Aspects of the Mathematical Analysis of Nonlinear Stratified Water Waves

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Abstract The mathematical analysis of stratified water waves, where the density distribution of the flow is free to fluctuate, is a highly intractable subject which is of great physical and geophysical importance. In this chapter, we present an overview of some recently derived rigorous analytical results for nonlinear steady periodic stratified water waves.

1 Introduction

The study of stratified water waves, or flows which exhibit a variable density distribution, is a subject which is of great physical interest, since fluid density may be caused to fluctuate by a plethora of factors – for example, salinity, temperature, pressure, topography, oxygenation [54, 55]. Furthermore, stratification plays a prominent role for geophysical fluid phenomena where the physical scales of the fluid motion are such that the effects of the Earth's rotation are significant [16]. Mathematically, allowing for heterogeneity in the fluid adds severe complications to the governing equations – for instance, stratified flows are inherently rotational [54, 55] – rendering them highly intractable to mathematical analysis. Accordingly, there has been a marked paucity of rigorous mathematical results for the full governing equations for stratified water waves, even in the inviscid regime and where the effects of the Earth's rotation are neglected.

This survey aims to give an overview of some recent analytical results for periodic, steady stratified water waves. The primary emphasis in this survey will be on existence results for the fully nonlinear governing equations in the inviscid regime. Due to space considerations, and due to the vastness of the subject, in the considerations of this survey chapter we must omit a number of important aspects of stratified water waves which are currently active areas of mathematical research, among these

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being the study of internal waves. Also, aside from outlining some Gerstner-type exact solutions for geophysical water waves in Sect. 4, we neglect the effect of the Earth's rotation. We note that the survey [54] and book [55] are good expositions of the vast subject of stratified water waves, while [16] provides a nice introduction to stratification in geophysical flows.

The first results concerning the existence of small-amplitude waves for stratified flows were obtained by Dubreil-Jacotin [19] in 1937. In Sect. 2, we survey recent rigorous existence results [21, 27, 28] where the authors use local and global bifurcation theory to prove the existence of both small-, and large-, amplitude stratified water waves. A novel feature of this work is that the existence of critical points and layers is not precluded from the flows. Stagnation points have long been a source of great interest and fascination in hydrodynamical research, dating back to Kelvin's work concerning "cat's eyes" [29] and Stokes conjecture on the wave of greatest height, cf. [5, 43] for an overview of Stokes' waves.

In Sect. 3, we outline the existence results of Walsh [48–50] for stratified water waves which do not contain critical points. In the absence of critical or stagnation points, the framework of local and global bifurcation which was presented in the breakthrough paper [12] for homogeneous water waves with vorticity, is successfully adapted to prove the existence of large-amplitude stratified gravity water waves in [48]. In [49, 50], Walsh then proved the existence of large-amplitude stratified capillary-gravity water waves, where surface tension is a significant restoration force at the free boundary.

In Sect. 4, we outline some recently derived exact Gerstner-type solutions for geophysical water waves which incorporate stratification in the underlying flow, and in Sect. 5, we present some results concerning qualitative properties of stratified flows.

1.1   The Mathematical Model

For the analytic study of stratified water waves, the Long-Yih formulation [31, 44, 52] for the motion of a two-dimensional inviscid, incompressible fluid with variable density is essential. The governing equations are formulated in terms of the unknown free-surface of the wave $y = \eta(t, x)$, the velocity field (u, v) of the fluid, the pressure distribution P, the variable density ρ, and the gravity constant g. The equations are defined in the fluid domain

$$\Omega_\eta := \{(x, y) \,:\, x \in \mathbb{R} \quad \text{and} \quad -d < y < \eta(t, x)\},$$

where d is a positive constant and $y = -d$ models a flat impermeable bed. Taking $y = 0$ to represent the location of the undisturbed water surface, we assume for any fixed time t that $\eta(t, \cdot)$ has zero integral mean over a period.

This survey concerns traveling wave solutions of the governing equations, with all functions having an (x, y, t) dependence of the form $(x - ct, y)$, and so

$$\eta(t, x) = \eta(x - ct), \qquad (\rho, u, v, P)(t, x, y) = (\rho, u, v, P)(x - ct, y),$$

with the positive constant $c > 0$ the *wavespeed*. Under the assumption of constant temperature and zero viscosity, from a reference frame moving with the wave the flow is steady and is described by the steady two-dimensional Euler equations:

$$\begin{cases} \rho(u - c)u_x + \rho v u_y = -P_x & \text{in } \Omega_\eta, \\ \rho(u - c)v_x + \rho v v_y = -P_y - g\rho & \text{in } \Omega_\eta, \\ \rho_x(u - c) + \rho_y v = 0 & \text{in } \Omega_\eta, \\ u_x + v_y = 0 & \text{in } \Omega_\eta. \end{cases} \tag{1a}$$

The first two relations of (1a) are the conservation of momentum equations, the last relation expresses the fact that the flow is incompressible, and the second last relation is a reformulation of the steady continuity equation. Since some of the results we mention herein apply in the setting of waves with surface tension effects, the following boundary conditions have to be imposed

$$\begin{cases} v = (u - c)\eta' & \text{on } y = \eta(x), \\ P = P_0 - \sigma \eta''/(1 + \eta'^2)^{3/2} & \text{on } y = \eta(x), \\ v = 0 & \text{on } y = -d. \end{cases} \tag{1b}$$

Here, P_0 is the constant atmospheric pressure and σ is the surface tension coefficient. In the cases where we neglect the effects of surface tension, we simply set $\sigma = 0$ in the above. The first and last equations of (1b) are kinematic conditions which express the fact that the wave surface moves along with the fluid and that the fluid bed is impermeable. The second relation of (1b) is the dynamic boundary condition which says that the pressure jump along the wave surface obeys the Laplace–Young law: it is proportional to the curvature of the surface and the constant of proportionality is the surface tension coefficient [30].

We reformulate system (1a, 1b) into a more useful, malleable form as follows. An important consequence of the last two equations of (1a) is that

$$\nabla \cdot (\sqrt{\rho}(u - c), \sqrt{\rho} v) = 0 \qquad \text{in } \Omega_\eta.$$

This relation enables us to associate to system (1a, 1b) the *pseudo-stream function* ψ defined by

$$\psi(x, y) := \lambda + \int_{-d}^{y} \sqrt{\rho}(x, s)(u(x, s) - c)\, ds, \qquad (x, y) \in \overline{\Omega}_\eta.$$

The constant λ is defined by the property that ψ vanishes at the wave surface, with $-\lambda$ being the *relative mass flux* of the flow. Since $\nabla \psi = (-\sqrt{\rho} v, \sqrt{\rho}(u - c))$ is orthogonal to the steady velocity field $(u - c, v)$ throughout the fluid domain, it follows that the level sets of the ψ function are the streamlines of the steady flow.

An important assumption which is often invoked when studying traveling water waves is that

$$\partial_y \psi = \sqrt{\rho}(u - c) < 0 \qquad \text{in } \overline{\Omega}_\eta, \tag{2}$$

a condition which excludes the presence of stagnation points in the flow, that is, that points for which $\nabla \psi = 0$. This is a physically reasonable assumption for water waves, without underlying currents containing strong nonuniformities, and which are not near breaking [30]. Nevertheless, stagnation points are a very interesting phenomenon and ideally we would not wish to exclude them from our picture. For fluid motions where (2) holds we have $\lambda > 0$, the relative mass flux being negative. The condition (2) can be used to formulate the hydrodynamical problem in the rectangular domain $\Omega_\lambda := \mathbb{R} \times (-\lambda, 0)$, where the streamlines become straight horizontal lines in the Ω_λ. This property provides us with a convenient means of analyzing how certain quantities change either along fixed streamlines or as we vary the streamlines. To present this new formulation, we first recall that the Dubreil-Jacotin [18] semi-hodograph transformation:

$$\mathcal{H}(x, y) := (q, p)(x, y) := (x, -\psi(x, y)), \qquad (x, y) \in \overline{\Omega}_\eta \qquad (3)$$

is a diffeomorphism $\mathcal{H} : \Omega_\eta \to \Omega_\lambda$. Using the second last equations of (1a), it is easy to see that the density is constant on each streamline, since

$$\partial_q \left(\rho \circ \mathcal{H}^{-1} \right) \circ \mathcal{H} = \rho_x + \rho_y \frac{v}{u - c} = \frac{(u - c)\rho_x + v\rho_y}{u - c} = 0.$$

This means that there exists a function $\overline{\rho} : [-\lambda, 0] \to (0, \infty)$, the so-called *streamline density function* such that $\rho \circ \mathcal{H}^{-1} = \overline{\rho}$, or equivalently $\rho(x, y) = \overline{\rho}(-\psi(x, y))$ for all $(x, y) \in \overline{\Omega}_\eta$. Furthermore, by Bernoulli's principle, the energy

$$E := P + \rho \frac{(u - c)^2 + v^2}{2} + g\rho y \qquad (4)$$

is constant along streamlines, that is $\partial_q \left(E \circ \mathcal{H}^{-1} \right) = 0$. Particularly, when $p = 0$, we obtain

$$|\nabla \psi|^2 - 2\sigma \frac{\eta''}{(1 + \eta'^2)^{3/2}} + 2g\overline{\rho}(0)y = Q \qquad \text{on } y = \eta(x),$$

for some constant $Q \in \mathbb{R}$. Moreover, a direct calculation shows that

$$\Delta \psi - gy\overline{\rho}' \circ \mathcal{H} = -\partial_p (E \circ \mathcal{H}^{-1}) \circ \mathcal{H} \qquad \text{in } \Omega_\eta.$$

Since $\partial_q \left(E \circ \mathcal{H}^{-1} \right) = 0$, there exists a function $\beta = \beta(p)$, called *Bernoulli's function*, such that $-\partial_p(E \circ \mathcal{H}^{-1}) = \beta$ in Ω_λ. Piecing all this together leads to the Long-Yih formulation for steady stratified water waves [31, 44, 52] where ψ solves the following free boundary problem:

$$\begin{cases} \Delta \psi = gy\overline{\rho}'(-\psi) + \beta(-\psi) & \text{in } \Omega_\eta, \\ \psi = 0 & \text{on } y = \eta(x), \\ \psi = \lambda & \text{on } y = -d, \\ |\nabla \psi|^2 - 2\sigma \dfrac{\eta''}{(1 + \eta'^2)^{3/2}} + 2g\overline{\rho}(0)y = Q & \text{on } y = \eta(x). \end{cases}$$

$$(5)$$

A further equivalent formulation of the problem (1a, 1b) and (2) can be derived in terms of the height function $h : \overline{\Omega}_\lambda \to \mathbb{R}$ given by

$$h(q, p) = y + d \qquad \text{for } (q, p) \in \overline{\Omega}_\lambda.$$

In terms of h, the system (5) can be recast as a set of equations in a fixed rectangular domain:

$$\begin{cases} (1 + h_q^2)h_{pp} - 2h_q h_p h_{pq} + h_p^2 h_{qq} - (g\overline{\rho}'(h - d) + \beta)h_p^3 = 0 & \text{in } \Omega_\lambda, \\ h = 0 & \text{on } p = -\lambda, \\ 1 + h_q^2 + h_p^2 \left[2g\overline{\rho}(0)(h - d) - 2\sigma \dfrac{h_{qq}}{(1 + h_q^2)^{3/2}} - Q \right] = 0 & \text{on } p = 0, \end{cases}$$

$$(6)$$

the relation (2) taking the form

$$\inf_{\Omega_\lambda} h_p > 0. \tag{7}$$

The problem (6) consists of a quasilinear elliptic equation subjected to nonlinear boundary conditions. This formulation gives an insight into the flow because the streamlines in the moving frame are parametrized by the mappings $x \mapsto h(x, p) - d$. The formulations (1a, 1b), (2), and (5) are equivalent in the setting of classical solutions when (6) and (7) holds, cf. e.g., [48].

2 Existence Results for Flows Admitting Stagnation Points

When allowing for stagnation points to exist, that is, when dropping assumption (2), the three formulations – the Euler equations, the stream function formulation, and the height function formulation – are no longer equivalent. However, it is not difficult to see that if we are given a Bernoulli function $\beta : \mathbb{R} \to \mathbb{R}$ and a streamline density function $\overline{\rho} : \mathbb{R} \to (0, \infty)$, then any solution (η, ψ) of (5) defines a solution (η, u, v, P) of (1a, 1b). Furthermore, if $\nabla \psi(P) = 0$ at some point $P \in \overline{\Omega}_\eta$, then P is a stagnation point of the flow. This aspect has been exploited to construct small-amplitude gravity waves with a linear density distribution and constant Bernoulli function that contain stagnation points in [21]. In the case of more general streamline density and Bernoulli functions, small- as well as large-amplitude capillary-gravity waves that may posses stagnation points where shown to exist in [27, 28]. In the references [21, 27, 28], the fluid bed is taken to be located at $y = -1$ and the waves that are found are 2π-periodic and bifurcate from laminar flows with the wave surface being located at $y = 0$. Nevertheless, the methods can be applied in the context of flows with arbitrary finite depth and arbitrary wavelength.

2.1   Stratified Flows with Linear Density Distribution and Constant Bernoulli Function

The analysis in [21] was devoted to the classification of the gravity waves (that is $\sigma = 0$) with stagnation points that bifurcate from laminar flows when considering a linear streamline density and a constant Bernoulli function. A precise description of the flow pattern for the classes of flows that were found was also given. To recall the results of [21] we assume that

$$\overline{\rho}(p) = Ap + B \quad \text{and} \quad \beta(p) = C \qquad \text{for } p \in \mathbb{R}.$$

Of course, the constants $A, B, C \in \mathbb{R}$ have to be chosen such that the bifurcating solutions (η, ψ) of (5), that is in this particular case of

$$
\begin{cases}
\quad \Delta\psi = Agy + C & \text{in } \Omega_\eta, \\
\quad \psi = 0 & \text{on } y = \eta(x), \\
\quad \psi = \lambda & \text{on } y = -1, \\
|\nabla\psi|^2 + 2Bgy = Q & \text{on } y = \eta(x),
\end{cases}
\tag{8}
$$

have positive density $\rho(x, y) = \overline{\rho}(-\psi(x, y)) > 0$ for $(x, y) \in \Omega_\eta$. As a result of this and recalling that $\psi = 0$ at the wave surface, the constant B has to be taken positive. The laminar flow solutions of (8) with the free surface given by $\eta = 0$ can be parametrized as a family $\{(0, \psi_\mu) : \mu \in \mathbb{R}\}$, with the constants λ and Q being also function of μ, that is

$$\psi_\mu(y) := \frac{Agy^3}{6} + \frac{Cy^2}{2} + \mu y, \quad Q = \mu^2, \quad \lambda = -\frac{Ag}{6} + \frac{C}{2} - \mu.$$

The constant μ satisfies $\mu = \partial_y \psi|_{y=0}$, so that it represents the horizontal speed in the moving frame of the laminar flows at the wave surface. In fact, the stream function ψ can be uniquely determined when knowing μ and η, so that (μ, η) are the only unknowns in (8) when considering Q and λ as functions of μ as defined above.

The dispersion relation for such waves is determined as follows:

$$\mu_k^\pm := \frac{C}{2} \frac{\tanh(k)}{k} \pm \sqrt{\frac{C^2}{4} \frac{\tanh^2(k)}{k^2} + \frac{2Bg}{2} \frac{\tanh(k)}{k}},$$

the integer $|k|$ representing the wave number. Each of the points $(\mu_k^\pm, 0)$ is a critical bifurcation point.

Theorem 1 ([21, Theorem 2.1]) *Let $k \in \mathbb{N} \setminus \{0\}$, $\mu_* \in \{\mu_k^\pm\}$, $\alpha \in (0, 1)$, and $B > 0$ be given. Then, there exists $\varepsilon > 0$ and a real-analytic curve*

$$(\mu, \eta) : (-\varepsilon, \varepsilon) \to \mathbb{R} \times C^{2+\alpha}(\mathbb{R}),$$

consisting only of solutions $(\mu(s), \eta(s))$ of (8) of minimal period $2\pi/k$, having exactly one crest and trough per period, and with a real-analytic and symmetric free surface.

These are the only nonlaminar solutions of (8) close to $(\mu_*, 0)$, and, for $s \to 0$,

$$\mu(s) = \mu_* + O(s^2),$$
$$\eta(s) = -s \cos(kx) + O(s^2) \quad in \ C^{2+\alpha}(\mathbb{R}).$$

There are no bifurcation points $(\mu, 0)$ other than $\{(\mu_k^{\pm}, 0) : k \in \mathbb{N} \setminus \{0\}\}$.

Because of the asymptotic expansion derived for the bifurcation curves, it is not difficult to see that the flow determined by $(\mu(s), \eta(s))$ contains no stagnation points if $\partial_y \psi_{\mu_*}$ does not vanish in $[-1, 0]$ and ε is small. Moreover, it is shown in [21] that if

$$\partial_y \psi_{\mu_*}(y) = \frac{Agy^2}{2} + Cy + \mu = 0 \tag{9}$$

has solutions in $[-1, 0]$, then the bifurcating nonlaminar flows will also contain stagnation points. Hence, the classification of the flow pattern of the bifurcating small-amplitude waves with stagnation points has to take into account the location of the solutions y_1, y_2 of (9). Only the following classes of stratified waves bifurcating from laminar flows with stagnation points are possible:

(I) Stratified flows with exactly one critical layer. They bifurcate from the laminar flows $(\mu_*, 0)$ with $\mu_* \in \{(\mu_k^{\pm}, 0) : k \in \mathbb{N} \setminus \{0\}\}$ in one of the following cases:

   (*a*) The equation $\partial_y \psi_{\mu_*} = 0$ has a unique solution in $(-1, 0)$.
   (*b*) The equation $\partial_y \psi_{\mu_*} = 0$ has $y = -1$ as the unique solution within $[-1, 0)$ (or the solutions are both equal to -1).
   (*c*) The equation $\partial_y \psi_{\mu_*} = 0$ has two solutions in $(-1, 0)$, and they coincide.

(II) Stratified flows with two critical layers. They bifurcate from the laminar flows $(\mu_*, 0)$ with $\mu_* \in \{(\mu_k^{\pm}, 0) : k \in \mathbb{N} \setminus \{0\}\}$ in one of the following cases.

   (*d*) The equation $\partial_y \psi_{\mu_*} = 0$ has two solutions $-1 = y_2 < y_1 < 0$.
   (*e*) The equation $\partial_y \psi_{\mu_*} = 0$ has two solutions $-1 < y_2 < y_1 < 0$.

These five classes of stratified gravity waves with stagnation points are pictured in Fig. 1. We note that the stagnation points are located either beneath the wave crest or beneath the wave trough. Very interesting is the degenerate case (*c*) where bifurcation occurs when the solutions of $\partial_y \psi_{\mu_*} = 0$ satisfy $y_1 = y_2 \in (-1, 0)$. In this case, there is only a compact (in Ω_η) critical layer consisting of closed streamlines right beneath the wave crest. Such a flow pattern is not encountered in homogeneous flows with constant or linear vorticity, cf. [13, 20, 46], but the situation slightly resembles to that of background flows for tsunamis which contain isolated regions of vorticity surrounded by still water [7, 23]. For the solutions in all five classes, the density has an extremum at the stagnation point located in the center of the vortices.
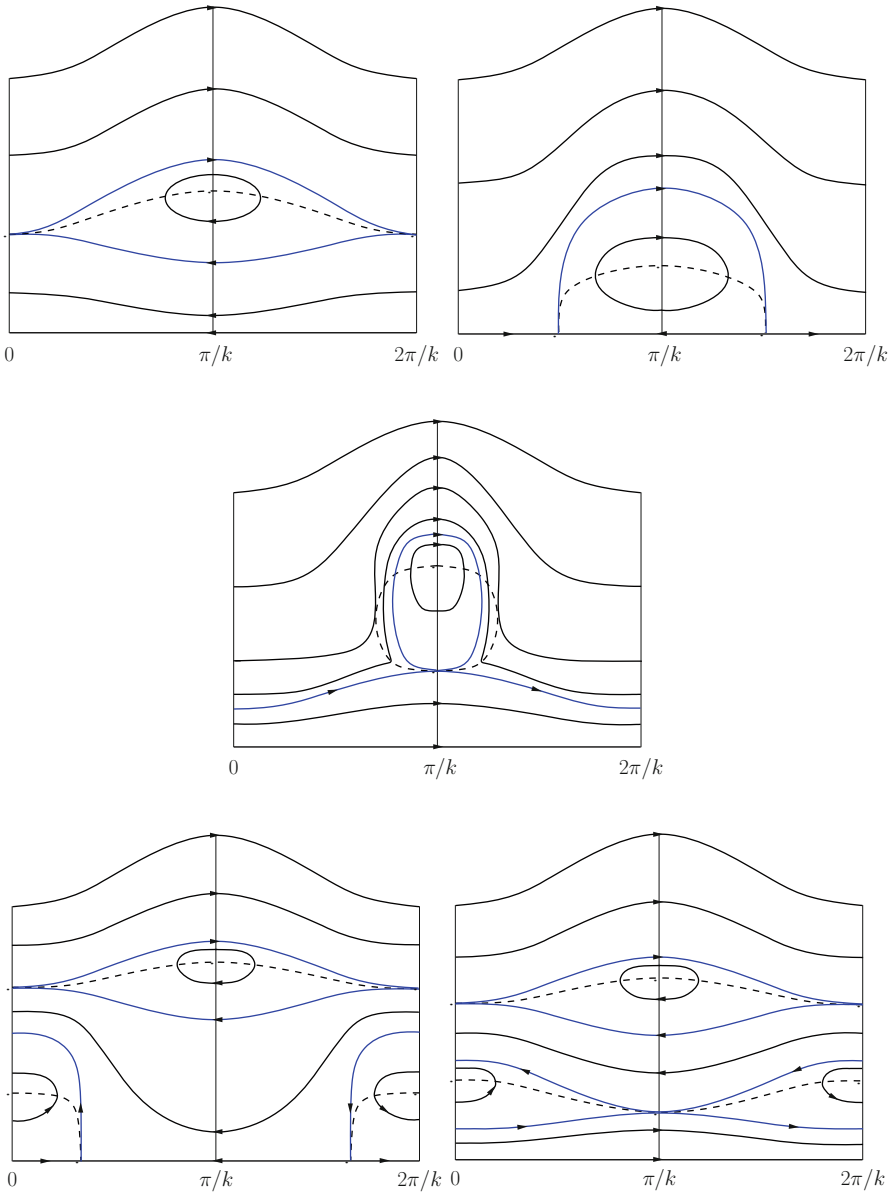
Fig. 1 The flow pattern for the stratified waves bifurcating from laminar flow solution with stagnation points for waves in the classes (*a*), (*b*), (*c*), (*d*), and (*e*) (from the *upper-left* corner to the *lower-right* corner). The dashed lines are curves where $\partial_y \psi$ vanishes, and the solid dots are the stagnation points. The blue curves connecting the stagnation points are separatrices that enclose (with the exception of the lower separatrix in the third figure) critical layers of closed streamlines

2.2 Stratified Flows with More General Density Distribution and Bernoulli Function

We address now the results established in [27, 28] in the context of more general streamline density and Bernoulli functions for waves with capillarity. To this end, we make the following notation:

$$f(y, \psi) := gy\overline{\rho}'(-\psi) + \beta(-\psi), \qquad (y, \psi) \in [-1, 1] \times \mathbb{R}.$$

The solutions (η, ψ) of (5) are to be found such that $\max_{\mathbb{R}} |\eta| < 1$. In fact, relying on the divergence structure of the curvature operator in Bernoulli's condition, the constant Q can be eliminated from the equations, if the integral mean of η is zero, by integrating the Bernoulli condition over $[0, 2\pi]$,

$$Q = \frac{1}{2\pi} \int_0^{2\pi} |\nabla \psi|^2|_{y=\eta(x)} \, dx.$$

One arrives at the following problem:

$$\begin{cases} \Delta \psi = f(y, \psi) & \text{in } \Omega_\eta, \\ \psi = 0 & \text{on } y = \eta(x), \\ \psi = \lambda & \text{on } y = -1, \\ |\nabla \psi|^2 - 2\sigma \dfrac{\eta''}{(1 + \eta'^2)^{3/2}} + 2g\overline{\rho}(0)y = \dfrac{1}{2\pi} \int_0^{2\pi} |\nabla \psi|^2|_{y=\eta(x)} \, dx & \text{on } y = \eta(x). \end{cases}$$
(10)

As a result of this choice for Q, for any solution (η, ψ) of (10), η has zero integral mean over $[0, 2\pi]$, and therefore the solutions obtained by bifurcation describe flows over the same volume of fluid (the mean depth of the fluid is constant along the bifurcation branch).

The following assumptions are made on $\overline{\rho}$, β, and f:

(A1) $\overline{\rho} > 0$.

(A2) $\overline{\rho} \in C^{4-}(\mathbb{R}), \beta \in C^{3-}(\mathbb{R})$, and $\beta, \overline{\rho}' \in BC^2(\mathbb{R})$.

(A3) $\partial_\psi f(y, \psi) \geq 0$ for all $(y, \psi) \in [-1, 1] \times \mathbb{R}$.

(A4) $2f(y, \psi) + g(1 + y)\overline{\rho}'(-\psi) \leq 0$ for all $(y, \psi) \in [-1, 0] \times (-\infty, 0]$.

In the context of homogeneous flows, we have that $\overline{\rho}' \equiv 0$ and the Bernoulli function β is identified with the vorticity function γ of the rotational flow. In this context, the conditions $(A1) - (A4)$ are equivalent to

$$\gamma \in C^{3-}(\mathbb{R}) \cap BC^2(\mathbb{R}), \quad \gamma \leq 0 \text{ in } [0, \infty), \quad \text{and} \quad \gamma' \leq 0.$$
(11)

While the first assumption is physical, the assumptions $(A2)$ and $(A3)$ ensure that the semilinear elliptic problem for ψ that consists of the first three equations of (10)

has a solution $\psi \in C^{2+\alpha}(\overline{\Omega}_\eta)$ ($\alpha \in (0,1)$ is fixed), that is uniquely determined by the pair $(\lambda, \eta) \in \mathbb{R} \times \text{Ad}$, whereby

$$\text{Ad} := \left\{ \eta \in C^{2+\alpha}(\mathbb{R}) \,:\, \eta \text{ is } 2\pi-\text{periodic}, \int_0^{2\pi} \eta \, dx = 0, \text{ and } \max_\mathbb{R} |\eta| < 1 \right\}.$$

Therefore, the pairs $(\lambda, \eta) \in \mathbb{R} \times \text{Ad}$ are the true unknowns of (10). This unique solvability property of the semilinear elliptic problem for ψ was used in [27], after transforming the problem on a fixed reference domain, to recast the hydrodynamical problem as a nonlinear and nonlocal equation with (λ, η) as unknowns. Because for (λ, η) with $\eta = 0$, the solution ψ of (10) depends only upon y, it is easy to see that $(\lambda, 0)$ is a solution of (10) for any $\lambda \in \mathbb{R}$. This trivial branch of solution contains countably many bifurcation points $(\lambda_*, 0)$ where branches consisting of nonlaminar solutions of (10) emerge.

Theorem 2 ([27, Theorem 4.6]). *Let $\sigma > 0$ be fixed and assume additionally to* $(A1) - (A4)$ *assume that:*

(B1) $\partial_{\psi\psi} f \geq 0$ *on* $[-1,0] \times (-\infty, 0]$.

(B2) $2\partial_\psi f(y, \psi) - g(1 + y)\overline{\rho}''(-\psi) \geq 0$ *for all* $(y, \psi) \in [-1,0] \times (-\infty, 0]$.

(B3) $\displaystyle\int_{-1}^0 \int_t^0 \partial_\psi f(s, 0) \, ds \, dt \leq 1$.

Then there exists a positive integer $K \in \mathbb{N}$ and for all $k \geq K$ a sequence $(\lambda_m)_{m \geq 1} \subset \mathbb{R}$ with $\lambda_m \to -\infty$ such that:

(i) *Given $m \geq 1$, there exists a curve $(\lambda_m, \eta_m) : (-\varepsilon, \varepsilon) \to \mathbb{R} \times (\text{Ad} \cap C_{2\pi/k}(\mathbb{R}))$ which is continuously differentiable in $\mathbb{R} \times (C^{2+\alpha}(\mathbb{R}) \cap C_{2\pi/k}(\mathbb{R}))$ and consists only of solutions $(\lambda_m(s), \eta_m(s))$ of (10). The wave determined by $(\lambda_m(s), \eta_m(s))$ with $s \neq 0$ has minimal period $2\pi/(mk)$, exactly one crest and trough per period and is symmetric with respect to the crest line.*

(ii) *For $s \to 0$,*

$$\lambda_m(s) = \lambda_m + O(s), \quad \eta_m(s) = -s\cos(mkx) + O(s^2).$$

All the solutions of (10) in $\mathbb{R} \times (\text{Ad} \cap C_{2\pi/k}(\mathbb{R}))$ that are close to $(\lambda_m, 0)$ are either laminar flows or belong to the curve (λ_m, η_m). Moreover, there exists a constant $\Lambda_- \in \mathbb{R}$ with the property that if $(\lambda, 0)$ is a bifurcation point of (10) with $\lambda \in (-\infty, \Lambda_-)$, then $\lambda \in \{\lambda_m : m \geq 1\}$.

Hereby, $C_{2\pi/k}(\mathbb{R})$ stands for the Banach space of $2\pi/k$-periodic functions. That the wave number k should be sufficiently large is a condition that appears frequently when dealing with waves with surface tension effects, cf. e.g., [32, 45, 49]. Lastly, let us note that in the context of homogeneous capillary-gravity waves the conditions $(B1) - (B3)$ mean that, additionally to (11), we assume

$$\gamma'' \geq 0 \ \text{ in } [0, \infty) \qquad \text{and} \qquad \gamma'(0) \geq -2. \tag{12}$$

The local bifurcation branches obtained in Theorem 2 can be continued by using global bifurcation theory, cf. [28]. The authors of [28] observed that the quasilinear curvature term in Bernoulli's condition can be inverted (see also [36]), the problem (10) being equivalent to an operator equation for a compact (nonlocal and nonlinear) perturbation of the identity. This facilitates the use of the Rabinowitz global bifurcation theorem, cf. e.g., Theorem II.3.3 in [23].

Theorem 3 ([28, Theorem 4.1]) *Let the assumptions of Theorem 2 be satisfied, let $k \geq K$, and fix $m \in \mathbb{N}$ with $m \geq 1$. Moreover, let $\mathcal{C}_M$ be the maximal connected component of the closure of the set:*

$$\left\{ (\lambda, \eta) \, : \, (\lambda, \eta) \in \mathbb{R} \times (\mathrm{Ad} \cap C_{2\pi/k}(\mathbb{R})) \text{ is a solution of (10) with } \eta \not\equiv 0 \right\}$$

in $\mathbb{R} \times (C^{2+\alpha}(\mathbb{R}) \cap C_{2\pi/k}(\mathbb{R}))$ that contains $(\lambda_m, 0)$. Then, we have:

(*i*) $\mathcal{C}_M$ *is unbounded in* $(-\infty, \Lambda_-) \times (C^{2+\alpha}(\mathbb{R}) \cap C_{2\pi/k}(\mathbb{R}))$, *or*

(*ii*) $\displaystyle\sup_{(\lambda,\eta)\in\mathcal{C}_M} \max_{\mathbb{R}} |\eta| = 1.$

Finally, let us emphasize that the Theorems 2 and 3 were stated for a fixed surface tension coefficient. There is a second possibility when dealing with the problem (10): To consider the mass flux fixed and use the surface tension coefficient as a bifurcation parameter. From mathematical point of view this choice is more useful: one can show [27, Theorem 4.3] that there exists a sequence of bifurcation points $(\sigma_m, 0)$ with $m \in \mathbb{N}$, $m \geq 1$, for (10) with $\sigma_m \searrow 0$. Each of the points $(\sigma_m, 0)$ belongs to a continuously differentiable local curve of nonlaminar solutions in $\mathbb{R} \times \mathrm{Ad}$. There is no restriction on the wave number any longer, and there is no need of the assumptions $(B1) - (B3)$. These local branches can be continued to global continua as in [28, Theorem 4.1].

3   Existence Results for Flows Without Critical Points

For flows without stagnation or critical points, the assumption (2) is valid throughout the fluid domain. Accordingly, the Dubreil-Jacotin semi-hodograph transformation (3) represents a change of variables and the stratified capillary-gravity water wave problem may be recast as system (6)—the inequality (7) ensures that system (6) is uniformly elliptic. In this framework, Walsh has recently proved the existence of finite amplitude steady periodic stratified water waves, in [48] for the setting of pure gravity waves, and in [49, 50] for capillary-gravity waves.

3.1   Gravity Stratified Waves

For the setting of pure gravity stratified water waves, which we may obtain by setting $\sigma = 0$ in (6), Walsh proved the existence of large-amplitude steady periodic

stratified water waves [48]. The approach used in [48] successfully implemented a local and global bifurcation analysis along the lines of the seminal work [12] which proved existence for homogeneous, gravity water waves with vorticity. However, the application of this method for stratified water waves is highly nontrivial, and one must overcome a number of significant technical hurdles induced by the fluid stratification. A benefit of this approach is the derivation of an explicit size condition (13) on the physical variables which is sufficient for bifurcation to occur. The local bifurcation analysis employs the Crandall–Rabinowitz bifurcation theorem [15], whereby a necessary and sufficient condition, denoted (L-B), for local bifurcation to occur is derived depending on $\lambda, \beta, \overline{\rho}$. An explicit version of this condition, which is sufficient for local bifurcation to occur, may be stated as follows. First, we define

$$B(p) = -\int_0^p \beta(s)\mathrm{d}s, \qquad -\lambda \leq p \leq 0,$$

and let

$$\varepsilon_0 = \max\left\{ \left(2g\|\overline{\rho}'\|_\infty \lambda^2 e^\lambda\right)^{2/3}, (2g\|\overline{\rho}'\|_\infty)^2, (4\|\overline{\rho}'\|_\infty)^2, (8g\lambda\overline{\rho}(0))^{2/3} \right\}.$$

Then the condition which is sufficient for local bifurcation to occur is given by

$$
\begin{aligned}
g\overline{\rho}(0)\lambda^2 > \int_{-\lambda}^0 \Bigg\{ & \frac{4\pi^2}{L^2}\left(2B(p) - 2B_{\min} + 2\varepsilon_0\right)^{3/2} \\
& + (p+\lambda)^2\left((2B(p) - 2B_{\min} + 2\varepsilon_0)^{1/2} + g\overline{\rho}'(p)\right) \Bigg\} \, \mathrm{d}p.
\end{aligned}
\tag{13}
$$

Following the local existence result, the bifurcation curve may then be extended to a global continuum using the global bifurcation approach of Healey-Simpson and Kielhofer, cf. [6] for an overview of this approach. Using the notation of this survey chapter, the main result for stratified gravity water waves in [48] may be stated as follows.

Theorem 4 ([48]) *Fix a wavespeed $c > 0$, wavelength $L > 0$, and $\lambda > 0$. Fix any $\alpha \in (0, 1)$, and let the functions $\beta \in C^{1+\alpha}([-\lambda, 0])$ and $\overline{\rho} \in C^{1+\alpha}([-\lambda, 0])$ be given such that the (L-B) condition holds. Also, we assume the streamline density function $\overline{\rho}$ is nonincreasing. Consider traveling solutions to the stratified water wave problem (1a, 1b) of speed c, relative mass flux $-\lambda$, Bernoulli function β, and streamline density function $\overline{\rho}$ such that $u < c$ throughout the fluid. There exists a connected set $\mathcal{C}$ of solutions (u, v, ρ, η) in the space $C_{per}^{2+\alpha}(\overline{\Omega}_\eta) \times C_{per}^{2+\alpha}(\overline{\Omega}_\eta) \times C_{per}^{2+\alpha}(\overline{\Omega}_\eta) \times C_{per}^{3+\alpha}(\mathbb{R})$ with the following properties:*

1. *$\mathcal{C}$ contains a laminar flow (with a flat surface $\eta \equiv 0$ and all streamlines parallel to the bed).*
2. *Along some sequence $(u_n, v_n, \rho_n, \eta_n) \in \mathcal{C}$, either $\max_{\overline{\Omega}_{\eta_n}} u_n \uparrow c$, $\min_{\overline{\Omega}_{\eta_n}} u_n \downarrow -\infty$, or $\mathcal{C}$ contains more than one distinct laminar solution.*

Furthermore, each nonlaminar flow $(u, v, \rho, \eta) \in \mathcal{C}$ *is regular in the sense that:*

1. *u, v, ρ, and η each have period L in x.*
2. *Within each period the wave profile η has a single crest and trough; say, the crest occurs at $x = 0$.*
3. *u, ρ, and η are symmetric, v antisymmetric across the line $x = 0$.*
4. *A water particle located at (x, y), with $0 < x < L$ and $y > d$ has positive vertical velocity $v > 0$.*
5. *$\eta'(x) < 0$ on $(0, L)$.*

The symmetry of solutions ensured by Theorem 4 is interesting, and it will be considered further in Sect. 5.

3.2 Capillary-Gravity Stratified Waves

When surface tension is taken into account in the governing equations (6), Walsh has also proven the existence of large-amplitude waves [50]. The global bifurcation analysis in [50] extends the local bifurcation curves derived in [49], adapting the methods first applied for the local existence of homogeneous capillary-gravity waves with vorticity in [45]. In [50], the author also uses Dancer's analytic global bifurcation theory [2, 6] to extend the local bifurcation curves. In [49], the author proves local bifurcation from both simple eigenvalues, and from double eigenvalues where additional non-degeneracy conditions are satisfied. For the simple eigenvalue setting, we let ε_0 be as above, then a sufficient condition for local bifurcation to occur is given by

$$(g\overline{\rho}(0) + \sigma)\,\lambda^2 > \int_{-\lambda}^{0} \left\{ \frac{4\pi^2}{L^2} \left(2B(p) - 2B_{\min} + 2\varepsilon_0\right)^{3/2} \right.$$

$$\left. + (p + \lambda)^2 \left((2B(p) - 2B_{\min} + 2\varepsilon_0)^{1/2} + g\overline{\rho}'(p)\right) \right\} \mathrm{d}p.$$

The main results for bifurcation from a simple eigenvalue in [49, 50] may then be stated (combined) as follows:

Theorem 5 *Fix any $\alpha \in (0, 1)$, and let the wavespeed $c > 0$, wavelength $L > 0$, and $\lambda > 0$ be given, along with $\beta \in C^{1+\alpha}([-\lambda, 0])$ and $\overline{\rho} \in C^{2+\alpha}([-\lambda, 0])$, such that the (L-B) condition holds. Also, let the given coefficient of surface tension $\sigma \in \Sigma_1$, where $(\sigma_c, \infty) \subset \Sigma_1 \subset (0, \infty)$, and we assume the streamline density function $\overline{\rho}$ is nonincreasing. Consider traveling solutions to the stratified water wave problem (1a, 1b) of speed c, relative mass flux $-\lambda$, Bernoulli function β, and streamline density function $\overline{\rho}$ such that $u < c$ throughout the fluid. There exists a connected set $\mathcal{C}$ of solutions (u, v, ρ, η) in the space $C_{per}^{2+\alpha}(\overline{\Omega}_\eta) \times C_{per}^{2+\alpha}(\overline{\Omega}_\eta) \times C_{per}^{2+\alpha}(\overline{\Omega}_\eta) \times C_{per}^{3+\alpha}(\mathbb{R})$ with the following properties:*

1. *$\mathcal{C}$ contains a laminar flow (with a flat surface $\eta \equiv 0$ and all streamlines parallel to the bed).*

2. Along some sequence $(u_n, v_n, \rho_n, \eta_n) \in \mathcal{C}$, either $\max_{\overline{\Omega}_{\eta_n}} u_n \uparrow c$, $\min_{\overline{\Omega}_{\eta_n}} u_n \downarrow -\infty$, or $\mathcal{C}$ contains more than one distinct laminar solution.

Additionally, there exists a path-connected subset $\mathcal{K} \subset \mathcal{C}$ such that:

1. $\mathcal{K}$ admits a global, locally injective continuous parameterization with a locally injective C_1 reparametrization.
2. Either $\mathcal{K}$ is a closed loop, or it is unbounded in the sense that along some sequence $(u_n, v_n, \rho_n, \eta_n) \in \mathcal{K}$, either $\max_{\overline{\Omega}_{\eta_n}} u_n \uparrow c$, $\min_{\overline{\Omega}_{\eta_n}} u_n \downarrow -\infty$.

Furthermore, each nonlaminar flow $(u, v, \rho, \eta) \in \mathcal{C}$ is regular in the sense of Theorem 4.

A mathematical idiosyncrasy of the effects of surface tension, rather than stratification, is the possibility of bifurcation from double eigenvalues, and we therefore refer the reader to [49, 50] for an exposition of this phenomenon in stratified capillary-gravity waves.

4   Stratification for Gerstner-Type Flows

Interestingly, prior to her work on the existence of small-amplitude stratified water waves [18, 19], Dubreil-Jacotin showed that the celebrated Gerstner's water wave solution [3, 24] could be adapted to allow for continuous stratification in the underlying fluid [17]. Gerstner's wave is one of the few examples of an explicit solution for the fully nonlinear governing equations, since the formulation of the solution is explicit in the Lagrangian formulation [1]. The form of Gerstner's solution can be modified in order to obtain an explicit exact solution for edge waves – this was first achieved in [4]. In [37, 40, 53], it was shown that stratification is also possible in the prescribed flow for edge waves.

Recently, a number of exact solutions for geophysical water waves have been derived, beginning with [8] and extending to [9, 25, 33, 34], which are explicit in the Lagrangian representation and which admit continuous stratification. We present the solution derived in [8] for three-dimensional trapped equatorial water waves, since it can be reduced to the setting of the stratified Gerstner's solution when the geophysical parameter is set to zero. The β-plane approximation of the geophysical governing equations, which applies in regions close to of the equator, is given by [16]:

$$(u - c)u_x + vu_y + wu_z + 2\Omega w - \beta yv = -\frac{1}{\rho}P_x, \tag{14a}$$

$$(u - c)v_x + vv_y + wv_z + \beta yu = -\frac{1}{\rho}P_y, \tag{14b}$$

$$(u - c)w_x + vw_y + ww_z - 2\Omega u = -\frac{1}{\rho}P_z - g. \tag{14c}$$

Here, we take the Earth to be a perfect sphere of radius $R = 6378$ km, which has a constant rotational speed of $\Omega = 73 \times 10^{-6}\ rad\ \mathrm{s}^{-1}$. Then

$g = 9.8 \ ms^{-2}$ is the standard gravitational acceleration at the Earth's surface, and $\beta = 2\Omega/R = 2.28 \times 10^{-11} \ m^{-1}s^{-1}$ is a geophysical parameter. The boundary conditions for the fluid are given by

$$w = (u - c)\eta_x + v\eta_y \qquad \qquad \text{on } y = \eta(x, y, t), \qquad (15)$$

$$P = P_0 \qquad \qquad \text{on } y = \eta(x, y, t). \qquad (16)$$

Then, the Eulerian coordinates of fluid particles (x, y, z) are expressed as functions of the Lagrangian labeling variables $(q, r, s) \in \mathbb{R} \times (-\infty, r_0) \times \mathbb{R}$, and time t, as follows:

$$x = q - \frac{1}{k}e^{k[r - f(s)]} \sin [k(q - ct)], \qquad (17a)$$

$$y = s, \qquad (17b)$$

$$z = r + \frac{1}{k}e^{k[r - f(s)]} \cos [k(q - ct)], \qquad (17c)$$

where $r_0 < 0$ and k is the wave number. The function $f(s)$ is given by

$$f(s) = \frac{c\beta}{2g}s^2, \qquad (18)$$

and it determines the decay of the particle oscillation as it moves in the latitudinal direction away from the equator. Let us prescribe the density function by

$$\rho(r, s) = F\left(\frac{e^{2k[r - f(s)]}}{2k} - r\right),$$

where $F : (0, \infty) \to (0, \infty)$ is continuously differentiable and nondecreasing. Then the formula (17) prescribes a solution of the governing equations (14) if we define the pressure function

$$P = g\mathcal{F}\left(\frac{e^{2k[r - f(s)]}}{2k} - r\right) + P_0 - g\mathcal{F}\left(\frac{e^{2kr_0}}{2k} - r_0\right), \qquad (19)$$

where $\mathcal{F}' = F$ and $\mathcal{F}(0) = 0$, cf. [8]. We observe that setting $\beta = 0$ in the previous considerations reduces us to the case of Gerstner's water wave, and taking F constant we get the homogeneous non-stratified setting. We finally mention that Gerstner-type solutions have been derived which model discontinuous stratification, [10, 39, 41, 42], with these solutions giving an explicit formulation for internal waves in a fluid.

5 Qualitative Properties for General Stratified Flows

In this section, we recall recent results concerning the regularity of the streamlines and the symmetry of the wave profile for stratified waves without stagnation points. In the first paper [26], it was shown that the streamlines and the wave profile of stratified

flows for which $\beta, \overline{\rho}' \in C^\alpha([-\lambda, 0])$, for some $\alpha \in (0, 1)$, are smooth curves. The streamlines are even real-analytic curves provided that the density decreases with depth. These results of [26] were improved by Wang [51, Theorems 2.2 and 5.1] as follows:

Theorem 6 (Regularity properties). *Let $\alpha \in (0, 1)$ and assume that $\sigma \geq 0$. If $\beta, \overline{\rho}' \in C^\alpha([-\lambda, 0])$ and $h \in C^{2+\alpha}(\overline{\Omega}_\lambda)$ is a classical solution of (6) and (7) then, the wave profile together with all the other streamlines are real-analytic curves.*

The symmetry of steady water waves is an old problem, investigated first by Garabedian [22]. It was recently shown that, when excluding stagnation, periodic gravity water waves that possess a single crest per period have to be symmetric [11, 14, 38]. Surprisingly, the symmetry property of such waves can be characterized intrinsically in terms of the flow beneath the wave [35, Theorem 2.1]: the wave profile is symmetric and has only one crest and trough per period if and only if there exists a vertical line within the fluid domain such that all the fluid particles located on that line minimize there their distance to the fluid bed.

Because maximum principles do not apply directly in this context, the symmetry problem for stratified flows is more delicate. An intrinsic characterization of the symmetric waves in terms of the underlying flow as in the case of gravity waves is not yet known. There exist though criteria which ensure the symmetry of stratified waves with a single crest per period [47]. To present them we introduce first some notation:

$$A_0 := \max_{\overline{\Omega}_\eta} \sqrt{\rho}(c - u), \quad \eta_M := \max_{\mathbb{R}} \eta + d, \quad \eta_m := \min_{\mathbb{R}} \eta + d,$$

$$a_0 := \min_{\overline{\Omega}_\eta} \frac{\sqrt{\rho}(c-u)}{2}\left(1 + |\nabla\psi|^2 - \sqrt{(1 + |\nabla\psi|^2)^2 - 4(\sqrt{\rho}(c-u))^2}\right),$$

$$M := \max\left\{\max_{\overline{\Omega}_\eta}\left|\frac{v}{u-c}\right|, \max_{\overline{\Omega}_\eta}\left|\left(\partial_x + \frac{v}{u-c}\partial_y\right)\frac{v}{u-c}\right|, \max_{\overline{\Omega}_\eta}\left|\frac{1}{\sqrt{\rho}(u-c)}\partial_y\frac{v}{u-c}\right|\right\},$$

$$\varepsilon := \frac{A_0^3 M(1 + 2M^2)}{a_0} + \frac{(1 + 3a_0)A_0^3}{4a_0^2}(\max|\beta| + g\eta_M\max|\overline{\rho}'|)$$

$$+ 2A_0^2 M(1 + a_0 M) + \frac{A_0^3 M^2}{2a_0^2}.$$

The result in Theorem 7 below was obtained by Walsh [47] and presents two criteria for symmetry.

Theorem 7 (Symmetry result). *Consider a stably stratified steady train propagating at fixed speed c over a flat bed at $y = -d$ with relative pseudo-mass flux $-\lambda < 0$. Let $\overline{\rho} \in C^2([-\lambda, 0])$ and $\beta \in C^1([-\lambda, 0])$ be the streamline density function and Bernoulli function associated with the flow, and let $(u, v) \in (C^2(\overline{\Omega}_\eta))^2$ be the vector field. Assume that the wave profile $y = \eta(x)$ is monotonic between crests and troughs with period L, has integral mean zero over a period, and that $\max_{\overline{\Omega}_\eta} u < c$. Each of*

the following is a sufficient condition for the wave to be symmetric:

$$(i) \quad \max \beta' + g\eta_M \max|\overline{\rho}''| < \max\left\{\frac{\pi^2}{\eta_M^2}, \exp\left(-\min\{L, \eta_m\}\right)\right\}, \quad or$$

$$(ii) \quad \begin{cases} \varepsilon\sqrt{\frac{\varepsilon}{a_0}} + g\max|\overline{\rho}'| < \exp\left(-a_0^{-1/2}\min\{L, \lambda\}\right), \\ \varepsilon < a_0. \end{cases}$$

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