Towards the Hanani-Tutte Theorem for Clustered Graphs

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Abstract. The weak variant of the Hanani–Tutte theorem says that a graph is planar, if it can be drawn in the plane so that every pair of edges cross an even number of times. Moreover, we can turn such a drawing into an embedding without changing the order in which edges leave the vertices. We prove a generalization of the weak Hanani–Tutte theorem that also easily implies the monotone variant of the weak Hanani–Tutte theorem by Pach and Tóth. Thus, our result can be thought of as a common generalization of these two neat results. In other words, we prove the weak Hanani-Tutte theorem for strip clustered graphs, whose clusters are linearly ordered vertical strips in the plane and edges join only vertices in the same cluster or in neighboring clusters with respect to this order.

Besides usual tools for proving Hanani-Tutte type results our proof combines Hall's marriage theorem, and a characterization of embedded upward planar digraphs due to Bertolazzi et al.

Keywords: Hanani–Tutte theorem · Hall's theorem · Upward planarity · C-planarity

1 Introduction

A *drawing* of G is a representation of G in the plane, where every vertex is represented by a unique point and every edge $e = uv$ is represented by a simple arc joining the two points that represent u and v. If it leads to no confusion, we do not distinguish between a vertex or an edge and its representation in the drawing and we use the words "vertex" and "edge" in both contexts. We assume that in a drawing no edge passes through a vertex, no two edges touch and every pair of edges cross in finitely many points. A drawing of a graph is an *embedding* if no two edges cross. A graph is *planar*, if it admits a planar embedding.

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1.1 Hanani–Tutte Theorem

The Hanani–Tutte theorem [13,22] is a classical result that provides an algebraic characterization of planarity with interesting algorithmic consequences. The (strong) Hanani–Tutte theorem says that a graph is planar as soon as it can be drawn in the plane so that no pair of independent edges crosses an odd number of times. Moreover, its variant known as the weak Hanani–Tutte theorem [3,14,17] states that if we have a drawing $\mathcal{D}$ of a graph G where every pair of edges cross an even number of times then G has an embedding that preserves the cyclic order of edges at vertices from $\mathcal{D}$. Note that the weak variant does not directly follow from the strong Hanani–Tutte theorem. For sub-cubic graphs, the weak variant implies the strong variant.

Other variants of the Hanani–Tutte theorem in the plane were proved for x-monotone drawings [10,15], partially embedded planar graphs, simultaneously embedded planar graphs [20], and two clustered graphs [9]. As for the closed surfaces of genus higher than zero, the weak variant is known to hold in all closed surfaces [18], and the strong variant was proved only for the projective plane [16]. It is an intriguing open problem to decide if the strong Hanani–Tutte theorem holds for closed surfaces other than the sphere and projective plane.

To prove a strong variant for a closed surface it is enough to prove it for all the minor minimal graphs (see e.g. [6] for the definition of a graph minor) not embeddable in the surface. Moreover, it is known that the list of such graphs is finite for every closed surface, see e.g. [6, Section 12]. Thus, proving or disproving the strong Hanani-Tutte theorem on a closed surface boils down to a search for a counterexample among a finite number of graphs. That sounds quite promising, since checking a particular graph is reducible to a finitely many, and not so many, drawings, see e.g. [21]. However, we do not have a complete list of such graphs for any surface besides sphere and projective plane.

On the positive side, the list of possible minimal counterexamples for each surface was recently narrowed down to vertex two-connected graphs [21]. See [19] for a recent survey on applications of the Hanani–Tutte theorem and related results.

1.2 Notation

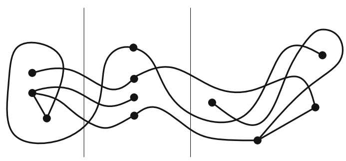
In the present paper we assume that $G = (V, E)$ is a (multi)graph. We refer to an embedding of G as to a *plane* graph G. The *rotation* at a vertex v is the clockwise cyclic order of the end pieces of edges incident to v. The *rotation system* of a graph is the set of rotations at all its vertices. Two embeddings of a graph are the *same*, if they have the same rotation system up to switching the orientations of all the rotations simultaneously. A pair of edges in a graph is *adjacent* or *independent*, if they do not share a vertex. An edge in a drawing is (independently) *even*, if it crosses every other (non-adjacent) edge an even number of times. A drawing of a graph is (independently) *even*, if all edges are (independently) even. Note that an embedding is an even drawing. Let $x(v)$ (resp. $y(v)$) denote the x-coordinate (resp. y-coordinate) of a vertex in a drawing.

1.3 Hanani–Tutte for Strip Clustered Graphs

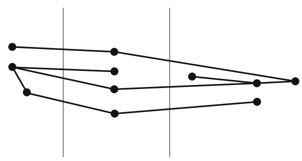
Borrowing the notation from [1] a *clustered graph*[1] is an ordered pair (G, T), where G is a graph, and $T = \{V_i | i = 1, \ldots, k\}$ is a partition of the vertex set of G into k parts. We call the sets V_i clusters. A *clustered graph* (G, T) is *strip clustered*, if $G = \left(V_1 \uplus \ldots \uplus V_k, E \subseteq \bigcup_i \binom{V_i \uplus V_{i+1}}{2}\right)$, i.e., the edges in G are either contained inside a part or join vertices in two consecutive parts. A drawing of a strip clustered graph (G, T) in the plane is *clustered*, if $i < x(v_i) < i + 1$ for all $v_i \in V_i$, and every vertical line $x = i$, $i \in \mathbb{Z}$, intersects every edge at most once. We use the term "cluster V_i" also, when referring to a vertical strip containing the vertices in V_i. A strip clustered graph (G, T) is *clustered planar* (or briefly *c-planar*) if (G, T) has a clustered embedding in the plane.

The notion of clustered planarity appeared for the first time in the literature in the work of Feng, Cohen and Eades [7,8] under the name of c-planarity. See, e.g., [5,7,8] for the general definition of c-planarity. Here, we consider only a special case of it. See, e.g., [5] for further references. We only remark that it has been an intriguing open problem for almost two decades to decide, if c-planarity is NP-hard, despite of considerable effort of many researchers and that already for strip clustered graphs the problem constitutes a challenge [1].

We show the following generalization of the weak Hanani–Tutte theorem for strip clustered graphs. See Fig. 1(a) and (b) for an illustration.



(a)



(b)

Fig. 1. (a) Even clustered drawing of a strip clustered graph; (b) Clustered embedding of the same clustered graph.

Theorem 1. *If a strip clustered graph (G, T) admits an even clustered drawing $\mathcal{D}$ then (G, T) is c-planar. Moreover, there exists a clustered embedding of (G, T) with the same rotation system as in $\mathcal{D}$.*

Due to the family of counterexamples in [9], Theorem 1 does not leave too much room for straightforward generalizations. Let (G, T) denote a clustered graph, and let $G' = G'(G, T)$ denote a graph obtained from (G, T) by contracting every cluster to a vertex and deleting all the loops and multiple edges. If (G, T) is a strip clustered graph, G' is a subgraph of a path. In this sense, the most general

[1] This type of clustered graphs is usually called flat clustered graph in the graph drawing literature. We chose this simplified notation in order not to overburden the reader with unnecessary notation.

variant of Hanani-Tutte, the weak or strong one, we can hope for, is the one for the class of clustered graphs (G, T), for which G' is an arbitrary tree.

By allowing G' to contain a cycle, c-planarity testing seems to be much harder than in the case, when it is acyclic. Already in the case of three clusters [4], if G (not G') is a cycle, the polynomial time algorithm for c-planarity is not trivial, while if G can be any graph, its existence is still open. For a comparison, if G is a cycle then a strip clustered graph (G, T) is trivially c-planar. We note that by an easy geometric argument a polynomial time algorithm for c-planarity in the case of three clusters would imply a polynomial time algorithm in the case of strip clustered graphs.

Our proof of Theorem 1 is slightly technical, and combines a characterization of upward planar digraphs from [2] and Hall's theorem [6, Sect. 2]. Using the result from [2] in our situation is quite natural, as was already observed in [1], where they solve an intimately related algorithmic question discussed below. The reason is that deciding the c-planarity for embedded strip clustered graphs is, essentially, a special case of the upward planarity testing. The technical part of our argument augments the even drawing with subdivided edges by using tricks from [10, 17] so that we are able to apply Hall's Theorem. Hence, the real novelty of our work lies in proving the marriage condition, which makes the characterization do the work for us. It took a considerable effort to make Hall's Theorem work here, and thus, we wonder if a more direct proof exists.

An edge e of a topological graph is *x-monotone*, if every vertical line intersects e at most once. Pach and Tóth [15] (see also [10] for a different proof of the same result) proved the following theorem.

Theorem 2. *Let G denote a graph, whose vertices are totally ordered. Suppose that there exists a drawing $\mathcal{D}$ of G, in which x-coordinates of vertices respect their order, edges are x-monotone and every pair of edges cross an even number of times. Then there exists an embedding of G, in which the vertices are drawn as in $\mathcal{D}$, the edges are x-monotone, and the rotation system is the same as in $\mathcal{D}$.*

We show that Theorem 1 easily implies Theorem 2. Our argument for showing that suggests a slightly different variant of Theorem 1 for not necessarily clustered drawings that directly implies Theorem 2 (see Sect. 2.1). The strong variant of Theorem 1, which we conjecture to hold, would imply the existence of a polynomial time algorithm for the corresponding variant of the c-planarity testing [9]. To the best of our knowledge, a polynomial time algorithm was given only in the case, when the underlying planar graph has a prescribed planar embedding [1]. Our weak variant gives a polynomial time algorithm if G is subcubic, and in the same case as [1]. Nevertheless, we think that the weak variant is interesting in its own right. To support our conjecture we prove the strong variant of Theorem 1 under the condition that the underlying abstract graph G of a clustered graph is a subdivision of a vertex three-connected graph. In general, we only know that it is true for two clusters [9].

Theorem 3. *Let G denote a subdivision of a vertex three-connected graph. If a strip clustered graph (G, T) admits an independently even clustered drawing $\mathcal{D}$ then (G, T) is c-planar.*[2]

The proof of Theorem 3 reduces to Theorem 1 by correcting the rotations at the vertices of G so that the theorem becomes applicable. As we noted above, the weak Hanani-Tutte theorem fails already for three clusters. Moreover, the underlying graph in the counterexample is a cycle [9], and thus, the strong variant fails as well in general clustered graphs without imposing additional restrictions.

The paper is organized as follows.
In Sect. 2 we introduce terminology and tools for proving our results, where in Subsect. 2.2 we outline the proof of our main result Theorem 1. In Sect. 3 we give the proof of Theorem 3. In Sect. 4 we derive Theorem 2 from Theorem 1. Open problems are stated in Sect. 5.

2 Preliminaries

2.1 Even Drawings

We will use the following fact about closed curves in the plane. Let C denote a closed (possibly self-crossing) curve in the plane.

Lemma 1. *The regions in the complement of C can be two-colored so that two regions sharing a non-trivial part of the boundary receive opposite colors.*

Let us two-color the regions in the complement of C so that two regions sharing a non-trivial part of the boundary receive opposite colors. A point not lying on C is *outside* of C, if it is contained in the region with the same color as the unbounded region. Otherwise, such a point is *inside* of C. As a simple corollary of Lemma 1 we obtain a well-known fact that a pair of closed curves in the plane cross an even number of times. We use this fact tacitly throughout the paper.

Let G denote a planar graph. Since in the problem we study connected components of G can be treated separately, we can afford to assume that G is connected. A *face* in an embedding of G is a walk that corresponds to the boundary of the connected component of the complement of G in the plane. A vertex or an edge is *incident* to a face, if it appears on the corresponding walk.

Given a drawing of a graph G, where every pair of edges crosses an even number of times, by the weak Hanani-Tutte theorem [3,14,17], we can obtain an embedding of G with the same rotation system, and hence, the facial structure of an embedding of G is already present in an even drawing. This allows us to speak about faces in an even drawing of G. Hence, a face in an even drawing of

[2] The argument in the proof of Theorem 3 proves, in fact, a strong variant even in the case, when we require the vertices participating in a cut or two-cut to have the maximum degree three. Hence, we obtained a polynomial time algorithm even in the case of sub-cubic cuts and two-cuts.

G is the walk bounding the corresponding face in the embedding of G with the same rotation system.

Let $\gamma : V \to \mathbb{N}$ be a labeling of the vertices of G by integers. Given a face f in an even drawing of G, a vertex v incident to f is the *local minimum* (resp. *maximum*) of f, if in the corresponding facial walk W of f the value of $\gamma(v)$ is not smaller (not bigger) than the value of its successor and predecessor on W. The minimal (resp. maximal) local minimum (resp. maximum) of f is called *global minimum* (resp. *maximum*) of f. The face f is *simple* with respect to γ, if f has exactly one local minimum and one local maximum. The face f is *semi-simple* (with respect to γ), if f has exactly two local minima and these minima have the same value, and two local maxima and these maxima have the same value. A path P is *(strictly) monotone with respect to* γ, if the labels of the vertices on P form a (strictly) monotone sequence if ordered in the correspondence with their appearance on P.

Given a strip clustered graph (G, T) we naturally associate with it a labeling γ that for each vertex v returns the number of the cluster v belongs to. We refer to the cluster, whose vertices get label k, as to the k-th cluster. Let $(\overrightarrow{G}, T)$ denote the directed strip clustered graph obtained from (G, T) by orienting every edge uv from the vertex with the smaller label to the vertex with the bigger label, and in case of a tie orienting uv arbitrarily. A *sink* (resp. *source*) of $\overrightarrow{G}$ is a vertex with no outgoing (resp. incoming) edges.

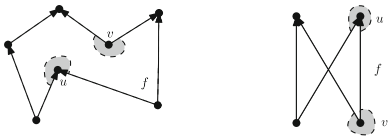


Fig. 2. Assignment of angles at u and v to f corresponding to an upward embedding (on the left), and assignment of angles that is not admissible in an upward embedding (on the right).

In our arguments we use a continuous deformation in order to transform a given drawing into a drawing with desired properties. Observe that during such transformation of a drawing of a graph the parity of crossings between a pair of edges is affected only when an edge e passes over a vertex v, in which case we change the parity of crossings of e with all the edges adjacent to v. Let us call such an event an *edge-vertex switch.*

Edge contraction and vertex split. A *contraction* of an edge $e = uv$ in a topological graph is an operation that turns e into a vertex by moving v along e towards u while dragging all the other edges incident to v along e. Note that by contracting an edge in an even drawing, we obtain again an even drawing.

We will also often use the following operation which can be thought of as the inverse operation of the edge contraction in a topological graph. A *vertex split* in a drawing of a graph G is the operation that replaces a vertex v by two vertices v' and v'' drawn in a small neighborhood of v joined by a short crossing free edge so that the neighbors of v are partitioned into two parts according to whether they are joined with v' or v'' in the resulting drawing, the rotations at v' and v'', resp., is inherited from the rotation at v, and the new edges are drawn in the small neighborhood of the edges they correspond to in G.

Bounded Edges. Theorem 1 can be extended to more general clustered graphs (G, T) that are not necessarily strip clustered, and drawings that are not necessarily clustered. The clusters $V_1, \ldots, V_k$ of (G, T) in our drawing $\mathcal{D}$ are still linearly ordered and drawn as vertical strips respecting this order. An edge $uv \in E(G)$, where $u \in V_i, v \in V_j$, can join any two vertices of G, but it must be drawn so that it intersects only clusters V_l such that $i \leq l \leq j$. We say that the edge uv is *bounded*, and the drawing *quasi-clustered*.

A similar extension of a variant of Hanani-Tutte theorem is also possible in the case of x-monotone drawings [10]. In the x-monotone setting instead of the x-monotonicity of edges in an (independently) even drawing it is only required that the vertical projection of each edge is bounded by the vertical projections of its vertices. Thus, each edge stays between its end vertices.

In the same vein as for x-monotone drawing the extension of our result to drawings $\mathcal{D}$ of clustered graphs with bounded edges can be proved by a reduction to the original claim, Theorem 1. To this end we just need to subdivide every edge e of (G, T) violating conditions of strip clustered drawings so that newly created edges join the vertices in the same or neighbouring clusters, and perform edge-vertex switches in order to restore the even parity of the number of crossings between every pair of edges. The reduction is carried out by the following lemma that is also used in the proof of Theorem 1.

Lemma 2. *Let $\mathcal{D}$ denote an even quasi-clustered drawing of a clustered graph (G, T). Let $e = uv$, where $u \in V_i, v \in V_j$ denote an edge of G. Let G' denote a graph obtained from G by subdiving e by $|i - j| - 1$ vertices. Let (G', T') denote the clustered graph, where T' is inherited from T so that the subdivided edge e is turned into a strictly monotone path w.r.t. γ. Then there exists an even quasi-clustered drawing $\mathcal{D}'$ of (G', T'), in which each new edge crosses the boundary of a cluster exactly once and in which no new intersections of edges with boundaries of the clusters are introduced.*

Proof. Refer to Fig. 3(a) and (b). First, we continuously deform e so that e crosses the boundary of every cluster it visits at most twice. During the deformation we could change the parity of the number of crossings between e and some edges of G. This happens when e passes over a vertex w. We remind the reader that we call this event an edge-vertex switch. Note that we can further deform e so that it performs another edge-vertex switch with each such vertex w, while introducing new crossings with edges "far" from w only in pairs. Thus, by performing the appropriate edge-vertex switches of e with vertices of G we maintain the parity of the number of crossings of e with the edges of G and we do not introduce intersections of e with the boundaries of the clusters.

Fig. 3. (a) Subdivision of the edge e by the vertex z resulting into odd crossing pairs; (b) Restoration of the evenness by performing edge-vertex switches with z.

Second, if e crosses the boundary of a cluster twice, we subdivide e by a vertex z inside the cluster thereby turning e into two edges, the edge joining u with z and the edge joining z with v. After we subdivide e by z, the resulting drawing is not necessarily even. However, it cannot happen that an edge crosses an odd number of times exactly one edge incident to z, since prior to subdividing the edge e the drawing was even. Thus, by performing edge-vertex switches of z with edges that cross both edges incident to z an odd number of times we restore the even parity of crossings between all pairs of edges. By repeating the second step until we have no edge that crosses the boundary of a cluster twice we obtain a desired drawing of G'. □

2.2 From Strip Clustered Graphs to the Marriage Condition

The main tool for proving Theorem 1 is [2, Theorem 3] of Bertolazzi et al. that characterizes embedded directed planar graphs, whose embedding can be straightened (the edges turned into straight line segments) so that all the edges are directed upward, i.e., every edge is directed towards the vertex with a higher y-coordinate. Here, it is not crucial that the edges are drawn as straight line segments, since we can straighten them as soon as they are y-monotone [15]. The theorem says that an embedded directed planar graph $\overrightarrow{G}$ admits such an embedding, if there exists an assignment of the sources and sinks of $\overrightarrow{G}$ to the faces of $\overrightarrow{G}$ that is easily seen to be necessary for such a drawing to exist (see Fig. 2 for an illustration).

Intuitively, a sink or source v is assigned to a face f, if and only if a pair of edges vw and vz, incident to f form in f a concave angle, i.e., an angle bigger than π in an upward embedding. Thus, a vertex can be assigned to a face only if it is incident to it. First, note that the number of sinks incident to a face f is the same as the number of sources incident to f. The mentioned easy necessary condition for the existence of an upward embedding is that an internal (resp. external) face with $2k$ sinks and sources have precisely $k-1$ (resp. $k+1$) of them assigned to it, and that the rotation at each vertex can be split into two parts consisting of incoming and outgoing, resp., edges. The embeddings satisfying the latter are dubbed *candidate embeddings* by [2].

Assuming that in (G,T) each cluster forms an independent set, we would like to prove that $(\overrightarrow{G},T)$ satisfy this condition, if (G,T) admits an even clustered drawing.

That would give us the desired clustered drawing by an easy geometric argument. However, we do not know how to do it directly, if faces have arbitrarily many sinks and sources. Thus, we first augment the given even drawing by adding edges and vertices so that the outer face in $\overrightarrow{G}$ is incident to at most one sink and one source, i.e., it is simple in G w.r.t. γ, and each internal face, that is not simple, is incident to exactly two sinks and two sources, i.e., it is semi-simple.[3] Let (G', T') denote the resulting strip clustered graph. This reduces the proof to showing that there exists a bijection between the set of internal semi-simple faces, and the set of sinks and sources in $\overrightarrow{G}'$ without the source and sink incident to the outer face.

By [2, Lemma 5] the total number of sinks and sources is exactly the total demand by all the faces in a candidate embedding, which is also a direct consequence of the discretized version [11] of the Poincaré-Hopf index theorem [12]. Hence, by Hall's Theorem the bijection exists, if every subset of internal semi-simple faces of size l is incident to at least l sinks and sources.

2.3 Crossing Paths

Just to give a glimpse of the proof of our main result we present two observations, whose combination plays an important role in the proof of the required marriage condition that we need in order to apply [2, Theorem 3]. The first one is a simple parity variant of the Pigeon hole principle.

Observation 1. *Let $C = v_1v_2 \dots v_{2a}$, $a \geq 2$, denote an even cycle. Let V' denote a subset of the vertices of C of size at least $a+2$. Then V' contains four vertices v_i, v_j, v_k and v_l, where $i < j < k < l$, such that i, k is odd and j, l is even (or vice versa).*

Proof. For the sake of contradiction we assume that V' does not contain four such vertices. Let V_0 and V_1, resp., denote the vertices of V with even and odd index. Similarly, let V_0' and V_1', resp., denote the vertices of V' with even and odd index. Suppose that $2 \leq |V_0'| \geq |V_1'|$ and fix a direction in which we traverse C. Between every two consecutive vertices of V_0' along C except for at most one pair of consecutive vertices we have a vertex in $V_1 - V_1'$. Thus, $|V_1 - V_1'| \geq |V_0'| - 1$. On the other hand, $|V_1 - V_1'| = a - |V_1'| \leq a - (a + 2 - |V_0'|) = |V_0'| - 2$ (contradiction). □

Let G denote a graph with a rotation system. We define the *crossing index* of paths P_1 and P_2 in G as follows. Let us orient all the edges of P_1 and P_2, resp., so that P_1 and P_2 has only one sink and one source. Let P denote the subgraph of G which is the union of P_1 and P_2. We define $cr(v) = +1$ (resp. $cr(v) = -1$), if v is a vertex of degree four in P such that the paths P_1 and P_2 alternate in the rotation at v and at v the path P_2 crosses P_1 from left to right (resp. right to left) in the direction of P_1. We define $cr(v) = +1/2$ (resp. $cr(v) = -1/2$), if v is a vertex of degree three in P such that at v the path P_2 is oriented towards P_1

[3] We would not have to do anything that follows, if we could turn all the faces into simple ones. However, this seems to be a difficult task.

from left, or from P_1 to right (resp. towards P_1 from right, or from P_1 to left) in the direction of P_1. The crossing index of P_1 and P_2 is then the absolute value of the sum of $cr(v)$ over all vertices of degree three and four in P. Let $G' \subseteq G$. Let $\max(G')$ and $\min(G')$, resp., denote the maximal and minimal value of $\gamma(v)$, $v \in V(G')$.

A path P in G is an *i-cap* and *i-cup*, resp., if for the end vertices u and v of P we have $\min(P) = \gamma(u) = \gamma(v) = i$ and $\max(P) = \gamma(u) = \gamma(v) = i$. Note that we can define the crossing index for any two subgraphs of maximum degree two. Then two cycles in an even drawing of G cannot have an odd crossing index, since they would correspond to two curves in the plane crossing an odd number of times. This observation can be easily extended to special pairs of an i-cap and j-cup.

An i-cap P_1 and j-cup P_2 *cross*, if

(A) their crossing index is odd; and
(B) $\min(P_1) < \min(P_2) \leq \max(P_1) < \max(P_2)$.

Observation 2. *The clustered graph (G, T) admitting an even clustered drawing does not contain a pair P_1 and P_2 of an i-cap and j-cup, $i + 1 < j$, that cross.*

3 The Proof of Theorem 3

First, we prove a lemma that allows us to get rid of odd crossing pairs by doing only local redrawings and vertex splits.

A drawing of a graph G is obtained from the given drawing of G by *redrawing edges locally at vertices* if the resulting drawing of G differs from the given one only in small pairwise disjoint neighborhoods of vertices not containing any other vertex. The proof of the following lemma is inspired by the proof of [17, Theorem 3.1].

Lemma 3. *Let G denote a subdivision of a vertex three-connected graph drawn in the plane so that every pair of non-adjacent edges cross an even number of times. We can turn the drawing of G into an even drawing by a finite sequence of local redrawings of edges at vertices and vertex splits.*

Proof. We process cycles in G containing an edge crossed by one of its adjacent edges an odd number of times one by one until no such cycle exists. Let C denote a cycle of G. By local redrawings at the vertices of C we obtain a drawing of G, where every edge of C crosses every other edge an even number of times. Let v denote a vertex of C.

First, suppose that every edge incident to v and starting inside of C crosses every edge incident to v and starting outside of C an even number of times. In this case we perform at most two subsequent vertex splits. If there exists at least two edges starting at v inside (resp. outside) of C, we split v into two vertices v' and v'' joined by a very short crossing free edge so that v' is incident to the neighbors of v formerly joined with v by edges starting inside (resp. outside) of C, and v''

is incident to the rest of the neighbors of v. Thus, v'' replaces v on C. Notice that by splitting we maintain the property of the drawing to be independently even, and the property of our graph to be three-connected. Moreover, all the edges incident to the resulting vertex v'' of degree three or four cross one another an even number of times. Hence, no edge of C will ever be crossed by another edge an odd number of times, after we apply appropriate vertex splits at every vertex of C.

Second, we show that there does not exist a vertex v incident to C so that an edge vu starting inside of C crosses an edge vw starting outside of C an odd number of times. Since G is a subdivision of a vertex three-connected graph, there exist two distinct vertices u' and w' of C different from v such that u' and w', resp., is connected with u and w by a path internally disjoint from C. Let uP_1u' and wP_2w', resp., denote this path. Note that u can coincide with u' and w can coincide with w'. Let vP_3u' denote the path contained in C no passing through w'. Let C' denote the cycle obtained by concatenation of P_1, P_3, and vu. Let C'' denote the cycle obtained by concatenating P_2 and the portion of C between w' and v not containing u'. Since vw and vu cross an odd number of times and all the other pairs of edges $e \in E(C')$ and $f \in E(C'')$ cross an even number of times, the edges of C' and C'' cross an odd number of times. It follows that their corresponding curves cross an odd number of times (contradiction).

Notice that by vertex splits we decrease the value of the function $\sum_{v \in V(G)} deg^3(v)$ whose value is always non-negative. Hence, after a finite number of vertex splits we turn G into an even drawing of a new graph G'. □

We turn to the actual proof of Theorem 3.
We apply Lemma 3 to the graph G thereby obtaining a clustered graph (G', T'), where each vertex obtained by a vertex split, belongs to the cluster of its parental vertex and the membership of other vertices to clusters is unchanged. By applying Theorem 1 to (G', T') we obtain a clustered embedding of (G', T'). Finally, we contract the pairs of vertices obtained by vertex splits in order to obtain a clustered embedding of (G, T).

4 Monotone Variant of the Weak Hanani–Tutte Theorem

In the present section we derive Theorem 2 from Theorem 1.

Given a graph G with a fixed order of vertices let $\mathcal{D}$ denote its drawing such that x-coordinates of the vertices of G respect their order, edges are drawn as x-monotone curves and every pair of edges cross an even number of times. We turn our drawing $\mathcal{D}$ of G into a clustered drawing $\mathcal{D}'$ of a strip clustered graph (G', T') which is still even.

We divide the plane by vertical lines such that each resulting strip contains exactly one vertex of G in its interior. Let (G, T) denote the clustered graph, in which every cluster consists of a single vertex, such that the clusters are ordered according to x-coordinates of the vertices. Thus, every vertical strip corresponds to a cluster of (G, T). Note that all the edges in the drawing of (G, T) are bounded, and hence, by Lemma 2 can be turned into paths so that the resulting clustered graph is strip clustered, and even drawing clustered. We denote the resulting strip clustered graph by (G', T') and drawing by $\mathcal{D}'$.

Applying Theorem 1 to $\mathcal{D}'$, we obtain an embedding of (G', T') that can be turned into an embedding of (G, T) by converting the subdivided edges in G' back to the edges of G. The obtained embedding is turned into an x-monotone embedding by replacing each edge e with a polygonal path whose bends are intersections of e with vertical lines separating clusters in (G, T).

5 Open Problems

We proved the weak variant of Hanani-Tutte theorem for strip clustered graphs, and verified the corresponding strong variant for three-connected graphs. Naturally, the main open problem we left open is to prove or disprove the strong variant, if the underlying abstract graph G is not a subdivision of a three-connected graph. We find the case, when G is guaranteed to be only two-connected, already quite challenging. A possible approach to prove the strong variant is to adapt the technique of "untangling" pairs of edges crossing an odd number times from [10, Sect. 3]. Another direction for further research would be the weak variant of Conjecture 1 from [9].

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