Unifying Duality Theorems for Width Parameters in Graphs and Matroids (Extended Abstract)

Reinhard Diestel[1(✉)] and Sang-il Oum[2]

[1] Mathematisches Seminar, Universität Hamburg, Hamburg, Germany
R.Diestel@math.uni-hamburg.de
[2] Department of Mathematical Sciences, KAIST, Daejeon, South Korea

Abstract. We prove a general duality theorem for width parameters in combinatorial structures such as graphs and matroids. It implies the classical such theorems for path-width, tree-width, branch-width and rank-width, and gives rise to new width parameters with associated duality theorems. The dense substructures witnessing large width are presented in a unified way akin to tangles, as orientations of separation systems satisfying certain consistency axioms.

1 Introduction

There are a number of theorems in the structure theory of sparse graphs that assert a duality between certain 'dense objects' and an overall tree structure. For example, a graph has small tree-width if and only if it contains no large-order bramble. The aim of this paper is to prove one such theorem in a general setting, a theorem that will imply all the classical duality theorems as special cases, but with a unified and simpler proof. Our theory will give rise to new width parameters as well, with dual 'dense objects', and conversely provide dual tree-like structures for notions of dense objects that have been considered before but for which no duality theorems were known.

Amini, Mazoit, Nisse, and Thomassé [1] have also established a theory of dualities of width parameters, which pursues (and achieves) a similar aim. Our theory differs from theirs in two respects: we allow more general separations of a given ground set than just partitions, including ordinary separations of graphs; and our 'dense objects' are modelled after tangles, while theirs are modelled on brambles. Hence while our main results can both be used to deduce those classical duality theorems for width parameters, they differ in substance. And so do their corollaries for the various width parameters, even if they imply the same

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classical results. Moreover, while the main results of [1] can easily be deduced from ours, the converse seems less clear. And finally, our theory gives rise to duality theorems for new width parameters that can only be expressed in our setup.

All we need in our set-up is that we have a notion of 'separation' for the combinatorial structure to be considered, by which we mean an ordered pair (A, B) of subsets of some ground set V such that $A \cup B = V$.[1] For example, V might be the vertex set of a graph or the ground set of a matroid, and 'separations' would be defined as is usual for graphs and matroids. In order to apply our theorem we may need in addition that there is a submodular function defined on these separations, such as their order, but our main result can be stated without such an assumption.

Our unified treatment of 'dense objects' is gleaned from the notion of tangles in graph minor theory [10], or of ultrafilters in set theory. The idea is as follows. Consider any set S of separations of a given graph or matroid. In order to deserve its name with respect to S, we expect of a 'dense object' that for every separation in S it lies on one side but not the other. For example, if S is the set of all separations (A, B) of a graph G such that $|A \cap B| < k$, then every K_n minor of G with $n \geq k$ will have a branch set in $A \setminus B$ or in $B \setminus A$, but not both. Our dense object $\mathcal{D}$ therefore *orients* every separation in S by choosing exactly one of the two ordered pairs $(A, B), (B, A)$ in such cases,[2] and our paradigm is that this orientation of S is the only information about $\mathcal{D}$ that we ever use. We formalize this by *defining* 'dense objects' as certain orientations of S.

To deserve their name, 'dense objects' cannot be arbitrary orientations of S but have to satisfy some consistency rules. For example, if in a graph G we have two separations $(A, B), (C, D)$ and their inverses in S, and $A \subseteq C$ and $B \supseteq D$, then $\mathcal{D}$ should not orient $\{A, B\}$ towards A by selecting (B, A) and $\{C, D\}$ towards D by selecting (C, D). While this rule will be common to all the 'dense objects' we shall consider, there may be further rules depending on the type of object, so that we can tell them apart. These additional rules will stipulate that the orientation of S given by a dense object $\mathcal{D}$ must not contain certain subsets of S, such as the set $\{(B, A), (C, D)\}$ in the above example. Thus, each type of dense object will be specified by a collection $\mathcal{F}$ of 'forbidden' subsets of S.

The tree-like structure that is dual to a dense object $\mathcal{D}$, i.e., which will exist in a graph or matroid if and only if it contains no instance of $\mathcal{D}$, will be defined by this same collection $\mathcal{F}$ of separation sets forbidden in $\mathcal{D}$. It will typically come as a subset of S that is nested, and which thus cuts up the underlying set in a tree-like way, and the 'stars of separations' by which this tree branches will be *required* to lie in $\mathcal{F}$. Tangles, for example, are defined in this way: with $\mathcal{F}$ the set of all

[1] In fact, we need even less. It would be enough to consider instead of 'separations' any poset with an involution that commutes with its ordering, just as the ordering of separations introduced below satisfies $(A, B) \leq (C, D) \Leftrightarrow (B, A) \geq (D, C)$. It is only for the sake of readability that we are writing this paper in terms of separations, as readers are likely to have graphs or matroids in mind.

[2] Our notational convention will be that we think of (A, B) as pointing towards B.

triples $(A_1, B_1), (A_2, B_2), (A_3, B_3)$ of separations whose 'small' sides A_1, A_2, A_3 cover the entire graph or matroid, and branch decompositions, their dual objects, as nested sets of separations branching at precisely such triples.

The following familiar dualities between dense objects and tree structures can be captured in this way, and their duality theorems will follow from our theorem. For graphs, we can capture path-decompositions and blockages [2], tree-decompositions and brambles [11], branch-decompositions and tangles of graphs [10]. For matroids, our framework captures branch-decompositions and tangles [4,10], as well as matroid tree-decompositions [5] and their dual objects proposed by Amini, Mazoit, Nisse, and Thomassé [1]. Our framework also captures branch-decompositions and tangles of symmetric submodular functions [4,10], which includes branch-width of graphs and matroids, carving-width of graphs [12], and rank-width of graphs [8].

Since blockages and brambles are not defined in terms of orientations of sets of separations, the duality theorems we obtain when we specify S and $\mathcal{F}$ to capture path- or tree-width (of graphs or matroids) will differ from their known duality theorems. But they will be easily interderivable with these. Since S and $\mathcal{F}$ can be chosen in many other ways too, our results also imply dualities for new width parameters.

Our unifying duality theorem comes in three flavours: as *weak*, *strong*, and *general* duality. In this extended abstract we only present the Strong Duality Theorem, along with applications indicating how to derive duality theorems for all the classical width parameters.

2 Terminology and Basic Facts

A *separation* of a set V is a pair (A, B) of subsets such that $A \cup B = V$. Its *inverse* is the separation (B, A). A set S of separations is *symmetric* if $(B, A) \in S$ whenever $(A, B) \in S$, and *antisymmetric* if $(B, A) \notin S$ whenever $(A, B) \in S$. A symmetric set of separations of a set V is a *separation system* on V.

The separation (A, B) is *proper* if $A, B \neq V$, and *improper* otherwise. The separations of V are partially ordered by

$$(A, B) \leq (C, D) :\Leftrightarrow A \subseteq C \text{ and } B \supseteq D.$$

Note that this is equivalent to $(D, C) \leq (B, A)$, and that (A, B) is proper if and only if (A, B) and (B, A) are incomparable with respect to $\leq$.

Informally, we think of (A, B) as *pointing towards* B and *away from* A. Similarly, if $(A, B) \leq (C, D)$, then (A, B) *points towards* (C, D) and (D, C), while (C, D) *points away from* (A, B) and (B, A).

A set S of separations of V is *nested* if each of them is comparable with every other or its inverse. Thus, two nested separations are either comparable, or point towards each other, or point away from each other. Two separations that are not nested are said to *cross*.

A set of separations is a *star* if they point towards each other (Fig. 1). Thus, S is a *star* if $(A, B) \leq (D, C)$ for distinct $(A, B), (C, D) \in S$. In particular, stars

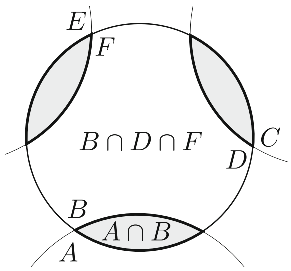

Fig. 1. The separations $(A,B),(C,D),(E,F)$ form a 3-star

are nested. They need not be antisymmetric, but if not they contain an inverse pair $(A,B),(B,A)$, then any other separation they contain must be improper.

Let $\mathcal{F} \subseteq 2^S$ be a collection of sets of separations in S, and $S^- \subseteq S$. An *S-tree over $\mathcal{F}$* and *rooted in S^-* is a pair (T,α) of a tree T with at least one edge and a function $\alpha\colon \vec{E}(T) \to S$ from the set

$$\vec{E}(T) := \{(s,t) : \{s,t\} \in E(T)\}$$

of all *orientations* of edges of T satisfying the following:

(i) For each edge xy of T, if $\alpha(x,y) = (A,B)$ then $\alpha(y,x) = (B,A)$.
(ii) For each internal node t of T, the set $\{\alpha(s,t) : st \in E(T)\}$ is in $\mathcal{F}$.
(iii) For each leaf s of T with neighbour t, say, $\alpha(s,t) \in S^-$.

We say that the separation $\alpha(s,t)$ in (iii) is *associated with*, or simply *at*, the leaf s. The separations at leaves are the *leaf separations* of (T,α).

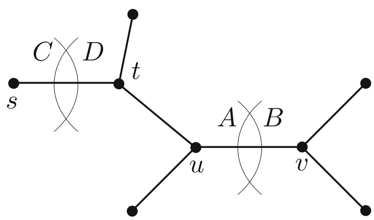

Fig. 2. An S-tree with $(C,D) = \alpha(s,t) \le \alpha(u,v) = (A,B)$

An important example are the S-trees *over stars*: the S-trees over some $\mathcal{F}$ all whose elements are stars of separations. In such an S-tree (T,α) the map α preserves the *natural partial ordering* on $\vec{E}(T)$ defined by letting $(s,t) \le (u,v)$ if the unique $\{s,t\}$–$\{u,v\}$ path in T starts at t and ends at u. Indeed, the images under α of the oriented stars

$$S_t = \{(s,t) : t \text{ an internal node of } T\},$$

which by (ii) are sets in $\mathcal{F}$, are then stars of separations. This means precisely that α preserves the partial ordering on the sets S_t as induced by $\vec{E}(T)$, which in turn easily implies that α preserves the ordering on all of $\vec{E}(T)$ (Fig. 2).

Let S be a separation system. An *orientation* of S is a subset $O \subseteq S$ that contains, for every $(A, B) \in S$, exactly one of (A, B) and (B, A). A *partial orientation* of S is an orientation of some symmetric subset of S.

A set P of separations is *consistent* if it contains no two separations pointing away from each other: if $(C, D) \leq (A, B) \in P$ implies $(D, C) \notin P$.[3] Note that this does not imply $(C, D) \in P$: it may also happen that P contains neither (C, D) nor (D, C). Note that consistent sets of separations are antisymmetric.

If $P \subseteq S$ is consistent, it is clearly a partial orientation of S. Conversely, if P is an orientation of all of S, it is consistent if and only if it is *closed down* in the partial ordering of S, i.e., if and only if $(C, D) \in P$ whenever $(C, D) \leq (A, B) \in P$ and $(C, D) \in S$.

Whenever $P \subseteq O \subseteq S$ we say that P *extends to* O, and O *extends* P.

Proposition 1. *Every consistent partial orientation of a separation system S extends to a consistent orientation of S.*

3 The (Strong) Duality Theorem

Our paradigm is to capture the notion of a 'dense object' $\mathcal{D}$ in a structure on a set V by orientations of suitable separation systems S on V. Here, 'suitable' means that for every separation in S the object $\mathcal{D}$ should 'lie on' one of its sides but not the other, and S should ideally contain all separations of V for which this is the case.

If $\mathcal{D}$ was a concrete subset X of V, for example, such as a set spanning a large complete subgraph in a graph, there would then be a unique orientation O of S that describes $\mathcal{D}$: the set $\{(A, B) \in S : X \subseteq B\}$. What makes the orientations paradigm so attractive, however, is that it is more general than this. For example, a large grid H in a graph G defines a high-order tangle $\mathcal{T}$ – for every small-order separation of G, most of H will lie on one side but not the other – yet the intersection of the 'large sides' B of all the oriented separations $(A, B) \in \mathcal{T}$ will be empty. What the existence of a large grid H in G does imply, however, is that G has no three low-order separations (A_i, B_i) $(i = 1, 2, 3)$ such that $H \subseteq G[A_1] \cup G[A_2] \cup G[A_3]$. So Robertson and Seymour [10] chose this latter property as the defining axiom for a *tangle*.

In this spirit, we seek to define our 'dense objects' as orientations of separation systems S that do not contain certain subsets of S. We say that a partial

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[3] It is a good idea to work with this formal definition of consistency, since the more intuitive notion of 'pointing away from each other' can be counterintuitive. For example, we shall need that no consistent set of separations of V contains a separation of the form (V, A); this follows readily from the formal definition, as $(A, V) \leq (V, A)$, but is less obvious from the informal.

orientation P of a separation system S *avoids* $\mathcal{F} \subseteq 2^S$ if P has no subset in $\mathcal{F}$, i.e., if $2^P \cap \mathcal{F} = \emptyset$.

Before we can state our duality theorem, we have to introduce the somewhat technical notion of 'shifting' an S-tree of a graph G across a separation (X, Y) of G to give, essentially, two S-trees of $G[X]$ and of $G[Y]$. The idea behind this is as follows.

We prove the duality theorem by inverse induction on the size of S^-. Given a separation $\{X, Y\}$ such that $\{(X, Y), (Y, X)\} \subseteq S \setminus S^-$, the induction hypothesis will give us an S-tree (T_Y, α_Y) of G rooted in $S^-_X = S^- \cup \{(X, Y)\}$ (which, if (X, Y) is indeed associated with a leaf and hence certifies that X is small, can be viewed as an S-tree of $G[Y]$), and another S-tree (T_X, α_X) of G rooted in $S^-_Y = S^- \cup \{(Y, X)\}$ (which can be viewed as an S-tree of $G[X]$). We shall then seek to combine these two trees to an S-tree (T, α) of G, rooted in the given S^-.

In order to make the separations associated with the two trees compatible (i.e., nested with each other), we have to regard the separations (of G) in the image of α_X as 'essentially separations of $G[X]$', which we shall do by adding all of Y to one of their sides. Similarly, we add X to one side of every separation in the image of α_Y. The next few paragraphs describe how exactly to do this.

Let $(X, Y) \le (U, W)$ be elements of a set S of separations of a set V. Assume that $U \ne V$, and that (W, U) is associated with a leaf w of an S-tree (T, α) over some set $\mathcal{F} \subseteq 2^S$ of stars. Our aim is to 'shift' (T, α) to a new S-tree (T, α') based on the same tree T, by shifting the separations in the image of α over to X.

Given a separation $(A, B) \le (U, W)$, let us define (Fig. 3, left)

$$f\!\downarrow^{(U,W)}_{(X,Y)} (A, B) := (A \cap X, B \cup Y) \quad \text{and} \quad f\!\downarrow^{(U,W)}_{(X,Y)} (B, A) := (B \cup Y, A \cap X).$$

This defines a *shifting map* $f\!\downarrow^{(U,W)}_{(X,Y)}$ on the set $S_{(U,W)}$ of separations $(A, B) \le (U, W)$ and their inverses. Since (W, U) is a leaf separation of (T, α) and $\mathcal{F}$ consists of stars, the image of α lies in $S_{(U,W)}$ (Fig. 3, right). Hence the concatenation

$$\alpha' := f\!\downarrow^{(U,W)}_{(X,Y)} \circ \alpha$$

is well defined. However it is not clear for now whether α' takes all its images in S.

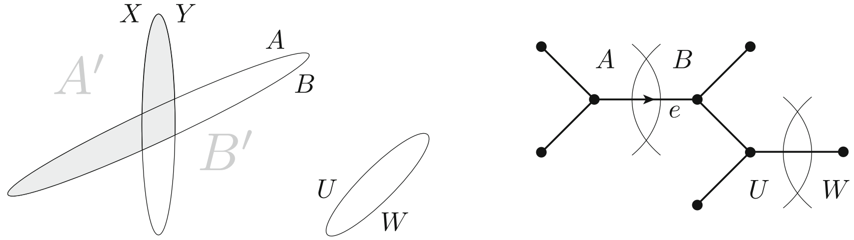

Fig. 3. Shifting $\alpha(\vec{e}) = (A, B)$ to $\alpha'(\vec{e}) = (A', B')$

What is immediate, however, is that $f\!\downarrow^{(U,W)}_{(X,Y)}$ maps stars to stars:

Lemma 1. *The map $f\!\downarrow^{(U,W)}_{(X,Y)}$ preserves the ordering $\le$ of separations.*

Lemma 1 not only implies that $f\!\downarrow^{(U,W)}_{(X,Y)}$ maps stars to stars,[4] it also implies that all leaf separations of (T,α), other than (W,U), get smaller in the transition to α'.[5] Indeed, if $\alpha(s,t) = (A,B)$ with $s \neq w$ a leaf of T, then $(A,B) \le (U,W)$ and hence

$$f\!\downarrow^{(U,W)}_{(X,Y)}(A,B) = (A\cap X, B\cup Y) \le (A,B).$$

It remains to ensure that α' takes its image in S if α does. The following condition on S will ensure the existence of a separation (X,Y) for which this is the case. Let us say that $(X,Y)\in S$ is *linked* to $(U,W)\in S$ if $(X,Y)\le(U,W)$ and

$$(A\cap X, B\cup Y)\in S$$

for all $(A,B)\in S$ with $(A,B)\le(U,W)$. Let us call S *separable* if for every pair $(W',U')\le(U,W)$ of separations in S there exists $(X,Y)\in S$ such that (X,Y) is linked to (U,W) and (Y,X) is linked to (U',W').

Finally, we need a condition on $\mathcal{F}$ to ensure that the shifts of stars that occur as images under α of oriented stars at nodes of T are not only again stars but are also again in $\mathcal{F}$ (see Footnote 4). Let us say that a separation $(X,Y)\in S$ is *$\mathcal{F}$-linked* to $(U,W)\in S$ with $U\neq V$ if (X,Y) is linked to (U,W) and the image under $f\!\downarrow^{(U,W)}_{(X,Y)}$ of any star $S'\subseteq S_{(U,W)}$ in $\mathcal{F}$ that contains a separation (A,B) with $(B,A)\le(U,W)$ is again in $\mathcal{F}$. We say that S is *$\mathcal{F}$-separable* if for every pair $(W',U')\le(U,W)$ of separations in S, with $U,U'\neq V$, there exists $(X,Y)\in S$ such that (X,Y) is $\mathcal{F}$-linked to (U,W) and (Y,X) is $\mathcal{F}$-linked to (U',W'). And a set $\mathcal{F}$ of stars in S is *closed under shifting* if whenever $(X,Y)\in S$ is linked to $(U,W)\in S$ with $U\neq V$ it is even $\mathcal{F}$-linked to (U,W).

The following observation is immediate from the definitions:

Lemma 2. *If S is separable and $\mathcal{F}$ is closed under shifting, then S is $\mathcal{F}$-separable.*

In Sect. 4 we shall see that for all sets $\mathcal{F}$ describing classical 'dense objects', such as tangles and brambles (as well as many others), the usual separation systems S are $\mathcal{F}$-separable. In many cases, $\mathcal{F}$ will even be closed under shifting, in which case we will simply prove this stronger property.

Theorem 1 (Strong Duality Theorem). *Let S be a separation system of a set V, and let $\mathcal{F}\subseteq 2^S$ be a set of stars. Let S^- be a down-closed subset of S containing all its separations of the form (A,V). If S is $\mathcal{F}$-separable, then exactly one of the following holds:*

(i) *There exists an S-tree over $\mathcal{F}$ rooted in S^-.*
(ii) *There exists a consistent $\mathcal{F}$-avoiding orientation of S extending S^-.*

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[4] This will help us show that (T,α') is over $\mathcal{F}$ if (T,α) is.
[5] This will help us show that (T,α') is rooted in S^- if (T,α) is.

Thus, in order to derive from Theorem 1 a specific duality theorem for some given width parameter, it remains to do two things: to specify the $\mathcal{F}$ that describes this parameter, and then to show that the set S of separations we are considering is $\mathcal{F}$-separable.

We shall do this in the next section for some standard examples.

4 Applications of Strong Duality

In this section we show that the separation systems usually considered for graphs and matroids are all separable, and that the collections $\mathcal{F}$ needed to capture 'dense objects' such as tangles, brambles and blockages are closed under shifting. This will make our strong duality theorem imply the classical duality theorems for graphs and matroids. We also obtain some interesting new such theorems.

Let us call a separation system a *universe* if for any two of its separations (A, B) and (C, D) it also contains $(A \cap C, B \cup D)$. For instance, the set of all partitions of the ground set of a matroid is a universe, and so is the set of all vertex separations of a graph (which does not normally include all its vertex partitions).

We shall call a real function $(A, B) \mapsto |A, B|$ on a universe $\mathcal{U}$ an *order function* if it is symmetric and submodular, that is, if $|A, B| = |B, A|$ and

$$|A \cap C, B \cup D| + |A \cup C, B \cap D| \leq |A, B| + |C, D|$$

for all $(A, B), (C, D) \in \mathcal{U}$. We then call $|A, B|$ the *order* of the separation (A, B). Given a universe $\mathcal{U}$ with an order function, our focus will often be on the subsystem

$$S_k = \{(A, B) \in \mathcal{U} : |A, B| < k\}$$

for some positive integer k.

Lemma 3. *Every such S_k is separable.*

For the remainder of this section, whenever we consider a graph $G = (V, E)$ we let $\mathcal{U}$ be its universe of *vertex separations*, the set of pairs (A, B) of vertex sets A, B such that $A \cup B = V$ and G has no edge between $A \setminus B$ and $B \setminus A$. We then take $|A, B| := |A \cap B|$ as our order function for $\mathcal{U}$, and put

$$S_k^- := \big\{(A, B) \in S_k : |A| < k\big\}.$$

This is obviously closed down in S_k, and S_k is separable by Lemma 3.

We remark that any consistent orientation O of S_k must extend the subset of S_k^- consisting of its separations of the form (A, V). This is because otherwise O would contain (V, A), with $(A, V) \leq (V, A) \in O$ violating consistency.

4.1 Branch-Width and Tangles

Let $G = (V, E)$ be a finite graph. A *tangle of order* k in G is (easily seen to be equivalent to) an $\mathcal{F}$-avoiding orientation of S_k extending S_k^- for

$$\mathcal{F} := \big\{\{(A_1, B_1), (A_2, B_2), (A_3, B_3)\} \subseteq S_k : G[A_1] \cup G[A_2] \cup G[A_3] = G\big\}.$$

(The three separations $(A_1, B_1), (A_2, B_2), (A_3, B_3)$ need not be distinct.) Notice that any orientation of S_k that avoids this $\mathcal{F}$ is consistent, since for any pair of separations $(C, D) \le (A, B)$ we have $G[D] \cup G[A] \supseteq G[B] \cup G[A] = G$ and hence $\{(D, C), (A, B)\} \in \mathcal{F}$.

Since our duality theorems, so far, only work with $\mathcal{F}$ consisting of stars of separations, let us consider the set $\mathcal{F}^*$ of those sets in $\mathcal{F}$ that are stars. Using submodularity one can easily show that an $\mathcal{F}^*$-avoiding orientation of S_k in fact avoids all of $\mathcal{F}$ – but only if we assume consistency:

Lemma 4. *Every consistent $\mathcal{F}^*$-avoiding orientation O of S_k avoids $\mathcal{F}$.*

It is easy to check that $\mathcal{F}^*$ is closed under shifting, and so we have our first application:

Theorem 2. *The following are equivalent for finite graphs $G \neq \emptyset$ and $k > 0$:*

(i) *G has a tangle of order k.*
(ii) *S_k has an $\mathcal{F}$-avoiding orientation extending S_k^-.*
(iii) *S_k has a consistent $\mathcal{F}^*$-avoiding orientation extending S_k^-.*
(iv) *G has no S_k-tree over $\mathcal{F}^*$ rooted in S_k^-.*
(v) *G has branch-width at least k, or $k \le 2$ and G is a disjoint union of stars and isolated vertices and has at least one edge.*

4.2 Tree-Width and Path-Width

We now apply our strong duality theorem to obtain a duality theorem for tree-width in graphs. Its dual 'dense objects' will be orientations of S_k, like tangles, and thus different from brambles (or 'screens'), the dual objects in the classical tree-width duality theorem of Seymour and Thomas [11].

This latter theorem, which ours easily implies, says that a finite graph either has a tree-decomposition of width less than $k-1$ or a bramble of order at least k, but not both. The original proof of this theorem is as mysterious as the result is beautiful. The shortest known proof is given in [3] (where we refer the reader also for definitions), but it is hardly less mysterious. A more natural, if slightly longer, proof was given recently by Mazoit [7]. The proof by our strong duality theorem, as outlined below, is perhaps the most basic proof one can have.[6]

Consider a finite graph $G = (V, E)$, with sets of vertex separations $S_k^- \subseteq S_k$ for some integer $k > 0$ as defined at the start of Sect. 4. Let

$$\mathcal{F}_k := \big\{\, S \subseteq S_k \mid S = \{(A_i, B_i) : i = 0, \dots, n\} \text{ is a star with } \big|\textstyle\bigcap_{i=0}^n B_i\big| < k \,\big\}.$$

[6] For example, we do not need Menger's theorem, as all the other proofs do.

We have seen that S_k is separable (Lemma 3). To apply Theorem 1 we thus only need the following lemma – whose very easy proof contains the only bit of magic now left in the tree-width duality theorem:

Lemma 5. *$\mathcal{F}_k$ is closed under shifting.*

It is easy to check that G has an S_k-tree (T, α) over $\mathcal{F}_k$ rooted in S_k^- if and only if it has a tree-decomposition $(T, \mathcal{V})$ of width less than $k-1$. The translation between orientations of S_k and brambles in a graph G is more interesting. Before the notion of a bramble was introduced in [11] (under the name of 'screen'), Robertson and Seymour had looked for an object dual to small tree-width that was more akin to our orientations of S_k: maps β assigning to every set X of fewer than k vertices one component of $G - X$. The question was how to make these choices consistent, so that they would define the desired 'dense object' dual to small tree-width. The obvious consistency requirement, that $\beta(Y) \subseteq \beta(X)$ whenever $X \subseteq Y$, is easily seen to be too weak, while asking that $\beta(X) \cap \beta(Y) \neq \emptyset$ for all X, Y turned out to be too strong. In [11], Seymour and Thomas then found a requirement that worked: that any two such sets, $\beta(X)$ and $\beta(Y)$, should *touch*: that either they share a vertex or G has an edge between them. Such maps β are now called *havens*, and it is easy to show that G admits a haven of *order* k (one defined on all sets X of less than k vertices) if and only if G has a bramble of order at least k.

The notion of 'touching' was perhaps elusive because it appeals directly to the structure of G, its edges: it is not be phrased purely in terms of set containment. It turns out, however, that it can be phrased in such terms after all, as the consistency of orientations of S_k:

Lemma 6. *G has a bramble of order at least k if and only if S_k has a consistent $\mathcal{F}_k$-avoiding orientation extending S_k^-.*

The next application of our duality theorem includes the tree-width duality theorem of Seymour and Thomas [11], and extends it by the new width parameter of S_k-trees over $\mathcal{F}_k$:

Theorem 3. *The following are equivalent for all finite graphs G and $k > 0$:*

(i) *G has a bramble of order at least k.*
(ii) *S_k has a consistent $\mathcal{F}_k$-avoiding orientation extending S_k^-.*
(iii) *G has no S_k-tree over $\mathcal{F}_k$ rooted in S_k^-.*
(iv) *G has tree-width at least $k - 1$.*

We can also bound the adhesion of a tree-decomposition independently from its width. For integers $k < w$, setting

$$\mathcal{F}_w = \big\{ S \subseteq S_k \mid S = \{(A_i, B_i) : i = 0, \dots, n\} \text{ is a star with } \big|\textstyle\bigcap_{i=0}^n B_i\big| < w \big\}$$

yields the following new duality theorem:

Theorem 4. *The following are equivalent for all graphs G and $w \geq k > 0$:*

(i) *S_k has a consistent $\mathcal{F}_w$-avoiding orientation extending S_k^-.*
(ii) *G has no S_k-tree over $\mathcal{F}_w$ rooted in S_k^-.*
(iii) *G has no tree-decomposition of width $< w - 1$ and adhesion $< k$.*

Similarly, setting

$$\mathcal{F}_k^{(2)} := \{\, S \subseteq S_k \mid S = \{(A_1, B_1), (A_2, B_2)\} \text{ is a star with } \big|B_1 \cap B_2\big| < k \,\}$$

yields a duality theorem for path-width:

Theorem 5. *The following are equivalent for all finite graphs G and $k > 0$:*

(i) *S_k has a consistent $\mathcal{F}_k^{(2)}$-avoiding orientation extending S_k^-.*
(ii) *G has no S_k-tree over $\mathcal{F}_k^{(2)}$ rooted in S_k^-.*
(iii) *G has path-width at least $k - 1$.*

4.3 Carving Width, Rank Width, and Matroid Tangles

The concepts of branch-width and tangles were introduced by Robertson and Seymour [10] not only for graphs but more generally for hypergraphs. As the order of a separation (A, B) they already considered, instead of $|A \cap B|$, also arbitrary symmetric submodular order functions $|A, B|$ and proved the relevant lemmas more generally for these. Geelen, Gerards, Robertson, and Whittle [4] applied this explicitly to a submodular connectivity function.

Our aim in this section is to derive from Theorem 1 a duality theorem for branch-width and tangles in arbitrary separation universes with an order function, as introduced at the start of Sect. 4. This will imply the above branch-width duality theorems for hypergraphs and matroids, as well as their cousins for carving width [12] and rank-width of graphs [8].

Let $\mathcal{U}$ be any universe of separations of some set E of at least two elements, with an order function $(A, B) \mapsto |A, B|$. Let $k > 0$ be an integer, and consider

$$S_k = \big\{(A, B) \in \mathcal{U} : |A, B| < k\big\} \quad \text{and} \quad S_k^- = \big\{(A, B) \in S_k : |A| \leq 1\big\}.$$

Let us call an orientation of S_k a *tangle of order k* if it extends S_k^- and avoids

$$\mathcal{F} = \big\{\{(A_1, B_1), (A_2, B_2), (A_3, B_3)\} \subseteq S_k : A_1 \cup A_2 \cup A_3 = E\big\},$$

where $(A_1, B_1), (A_2, B_2), (A_3, B_3)$ need not be distinct; in particular, tangles are consistent. This extends the existing notions of tangles for hypergraphs and matroids, with their edge set or ground set as E, partitions as separations, and the appropriate order functions.

Let $\mathcal{F}^* \subseteq \mathcal{F}$ be the set of stars in $\mathcal{F}$. It is easy to prove that $\mathcal{F}^*$ is closed under shifting, and we have the following analogue of Lemma 4:

Lemma 7. *Every consistent $\mathcal{F}^*$-avoiding orientation of S_k avoids $\mathcal{F}$.*

Let us say that $\mathcal{U}$ has *branch-width* $< k$ if there exists an S_k-tree over $\mathcal{F}^*$ that is rooted in S_k^-. As before, this definition agrees with the usual ones when $\mathcal{U}$ is a hypergraph or matroid. By Lemmas 3 and 7, Theorem 1 now specializes as follows:

Theorem 6. *Given a separation universe $\mathcal{U}$ with an order function, and $k > 0$, the following assertions are equivalent:*

(i) *$\mathcal{U}$ has a tangle of order k.*
(ii) *S_k has a consistent $\mathcal{F}^*$-avoiding orientation extending S_k^-.*
(iii) *$\mathcal{U}$ does not have branch-width $< k$.*

4.4 Matroid Tree-Width

Hliněný and Whittle [5,6] generalized the notion of tree-width from graphs to matroids.[7] Our aim in this section is to specialize our strong duality theorem to a duality theorem for tree-width in matroids.

Let $M = (E, I)$ be a matroid with rank function r. Its *connectivity function* is defined as

$$\lambda(X) := r(X) + r(E \setminus X) - r(M).$$

We consider the universe $\mathcal{U}$ of all bipartitions (X, Y) of E. Since

$$|X, Y| := \lambda(X) = \lambda(Y)$$

is submodular and symmetric, it is an order function on $\mathcal{U}$.

A *tree-decomposition* of M is a pair (T, τ), where T is a tree and $\tau\colon E \to V(T)$ is any map. Let t be a node of T, and let $T_1, \dots, T_d$ be the components of $T - t$. Then the *width* of t is the number

$$\sum_{i=1}^{d} r(E \setminus F_i) - (d-1)\, r(M),$$

where $F_i = f^{-1}(V(T_i))$. (If t is the only node of T, we let its width be $r(M)$.) The *width* of (T, τ) is the maximum width of the nodes of T. The *tree-width* of M is the minimum width over all tree-decompositions of M.

Matroid tree-width generalizes the tree-width of graphs in the expected way:

Theorem 7 (Hliněný and Whittle [5,6]). *The tree-width of a finite graph containing at least one edge equals the tree-width of its cycle matroid.*

In order to specialize Theorem 1 to a duality theorem for tree-width in matroids, we consider

$$S_k = \{\, (A, B) \in \mathcal{U} : |A, B| < k \,\} \quad \text{and} \quad S_k^- = \{\, (A, B) \in \mathcal{U} : r(A) < k \,\} \subseteq S_k.$$

[7] In our matroid terminology we follow Oxley [9].

Since λ is symmetric and submodular, S_k is separable by Lemma 3. Let

$$\mathcal{F}_k := \Big\{ S \subseteq \mathcal{U} \;\Big|\; S = \{(A_i, B_i) : i = 0, \ldots, n\} \text{ is a star with } n \geq 1 \text{ and } \sum_{i=0}^{n} r(B_i) - n\, r(M) < k \Big\}.$$

Even without requiring this in the definition, one can show that every $S \in \mathcal{F}_k$ is a subset of S_k, and that S_k is $\mathcal{F}_k$-separable.

Theorem 1 now yields the following duality theorem for matroid tree-width.

Theorem 8. *Let $M = (E, I)$ be a matroid with the rank function r, and let k be an integer. Then the following statements are equivalent:*

(i) *M has tree-width at least k.*
(ii) *M has no S_k-tree over $\mathcal{F}_k$ rooted in S_k^-.*
(iii) *S_k has a consistent $\mathcal{F}_k$-avoiding orientation extending S_k^-.*

5 Algorithms

We have not considered the algorithmic task of finding, given fixed types of S, S^- and $\mathcal{F}$, for any input graph G the right one of the two alternatives from our duality theorem: an S-tree over $\mathcal{F}$ rooted in S^-, or a consistent $\mathcal{F}$-avoiding orientation of S extending S^-. As far as we are aware, such algorithmic results do not even exist for the classical duality theorems, such as the one for tree-width and brambles. Our setup makes it possible to treat this in greater generality, which seems like a fitting challenge for this conference's academic environment.

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