Deciding the Bell Number for Hereditary Graph Properties
(Extended Abstract)

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Abstract. The paper [J. Balogh, B. Bollobás, D. Weinreich, A jump to the Bell number for hereditary graph properties, *J. Combin. Theory Ser. B* 95 (2005) 29–48] identifies a jump in the speed of hereditary graph properties to the Bell number B_n and provides a partial characterisation of the family of minimal classes whose speed is at least B_n. In the present paper, we give a complete characterisation of this family. Since this family is infinite, the decidability of the problem of determining if the speed of a hereditary property is above or below the Bell number is questionable. We answer this question positively for properties defined by finitely many forbidden induced subgraphs. In other words, we show that there exists an algorithm which, given a finite set $\mathcal{F}$ of graphs, decides whether the speed of the class of graphs containing no induced subgraphs from the set $\mathcal{F}$ is above or below the Bell number.

Keywords: Hereditary class of graphs · Speed of hereditary properties · Bell number · Decidability

1 Introduction

A *graph property* (or a *class of graphs*[1]) is a set of graphs closed under isomorphism. Given a property $\mathcal{X}$, we write $\mathcal{X}_n$ for the number of graphs in $\mathcal{X}$ with vertex set $\{1, 2, \ldots, n\}$ (that is, we are counting *labelled* graphs). Following [5], we call $\mathcal{X}_n$ the *speed* of the property $\mathcal{X}$.

A property is *hereditary* if it is closed under taking induced subgraphs. It is well-known (and can be easily seen) that a graph property $\mathcal{X}$ is hereditary if and only if $\mathcal{X}$ can be described in terms of forbidden induced subgraphs. More formally, for a set $\mathcal{F}$ of graphs we write $\mathrm{Free}(\mathcal{F})$ for the class of graphs containing no induced subgraph isomorphic to any graph in the set $\mathcal{F}$. A property $\mathcal{X}$ is

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[1] Throughout the paper we use the two terms – graph property and class of graphs – interchangeably.

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hereditary if and only if $\mathcal{X} = \text{Free}(\mathcal{F})$ for some set $\mathcal{F}$. We call $\mathcal{F}$ a set of *forbidden induced subgraphs* for $\mathcal{X}$ and say that graphs in $\mathcal{X}$ are $\mathcal{F}$*-free*.

The speeds of hereditary properties and their asymptotic structure have been extensively studied, originally in the special case of a single forbidden subgraph [9–11,13–15], and more recently in general [1,4–7,16]. These studies showed, in particular, that there is a certain correlation between the speed of a property $\mathcal{X}$ and the structure of graphs in $\mathcal{X}$, and that the rates of the speed growth constitute discrete layers. The first four lower layers have been distinguished in [16]: these are constant, polynomial, exponential, and factorial layers. In other words, the authors of [16] showed that some classes of functions do not appear as the speed of any hereditary property, and that there are discrete jumps, for example, from polynomial to exponential speeds.

Independently, similar results were obtained by Alekseev in [2]. Moreover, Alekseev provided the first four layers with the description of all minimal classes, that is, he identified in each layer the family of all classes every proper hereditary subclass of which belongs to a lower layer (see also [5] for some more involved results). In each of the first four lower layers the set of minimal classes is finite and each of them is defined by finitely many forbidden induced subgraphs. This provides an efficient way of determining whether a property $\mathcal{X}$ belongs to one of the first three layers.

One more jump in the speed of hereditary properties was identified in [7] and it separates – within the factorial layer – the properties with speeds strictly below the Bell number B_n from those whose speed is at least B_n. The importance of this jump is due to the fact that all the properties below the Bell number are well-structured. In particular, all of them have bounded clique-width [3] and all of them are well-quasi-ordered by the induced subgraph relation [12]. From the results in [5,12] it follows that every hereditary property below the Bell number can be characterised by finitely many forbidden induced subgraphs and hence the membership problem for each of them can be decided in polynomial time.

Even so, very little is known about the boundary separating the two families, that is, very little is known about the *minimal* classes on or above the Bell number. Paper [7] distinguishes two cases in the study of this question: the case where a certain parameter associated with each class of graphs is finite and the case where this parameter is infinite. In the present paper, we call this parameter *distinguishing number*. For the case where the distinguishing number is infinite, [7] provides a complete description of minimal classes, of which there are precisely 13. For the case where the distinguishing number is finite, [7] mentions only one minimal class above the Bell number (linear forests) and leaves the question of characterising other minimal classes open.

In the present paper, we give a complete answer to the above open question: we provide a structural characterisation of all minimal classes above the Bell number with a finite distinguishing number. This family of minimal classes is infinite, which makes the problem of deciding whether a hereditary class is above or below the Bell number questionable. Nevertheless, for properties defined by *finitely many* forbidden induced subgraphs, our characterisation allows us to prove decidability of this problem: we show that there exists an algorithm which, given a finite set $\mathcal{F}$ of graphs, decides whether the class Free($\mathcal{F}$) is above or below the Bell number.

2 Preliminaries and Preparatory Results

2.1 Basic Notation and Terminology

All graphs we consider are undirected without multiple edges. The graphs in our hereditary classes have no loops; however, we allow loops in some auxiliary graphs, called "density graphs" and denoted usually by H, that are used to represent the global structure of our hereditary classes.

If G is a graph, $V(G)$ stands for its vertex set, $E(G)$ for its edge set and $|G|$ for the number of vertices (the *order*) of G. The edge joining two vertices u and v is uv (we do not use any brackets); uv is the same edge as vu.

If $W \subseteq V(G)$, then $G[W]$ is the subgraph of G induced by W. For W_1, W_2 disjoint subsets of $V(G)$ we define $G[W_1, W_2]$ to be the bipartite subgraph of G with vertex set $W_1 \cup W_2$ and edge set $\{uv : u \in W_1,\ v \in W_2,\ uv \in E(G)\}$. The *bipartite complement* of $G[W_1, W_2]$ is the bipartite graph in which two vertices $u \in W_1$, $v \in W_2$ are adjacent if and only if they are not adjacent in $G[W_1, W_2]$.

The *neighbourhood* $N(u)$ of a vertex u in G is the set of all vertices adjacent to u, and the *degree* of u is the number of its neighbours. Note that if (and only if) there is a loop at u then $u \in N(u)$.

As usual, P_n, C_n and K_n denote the path, the cycle and the complete graph with n vertices, respectively. Furthermore, $K_{1,n}$ is a star (i.e., a tree with $n+1$ vertices one of which has degree n), and $G_1 + G_2$ is the disjoint union of two graphs. In particular, mK_n is the disjoint union of m copies of K_n.

A *forest* is a graph without cycles, i.e., a graph every connected component of which is a tree. A *star forest* is a forest every connected component of which is a star, and a *linear forest* is a forest every connected component of which is a path.

A *quasi-order* is a binary relation which is reflexive and transitive. A *well-quasi-order* is a quasi-order which contains neither infinite strictly decreasing sequences nor infinite antichains (sets of pairwise incomparable elements). That is, in a well-quasi-order any infinite sequence of elements contains an infinite increasing subsequence.

Recall that the Bell number B_n, defined as the number of ways to partition a set of n labelled elements, satisfies the asymptotic formula $\ln B_n/n = \ln n - \ln\ln n + \Theta(1)$.

Balogh, Bollobás and Weinreich [7] showed that if the speed of a hereditary graph property is at least $n^{(1-o(1))n}$, then it is actually at least B_n; hence we call any such property a *property above the Bell number*. Note that this includes hereditary properties whose speed is exactly equal to the Bell numbers (such as the class of disjoint unions of cliques).

2.2 (ℓ, d)-graphs and Sparsification

Given a graph G and two vertex subsets $U, W \subset V(G)$, define $\Delta(U,W) = \max\{|N(u) \cap W|, |N(w) \cap U| : u \in U, w \in W\}$. With $\overline{N}(u) = V(G) \backslash (N(u) \cup \{u\})$, let $\overline{\Delta}(U,W) = \max\{|\overline{N}(u) \cap W|, |\overline{N}(w) \cap U| : w \in W, u \in U\}$. Note that $\Delta(U,U)$ is simply the maximum degree in $G[U]$.

Definition 2.1. *Let G be a graph. A partition $\pi = \{V_1, V_2, \ldots, V_{\ell'}\}$ of $V(G)$ is an* (ℓ, d)-partition *if $\ell' \le \ell$ and for each pair of not necessarily distinct integers $i, j \in \{1, 2, \ldots, \ell'\}$ either $\Delta(V_i, V_j) \le d$ or $\overline{\Delta}(V_i, V_j) \le d$. We call the sets V_i* bags*. A graph G is an* (ℓ, d)-graph *if it admits an (ℓ, d)-partition.*

It should be clear that, given an (ℓ, d)-partition $\{V_1, V_2, \ldots, V_{\ell'}\}$ of $V(G)$, for each $x \in V(G)$ and $i \in \{1, 2, \ldots, \ell'\}$ either $|N(x) \cap V_i| \le d$ or $|\overline{N}(x) \cap V_i| \le d$. In the former case we say that x is *d-sparse* with respect to V_i and in the latter case we say x is *d-dense* with respect to V_i. Similarly, if $\Delta(V_i, V_j) \le d$, we say V_i is *d-sparse* with respect to V_j, and if $\overline{\Delta}(V_i, V_j) \le d$, we say V_i is *d-dense* with respect to V_j. We will also say that the pair (V_i, V_j) is *d-sparse* or *d-dense*, respectively. Note that if the bags are large enough (i.e., $\min\{|V_i|\} > 2d + 1$), the terms d-dense and d-sparse are mutually exclusive.

Definition 2.2. *A* strong (ℓ, d)-partition *is an (ℓ, d)-partition each bag of which contains at least $5 \times 2^\ell d$ vertices; a* strong (ℓ, d)-graph *is a graph which admits a strong (ℓ, d)-partition.*

Given any strong (ℓ, d)-partition $\pi = \{V_1, V_2, \ldots, V_{\ell'}\}$ we define an equivalence relation $\sim$ on the bags by putting $V_i \sim V_j$ if and only if for each k, either V_k is d-dense with respect to both V_i and V_j, or V_k is d-sparse with respect to both V_i and V_j. Let us call a partition π *prime* if all its $\sim$-equivalence classes are of size 1. If the partition π is not prime, let $p(\pi)$ be the partition consisting of unions of bags in the $\sim$-equivalence classes for π.

In the full version of this paper we prove that the partition $p(\pi)$ of a strong (ℓ, d)-graph is an $(\ell, \ell d)$-partition whose dense (sparse) pairs correspond to the dense (sparse) pairs of π, and that it does not depend on the choice of a strong (ℓ, d)-partition π:

Theorem 2.3. *Let G be a strong (ℓ, d)-graph with strong (ℓ, d)-partitions π and π'. Then $p(\pi) = p(\pi')$.* □

With any strong (ℓ, d)-partition $\pi = \{V_1, V_2, \ldots, V_{\ell'}\}$ of a graph G we can associate a *density graph* (with loops allowed) $H = H(G, \pi)$: the vertex set of H is $\{1, 2, \ldots, \ell'\}$ and there is an edge joining i and j if and only if (V_i, V_j) is a d-dense pair (so there is a loop at i if and only if V_i is d-dense).

For a graph G, a vertex partition $\pi = \{V_1, V_2, \ldots, V_{\ell'}\}$ of G and a graph with loops allowed H with vertex set $\{1, 2, \ldots, \ell'\}$, we define (as in [5]) the H, π-*transform* $\psi(G, \pi, H)$ to be the graph obtained from G by replacing $G[V_i, V_j]$ with its bipartite complement for every pair (V_i, V_j) for which ij is an edge of H, and replacing $G[V_i]$ with its complement for every V_i for which there is a loop at the vertex i in H.

Moreover, if π is a strong (ℓ, d)-partition we define $\phi(G, \pi) = \psi(G, \pi, H(G, \pi))$; recall that both $p(\pi)$ and $p(\pi')$ are (not necessarily strong) $(\ell, \ell d)$-partitions of G. Note that π is a strong (ℓ, d)-partition for $\phi(G, \pi)$ and each pair (V_i, V_j) is d-sparse in $\phi(G, \pi)$. We now show that the result of this "sparsification" does not depend on the initial strong (ℓ, d)-partition.

Proposition 2.4. *Let G be a strong (ℓ, d)-graph. Then for any two strong (ℓ, d)-partitions π and π', the graph $\phi(G, \pi)$ is identical to $\phi(G, \pi')$.*

Proof. Suppose that $\pi = \{U_1, U_2, \ldots, U_{\hat{\ell}}\}$ and $\pi' = \{V_1, V_2, \ldots, V_{\hat{\ell}'}\}$. By Theorem 2.3, $p(\pi) = p(\pi') = \{W_1, W_2, \ldots, W_{\hat{\ell}''}\}$. Consider two vertices x, y of G. Let i, j, i', j', i'', j'' be the indices such that $x \in U_i$, $x \in V_{i'}$, $x \in W_{i''}$, $y \in U_j$, $y \in V_{j'}$, $y \in W_{j''}$. As the partitions have at least $5 \times 2^\ell d$ vertices in each bag, ℓd-dense and ℓd-sparse are mutually exclusive properties. Hence the pair (U_i, U_j) is d-sparse if and only if $(W_{i''}, W_{j''})$ is ℓd-sparse if and only if $(V_{i'}, V_{j'})$ is d-sparse; and analogously for dense pairs. Therefore xy is an edge of $\phi(G, \pi)$ if and only if it is an edge of $\phi(G, \pi')$. □

Proposition 2.4 motivates the following definition, originating from [5].

Definition 2.5. *For a strong (ℓ, d)-graph G, its* sparsification *is $\phi(G) = \phi(G, \pi)$ for any strong (ℓ, d)-partition π of G.*

2.3 Distinguishing Number $k_{\mathcal{X}}$

Given a graph G and a set $X = \{v_1, \ldots, v_t\} \subseteq V(G)$, we say that the disjoint subsets $U_1, \ldots, U_m$ of $V(G)$ are *distinguished* by X if for each i, all vertices of U_i have the same neighbourhood in X, and for each $i \neq j$, vertices $x \in U_i$ and $y \in U_j$ have different neighbourhoods in X. We also say that X *distinguishes* the sets $U_1, U_2, \ldots, U_m$.

Definition 2.6. *Given a hereditary property $\mathcal{X}$, we define the* distinguishing number *$k_{\mathcal{X}}$ as follows:*

(a) If for all $k, m \in \mathbb{N}$ we can find a graph $G \in \mathcal{X}$ that admits some $X \subset V(G)$ distinguishing at least m sets, each of size at least k, then put $k_{\mathcal{X}} = \infty$.
(b) Otherwise, there must exist a pair (k, m) such that any vertex subset of any graph $G \in \mathcal{X}$ distinguishes at most m sets of size at least k. We define $k_{\mathcal{X}}$ to be the minimum value of k in all such pairs.

In [5] Balogh, Bollobás and Weinreich show that the speed of any hereditary property $\mathcal{X}$ with $k_{\mathcal{X}} = \infty$ is above the Bell number. To study the classes with $k_{\mathcal{X}} < \infty$ in the next sections we will use the following results from their paper:

Lemma 2.7 ([5], Lemma 27). *If $\mathcal{X}$ is a hereditary property with finite distinguishing number $k_{\mathcal{X}}$, then there exist absolute constants $\ell_{\mathcal{X}}$, $d_{\mathcal{X}}$ and $c_{\mathcal{X}}$ such that for all $G \in \mathcal{X}$, the graph G contains an induced subgraph G' such that G' is a strong $(\ell_{\mathcal{X}}, d_{\mathcal{X}})$-graph and $|V(G) \backslash V(G')| < c_{\mathcal{X}}$.* □

Theorem 2.8 ([5], Theorem 28). *Let $\mathcal{X}$ be a hereditary property with $k_{\mathcal{X}} < \infty$. Then $\mathcal{X}_n \geq n^{(1+o(1))n}$ if and only if for every m there exists a strong $(\ell_{\mathcal{X}}, d_{\mathcal{X}})$-graph G in $\mathcal{X}$ such that its sparsification $\phi(G)$ has a component of order at least m.* □

3 Structure of Minimal Classes Above Bell

In this section, we describe minimal classes with speed above the Bell number. In [7], Balogh, Bollobás and Weinreich characterised all minimal classes with infinite distinguishing number. In Sect. 3.1 we report this result and show additionally that each of these classes can be characterised by finitely many forbidden induced subgraphs. Then in Sect. 3.2 we move on to the case of finite distinguishing number, which had been left open in [7].

3.1 Infinite Distinguishing Number

Theorem 3.1 (Balogh–Bollobás–Weinreich [7]). *Let $\mathcal{X}$ be a hereditary graph property with $k_{\mathcal{X}} = \infty$. Then $\mathcal{X}$ contains at least one of the following (minimal) classes:*

(a) the class $\mathcal{K}_1$ of all graphs each of whose connected components is a clique;
(b) the class $\mathcal{K}_2$ of all star forests;
(c) the class $\mathcal{K}_3$ of all graphs whose vertex set can be split into an independent set I and a clique Q so that every vertex in Q has at most one neighbour in I;
(d) the class $\mathcal{K}_4$ of all graphs whose vertex set can be split into an independent set I and a clique Q so that every vertex in I has at most one neighbour in Q;
(e) the class $\mathcal{K}_5$ of all graphs whose vertex set can be split into two cliques Q_1, Q_2 so that every vertex in Q_2 has at most one neighbour in Q_1;
(f) the class $\mathcal{K}_6$ of all graphs whose vertex set can be split into two independent sets I_1, I_2 so that the neighbourhoods of the vertices in I_1 are linearly ordered by inclusion (that is, the class of all chain graphs*);*
(g) the class $\mathcal{K}_7$ of all graphs whose vertex set can be split into an independent set I and a clique Q so that the neighbourhoods of the vertices in I are linearly ordered by inclusion (that is, the class of all threshold graphs*);*
(h) the class $\overline{\mathcal{K}_i}$ of all graphs whose complement belongs to $\mathcal{K}_i$ as above, for some $i \in \{1, 2, \ldots, 6\}$ (note that the complementary class of $\mathcal{K}_7$ is $\mathcal{K}_7$ itself).

Before showing the characterisation of the classes $\mathcal{K}_1$–$\mathcal{K}_6$ in terms of forbidden subgraphs, we introduce some of the less commonly appearing graphs: the *claw* $K_{1,3}$, the *3-fan* F_3, the *diamond* K_4^-, and the *H-graph* H_6 (Fig. 1).

Theorem 3.2. *Each of the classes of Theorem 3.1 is defined by finitely many forbidden induced subgraphs, namely*

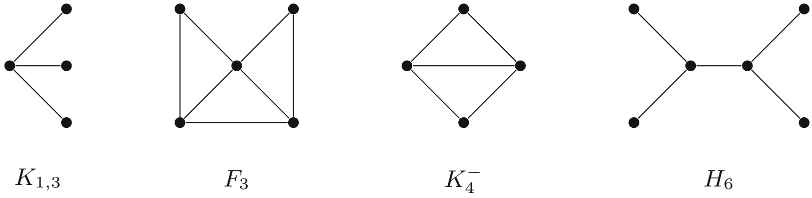

Fig. 1. Some small graphs

(a) $\mathcal{K}_1 = \text{Free}(P_3)$,
(b) $\mathcal{K}_2 = \text{Free}(K_3, P_4, C_4)$,
(c) $\mathcal{K}_3 = \text{Free}(\mathcal{F})$ *for* $\mathcal{F} = \{2K_2, C_4, C_5, K_{1,3}, F_3\}$,
(d) $\mathcal{K}_4 = \text{Free}(\mathcal{F})$ *for* $\mathcal{F} = \{2K_2, C_4, C_5, K_4^-\}$,
(e) $\mathcal{K}_5 = \text{Free}(\mathcal{F})$ *for* $\mathcal{F} = \{3K_1, C_5, \overline{P_4 + K_1}, \overline{2K_2 + K_1}, \overline{C_4 + K_2}, \overline{C_4 + 2K_1}, \overline{H_6}\}$,
(f) $\mathcal{K}_6 = \text{Free}(2K_2, K_3, C_5)$ *[17]*,
(g) $\mathcal{K}_7 = \text{Free}(2K_2, P_4, C_4)$ *[8]*,
(h) $\overline{\text{Free}(\mathcal{F})} = \text{Free}(\overline{\mathcal{F}})$.

3.2 Finite Distinguishing Number

In this section we provide a characterisation of the minimal classes for the case of finite distinguishing number $k_{\mathcal{X}}$. It turns out that these minimal classes consist of $(\ell_{\mathcal{X}}, d_{\mathcal{X}})$-graphs, that is, the vertex set of each graph is partitioned into at most $\ell_{\mathcal{X}}$ bags and dense pairs are defined by a density graph H (see Lemma 2.7). The condition of Theorem 2.8 is enforced by long paths (indeed, an infinite path in the infinite universal graph). Thus actually $d_{\mathcal{X}} \leq 2$ for the minimal classes $\mathcal{X}$.

Let A be a finite alphabet. A *word* is a mapping $w : S \to A$, where $S = \{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$ or $S = \mathbb{N}$; $|S|$ is the *length* of w, denoted by $|w|$. We write w_i for $w(i)$, and we often use the notation $w = w_1 w_2 w_3 \ldots w_n$ or $w = w_1 w_2 w_3 \ldots$. For $n \leq m$ and $w = w_1 w_2 \ldots w_n$, $w' = w'_1 w'_2 \ldots w'_m$ (or $w' = w'_1 w'_2 \ldots$), we say that w is a *factor* of w' if there exists a non-negative integer s such that $w_i = w'_{i+s}$ for $1 \leq i \leq n$; w is an *initial segment* of w' if we can take $s = 0$.

Let H be an undirected graph with loops allowed and with vertex set $V(H) = A$, and let w be a (finite or infinite) word over the alphabet A. For any increasing sequence $u_1 < u_2 < \cdots < u_m$ of positive integers such that $u_m \leq |w|$, define $G_{w,H}(u_1, u_2, \ldots, u_m)$ to be the graph with vertex set $\{u_1, u_2, \ldots, u_m\}$ and an edge between u_i and u_j if and only if

- either $|u_i - u_j| = 1$ and $w_{u_i} w_{u_j} \notin E(H)$,
- or $|u_i - u_j| > 1$ and $w_{u_i} w_{u_j} \in E(H)$.

Let $G = G_{w,H}(u_1, u_2, \ldots, u_m)$ and define $V_a = \{u_i \in V(G) : w_{u_i} = a\}$ for any $a \in A$. Then $\pi = \pi_w(G) = \{V_a : a \in A\}$ is an $(|A|, 2)$-partition, and so G is an $(|A|, 2)$-graph. Moreover, $\psi(G, \pi, H)$ is a linear forest whose paths are formed by the consecutive segments of integers within the set $\{u_1, u_2, \ldots, u_m\}$. This partition $\pi_w(G)$ is called the *letter partition* of G.

Define $\mathcal{P}(w, H)$ to be the hereditary class of graphs consisting of graphs $G_{w,H}(u_1, u_2, \ldots, u_m)$ for all finite increasing sequences $u_1 < u_2 < \cdots < u_m$ of positive integers.

It can be shown that any such class $\mathcal{P}(w, H)$ has finite distinguishing number. Our goal here is to prove that any hereditary class above the Bell number with finite distinguishing number contains the class $\mathcal{P}(w, H)$ for some word w and graph H. Moreover, we describe sufficient conditions on the graph H and the word w so that $\mathcal{P}(w, H)$ is a minimal class above the Bell number.

Definition 3.3. *If $u_1, u_2, \ldots, u_m$ is a sequence of consecutive integers (i.e., $u_{k+1} = u_k + 1$ for each k), we call the graph $G_{w,H}(u_1, u_2, \ldots, u_m)$ an* $|H|$-factor. *Notice that each $|H|$-factor is an $(|H|, 2)$-graph; if its letter partition is a strong $(|H|, 2)$-partition, we call it a* strong $|H|$-factor.

Note that if $G = G_{w,H}(u_1, u_2, \ldots, u_m)$ is a strong ℓ-factor, then its sparsification $\phi(G) = \psi(G, \pi_w(G), H)$ is an induced path of length $m - 1$.

Proposition 3.4. *If w is an infinite word over a finite alphabet A and H is a graph on A, with loops allowed, then the class $\mathcal{P}(w, H)$ is above the Bell number.*

Proof. We may assume that every letter of A appears in w infinitely many times: otherwise we can remove a sufficiently long starting segment of w to obtain a word w' satisfying this condition, replace H with its induced subgraph H' on the alphabet A' of w', and obtain a subclass $\mathcal{P}(w', H')$ of $\mathcal{P}(w, H)$. Then for sufficiently large k, the $|A|$-factor $G_k = G_{w,H}(1, \ldots, k)$ is a strong $|A|$-factor; thus $\phi(G_k)$ is an induced path of length $k - 1$. Hence by Theorem 2.8, the class $\mathcal{P}(w, H)$ is above the Bell number. □

Definition 3.5. *A word w is called* almost periodic *if for any factor f of w there is a constant k_f such that any factor of w of size at least k_f contains f as a factor.*

The next theorem asserts that any class with finite distinguishing number, if it is above Bell, contains one of the classes $\mathcal{P}(w, H)$. Consequently any minimal class will be of the form $\mathcal{P}(w, H)$.

Theorem 3.6. *Suppose $\mathcal{X}$ is a hereditary class above the Bell number with $k_{\mathcal{X}}$ finite. Then $\mathcal{X} \supseteq \mathcal{P}(w, H)$ for an infinite almost periodic word w and a graph H of order at most $\ell_{\mathcal{X}}$ with loops allowed.*

Sketch of proof. From Theorem 2.8 it follows that for each m there is a graph $G_m \in \mathcal{X}$ which admits a strong $(\ell_{\mathcal{X}}, d_{\mathcal{X}})$-partition $\{V_1, V_2, \ldots, V_{\ell_m}\}$ with $\ell_m \leq \ell_{\mathcal{X}}$ such that the sparsification $\phi(G_m)$ has a connected component C_m of order at least $(\ell_{\mathcal{X}} d_{\mathcal{X}})^m$. Fix an arbitrary vertex v of C_m. As C_m is an induced subgraph of $\phi(G_m)$, the maximum degree in C_m is bounded by $d = \ell_{\mathcal{X}} d_{\mathcal{X}}$. Therefore C_m contains an induced path $v = v_1, v_2, \ldots, v_m = v'$ of length $m - 1$. Then the induced subgraph $G_m[v_1, v_2, \ldots, v_m]$ is an $\ell_{\mathcal{X}}$-factor of order m contained in $\mathcal{X}$.

The existence of arbitrarily large $\ell_{\mathcal{X}}$-factors in $\mathcal{X}$ implies that $\mathcal{X}$ contains arbitrarily large *strong* $\ell_{\mathcal{X}}$-factors: It can be shown that by removing the small bags (repeatedly, if necessary) we cannot decrease the size of the $\ell_{\mathcal{X}}$-factor too much.

Having established that each class $\mathcal{X}$ with speed above the Bell number with finite distinguishing number $k_{\mathcal{X}}$ contains an infinite set $\mathcal{S}$ of strong $\ell_{\mathcal{X}}$-factors of increasing order, we can assume that each of the strong $\ell_{\mathcal{X}}$-factors is of the form $G_{w,H}(1,\ldots,m)$ for some prime graph H and that its letter partition is prime. For each H on $\{1,2,\ldots,\ell\}$ with $1 \le \ell \le \ell_{\mathcal{X}}$ let $\mathcal{S}_H = \{G_{w,H}(1,\ldots,m) \in \mathcal{S}\}$ be the set of all $\ell_{\mathcal{X}}$-factors in $\mathcal{S}$ whose adjacencies are defined using the density graph H. Then for some (at least one) fixed graph H_0 the set $\mathcal{S}_{H_0}$ is infinite. Hence also $L = \{w : G_{w,H_0}(1,\ldots,m) \in \mathcal{X}\}$ is an infinite language. As $\mathcal{X}$ is a hereditary class, the language L is closed under taking word factors (it is a *factorial language*).

It is not hard to prove that any infinite factorial language contains a minimal infinite factorial language. So let $L' \subseteq L$ be a minimal infinite factorial language. It follows from minimality that L' is well quasi-ordered by the factor relation, because removing one word from any infinite antichain and taking all factors of the remaining words would generate an infinite factorial language strictly contained in L'. Thus there exists an infinite chain $w^{(1)}, w^{(2)}, \ldots$ of words in L' such that for any $i < j$, the word $w^{(i)}$ is a factor of $w^{(j)}$. More precisely, for each i there is a non-negative integer s_i such that $w^{(i)}_k = w^{(i+1)}_{k+s_i}$. Let $g(i,k) = k + \sum_{j=1}^{i-1} s_i$. Now we can define an infinite word w by putting $w_k = w^{(i)}_{g(i,k)}$ for the least value of i for which the right-hand side is defined. (Without loss of generality we get that w is indeed an infinite word; otherwise we would need to take the reversals of all the words $w^{(i)}$.)

Observe that any factor of w is in the language L'; if w is not almost periodic, then there exists a factor f of w such that there are arbitrarily long factors f' of w not containing f. These factors f' generate an infinite factorial language $L'' \subseteq L'$ which does not contain $f \in L'$, contradicting the minimality of L'.

Because any factor of w is in L, any $G_{w,H}(u_1,\ldots,u_m)$ is an induced subgraph of some $\ell_{\mathcal{X}}$-factor in $\mathcal{X}$. Therefore $\mathcal{P}(w,H) \subseteq \mathcal{X}$. □

As a matter of fact, we can also show that if H is a graph with loops allowed and w is an almost periodic infinite word, then $\mathcal{P}(w,H)$ is a minimal property above the Bell number. This implies the following characterisation.

Theorem 3.7. *Let $\mathcal{X}$ be a class of graphs with $k_{\mathcal{X}} < \infty$. Then $\mathcal{X}$ is a minimal hereditary class above the Bell number if and only if there exists a finite graph H with loops allowed and an infinite almost periodic word w over $V(H)$ such that $\mathcal{X} = \mathcal{P}(w,H)$.*

4 Decidability of the Bell Number

Our main goal is to provide an algorithm that decides for an input consisting of a finite number of graphs $F_1,\ldots,F_n$ whether the speed of $\mathcal{X} = \mathrm{Free}(F_1,\ldots,F_n)$ is above the Bell number. That is, we are interested in the following problem.

Problem 4.1. INPUT: *A finite set of graphs* $\mathcal{F} = \{F_1, F_2, \ldots, F_n\}$
OUTPUT: *Yes, if the speed of* $\mathcal{X} = \text{Free}(\mathcal{F})$ *is above the Bell number; no otherwise.*

Our algorithm, following the characterisation of minimal classes above the Bell number, distinguishes two cases depending on whether the distinguishing number $k_{\mathcal{X}}$ is finite or infinite. First we show how to discriminate between these two cases.

Problem 4.2. INPUT: *A finite set of graphs* $\mathcal{F} = \{F_1, F_2, \ldots, F_n\}$
OUTPUT: *Yes, if* $k_{\mathcal{X}} = \infty$ *for* $\mathcal{X} = \text{Free}(\mathcal{F})$*; no otherwise.*

Theorem 4.3. *There is a polynomial-time algorithm that solves Problem 4.2.*

Proof. By Theorem 3.1, $k_{\mathcal{X}} = \infty$ if and only if $\mathcal{X}$ contains one of the thirteen minimal classes listed there. By Theorem 3.2, each of the minimal classes is defined by finitely many forbidden induced subgraphs; thus membership can be tested in polynomial time. Then the answer to Problem 4.2 is no if and only if each of the minimal classes given by Theorem 3.1 contains at least one of the graphs in $\mathcal{F}$, which can also be tested in polynomial time. □

By Theorem 3.7, the minimal hereditary classes with finite distinguishing number with speed above the Bell number can be described as $\mathcal{P}(w, H)$ with an almost periodic infinite word w. Here we give a more precise characterisation restricted to classes defined by finitely many forbidden induced subgraphs.

Definition 4.4. *Let* $w = w_1 w_2 \ldots$ *be an infinite word over a finite alphabet* A*. If there exists some* p *such that* $w_i = w_{i+p}$ *for all* $i \in \mathbb{N}$*, we call the word* w periodic *and the number* p *its* period*. If, moreover, for some period* p *the letters* $w_1, w_2, \ldots, w_p$ *are all distinct, we call the word* w cyclic*.*

If w *is a finite word, then* $w' = (w)^\infty$ *is the periodic word obtained by concatenating infinitely many copies of the word* w*; thus* $w'_i = w_k$ *for* $k = i \bmod |w'|$*.*

A class $\mathcal{X}$ *of graphs is called a* periodic class *(*cyclic class*, respectively) if there exists a graph* H *with loops allowed and a periodic (cyclic, respectively) word* w *such that* $\mathcal{X} = \mathcal{P}(w, H)$*.*

Definition 4.5. *Let* $A = \{1, 2, \ldots, \ell\}$ *be a finite alphabet,* H *a graph on* A *with loops allowed, and* M *a positive integer. Define a graph* $S_{H,M}$ *with vertex set* $V(S_{H,M}) = A \times \{1, 2, \ldots, M\}$ *and an edge between* (a, j) *and* (b, k) *if and only if one of the following holds:*

- $ab \in E(H)$ *and either* $|a - b| \neq 1$ *or* $j \neq k$*;*
- $ab \notin E(H)$ *and* $|a - b| = 1$ *and* $j = k$*.*

The graph $S_{H,M}$ *is called an* (ℓ, M)-strip*.*

Notice that a strip can be viewed as the graph obtained from the union of M disjoint paths $(1, j)-(2, j)-\cdots-(\ell, j)$ for $j \in \{1, 2, \ldots, M\}$ by swapping edges with non-edges between vertices (a, j) and (b, k) if $ab \in E(H)$.

Theorem 4.6. *Let* $\mathcal{X} = \text{Free}(F_1, F_2, \ldots, F_n)$ *with the distinguishing number* $k_{\mathcal{X}}$ *finite. Then the following conditions are equivalent:*

(a) The speed of $\mathcal{X}$ *is above the Bell number.*
(b) $\mathcal{X}$ *contains a periodic class.*
(c) For every $p \in \mathbb{N}$, $\mathcal{X}$ *contains a cyclic class with period at least* p.
(d) There exists a cyclic word w *and a graph* H *on the alphabet of* w *such that* $\mathcal{X}$ *contains the* ℓ*-factor* $G_{w,H}(1, 2, \ldots, 2\ell m)$ *with* $\ell = |V(H)|$ *and* $m = \max\{|F_i| : i \in \{1, 2, \ldots, n\}\}$.
(e) For any positive integers ℓ, m, *the class* $\mathcal{X}$ *contains an* (ℓ, m)*-strip.*

We omit the proof of Theorem 4.6 here (it can be found in the full version of this paper). Finally, we are ready to tackle the decidability of Problem 4.1.

Algorithm 4.7. Input: A finite set of graphs $\mathcal{F} = \{F_1, F_2, \ldots, F_n\}$
Output: Yes, if the speed of $\mathcal{X} = \text{Free}(\mathcal{F})$ is above the Bell number; no otherwise.

(1) Using Theorem 4.3, decide whether $k_{\mathcal{X}} = \infty$. If it is, output *yes* and stop.
(2) Set $m := \max\{|F_1|, |F_2|, \ldots, |F_n|\}$ and $\ell := 1$.
(3) Loop:
  (3a) For each graph (with loops allowed) H on $\{1, 2, \ldots, \ell\}$ construct the (ℓ, ℓ)-strip $S_{H,\ell}$. Check if some F_i is an induced subgraph of $S_{H,\ell}$. If for each H the strip $S_{H,\ell}$ contains some F_i, output *no* and stop.
  (3b) For each graph (with loops allowed) H on $\{1, 2, \ldots, \ell\}$ and for each word w consisting of ℓ distinct letters from $\{1, 2, \ldots, \ell\}$ check if the ℓ-factor $G_{w^\infty,H}(1, 2, \ldots, 2\ell m)$ contains some F_i as an induced subgraph. If one of these ℓ-factors contains no F_i, output *yes* and stop.
  (3c) Set $\ell := \ell + 1$ and repeat.

It remains to prove the correctness of this algorithm.

Theorem 4.8. *Algorithm 4.7 correctly solves Problem 4.1.*

Proof. We show that if the algorithm stops, it gives the correct answer, and furthermore that it will stop on any input without entering an infinite loop. First, if it stops in step (1), the answer is correct by [7], since any class with infinite distinguishing number has speed above the Bell number.

Assume that the algorithm stops in step (3a) and outputs *no*. This is because every (ℓ, ℓ)-strip contains some forbidden subgraph F_i, hence no (ℓ, ℓ)-strip belongs to $\mathcal{X}$. By Theorem 4.6(e), the speed of $\mathcal{X}$ is below the Bell number.

Next suppose that the algorithm stops in step (3b) and answers *yes*. Then $\mathcal{X}$ contains the ℓ-factor $G_{w^\infty,H}(1, 2, \ldots, 2\ell m)$, where w^∞ is a cyclic word. Hence by Theorem 4.6(d) the speed of $\mathcal{X}$ is above the Bell number.

Finally, if $k_{\mathcal{X}} = \infty$ the algorithm stops in step (1). If $k_{\mathcal{X}} < \infty$ and the speed of $\mathcal{X}$ is above the Bell number, then by Theorem 4.6(d) the algorithm will stop in step (3b). If, on the other hand, the speed of $\mathcal{X}$ is below the Bell number, then by

Theorem 4.6(e) there exist positive integers ℓ, M such that $\mathcal{X}$ contains no (ℓ, M)-strip. Let $N = \max\{\ell, M\}$. Obviously, $\mathcal{X}$ contains no (N, N)-strip, because any (N, N)-strip contains some (many) (ℓ, M)-strips as induced subgraphs and $\mathcal{X}$ is hereditary. Therefore the algorithm will stop in step (3a) after finitely many steps. □

Our result leaves many open questions. For instance, what is the computational complexity of Problem 4.1?

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