A Note on the Local Well-Posedness for the Whitham Equation

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Abstract We prove local well-posedness for the Whitham equation in H^s, $s > \frac{3}{2}$, for both solitary and periodic initial data.

1 Introduction

The Whitham equation,

$$u_t + 2uu_x + Lu_x = 0, \tag{1}$$

with the Fourier multiplier L given by

$$\mathscr{F}(Lf)(\xi) = \left(\frac{\tanh(\xi)}{\xi}\right)^{\frac{1}{2}} \hat{f}(\xi), \tag{2}$$

has recently received renewed attention due to its generically nonlocal properties. Originally introduced by Whitham in 1967 as an alternative to the Korteweg-de Vries (KdV) equation for modelling shallow water waves [17], it is known to feature the exact linear dispersion relation for travelling gravity water waves (KdV, in contrast, features a local approximation of this relation, making it a less-suited model for large wave numbers). Some of the intriguing properties of (1) is the nonlocal, fractional, and inhomogeneous character of the symbol $m(\xi) := (\frac{\tanh(\xi)}{\xi})^{\frac{1}{2}}$, as well as its allowance for qualitative wave breaking, i.e., for bounded solutions whose first spatial

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derivate blows up in finite time (we refer to [18] for the concept of wave breaking, and [1, 12] for a proof of this in the case of the Whitham equation). The (1) is also an equation whose balance of dispersive and nonlinear effects admits for the existence of solitary-wave solutions [3].

The operator L in (1) is singular—its convolutional kernel blows up at the origin as $|x|^{-1/2}$, cf. [5]—and not thoroughly understood; even the positivity and single-sided monotonicity of the inverse Fourier transform of $m(\xi)$ are unknown. Although it was conjectured already by Whitham himself that (1) admits cusp-like solutions [17], so far only numerical evidence—and partial analytic results supporting this conjecture—exists, see [4]. In addition, the dispersion determined by (2) is unusually weak, making global estimates and related well-posed results rather difficult. In fact, even for the homogenous symbol $|\xi|^{-1/2}$ very little is known (see [14], and [6] for a related investigation).

The main motivation for this contribution is [3], in which the existence of solitary waves and a conditional stability result were proved for a large class of equations connected to (1). It is noted in [3], as well as in [8], that a local well-posedness result for the Whitham equation can be obtained using Kato's method. Since the proof has never been presented, and since the Whitham equation in important respects differs from many other model equations investigated in the literature, both what concerns its generic nonlocal properties and the weak dispersion associated to L, we give here the details in form of a short, but rigorous, proof of the local well-posedness of the Whitham equation, on the line as well as in the periodic case. The proof is valid for initial data in the Sobolev space H^s, $s > 3/2$, with the natural and best possible regularity for the solutions in terms of their time and space dependence. Note that the wave-breaking results [1, 12] for (1) imply that a general global well-posedness result is excluded. A conditional global existence result is still possible, but so far out of reach. For earlier treatments of the Whitham equation, see also [7] and [19].

The structure of the note is as follows. First, in Section 2, some general properties of the Whitham kernel Eq. 2 and related concepts are presented. Section 2 continues with the main proof for the case of initial data on the line, i.e. for data in $H^s(\mathbb{R})$, $s > 3/2$. The method of proof follows that of [2], which in turn is based on Kato's method, see [10]. Note, however, that the weakly dispersive, nonlocal, term in the Whitham (1) differs from those in the equations treated in [2] and [10], so that a thorough analysis is necessary. Once the arguments are in place, functional-analytic arguments can be employed to establish the result for periodic initial data, in the appropriate spaces. That is the content of Section 3.

2 General Preliminaries and the Case of Initial Data on $\mathbb{R}$

In (1) and (2), the Fourier transform $\mathcal{F}(f)$ of a function f is defined by the formula

$$\mathcal{F}(f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x} f(x)dx,$$

extended by duality from $S(\mathbb{R}) \subset L^2(\mathbb{R})$ to $S'(\mathbb{R})$, the space of tempered distributions. Here, $S(\mathbb{R})$ is the Schwarz space of rapidly decaying smooth functions on $\mathbb{R}$, and $H^s(\mathbb{R})$, $s \in \mathbb{R}$, is the Sobolev space of tempered distributions whose Fourier transform satisfies

$$\int_{\mathbb{R}} (1 + |\xi|^2)^s |\mathcal{F}(f)(\xi)|^2 \, \mathrm{d}\xi < \infty,$$

with the standard inner product. For convenience, we will sometimes omit the domain in the notation for function spaces in the following. Throughout the chapter, the notation $\hat{f}$ will also be used interchangeably with $\mathcal{F}(f)$. Note that although the functions and operators of interest in this chapter are all real-valued, and although the operator L defined in (2) maps real data to real data, the Fourier transform is naturally defined in complex-valued function spaces; hence, the function spaces used in this investigation should in general be understood as the complexifications of corresponding real-valued function spaces.

As usual, C^k denotes the space of k times continuously differentiable functions, and BC^k the corresponding space of functions whose derivatives up to order k are also bounded.

Before considering some properties of the Fourier multiplier operator L, let us state our main theorem.

Theorem 1 *Let $s > \frac{3}{2}$. Given $u_0 \in H^s(\mathbb{R})$ there is a maximal $T > 0$ depending only on $\|u_0\|_{H^s}$, and a unique solution $u = u(\cdot, u_0)$ to (1) in $C([0,T); H^s(\mathbb{R})) \cap C^1([0,T); H^{s-1}(\mathbb{R}))$. The solution depends continuously on the initial data, i.e., the map $u_0 \mapsto u(\cdot, u_0)$ from $H^s(\mathbb{R})$ to $C([0,T); H^s(\mathbb{R})) \cap C^1([0,T); H^{s-1}(\mathbb{R}))$ is continuous.*

Remark 1 It is clear from the method behind Theorem 1 as well as Theorem 3 that the results can be generalised to a larger class of Whitham-like equations (cf. [3]). For example, similar results could be obtained if the linear term L is bounded $L^2 \to L^2$ and the nonlinear term is temperate (slowly growing) and belongs to C^k for $k > s+2$, although the Whitham equation gives some more regularity than this general case. These conditions, however, are not optimal, and a larger class of interest requires results on the Nemytskii operator n that are still only partially available (the periodic case is open, although arguments are likely available [15]). Results in this direction, including the possibility of L being a Fourier integral operator, are under preparation.

In order to prove Theorem 1, we first make some basic assumptions and observations. Let

$$X = L^2(\mathbb{R}) \qquad \text{and} \qquad Y = H^s(\mathbb{R}),$$

for some fixed $s > \frac{3}{2}$. Note that for $f \in L^2(\mathbb{R})$,

$$\|Lf\|_{L^2} = \|\mathscr{F}(Lf)\|_{L^2} = \left\| \left(\frac{\tanh(\xi)}{\xi} \right)^{\frac{1}{2}} \hat{f}(\xi) \right\|_{L^2} \le \|\hat{f}\|_{L^2} = \|f\|_{L^2},$$

so that L is a bounded linear operator on $L^2(\mathbb{R})$. In fact, it is shown in [3] that

$$L \in C^\infty(H^s(\mathbb{R}), H^{s+\frac{1}{2}}(\mathbb{R})) \cap C^\infty(S(\mathbb{R}), S(\mathbb{R})), \tag{3}$$

for all $s \geq 0$. We note that the smooth and even function $\xi \mapsto (\frac{\tanh(\xi)}{\xi})^{\frac{1}{2}}$ is increasing in $(-\infty, 0)$ and decreasing in $(0, \infty)$, reaching its global maximum of unit size at $x = 0$. As $|\xi| \to \infty$, it vanishes with the rate $|\xi|^{-\frac{1}{2}}$.

The operator L is furthermore symmetric on $L^2(\mathbb{R})$, since for $f, g \in L^2(\mathbb{R})$ we have that

$$\begin{aligned}
(Lf, g)_{L^2} &= (\mathcal{F}(Lf), \mathcal{F}(g))_{L^2} \\
&= \int_{\mathbb{R}} \mathcal{F}(Lf)\overline{\mathcal{F}(g)}\, d\xi \\
&= \int_{\mathbb{R}} (\frac{\tanh(\xi)}{\xi})^{\frac{1}{2}} \hat{f}(\xi)\overline{\hat{g}(\xi)}\, d\xi \\
&= \int_{\mathbb{R}} \mathcal{F}(f)\overline{\mathcal{F}(Lg)}\, d\xi \\
&= (f, Lg)_{L^2},
\end{aligned}$$

so it follows that L is a symmetric bounded linear operator on the space X.

To proceed, rewrite Theorem 1 as

$$\begin{aligned}
u_t + A(u)u = 0, &\qquad t \geq 0, \\
u(0) = u_0. &
\end{aligned} \tag{4}$$

Here,

$$A(y) = (2y + L)\partial_x, \qquad \operatorname{dom}(A(y)) := \{v \in L^2 \mid 2yv + Lv \in H^1\},$$

for some $y \in H^s$ with $s > \frac{3}{2}$. In view of (3), and since H^s is a Banach algebra for $s > \frac{1}{2}$, $\operatorname{dom}(A(y))$ is dense in $X = L^2$.

In order to state Kato's theorem in a form suitable for our purposes, we also need the concept of *accretiveness* (cf. [11]). Let T be an operator on a Hilbert space H, and let $\mathcal{L}$ denote the space of all bounded linear operators from some space to another, in this case $H \to H$. We then say that:

- T is *accretive* if $\operatorname{Re}(Tv, v)_H \geq 0$ for all $v \in \operatorname{dom}(T)$.
- T is *quasi-accretive* if $T + \alpha$ is accretive for some scalar $\alpha > 0$.
- T is *m-accretive* if $(T + \lambda)^{-1} \in \mathcal{L}(H)$ with $\operatorname{Re}(\lambda)\|(T + \lambda)^{-1}\| \leq 1$ for $\operatorname{Re}(\lambda) > 0$.
- T is *quasi-m-accretive* if $T + \alpha$ is m-accretive for some scalar $\alpha > 0$.

We shall make use of the following version of Kato's theorem to establish the local well-posedness for the problem (4), adapted from [2].

Theorem 2 *(Cf. [2]) Consider the abstract quasi-linear evolution (4). Let X and Y be as above (Hilbert spaces such that Y is continuously and densely injected into X), and let $Q: Y \to X$ be a topological isomorphism. Assume that:*

(i) $A(y) \in \mathcal{L}(Y, X)$ for $y \in Y$ with

$$\|(A(y) - A(z))w\|_X \le \mu_A \|y - z\|_X \|w\|_Y, \qquad y, z, w \in Y,$$

and $A(y)$ is quasi-m-accretive, uniformly on bounded sets in Y.

(ii) $QA(y)Q^{-1} = A(y) + B(y)$, where $B(y) \in \mathcal{L}(X)$ is bounded, uniformly on bounded sets in Y. Moreover,

$$\|(B(y) - B(z))w\|_X \le \mu_B \|y - z\|_Y \|w\|_X, \qquad y, z \in Y, \quad w \in X.$$

Here, the constants μ_A and μ_B depend only on $\max\{\|y\|_Y, \|z\|_Y\}$.
Then, for any given $v_0 \in Y$, there is a maximal $T > 0$ depending only on $\|v_0\|_Y$ and a unique solution v to (4) such that

$$v = v(\cdot, v_0) \in C([0, T); Y) \cap C^1([0, T); X).$$

The map $v_0 \mapsto v(\cdot, v_0)$ is continuous from Y to $C([0, T); Y) \cap C^1([0, T); X)$.

To continue, we need to study the operator $A(y)$ for a fixed $y \in H^s$, with $s > \frac{3}{2}$. Later y will be taken in a bounded subset of $Y = H^s$ for $s > \frac{3}{2}$, but note that all estimates to come are uniform with respect to any such bounded subset.

For $y \in Y$, we define

$$Dv := (2yv + Lv)_x - 2y_x v, \qquad \operatorname{dom}(D) := \{v \in L^2 \mid 2yv + Lv \in H^1\},$$

and

$$D_0 v := -(2yv + Lv)_x, \qquad \operatorname{dom}(D_0) := \operatorname{dom}(D).$$

Note that the choice of these domains makes both D and D_0 closed operators in $X = L^2$. In view of the embedding $H^s \subset BC$ for $s > \frac{1}{2}$, we furthermore have

$$(2yv + Lv)_x = 2y_x v + 2yv_x + (Lv)_x \quad \text{in } H^{-1}, \tag{5}$$

so that, in that space,

$$Dv = 2yv_x + Lv_x,$$
$$D_0 v = -2y_x v - 2yv_x - (Lv)_x,$$

and

$$2yv_x + (Lv)_x = (2yv + Lv)_x - 2y_x v \in L^2,$$

for y and v as considered here.

We will prove that D satisfies the condition (i) in Theorem 2 with the help of a few consecutive lemmata. To state the first of these, let C_0^∞ be the space of compactly supported smooth functions on some open set, in this case $\mathbb{R}$.

Lemma 1 *Given $v \in \mathrm{dom}(D)$, there exists a sequence $\{v_n\}_n \subset C_0^\infty$ such that*

$$v_n \to v \quad \textit{and} \quad (2yv_n + Lv_n)_x \to (2yv + Lv)_x$$

in L^2 as $n \to \infty$.

Proof Pick $\rho \in C_0^\infty$ with $\rho(x) \geq 0$ and $\int_{\mathbb{R}} \rho \, dx = 1$. For $n \geq 1$, let $\rho_n(x) := n\rho(nx)$ be a mollifier on $\mathbb{R}$. Denoting by v_n the convolution $v * \rho_n$, we have $v_n \in C^\infty$ and $v_n \to v$ in L^2 as $n \to \infty$. This proves the first part of the lemma due to the density of C_0^∞ in L^2.

As what concerns the second part, we have

$$\begin{aligned}
&(2yv_n + Lv_n)_x - (2yv + Lv)_x \\
&= ((2y(v_n)_x + L(v_n)_x) - (2yv_x + Lv_x)) + (2y_x v_n - 2y_x v) \\
&= (2y(v_n)_x + L(v_n)_x - (2yv_x + Lv_x) * \rho_n) \\
&\quad + ((2yv_x + Lv_x) * \rho_n - (2yv_x + Lv_x)) + (2y_x v_n - 2y_x v) \\
&=: \mathrm{I}_n(v) + \mathrm{II}_n(v) + \mathrm{III}_n(v).
\end{aligned}$$

By observing that $2yv_x + Lv_x \in L^2$ with $y_x \in H^{s-1} \subset BC$ for $s > \frac{3}{2}$, one naturally gets

$$\begin{aligned}
\mathrm{II}_n(v) &= (2yv_x + Lv_x) * \rho_n - (2yv_x + Lv_x) \to 0, \\
\mathrm{III}_n(v) &= 2y_x v_n - 2y_x v \to 0,
\end{aligned}$$

in L^2, as $n \to \infty$.

It remains to prove that $\mathrm{I}_n(v) \to 0$ in L^2 as $n \to \infty$. Since this clearly holds for $v \in C_0^\infty$, and since C_0^∞ is densely and continuously embedded in L^2, we only need to prove that $\|I_n(v)\|_{L^2} \leq \|v\|_{L^2}$, for $v \in L^2$. Then the result follows from continuity.

To prove this, note that for any $v \in \mathrm{dom}(D)$, we have

$$\begin{aligned}
\mathcal{F}(I_n(v)) &= \mathcal{F}(2y(v_n)_x + (Lv_n)_x - ((2yv + Lv)_x - 2y_x v) * \rho_n) \\
&= \mathcal{F}(2y(v_n)_x) + \mathcal{F}((Lv_n)_x) - \mathcal{F}((2yv + Lv) * (\rho_n)_x) + \mathcal{F}((2y_x v) * \rho_n) \\
&= \mathcal{F}(2y(v_n)_x) + i\xi \left(\frac{\tanh(\xi)}{\xi}\right)^{\frac{1}{2}} \mathcal{F}(v)\mathcal{F}(\rho_n) - \mathcal{F}((2yv) * (\rho_n)_x) \\
&\quad - \left(\frac{\tanh(\xi)}{\xi}\right)^{\frac{1}{2}} \mathcal{F}(v) i\xi \mathcal{F}(\rho_n) + \mathcal{F}((2y_x v) * \rho_n) \\
&= \mathcal{F}(2y(v_n)_x - (2yv) * (\rho_n)_x + (2y_x v) * \rho_n),
\end{aligned}$$

where we have used $(2yv + Lv)_x * (\rho_n) = (2yv + Lv) * (\rho_n)_x$ for $v \in \mathrm{dom}(D)$.

Then

$$\begin{aligned}
\mathrm{I}_n(v) &= 2y(v*(\rho_n)_x) - (2yv)*(\rho_n)_x + (2y_x v)*\rho_n \\
&= 2\int_{\mathbb{R}} (y(x)-y(x-s))v(x-s)(\rho_n)_x(s)\,ds + (2y_x v)*\rho_n \\
&= 2n^2\int_{\mathbb{R}} (y(x)-y(x-s))v(x-s)\rho_x(ns)\,ds + (2y_x v)*\rho_n \\
&=: \hat{\mathrm{I}}_n(v) + (2y_x v)*\rho_n.
\end{aligned}$$

Note that $2y_x v \in L^2$ implies that $(2y_x v)*\rho_n \to 2y_x v$ in L^2, whence we only need to prove $\|\hat{\mathrm{I}}_n(v)\|_{L^2} \le \|v\|_{L^2}$. For this purpose, suppose $\mathrm{supp}(\rho) \subset [-\lambda, \lambda]$ for some $\lambda > 0$. Then, by Hölder's inequality, we have

$$\begin{aligned}
\|\hat{\mathrm{I}}_n(v)\|_{L^2} &= \|2n\int_{-\lambda}^{\lambda} (y(x)-y(x-s/n))v(x-s/n)\rho_x(s)\,ds\|_{L^2} \\
&\le \|2\sup_{s\in\mathbb{R}} |y_x(s)| \int_{-\lambda}^{\lambda} |sv(x-s/n)\rho_x(x-s)|\,ds\|_{L^2} \\
&\le 2\sup_{s\in\mathbb{R}} |y_x(s)| \|(\int_{-\lambda}^{\lambda} |s\rho_x(s)|^2\,ds)^{\frac{1}{2}}(\int_{-\lambda}^{\lambda} |v(x-s/n)|^2\,ds)^{\frac{1}{2}}\|_{L^2}
\end{aligned}$$

Let $M = 2\sup_{s\in\mathbb{R}} |y_x(s)|(\int_{-\lambda}^{\lambda} |s\rho_x(s)|^2\,ds)^{\frac{1}{2}} < \infty$. Then, by Fubini's theorem, we have

$$\begin{aligned}
\|\hat{\mathrm{I}}_n(v)\|_{L^2} &\le M\Big(\int_{\mathbb{R}}\int_{-\lambda}^{\lambda} |v(x-s/n)|^2\,ds\,dx\Big)^{\frac{1}{2}} \\
&= M\Big(\int_{-\lambda}^{\lambda}\int_{\mathbb{R}} |v(x-s/n)|^2\,dx\,ds\Big)^{\frac{1}{2}} \\
&= (2\lambda)^{\frac{1}{2}} M\|v\|_{L^2},
\end{aligned}$$

which completes the proof. □

Lemma 2 *The operators D and D_0 are both quasi-accretive in L^2.*

Proof By definition, D is quasi-accretive in L^2 if and only if

$$\mathrm{Re}((D+\alpha I)v, v)_{L^2} \ge 0,$$

for all $v \in \mathrm{dom}(D)$ and some scalar $\alpha > 0$. In view of Lemma 1 and (5), we can find a sequence $v_n \in C_0^\infty$ such that

$$(Dv, v)_{L^2} = \lim_{n\to\infty} (Dv_n, v_n)_{L^2} = \lim_{n\to\infty} \int_{\mathbb{R}} (2y(v_n)_x + L(v_n)_x)v_n\,dx.$$

As L is symmetric the operator $L\partial_x$ is skew-symmetric, and we have $(L\partial_x v_n, v_n)_{L^2} = -(v_n, L\partial_x v_n)_{L^2}$. Since both L and v_n are real-valued, the term $\int_{\mathbb{R}} L(v_n)_x v_n\,dx$ thus vanishes completely for all $n \ge 1$.

Since, in addition,

$$\int_{\mathbb{R}} 2y(v_n)_x v_n dx = \int_{\mathbb{R}} y(v_n^2)_x \, dx = -\int_{\mathbb{R}} y_x v_n^2 \, dx,$$

we have

$$(Dv_n, v_n)_{L^2} = -\int_{\mathbb{R}} y_x v_n^2 \, dx. \tag{6}$$

Now, for $y_x \in H^{s-1} \subset BC$, $s > \frac{3}{2}$, we can select $\alpha > 0$ such that $\alpha \geq \|y_x\|_{L^\infty}$. Then

$$\operatorname{Re}((D + \alpha I)v_n, v_n)_{L^2} = \int_{\mathbb{R}} (\alpha - y_x)v_n^2 \, dx \geq 0,$$

implying that

$$\operatorname{Re}((D + \alpha I)v, v)_{L^2} = \lim_{n\to\infty} \operatorname{Re}((D + \alpha I)v_n, v_n)_{L^2} \geq 0.$$

Hence, D is quasi-accretive.

As what concerns the operator D_0, note that $D_0 v = Dv + 2y_x v$ and $(2y_x v, v)_{L^2}$ can be bounded with same technique as used in connection to (6). Thus, the quasi-accretiveness of D_0 follows from that of D. □

For use in the following, denote by

$$[T_1, T_2] = T_1 T_2 - T_2 T_1$$

the commutator of two operators T_1 and T_2. Note that both ∂_x and L are Fourier multiplier operators, so that $[\partial_x, L] = 0$ on H^s for all $s \in \mathbb{R}$.

Lemma 3 *The adjoint of D in L^2 is D_0.*

Proof For $v \in C_0^\infty \subset \operatorname{dom}(D)$ and any $\omega \in \operatorname{dom}(D^*)$, we have

$$\begin{aligned}
(v, D^*\omega)_{L^2} &= (Dv, \omega)_{L^2} \\
&= \int_{\mathbb{R}} [(2yv + Lv)_x - 2y_x v]\omega \, dx \\
&= \int_{\mathbb{R}} (2yv_x + Lv_x)\omega \, dx \\
&= \int_{\mathbb{R}} (2y\omega + L\omega)v_x \, dx,
\end{aligned}$$

so that $D^*\omega \in L^2$ is the weak derivative of $2y\omega + L\omega \in L^2$. Consequently, $2y\omega + L\omega \in H^1$ and $\omega \in \operatorname{dom}(D_0)$. Then

$$\begin{aligned}
(v, D^*\omega)_{L^2} &= \int_{\mathbb{R}} -(2y\omega + L\omega)_x v \, dx \\
&= (v, D_0\omega)_{L^2},
\end{aligned}$$

so that $D^* \subset D_0$.

Assume now that $v \in \text{dom}(D_0) \subset L^2$ and note that for any $\tilde{v} \in \text{dom}(D^*)$, which by the above calculation belongs to $\text{dom}(D_0) = \text{dom}(D)$, we can always find a sequence $\{\tilde{v}_n\} \subset C_0^\infty$ such that Lemma 1 holds. Therefore, we have that

$$\begin{aligned}(D\tilde{v}, v)_{L^2} &= \lim_{n\to\infty} (D\tilde{v}_n, v)_{L^2} \\ &= \lim_{n\to\infty} \int_{\mathbb{R}} ((2y(\tilde{v}_n)_x + L(\tilde{v}_n)_x)v \, dx \\ &= \lim_{n\to\infty} \int_{\mathbb{R}} (2yv + Lv)(\tilde{v}_n)_x \, dx.\end{aligned}$$

Since $v \in \text{dom}(D_0)$ implies that $2yv + Lv \in H^1$ by our assumptions, we also have that

$$\begin{aligned}(D\tilde{v}, v)_{L^2} &= \lim_{n\to\infty} \int_{\mathbb{R}} (2yv + Lv)(\tilde{v}_n)_x \, dx \\ &= -\lim_{n\to\infty} \int_{\mathbb{R}} (2yv + Lv)_x \tilde{v}_n \, dx \\ &= (\tilde{v}, -(2yv + Lv)_x)_{L^2} \, dx \\ &= (\tilde{v}, D_0 v)_{L^2}.\end{aligned}$$

Thus, it follows that $v \in \text{dom}(D^*)$ and $D_0 \subset D^*$. In view of the above, it is then clear that $D_0 = D^*$.

By Lemmata 2 and 3, both D and D^* are quasi-accretive. A classical argument (cf. [13], Corollary 4.4) then gives the following.

Lemma 4 *For the closed linear operator D, densely defined on the Banach space X, with both D and its adjoint D^* quasi-accretive, there exists a scalar $\alpha \in \mathbb{R}$ such that the operator $-(D + \alpha)$ is the infinitesimal generator of a C_0-semigroup of contractions on X, i.e., D is a quasi-m-accretive operator.*

This proposition shows that the operator A in our Cauchy problem satisfies condition (i) in Theorem 2.

We next turn to condition (ii). Let

$$B(y) := Q(A(y))Q^{-1} - A(y) = [Q, A(y)]Q^{-1},$$

where $A(y) = (2y + L)\partial_x$ and $Q := \Lambda^s = (1 - \partial_x^2)^{\frac{s}{2}}$ is an isomorphism from H^s to L^2. We then have the following lemma.

Lemma 5 *For $y \in Y$, the operator $B(y)$ satisfies* condition (ii) *in* Theorem 2.

Proof Note that

$$\begin{aligned}[Q, A(y)] &= [\Lambda^s, A(y)] \\ &= [\Lambda^s, (2y + L)\partial_x] \\ &= 2[\Lambda^s, y]\partial_x + [\Lambda^s, L]\partial_x \\ &= 2[\Lambda^s, y]\partial_x,\end{aligned} \tag{7}$$

where we have used the commutation properties $[\Lambda^s, \partial_x] = 0$ and $[\Lambda^s, L] = 0$.

In order to prove uniform boundedness of $B(y)$ for y in a bounded subset of H^s, we assume without loss of generality that $y \in \mathscr{W} \subset H^s$, where $\mathscr{W}$ is an open ball in H^s with radius $R > 0$. Using classical estimates for (7) (cf. [10]), we get

$$\|[\Lambda^s, y]\Lambda^{1-s}\| \leq C_0\|\partial_x y\|_{H^{s-1}} \leq C_0\|y\|_{H^s} \leq \alpha_0(R),$$

where C_0 is a constant relying only on s, and $\alpha_0(R)$ is a constant relying only on $\mathscr{W}$. Then, for any $z \in L^2$, we have

$$\begin{aligned}
\|B(y)z\|_{L^2} &= \|[\Lambda^s, y]\Lambda^{1-s}\Lambda^{s-1}\partial_x\Lambda^{-s}z\|_{L^2} \\
&\leq \|[\Lambda^s, y]\Lambda^{1-s}\| \, \|\Lambda^{s-1}\partial_x\Lambda^{-s}z\|_{L^2} \\
&\leq \alpha_0(R)\|\partial_x\Lambda^{-1}z\|_{L^2} \\
&\leq \alpha_0(R)\|z\|_{L^2},
\end{aligned}$$

where the last step is due to the fact that

$$\|\partial_x\Lambda^{-1}z\|_{L^2} \leq \|\Lambda^{-1}z\|_{H^1} = \|z\|_{L^2}.$$

Hence, $B(y)$ is a bounded linear operator on L^2 for $y \in Y$. In addition, for any $y, z \in \mathscr{W}$ and $w \in X$,

$$\begin{aligned}
\|B(y)w - B(z)w\|_{L^2} &= \|[\Lambda^s, y - z]\partial_x\Lambda^{-s}w\|_{L^2} \\
&\leq \alpha_1(R)\|[\Lambda^s, y - z]\Lambda^{1-s}\| \, \|\Lambda^{s-1}\partial_x\Lambda^{-s}w\|_{L^2} \\
&\leq \alpha_2(R)\|y - z\|_{H^s}\|w\|_{L^2},
\end{aligned}$$

where $\alpha_1(R)$ and $\alpha_2(R)$ are constants depending only on $\mathscr{W}$. Thus, $B(y)$ satisfies condition (ii) for all $y \in \mathscr{W} \subset H^s$. □

We are now ready to prove the main theorem for initial data $u_0 \in H^s(\mathbb{R})$.

Proof of Theorem 1 According to Lemmata 1–5 we can apply Theorem 2 to find a solution u as described in Theorem 1, although in the solution class $C([0, T); H^s(\mathbb{R})) \cap C^1([0, T); L^2(\mathbb{R}))$. In view of that $H^{s-1}(\mathbb{R})$ is an algebra with respect to pointwise multiplication, and that L maps $H^s(\mathbb{R})$ continuously into $H^{s+1/2}(\mathbb{R})$, one, however, sees that for the Whitham equation,

$$u_t = -2uu_x - Lu_x \in H^{s-1}(\mathbb{R}).$$

Hence, $u \in C^1([0, T); H^{s-1}(\mathbb{R}))$.

Also, since $[u_0 \mapsto u] \in C(H^s(\mathbb{R}), C([0, T), H^s(\mathbb{R}))$, and ∂_x maps $H^s(\mathbb{R})$ continuously into $H^{s-1}(\mathbb{R})$, the same argument can be used to conclude that $[u_0 \mapsto u] \in C(H^s(\mathbb{R}), C^1([0, T), H^{s-1}(\mathbb{R})))$. This yields the desired solution class and concludes the proof of Theorem 1.

3 The Periodic Case

Consider now the Cauchy problem (1) in the periodic setting. We first define the periodic Sobolev space $H^s_{2\pi\text{-per}}$ for $s \in \mathbb{R}$ as the set of all 2π-periodic distributions $f \in S'$ such that

$$\|f\|_{H^s_{2\pi\text{-per}}} = \Big(\sum_{k\in\mathbb{Z}} (1+|k|^2)^s |\hat{f}(k)|^2\Big)^{\frac{1}{2}} < \infty,$$

with inner product

$$(f|g)_{H^s_{2\pi\text{-per}}} = \sum_{k\in\mathbb{Z}} (1+|k|^2)^s \hat{f}(k)\overline{\hat{g}(k)}.$$

Here $\hat{f}(k)$ are generalized Fourier coefficients, see [16]. In what follows, we will be working with H^s for $s \geq 0$ and $H^0_{2\pi\text{-per}}$ will be denoted by $L^2_{2\pi\text{-per}}$. If we define the Fourier coefficients of $f \in H^s_{2\pi\text{-per}}$ to be

$$c_k = \hat{f}(k) = \frac{1}{\sqrt{2\pi}}(f|e^{ikx})_{L^2((-\pi,\pi))} = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x)e^{-ikx}\,dx,$$

then each $f \in H^s_{2\pi\text{-per}}$ is uniquely determined by its Fourier coefficients, and we have

$$f(x) = \sum_{k\in\mathbb{Z}} \hat{f}(k)e^{ikx} \qquad \text{in} \quad L^2_{2\pi\text{-per}}. \tag{8}$$

Note that since $H^s_{2\pi\text{-per}} \subset S'$, the Fourier transform $\mathcal{F}f \in S'$ is well-defined for all $f \in H^s_{2\pi\text{-per}}$, but in view of (8) it can also be identified with an element in l^s for all $s \geq 0$. From Parseval's identity,

$$\|\hat{f}\|^2_{l^2} = \|f\|^2_{L^2((-\pi,\pi))}, \quad f \in H^s_{2\pi\text{-per}},$$

and

$$(\hat{f}|\hat{g})_{l^2} = \sum_{k\in\mathbb{Z}} \hat{f}(k)\overline{\hat{g}(k)} = \int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx = (f|g)_{L^2((-\pi,\pi))},$$

for all $f, g \in H^s_{2\pi\text{-per}}$. It follows that all Fourier integral formulas in the case on the line immediately translate into Fourier series in the periodic case. In particular (cf. [3]),

$$\mathscr{F}(Lf)(k) = \Big(\frac{\tanh(k)}{k}\Big)^{\frac{1}{2}} \hat{f}(k), \quad k \in \mathbb{Z},$$

$$L \in C^\infty(H^s_{2\pi\text{-per}}, H^{s+\frac{1}{2}}_{2\pi\text{-per}}) \cap C^\infty(\mathcal{S}'(\mathbb{R}), \mathcal{S}'(\mathbb{R})), \quad s \geq 0,$$

and L is a bounded linear symmetric operator on $L^2_{2\pi\text{-per}}$. For more information on the periodic Sobolev spaces, see also the monograph [9].

As a consequence of this relationship between the periodic Sobolev spaces $H^s_{2\pi\text{-per}}$ and their non-periodic counterparts $H^s(\mathbb{R})$, the proof of Theorem 1 can be followed in detail with $H^s(\mathbb{R})$ replaced by $H^s_{2\pi\text{-per}}$, and the Fourier transform replaced accordingly, as described above[1]. Note here that the crucial embedding $H^s_{2\pi\text{-per}} \hookrightarrow BC^1$ for $s > 3/2$ is equally valid in the periodic case. Analogous to the case on $\mathbb{R}$, we thus obtain the following result.

Theorem 3 *Let $s > \frac{3}{2}$. Given $u_0 \in H^s_{2\pi\text{-per}}$, there is a maximal $T > 0$ and a unique solution u to 1 such that $u = u(\cdot, u_0) \in C([0,T); H^s_{2\pi\text{-per}}) \cap C^1([0,T); H^{s-1}_{2\pi\text{-per}})$. The solution depends continuously on the initial data, i.e., the map $u_0 \mapsto u(\cdot, u_0)$ is continuous from $H^s_{2\pi\text{-per}}$ to $C([0,T); H^s_{2\pi\text{-per}}) \cap C^1([0,T); H^{s-1}_{2\pi\text{-per}})$.*

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[1] For questions related to global existence questions, the difference between the periodic and non-periodic case might play a bigger role.

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