Chapter 11
Piecewise Expanding Maps and Conjugacy Equations

Cristina Serpa and Jorge Buescu

Abstract Topological invariants of interval maps are preserved by conjugacy. We investigate some features of the conjugacy equations associated to piecewise expanding maps. For special cases, it is possible to construct explicitly a conjugacy function in terms of the a-base expansion of numbers through a solution of the corresponding functional equations.

One possible interest of conjugacy equations is to simplify the study of a family of maps by considering the simplest possible cases while preserving topological properties. In our case these will be the piecewise linear and expanding interval maps.

The conjugacy equation $h \circ g = f \circ h$ (where h is the unknown function) is the subject of much research in the field of functional equations. The main results already obtained for this kind of equation are those for invertible functions f, g (see [9]), where f is a scalar or a linear operator on the range of h (see the Schröder equation in [9, 10]). Other cases of interest arise when f, g are continuous functions with real domain and real range, strictly increasing and fixed-point free (see [11]), or f strictly decreasing continuous and g continuous (maybe nonmonotonic; see [14]). Usual references of one-dimensional dynamics [2, 12, 13] treat the case where f is continuous. In our case f is piecewise continuous.

We focus our attention on particular cases of the equation $h \circ g = f \circ h$, which correspond to a conjugacy equation involving the piecewise linear case. From the functional point of view we refer to results of de Rham [5] and their generalization by Girgensohn [8]. Since this last generalization provides an explicit solution in terms of the a-base expansion of numbers it is possible to construct explicitly a solution of our equation.

Definition 11.1 (see [3]) A map $f : I \to I$ is a *horseshoe map* if it has more than one lap and each lap is mapped onto the whole of I.

C. Serpa (✉) · J. Buescu
Centro de Matemática e Aplicações Fundamentais, Departamento de Matemática, Faculdade de Ciências, Universidade de Lisboa, 1749-016, Lisboa, Portugal
e-mail: mcserpa@fc.ul.pt

J. Buescu
e-mail: jsbuescu@fc.ul.pt

R. López-Ruiz et al. (eds.), *Nonlinear Maps and their Applications,*
Springer Proceedings in Mathematics & Statistics 112, DOI 10.1007/978-3-319-12328-8_11

The next Lemma is a generalization of a result of Ciepliński and Zdun [4] to noncontinuous functions (only piecewise continuity is required).

Lemma 11.1 *Let $a \geq 2$, be an integer, $g : [\alpha_1, \beta_1] \to [\alpha_1, \beta_1]$, $G : [\alpha_2, \beta_2] \to [\alpha_2, \beta_2]$ be piecewise continuous functions, respectively, with laps $I_i = \left[x_i, x_{i+1}\right]$, $J_i = \left[y_i, y_{i+1}\right]$, $i \in \{0, 1, \ldots, a-1\}$. Consider the conjugacy equation*

$$h(g(x)) = G(h(x)), \; x \in [\alpha_1, \beta_1], \tag{11.1}$$

where $h : [\alpha_1, \beta_1] \to [\alpha_2, \beta_2]$ is the unknown function.

Suppose g, G are horseshoe maps, and φ is a monotone and surjective solution of (11.1). If φ is increasing, then $\varphi(x_i) = y_i$, and $\varphi[I_i] = J_i$ for $i \in \{0, 1, \ldots, a-1\}$. If φ is decreasing, then $\varphi(x_i) = y_{a-i}$, and $\varphi[I_i] = J_{a-i-1}$ for $i \in \{0, 1, \ldots, a-1\}$.

Proof Since g is a horseshoe map and φ is a monotone and surjective solution of Eq. (11.1), we have for each $i \in \{0, 1, \ldots, a-1\}$,

$$\begin{aligned}\varphi(g(x_i)) = G(\varphi(x_i)) &\Rightarrow G(\varphi(x_i)) \in \{\varphi(\alpha_1), \varphi(\beta_1)\}\\ &\Rightarrow G(\varphi(x_i)) \in \{\alpha_2, \beta_2\}\\ &\Rightarrow \varphi(x_i) \in \{y_0, y_1, \ldots, y_a\}.\end{aligned}$$

Suppose for each $i \in \{0, 1, \ldots, a-1\}$, $\varphi(x_i) = \varphi(x_{i+1})$. Since φ is monotone, $\varphi[I_i]$ is a single point, as is $G[\varphi[I_i]]$. Again by Eq. (11.1) and surjectivity of φ we obtain $G[\varphi[I_i]] = \varphi[g[I_i]] = \varphi[\alpha_1, \beta_1] = [\alpha_2, \beta_2]$. Then $\varphi(x_i) \neq \varphi(x_{i+1})$, $i \in \{0, 1, \ldots, a-1\}$.

If φ is increasing, then $\varphi(x_0) < \varphi(x_1) < \cdots < \varphi(x_a)$, implying $\varphi(x_i) = y_i$, because $\varphi(x_i) \in \left\{y_0, y_1, \ldots, y_p\right\}$. Then $\varphi[I_i] = J_i$, $i \in \{0, 1, \ldots, a-1\}$. If φ is decreasing, then $\varphi(x_0) > \varphi(x_1) > \cdots > \varphi(x_a)$, implying $\varphi(x_i) = y_{a-i}$, because $\varphi(x_i) \in \{y_0, y_1, \ldots, y_a\}$. Then $\varphi[I_i] = J_{a-i-1}$, $i \in \{0, 1, \ldots, a-1\}$.

We will restrict attention to the family $\mathcal{M}$ of piecewise monotone and expanding interval maps $f : [0, 1] \to [0, 1]$ where there exists a partition $0 = a_0 < a_1 < \cdots < a_r = 1$, with $r \geq 2$, of $[0, 1]$ such that $f_{|[a_{i-1}, a_i]}$, for $i = 1, 2, \ldots, r$, is a monotone piecewise continuous function for which there exists $\lambda > 1$ such that $\left|f'(x)\right| \geq \lambda$, for almost every $x \in [0, 1]$.

Note that the expansivity condition does not necessarily require that the function be differentiable. The definition of $\mathcal{M}$ may be weakened to the following expansivity condition.

Definition 11.2 A continuous map $f : X \to X$ on a metric space (X, d) is expanding if there exist constants $\varepsilon > 0$ and $\lambda > 1$ such that, for all $x, y \in X$,

$$d(x, y) < \varepsilon \Rightarrow d(f(x), f(y)) \geq \lambda d(x, y),$$

λ is called the expansion factor of f.

Let $f \in \mathcal{M}$ with partition $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_a = 1$ of $[0, 1]$ such that $f_{|[\alpha_{i-1}, \alpha_i]}$, for $i = 1, 2, \ldots, a$, is an increasing, continuous and expanding function satisfying $f(\alpha_{i-1}, \alpha_i) = (0, 1)$, for every $i = 1, 2, \ldots, a$. We will see, in the piecewise linear case, that f is topologically conjugate to the map,

$$g_a(x) = \begin{cases} ax \,(\text{mod } 1)\,, & \text{if } x \in [0, 1) \\ 1, & \text{if } x = 1, \end{cases}$$

i.e, there exists a homeomorphism h such that

$$h \circ g_a = f \circ h. \tag{11.2}$$

Let $a \geq 2$. Define $\mu_0, \mu_1, \ldots, \mu_{a-1} \in (0, 1)$ by $\mu_i = \alpha_{i+1} - \alpha_i$ for $i \in \{0, 1, \ldots, a-1\}$. Clearly $\sum_{j=0}^{a-1} \mu_j = 1$.

In the linear case, is given by

$$f(x) = \begin{cases} \frac{1}{\mu_i} x - \frac{\alpha_i}{\mu_i} & , \text{if } x \in [\alpha_{i-1}, \alpha_i)\,,\ i \in \{0, 1, \ldots, a-1\} \\ 1 & , \text{if } x = 1, \end{cases}$$

it is possible to construct an explicit solution using the following results.

Theorem 11.1 *Any monotone increasing and surjective solution of the conjugation equation* $h \circ g_a = f \circ h$ *satisfies the functional equation*

$$h(x) = \mu_i h(ax - i) + \alpha_i, \text{for each } i \in \{0, 1, \ldots, a-1\}\,,\ x \in \left[\frac{i}{a}, \frac{i+1}{a}\right]. \tag{11.3}$$

Proof Let M be an increasing and surjective solution of Eq. (11.3). Then

$$M(ax - i) = \frac{1}{\mu_i} M(x) - \frac{\alpha_i}{\mu_i}, i \in \{0, 1, \ldots, a-1\}\,,\ x \in \left[\frac{i}{a}, \frac{i+1}{a}\right].$$

By direct computation,

$$M \circ g_a(x) = \begin{cases} M(ax - i) & , \text{if } x \in \left[\frac{i}{a}, \frac{i+1}{a}\right)\,,\ i \in \{0, 1, \ldots, a-1\} \\ 1 & , \text{if } x = 1. \end{cases}$$

$$= \begin{cases} \frac{1}{\mu_i} M(x) - \frac{\alpha_i}{\mu_i} & , \text{if } x \in \left[\frac{i}{a}, \frac{i+1}{a}\right)\,,\ i \in \{0, 1, \ldots, a-1\} \\ 1 & , \text{if } x = 1, \end{cases}$$

and

$$f \circ M(x) = \begin{cases} \frac{1}{\mu_i} M(x) - \frac{\alpha_i}{\mu_i} & , \text{if } M(x) \in [\alpha_{i-1}, \alpha_i)\,,\ i \in \{0, 1, \ldots, a-1\} \\ 1 & , \text{if } M(x) = 1. \end{cases}$$

By Lemma 11.1, if M is monotone increasing and surjective, then for each $i \in \{0, 1, \ldots, a-1\}$ $M(x) \in [\alpha_{i-1}, \alpha_i)$ is equivalent to $x \in [i/a, (i+1)/a)$. □

Remark 11.1 An analogous result holds for decreasing and surjective solutions as a consequence of Lemma 11.1.

Example For $a = 3$, Eq. (11.3) is

$$h(x) = \begin{cases} \mu_0 h(3x) & \text{, if } x \in \left[0, \frac{1}{3}\right] \\ \mu_1 h(3x-1) + \alpha_1 & \text{, if } x \in \left[\frac{1}{3}, \frac{2}{3}\right] \\ \mu_2 h(3x-2) + \alpha_2 & \text{, if } x \in \left[\frac{2}{3}, 1\right]. \end{cases}$$

Lemma 11.2 *Let $a \geq 2$ be an integer. Consider the system of functional equations*

$$h\left(\frac{k+t}{a}\right) = F_k h(t), \tag{11.4}$$

where $k \in \{0, 1, 2, \ldots, a-1\}$, F_k are contractions mappings and $h : [0,1] \to [0,1]$ is the unknown function.

Then the system of functional equations (11.4) is equivalent to the functional equation

$$h(t) = F_k h(at-k), t \in \left[\frac{k}{a}, \frac{k+1}{a}\right], k \in \{0, 1, 2, \ldots, a-1\}.$$

Theorem 11.2 *(Girgensohn [9]) Let $a \geq 2$ be an integer. Let $s_k : [0,1] \to \mathbb{R}$ be continuous, $|r_k| < 1$ for $0 \leq k \leq a-1$ and assume*

$$\frac{r_{k-1}}{1-r_{a-1}} s_{a-1}(1) + s_{k-1}(1) = \frac{r_k}{1-r_0} s_0(0) + s_k(0), \ 1 \leq k \leq a-1. \tag{11.5}$$

Then there exists exactly one bounded function $f : [0,1] \to \mathbb{R}$ which satisfies the system

$$f\left(\frac{x+k}{a}\right) = r_k f(x) + s_k(x), \ x \in [0,1], \ 0 \leq k \leq a-1. \tag{11.6}$$

The function f is continuous and given in terms of the a-base expansion of x by

$$f\left(\sum_{n=1}^{\infty} \frac{\xi_n}{a^n}\right) = \sum_{n=1}^{\infty} \left(\prod_{k=1}^{n-1} r_{\xi_k}\right) s_{\xi_n}\left(\sum_{k=1}^{\infty} \frac{\xi_{k+n}}{a^k}\right). \tag{11.7}$$

We now return to the study of the general Eq. (11.2).

Lemma 11.3 *Any monotone increasing and surjective solution of the conjugation equation $h \circ g_a = f \circ h$ satisfies the functional equation*

$$h(x) = \mu_i h(ax - i) + \alpha_i, i \in \{0, 1, \ldots, a-1\}, \ x \in \left[\frac{i}{a}, \frac{i+1}{a}\right],$$

i.e.,

$$h\left(\frac{x+i}{a}\right) = \mu_i h(x) + \alpha_i, i \in \{0, 1, \ldots, a-1\}, \ x \in \left[\frac{i}{a}, \frac{i+1}{a}\right].$$

The condition

$$\frac{r_{k-1}}{1 - r_{a-1}} s_{a-1}(1) + s_{k-1}(1) = \frac{r_k}{1 - r_0} s_0(0) + s_k(0), \ 1 \le k \le a-1$$

in this case assumes the form

$$\frac{\mu_{i-1}}{1 - \mu_{a-1}} \alpha_{a-1} + \alpha_{i-1} = \frac{\mu_i}{1 - \mu_0} 0 + \alpha_i, \ 1 \le i \le a-1,$$

which is equivalent to

$$\mu_{i-1} = \alpha_i - \alpha_{i-1}, \ 1 \le i \le a-1,$$

coinciding with the original hypothesis.

Applying Theorem 11.2, we obtain the following explicit solution in terms of the a-base expansion of numbers.

Theorem 11.3 *Let $a \ge 2$, $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_a = 1$, and $\mu_0, \mu_1, \ldots, \mu_{a-1} \in (0, 1)$, such that $\mu_i = \alpha_{i+1} - \alpha_i$, for $i \in \{0, 1, \ldots, a-1\}$.*

Given f and g_a defined above, there exists exactly one increasing homeomorphism $h : [0, 1] \to [0, 1]$ such that $h \circ g_a = f \circ h$, defined by

$$h\left(\sum_{n=1}^{\infty} \frac{\xi_n}{a^n}\right) = \sum_{n=1}^{\infty} \left(\prod_{i=1}^{n-1} \mu_{\xi_i}\right) \alpha_{\xi_n}. \tag{11.8}$$

Proof By Lemma 11.3 and Theorem 11.2, there exists exactly one-bounded monotone increasing and surjective $h : [0, 1] \to [0, 1]$, such that $h \circ g_a = f \circ h$ which is defined by formula (11.8). Lemma 1 in [18] by Zdun shows that function h is a homeomorphism. □

Remark 11.2 An analogous result may be proved for the case of a decreasing homeomorphism.

We next provide two examples of application of the explicit formula (11.8) to problems of Number Theory and Probability Theory.

11.1 Applications: Variable Base Expansions and Bold Play

11.1.1 Q-Representation of Real Numbers

The usual a-base representation of a number is given by

$$x = \sum_{n=1}^{\infty} \frac{\xi_n}{a^n}. \tag{11.9}$$

Turbin and Prats'ovytyi [15, 16] introduced a more general representation, where the length of the intervals defining the expansion is not uniform.

Let $a \geq 2$ be a fixed positive integer, and $q_0, q_1, \ldots, q_{p-1} \in (0, 1)$, such that $\sum_{j=0}^{p-1} q_j = 1$. Let $r_0 = 0$, $r_j = \sum_{k=0}^{j} q_{k-1}$, for $j \in \{1, 2, \ldots, a\}$, $A = \{0, 1, 2, \ldots, a-1\}$, and $Q = \left\{q_0, q_1, \ldots, q_{p-1}\right\}$.

Theorem 11.4 *(Turbin and Prats'ovytyi) For any number $x \in [0, 1]$, there exists a sequence of numbers $\nu = (\nu_n) \in A$ such that*

$$x = \sum_{n=1}^{\infty} \left(\prod_{k=1}^{n-1} q_{\nu_k} \right) r_{\nu_n}. \tag{11.10}$$

Remark 11.3 Obviously, for any real number x there exists an expansion

$$x = [x] + \sum_{n=1}^{\infty} \left(\prod_{k=1}^{n-1} q_{\nu_k(x)} \right) r_{\nu_n(x)}. \tag{11.11}$$

Definition 11.3 Given $x \in \mathbb{R}$, the representation by series (11.10) or (11.11) is called a-symbol Q-representation or a-symbol Q-expansion of x. For $x \in [0, 1]$ we use the notation given by

$$\triangle_{\nu}^{Q} = \triangle_{\nu_1 \nu_2 \cdots \nu_n \cdots}^{Q} := \sum_{n=1}^{\infty} \left(\prod_{k=1}^{n-1} q_{\nu_k} \right) r_{\nu_n}. \tag{11.12}$$

Remark 11.4 An algorithm to find an a-symbol Q-representation of a number $x \in [0, 1]$ is:

0. Let $n = 1$ and $x_1 = x$.
1. Find $\nu_n \in \{0, 1, \ldots, p-1\}$, such that $r_{\nu_n} \leq x_n < r_{\nu_n + 1}$. If $x = r_{\nu_n}$, do $\nu_N = 0$, for $N > n$, and the process is finished. Else do step 2.
2. Find the difference $x_n - r_{\nu_n}$ and divide by q_{ν_n}:

$$x_{n+1} = \frac{x_n - r_{\nu_n}}{q_{\nu_n}}.$$

3. Do step 1 for $n + 1$.

The sequence $\nu = (\nu_n) \in A$ determines the a-symbol Q-representation of x. We now show that representations (11.9) and (11.12) are related by a homeomorphism which is a solution of an equation of the form (11.2).

Theorem 11.5 *There is a homeomorphism φ such that the image of each $x \in [0, 1]$ by φ is the a-symbol Q-expansion of x, where the sequence of numbers $(\nu_n) \in A$ of the image obtained is the same as the a-base representation of x, i.e., $\nu_n = \xi_n$, $\forall n \in \mathbb{N}$. Moreover, φ is the unique bounded solution of the system of equations*

$$\varphi\left(\frac{x+k}{a}\right) = q_k\varphi(x) + r_k,\ x \in [0, 1],\ 0 \leq k \leq a-1. \tag{11.13}$$

Proof By definition of q_k and r_k, condition (11.5) is satisfied. Applying Theorem 11.2 there exists a unique bounded solution of (11.13) given, in terms of the a-base representation, by

$$\varphi\left(\sum_{n=1}^{\infty}\frac{\xi_n}{a^n}\right) = \sum_{n=1}^{\infty}\left(\prod_{k=1}^{n-1} q_{\xi_k}\right) r_{\xi_n}. \tag{11.14}$$

□

This result may be viewed as a special case of Lemma 1 in [18] by Zdun which shows that the function φ is a homeomorphism. Girgensohn's result (Theorem 11.2) is more general, since in its statement the r_k are allowed to be continuous functions of x, $r_k : [0, 1] \to \mathbb{R}$, instead of constants.

Remark 11.5 The function φ in (11.14) may be given in the equivalent forms

$$\varphi\left(\sum_{n=1}^{\infty}\frac{\xi_n}{a^n}\right) = \triangle_{\xi}^{Q}, \tag{11.15}$$

$$\varphi^{-1}\left(\triangle_{\xi}^{Q}\right) = \sum_{n=1}^{\infty}\frac{\xi_n}{a^n}. \tag{11.16}$$

In a similar fashion, the function that transforms a-symbol Q-expansions in other a-symbol Q-expansions was given by Prats'ovytyi and Kalashnikov [17]. If we denote by $\triangle_{\nu}^{Q[q,r]} = \triangle_{\nu_1\nu_2\cdots\nu_n\cdots}^{Q[q,r]}$ the a-symbol Q-expansion in terms of q_j, r_j, and ν, defined above, the system of functional equations

$$\psi\left(\triangle_{k\nu_1\nu_2\cdots\nu_n\cdots}^{Q[q,r]}\right) = t_k\psi\left(\triangle_{\nu_1\nu_2\cdots\nu_n\cdots}^{Q[q,r]}\right) + s_k,\ k \in \{0, 1, \ldots, a-1\}$$

has a unique bounded solution ψ defined by

$$\psi\left(\triangle_{\nu}^{Q[q,r]}\right) = \triangle_{\nu}^{Q[t,s]}. \tag{11.17}$$

We note that our results imply immediately construction (11.17). In fact, let $a \geq 2$ be an integer, and let $(a, Q[q,r])$ and $(a, Q[t,s])$ be bases of a-symbol Q-representations. Remark 11.5 implies that the diagram

$$\begin{array}{ccc} \sum_{n=1}^{\infty} \frac{\xi_n}{a^n} & \underset{\mathrm{id}_{[0,1]}}{\rightarrow} & \sum_{n=1}^{\infty} \frac{\xi_n}{a^n} \\ \varphi_{q,r} \downarrow & & \downarrow \varphi_{t,s} \\ \triangle_{\xi}^{Q[q,r]} & \underset{\psi}{\rightarrow} & \triangle_{\xi}^{Q[t,s]} \end{array}$$

is commutative, where the indices of φ correspond to the parameters of the system (11.13) for which φ is solution.

Thus the required homeomorphism ψ in (11.17) is given by

$$\psi = \varphi_{t,s} \circ \varphi_{q,r}^{-1}. \tag{11.18}$$

Thus Theorem 11.2 allows us to construct an alternative, equivalent representation of the homeomorphism ψ.

11.1.2 Bold Play Gambling

Our second example, bold play gambling, originates in casino games (see [1] for a detailed description). Consider a gambler playing roulette, staking the amount of money s at each turn of the wheel. The probability of winning s is p and the probability of losing s is $q = 1 - p$.

Suppose the initial capital is C and the gambler's goal is G. The gambler will play until his fortune has reached G or has dwindled to nothing. The game strategy called *bold play* is the following: in each turn of the wheel the gambler either stakes his entire fortune, if this fortune does not exceed half the goal, or bets the difference between the goal and his current fortune. For a normalized problem, consider $G = 1$, and the domain will be the interval [0, 1].

Formalizing the problem, denote the gambler's current fortune by x, so that $0 \leq x \leq 1$. If $0 \leq x \leq 1/2$, he bets x (all the money); a win gives him a new fortune of $2x$, and a loss ruins him (lose everything). If $1/2 \leq x \leq 1$, he bets $1 - x$, just enough to carry him to his goal of 1; a win gives him a new fortune of 1 (a success), and a loss leaves him with $2x - 1$.

Proposition 11.1 *Let $p + q = 1$, $p > 0$, $q > 0$. The probability of success under bold play $f(x)$ for an initial fortune x is the unique solution of the system of equations*

$$\begin{cases} f(x) = pf(2x) & ,\text{if } 0 \leq x \leq \frac{1}{2} \\ f(x) = p + qf(2x-1) & ,\text{if } \frac{1}{2} \leq x \leq 1, \end{cases} \tag{11.19}$$

constrained to $f(0) = 0$, $f(1) = 1$.

Remark 11.6 If $p = q = 1/2$, it is immediate that the identity is a solution of (11.19). From Proposition 11.1 this solution is unique.

Proof (see [1]) For the case $p = q = 1/2$, a classical result in probability applies (see e.g. [7]). For $x = 0$, $f(x) = 0$, and equation $f(x) = pf(2x)$ is satisfied. For $x = 1$, $f(x) = 1$, and equation $f(x) = p + qf(2x - 1)$ is verified. For $0 < x \leq 1/2$, under bold play the gambler stakes the amount x. In case of success in the first turn, with probability p, his new fortune is $2x$. Since each turn has independent outcomes, the probability of success in the second turn $f(2x)$ multiplied by p equals the initial probability of success $f(x)$, proving the first equation of (11.19).

For $1/2 \leq x < 1$, the first stake is $1 - x$. In case of success in the first turn, with probability p, his new fortune is 1. In case of loss, with probability q, his new fortune is $2x - 1$. The probability of success after a loss is $f(2x - 1)$. Since each turn has independent outcomes, the probability of success given by $p + qf(2x - 1)$ equals the initial probability of success $f(x)$, proving the second equation of (11.19). Uniqueness of the solution now follows from Theorem 11.3 given initial conditions.

The probability of success of a strategy of bold play gamble satisfies a system of equations of type (11.3). Applying Theorem 11.3 we have the following explicitly defined function of probability of success, in terms of binary representation of real numbers.

Corollary 11.1 *The probability of success under bold play for an initial fortune*

$$x = \sum_{n=1}^{\infty} \frac{\xi_n}{2^n}$$

is

$$f\left(\sum_{n=1}^{\infty} \frac{\xi_n}{2^n}\right) = \sum_{n=1}^{\infty} \left(\prod_{i=1}^{n-1} \mu_{\xi_i}\right) \alpha_{\xi_n},$$

where $\alpha_0 = 0$, $\alpha_1 = p$, $\mu_0 = p$, *and* $\mu_1 = q$.

Dubins and Savage [6] showed that in case where the game is unfair for the player $(p < q)$, the bold play strategy is optimal, although it is not the only optimal strategy.

Acknowledgements The first author acknowledges support by Fundação para a Ciência e Tecnologia, grant SFRH/BD/77623/2011. The second author acknowledges partial support by Fundação para a Ciência e Tecnologia, PEst-OE/MAT/UI0209/2013. The authors are also indebted to the anonymous referees for their sharp suggestions.

References

1. Billingsley, P.: The singular function of bold play. Am. Sci. **71**, 4 (1983)
2. Block, L. S., Coppel, W. A.: Dynamics in One Dimension, Lecture Notes in Mathematics, 1513, Springer-Verlag, Berlin, Heidelberg (1992)

3. Blokh, A., Coven, E., Misiurewicz, M., Nitecki, Z.: Roots of continuous piecewise monotone maps of an interval. Acta Math. Univ. Comenian. LX **1**, 3–10 (1991)
4. Ciepliński, K., Zdun, M.: On uniqueness of conjugacy of continuous and piecewise monotone functions. Fixed Point Theory A **2009**, 230414 (2009)
5. de Rham, G.: Sur quelques courbes définies par des équations fonctionnelles. Rend. Sem. Math. Torino **16**, 101–113 (1956)
6. Dubins, L. E., Savage, L. J.: Optimal gambling systems. PNAS **46**, 1597–1598 (1960)
7. Feller, W.: An Introduction to Probability Theory and Its Applications , I, 3rd edn., John Wiley & Sons, Inc., New York (1968)
8. Girgensohn, R.: Functional equations and nowhere differentiable functions. Aequationes Math. **48**, 243–256 (1993)
9. Kuczma, M.: Functional Equations in a Single Variable. PWN Warszawa (1968)
10. Kuczma, M., Choczewski, B., Ger, R.: Iterative Functional Equations. Encyclopedia Math. Appl., Cambridge University Press, New York, Melbourne (1990)
11. Laitochová, J.: On conjugacy equations for iterative functional equations. Int. J. Pure Appl. Math. **26**(3), 421–430 (2006)
12. May, R. M.: Simple mathematical models with very complicated dynamics. Nature **261**, 459–467 (1976)
13. Melo, W., van Strien, S.: One-Dimensional Dynamics. Springer-Verlag, Berlin, Heidelberg (1993)
14. Shi, Y.-G.: Non-monotonic solutions and continuously differentiable solutions of conjugacy equations. Appl. Math. Comput. **215**, 2399–2404 (2009)
15. Turbin, A. F., Prats'ovytyi, M. V.: Fractal Sets, Functions and Distributions. Naukova Dumka, arXiv:math/0308007v2 (1992)
16. Prats'ovytyi, M. V.: Fractal approach to investigations of singular probability distributions, Kiyv, Drahomanov National Pedagogical Univ. (1998)
17. Prats'ovytyi, M. V., Kalashnikov, A. V.: Self-affine singular and monotone functions related to the Q-representation of real numbers. Ukr. Math. J. **65**, 448–462 (2013)
18. Zdun, M.: On conjugacy of some systems of functions. Aequationes Math. **61**, 239–254 (2001)