Proclus' Conception of Geometric Space and Its Actuality

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The main aim of this paper is to present Proclus' philosophy of geometric extension not so much from the point of view of what he says about it as what he does with it. I will henceforth pay particular attention to the role of spatial configurations in the *practice* which he describes. My motivations are twofold. First, although Proclus' philosophy of geometry has received quite a lot of attention in the scholarship, this attention has remained mainly inspired by the philosophical doctrine expounded in the *Prologues.*[1] It did not engage much, at least in a systematic way, with the material given in the actual commentary of Euclid's propositions and the mathematical practice there described. As a consequence, the complexity and the flexibility of Proclus' views on the geometric imagination were not always well rendered. I would like to complete existing descriptions by paying more attention to these details, although, as I will indicate, they may sometimes introduce important nuances, if not tensions, in the philosophical system. Second, Proclus provides indications throughout his commentary about the geometric practice which go far beyond his own specific philosophical agenda. He deals, for example, with objections that other mathematicians and philosophers raised against Euclid's proofs. These objections, which sometimes stem from views opposite to his own (typically, Epicurean objections of an "empiricist" flavor), were taken seriously enough to ask for answers which Proclus also mentions (or sometimes even initiates). For the modern reader, these passages are precious because they provide, by

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[1] See Breton (1969), Charles-Saget (1982), Mueller (1987), O'Meara (1989), Cleary (2000), Nikulin (2002) and Lernould (2010).

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contrast, testimonies about certain conditions of the practice which seem to have been accepted by the various interlocutors (whatever their philosophical background may have been). I would like to reconstitute some of these conditions and compare these reconstructions with recent works dealing with ancient Greek geometric practice, especially as regards the use of diagrams.[2]

My approach to Proclus will therefore be partly instrumental. Instead of focusing on the quite rigid Neoplatonic framework set up in the *Prologues*, which has already received considerable attention and which in Proclus' eyes was probably the most important aspect of his interpretative enterprise, I will try to extract a more flexible approach of geometric practice found throughout his commentary.[3] My handling of the question will be focused on the role of "impossible" diagrams, be they related to the picturing of infinite or indivisible entities, to objections raised against this or that construction or to proofs by *reductio*. In the first part of the paper, I will motivate this choice with the first series of examples. As we will see, it so happens that Proclus gives important indications on these cases about the role of the geometric imagination. He insists on this occasion, more so than in the *Prologues*, on the *autonomous* activity of the imagination and on its relative *opacity* to the understanding. These two features will constitute the *leitfaden* of my study. In the second part of the paper, I will show that this situation is not specific to the first set of examples and conforms to the frequent use of other "impossible" representations in Ancient Geometry (a fact which indicates accordingly that they are by no means "impossible"). I will document some of their appearances in Proclus, detail how the geometric imagination is involved in these situations and indicate several related philosophical issues. In the last part of the paper, I will focus on these philosophical issues and compare them with recent attempts to characterize Ancient Geometric Practice. As a conclusion, I will try to show how this description may help us tackle broader issues dealt with in contemporary philosophy of mathematics.

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[2] In particular the studies by Manders (2008b), and Netz (1999). As regards the characterization of geometric "practice", I follow here an insightful remark made by Netz in passing (p. 2): "what unites a scientific community need not be a set of beliefs. Shared beliefs are much less common than shared practices. This will tend to be the case in general, because shared beliefs require shared practices, but not vice versa. And this must be the case in cultural settings such as the Greek, where polemic is the rule, and consensus is the exception. Whatever is an object of belief, whatever is verbalisable, will become visible to the practitioners. What you believe, you will sooner or later discuss; and what you discuss, especially in a cultural setting similar to the Greek, you will sooner or later debate. But the real undebated, and in a sense undebatable, aspect of any scientific enterprise is its non-verbal practices".

[3] Although the extant text of Proclus' commentary deals only with the first book of Euclid's *Elements*, my interest will not be about the practice attributed to a particular mathematician ("Euclid") and developed into a text entitled the *Elements*, but on the common practice spanning centuries (say, at least until Proclus) by practitioners of Euclidean Geometry and which remained quite stable through various interpretations (involving different cultural settings, conceptual debates, ideological criticisms, etc.). Moreover, what I will describe easily extends to other classical authors such as Archimedes and Apollonius.

1 Proclus on Geometric Space

Before entering into the flesh of Proclus' views, a word of justification about the choice of the topic seems to be in order. Indeed it may sound strange to aim at studying "geometric space" in Proclus, as it does more generally for ancient Greek mathematics. A usual claim would be, quite on the contrary, that ancient Geometry was a science of figures in contrast with our modern Geometry which takes space and transformations in space as its primary object of inquiry. In his seminal study on Euclid's *Elements*, Ian Mueller stresses this difference between the modern "structural" viewpoint and that of the Ancients:

> For Hilbert geometric axioms are characterized by an existent system of points, straight lines, etc. At no time in the *Grundlagen* is an object brought into existence, constructed. Rather its existence is inferred from the axioms. In general Euclid produces, or imagines produced, the objects he needs for a proof (…). It seems fair to say then that in the geometry of the *Elements* there is no underlying system of points, straight lines, etc. which Euclid attempts to characterize. Rather, geometric objects are treated as isolated entities about which one reasons by bringing other entities into existence and into relation with the original objects and one another.[4]

A source of great confusion on this issue was the description of the birth of "Modern Science" as characterized, to repeat Koyré's famous wording by "the replacement of the Aristotelian conception of space—a differentiated set of inner worldly places, by that of Euclidean geometry—an essentially infinite and homogenous extension—from now on considered as identical with the real space of the world".[5] Indeed, in this picture, it seems that Euclidean Geometry is already, by contrast to Aristotelian philosophy, a science of infinite (homogeneous) extension. But, as emphasized by E. Grant in a manner very close to that of Mueller:

> There is nothing in Euclid's geometry to suggest that he assumed an independent, infinite, three-dimensional, homogeneous space in which the figures of his geometry were located. In a purely geometric sense, such a space would have been superfluous because every geometric figure has its own internal space. Moreover, if the space of the geometric figure and the independent space it is alleged to occupy are conceived as indistinguishable, an infinity of spaces could be postulated in one and the same place. (…) Euclidean geometric space was the space of geometric figures of any size whatever and when applied to material bodies was conceived as an internal space.[6]

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[4] Mueller (1981, p. 14).

[5] Koyré (1957, Preface, p. viii).

[6] Grant (1981, p. 17). The last sentence of the quote is intended to show the proximity between Euclid and Aristotle's concept of place. This goes along with another important historical rectification emphasized by Grant: "the adoption of an infinite space in the seventeenth century resulted primarily from the divinization of space—a process begun in the fourteenth century—and to a lesser extent, from the needs from physics and cosmology. But it did not arise from any straightforward application of an alleged Euclidean geometric space to the physical world" (note 49, p. 273).

Considering these debates, it might seem that the attribution of a doctrine of "geometric space" to Greek mathematicians amounts to some form of anachronistic projection of our modern "structural" viewpoint (i.e. what we *now* call "Euclidean Geometry"). It is therefore worth recalling that although we have no way of knowing what Euclid thought about the subject, there was at least one Ancient Greek (although much later than Euclid) who had no trouble finding infinite space in his geometry: Proclus! Moreover, he gave strong justifications for this claim.

The discussion on this topic occurs mainly in the commentary of proposition I.12, which asks: "To a *given infinite straight line*, from a given point which is not on it, to draw a perpendicular straight line" (my emphasis). Proclus remarks that the condition ("from a given point which is not on it") cannot be known to be satisfied if we take an arbitrary finite segment and a point at random in the plane. Indeed we would have no way of knowing whether or not the given point would be on the line when produced (a situation dealt with in the previous proposition and asking for another construction cf. *In Eucl.* 284).[7] This is, according to him, why Euclid was led to posit an infinite straight line given *in actuality* (κατ᾽ ἐνέργειαν). He insists then on the fact that "if there is an infinite line, there will also be an infinite plane, and infinite in actuality if the problem is to be a real one" (*In Eucl*. 284, 19–21). This last stance transfers immediately to three dimensions, since proposition XI.11 of the *Elements* asks for the construction of a perpendicular to a plane and relies directly on I.12. If the plane in I.12 is infinite in actuality, so will be the three dimensional space containing it (by repeating the construction twice, if needed).

One could wonder if too much emphasis can be put on a single mention which remains isolated in Euclid and, by consequence, in Proclus. But the commentator also emphasizes that the *Elements* express at several occasions the fact that a straight line is given as *finite* (prop. I.1, I.10), a specification which would be pointless if there were no possibility for a straight line to be given as *not* finite. He also indicates that the condition is not specified in certain propositions, because it is implicit in the position of the problem (see for example *In Eucl.* 208, 11–12; 223, 16; 224, 4; 277, 18–24). Although these terminological remarks are not enough to tell us about Euclid's own conception, they certainly indicate that the acceptance of an actual infinite in Geometry was more than a strange *hapax legomenon* in Proclus' discourse. Moreover, one has to remember that the construction involved in I.12 is ubiquitous in the *Elements* (and more generally in Ancient Greek Geometry).

The epistemological problem raised by this description is of course that an actual infinite space seems inconceivable if one looks either in the realm of sensible things, in which only finite magnitudes have existence (culminating according to Proclus with the last sphere which gives a boundary to all material entities), or in the realm of intelligible where there is a platonic 'idea' of infinity, but not endowed

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[7] The Greek text is Proclus (1873). I will follow the English translation by Morrow (1992). For a contemporary discussion of the problem raised by the choice of arbitrary points in the *Elements* and their relation to the presupposition of an infinite system of objects, see Mumma (2012, p. 117; for propositon I. 11 and 12, see in particular note 14).

with spatial extension (*In Eucl.* 284–285). To this traditional dilemma, Proclus proposes a solution, which is of particular interest to us:

> It remains, then, that the infinite exists in the imagination, only without the imagination knowing the infinite. For when imagination knows, it simultaneously assigns to the object of its knowledge a form and limit, and in knowing brings to an end its movement through the imagined object; it has gone through it and comprehend it. The infinite therefore is not the object of knowing imagination, but of imagination that is uncertain about its object, suspends further thinking and calls infinite all that it abandons, as immeasurable and incomprehensible to thought. Just as sight recognizes darkness by the experience of not seeing, so imagination recognizes the infinite by not understanding it. It produces it indeed, because it has an indivisible power of proceeding without end. (*In Eucl.* 285, 5–17)

This does not contradict what Mueller told us.[8] Space does not appear here as an object of inquiry or a system of objects which would be treated mathematically as such. Indeed Proclus insists on the fact that it is given by imagination as something that the understanding *cannot grasp*. In this sense, geometric space acts only in the background of the theory. But this background, although there is *stricto sensu* no concept of it, is nonetheless presented as a necessary condition for a proper study of geometric objects. It is not something that we know, but something that we need in order to know.[9] Hence one should not confuse the acceptance of an infinite geometric space and the fact of taking it as an object of mathematical study. The purpose of this paper is to elucidate the first of these directions by embedding it, following Proclus, in a more general view on the role of the imagination in Geometry and on the way space is *used* not as an object of study, but as a tool for the study of mathematical objects.

These remarks are of great importance from a historiographical point of view. Even if modern scholars are more cautious than their predecessors with regards to diagnosing "revolutions" in science, they often leave untouched the above mentioned thesis, according to which the birth of an infinite *mathematical* space in the Renaissance was an important breakthrough (the existence of an infinite physical space being already present in certain Ancient philosophers). In this regard, it

[8] It does seem, however, to directly contradict Grant's declarations: according to Proclus, we certainly need an actual infinite extension to perform Euclidean geometry.

[9] "The understanding from which our ideas and demonstrations proceed does not use the infinite for the purpose of knowing it, for the infinite is altogether incomprehensible to knowledge; rather it takes it hypothetically and uses only the finite for demonstration; that is, it assumes the infinite not for the sake of the infinite, but for the sake of the finite" (*In Eucl.* 285–286). Compare with Mumma (2012, p. 117): "In the construction stage, most steps produce unique geometric objects from given ones and so can be represented logically by functions. Yet one kind does not: the free choice of a point satisfying non-metric positional conditions. Such points have an *indefinite* character within proofs. Their precise identity is not fixed relative to other given objects in the configuration. The natural logical representation for what licenses their introduction are thus existential statements, asserting the existence of a point satisfying certain non-metric positional conditions. *And so, though the geometric reasoning in* ***Eu*** *is always performed with a particular finite diagram, it still seems to presuppose a domain of geometric objects, i.e. the domain over which the quantifiers of these propositions range*" (my emphasis). The link between "indefinite" objects and the presupposition of an infinite domain of objects is of particular interest.

should be recalled not only that it was a possibility foreseen in Ancient times, but that, symmetrically, such an idea was controversial amongst early modern thinkers. Moreover the Proclean position of the problem, which distinguishes between what is given to understanding and what is given to imagination, has strong echoes in these discussions. In particular, one has to keep in mind that Descartes, one of the alleged "heroes" of the modern identification between mathematical space and the real world, *rejected* the idea of an infinite extension as conceptually given and introduced the idea of "indefinite" for what is given as infinite to the imagination.[10] It should also be stressed that in Proclus one can find not only the acceptance of an infinite extension given by imagination, but a dynamical conception of geometric space in which *transformations* are brought to the fore (in contrast to the study of properties of static figures, often presented as a characteristic of Ancient geometry). In fact, the history of the "Euclidean" tradition shows continuous evolution toward this view[11] and produced as early as Ibn al-Haytham and al-Sijzi presentations of Geometry in which space and transformations became of primary importance.[12]

It is no big surprise that geometric space makes its appearance in the case of entities such as the infinite straight line or points taken at random in the plane (i.e. not as intersection of given lines or circles).[13] It provides us with a typical situation in which the spatial proxy acting in the background of the theory and the conceptual apparatus operating on its surface are not in perfect accordance—whereas other regular situations (but hardly all of them, as I will argue in Sect. 2) could let us think that they evolve in a perfect parallelism. There are spatial configurations which, so to speak, do not "correspond" to conceptual configurations. This relative opacity of space, as I will call it following the Proclean metaphor of darkness and sight, will be the leading topic of my inquiry. Note however that the problem here is not presented in the form which a modern reader could expect: it does not come from the fact that we, human beings with finite resources, are unable to properly represent an actual infinite. Quite on the contrary since Proclus credits human knowledge with being able to represent infinity thanks to imagination! What is

[10] "I do distinguish here between 'indefinite' and 'infinite'; strictly speaking, I designate only that thing to be 'infinite' in which no limits of any kind are found. In this sense God alone is infinite. However, there are things in which I discern no limit, but only in a certain respect (such as the extension of imaginary space, a series of numbers, the divisibility of the parts of a quantity, and the like). These I call 'indefinite' but not 'infinite,' since such things do not lack a limit in every respect." ("Reply to First Set of Objections", translation in Ariew 2000, p. 155; AT 7: 113). The close connection between the concept of 'indefinite' and imagination is emphasized as early as *Le Monde* (AT 11: 31–33, transl. Ariew 2000, p. 36) and persists until the *Principles of Philosophy* (I § 26–27, AT 8a: 15, transl. Ariew 2000, pp. 237–238). On Newton's proximity with Proclus at the time of the *De gravitatione*, see Domski (2012).

[11] See Vitrac (2005, pp. 1–56), which gives a prominent role to Proclus.

[12] See Rashed (2001, Introduction, pp. 1–11 and Chap III, pp. 655–685: "Ibn Al-Haytham et la géométrisation du lieu"), Crozet (2010), De Vittori (2009).

[13] The word used in the *Elements* is τυχὸν σημεῖον. Before I.12, it appears in I.5, I.9 and I.11.

particularly interesting is precisely that we are able to represent something to which we have no conceptual access.

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The commentary of Eucl. I.12 provides the occasion to recall the general epistemological framework set up by Proclus in the *Prologues*. It also allows us to identify important nuances introduced in the details of the commentary. This is particularly the case with the *autonomous* activity which the imagination acquires with increasing clarity as the commentary unfolds.

In the beginning of the book (*Prologue* 1), under the guise of an orthodox Neoplatonism, Proclus has emphasized the intermediate nature of mathematical entities which are neither simple nor dispersed in the uncontrolled diversity of Becoming, neither mere objects of intellect (*nous*) nor of opinion (*doxa*) (*In Eucl.* 3–5). In the second *Prologue*, however, paying closer attention to geometric objects, he introduces a more original view in which this intermediary position is not linked solely to the position of discursive thinking (*dianoia*), but to the intervention of a kind of "matter" identified with imagination (*hylè phantastikè* cf. *In Eucl.* 51–52). Here, Proclus is very clear about the fact that he is not following an orthodox platonic doctrine.[14] He then goes so far as to assert that *dianoia* is *unable* to access geometric ideas by itself and *needs* the help of imagination to seize them:

> When, therefore, geometry says something about the circle or its diameter, or about its accidental characteristics, such as tangents to it or segments of it and the like, let us not say that it is instructing us either about the circles in the sense world, for it attempts to abstract from them, or about the form in the understanding. For the circle [in the understanding] is one, yet geometry speaks of many circles, setting them forth individually and studying the identical features in all of them; and that circle [in the understanding] is indivisible, yet the circle in geometry is divisible. Nevertheless we must grant the geometer that he is investigating the universal, only this universal is obviously the universal present in the imagined circles. Thus while he sees one circle [the circle in imagination], he is studying another, the circle in the understanding, yet he makes his demonstrations about the former. For the understanding contains the ideas but, *being unable to see them when they are wrapped up, unfolds and exposes them and presents them to the imagination sitting in the vestibule*; and in imagination, or with its aid, it explicates its knowledge of them, happy in their separation from sensible things and finding in the matter of imagination a medium apt for receiving its forms. (*In Eucl.* 54–55, my emphasis)

We find here for the first time the crucial idea that imagination provides *dianoia* with a knowledge which it is unable to access by itself. At this stage, however, it would still be possible to understand this doctrine as a mere complement to what has been stated in the first prologue. One could claim that the discursive movement remains attached only to *dianoia*, imagination being just a proxy on which this

[14] "We are not unaware of what the philosopher Porphyry in his *Miscellaneous Inquiries* and most of the Platonists have set forth, but we believe that what we have said is more in agreement with the principles of geometry and with Plato's declaration that the objects of geometry are understandables" (*In Eucl.* 56).

movement is transcribed and which allows the manifestation of pure and simple ideas. The famous metaphor of the *projection* on the “receptacle” of imagination seems to go in this direction:[15]

> We invoke the imagination and the intervals that it furnishes, since the form itself is without motion or genesis, indivisible and free of all underlying matter, though the elements latent in the form are produced distinctly and individually on the screen of imagination. What projects the images is the understanding; the source of what is projected is the form in the understanding; and what they are projected in is this “passive nous”[16] that unfolds in revolution about the partlessness of genuine *Nous*. (*In Eucl.* 56)

This seems even clearer in the metaphor of the surface on which the understanding *writes* its mathematical concepts (or “ratios” since Proclus designates both as *logoi*). Imagination is then presented as a plane mirror[17] on which discursive knowledge contemplates itself: “We must think of the plane as projected and lying before our eyes and the understanding as writing everything upon it, the imagination becoming something like a plane mirror to which the ideas of the understanding send down impressions of themselves” (*In Eucl.* 121).

One important thing to notice in these various declarations is that imagination is identified with the spatial support, Proclus establishing a strong connection between what could appear as a faculty of the soul and what I have designated above as a form of “geometric space”. In this sense, space is not just a screen on which the *dianoia* projects its concepts, it is also, and by the same token, a screen on which the soul recognizes its rational activity. But, in any case, the activity seems situated on the discursive side, the apparent activity on the surface being just a reflection of the dianoetic activity. This idea is well expressed in the description of the geometric “figure” where the mirror is presented as a surface on which the seer and the seen coincide (because in the mirror I see myself seeing):

> Therefore just as nature stands creatively above the visible figures, so the soul, exercising her capacity to know, projects on the imagination, as on a mirror, the ideas of the figures; and the imagination, receiving in pictorial form these impressions of the ideas within the soul, by their means affords the soul an opportunity to turn inward from the pictures and attend to herself. It is as if a man looking at himself in a mirror and marveling at the power of nature and at his own appearance should wish to look upon himself directly and possess such a power as would enable him to become at the same time the seer and the object seen. (*In Eucl.* 141)

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[15] On Proclus’s “projectionism”, see Mueller in Morrow (1992, p. xxvi), Mueller (1987) and O’Meara (1989).

[16] This is the way Aristotle designates imagination in *De Anima* 430a24. As explained a few pages earlier by Proclus, this expression should be taken *cum grano salis* since there is no such thing as a *passive* nous in his view (*In Eucl.* 52).

[17] A metaphor coming from Plato (especially *Timaeus* 70e–f) and elaborated by Plotinus, see Claessens (2012, where the relevant literature is mentioned).

However, as we have seen in I.12, imagination does not limit itself to reflecting a dynamic coming from outside. It is also endowed with a *dynamic of its own*.[18] In the case of infinity, no reflexivity is allowed since the dynamic attached to it is presented as *irreducible* to conceptualization: with infinite entities (be they straight lines, planes, three dimensional space), imagination provides knowledge with forms of representation which must remain opaque to it. This is why I talked about important nuances introduced in the course of the commentary.

This autonomous and irreducible activity of imagination appears in other situations. In fact, it could be detected as early as the first definition of the first Book, which will serve me as a second basic example. As is well known, def. 1 of the *Elements* defines the *point* as "what has not part". A traditional difficulty here is to determine how the geometer could represent in a spatial diagram, endowed *de jure* with infinite divisibility, an indivisible entity (a problem which, like for the infinite line, immediately transfers to higher dimensions: length without breadth, surface with length and breadth only). This is a symmetrical, and more traditional, problem than the one posed by I.12: we have here a clear concept, but no possible image of it. What is interesting is Proclus' answer, which once again amounts to stress the irreducible role of imagination in its dynamical aspect (*phantastikès kinesis*):

> But someone may object: How can the geometer contemplate a partless something, a point, within the imagination if the imagination always apprehends things as shaped and divisible? For not only ideas in the understanding, but also the impressions of intellectual and divine forms, are accepted by the imagination in accordance with its peculiar nature, which furnishes forms to the formless and figures to what is without figure. To this difficulty we reply that the imagination in its activity is not divisible only, neither is it indivisible. Rather it moves from the undivided to the divided, from the unformed to what is formed. (*In Eucl.* 94–95)

Although it is not entirely clear what Proclus has in mind when claiming that imagination is "not divisible, neither indivisible" (literally: "not divided, neither undivided"), the insistence on the dynamical aspect is striking. It brings to the fore the role of the action (in this case: division) which generates objects (the indivisible), in contrast to the object given as such to *dianoia* and necessarily endowed with non-incompatible properties (either divided, or undivided). As images (unlike concepts-*logoi*, given solely by definitions), "points" carry information about what we can or cannot do (in this case: divide any further), although they are endowed with properties allowing us to do what they forbid.[19] What I would like to stress is,

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[18] Reflecting in the first *Prologue* on the famous Aristotelian metaphor of the soul as a wax tablet, Proclus objects that it is rather a tablet "writing itself", a first occurrence of the idea of an active surface (*In Eucl*. 16, 10: γράφον ἑαυτὸ). On the autonomy of imagination in Proclus, see Claessens (2012).

[19] For a modern reading of "points" in Euclidean Plane Geometry insisting on the role of division, see Panza (2012, p. 73). According to Panza: "geometrical points are not represented by elementary diagrams. They are rather represented by extremities or intersections of lines, some of which are possibly elementary diagrams, whereas others are parts of such elementary diagrams resulting from dividing them through intersection". For the problems arising from such a view see Mumma (2012) quoted note 10.

once again, the relative opacity attributed to geometric imagination. It manifests itself in the fact that the dynamic of imagination is not presented as a projection of some discursive reasoning here. Quite the contrary since imagination allows us to circulate between properties ("not divided, neither undivided") which are conceptually incompatible.

In this regard, the discussion on the bisection of a line segment (prop. I.10) is also interesting. Proclus begins by recalling that the very act of bisecting seems to contradict a view on geometric entities as constituted by multiplicities of points.[20] A multiplicity being something which can be numbered, it would not be possible to cut in two parts a segment composed of an odd number of points (*In Eucl.* 278). But Proclus also refuses to consider infinite divisibility as an assumption entailed in the position of continuous magnitudes. All we need, according to him, is mere continuity described in terms of contact and possibility of action (division).[21] Infinite divisibility is then inferred as a consequence of this possibility by the (provable) fact that there exist incommensurable magnitudes (i.e. for which the alternate subtraction of one from the other cannot be a finite process cf. *Elements* X.2). Once again, we see imagination circulating between indivisibility and divisibility, but refraining from fixing these features into objective forms (a line constituted of points or an actual infinite division attached to continuous magnitude in and of itself). This makes it possible to relieve the traditional paradox of positing at the same time divisibility with no end and indivisible entities—or, better, in parallel to I.12, it transfers the paradoxical aspect to a dynamic of imagination which does not have to be fully transparent to conceptual determinations.

The infinite straight line and the point are however a very particular type of representations. If along with Peirce we define a diagram as a type of "icon" characterized by the fact that certain relations in the situation represented are carried by the *representamen*, they lead to the following paradoxical situation: the very nature of the *representamen forbids* us from mapping what are supposed to be the *characterizing* properties of the objects under study (non-divisibility for the point; actual infinity for the infinite straight line). For this very reason, it may look as if they constitute exceptions, related to the mysterious treatment of the infinite and indivisibility. As such, they could be treated with particular conventions of representation. What I would like to argue in the next section is that they are far from being isolated cases, although they may have pushed Proclus to explicate things which would have otherwise remained implicit (the autonomous position of geometric space and its relative opacity). Proclus' commentary is full of other diagrams which raise the same type of issues, and for a very good reason: so is ancient Greek geometry.

[20] An argument already put forward by Sextus Empiricus (*Adv. Math.* IX 282–283) and coming from the Epicureans, although they used it in an opposite strategy (in order to show that one cannot follow geometers in their claims) cf. Benatouïl (2010, pp. 156–157).

[21] This would go in the same direction as Panza's general discussion on continuity (see note 19 above and Panza 2012, Sect. 1.3.1, p. 72 sq.).

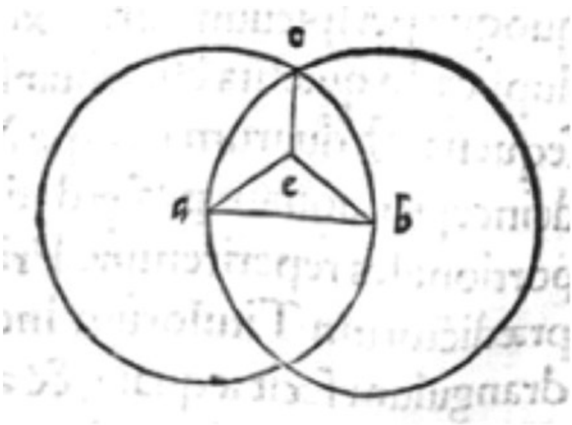

Fig. 1 Zeno's objection to I.1 (diagram from Barozzi 1560, p. 122)

2 Geometric Imagination in Practice

Let us look at the very first proposition of the first book of Euclid's *Elements* and what Proclus has to tell us about it. As is well known, this proposition asks to construct an equilateral triangle on a given finite straight line. The resolution goes on to construct the two circles whose centres are the extremities of the given segment and whose radius is the given segment. As was made famous by objections from modern geometers such as Pasch or Hilbert, a hidden assumption in the construction is that the two circles meet in a point which will serve as apex for the sought triangle (in Euclid's construction, only one of the two points of intersection is considered). What is less well known, however, is that a similar objection was already raised in ancient times. It is attributed by Proclus to Zeno of Sidon (*In Eucl.* 214), an Epicurean philosopher from the first century BC, and it deals not so much with the existence of the point of intersection as with its being well determined. How are we to be sure that we are not in a situation in which AC and AB have a segment in common and the apex of the sought triangle is not well determined? (Fig. 1).

The objection may strike us as odd, since it relies on a diagram in which straight lines are represented as not straight. Of course we know that geometric drawings are not to be taken as exact representations. But even if we don't pay attention to the drawing itself (as was certainly the case for Proclus who repeatedly dismisses any direct relation between mathematics and perception), the situation would remain problematic. Indeed one main difference between ancient and modern geometry is that some information has nonetheless to be retrieved from the diagram, be it interpreted as the concrete drawing or as some form of idealized counterpart.[22] These pieces of information typically feature inclusions of one region into

[22] For a survey on this question, see Manders (2008a).

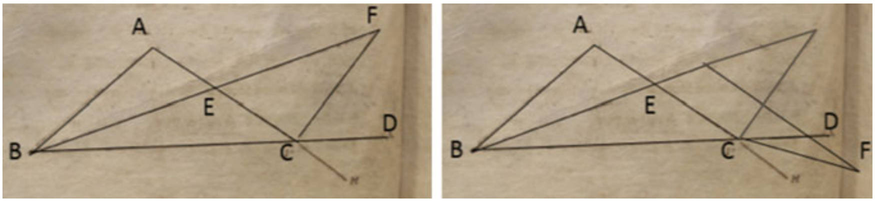

Fig. 2 *Elements* I.16 (*left*) and an alternative diagram with broken lines (*right*). As is well known, we do not have access to ancient Geometric diagrams and have to rely on late copies. For a survey on the question in the first books of Euclid's *Elements*, see Saito (2006) and more recently Saito and Sidoli (2012).

another.[23] Yet by allowing straight lines to be imagined as broken, we seem to run into serious difficulties when conducting this type of argument.

To make this point clear, let us read a typical Euclidean proposition. Consider, for example, prop. I.16 which states that "in any triangle, if one side is produced, then the exterior angle is greater than either of the interior and opposite angles". The demonstration asks us to bisect AC at E, join BE and produce it "in a straight line" to F. We then make EF equal to BE to obtain a triangle EFC similar to ABE, so that the angle BAE is equal to ECF. At that stage, we rely on the fact that ECD is greater than the angle ECF and we can therefore conclude that the angle ACD is greater than ECF (equal to BAE). The last information (ECD is greater than ECF) is a typical example of diagrammatic attribution (inclusion from one region into another leading to the conclusion that one is greater than the other). Let now suppose that we imagine another diagram allowing the "straight line" EF to be broken (as in the diagram on the right). In this case, angle ECF will contain angle ECD and a crucial step in the demonstration would not hold (Fig. 2).

Considering this situation, it may sound safe to reject the alternative diagram, and by the same token Zeno's objection, on the ground that the diagrams are not admissible. This is not, however, what Proclus does (nor, apparently, any of his predecessors).[24] He takes Zeno's objection seriously and answers by a proof. In order to do so, he first relies on the Euclidean definition of the straight line, which he takes to involve that the line is the shortest path and hence unique

[23] This belongs to the class of what Ken Manders has described as "co-exact" attributes: "The only claims based on diagram appearance in a demonstration recognize conditions that are insensitive to the effects of a range of variation in diagram entries: lines and circles that are not perfectly straight or circular, and cannot be taken to be without thickness. As we distort the 'circles' in I.1, their intersection point C may shift but it does not disappear. Such conditions I call co-exact. They include: part-whole relations of regions, segments bounding regions, and lower-dimensional counterparts" (Manders 2008a, p. 6).

[24] Other authors before him considered Zeno's objections and Proclus mentions a complete book written by Posidonius on this issue (see *In Eucl.* 216, 20).

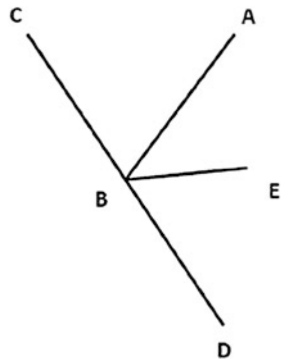

Fig. 3 *Elements* I.14

(*In Eucl.* 215).[25] The reason for this strategy, in which the diagram is not rejected from the outset, seems to be not so much that Proclus realizes the difficulty hidden in the question of intersection and the interest of Zeno's objection, but that such a diagram *is* admissible as such in Euclid (and more generally in Ancient Geometry). Indeed in many demonstrations in which the *Elements* proceed *ad absurdum*, they involve situations in which straight lines have to be imagined as not straight.[26]

Take for example *El.* I.14 which states that: "If with any straight line, and at a point on it, two straight lines not lying on the same side make the sum of the adjacent angles equal to two right angles, then the two straight lines are in a straight line with one another". The demonstration proceeds *ad absurdum* by supposing that the two lines CB and BD, touching AB in B and satisfying the conditions on angles, are not on a straight line (although they are drawn in a straight line in the extant diagrams!). Then a line BE is introduced, which is supposed to be "in a straight line" with CB (but not represented as such!). One then shows that the resulting conditions on the angles are contradicted by the fact that the angle BE *is contained in* the angle BD. This last condition seems to work only because the "straight line" CBE has been imagined as broken and CBD not. It does not seem possible to get rid of this representation (for example, exchange the representation of CBD with CBE or represent both of them as broken lines) and maintain the argument as it stands (Fig. 3).

Another famous example of *reductio* is given by prop. I.27 which states that "if a straight line falling on two straight lines makes the alternate angles equal to one another, then the straight lines are parallel to one another". In the course of the

[25] After this first rejection, Proclus also proposes a *reductio ad absurdum* of the fact that two lines may have a common segment by showing that this contradicts the fact, which he demonstrated before, that a circle will be cut in two by its diameter (*In Eucl.* 216).

[26] In what follows, one should keep in mind that proofs by *reductio* are widespread in the *Elements*. By modern standards, their ratio in the total number of proofs seems even very high (around one fourth). Although not all of them involve absurd representations (see note 58 below), the latter constitute a significant number of them, especially in *Elements* Book III where they are numerous (nearly 50 % of the proposition are proven by *reductio*), see Vitrac (2012).

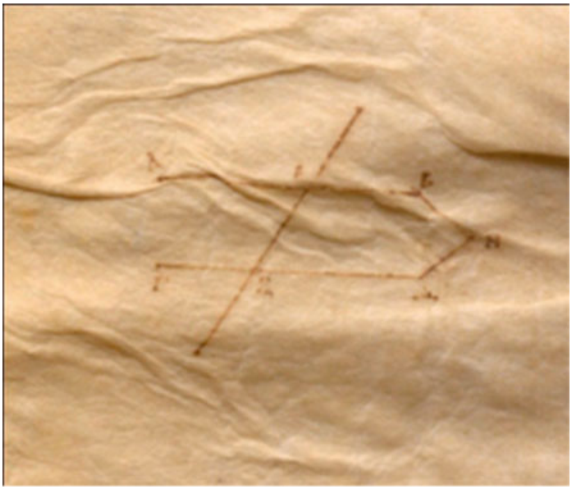

Fig. 4 *Elements* I.27 from the Bodleian copy (MS D'Orville 301)

proof, one assumes that the two straight lines meet and that they hence form a triangle with the one falling on them. The most common way to represent this situation is that presented in Fig. 4. In this representation, we see that considering a "triangle" whose sides are represented by broken lines, as I did in my alternative diagram to I.16 and as Zeno did in his objection to I.1, does not seem inadmissible.

At this point, one could object that this is not necessarily related to Proclus' view on the role of geometric imagination, since he is just here collecting objections without endorsing them or assuming that they are legitimate.[27] On this issue, let me first recall that "objection" is presented by him as a technical term of geometric discourse, on a par with "lemma", "case", "diorism", etc. (*In Eucl.* 212). According to the characterization given here, "objection" involves accepting counter-arguments attacking either the demonstration or the construction *without proof*.[28] In this sense, it is already unclear what an "illegitimate" objection would be and what grounds Proclus may have to reject an objection. It seems that any counter-argument based on an alternative diagram has to be accepted by the geometer, with the burden of proof lying on him. As we just have seen, there are good reasons for this principle of tolerance: considering the very functioning of *reductio ad absurdum* in the ancient geometric context, there seems to be no way of rejecting a diagram as illegitimate once and for all. A geometer who would answer to Zeno's argument by

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[27] Think of the objections related to the fact that there may not be "enough room" around a given diagram (*In Eucl*. 225, 16; 275, 7; 289, 21), which seems to come from an "empiricist" interlocutor.

[28] "An 'objection' (*enstasis*) prevents an argument from proceeding on its way by opposing either the construction or the demonstration. Unlike the proposer of a case, who has to show that the proposition is true of it, he who makes an objection does not need to prove anything; rather it is necessary [for his opponent] to refute the objection and show that he who uses it is in error (*In Eucl.* 212).

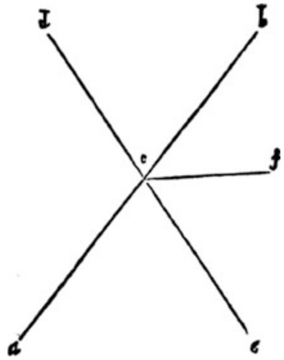

Fig. 5 Proclus's lemma in I.15 (Barozzi 1560, p. 172)

saying that what he represented as a triangle (or straight lines) could not be a triangle (or straight lines) would likewise have to block "regular" Euclidean proofs such as I.14 or I.27. On the other hand, it is clear that not all the alternative diagrams could be admissible, if we want the geometer to do something other than painstakingly answer any absurd configurations which could cross the mind of his objector.

I will come back to this issue below. Let me just note for now that even if it were the case that Proclus did not assume some of these objections as legitimate, it so happens that he also appeals in his own name to situations in which broken "straight lines" are involved. Commenting on propositon I.15, for example, he proposes to prove its converse by a *reductio* involving exactly the same type of diagram as the one used in I.14 (*In Eucl.* 302–305). In the diagram below CF is supposed to lie "on a straight line" with CD. Accepting this kind of representation naturally leads to accepting Zeno's objection as a serious one. What Zeno is doing is just transporting this type of representation, used by the geometer in some proof, in others (Fig. 5).

Another example of straight line represented as "non straight" and used by Proclus is given in the course of the commentary on I.4. There he criticizes the fact that Euclid assumed without proof that two lines cannot enclose a space (this is a crucial assumption in proofs by superposition such as I.4 or I.8 and was even incorporated in the "common notions" in some versions of the *Elements*).[29] Proclus undertakes to provide the missing argument and proceeds *ad absurdum* by relying on the following diagram (Fig. 6). In this case, straight lines are represented by arcs,

[29] Note that the problem is not unrelated to Zeno's objection. If we accept triangles with sides represented as "unstraight", we could make two triangles coincide on two of their sides and the angle contained by them without coinciding on the third. This will ruin the demonstrations "by superposition". The admissibility of such a diagram seems to be attested in the discussion about the curious "four sided" triangle, to which Proclus alludes in *In Eucl.* 329. Barozzi (1560, p. 189)

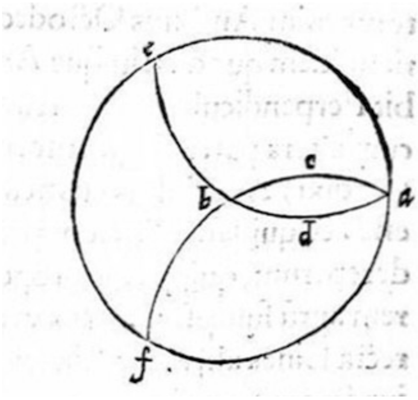

Fig. 6 Two "straight lines" which enclose a space (Barozzi 1560, p. 136)

a situation which is once again to be found in Euclidean proofs (see, for example, III.2)[30] and raises similar issues as broken lines do.

What I would like to emphasize with these examples is the following. If we pay close attention to the practice described by Proclus, be it in ordinary Euclidean proofs which he comments or in objections raised by him or by others, we see that there is nothing exceptional about the situation sketched in the preceding section. The very functioning of a *reductio* proof involves a clear distinction between two regimes which are not fully transparent to each other: geometric imagination on the one hand, which provides diagrammatic configurations which are admissible *prima facie*, and discursive reason on the other hand, which analyses these configurations.[31] The same holds for objections and, as I have shown above, there is an interesting interplay between the two situations: it does not seem possible to reject certain obviously "absurd" objections at first glance precisely because we need

(Footnote 29 continued)

With such a diagram, we could easily demonstrate the contradictory of I.4 by supposing that two triangles coincide on two sides (and the angle containing them) and then realize that one is nonetheless *contained* in the other. This possibility is ruled out by the fact that it would presuppose admitting two straight lines enclosing a space.

[30] Diagram in Saito (2008).

[31] I deliberately mimic Proclus' terminology (distinction between *phantasia* and *dianoia*, realm of the *logoi*). However, it should be clear at this point that "conceptual analysis" may involve information taken from the diagrams.

them to be admissible, at least hypothetically, in order to conduct regular *reductio* proofs.

All of these cases provide us with a geometric imagination which is therefore at the same time necessary, autonomous *and* opaque.[32] This could be understood in a quite straightforward way: imagination produces configurations which the conceptual process cannot recognize as legitimate. This is precisely why it may reject them after analysis.[33] We saw a more positive side of this opacity in the first section (it provides a possibility for representing infinite or indivisible entities), we now see its more negative side (although very useful for certain kind of proofs) and the dangerous game of objections without proof into which the geometer enters by allowing room for it in its practice.

If we take "figures" in a broad sense (including straight lines, points, angles, etc.) to be either the objects of ancient geometry or faithful representations of these objects,[34] the diagrams provided in our examples cannot be "figures" (or collections of "figures"): there is no such thing as a "triangle" formed by two straight lines and a third making equal alternate angles with them (*El.* I.27), no "circle" which cuts another in more than two points (*El.* III.10), no "straight line" joining two points of a circle and falling outside it (*El.* III.2), no "straight lines" enclosing a space or

[32] Considering the above example, it does not seem possible to claim, as Nikulin (2008, p. 160) does when presenting Proclus' concept of imagination, that "geometrical figures must be perfect, i.e. adequately represent their corresponding properties. Thus, a straight physical line is never straight, and a bodily circle is never round, whereas a geometrical straight line cannot be anything else but straight, and a circle nothing but perfectly round, which follows from their definitions. Therefore, sense perception, or *aisthesis*, cannot be the faculty responsible for the adequate representation of geometrical figures. Discursive reason, however, conceives geometrical objects in their properties as *logoi* which are not extended. This means that there has to be a distinct cognitive faculty capable of representing geometrical objects as figures, i.e. as extended and perfect.".

[33] This discrepancy between imagination and *dianoia* is paradoxically rendered obvious by recent reconstructions of Euclid's theory as a system in which the conceptual and the diagrammatic regimes are supposed to evolve in a perfect parallelism. In M. Panza (2012), for example, it is stated that the determination of the centre of a circle in *El*. III.1 is "in tension" with the rule of EPG (Panza's reconstruction of Euclidean Plane Geometry), because EPG provides "no possibility of constructing a circle without having previously constructed its centre, unless a rule for circles analogous to R.0 is admitted" (i.e. unless we accept that circles can be given as such, without any underlying construction). The problem is that we *need* the centre of a circle *not to be given* to perform the *reductio* proof in III.5 and III.6. In fact, the diagrams of III.5 and III.6 are simply not compatible with what Panza presents as a rule of construction for admissible diagrams of circles: "If two points are given, then two and only two concrete lines, each of which represents a circle having its centre in one of the given points and passing through the other, can be drawn" (p. 89). In Mumma (2006), many *reductio* proofs of book III concerning intersection and tangency (such as III.2 and III.13) are rejected from the outset, since the properties which they describe are consequences of the rule of construction of circles (see for example p. 26 where the convexity condition on representation of circles is stated). Same thing in the system proposed by Miller (2001), where III.2 and III.10 are presented as rules of formation for admissible diagrams (what Miller calls "nicely well-formed" diagrams, see Definition 2.1.5 p. 20 and Miller 2007, p. 26, Sect. 2, Definition 5).

[34] Calling a representation "faithful" when it preserves what are considered as the *characterizing* properties of the *representatum*.

having a common segment (without coinciding), etc.[35] This is precisely what *reductio* proofs prove. It is therefore legitimate to distinguish the figure, in the sense mentioned above, from the diagrammatic configuration presented to our imagination in order to conduct the proof.[36] The spatial proxy in which the *latter* occur is what Proclus calls geometric imagination and what he identifies with geometric space (usually two-dimensional space, since he deals here with plane geometry). One difficulty for the modern reader is that we tend to put geometric space and its various determinations in an objective position, fully captured by conceptual determinations (precisely because we tend to identify the consideration of space in mathematics with the fact of taking it as a proper object of study, see Sect. 1). This tendency can result in a considerable amount of confusion when attempting to comprehend a practice in which this was not done. Moreover, as I will try to indicate in the conclusion of this paper, it is not clear that our modern "structural" geometry relates to space *only* as an object or a system of objects. Quite on the contrary, one main feature of modern usages of space may well be that spatial configurations are used very generally not only as objects of study per se but as ways of acquiring information about objects of a non-geometric nature (in the sense in which, for example, a given non-geometric structure can be "equipped" with a topology).

*

Another interest of Proclus' testimony is to present us with a context of "objections and replies" which seems, from the examples collected and the names mentioned by him, quite widespread. The continuity between this dialogical context and the practice of *reductio* is nicely expressed in the discussion of I.7 where the Euclidean proof *ad absurdum* gives rise to the following discussion: "maybe perhaps some persons, notwithstanding all these scientific restrictions, will be bold enough to object and say that what our geometer calls impossible is possible" (*In Eucl.* 262, 5–6). The core of the objection is simply to present another diagram in which the conditions expressed in the Euclidean proof are not satisfied. Symmetrically, Proclus often replies to objections by showing that they lead to absurdity

[35] Note that this has nothing to do with the usual distinction between an imperfect physical drawing, either actually drawn or imagined, and a figure which would be its ideal counterpart. In our case, there is simply no ideal counterpart.

[36] See similar remarks by Manders, mentioned in the next section, against the "semantic" role spontaneously ascribed to diagrams and contradicted by the practice of *reductio*. See also Netz's remarks about the "make believe" elements contained in these proofs: one natural way to describe the situation is that we have a spatial configuration *pretending* to be such and such (a circle, a straight line, a parallel…) and conceptual analysis ruling out this hypothesis. Interestingly enough, Arab mathematicians had two different words for designating these two entities: the figure as representation and the figure as object, see Crozet (1999). This is related to an issue which I will not tackle in this paper, but which is another entrance in the question of spatial representations, that of the possibility of various configurations ("cases") representing one and the same geometric proposition.

(see the answer to Zeno *In Eucl.* 216, but also the answer to the objection to I.7 just mentioned *In Eucl.* 262–263).

More generally, this situation seems to be related to a common practice amongst practitioners in which one imagines a problem solved or a theorem proved and represents it in a diagram without knowing whether it is even possible to solve or to prove it—one of the meanings of "analysis" in a broad sense. In this regard, it is very striking that Proclus embeds his commentary on *reductio* in the following context:

> Every reduction to impossibility takes the contradictory of what it intends to prove and from this as a hypothesis proceeds until it encounters something admitted to be absurd and, by thus destroying its hypothesis, confirms the proposition it set out to establish. In general, we must understand that all mathematical arguments proceed either from or to the starting-points, as Porphyry somewhere says. (…) Those that proceed to the starting-points are either affirmative of them or destructive. But those that affirm first principles are called 'analyses' (…); when they are destructive, they are called 'reductions to impossibility', for it is the function of this procedure to show that something generally accepted and self-evident is overthrown. (*In Eucl.* 255–256)

Here we might also recall Pappus' famous description of "problematic analysis" in which he evokes the fact that, even in the process of analysis *stricto* sensu, one can stumble upon impossibilities.[37] An interesting piece of evidence for the continuity between these various aspects is given by "analyses" leading to cases under which a problem reveals itself "impossible" to solve. Although we do not have many testimonies on this practice in Ancient Greek Geometry (since we do not have many testimonies on analyses anyway), we do have some.[38] They tend to indicate an important role attributed to impossible configurations in the determination of the conditions under which a problem would be solvable. This corresponds to one of the senses of "diorism" whose purpose, according to Proclus, was "to determine when a problem under investigation is capable of solution and when it is not" (*In Eucl.* 66, 22).[39]

[37] Pappus, *Collection* 7.2.24–12: "We assume the proposition as something we know, then, proceeding through its consequences, as if true, to be something established, if the established thing is possible and obtainable, which is what mathematicians call 'given', the required thing will also be possible, and again the proof will be the reverse of the analysis; but should we meet with something established to be impossible, then the problem too will be impossible" (translation Jones 1986, pp. 82–83). Note that according to this description, only a *possible* geometric configuration can be considered as "given".

[38] Examples of this kind of analyses can be found in Apollonius (see, for example *Conica* II. 54) or in Eutocius commenting Archimedes (*On the Sphere and the Cylinder* II.4). On many occasions, including I.7 already mentioned, Proclus insists on the precision of the conditions expressed in the Euclidean propositions and mentions to this effect the impossibility encountered when they are not specified (*In Eucl.* 260–261).

[39] See also *In Eucl*. 202, 3–8 where Proclus explains that this is the place where geometry asks questions such as: "does the object exist as defined?" This passage is important to counter a widespread view on ancient diagrams according to which they are supposed to attest to the existence of the objects: if I can ask if the object exists as characterized in a proof, it may happen that the answer is 'no' (if not, why ask?). I shall come back to this issue later on.

I emphasize these aspects, because they indicate that the opacity of geometric space is not the result of some metaphysical views here, but of a certain practice. Opacity is a natural outcome of ignorance and ignorance a condition of discovery and progress. When we pose a question without knowing the answer, it is normal procedure to imagine the question solved and see what will result from there. It may happen that we can reduce opacity into knowledge, but it may happen that we cannot. In other words, we have represented something which is *stricto sensu* impossible to conceive. This is still a very useful situation, since it can help us either to show that the contradictory proposition of an assumed theorem is true, or to rule out a problem as impossible to solve, or to specify conditions under which a more specific proposition/problem may be true/solvable. This general setting may help us to explain why geometers may have been led to accept objections *without proofs*. It is normal standard in geometric practice to propose situations in which we do not know in advance if they admit a solution or under which conditions they do.[40] This is part of the game and a condition of progress. Hence it seems a normal aspect of ancient geometric practice that one can produce a diagram without *knowing* if this diagram is consistent with the conceptual determinations given by the geometric discourse. This spatial configuration which is not already known in every respect to us is another way to designate what I have pointed to repeatedly as a form of "opacity".

It is worth noting that Proclus expresses no reservation at all regarding the practice described in the present section. As I explained before, he takes various objections seriously and accepts that an objection does not even need the support of a proof. When explaining the process of *reductio*, he raises no criticism whatsoever against this way of proving (*In Eucl.* 255–256). He even presents it as the most natural way to prove converse theorems.[41] This may sound puzzling at first, since

[40] Note that it seems to remain true in a modern setting. One can ask, for example, what is the shape of the right angle triangle built on the bisectors of a given isosceles triangle. It also looks isosceles, but is it *really*?

With a little reflection, one may come to the conclusion that such a triangle is, in fact, "impossible". This problem, which I took from a study in Mathematics Education, works in Ancient *and* "structural" presentations of Euclidean Geometry, see Richard (2000).

[41] See especially the commentary on *El*. I.19: "It was obviously from a desire to avoid complexity in the order of demonstration that the author of the *Elements* avoided this method of proof [scil. an alternative direct proof mentioned by Proclus], preferring to proceed by division and reduction to impossibility, because he wished to establish the converse of the preceding theorem without anything intervening. (…). It is preferable to prove a converse theorem by the reduction to impossibility while preserving continuity than to break the continuity with the preceding demonstration. This is why he almost always proves a converse by reduction to impossibility" (*In Eucl.* 321.9–20).

Proclus is also well known for having emphasized the role of geometric constructions as a testimony for the *existence* of geometric objects.[42] This is often taken as being a distinctive feature of the ancient epistemology of mathematics, as opposed to "modern" approaches in which the symbolic means no longer give us evidence for the existence of objects.[43] Moreover, Proclus strongly correlated the constructive aspect of geometric proofs with their explicative power and did not hesitate to criticize Euclid in that regard.[44] This question played a very important historical role in the debate over the nature of mathematical explanation, especially at the beginning of the Early Modern Age, and does not seem without relation to (the first?) strong rejections of *reductio* proofs.[45] It also plays a pivotal role in the widespread parallel drawn by modern commentators between Proclus' and Kant's forms of "productive imagination" (sometimes along with Descartes).[46]

Hence the need to remember that criticisms against *reductio* are *not* to be found in Proclus (whereas they play a crucial role in Descartes and Kant, for example). Moreover Proclus' "constructivism" should be balanced by the fact that it appears only in very specific examples. In the discussion on the fact that geometric proofs *can* be causal (202, 9–25), Proclus explicitly states, without expressing any form of discontent, that this is *not* the case for the proofs *by absurdum* so widely used by geometers.[47] When mentioning the fact that Euclid used "both proof founded on causes and proof based on signs", he hastens to add: "but all of them impeccable, exact *and appropriate to science*" (*In Eucl*. 69, 10–13, my emphasis).[48]

An interesting passage on the role of "porism" makes it clear that one should not conflate the role of geometric imagination in general with that of *construction*, if we understand the latter in terms of "geneses" expressing causal processes:

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[42] See the famous comment on the fact that in the *Elements* the problems concerning construction of triangles precede the first Theorem (I.4), which Proclus comments in this way: "For unless he had previously shown the existence of triangles and their mode of construction, how could he discourse about their essential properties?" (*In Eucl.* 233–235). In her paper critically discussing the widespread "existential" interpretation of constructions in Euclid, Harari (2003, p. 5) recalls that "the main evidence in supporting the existential interpretation is found in Proclus' commentary on the first book of Euclid's *Elements*, where he accounts for the sequential priority of problems over theorems in existential terms".

[43] See Detlefsen (2005).

[44] See the famous discussion on *El.* I.32, mentioned in *In Eucl.* 206.12–26, and the related issue in the commentary of *El*. I.16 and I.17 (*In Eucl*. 309–312), cf. Harari (2008).

[45] Mancosu (1996).

[46] "The part played by imagination is Proclus' main addition to the Platonic theory, an addition which anticipates, it need hardly be pointed out, Kant's doctrine of schematism of the understanding" (Morrow 1992, p. lix). See also Bouriau (2000).

[47] "It is true that, when the reasoning employs reduction to impossibility, geometers are content merely to discover an attribute" (by contrast to establishing the reason for a given fact, *In Eucl.* 202.19–21).

[48] Note also what he says further: "if you add or take away any detail whatever, are you not inadvertently leaving the way of science and being led down the opposite path of error and ignorance?" (*In Eucl*. 69.27–70.1).

> Bisecting an angle, constructing a triangle, taking away or adding a length—all these require us to make something. But to find the centre of a given circle, or the greatest common measure of two given commensurable magnitudes, and the like—these lie in a sense between problems and theorems. For in these inquiries, there is no construction (γενέσεις) of the things sought, but a finding of them.[49] Nor is the procedure purely theoretical; for it is necessary to bring what is sought into view and to exhibit it before the eyes (δεῖ γὰρ ὑπ᾽ ὄψιν ἀγαγεῖν καὶ πρὸ ὀμμάτων ποιήσασθαι τὸ ζητούμενον). Such are the porisms that Euclid composed and arranged in three books. (302, 3–13)

This offers a nice setting not only to develop a more nuanced view of Proclus's concept of imagination, but also to understand why it appears at once in an active and a passive role.[50] Imagination allows the representing and exploring of the conceptual realm of objects with the resources proper to it. But at the same time it may offer some resistance to conceptualization. This last feature seems incompatible with the kind of productive imagination which was put forward by later philosophers—and is still quite widespread in various "constructivist" readings of Ancient geometry.

3 Philosophical Issues

Reflecting on the case of the *reductio* proof in Ancient Geometry, Ken Manders has claimed that traditional philosophical questions about the nature of geometric objects were ill posed because they assumed a semantic role of the diagrams which is simply *incompatible* with the practice of ancient geometers:

> Artifacts in a practice that gives us a grip on life are sometimes thought of in semantic terms —say, as representing something in life. There is, of course, an age-old debate on how geometrical diagrams are to be treated in this regard. Long-standing philosophical difficulties, on the nature of geometric objects and our knowledge of them, arise from the assumption that the geometrical text is in an ordinary sense true of the diagram or a 'perfect counterpart'. These difficulties aside, a genuinely semantic relationship between the geometrical diagram and text is incompatible with the successful use of diagrams in proof by contradiction: reductio contexts serve precisely to assemble a body of assertions which patently could not together be true; hence no genuine geometrical situation could in a serious sense be pictured in which they were. (Manders 2008b, p. 84)

From this "simple minded objection", Manders concluded that "the problem of the relationship between diagram and geometric inference here turns out to be one of standards of inference *not reducible in a straightforward way to an interplay of ontology, truth, and approximate representation*" (Manders 2008b, p. 86, my emphasis). This went along with a program of "inferential analysis of diagram-based geometrical reasoning" which resulted in a very nice and clear-cut result. Indeed, some inferences appear to be licensed by the diagrams and by the diagrams

[49] Note that the first example, the finding of the centre of a circle (*El*. III.1) is precisely the one which Panza found to be "in tension" with his reconstruction of Euclidean Plane Geometry (see note 33).

[50] On this issue, see Claessens (2012, especially p. 6, where the relevant literature is mentioned); Nikulin (2008, p. 164).

only (Manders coined the attribution on which they rely "co-exact"), whereas other inferences are licensed by the discourse and by the discourse only ("exact").[51] The main question is then to understand how the two inferential regimes (diagrams and text) can adjust so that one can have a good control over their interplay (especially as regards the range of admissible variations in the diagrams since the "co-exactness" criterion is not precise enough to rule out many apparently inappropriate configurations). We know that ancient geometry was not only a success, but also an efficient piece of machinery for producing results which are still considered, for good or bad reasons, to be true. But we do not know exactly why. This is one important aspect of what Manders has called the problem of "diagram control".[52]

Reviel Netz, in his seminal study on the shaping of deduction in Greek Mathematics, also put some emphasis on the *reductio* proof in order to discard a naïve "semantic" view of diagrams. But he reached a slightly more dramatic conclusion: "We seem to have reached a certain impasse. On the one hand, the Greeks speak as if the object of the proposition is the diagram. Verbs signifying spatial action must be taken literally. On the other hand, Greeks act in a way which precludes this possibility (quite regardless of what their ontology may have been!), and the verbs signifying spatial action must, therefore, be counted as metaphors" (Netz 1999, p. 54). He proposed to solve this puzzle by emphasizing the "make believe" element entailed in this kind of proof:

> The proof, of course, proceeds with the aid of a diagram. But this is a strange diagram: for good geometric reasons, proved *in this very proposition*, such a diagram is impossible. Euclid draws what is impossible; worse, what is patently impossible. For, let us remember, there is reason to believe a circle is one of the few geometric objects a Greek diagram could represent in a satisfying manner. The diagram cannot be; it can only survive thanks to the make-believe which calls a 'circle' something which is similar to the oval figure in Fig. 7. By the force of the make-believe, this oval shape is invested with circlehood for the course of the *reductio* argument. The make-believe is discarded at the end of the argument, the bells of midnight toll and the circle reverts to a pumpkin. (Netz 1999, p. 55)

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[51] Exact attributes "are those which, for at least some continuous variation of the diagram, obtain only in isolated cases". The latter "are those […] which are unaffected by some range of every continuous variation of a specified diagram" (Manders 2008b, p. 92). For a detailed and critical discussion of these criteria, see Panza (2012).

[52] The whole Sect. (4.1) is entitled "Euclidean diagrams: artifacts of control or semantics?" (pp. 82–87). It is introduced in the following manner: "At its most basic, a mathematical practice is a structure for cooperative effort in *control* of self and life. In geometry, this takes many forms, starting with the acceptance of postulates, and the unqualified assent to stipulations—and as it appears, for now, to conclusions—required of participants. Successes of control may be seen in the way we can expect the world to behave according to the geometer's conclusions; the way one geometer centuries later can pick up where another left off; the way geometers can afford not to accept contradiction. When the process fails to meet the expectations of control to which the practice gives rise, I speak of *disarray*, or occasionally, *impotence*. Such occurrences are disruptive of mathematical practices; they tend to reduce the benefits to participants and to deter participation. At best, they motivate adjusting artefact use, modifying the practice to give similar benefits with less risk of disarray.".

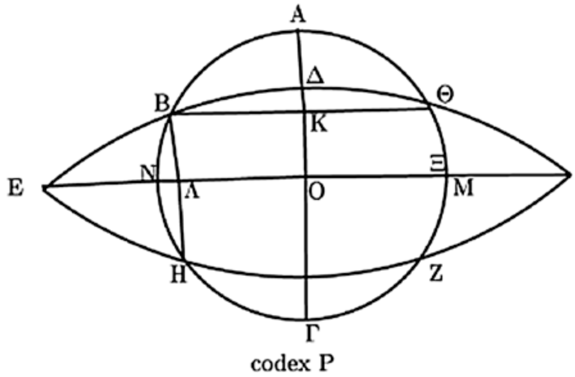

Fig. 7 Euclid's *Elements* III.10 (diagram from Saito 2008, p. 59)

But the "make believe" element, although undeniable and often mentioned when dealing with the *reductio* context, gives us no clue about the rules of the game.[53] Remember my alternative diagram to I.16: when are we to accept such a representation and when are we to reject it? There seem to be some fictions harder to swallow than others.

It is not sufficient to answer that "impossible" diagrams are only admissible in the framework of *reductio* (as does the "make believe" argument), since this is just begging the question: why not forge new proofs with new "impossible" diagrams and prove in this manner some propositions *incompatible* with Euclid's ones? This is the real question raised by my alternative diagram: by mimicking Euclid's reasoning, one could now show by *absurdum* that the exterior angle is *not* greater than one of either the interior and opposite angles. One just needs to suppose that it is greater, draw the alternative diagram following Euclid' instructions (but with the representation of a broken line) and reach the absurd conclusion that ECD is at the same time greater than ECF (equal to BAE) and contained in it. If one objects that we do not have the right to retrieve this last information from the diagram, we will reply that this is precisely the kind of information that Euclid retrieves from his own configuration. If one objects that we do not have the right to imagine a straight line in such a way, we will reply that this is what Euclid does in I.14. There is clearly a problem of "diagram control" here and this problem occurs *inside* the general regime of fictitious configurations (Fig. 8).

An hypothesis often mentioned is that such "impossible" diagrams may have been *temporarily* admissible precisely because they allowed the *excluding of* forms of representation and therefore progressively the limiting of the range of admissible variations. Diagram control and the *reductio* proof would then go hand by hand.[54]

[53] In a subsequent paper, Netz himself emphasized the fact that the role of imagination in ancient mathematics is quite widespread and not limited to impossible diagrams. In all of these cases, the "make believe" elements are of first importance (Netz 2009).

[54] "Diagram control theory invokes our ability, using geometrical constructions, to produce reasonably accurate physical diagrams, and so limit the diagram appearance outcomes to be considered by physical diagram production rather than discursive argument. Conversations with

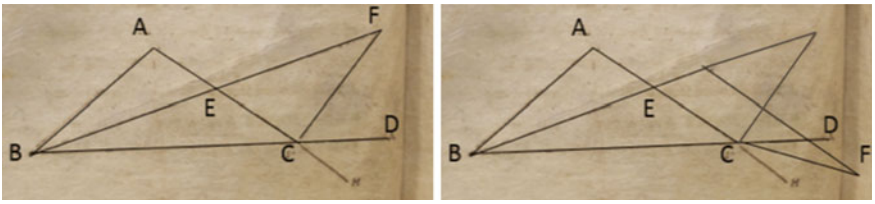

Fig. 8 *Elements* I.16 (*left*) and an alternative diagram with broken lines (*right*)

But we already know by the example of Zeno's objection that this does not seem to be the case: if we exclude the diagram in I.14 and not before, we will solve the difficulty for I.16, but not for I.1 (Zeno's objection); and if we reject it as early as I.1 (for example, following the reasons developed by Proclus), we would have to reject it in I.14. Helped by the larger view presented in the preceding section, we can easily generalize this dilemma. We just have to play the same trick as Zeno's by retrojecting one of the "impossible" diagrams taken from a *reductio* in a previous proof where it is still supposed to be, according to the above mentioned view, admissible. If one then objects that what it shows is that the diagram was already non admissible at that stage, the later proof from which it was taken will become itself non-admissible.

Take, for example, III.13 which states that "A circle does not touch another circle at more than one point whether it touches it internally or externally". It relies on a diagram of the following type (Fig. 9) in which we suppose circles touching in two points.

We can plug this diagram (for example for the case of circles touching internally) into a previous proposition such as III.6 which states that "if two circles touch another, they do not have the same centre". The result will amount to blocking what produces the contradiction in the known Euclidean proof, that is to say that ZE is shown to be at the same time equal to ZB (by the fact that the two circles have the same centre Z) and less than ZB (by inclusion in the diagram: this condition is satisfied in the diagram on the left, but not in the alternative diagram on the right, see Fig. 10).[55]

I leave it to the reader to play this game with other propositions. We have already seen that we can shortcut with broken "straight lines" (or "four sided triangles") the extant proof of I.4 or I.8, which are of pervasive use in the *Elements*; the same for I.16, which is needed for I.17 and I.18, themselves used in I.19 and

(Footnote 54 continued)
specialists suggest this is the basic tool of ancient practice, *with reductio argument* for the exclusion of putative alternatives as backup" (Manders 2008b, pp. 70–71, my emphasis).

[55] There may, of course, be discussions about whether to exclude the alternative diagram, for example by emphasizing that ZB and ZE are chosen in a particular position. But, even in this case, the demonstration would be different than the one given in the *Elements*.

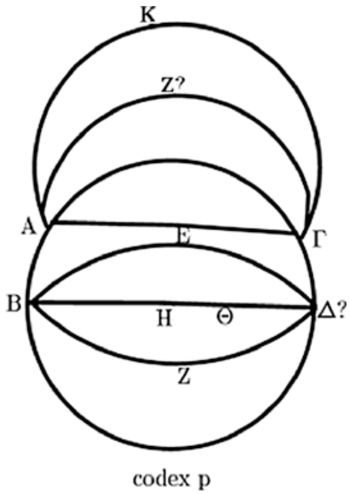

Fig. 9 *Elements.* III.13 (diagram from Saito (2008), p. 62)]

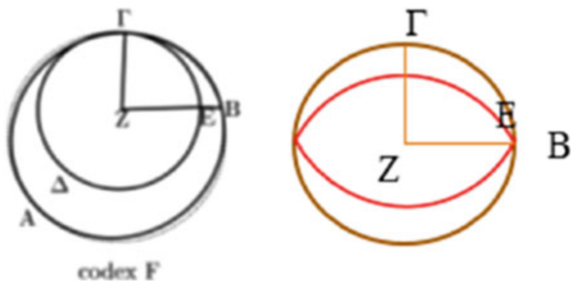

Fig. 10 *Elements*. III.6 and an alternative diagram derived from III. 13

I.20, etc. Many propositions of Book III rely on these results and when they do not, they are subject to the problems presented above in the case of III.6, which is easily generalized. Bit by bit, the whole edifice of the first books of the *Elements* appears in danger of collapsing, if not by contradiction, at least by multiplication of "impossible" diagrams to rule out at each step.

Of course, there is no real danger here. One will always be in the position to reject one of the diagrams presented so far through conceptual analysis. This is the very reason why we called them "absurd" or "impossible": they exhibit features which contradict the characteristic properties of the geometric configurations under study (in III.6, for example, it is easy to indicate by inclusion of one segment in another that we have *radii* in the inside circle whose distance to the centre have to be unequal). This is what Proclus first undertakes to show when he wants to answer an objection. But something is worth stressing at this point: if the result of conceptual analysis leading to this rejection is then considered to be a *rule* for the admissibility of diagrams, then many of the *reductio* used by Euclid will hence have to be *rejected* according to that very standard.[56] If, on the contrary, some other norm *allows* the introduction of the diagram, it will also allow it in situations *in which it has to be rejected*. This may be, according to my account, one of the reasons why objections were taken seriously. This is also why it seems so difficult, if at all possible, to determine a set of rules that precisely fix the range of variation for diagrams in ancient Geometric practice. If no information were retrieved from absurd diagrams, this would be unproblematic. Unfortunately, as we have now seen in many examples, this is not the case.

Neglecting however the quest for a complete system of rules, there may be an easy way out of the preceding dilemma. Indeed one can always introduce an absurd diagram and answer to possible criticisms that one *knows* ("by concepts") that this diagram is absurd. In the case of *reductio*, the objection against the diagram would then not be sufficient to block the proof. Indeed we could easily reply that this absurdity is precisely what we want to establish. In other words, we would *agree* with our interlocutor on the "absurdity" of the diagram, which is just an illustration of the absurdity of the premise. If the objector still maintains that we don't have the right to assume such an absurdity, what she or he will be criticizing will not be the representation, but the structure of the proof. But this seems to imply that the "control" over diagrams will involve in a substantial manner some form of semantic regulation (an "illustrative" role for diagrams). In particular, it would impose further constraints on the "make believe" component: the absurdity of the diagram will be admissible if and only if it is directly related to (if it "illustrates") the claim which we want to refute. Yet it so happens that this condition is *not* fulfilled in my alternative diagram to I.16 (even when the proposition is recast into the form of a *reductio*) or in Zeno's objection to I.1. In both cases, the absurdity of the diagram is not related to the particular claim in play. We hence have no reason to accept it if it enters in conflict with conceptual analysis and this is precisely what Proclus intends to show.[57]

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[56] Not all of them however, since they do not all rely on "impossible" diagrams in the sense mentioned here. I.6, for example, represents two equal segments as unequal, which is *normal* practice. Even I.27 does not seem to be inadmissible since we do not really need the absurd diagram (with broken lines) and can just draw two straight lines which meet in a "regular" triangle (as is drawn in one of the diagrams in the Bodleian copy, MS D'Orville 301). These examples should not be put on the same footing as I.14 or III.2 (*pace* Panza (2012), p. 82, n. 53).

[57] Let me make this argument more explicit. In the case of Zeno's objection, Proclus' first reaction is to reply that the diagram contradicts the condition imposed on what it is to be a straight line. As I have tried to argue above, this reply does not justify an immediate rejection of the diagram and this is no surprise since we need this diagram to be admissible in other cases (typically I.14). The question is then: how is it that the diagram of I.14 is not ruled out on the same grounds? My interpretation is that the diagram remains admissible in I.14 after conceptual analysis, because what it shows is precisely what the proposition *claims*: it is not possible to draw a line satisfying the conditions on angles and not being "in a straight line" with the given one. In any *reductio* of that form, we will be forced to represent a "straight line" by a non-straight one: this is no objection, this is the claim!.

Of course, one will always be in a position to emphasize the "make believe" elements and claim that there is no real absurdity involved in this kind of representation. Of course, one is just *pretending* that what is apparently not a circle or a straight line in the diagram is a circle or a straight line. This does not seem different from claiming that what does not appear as an indivisible entity or an infinite one is. Since points without part and lines without breadth are pervasive in plane Geometry, what we are describing here is just the *regular* functioning of the geometric imagination. These are strange representations, since they exhibit features incompatible with what they represent, but this is simply the *general* problem of geometric representation. Moreover, the "inferential analysis of diagram-based geometrical reasoning" helps give a precise contour to a fact which is widely accepted: in geometry, we do not need representations to be faithful, but trustworthy. For that purpose we only need diagrams to carry certain kind of information. There is no more difficulty in the fact that they exhibit properties incompatible with the property of the objects they represent than in the fact of writing "red" in green ink and "green" in red ink. So far so good. But I would like to emphasize that this general view leaves the problems raised above *untouched*. What seems to have escaped attention until now is that *the same diagram* must be admitted in some proofs and rejected in others. In other words, diagrams of the sort that we are dealing with are *not* trustworthy. As a consequence, the problem of "diagram control" does not seem to be solvable by fixing a range of variation once and for all (since the same variant will have to be admissible in some proofs and not in others); neither will it be solved by stating a set of fixed rules of construction.[58]

More generally, many questions are raised by our detour through Proclus about the role of these "constructions" in Euclidean Geometric practice. If we take *El.* III.6 seriously, we can draw a circle without knowing where its centre is. If we take *El.* III.2 seriously, we can join two points on a circle by a straight line without knowing where the straight line stands relative to the given circle. How would these claims be consistent with a constructive reading of lines and circles?[59] More generally what do I "construct" when imagining two "circles" intersecting in more than two points or touching internally in more than two points? What do I "construct" when representing a "straight line" which does not coincide with the line making angles equal to two right angles at a given point on another straight line? Many modern philosophical interpretations of Ancient Geometry seem too imbued with a view in which constructions are essential because they are supposed to give evidence for the possibility of concepts and/or existence of objects. But this is far from obvious as soon as one realizes that it was standard practice in Ancient

[58] This is also why we cannot be fully satisfied with overly general rules such as the Euclidean postulates, which allow "impossible" diagrams without specifying the cases in which they have to be rejected. If the first postulate allows us to represent a "straight line" by a broken line, as it seems, or if the third postulate allows us to represent a circle with an eccentric "centre", they allow "constructions" which will have to be *prohibited* in some other proofs.

[59] Remember that postulate III asks us to draw a circle *from any centre* and radius: Καὶ παντὶ κέντρῳ καὶ διαστήματι κύκλον γράφεσθαι.

Geometry (and not only in Ancient Geometry!) to imagine a situation which is not known in advance to be even *possible* (i.e. "constructible", if possibility and construction of objects are equated). In my view, a recognition of what I have designated (following Proclus) as the "opacity" of space and the autonomous functioning of the geometric imagination offers a more promising account.

Another way to designate this opacity would be to stress that ("spatial") images and ("geometric") concepts do not evolve in a perfectly parallel manner and that this is not something that we have to fix. It is true that this discrepancy may sound strange at first. As I have recalled above, the usual expectation would be that spatial images ("diagrams") are, if not faithful, at least, trustworthy. Unfortunately, this does not seem to be the case, at least in a straightforward way. But much of this strangeness may vanish if one realizes that the coupling of heterogeneous entities that relate one to the other, but do not evolve in perfect parallelism is a common situation in semiotic systems.[60] In this regard, it is very interesting to note that, up to now, nobody seems to have succeeded in coming up with a complete system of rules fixing the range of the tremendous variations occurring in the writing and/or pronouncing of one and the same phoneme in a given language. The complexity of automatic recognition systems are good evidence for that. They still rely massively on the statistical method and on a control operated a posteriori by the knowledge of the linguistic "content"—as opposed to systems of rules fixing a priori the variation of graphemes or phonemes in and of themselves.[61] This is not the place to enter into the fascinating questions related to these systems of co-variation, but let me just emphasize here that stabilization does not seem to occur in symbolic systems *only* through a complete and fixed system of parallel rules (although there *are* some structural rules on both levels). It also usually involves in a substantial manner the *interplay* between the different levels. As I have tried to indicate in this section, this seems to be the kind of "control" which is at stake in Ancient Geometric practice. The fascination for "formal language", in which this interplay is carefully shortcut, and the (false) conviction that it offers a model for any symbolic systems used in mathematical sciences may explain why we still have difficulty applying this description to mathematical symbolic systems.

This way of approaching Proclus by laying emphasis on the interplay between image and concept is of particular interest when faced with the epistemological issues which we have encountered so far. Let me first summarize them. On the one hand, we have the fact that geometric texts talk about something which, on their

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[60] Beginning with the coupling of acoustic images and concepts in natural languages, one of Saussure's deep insights when launching the "structuralist" approach to language, see Maniglier (2006).

[61] Crettez and Lorette (1998). Even in very standardized systems such as typographical ones, we encounter not only important variations for the same grapheme, but also graphical signs which can be very similar for different graphemes (such as d and ∂, for example). Symmetrically, one could note the similarity of graphical signs used in one and the same system for representing different graphemes (such as d and a).

surface grammar, *is* the diagram;[62] the *reductio* proofs, however, make it clear that the surface grammar is misleading here and that geometric texts cannot talk about the actual configuration which they exhibit; at the end of the argument, as Netz puts it, "the bells of midnight toll and the circle reverts to a pumpkin". Call this the semantic problem or Netz's "impasse". On the other hand, one could declare the approach in terms of "objects" to be misleading and consider that one has to focus mainly on the inferences carried with the help of our two resources: texts and diagrams. In the absence of semantic rigidity, what we need now is a good adjustment of two inferential regimes which are presented as evolving in a parallel and complementary way. The problem, which I have been pointing out regularly in this paper, is that the variation of diagrams does not appear to be *intrinsically* regulated: one and the same diagram has to be accepted in one proof and rejected in another; moreover, this seems to depend on the "content" of the proposition in play, that is to say… on a "semantic" regulation. Call this a (dramatic) variant of the problem of "diagram control".

Proclus's position falls so to speak in between these two options. It may even help us to rephrase them in a more positive way. Let me, for example, restate the first dilemma in a Proclean manner: contrary to a lazy and narrow Platonist interpretation, it is not possible to say that Geometry deals only with ideal figures, the diagram being just a dispensable auxiliary. This is made obvious, amongst other reasons, by cases in which one has to represent a geometric situation to which *no* ideal entities correspond in the conceptual setting (be it an infinite straight line, a circle intersecting another circle in more than two points, two straight lines enclosing a space, etc.). However, and for the very same reason, it is not possible to avoid the *distinction* between entities characterized by concepts and diagrams. Remember that at the end of the argument, the circle reverts to a pumpkin. In other words, the circle, as presented by the definition in the text (and propositions exhibiting such and such of its properties), has to be something *different* from what was represented as such in the diagram. There is no impasse here if one accepts that mathematical knowledge stabilizes itself in the *interplay* between these two regimes which Proclus calls "discursive reasoning" and "imagination".[63] This means to accept that imagination has a form of autonomy and does not limit itself to

[62] This aspect is documented in great detail in Netz's book. This relation is the basis of the "ontological" issues then engaged in order to assess the link between concrete diagrams and possible abstract objects.

[63] The interest of a semiotic approach is to detach the role of imagination from its geometric origin and reveal its more general nature. When, for example, one supposes in abstract algebra that the centre of a p-group is trivial (reduced to identity) in order to show that it leads to absurdity, the usual proof proceeds as follows: one decomposes the group in its conjugacy classes and obtains an equation of the form $p^k = 1 + p^i + p^j + \cdots + p^m$ (p^k being the order of the group). This latter formula is an "impossible diagram" (since it represents a p which is supposed to divide both side of the equation) in much the same sense as the one we encountered in this study (see van der Waerden 2003, p. 153 for this classical proof in abstract algebra). The opacity of geometric space transfers immediately to the opacity of symbolic writings, as was remarkably seen by Leibniz who called both of them "characters" and associated them with "symbolical" or "blind" knowledge.

illustrating some purely conceptual process. As regards the second problem, Proclus clearly emphasizes the dynamical aspect internal to the two regimes and the fact that they provide complementary systems of inferences, *but* he also points out another aspect of particular importance: the fact that imagination occasionally provides situations which are opaque to knowledge. As I have tried to document above, this is another very important role of diagrams: they not only help us to prove propositions, but also to represent situations which we do not know in advance to be possible. In this sense, the control cannot stem from an internal system of rules fixing the kind of information which can be retrieved from diagrams. It also needs the regulation of conceptual knowledge.

4 Conclusion

As a conclusion, I would like to sketch, as announced in the title of this paper, what I take to be the actuality of some of Proclus's insights on geometric space. Before doing so, let me summarize the principle results of this study. We have seen that geometric imagination (*phantasia*), which Proclus identifies with geometric space or a form of "receptacle" on which discursive thinking (*dianoia*) projects its conceptual determinations, is not limited to a passive role of illustrating or picturing. It also has an autonomous activity, which manifests itself in the form of a possible opacity to conceptual knowledge. The metaphor comes directly from Proclus who mentions that sight can paradoxically recognize what escapes its power: darkness. Between complete obscurity, evoked in the case of the diagrammatic representation of actual infinity, and the bright light of concepts stand many other situations in which some parts of our representations are clear and some are not, many forms of *chiaroscuro*, so to speak. Moreover, not only can we picture darkness, but we can picture *with* darkness in order to make what has to be visible visible. What I have tried to do in the second section of the paper is to document this art of *chiaroscuro* in Proclus and show that it seems to correspond to a standard practice in Ancient Geometry. This allows us to provide to Proclus' conceptions not only with an immediate context, but also with a first actuality. When related to geometric practice, Proclus conception of space appears as a fruitful framework that can be used to solve certain difficulties encountered in philosophical reconstructions of ancient geometry, because they don't pay enough attention to this specific functioning of geometric imagination. What I have tried to emphasize in the third section is that this may even help us to overcome certain epistemological "impasses" in which recent studies seem to be stuck.

This overall picture conforms to two features of Proclus' philosophy which I had no space to develop in this paper, but which are of tremendous importance to situate his thought. First, Proclus, along with Jamblichus, disagreed with other members of the Neoplatonic School, such as Porphyry, on the role of logic in its relation to mathematics. The latter considered, in an Aristotelian vein, that logic is a universal science, attached to the structure of predication and governing the rules of reasoning, which has therefore to be learned *before* any particular science, such as mathematics. The former argued that mathematics is the proper place to learn the general art of reasoning ("dialectic"), which extends far beyond what logic in the above sense can teach us.[64] This is related to the fact that mathematics involves other elements of discursive thinking than mere deduction, such as definitions, divisions (of cases) or analyses.[65] A second distinctive feature of Proclus' philosophy concerns where he diverges from Jamblichus, that is: on where to situate the universal mathematical science. Whereas the former conceived of it, following a Neopythagorean tradition, in close connection to arithmetic, Proclus and his pupils considered it to be closely connected to Geometry. This is made clear in Marinus' commentary on Euclid's *Data* where he states that this treatise does not belong to any particular mathematical theory, but to the "universal mathematics"; he then emphasizes the fact that Euclid has presented other aspects of this general mathematics in Book V of the *Elements* (dealing with ratios and proportions), but *under the guise of a geometric presentation.*[66]

As I have tried to argue elsewhere, we have here different visions of what it means to give "foundations" to mathematics.[67] It would not seem exaggerated to state that the first direction was the leading option in terms of foundations from the middle of the nineteenth century to the middle of the twentieth century. It is not surprising that it accompanied a debate focusing on the disagreement between those who held that mathematics starts with logic and those who held that it starts with (basic) arithmetic and systems of numbers. Although there were always mathematicians who resisted that general tendency (and even more mathematicians not interested in foundational issues!), not many of them protested that geometry was the proper place for foundations. Things began to change significantly in the 1960s, when the unexpected relationship between geometry and logic began to emerge. In a philosophical "manifesto" entitled "Logic as a geometry of cognition" Jean-Yves Girard, a leading protagonist of this evolution, has called it—following a suggestion

[64] See Jamblichus *De com. Math.*, Chap. 29; Proclus, *In Eucl*. Chap. XIV and 69, 8 sq; for a commentary: O'Meara (1989), pp. 47–48 and Chap. 8.

[65] One may find surprising that Proclus credits Euclid's *Elements* for exhibiting forms of analysis, since according to the standard picture of the treatise, it seems the prototype of the *synthetic* method. But I hope to have given elements to better understand this claim: in a broad sense, analysis designates any way back to the principles. This is not what Proclus calls *analysis* strictly speaking in *In Eucl*. 255–256, but this broader sense is clearly stated in other places such as *In Eucl.* 8.9 and 57.19. In these passages, analysis is characterized more loosely as the method of proceeding from complex to simple, from things we seek to know to things better known. As regards division of cases, it should be noted that it is another entry into the question of the discrepancy between spatial representation and geometric object, which I could not deal with in this paper. Interestingly enough, this path was followed by Arab mathematicians who distinguished several meanings of "figures", see Crozet (1999).

[66] Marinus, *Com. in Euclidis Data*, 254, 5–27 and Rabouin (2009).

[67] Rabouin (2009), Chaps. I–III.

by Samuel Tronçon—a "geometric turn" (by contrast to the "linguistic turn").[68] I cannot resist quoting the incipit of this provocative paper:

> I. 'Les Grands cimetières sous la lune'
> To place philosophy again at the centre of scientific activity, to rehabilitate philosophy of science, what a program! In order to do so, we propose to reactivate the central tool of *logic* by extracting it from the narrow path of the 'linguistic turn'; this reactivation would operate with the tool of *geometry*, a 'geometric turn' so to speak (p. 15).[69]

This is not the place to comment upon this evolution, which has many facets and parallel developments. But it is certainly a task for the philosophers of our time to understand what it might signify. Supporters of the "logico-arithmetical view" had a very nice story to tell about their construction, a story going all the way back to Aristotle and justifying the foundation in terms of building from the more general to the more particular (whether in terms of sciences dealing with general forms, as opposed to science dealing with specific domains of objects, or in terms of domains of objects simpler than others and needed for their construction). First logic, then arithmetic, then geometry—the pending question being when exactly does mathematics start in this overall picture (think of the ambiguous status of Set Theory). But what kind of story, if any, could support the universality of geometry?

Proclus is not very explicit about this issue, but his general strategy makes it clear that in his eyes, geometry exhibits universal structures which not only can be useful for dealing with continuous magnitudes *and* numbers, but can also enrich the tools considered by logic (tools which remain invisible as long as one only considers the structure of predication and of direct deduction). This directly contradicts the picture according to which the "above" science is completely independent from the sciences "below". This is, so to speak, a universality "from below", from the point of view of what is *transversal* to the whole science considered. The point which I would like to insist on is that this view seems to be related to the issues tackled in this paper. Indeed, it amounts to considering spatial configurations not only as related to domains of objects, as is the case in geometry *stricto sensu*, but also as useful *tools* for studying other kinds of mathematical objects, *including* geometric objects themselves! In this picture, space acts in an ambivalent position: either as a framework in which one studies objects or as a means with which we study them. Call it the passive and the active role of geometric space. Moreover, as a tool, space need not to be fully transparent to one of the identified domains of objects.

It is worth noting that this kind of story about the universality of space was heavily emphasized in recent times by mathematicians involved in the "geometric turn". Alexander Grothendieck, when commenting on his idea of introducing "generalised spaces" declared:

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[68] Girard (2007).

[69] My translation, except for the title of the section which is taken from a book by Georges Bernanos and which I left in French.

> La notion d'‘espace’ est sans doute une des plus anciennes en mathématique. Elle est si fondamentale dans notre appréhension ‘géométrique’ du monde, qu'elle est restée plus ou moins tacite pendant plus de deux millénaires. C'est au cours du siècle écoulé seulement que cette notion a fini, progressivement, par se détacher de l'emprise tyrannique de la perception immédiate (d'un seul et même ‘espace’ qui nous entoure), et de sa théorisation traditionnelle (‘euclidienne’), pour acquérir son autonomie et sa dynamique propres. De nos jours, elle fait partie des quelques notions les plus universellement et les plus couramment utilisées en mathématique, familière sans doute à tout mathématicien sans exception. Notion protéiforme d'ailleurs s'il en fut, aux cents et mille visages, selon le type de structures qu'on incorpore à ces espaces (Grothendieck (1985–86), p. 52).[70]

Interestingly enough, it so happened that the concept of *topos* revealed itself as one of the forms in which logic and geometry appear as intimately related one to each other.[71] My intention, once again, is certainly not to comment upon these highly technical questions, which I am far from fully understanding, nor to claim that we have here the "right" candidate for foundations in mathematics. The mere fact that we have already two very different proposals (Girard, as he explains in his paper, is influenced by non-commutative geometry, not by topos theory),[72] and in fact two *amongst many others*, should make it clear that "foundations" is a *locus* of an internal debate in mathematics—a debate which structurally appears to have no definitive winner—and not a place for philosophers to validate (let alone dictate!) what should be the "right" set of choices. My only aim is to give a context where the questions raised in this paper could find interesting prolongations. In the quote from Grothendieck, one is surprised by the presence of many ideas, which are not so common in philosophical views on space, but are presented as widely accepted by mathematicians: the fact that space is *universally* present in mathematics; the fact that it has an *autonomy* and a *proper dynamic*; the fact that it is *proteiform*, precisely because it is not limited to such and such a structure under study, but changes faces by "incorporation" with other structures. All of these insights played a crucial role in the arguments developed in this paper.

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[70] The new idea of space or topos is then clearly presented as unifying the realm of continuous magnitudes and numbers: "Cette idée englobe, dans une intuition topologique commune, aussi bien les traditionnels espaces (topologiques), incarnant le monde de la grandeur continue, que les (soi-disant) ‘espaces’ (ou ‘variétés’) des géomètres algébristes abstraits impénitents, ainsi que d'innombrables autres types de structures, qui jusque-là avaient semblé rivées irrémédiablement au ‘monde arithmétique’ des agrégats ‘discontinus’ ou ‘discrets’." (Grothendieck (1985–86), p. 54).

[71] "A startling aspect of topos theory is that it unifies two seemingly wholly distinct mathematical subjects: on the one hand, topology and algebraic geometry, and on the other hand, logic and set theory. Indeed a topos can be considered both as a ‘generalized space’ and as a ‘generalized universe of sets’. These different aspects arose independently around 1963: with A. Grothendieck in his reformulation of sheaf theory for algebraic geometry, with William F. Lawvere in his search for an axiomatization of the category of sets and that of ‘variable‵ sets’, and with Paul Cohen in the use of forcing to construct new models of Zermelo-Fraenkel set theory." (Mac Lane and Moerdijk 1992, p. 1).

[72] He also makes it clear that he is not criticising the linguistic turn, which was so fruitful for logic, but is proposing to complete it.

Another way to state the "proteiform" nature of the notion of space would be to simply say that we do not know exactly what space is. Contrary to what is too often assumed by philosophers (by relying on such and such examples: metric spaces, manifolds, topological spaces, *topos*…), space does not have "a" fixed structure.[73] It is not transparent to concepts. As we can see in Grothendieck's passage, this has nothing to do with some form of metaphysical assumption, but with the very functioning of the geometric imagination which pervades all of mathematics. As soon as we use space in order to know some other structure, we open two important possibilities: first of all (and this was the general case in ancient Geometry), we may us space in order to know… spatial entities (think of my example with the real line above); this leads immediately to an irreducible *duality* between space as represented and space as tool of representation; second, the fact that we use space in order to know other mathematical objects structurally involves a form of opacity: in the case when we do not obtain a fully transparent representation, there is no way to know if the difficulty is due to the tools or is a structural impossibility, a problem coming from some restricted condition or, on the contrary, some excessively loose stipulations in the data of the problem, etc.[74]

In this regard, I would like to conclude this paper by mentioning a very interesting inquiry undertaken by the mathematicians John Gratus and Timothy Porter in a paper entitled: "A Geometry of Information".[75] Let me just quote the very first sentence of their study: "Spatial representation has two contrasting but closely related aspects: (i) representation *of* spaces and (ii) representation *by* spaces".[76] The paper is very interesting for what it proposes as a unifying structure for these "spaces" of various sorts. But it is also interesting as symptom: in 2005, it was still possible to consider that mathematical space has a dual nature, neatly expressed by the distinction between "representation *of* spaces" and "representation *by* spaces";

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[73] A very basic example may render this clearer: if we take the usual "real line", we may have the impression of being faced with a simple form of one-dimensional metric space. But this depends on the choice of what we take as basic open sets. If, for example, we consider the topology based on open sets of the form $[0, a]$ for any real a in the unit interval $[0, 1]$, we obtain a simple example of a topology which is not separated (hence not metric, any metric space being separated). This last feature contradicts what were often considered as the characterizing properties of "spaces" (being *partes extra partes*). Moreover, it raises the question: what is *the* spatial form of the real line?

[74] One can think, for example, of the story told by Grothendieck in the passage quoted above. Its background was the way in which algebraic geometers stumbled upon difficulties in their use of usual topologies and were led to introduce more exotic ones, such as Zariski's topology. At the end of the process, it is topology itself which revealed itself too narrow and had to make room for "topos" and "sites".

[75] Gratus and Porter (2005).

[76] This duality is then explained in the expected manner: "The first is, classically, based firmly in geometry, and topology and assumes some 'space' is given, whilst its aim is to study the 'attributes' of the space - essentially its geometry and topology, or more precisely those parts that are amenable to study by the usual tools of geometry and topology! The other aspect represents some configuration by a space. This 'configuration' may be a formal situation modelling some relationship between some objects and attributes, or perhaps a physical context such as the space of physical configurations of a molecule".

but, more interestingly, one could realize on those grounds that it still has no unified conceptualization. Although it would be wildly exaggerated—if not totally ridiculous—to claim that Proclus' philosophy could help us answer these questions, which are, of course, the prerogative of mathematicians, it can nonetheless aid us in understanding *why* they are still pertinent.

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References

Ariew, R., & Descartes, R. (Eds.). (2000). *Philosophical essays and correspondence*. Indianapolis: Hackett-Publishing.

Barozzi, F. (1560). *Procli Diadochi Lycii … In primum Euclidis elementorum librum commentariorum ad vniuersam mathematicam disciplinam principium eruditionis tradentium libri IIII.* Padua: Gratiosus Perchacinus.

Benatouïl, T. (2010). Les critiques épicuriennes de la géométrie. In P. E. Bour, L. Rebuschi, & L. Rollet (Eds.), *Construction, Festschrift for Gerhard Heinzmann*. London: College Publication.

Bouriau, C. (2000). *Aspects de la finitude, Descartes et Kant*. Pessac: Presses Universitaires de Bordeaux.

Breton, S. (1969). *Philosophie et mathématique chez Proclus*, suivi de *Principes philosophiques des mathémathiques d'après le commentaire de Proclus aux deux premiers livres des Éléments d'Euclide* (transl. G. De Pesloüan). In N. Hartmann (Ed.) Paris: Beauchesne.

Charles-Saget, A. (1982). *L'Architecture du divin*. Paris: Les Belles Lettres.

Claessens, G. (2012). Proclus: Imagination as a symptom. *Ancient Philosophy, 32*(2), 394–395.

Cleary, J. J. (2000). Proclus' philosophy of mathematics. In D. J. O'Meara & G. Bechtle (Eds.), *La philosophie des mathématiques de l'Antiquité tardive* (pp. 85–101). Fribourg: Éditions universitaires de Fribourg.

Crettez, J.P., & Lorette, G. (1998). Reconnaissance de l'écriture manuscrite. In *Traité informatique*, *Technique de l'Ingénieur,* (pp. 1–15). http://www.techniques-ingenieur.fr/base-documentaire/archives-th12/archives-documents-numeriques-gestion-de-contenu.

Crozet, P. (1999). A propos des figures dans les manuscrits arabes de géométrie: l'exemple de Sijzi. In Y. Ibish (Ed.), *Editing Islamic manuscripts on science* (pp. 131–163). London: al-Furqan.

Crozet, P. (2010). De I'usage des transformations géométriques à la notion d'invariant: la contribution d'al-sijzī. *Arabic Sciences and Philosophy, 20*(1), 53–91.

Descartes, R. (1969–75) [AT], *Oeuvres de Descartes* (New ed., 11 Vols.). In C. Adam & P. Tannery (Eds.). Paris: Vrin.

Detlefsen, M. (2005). Formalism. In S. Shapiro (Ed.), *The oxford handbook of philosophy of mathematics and logic* (pp. 236–317). Oxford: Oxford University Press.

De Vittori, T. (2009). *Les notions d'espace en géométrie, de l'Antiquité à l'Âge Classique*. Paris: L'Harmattan.

Domski, M. (2012). Newton and Proclus: Geometry, imagination, and knowing space. *The Southern Journal of Philosophy, 50*(3), 389–413.

Girard, J.-Y. (2007). La logique comme géométrie du cognitif. In J.-B. Joinet (Ed.), *Logique, dynamique et cognition* (pp. 15–29). Paris: Publications de la Sorbonne.
Grant, E. (1981). *Much ado about nothing: Theories of space and vacuum from the middle ages to the scientific revolution*. Cambridge: Cambridge University.
Gratus, J. & Porter, T. (2005). A geometry of information, I: Nerves, posets and differential forms. In R. Kopperman, M.B. Smyth, D. Spreen & J. Webster (Eds.), *Spatial representation: discrete vs. continuous computational models*, no. 04351 in Dagstuhl Seminar Proceedings, IBF Germany: Schloss Dagstuhl. http://drops.dagstuhl.de/opus/volltexte/2005/126/.
Grothendieck, A. (1985–86). Récoltes et Semailles. http://www.math.jussieu.fr/~leila/grothendieckcircle/RetS.pdf.
Harari, O. (2003). The concept of existence and the role of constructions in euclid's elements. *Archive for History of Exact Sciences, 57*, 1–23.
Harari, O. (2008). Proclus' account of explanatory demonstrations in mathematics and its context. *Archiv für Geschichte der Philosophie, 90*(2), 137–164.
Jones, A. (1986). *Pappus of Alexandria: Book 7 of the collection*. New York: Springer.
Koyré, A. (1957). *From the closed world to the infinite universe*. Baltimore: The John Hopkins Press.
Lernould, A. (Ed.). (2010). *Etudes sur le Commentaire de Proclus au premier livre des Eléments d'Euclide*. Lille: Presses du Septentrion.
Mac Lane, S., & Moerdijk, I. (1992). *Sheaves in geometry and logic: A first introduction in topos theory*. New York: Springer.
Mancosu, P. (1996). *Philosophy of mathematics and mathematical practice in the seventeenth century*. New York: Oxford University Press.
Manders, K. (2008a). Diagram-based geometric practice. In P. Mancosu (Ed.), *The philosophy of mathematical practice* (pp. 65–79). Oxford: Clarendon Press.
Manders, K. (2008b). The Euclidean diagram. In P. Mancosu (Ed.), *The philosophy of mathematical practice* (pp. 80–133). Oxford: Oxford University Press.
Maniglier, P. (2006). *La vie énigmatique des signes. Saussure et la naissance du structuralisme*. Paris: Leo Scheer.
Miller, N. (2001). *A diagrammatic formal system for Euclidean geometry.* Ph.D. Dissertation, Cornell University.
Miller, N. (2007). *Euclid and his twentieth century rivals: Diagrams in the logic of Euclidean geometry*. Stanford: Centre for the Study of Language and Information Press.
Morrow, G. R. (1992). *A commentary on the first book of Euclid's element*. Princeton: Princeton University Press.
Mueller, I. (1981). *Philosophy of mathematics and deductive structure in Euclid's 'Elements'*. Cambridge Mass: The MIT Press.
Mueller, I. (1987). Mathematics and philosophy in Proclus' commentary on book I of Euclid's elements. In J. Pépin & H. D. Saffrey (Eds.), *Proclus lecteur et interprète des anciens* (pp. 305–318). Paris: Edition du CNRS.
Mumma, J. (2006). *Intuition formalized: Ancient and modern methods of proof in elementary Euclidean geometry.* (Ph.D. Dissertation, Carnegie Mellon University).
Mumma, J. (2012). Constructive geometrical reasoning and diagrams. *Synthese, 186*, 103–119.
Netz, R. (1999). *The shaping of deduction in Greek mathematics: A study in cognitive history*. Cambridge: Cambridge University Press.
Netz, R. (2009). Imagination and layered ontology in Greek mathematics. *Configurations, 17*(1), 19–50.
Nikulin, D. (2002). *Imagination and geometry: Ontology, natural philosophy and mathematics in Plotinus, Proclus and Descartes*. Aldershot: Ashgate.
Nikulin, D. (2008). Imagination and mathematics in Proclus. *Ancient Philosophy, 28*(1), 153–172.
O'Meara, D. J. (1989). *Pythagoras revived.* Oxford: Clarendon Press.
Panza, M. (2012). The twofold role of diagrams in Euclid's plane geometry. *Synthese, 186*, 55–102.

Proclus (1873). *Procli Diadochi in primum Euclidis elementorum librum commentarii ex recognitione Godofredi Friedlein.* Leipzig: Teubner (reprint Olms, 1992).

Rabouin, D. (2009). *Mathesis universalis. L'idée de 'mathématique universelle' d'Aristote à Descartes*. Paris: P.U.F.

Rashed, R. (2001). *Les Mathématiques infinitésimales du IXe au XIe siècle* (Vol. IV). London: Al-Furqân Islamic Heritage.

Richard, P.R. (2000). Diagnostic sur la structure et la qualité de preuves inadmissibles, *La lettre de la preuve*. *International newsletter on the teaching and learning of mathematical proof*, November/December (published online).

Saito, K. (2006). A preliminary study in the critical assessment of diagrams in Greek mathematical works. *Sciamus, 7*, 81–144.

Saito, K. (2008). Diagrams in Greek mathematical texts. http://www.greekmath.org/index.html.

Saito, K., & Sidoli, N. (2012). Diagrams and arguments in ancient Greek mathematics: Lessons drawn from comparisons of the manuscript diagrams with those in modern critical editions. In K. Chemla (Ed.), *The history of mathematical proof in ancient traditions* (pp. 135–162). Cambridge: Cambridge University Press.

Vitrac, B. (2005). Quelques remarques sur l'usage du mouvement en géométrie dans la tradition euclidienne: de Platon et Aristote à 'Umar Khayyam. *Fahrang, 18*, 1–56.

Vitrac, B. (2012). *Les démonstrations par l'absurde dans les Éléments d'Euclide : Inventaire, formulation, usages.* http://hal.archives-ouvertes.fr/docs/00/68/55/20/PDF/Les_dA_monstrations_par_l_absurde.pdf.

van der Waerden, B. L. (2003). *Algebra*. New York: Springer.