^{-\frac{x}{2}}h_{i-1}(\sqrt{2}x)$ are Hermite functions, $h_i(x) = (-1)^i e^{\frac{x^2}{2}}\frac{d^i}{dx^i}e^{-\frac{x^2}{2}}$ are Hermite polynomials, and $\mathcal{T}$ is the set of all possible finite multi-indexes. These spaces contain an $\mathbb{H}$-valued Q-Wiener process $\{W(t) := \sum_{j\in\mathbb{N}} \sigma_j\beta_j e_j, t \geq 0\}$:

$$W(t) = \sum_{i,j\in\mathbb{N}} \sigma_j \int_0^t \xi(s)ds(\mathbf{h}_{\epsilon_{n(i,j)}}e_j)$$
$$= \sum_{n=1}^\infty \sigma_{j(n)}\left(\int_0^t \xi_{i(n)}(s)ds\, e_{j(n)}\right)\mathbf{h}_{\epsilon_n} \in (L_2)(\mathbb{H}),$$

as well as a Q-white noise $\{\mathbb{W}(t), t \geq 0\}$:

$$\mathbb{W}(t) = \sum_{i,j\in\mathbb{N}} \sigma_j \xi_i(t)(\mathbf{h}_{\epsilon_{n(i,j)}} e_j) = \sum_{n\in\mathbb{N}} \sigma_{j(n)} \mathbb{W}_{\epsilon_n}(t)\mathbf{h}_{\epsilon_n} \in (\mathcal{S})^*(H), \tag{1.7}$$

where $Qe_j = \sigma_j^2 e_j$, $\mathbb{W}_{\epsilon_n}(t) = \xi_{i(n)}(t)e_{j(n)}$, $n(i(n), j(n)) = n$, and $\epsilon_n := (0, 0, \dots, \underset{n}{1}, 0, \dots)$.

A generalized (in ω) solution of the Cauchy problem (0.1) with $\mathbb{W}(t) \in (\mathcal{S})^*(\mathbb{H})$ is constructed in [1] as follows.

Theorem 1.2 *Let A be the generator of a C_0-class semigroup $\{S(t), t \geq 0\}$ in a Hilbert space H, let $B \in \mathcal{L}(\mathbb{H}, H)$ and let $\mathbb{W}$ be a white noise process given by* (1.7). *Then*

$$X(t) = \sum_{\alpha} X_\alpha(t)h_\alpha \in (\mathcal{S})^*(H), \quad t \geq 0, \tag{1.8}$$

where

$$X_\alpha(t) = \begin{cases} S(t)\zeta_{\epsilon_n} + \int_0^t S(t-s)B\mathbb{W}_{\epsilon_n}(s)ds, & \alpha = \epsilon_n \\ S(t)\zeta_\alpha, & \alpha \neq \epsilon_n, \end{cases}$$

is a unique solution of the Cauchy problem (0.1) *in $(\mathcal{S})^*(H)$ for any $\zeta = \sum_\alpha \zeta_\alpha h_\alpha \in$* (dom A).

Remark 1.3 Note, that a cylindrical Wiener process and a singular white noise are defined in the space $(\mathcal{S})^*(\mathbb{H})$ and Theorem 1.2 is true for the singular white noise, but in this paper we do not consider them.

2 Connections Between Weak and Generalized in *t*-Variable Solutions

In this section we prove the coincidence of solutions to the generalized Cauchy problem (1.4) with weak solutions of the Cauchy problem (0.2) in the case of C_0-class, n-times integrated and K-convoluted semigroups. In the next section we will prove the coincidence of the obtained solution in the space $(\mathcal{S})^*(H)$ with a weak solution of the Cauchy problem (0.2).

Theorem 2.1 *Let A be the generator of a C_0-class semigroup $\{S(t), t \geq 0\}$ under condition* (1.2) *with $\Psi(s) = S(t-s)B$, $t \geq 0$. Then weak solution* (1.1) *is a solution of the generalized Cauchy problem* (1.4), *where Q-white noise $\mathbb{W}$ is the generalized derivative of a Q-Wiener process W. Conversely: the generalized solution* (1.5) *with $G = \mathbf{S}$ and $\mathbb{W}$ defined by* (1.3) *is a weak solution of the Cauchy problem* (0.2).

Proof Further the following equality

$$-\int_0^\infty W(t)\varphi'(t)dt = \int_0^\infty \varphi(t)dW(t), \quad \varphi \in \mathcal{D}, \tag{2.1}$$

resulting from a generalization of the Ito formula to the infinite-dimensional case will be important.

Now we check that $\mathbf{X}$ defined as weak solution $X(t) = S(t)\zeta + \int_0^t S(t-s)BdW(s)$, $t \geq 0$, extended by zero for $t < 0$, satisfies (1.4) for each H-valued $\mathcal{F}_0$-measurable random variable ζ. For this we multiply X by a function $\varphi \in \mathcal{D}$ and integrate with respect to t from zero to infinity. From (2.1) we obtain the following equalities:

$$\begin{aligned}\langle \varphi, \mathbf{X} \rangle &= \int_0^\infty \varphi(t)S(t)\zeta dt + \int_0^\infty \varphi(t)\int_0^t S(t-s)BdW(s)dt \\ &= \langle \varphi, S\zeta \rangle - \left\langle \varphi'(t), \int_0^t S(t-s)BW(s)ds \right\rangle, \quad a.s.\ in\ \omega.\end{aligned} \tag{2.2}$$

The equalities can be written as follows:

$$\langle \varphi, \mathbf{X} \rangle = \langle \varphi, \mathbf{S}\zeta \rangle - \langle \varphi', \mathbf{S} * BW \rangle = \langle \varphi, \mathbf{S}\zeta \rangle + \langle \varphi, \mathbf{S} * B\mathbb{W} \rangle, \tag{2.3}$$

where the (regular) distribution $\mathbf{S} \in \mathcal{D}_0'(\mathcal{L}(H, [\operatorname{dom} A]))$ is the semigroup S extended by zero for $t < 0$. The obtained equality (2.3) means that $\mathbf{X}$ coincides with the generalized solution (1.5) of the problem (1.4).

Conversely, we go from bottom to top; then it follows from (2.3)–(2.2) that the generalized solution $\mathbf{X} = \mathbf{S}\zeta + \mathbf{S} * B\mathbb{W}$ of (1.4) coincides with the weak solution of (0.2) $X(t)$, $t \geq 0$, continued by zero for $t < 0$. □

Analyzing the structure of the solutions and the relationship between generalized and weak solutions obtained, we see that in the case of the generator of a C_0-class semigroup the sum of two components $S(t)\zeta + W_A(t), t \geq 0$, is a weak solution for each H-valued $\mathcal{F}_0$-measurable random variable ζ; this is due to the fact that we do not need to apply A to $S(t)\zeta$ and $W_A(t)$, instead A^* is applied to elements $y \in \operatorname{dom} A^*$.

The sum $\mathbf{X} = \mathbf{S}\zeta + \mathbf{S} * B\mathbb{W}$ is a generalized solution due to the equality $A\langle \varphi, S\zeta \rangle = -\varphi(0)\zeta - \int_0^\infty \varphi'(t)S(t)dt$, which in the case of a C_0-class semigroup is proved to hold on the whole space H and implies that the action of A is "smoothed" by test functions φ'.

It is important to note, that by the proof of the coincidence of weak and generalized in t solutions, we have actually proved the coincidence of the stochastic convolution defined by Ito integral and the one defined in the space of abstract distributions: $W_A = \mathbf{S} * B\mathbb{W}$.

In the case of a n-times integrated semigroup $S = \{S(t), t \geq 0\}$ we show that a generalized solution coincides with the n-th order derivative of a weak n-times integrated solution.

Theorem 2.2 *Let A be the generator of a n-times integrated semigroup S under condition* (1.2). *Then the n-th order generalized derivative of $X(t) = S(t)\zeta + \int_0^t S(t-s)BdW(s)$, weak n-times integrated solution of* (0.1), *is the solution of* (0.2). *Conversely, a solution of the generalized problem* (1.4) *is the n-th derivative of a weak n-times integrated solution.*

Proof Let $X(t), t \geq 0$ be a weak n-times integrated solution and $\mathbf{X}$ be equal to the $X(t)$ continued by zero as $t < 0$. Then

$$\langle \varphi, \mathbf{X}^{(n)} \rangle = (-1)^n \left[\int_0^\infty \varphi^{(n)}(s) S(s)\zeta ds + \int_0^\infty \varphi^{(n)}(t)dt \int_0^t S(t-s)BdW(s) \right]$$
$$= (-1)^n \left[\int_0^\infty \varphi^{(n)}(t) S(t)\zeta dt - \int_0^\infty \varphi^{(n+1)}(t)dt \int_0^t S(t-s)BW(s)ds \right].$$

Hence by (2.1), the distribution $\mathbf{X}^{(n)}$ is a generalized solution of (1.4) ω a.s. From the obtained equality and the convolution properties the converse statement follows. □

The theorem for the case of K-convoluted semigroups we present without a proof, due to the limited size of the article.

Theorem 2.3 *Let A be the generator of a K-convoluted semigroup $\{S(t), t \geq 0\}$. Then the process $\{P_{ult}(\frac{d}{dt})X(t), t \geq 0\}$, where X is a weak K-convoluted solution of the problem* (0.2), *is a solution of the problem* (1.4). *Conversely, a generalized solution of the problem* (1.4) *is the result of applying ultra-differential operator $P_{ult}(\frac{d}{dt})$ to a weak K-convoluted solution.*

3 Connections Between Weak and Generalized in ω-Variable Solutions

We consider the solution (1.8) obtained for the stochastic Cauchy problem in spaces of stochastic distributions $(\mathcal{S})^*(H)$ with the generator of a C_0-class semigroup, a white noise $\mathbb{W}$ defined by (1.7), and an initial data $\zeta \in (\operatorname{dom} A)$. As it follows from Remark 1.3, solutions in the space can be constructed for the problem with a singular white noise, not only with a Q-white noise, but with irregular white noise; nevertheless, we have a restriction on the initial data: $\zeta \in (\operatorname{dom} A)$.

Now we show that a generalized in ω solution coincides with a weak solution, under the existence conditions for both of them.

Theorem 3.1 *Let A be the generator of a C_0-class semigroup $\{S(t), t \geq 0\}$, $\Psi(s) = S(t-s)B$ satisfy* (1.2), $\zeta \in (\operatorname{dom} A)$, *and W be a Q-Wiener process. Then the generalized in ω solution* (1.8) *for the Cauchy problem* (0.1) *and a weak solution for* (0.2) *coincide.*

Proof The solution obtained by (1.8) can be written in the following form: $X(t) = S\zeta + \int_0^t S(t-s)B\mathbb{W}(s)ds$, $t \geq 0$, $X(t) \in (\mathcal{S})^*(H)$, where

$$\int_0^t S(t-s)B\mathbb{W}(s)ds := \sum_{i,j\in\mathbb{N}} \sigma_j \int_0^t S(t-s)B\xi_i(s)ds e_j \mathbf{h}_{\epsilon_{n(i,j)}}.$$

Hence, in the case under consideration to prove the coincidence of the solutions (1.8) and (1.1), is enough to show the coincidence of the integrals:

$$\int_0^t S(t-s)BdW(s) = \int_0^t S(t-s)B\mathbb{W}ds. \tag{3.1}$$

First, for $\mathbb{W}$ defined by (1.7) we show that the right hand integral in (3.1) belongs to the space $(L_2)(H) = L_2(\Omega, \mathcal{F}; H)$, where $\Omega = \mathcal{S}'$ and $\mathcal{F} = \mathcal{B}(\mathcal{S}')$. For this we check up that $\| \int_0^t S(t-s)B\mathbb{W}ds\|_{(L_2)(H)} < \infty$ for each $t < \infty$. It follows from (1.2) for $\Psi(s) = S(t-s)B$ and the following equalities (where $\{g_k\}$ is a basis in H):

$$\begin{aligned}
&\sum_{i,j\in\mathbb{N}} \sigma_j^2 \left\| \int_0^t S(t-s)Be_j\xi_i(s)ds \right\|_H^2 \\
&\quad = \sum_{j,k\in\mathbb{N}} \sum_{i\in\mathbb{N}} \left(\int_0^t \xi_i(s) \big(\sigma_j S(t-s)Be_j, g_k\big)_H ds \right)^2 \\
&\quad = \sum_{j,k\in\mathbb{N}} \left\| \mathbf{1}_{[0,t]} \big(\sigma_j S(t-\cdot)Be_j, g_k\big) \right\|_{L_2(\mathbb{R})}^2 \\
&\quad = \sum_{j,k\in\mathbb{N}} \int_0^t \left| \big(S(t-\cdot)BQ^{\frac{1}{2}}, g_k\big)_H \right|^2 ds = \sum_{j\in\mathbb{N}} \int_0^t \sum_{k\in\mathbb{N}} \big(S(t-\cdot)BQ^{\frac{1}{2}}e_j, g_k\big)_H^2 ds \\
&\quad = \int_0^t \sum_{j\in\mathbb{N}} \left\| S(t-s)BQ^{\frac{1}{2}}e_j \right\|_H^2 ds = \int_0^t \left\| S(t-s)B \right\|_{\mathrm{HS}}^2 ds.
\end{aligned}$$

The fact that the integrals in (3.1), being in $(L_2)(H)$ coincide, can be proved by obtaining this equality on elementary functions and then going to the limit. Show (3.1) on elementary operator-functions $\Psi_m(s) = \{\chi_{[s_k,s_{k+1})}\Psi_{mk}\}_k^m$ approximating $S(t-s)B$:

$$\begin{aligned}
\int_0^t \Psi_(s)\mathbb{W}ds &= \sum_{i,j\in\mathbb{N}} \sigma_j \int_0^t \Psi_m(s)e_j\xi_i(s)ds\mathbf{h}_{\epsilon_{n(i,j)}} \\
&= \sum_{i,j\in\mathbb{N}} \sigma_j \sum_{k=0}^{m-1} \int_{t_{k-1}}^{t_k} \Psi_{mk}\xi_i(s)ds e_j \mathbf{h}_{\epsilon_{n(i,j)}}
\end{aligned}$$

$$= \sum_{k=0}^{m-1} \Psi_{nk} \sum_{i,j \in \mathbb{N}} \sigma_j \int_{t_{k-1}}^{t_k} \xi_i(s) ds e_j \mathbf{h}_{\epsilon_{n(i,j)}}$$

$$= \sum_{k=0}^{m-1} \Psi_{mk} \big[W(t_k) - W(t_{k-1}) \big] = \int_0^t \Psi_n(t) dW(t).$$

Going to the limit as $n \to \infty$, we obtain (3.1). It follows that generalized in ω and the corresponding weak solution under the existence conditions for both of them. □

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Biomechanical Model of the Human Eye on the Base of Nonlinear Shell Theory

Vladimir Yakushev

Abstract The goal of this work is the development of a biomechanical model of the human eye and to prove software simulation systems of measuring the intraocular pressure (IOP) by an optical analyzer. We numerically simulate the eye deformation when the IOP is measured using the Ocular Response Analyzer developed by the USA company Reichert. The biomechanical model includes a cornea and a sclera, which are considered as axisymmetrically deformable shells of revolution with fixed boundaries; the space between these shells is filled with incompressible fluid. Nonlinear shell theory is used to describe the stressed and strained state of the cornea and sclera. The optical system is calculated from the viewpoint of the geometrical optics. Dependences between the pressure in the air jet and the area of the surface reflecting the light into a photo detector for the different thickness of the cornea were obtained. Three problems with different boundary conditions were considered. The shapes of the regions on the cornea surface were found from which the reflected light falls on the photo detector. First, the light is reflected from the center of the cornea, but then, as the cornea deforms, the light is reflected from its periphery. The numerical results make it possible to better interpret the measurement data. This work was supported by a grant No. 13-01-00801 from the Russian Foundation for Basic Research.

Keywords Nonlinear shell theory · Computational mathematics · Geometrical optics · Human eye · Intraocular pressure

Mathematics Subject Classification (2010) Primary 74K25 · Secondary 65M06

1 Introduction

Modeling of the cornea and the sclera is important for clinical applications and as a tool to deepen our understanding of how an eye behaves from a point of view of

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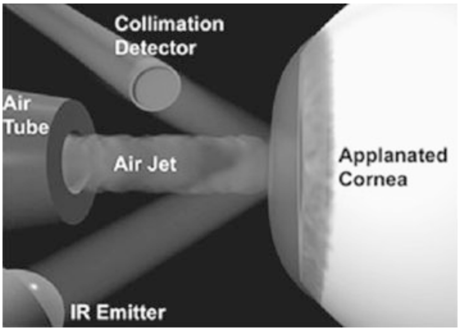

Fig. 1 Scheme for measuring the intraocular pressure

measuring the intraocular pressure. Some ocular instruments, such as the CORVIS ST (Oculus, Inc.) or the ORA (Reichert, Inc.) [1] can be used to investigate biomechanical properties of eye and are becoming useful tools for assessing refractive surgery qualification and outcomes. These devices use a brief and intense air pulse to rapidly deform the eye and detect its shape during deformation. The pressure pulse exerted on the cornea during loading is measured throughout the test at all times.

Glaucoma, an eye disease characterised by degeneration of the optic nerve, is often associated with increased IOP. The only effective therapy against disease progression is lowering of IOP. However, our limited understanding of outflow resistance generation by the trabecular outflow pathway impedes the development of new IOP-lowering therapies.

The outer coat (sclera and cornea) provides structural integrity and has key relationships with the systems that generate and control intraocular pressure. The latter is a key risk factor for glaucoma and if too low also can lead to catastrophic loss of vision.

The scheme of the device can be described as follows. A patient presses his or her forehead to the device. A narrow beam of light is directed to the cornea center at a certain angle using a special positioning system. The light passes through the aperture, as a result, a part of the light flux is cut off and an illuminated area S emerges on the cornea (Fig. 1). Then, an air jet in which the pressure increases from zero to a certain magnitude is directed to the center of the cornea. As a result, the cornea is deformed, and the reflected light flux changes depending on the cornea shape. As a result of the measurements, a curve is shown on the device display that plots the reflected light flux against the pressure in the jet. Figures 1 and 2 were taken by the author from the Internet.

2 Basic Equations

Eye schematically depicted in Fig. 2. We assume that the cornea and the sclera are axisymmetric about the longitudinal axis and the pressure at the center is also distributed symmetrically about this axis. Thus, we have an axisymmetric problem for

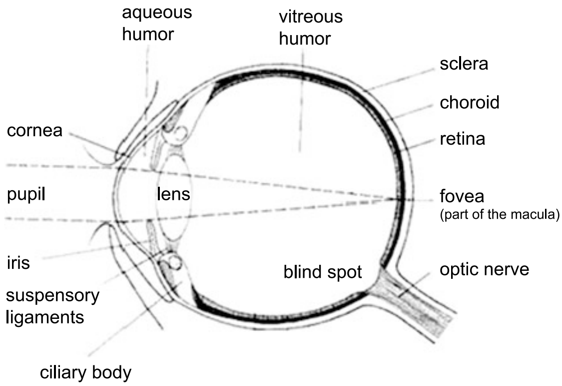

Fig. 2 The scheme of the cornea and the sclera

the calculation of the deformation of both shells (Fig. 3). The coordinates in the section are denoted by the capital letters X and Z. The cornea and the sclera are considered as elastic shells whose deformation is described by a geometrically nonlinear theory under finite displacements and rotation angles. So, we have the basic equations of the theory of shells of revolution under the influence of an axisymmetric load. These equations are the same for the cornea and the sclera; only their geometric and mechanical parameters are different. For that reason, we write these equations in the general form [2].

The shell surface can be obtained by rotating the plane curve $X = X(s_0)$, $Z = Z(s_0)$, around the axis Z, where s_0 is the arc length along the cornea surface measured from its center (see [2]). The angle between the tangent to the surface and the axis X is $\varphi(s_0)$. The values $X(s_0)$, $Z(s_0)$ and $\varphi(s_0)$ are obtained by solving the problem of the simultaneous deformation of the cornea and the sclera. Such problems were studied in [2–5].

Nonlinear shell theory is used to describe the stressed and strained state of the cornea and sclera. The spatial problem was decided by a method of finite differences. For a solution of the nonlinear problems of deformation and stability of shells a method of additional viscosity was used [2]. It permits to build converging iterative processes, including those near critical loads. In this case, there is no necessity to change solution parameters and to chose specified procedures for bypassing singular points.

As a result, a system of six partial differential equations was obtained. It has the form of the canonical hyperbolic system (see [2])

$$\frac{\partial^2 \Phi}{\partial s_0 \partial t} + \mathrm{A}\frac{\partial \Phi}{\partial t} + \frac{1}{\tau}\left\{\frac{\partial \Phi}{\partial s_0} + \mathrm{B}\right\} = 0. \tag{2.1}$$

The components of Φ are functions of the spatial coordinate s_0 and the time t; they are determined by the relation

$$\Phi = [\gamma, \varepsilon_1, k_1, \varphi, X, Z]^T. \tag{2.2}$$

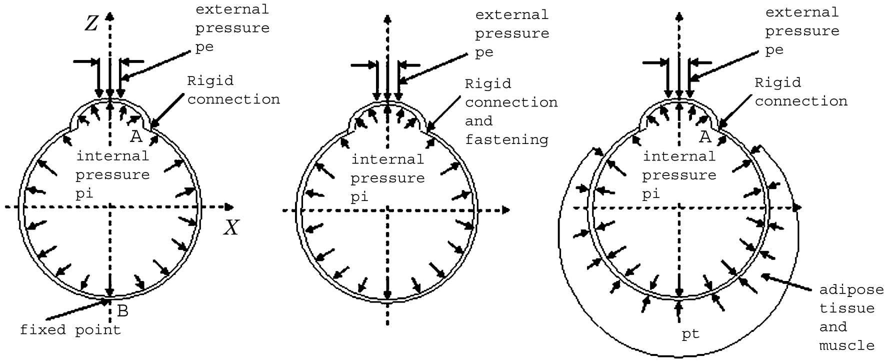

Fig. 3 Three problems with different boundary conditions

The matrices A and B of size 6 × 6 and 6 × 1 are functions of the components of Φ, which, in turn, are functions of the time t and the coordinate s_0.

The static problem is described by Eq. (2.1) for the zero velocities $\dot{\varepsilon}_1 = \dot{\varepsilon}_2 = \dot{k}_1 = \dot{k}_2 = \dot{\gamma} = 0$. Therefore, the static problem is described by the equations in braces:

$$\frac{\partial \Phi}{\partial s_0} + \mathrm{B} = 0.$$

Its solution is obtain from system (2.1) asymptotically when $\dot{\gamma}$, $\dot{\varepsilon}_1$, $\dot{k}_1$, $\dot{\varphi}$, $\dot{x}$, $\dot{y}$ tend to zero.

This problem was solved using the step-by-step method. For every load value p_e, the iterative process was carried out until stabilization, i.e., until the velocity $|\partial \Phi / \partial t|$ became less than a prescribed value determining the computation error.

Since we solved the problem numerically, the shapes of the cornea and the sclera could be defined by coordinates of some of their points. However, there is not enough data to do so; for that reason, we assumed that their shapes were sphere segments [2].

Three problems with different boundary conditions were considered (Fig. 3). In the first of them, the point where the cornea and sclera meet was fixed; denote this point by A. The lower point of the cornea can move freely. On the contrary, in the second problem considered here, the point A can move in the vertical direction, while the lower point is fixed. In the third problem sclera is surrounded with muscles and fatty tissue. They are modelled by Winkler foundation.

To analyze the influence of the initial internal pressure on the location of the curves in the dependence S–p_e, calculations for eight values of p_i were performed with the cornea 0.45 mm thick (see Fig. 4). It is seen that the maximum of the dependence S–p_e moves to the right as p_i increases. This fact can be used to interpret the measurement results.

The dependencies between the area S (sq. mm) and the pressure in the jet p_i (mm Hg) are shown in Fig. 5. It gives a comparison of the first and second boundary conditions. The continuous line corresponds to the second task, and dotted—the

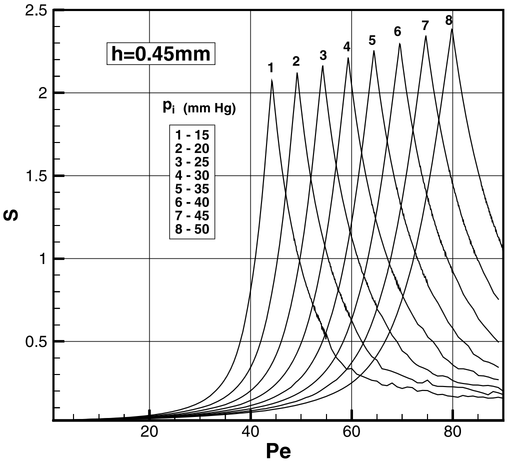

Fig. 4 The dependence S–p_e for the cornea 0.45 mm thick

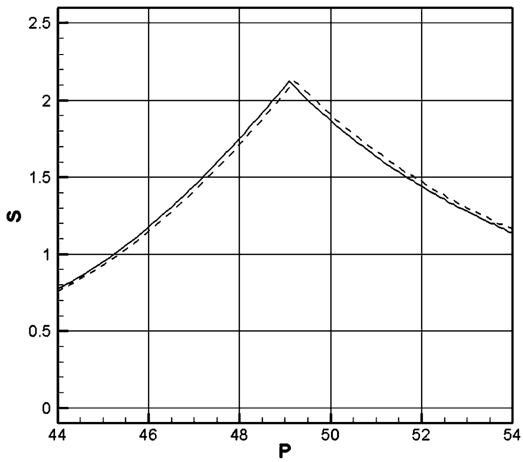

Fig. 5 The scheme of the cornea and the sclera

first. From these results it is visible that change of boundary conditions didn't lead to considerable change of the results.

The third type of boundary conditions gives the most realistic scheme of loading of an eye at measurement of intraocular pressure. Figure 6 displays the pressure in the air jet p_i (mm Hg) against the area S (sq. mm) from which the reflected light comes to the photodetector for various sizes of maximum shift down Z_c of a point A in millimeters. Apparently insignificant point shifts strongly influences the provision of a maximum. In Fig. 7 figure value of a maximum S for the area and pressure p_i depending on vertical shift is shown.

The optical system is calculated from the viewpoint of the geometrical optics. The dependences between the pressure in the air jet and the area of the surface reflecting the light into a photodetector are obtained. The shapes of the regions on the cornea surface are found from which the reflected light falls on the photodetector. First, the light is reflected from the center of the cornea, but then, as the cornea de-

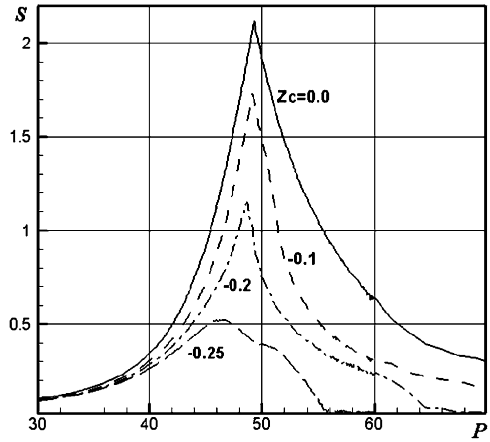

Fig. 6 The scheme of the cornea and the sclera

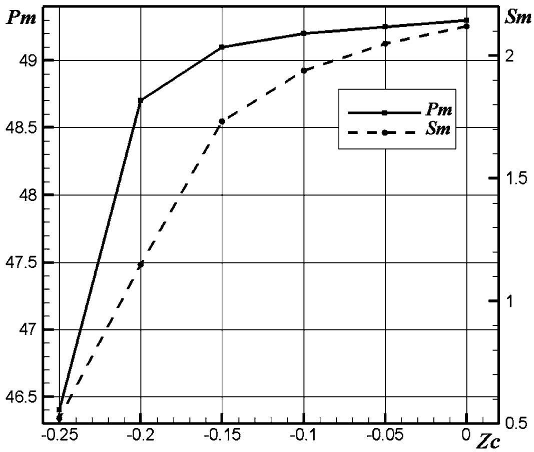

Fig. 7 The scheme of the cornea and the sclera

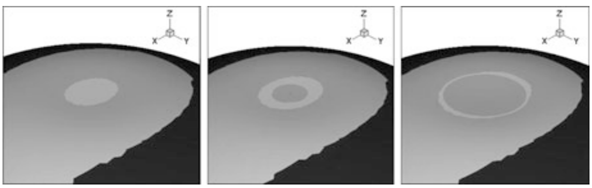

Fig. 8 The scheme of the cornea and the sclera

forms, the light is reflected from its periphery (Fig. 8). Figure shows the shapes of the regions in the central part of the cornea corresponding to the unlit part (black), the illuminated part that reflects the light not hitting the photodetector (gray), and the illuminated part that reflects the light hitting the photodetector (white). This is a very important result because no experiments on investigating the shape of the region for the reflected light were earlier carried out, while this is necessary for the correct interpretation of measurement results.

3 Conclusions

Knowledge on the biomechanical properties of human eye is essential in treatment and diagnosis of ophthalmic diseases. A numerical model was proposed to simulate the interaction between sclera and cornea. Better understanding the biomechanical characteristics of the eye can lead to improvements in treatments of IOP.

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Appendix
Impressions of 9th ISAAC Congress, Kraków 2013

V.V. Mityushev, M.V. Ruzhansky (eds.), *Current Trends in Analysis and Its Applications*, Trends in Mathematics, DOI 10.1007/978-3-319-12577-0


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