Some Topics in Probability Theory

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Abstract We present a succinct discussion of a number of topics in Probability Theory which have been of interest in recent years.

1 The Set of Martingale Laws

Consider, on the Skorokhod space of càdlàg functions, all probabilities $\mathbb{P}$ which make the canonical process of coordinates a martingale. Call $\mathscr{M}$ this set. Clearly, it is a convex set, and it may be of interest to characterize its extremal points. An application of Hahn-Banach theorem (to the pair H^1-BMO, and the fact that a BMO martingale is locally bounded) allows to show that $\mathbb{P}$ in $\mathscr{M}$ is extremal if and only if any martingale under $\mathbb{P}$ may be written as the sum of a constant and of a stochastic integral with respect to the canonical (martingale) process. A particularly illustrative example is that of $\mathbb{P} = \mathbb{W}$, Wiener measure. Indeed, on one hand, from Lévy's martingale characterization of Brownian motion, it is easily shown that $\mathbb{W}$ is extremal in $\mathscr{M}$. On the other hand, it is a theorem (due to Itô) that all Brownian martingales may be written as the sum of a constant and of a stochastic integral with respect to Brownian Motion. That these two properties hold for $\mathbb{W}$ is not a mere coincidence, but is explained by the general statement above (Jacod and Yor 1977).

To our knowledge, the first author who tried to connect the two properties, namely: extremality of $\mathbb{P}$, and martingale representation property under $\mathbb{P}$ is Dellacherie (1974, 1975). Dellacherie (1975) corrects Dellacherie (1974) partially, but the local boundedness property which seems necessary for a correct proof is only found in Jacod and Yor (1977).

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The use of the H^1-BMO duality in this topic is reminiscent to that of the L^1-L^∞ duality in the characterization of extremal probabilities, solutions of a (generalized) moment problem. In fact, it is a theorem, according to Douglas and Naimark (independently) the extremal points $\mathbb{P}$ of such a moment problem are those for which the vector space generated by the functions defining the problem and the constant function 1 is dense in $L^1(\mathbb{P})$. Yor (1978) explains how to relate the two frameworks and extremality results.

2 Strong and Weak Brownian Filtrations

We shall say that a filtration $\mathscr{F}_t$ is strongly Brownian if it is the natural filtration of a Brownian Motion. On the other hand, we shall say $\mathscr{F}_t$ is weakly Brownian if there exists a Brownian Motion B for this filtration such that all martingales for this filtration may be written as the sum of a constant and a stochastic integral with respect to B (but the integrand is predictable with respect to $\mathscr{F}_t$). Any strongly Brownian filtration is weakly Brownian (Itô's theorem recalled in Sect. 1). It is natural to ask whether any weakly Brownian filtration is strongly Brownian. The answer turns out to be negative:

- it is easily shown that on the canonical space of continuous functions, endowed with any probability $\mathbb{Q}$ equivalent to Wiener measure $\mathbb{W}$, the canonical filtration is weakly Brownian; however, it has been shown by Dubins et al. (1996) that there are infinitely many $\mathbb{Q}$'s such that $\mathscr{F}_t$ is not strongly Brownian under $\mathbb{Q}$;
- the filtration of Walsh's Brownian Motion with N rays, for $N \geq 3$, is weakly but not strongly Brownian, another result due to Tsirelson (1997). A posteriori, a clear explanation of this result emerged as it was shown that M. Barlow's conjecture holds: for g, the end of a predictable set in a strong Brownian filtration, the progressive σ-field up to g can only differ from the predictable one by, at most the addition of a set. This is clearly not the case for Walsh's Brownian motion with N rays, $N \geq 3$, and g the last zero of this process before time 1;
- there exist time changes of the canonical Brownian filtration such that the time changed filtration is weakly, but not strongly Brownian, a result due to Émery and Schachermayer (1999).

3 Weak Brownian Motions of Any Given Order

Although the adjective weak is used again here, this topic has nothing to do with topic in Sect. 2. It was suggested by a question of Stoyanov in his book of counter examples (Stoyanov 1987): does there exist, for a given integer k, a process which has the same k-dimensional marginals as Brownian Motion? The answer is yes, as was proven by Föllmer et al. (2000), by constructing probabilities $\mathbb{Q}$ equivalent to $\mathbb{W}$, the Wiener measure, such that the k-dimensional marginals of the canonical process under $\mathbb{Q}$ are those under $\mathbb{W}$.

4 Martingales with One-Dimensional Brownian Marginals

Note that this topic differs from Sect. 3, where the processes constructed there are not martingales, but, in general, semimartingales. For constructions of martingales, see Albin (2008), Baker et al. (2011), Hamza and Klebaner (2007), Madan and Yor (2002). There are at least two versions of these constructions, one where it is required that the martingale is continuous, e.g., Albin (2008); the other where discontinuity is allowed, e.g., Madan and Yor (2002).

5 Explicit Skorokhod Embedding

The problem is now well known: given a centered probability μ on $\mathbb{R}$, find a stopping time T of Brownian motion B, such that B_T is μ distributed and $B_{t\wedge T}$ is a uniformly integrable martingale. Although J. Obłój found 21 different solutions scattered in the literature (Obłój 2004), few of them are explicit, as in general, the authors proceed by finding solutions for simple μ's then pass to the limit.

Azéma-Yor found that if $T_\mu := \inf\{t : S_t \geq H_\mu(B_t)\}$, where $S_t = s \overset{\sup}{\leq} t\ B_s$, and the Hardy-Littlewood function $H_\mu(x)$ is defined as:

$$H_\mu(x) = \frac{1}{\mu([x,\infty))} \int_{[x,\infty)} t d\mu(t),$$

then T_μ solves Skorokhod problem for μ (Azéma and Yor 1979). To prove this result, Azéma and Yor (1979) use first-order stochastic calculus, whereas Rogers (1981) uses excursion theory. Madan and Yor (2002) remarked that for a family μ_t such that the corresponding Hardy-Littlewood family is pointwise increasing in t, the Brownian motion B taken at those stopping times is a martingale.

6 Peacocks and Associated Martingales

We say that a process X_t is a peacock (:PCOC) if, when composed with any convex function, the expectation of the obtained process is increasing in t. It is a consequence of Jensen's inequality that a martingale is a peacock. Conversely, it is a deep theorem due to Kellerer (1972) that a peacock is a process which has the same one-dimensional marginals as a martingale. Moreover, this martingale may be chosen Markovian. Thus, at least, two questions arise:

1. How to create peacocks in a systematic way? One answer is: the arithmetic average of a martingale is always a peacock. The original example of this seems to be due to Carr et al. (2008) who took for a martingale the geometric Brownian motion;

2. Given a peacock, how to associate to it a martingale with the same marginals? So far, there does not seem to exist a general answer. But, in their monograph, Hirsch, Profeta, Roynette, and Yor exhibit a number of general cases where some construction may be done (Hirsch 2011).

7 (Brownian) Penalisations

Consider $\mathbb{W}$, the Wiener measure and H_t, positive, a family of adapted probability densities (with respect to the canonical filtration). This allows to create a family $\mathbb{W}_t$ of probabilities on $\mathscr{F}_t$. The penalisation problem is to find whether, as $t \to \infty$, $\mathbb{W}_t$ when restricted to $\mathscr{F}_s$, for fixed s, converges weakly, and if so to describe the limit law. Two monographs have been devoted to this problem: Roynette and Yor (2009) and Najnudel et al. (2009), the first is a collection of examples, the second aims at finding general convergence criterions.

8 Martingales with the Wiener Chaos Decomposition

It is a well-known result, due to Wiener, that every L^2-martingale for the Brownian filtration may be written as the sum of a series of multiple integrals with respect to Brownian motion, with the series of squares of (deterministic) integrands, integrated with respect to Lebesgue measure on their corresponding sets of definitions, converging. A similar result is true for the martingale of the compensated Poisson process. For a long time, it was thought that these were the only two martingales with Wiener chaos decomposition. But, Émery (1989) showed that Azéma's martingale, i.e., the projection of Brownian motion on the filtration of Brownian signs up to time t, also satisfies this property. See also Azéma and Yor (1989) for another proof. Émery (1989) considered more generally some martingales solutions of so-called structure equations, some of which also enjoy the Wiener chaos decomposition; he also wrote a synthesis Émery (1991).

9 Asymptotics of Planar Brownian Windings

A number of limit theorems (in law) for additive functionals of one- or two-dimensional Brownian motion have been obtained throughout the years. This is in particular the case for the winding number of planar Brownian motion up to time t, which, when multiplied by $\frac{2}{\log(t)}$ converges in law toward a standard Cauchy variable Spitzer (1958). This result admits a number of multivariate extensions, in particular: with the same normalization $\frac{2}{\log(t)}$,the vector of n Brownian winding numbers around different points converges in law toward a random vector with (linked) Cauchy marginals Pitman and Yor (1986). The dependence between the different Cauchy marginals may be explained from the Kallianpur and Robbins (1953) asymptotic theorem: normalized by $\frac{1}{\log(t)}$, the time spent in an integrable Borel set by two-dimensional Brownian motion up to time t is asymptotically exponentially distributed.

10 How to Modify the Burkholder-Davis-Gundy Inequalities up to Any Time?

A version of the BDG inequalities is: for any positive p, the supremum of the absolute value of Brownian motion up to a stopping time T has L^p moment which is equivalent to that of $\sqrt{T}$. How could one modify this result when T is replaced by any random time L? A technique consists in making L a stopping time and to consider the semimartingale decomposition of Brownian motion stopped at L. Then, an extension of Fefferman's inequality allows to obtain the desired variants. For details, see Yor (1985).

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