Chapter 6
Lebesgue Measure of Recurrent Scrambled Sets

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Abstract It was proved by M. Babilonová-Štefánková (Int J Bifurc Chaos 13(7):1695–1700, 2003) that each bitransitive continuous map f of the interval is conjugated to a map g which is distributionally chaotic with a distributionally scrambled set D. The goal of this chapter is to improve this result, by showing that D is formed by points that are recurrent but not almost periodic. Moreover, as a main result it will be proved that any bitransitive map $f \in C(I, I)$ is topologically conjugate to a map $g \in C(I, I)$ which satisfies the following conditions: (i) g is extremally Li–Yorke chaotic with Li–Yorke scrambled set S with full Lebesgue measure and $S \subset R(g)\setminus A(g)$, (ii) g is ω chaotic and every ω scrambled set Ω has zero Lebesgue measure and $\Omega \subset R(g)\setminus A(g)$, (iii) g is distributionally chaotic with a distributionally scrambled set D with full Lebesgue measure and $D \subset R(g)\setminus A(g)$.

6.1 Introduction

Within the last 40 years numerous papers and books have been devoted to the research of discrete dynamical systems. The main aim of the theory of discrete dynamical systems is focused the understanding of what the trajectories of all points from the state space look like. Mostly the periodic structures and asymptotic properties of the orbit were studied. Many authors were fascinated by those motions which are not only periodic but also are not quasiperiodic. These movings were assumed to be *unpredictable* or *sensitive to initial conditions*, later named as *chaotic*.

The first and crucial development in the field of dynamical systems was made by H. Poincaré in 1890 [25] by *recurrence*, that is a point returns to itself arbitrarily close under the actions (iterations), or equivalently the point belongs to its omega limit set. Hence, a dynamical system preserves volume, all trajectories return arbitrarily close to their initial position and they do this an infinite number of times. More precisely, H. Poincaré discovered: *If a flow preserves volume and has only bounded orbits then for each open set there are orbits that intersect the set infinitely often*.

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As a consequence of the orbit observation situations where the trajectory is dense in the state space appear. Such a property is called *transitivity* and could be defined equivalently for action F (under some assumptions of the state space): for any two nonempty open sets U, V, that are subsets of the state space, there is $n \in \mathbb{N}$ such that $F^n(U) \cap V \neq \emptyset$. The notion of transitivity was introduced by G. D. Birkhoff in 1920 for flows [5].

Consequently, a transitive dynamical system has points which eventually move under iteration from one arbitrarily small open set to any other. Such a dynamical system cannot be decomposed into two disjointed sets with nonempty interiors which do not interact under the transformation. So, the notion of transitivity is still too rough for the observation of a local dynamics and moreover it is not possible to quantify (and compare) the complexity of systems. The *topological entropy* (defined by R. L. Adler, A. G. Konheim, and M. H. McAndrew in 1965 [1]) measures the complexity of the dynamical system. Later on the notion of topological entropy was equivalently formulated for compact metric spaces by R. Bowen in 1971 [9].

The periodic structure for continuous maps on the interval was also investigated and the crucial and well-known result on periodic structure was proved by A. N. Sharkovskiĭ [27] in 1965 where *chaos* is due to infinitely many repulsive cycles of increasing period, and their limit sets of increasing “classes” in the sense of C. P. Pulkin [26] form 1950.

There appeared many notions of chaos, starting with the famous paper by T. Y. Li and J. Yorke [19] in 1975. Later on, several notions of chaos motivated by diverse aspects were introduced (for more see, e.g., [6, 11] and references therein). Many natural questions arose. Which notion of chaos is the best one, or stronger than others (see, e.g., [14] and references therein)? Or, if the map is chaotic in some sense, how big is the scrambled set (in the sense of Lebesgue measure or Baire category, see, e.g., [6])? Motivated by these questions, the goal in this chapter is to study scrambled sets of Li–Yorke chaos, ω chaos, and distributional chaos for continuous maps on the interval. For this purpose, the following conventions are recalled.

Let (X, d) be a compact metric space with metric d and $C(X, X)$ the set of all continuous maps $f : X \to X$. Let $f \in C(X, X)$, $x \in X$ and n be a positive integer. The nth iteration of x under f is denoted by f^n, the set of all fixed points of f by $\mathrm{Fix}(f)$, the set of periodic points of f by $Per(f)$. The sequence $\{f^n(x)\}_{n=0}^{\infty}$ is the trajectory of x, and the set $\omega_f(x)$ of all limit points of the trajectory is the *ω-limit set* of x. An ω-limit set is *maximal* if it is not properly contained in any other ω-limit set. A point $x \in X$ is said to be *recurrent* for f if $x \in \omega_f(x)$, this means that for each neighborhood U of the point x there is positive n such that $f^n(x) \in U$. A point $x \in X$ is called an *almost periodic* point of f provided that for any neighborhood U of $x \in X$, there exists $N \in \mathbb{N}$ such that $\{f^{n+i}(x) : \ i = 0, 1, 2, \dots, N\} \cap U \neq \emptyset$ for all $n \in \mathbb{N}$. $R(f)$ denotes the set of all recurrent points of $f \in C(X, X)$ and $\mathrm{A}(f)$ the set of all almost periodic points of $f \in C(X, X)$ (for more about recurrence see, e.g., [12] or [29]). A map $f \in C(X, X)$ is *conjugate* to $g \in C(X, X)$ if there is a bijective map $h \in C(X, X)$ such that $h \circ f = g \circ h$. Let $[0, 1]$ be the closed unite interval I.

It is worthy of noticing that results of this chapter, and references, stand for a “macroscopic” point of view, that correspond to what can be either called as *stable* *chaos* (*strange attractors* or *invariant bounded sets*), or *chaotic transients* in other disciplines, see, e.g., [21] and [22].

In this chapter notions of Li and Yorke chaos, ω chaos, and distributional chaos are researched. These notions of chaos were introduced by T. Y. Li and J. Yorke [19], S. Li [18], and B. Schweizer and J. Smítal [28] in 1975, 1993, and 1994, respectively, which are defined as follows.

The map $f \in C(X, X)$ is *Li and Yorke chaotic* (briefly, *LYC*) if there is an uncountable set $S \subset X$ such that for any two different points x, y is

$$\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0$$

and

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0.$$

This set is called an *LY-scrambled set*. Moreover, $f \in C(X, X)$ is *extremally LYC* if $\limsup_{n\to\infty} d(f^n(x), f^n(y)) = \mathrm{diam}(X)$.

The map $f \in C(X, X)$ is *ω-chaotic* (briefly, *ωC*) if there is an uncountable set $\Omega \subset X$ such that for any two different points x, y holds:

1. $\omega_f(x)\backslash\omega_f(y)$ is uncountable,
2. $\omega_f(x) \cap \omega_f(y) \neq \emptyset$ and
3. $\omega_f(x)\backslash Per(f) \neq \emptyset$.

The set Ω is called *ω-scrambled*. Remember that the third condition from the definition of ωC is not needed if $X = I$ (see, e.g., [18] and for more about ωC see [17]).

For $f \in C(X, X)$, $x, y \in X$, $t \in \mathbb{R}$ and a positive integer n, let

$$\xi(x, y, n, t) = \sharp\{i;\ 0 \le i < n \text{ and } |f^i(x) - f^i(y)| < t\}.$$

Put

$$F^*_{xy}(t) = \limsup_{n\to\infty} \frac{1}{n}\sharp\xi(x, y, n, t),$$

and

$$F_{xy}(t) = \liminf_{n\to\infty} \frac{1}{n}\sharp\xi(x, y, n, t).$$

Then both F^*_{xy} and F_{xy} are nondecreasing maps, with $0 \le F_{xy} \le F^*_{xy} \le 1$, $F^*_{xy}(t) = 0$ for $t < 0$, and $F_{xy}(t) = 1$ for $t > 1$. F^*_{xy} and F_{xy} are referred to the *upper* and *lower distribution map* of x and y, respectively. The map f is *distributionally chaotic* (briefly, *dC*) if there is a set $S \subset X$ containing at least two points such that, for any $x \neq y$ in S, $F_{xy} < F^*_{xy}$ (by this it is meant that $F_{xy}(t) < F^*_{xy}(t)$ for all t in an interval), this set is called a *d-scrambled set* for the map f. There were stronger forms of distributional chaos introduced in [4] (for progress paper comparing dC and LYC see [30] and references therein).

A pair of points (x, y), $x, y \in I$, is called *isotectic* if for every positive integer n the ω-limit set $\omega_{f^n}(x)$ and $\omega_{f^n}(y)$ are subsets of the same maximal ω-limit set of f^n.

The *spectrum* of $f \in C(I, I)$, denoted by $\Sigma(f)$, is the set of minimal elements of the set $\{F_{xy};\ (x, y) \text{ is isotectic}\}$. By [10], $\Sigma(f)$ is always a finite nonempty set, and if f is bitransitive then $\Sigma(f)$ is singleton.

In the following section the result by M. Babilonová-Štefánková from [3] will be improved and the analogous results by M. Babilonová from [2] and M. Lampart from [15] will be summarized. The main result of the paper was announced (without proof) by M. Lampart in [16]. Here complete proofs will be given.

As main result it will be proved that any bitransitive map $f \in C(I, I)$ is topologically conjugate to a map $g \in C(I, I)$ which satisfies the following conditions:

i. g is extremally LYC with LY-scrambled set S with full Lebesgue measure and $S \subset R(g)\setminus \mathrm{A}(g)$,
ii. g is ωC and every ω-scrambled set Ω has zero Lebesgue measure and $\Omega \subset R(g)\setminus \mathrm{A}(g)$,
iii. g is dC with d-scrambled set D with full Lebesgue measure and $D \subset R(g)\setminus \mathrm{A}(g)$.

6.2 Properties of Bitransitive Maps

A map $f \in C(I, I)$ is *(topologically) transitive* if for any open intervals $U, V \subset I$ there is a positive integer n such that $f^n(U) \cup V \neq \emptyset$; f is *bitransitive* if f^2 is transitive.

Proposition 6.1 **([7] or [10])**. *Let $f \in C(I, I)$ be a bitransitive map, and $J, K \subset (0, 1)$ compact intervals. Then $f^n(J) \supset K$, for any sufficiently large n.*

Let $f \in C(I, I)$, $A, B \subset I$. It is said that f*−approximates* A if, for any $\varepsilon > 0$ and $\mu \in (0, 1)$, there is a $K > 0$ such that, for any $x \in A$ and any integer $m > K$ there is a $y \in B$ with $\sharp\{i, 0 \leq i \leq m$ and $|f^i(x) - f^i(y)| < \varepsilon\} > \mu m$.

Proposition 6.2 **([3])**. *Let $f \in C(I, I)$ be bitransitive. Then $Per(f)\setminus\{0, 1\}$ f-approximates $Per(f)$.*

Proposition 6.3 **([3])**. *Let $f \in C(I, I)$ and $A \subset Per(f)$. Then there is a countable set $B \subset A$ which f−approximates A.*

Proposition 6.4 **([28])**. *If $f \in C(I, I)$ is bitransitive then $\Sigma(f) = F$, i.e., the spectrum of f is a singleton.*

Proposition 6.5 **([3])**. *Let $f \in C(I, I)$ and $\Sigma(f) = F$. If $A \subset I$ f−approximates I then F is the pointwise infimum of $\{F_{xy};\ x, y \in A\}$.*

Formulas (6.2), (6.3), and (6.4) from the following Lemma 6.1 correspond with those in Lemma 3.1 from [3]. Conclusion *i.* from Theorem 6.1 equates with the property of Theorem 3.2 from [3]. These formulas are recalled for completeness since they are needed as well for the proof of new parts of Lemma 6.1 and Theorem 6.1 as entire proof of the main statement, Theorem 6.2.

For simplicity the following special notation will be used. If a sequence α is a subsequence of β, it is denoted $\alpha \prec \beta$; sequences may be finite or infinite. So it can be written, e.g., $\{a_n\}_{n=1}^{\infty} \prec \{b_n\}_{n=1}^{\infty}$.

Throughout this section, $\{A_n\}_{n=1}^{\infty}$ is a fixed sequence of blocks of positive integers determined by a division of the sequence $\{n\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} \sum_{i=1}^{n-1} \frac{|A_i|}{|A_n|} = 0, \tag{6.1}$$

where $|A_k|$ denotes the number of elements of A_k. Let us denote by m_k the first element of A_k.

Lemma 6.1 *Let $f \in C(I, I)$ be bitransitive, $\{A_n\}_{n=1}^{\infty}$ a sequence satisfying (6.1), $X \subset (0, 1)$ a nonempty countable set, and $\{p_n\}_{n=1}^{\infty}$ a sequence in X containing any member of X infinitely many times, and let $\{r_n\}_{n=1}^{\infty}$ be a sequence of all rational numbers in I.*

Then, for any compact interval $J \subset (0, 1)$ and any sequence $\alpha = \{a_n\}_{n=1}^{\infty} \prec \{n\}_{n=1}^{\infty}$ there is a nonempty nowhere dense perfect set $P \subset J$, and a sequence $\beta = \{b_n\}_{n=1}^{\infty} \prec \alpha$ with the following properties:

$$\{(b_n, p_n)\}_{n=1}^{\infty} \prec \{(a_n, p_n)\}_{n=1}^{\infty}, \tag{6.2}$$

$$f^k(P) \subset B\left(f^{k-m_{b_n}}(p_n), \frac{1}{n}\right), \ for\ k \in A_{b_n}, \tag{6.3}$$

where $B(x, r)$ stands for $[x - r, x + r]$,

$$F_{xy} \leq \inf\{F_{pq};\ \ p, q \in X\}, \ for\ each\ x, y \in P, x \neq y, \tag{6.4}$$

and

$$\omega_f(x) = I, \ for\ any\ x \in P. \tag{6.5}$$

Proof Let $\{p_n\}_{n=1}^{\infty}$ be a sequence in X such that the sequence $\{(p_n, q_n)\}_{n=1}^{\infty}$ contains every pair of points of X infinitely many times. Let the set P be in the form $P = \bigcap_{n=1}^{\infty} P_n$ where, for any n, P_n is the union of pairwise disjoint compact intervals $U_s, s \in \{0, 1\}^n$, and $P_{n+1} \subset P_n$. The intervals U_s are defined inductively by n.

Stage 1: Let U_0, U_1 be disjoined closed subintervals of J. Put $P_1 = U_0 \cup U_1$, and let $k(1, 0) < k(1, 1)$ be positive integers such that $(k(1, 0), p_1, q_1)$ is a member of a sequence $\{(a_n, p_n, q_n)\}_{n=1}^{\infty}$.

Stage $n + 1$: Sets $P_1, \ldots, P_n$ and positive integers $k(1, 0) < k(1, 1) < k(2, 0) < k(2, 1) < k(2, 2) < \cdots < k(n, 0) < k(n, 1) < \cdots k(n, n)$ are available from stage n such that, for any $s = s_1, \ldots, s_v \in \{0, 1\}^v$, $1 \leq v \leq n$, $1 \leq j \leq v + 1$,

$$\{(k(j, 0), p_j, q_j)\}_{j=1}^{\infty} \prec \{(a_i, p_i, q_i)\}_{i=1}^{\infty}, \tag{6.6}$$

for any $s = s_1, \ldots, s_v \in \{0, 1\}^v$, $1 \leq v \leq n, 0 \leq j \leq v$,

$$|U_s| \le \frac{1}{v}, \tag{6.7}$$

$$f^k(U_s) \subset B\left(f^{k-m_{k(v,j)}}(p_v), \frac{1}{v}\right) \text{ for } k \in A_{k(v,j)}, \text{ if } j = 0 \text{ or } s_j = 0, \tag{6.8}$$

$$f^k(U_s) \subset B\left(f^{k-m_{k(v,j)}}(q_v), \frac{1}{v}\right) \text{ for } k \in A_{k(v,j)}, \text{ if } s_j = 1, \tag{6.9}$$

$$f^{k(v,0)}(U_s) \subset B\left(r_v, \frac{1}{v}\right). \tag{6.10}$$

By Proposition 6.1 there is an integer $k(n+1,0) > k(n,n)$ such that, for any

$$s \in \{0,1\}^n, r_{n+1} \in f^{k(n+1,0)}(U_s).$$

Hence, for any $s \in \{0,1\}^n$ there is a compact interval $V_s \subset U_s$ such that

$$f^{k(n+1,0)}(V_s) \subset B\left(r_{n+1}, \frac{1}{n+1}\right). \tag{6.11}$$

Again by Proposition 6.1 there is an integer $k(n+1,1) > k(n+1,0)$ such that

$$\{(k(j,0), p_j, q_j)\}_{j=1}^{\infty} \prec \{(a_i, p_i, q_i)\}_{i=1}^{\infty}, \tag{6.12}$$

and, for any $s \in \{0,1\}^n$, $p_{n+1} \in f^{m_{k(n+1,0)}}(U_s)$. Hence, for any $s \in \{0,1\}^n$, there is a compact interval $V_s^1 \subset V_s$ such that

$$f^k(V_s^1) \subset B\left(f^{k-m_{k(n+1,0)}}(p_{n+1}), \frac{1}{n+1}\right) \text{ for } k \in A_{k(n+1,0)}. \tag{6.13}$$

Also for any $s \in \{0,1\}^n, r_{n+1} \in f^{t(n+1,0)}(U_s)$. Therefore, for any $s \in \{0,1\}^n$, there is a compact interval $V_s \subset U_s$ such that

$$f^{t(n+1,0)}(V_s) \subset \left(r_{n+1}, \frac{1}{n+1}\right). \tag{6.14}$$

Next, there is an integer $k(n+1,2) > k(n+1,1)$ such that, for any $s \in \{0,1\}^n$, $\{p_{n+1}, q_{n+1}\} \subset f^{m_{k(n+1,1)}}(V_s^1)$. Thus, for any $s = s_1, s_2, \dots, s_n \in \{0,1\}^n$ there is a compact interval $V_s^2 \subset V_s^1$ such that $z \in f^{m_{k(n+1,1)}}(V_s^2)$, where $z = p_{n+1}$ if $s_1 = 0$ and $z = q_{n+1}$ otherwise, and such that $|f^k(V_s^2)| \le 1/(n+1)$ where $k \in A_{k(n+1,1)}$. By applying this process n times, integers $k(n+1,2) < k(n+1,3) < \cdots < k(n+1,n+1)$ and compact intervals $V_s^2 \supset V_s^3 \supset \cdots \supset V_s^{n+1}$ are obtained such that, for any $s = s_1 s_2 \cdots s_n \in \{0,1\}^n$ and any $2 \le j \le n+1$,

$$f^k(V_s^j) \subset B\left(f^{k-m_{k(n+1,j)}}(p_{n+1}), \frac{1}{n+1}\right) \text{ for } k \in A_{k(n+1,j)} \text{ if } s_j = 0, \tag{6.15}$$

and

$$f^k(V_s^j) \subset B\left(f^{k-m_{k(n+1,j)}}(q_{n+1}), \frac{1}{n+1}\right) \text{ for } k \in A_{k(n+1,j)} \text{ if } s_j = 1. \tag{6.16}$$

Finally, let $k(n+1, n+2) > k(n+1, n+1)$ be such that, for any $s \in \{0,1\}^n$, $f^{m_{k(n+1,n+2)}}(V_s^{n+1}) \supset \{p_{n+1}, q_{n+1}\}$. Then there are disjoint compact intervals $U_{s0}, U_{s1} \subset V_s^{n+1}$ such that

$$|U_{s0}|, |U_{s1}| \leq \frac{1}{n+1}, \tag{6.17}$$

$$f^k(U_{s0}) \subset B\left(f^{k-m_{k(n+1,n+2)}}(p_{n+1}), \frac{1}{n+1}\right) \text{ for } k \in A_{k(n+1,n+2)}, \tag{6.18}$$

and

$$f^k(U_{s1}) \subset B\left(f^{k-m_{k(n+1,n+2)}}(q_{n+1}), \frac{1}{n+1}\right) \text{ for } k \in A_{k(n+1,n+2)}. \tag{6.19}$$

Thus sets U_s are defined for any $s \in \{0,1\}^{n+1}$. They satisfy the formulas (6.6), (6.7), (6.8), and (6.9), for $n := n+1$, by (6.12), by (6.17), by (6.13), (6.15), (6.18), or by (6.16), (6.19), respectively. This completes the induction.

For any n put $b(n) = k(n,1)$. Then (6.6) implies (6.2). Let $P = \bigcap_{n=1}^{\infty} \bigcup_{s \in \{0,1\}^n} U_s$. Then P is a nowhere dense perfect set; this follows by (6.7). By (6.8), P satisfies (6.3). It remains to prove (6.4) and (6.5).

Let x and y be distinct points in P. Then for any positive integer K there are $s, s' \in \{0,1\}^K$ such that $x \in U_s, y \in U_{s'}$. Take $K > 1/(2|x-y|)$. Thus, by (6.7), $U_s \cup U_{s'} = \emptyset$ and hence $s \neq s'$. Consequently, $s_r \neq s'_r$ for some r. Without loss of generality it can be assumed that $s_r = 0$ and $s'_r = 1$.

Now, to prove (6.4) it suffices to show that, for any positive integer N, $F_{xy} \leq F_{p_N q_N}$. Let $n > \max\{N, K\}$ be such that $(p_n, q_n) = (p_N, q_N)$. By (6.8) and (6.9) it follows that, for $k \in A_{k(n,r)}, |f^k(x) - f^{k-m_{k(n,r)}}(p_N)| \leq 1/n$, and $|f^k(y) - f^{k-m_{k(n,r)}}(q_N)| \leq 1/n$. Hence, $\{k \in A_{k(n,r)}; |f^k(x) - f^k(y)| < t\} \subset \{k \in A_{k(n,r)}; |f^{k-m_{k(n,r)}}(p_N) - f^{k-m_{k(n,r)}}(q_N)| < t + 2/n\}$, and consequently, by (6.1), since n can be taken arbitrarily large,

$$F_{xy}(t) \leq F_{p_N q_n}(t+\delta),$$

for any $\delta > 0$ and $t \in \mathbb{R}$. Thus, if t is a point of continuity of $F_{p_N q_N}$, then $F_{xy}(t) < F_{p_N q_N}(t)$ which proves (6.4).

Finally, it remains to prove (6.5). Since $(\{r_n\}_{n=1}^{\infty})' = I$ the condition (6.10) is fulfilled, consequently the property (6.5) is satisfied.

6.3 Main Results

Denote the upper and lower distributional map of x and y for a map g by G_{xy} and G^*_{xy}, respectively.

Theorem 6.1 *Any bitransitive map $f \in C(I, I)$ is topologically conjugate to a map $g \in C(I, I)$ which is dC almost everywhere. More precisely, there is a set H with $\lambda(H) = 1$ such that:*

i. for any distinct x and y in H, $G_{xy} = G$ where G is the unique member of $\Sigma(g)$, and $G^*_{xy} \equiv 1$,
ii. $H \subset R(g) \setminus A(g)$.

Proof By Proposition 6.4, f has a one-point spectrum, $\Sigma(f) = \{F\}$, and by Propositions 6.2, 6.3, and 6.5 there is a countable set $X \subset Per(f)\setminus\{0, 1\}$ such that $F = \inf\{F_{pq};\ p, q \in X\}$ (note that the relation of f-approximability is transitive). Let $\{p_n, q_n\}_{n=1}^{\infty}$ be a sequence in X^2 containing any pair of points o X infinitely many times, and $\{A_n\}_{n=1}^{\infty}$ blocks satisfying (6.1).

Firstly, to prove the first condition of Theorem 6.1, it suffices to define an increasing sequence $S_1 \subset S_2 \subset \cdots \subset (0, 1)$ of perfect sets, and a decreasing sequence $\{n\}_{n=1}^{\infty} = \alpha_0 \succ \alpha_1 \succ \alpha_2 \succ \cdots$ of sequences of positive integers with the following properties:

$$f^k(S_m) \subset B\left(f^{k-m_{a_n}}(p_n), \frac{1}{n}\right) \text{ for } k \in A_{a_n}, \tag{6.20}$$

where $\{a_n\}_{n=1}^{\infty}$ is the sequence of α_m,

$$F_{xy} \le \inf\{F_{uv};\ u, v \in X\} \text{ for } x \neq y \text{ in } S_m, \tag{6.21}$$

$$S = \bigcup_{m=1}^{\infty} S_m \text{ is dense and hence } c\text{-dense in I.} \tag{6.22}$$

Indeed, by [13], any c-dense set F_σ is homeomorphic to a set of full Lebesgue measure. So let ϕ be a homeomorphism of I such that $\lambda(\phi(S)) = 1$. Put $g = \phi \circ f \circ \phi^{-1}$ and $H = \phi(S)$. It is easy to see that $\phi(X) = Y \subset Per(g)\setminus\{0, 1\}$ g-approximates I. Hence, by (6.21) and Proposition 6.5, for any m and any $x, y \in H_m = \phi(S_m)$, $G_{xy} = \inf\{G_{pq};\ p, q \in Y\} = G$, where G is the unique distribution function in the spectrum of g. On the other hand, by (6.20),

$$g^k(H_m) \subset B\left(g^{k-m_{a_n}}(\phi(p_n)), \nu_\phi\left(\frac{1}{n}\right)\right) \text{ for } k \in A_{a_n}, \tag{6.23}$$

where ν_ϕ is defined by $\nu_\phi(d) = \sup_{|x-y|\le d} |\phi(x) - \phi(y)|$. This gives $G^*_{xy} \equiv 1$ for any $x, y \in H_m$ and hence for any $x, y \in H$.

Thus, it remains to define S_m and α_m for any m. Apply Lemma 6.1 to $J = [1/3, 2/3]$, and $\alpha = \alpha_0$ to get P and $\beta = \alpha_1$, and put $S_1 = P$.

Now assume by induction that S_m and α_m satisfy (6.20) and (6.21). Let V be the component interval of $I\backslash S_m$ of the maximal length and let $J \subset \{0, 1\}$ be a compact interval containing the center of V. Apply Lemma 6.1 to J, $\alpha = \alpha_m$ and to the sequence $\{q_n\}_{n=1}^{\infty}$ instead of $\{p_n\}_{n=1}^{\infty}$ to get a set P, and a sequence $\beta = \{b_n\}_{n=1}^{\infty}$, and put $S_{m+1} = S_m \cup P$. By (6.2) and (6.20) it derives

$$f^k(S_m) \subset B\left(f^{k-m_{b_n}}(p_n), \frac{1}{n}\right) \text{ for } k \in A_{b_n}, \tag{6.24}$$

and by (6.2) and (6.3)

$$f^k(P) \subset B\left(f^{k-m_{b_n}}(q_n), \frac{1}{n}\right) \text{ for } k \in A_{b_n}. \tag{6.25}$$

Hence, for $x \in S_m$ and $y \in P$, $F_{xy} \leq \inf\{F_{uv}; u, v \in X\}$ and $F^*_{xy} \equiv 1$ (see the final part of the proof of the Lemma 6.1). Thus, (6.21) is true for $m := m + 1$. Finally, let $\{d_n\}_{n=1}^{\infty} \prec \beta$ be such that $p_{d_n} = q_{d_n}$ for any n, and let $\alpha_{m+1} = \{d_n\}_{n=1}^{\infty}$. This implies (6.20) for $m := m + 1$. Finally, condition (6.22) follows as a consequence of properties described above.

Finally, the main result, the following theorem, can be proved by using the main results of [2], [15], and Theorem 6.1.

Theorem 6.2 *Any bitransitive map $f \in C(I, I)$ is topologically conjugate to a map $g \in C(I, I)$ which satisfies the following conditions:*

i. g is extremally LYC with LY-scrambled set S with full Lebesgue measure and $S \subset R(g)\backslash \mathrm{A}(g)$,
ii. g is ωC and every ω-scrambled set Ω has zero Lebesgue measure and $\Omega \subset R(g)\backslash \mathrm{A}(g)$,
iii. g is dC with d-scrambled set D with full Lebesgue measure and $D \subset R(g)\backslash \mathrm{A}(g)$.

As a consequence of Theorem 6.1 and a result by A. M. Blokh [8], that any continuous map of the interval with positive entropy has an iteration which is semiconjugate to a bitransitive map, the next corollary follows.

Corollary 6.1 *Let $f \in C(I, I)$ be a map with positive topological entropy. Then, for some $k \geq 1$, f^k is semiconjugate to a map $g \in C(I, I)$ which satisfies the following conditions:*

i. g is extremally LYC with LY-scrambled set S with full Lebesgue measure and $S \subset R(g)\backslash \mathrm{A}(g)$,
ii. g is ωC and every ω-scrambled set Ω has zero Lebesgue measure and $\Omega \subset R(g)\backslash \mathrm{A}(g)$,
iii. g is dC with d-scrambled set D with full Lebesgue measure and $D \subset R(g)\backslash \mathrm{A}(g)$.

Remark 6.1 It is worthy to note that it is possible to construct scrambled sets using residual relations and Mycielski's theorem [23], that is the scrambled set consists of pairs which are proximal but not asymptotic. Additionally, in a transitive nonminimal system the set of points (or pairs in a weakly mixing system) which are recurrent but not almost periodic is residual (points with dense orbit are residual). Now, applying this to our set adds that additional property to the scrambled set (all points will be recurrent but not almost periodic). Unfortunately, it is not possible to get the properties of ω - limit sets, hence ω chaos directly, which makes our construction essential (for further reading compare with [20] and [24]).

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