A Note on Lines and Planes in Euclid's Geometry

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The purpose of this note is to remind readers of information at times well-known and at times almost forgotten, namely that for several centuries in the modern West Euclid's *Elements* was simultaneously regarded as the epitome of knowledge and as flawed and confused. It is well known that many mathematicians brought up on Euclid and other Greek geometers complained that they found themselves compelled to accept the conclusions but not instructed in how to do geometry, and the long struggle with the parallel postulate has also been frequently discussed. The confusion discussed here is different, and relates to the concepts of straightness and shortest distance. It will also be suggested that the growing awareness of the defects in Euclid's presentation by the end of the 18th century contributed to the creation of the new geometries of the 19th century: projective geometry and non-Euclidean geometry.

It is, of course, notoriously difficult to give good definitions of fundamental concepts, and in any system of ideas some are going to have to be left undefined. The view to be taken here, in line with Lambert's opinion in his (1786), is that a careful reading of Euclid's *Elements* should equip the student to understand it and to use its ideas correctly, and that it does not adequately do so. The confusion can be introduced by comparing two early theorems in the *Elements*. Proposition I.2 asserts that any given line segment in a plane may be copied exactly with one of its end points at any prescribed point in the plane. The proof is quite long, it is scrupulous, and far from obvious. Proposition I.4, in Heath's translation, asserts that:

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> If two triangles have the two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, they will also have the base equal to the base, the triangle will be equal to the triangle, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend. (Heath 1956, vol. 1, p. 155)

In the proof, two triangles *ABC* and *DEF* are produced meeting the specified conditions, so $AB = DE$, $AC = DF$ and angle *BAC* equals angle *EDF*. Then the triangle *ABC* is "applied to the triangle *DEF*" and it is proved that the points *B* and *E* coincide, as do the points *C* and *F*. Thus applying one triangle to another has the effect of copying a given angle *BAC* correctly in another place. This claim is stated and proved as a theorem much later, in I.23, which, however, builds on these earlier results. The application of figures, as used in I.4, caused numerous commentators to complain that if it is a valid principle then there was no need to expend such care on I.2, and the whole of geometry could be full of such proofs.[1]

The difficulty with I.4 may be traced back to Common Notion 4, where Euclid states that if two segments coincide then they are equal. Heath carefully explains that coincides means, in the Greek, an exact fit, and the assertion can be read as saying that if two segments can be made to coincide then they are equal. How then are segments made to coincide?

We can imagine at least two ways. One corresponds to a physical motion, and the other to copying the same set of instructions in different places. With Common Notion 4, and with Proposition I.4, the idea would seem to be that of a motion.

But the difficulties continue. In Proposition I.4 Euclid used the converse of Common Notion 4, that if two segments are equal then they can be made to coincide, which should have been made explicitly as an assumption, perhaps as part of Common Notion 4. Heath (1956, vol. 1, p. 225) commented here that modern editions of Euclid's *Elements* remark that when *B* coincides with *E*, and *C* with *F*, the lines *BC* and *EF* have been made to coincide, else two straight lines would enclose an area. Euclid never stated that this is impossible, and Heath speculates that either this remark is an interpolation of a later commentator or Common Notion 4 may be an interpolation.

We are not done with the problems in Proposition I.4, but we need now to look at the proffered definition of a straight line. It is well known that the concept of a straight line receives only a most unsatisfactory definition. Indeed, it has recently been suggested that the first seven definitions in the *Elements* may be a later interpolation by a scribe who was impressed by the careful definitions of many types of figure rushed into give definitions of the point, the straight line, and the plane that were better left unattempted (see Russo 1998). Be that as it may, a line is said to be "a breathless length", and a straight line to be a line "which lies evenly with the points on itself". This may help convince readers that they share a common conception of the straight line, but it is no use if unexpected difficulties arise in the creation of a theory, as we have already seen.

[1] See Heath's commentary at this point (1956, vol. 1, pp. 249–250).

The plane is also given an inadequate definition, closely akin to that of the line: "a plane surface is a surface which lies evenly with the straight lines on itself" (a surface "is that which has length and breadth only"). After that, the word 'plane' is not mentioned in the first four Books, although they are solely concerned with plane geometry. Solid geometry enters the Euclid's *Elements* in Book XI, which opens with three theorems that purport to show successively that a straight line cannot lie partly in a plane and partly not, that if two straight lines cut one another they lie in a plane and every triangle lies in a plane, and that if two planes meet then they do so in a line. But Euclid's definition of a plane is too weak to allow any of these arguments to count as a proof. Euclid would have been better off assuming them and thus de facto defining a plane. These 'theorems' do form a suitable basis for the results that follow: XI.4 there is a perpendicular to a plane at any point of the plane, and XI.5 all the lines perpendicular to a given line at a given point form a plane.

But properties of the plane have been used tacitly much earlier in the *Elements*. Indeed, I.4 is again problematic. To show that, when AC and DF are made to coincide and the points B and E lie on the same side of the $AC = DF$, it follows that the points B and E coincide it must be assumed that the triangles ABC and DEF lie in the same plane. A good definition of a plane is required, one that allows this result to be proved.

Problems with the *Elements* are not confined to Proposition I.4, or even with the confusions behind it. Nor, indeed, is the much more famous issue of the parallel postulate all that remains to elucidate. Readers who read the first book and tried to be sure what they could correctly say would be puzzled on a number of counts.

They would find that there is a limited vocabulary in the *Elements*, but that the meaning of the most basic terms is unclear. Surely some understanding of the space around us is being captured in this geometry: there is a largely undiscussed ability to bring some intuitions about lines and planes in space to a study of geometry, and some intuitions about lines and planes in space relate to a study of geometry in only two dimensions. However, there is no proof that the geometry on a plane is the geometry described in the earlier books in the *Elements*.

Points and straight line segments can be moved around in the mathematical plane and in mathematical space, or perhaps it would be better to say that they can be copied exactly in any position. But the concept of motion, in the sense of bringing figures into coincidence, is left unexamined.

There is a concept of equality of line segments, which, as we have seen, is not properly tied to the notion of bringing into coincidence. From it follows the definition of the circle: "a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another", which point is the centre of the circle. Curiously, the definition of the sphere in Book XI is different in kind, and directly involves motion. It reads: "When the diameter of a semicircle remaining fixed, the semicircle is carried round and restored again to the same position from which it began to be moved, the figure so comprehended is the sphere".

However, the most basic and obscure figures (the straight line and plane) are not defined using the concept of distance. The most basic terms seem to be connected to straightness and flatness. Contrary to the case with what is called Euclidean geometry today, Euclid did not take distance as a fundamental concept. He did not define the straight line segment joining two points as the shortest curve joining them. Rather, in Proposition I.20 Euclid showed that "in any triangle two sides taken together in any manner are greater than the remaining one". This result has become known as the triangle inequality, and it goes a long way to proving that the line segment joining any two distinct points is the shortest curve through those points, although Euclid did not even hint at that consequence. It is also worth noting that there is no theorem in Euclid's *Elements* that depends on the actual size of a figure: any theorem that applies to one figure applies to all similar copies.

A plausible reading of *Elements* Book I is that a straight line can be understood as having a direction, so that at every point there is a straight line in every direction and only one straight line at a given point in a given direction. Thus a plane angle is defined as "the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line", and an angle is called rectilineal when the lines containing the angle are straight. Proposition I.27 implies that if there are lines that cross a given line in equal angles then these lines will never meet, and so could be said to point in the same direction. Once the parallel postulate is assumed it follows there are such pairs of lines.

The earlier attempts at defining a straight line that are discussed by Heath (1956, vol. 1, pp. 165–169) also generally appeal to the idea of direction. Plato (in *Parmenides* 137 E) says "straight is whatever has the middle in front of both its ends", and Aristotle says something equivalent to this. Archimedes seems to have gone boldly for the alternative, that the straight line segment is the curve of shortest length between its endpoints, when he wrote at the start of *On the sphere and the cylinder* that "of all the lines which have the same extremities the straight line is the least". This may be the source for Proclus's definition of the straight line as the only curve that occupies a distance equal to that between the points on it, and also as a line stretched to the utmost.

However, interpreting the *Elements* in terms of direction must be regarded as an interpretation, and one that requires quite some work to make precise. For example, as Gauss pointed out (in a book review published in 1816, see *Werke* IV, 365) if one says that two lines are parallel if they cross a third in such a way as to make included angles with the third line that sum to π, then it must be proved that they cross any other line in the same way. Heath (1956, vol. 1, p. 194) regards such an interpretation as having been decisively refuted by Charles Dodgson in his *Euclid and his modern rivals* (1879).

As is well known, for whatever reason, Euclid held back from assuming the existence of parallel lines until Proposition I.29. Then, once the parallel postulate has been introduced Euclid showed in I.34 that opposite sides of a parallelogram are equal. This is as close as he got to saying that the distance between a pair of parallel lines is a constant.

We can conclude that three fundamental concepts are in play, without there being sufficient clarity concerning any of them. There is straightness (and flatness);

equality of line segments (the surrogate for distance); and the ability to bring certain figures into coincidence, and also to make arbitrary similar copies.

It must be admitted that these are difficult concepts to elucidate, still more so when it is not clear whether to read Euclid's *Elements* as a formal system or as an account of the space around us. Indeed, it is neither, but some mixture of both, and for as long as there was no reason to suspect that things could be otherwise, for as long as it was accepted that there was one space and it was described by the one geometry that can be devised, there was no good reason to force a distinction. As a result, mathematicians on occasion cheerfully ran these concepts together, and on other occasions struggled to keep them apart. For example, Clavius, in his edition of Euclid's *Elements*, simply writes "when two or more lines are said to be parallel, or equidistant, …" (first edition 1589, this quote from the 1607 edition).

In order to keep things tidy, we can make two ad hoc definitions. When straightness and flatness are taken as primitive concepts, equality of segments (perhaps in the sense of being brought into coincidence) is used rather than distance, and the language is reluctant to re-place statements in geometry with statements about numbers (say, in the form of coordinates) although coordinate geometry can be erected upon it, we shall speak of purely synthetic geometry. When distance is a primitive concept, line segments are said to have the same or different lengths, congruent figures to have corresponding sides equal in length, and geometrical trans-formations to preserve lengths we shall speak of a metrical geometry. We presently leave the idea of similarities in an ambiguous position.

Elementary geometry in the modern West moved in a confused way towards making distance the primary primitive concept, while often maintaining the Euclidean emphasis on straightness, thus frequently muddling the implications of the different concepts.

A notable example of this being nonetheless productive was Wallis's argument (see Wallis 1693) in defence of the parallel postulate, which rested, as he realised, on the ability to make arbitrary scale copies of a triangle. No-one had thought to doubt that similar, non-congruent figures exist, and it is to Wallis's credit that saw that he had not proved the parallel postulate but only established the equivalence of these two systems:

1. Euclid's *Elements*
2. Euclid's Elements with the parallel postulate removed and the assumption that arbitrary similar figures exist added.

It seems to be the first time that this equivalence was recognised.

In a mass of unpublished work expertly analysed by De Risi (2007), Leibniz made numerous attempts to present a coherent system of geometry that met his own, shifting, philosophical standards. Unlike Euclid, he started with three-dimensional space and a primitive relation of congruence of segments. The plane is then defined as all the points equidistant from two given points, and the straight line as the set of points equidistant from three given points. Somewhat later in life he produced what De Risi (2007, p. 220) called Leibniz's "finest ever definitions of a plane and a straight line". A point *Y* lies on the straight line through *A* and *B* if it is "unique in situation" with respect to *A* and *B*.[2]

Leibniz's definitions remained in his desk drawers, but others later adopted the first of these positions. Fourier, in a discussion with Monge, also took the concept of distance as fundamental, and began with three-dimensional space. He then defined successively the sphere, the plane (as the points equidistant from two given points) and the line (as the points equidistant from three given points).[3] Almost certainly independently, and to no avail, Lobachevskii did the same in his long presentation of non-Euclidean geometry (1835).

A more Archimedean position was adopted by d'Alembert in the *Encylopedie Methodique* (1784, p. 538). There he defined geometry as the science that teaches us to know the extent, position, and solidity of bodies. A line (i.e. a curve of any kind) is one-dimensional, and the shortest line joining two points is the straight line. Parallel lines are lines that, however far they are extended will never meet because they are everywhere equidistant.

This put d'Alembert at the risk of over-defining. When a figure is characterised by possessing several properties, it may be that some parts of the definition imply others. In this case the characterisation is harmlessly over-defined. But if some parts of the definition contradict others then there is no figure with all these properties, and the over-definition is fatal. In the present case, d'Alembert needed to show that the curve everywhere equidistant from a straight line is itself straight, which he did not try to do, believing as he did that the principles of geometry are founded on truths so evident that it is not possible to contest them. Gauss, however, in a manuscript that may date from 1805 (in *Werke*, VIII, 163–164) observed that "The parallels to a straight line are those from which perpendiculars to the given line all have the same length. If the parallels are themselves straight remains undecided."

The approach to the difficulties in Euclid's *Elements* that was taken by Lambert (1786) is instructive. He noted the high level of Euclidean rigour, and that many trivial, indubitable propositions are proved with great attention to detail, while the parallel postulate is assumed, observing also that without the postulate much of Euclid's *Elements* lapses. The resolution of some of the problems, he suggested, was to admit that the difficulties in the *Elements* are insuperable unless one is allowed to suppose that the objects of geometry are representations. As he put it, "If we may neither see nor make a representation of the thing itself in Euclid's geometry, then there is a problem: To show that two lines do not enclose a space".[4] If, however, representation is admitted (Sect. 3), "one learns to know the thing itself (of which the axiom speaks) and also to add in thought that which seems to be missing in the axiom and in its representation, even if one cannot express this in

[2] See De Risi's analysis for what this means, but roughly speaking points not on the line cannot be unique in situation with respect to *A* and *B* because they cannot be distinguished from their mirror images in the line.

[3] See Bonola (1906, 54) who cites *Seance de l'Ecole Normale*, 1, pp. 28–33, reprinted in *Mathesis* 9, pp. 139–141 (1883).

[4] Translation from Ewald (1996 vol. 1, 159).

words." In this way, Lambert continued, one learns that one line may not approach another asymptotically, nor two straight lines enclose an area, not because these results can be proved but because they make claims false to the representation of the thing itself. So, by presupposing the representation of the actual thing, and not demanding only words, Euclid's procedure can be justified.

However, Lambert went on (Sect. 10), the problem with the parallel postulate concerns neither the truth nor the think ability of the axiom. We learn to think about straight lines by under-standing how they are used in Euclid's Elements, and the parallel postulate is plainly regarded as true, as the evidence of its consequences shows. Therefore, the task is to derive the parallel postulate from the other assumptions of the Elements, or, if that fails, to find other equally evident postulates that do imply the parallel postulate. Now one must cease to appeal to representations of the things themselves, but work entirely in words, as Euclid did in making his postulates, and "the proof should be carried out entirely symbolically—when this is possible" (in Ewald 1996, vol. 1, p. 166). That is to say, in terms of synthetic geometry.

In this case the approach put forward by Wolff, who defined parallel lines as equidistant, fails precisely because he forgot that "arbitrarily conjoined concepts must be established" (in Ewald 1996, vol. 1, p. 161).

Lambert ultimately abandoned his attempts to do give an entirely symbolic defence of Euclid's Elements, after establishing several novel theorems in a geometry in which the parallel postulate was replaced by the assumption that the angle sum of every triangle is less than two right angles. His *Theorie der Parallellinien* was published posthumously in 1786.

Legendre was a mathematician sympathetic to the didactic aims of the Elements but not to its original formulations. He wrote several different versions of his *Éléments de géométrie* (1794) with a view to restoring Euclidean rigour in the teaching of geometry, which in his view had been corroded by texts, such as one by Clairaut, that relied on motions of self-evidence. They differ largely, as he had to admit, in their unsuccessful attempts to deduce the parallel postulate. Its chief significance for present purposes is that it exemplifies the attempt to ground elementary geometry on a concept of distance, or rather, and more precisely, on the idea that a straight line is the curve of shortest distance between any of its points.

In all these editions Legendre took a firmly metrical point of view. His opening definition of the first edition proclaimed that "Geometry is a science that has as its object the measure of extent". Extent, he explained, has three dimensions, length breadth, and height; a line is a length without breadth, its extremities are called points and a point therefore has no extent. A straight line is the shortest path from one point to another; surfaces have length and breadth but no height or depth; and a plane is a surface in which if two arbitrary points are joined by a straight line this line lies entirely in the surface. Distance itself is not defined.

Legendre then set out to prove the theorems of the Elements together with some results Euclid had preferred to assume, such as (Legendre's first result): any two right angles are equal. His Theorem 3 proved that the line joining two distinct points is unique (its existence having been tacitly assumed to be a consequence of the definition of a straight line). Familiar congruence theorems follow in each edition until the parallel postulate could no longer be ignored. Once the existence of parallel lines was assured Legendre showed that they were equidistant. In fact, Legendre's attempts to restore rigour to the treatment of elementary geometry was no better than Euclid's, and in some ways worse, not only because his attempts to prove the parallel postulate inevitably failed, but because he smuggled more into his account than he realised.

Historians of geometry remember Gauss for many reasons, not only for his investigations of the parallel postulate and his ideas about non-Euclidean geometry (see Gray 2011). But Gauss also continued his investigations into the proper foundations of Euclidean geometry for many years. He kept a mathematical diary, in which he recorded new ideas and results.[5] One entry (number 72, from 28 July 1797) records "I have demonstrated the possibility of the plane." How he did this, and whether his demonstration continued to satisfy him we do not know, but thirty years later he wrote to Bessel about problems in the foundations of geometry and commented that, apart from the well-known problem with the parallel postulate "… there is another omission that to my knowledge no-one has criticised and will in no way be easy (although possible) to put right. This is the definition of the plane as a surface in which a straight line joining any two points in the surface lies in it entirely. This definition contains more than is necessary for the determination of the surface, and tacitly involves a theorem that must first be proved…" (*Werke* VIII, 200). Then, in pages that the editors of Gauss's *Werke* date to March 1831, and in letters to Schumacher in May 1831, Gauss set out some of his ideas. They are synthetic in character, and inconclusive.

L.A. Sohnke, writing in the *Allgemeine Encyclopaedie der Künste und Wissenschaften* on the term 'Parallel' in 1837 found no less than 91 attempts on the theory of parallels, which he surveyed by dividing them into those where 'parallel' means never meeting; where 'parallel' means equidistant; and where 'parallel' means cross a third line in equal angles. A reasonable conclusion for us to draw from all this unsuccessful activity is that metrical geometry needed to put its house in order, and it probably could not do so taking a strictly synthetic route, nor by grafting the concept of distance onto a structure modelled on Euclid's *Elements*. This is an awkward position for traditional geometry to be in, and it may have opened people's minds to the possibilities of alternatives. Certainly, two were to be produced. One, projective geometry, amplified and improved the synthetic side of geometry. The other, non-Euclidean geometry, was a new and challenging metrical geometry.

This note is not the place to explore how these new geometries may have come about.[6] But it interesting to note that there was an abundance of theorems in Euclid's Elements, even in Legendre's presentations, that drew on the straightness of the line, and Poncelet's ideas, as presented in his book (1822) stressed the

[5] In Gauss *Werke*, X.1, 483–574. There is an English translation in Dunnington (2004, 469–496).

[6] The non-Euclidean story is much better known, see e.g. (Gray 2011); the history of projective geometry needs to be written, and a start has been made in (Bioesmat-Martagon 2011).

incidence properties of lines (two lines meet in a point, two points determine a line) in creating what he provocatively called a non-metrical geometry. Equally, in their publications of the 1830s, Bolyai and Lobachevskii emphasised the metrical aspects of geometry, and indeed the capacity to express geometrical results in the language of novel trigonometrical formulae.

References

Bioesmat-Martagon, L. (2011). *Éléments d'une biographie de l'espace projectif* (Vol. 2). Nancy: Presses Universitaires de Nancy, Collection histories de geometries.

Bonola, R. (1906). *La geometria non-Euclidea*. (H. S. Carslaw, English Trans.), preface by F. Enriques. *History of non-Euclidean geometry.* Chicago: Open Court, 1912; reprint New York: Dover 1955.

Clavius, C. (1607). *Euclidis Elementorum Libri XV.* Frankfurt: Rhodius.

d'Alembert, J. (1784). *Encyclopédie Méthodique: Mathématique.* Paris: Panckoucke.

De Risi, V. (2007). *Geometry and Monadology: Leibniz's analysis situs and philosophy of space*. Basel: Birkhäuser.

Dodgson, C. (1879). *Euclid and his modern rivals*. London: Macmillan.

Dunnington, G. W. (2004). Introduction and appendices. In J. J. Gray (Ed.), *Gauss: Titan of science*. Washington: Mathematical Association of America.

Ewald, W. (1996). *From Kant to Hilbert: A source book in the foundations of mathematics* (2 Vols.) Oxford: Clarendon Press.

Gauss, C. F. (Ed.). [Werke], *Werke*. vols. IV (1880), VIII (1900), X.1 (1912). Königlichen Gesellschaft der Wissenschaften zu Göttingen. Göttingen: Dieterich; later Berlin: Springer, Leipzig: Teubner.

Gray, J. J. (2011). *Worlds out of nothing; a course on the history of geometry in the 19th century* (2nd ed.). London: Springer.

Heath Sir T. L. (1956). *Euclid's Elements* (3 Vols.) Cambridge: Cambridge University Press, reprint New York: Dover.

Lambert, J. H. (1786). Theorie der Parallellinien. In F. Engel & P. Stäckel (Eds.), Theorie der Parallellinien von Euklid bis auf Gauss (1895). Leipzig: Teubner.

Legendre, A.-M. (1794). *Éléments de géométrie.* Paris: F. Didot, several editions.

Lobachevskii, N. I. (1835). New Elements of Geometry, with a Complete Theory of Parallels (Russian). *Gelehrten Schriften der Universität Kasan*. German translation in Lobachetschefskij *Zwei geometrische Abhandlungen*, (F. Engel, Trans.), Leipzig: Teubner 1898.

Poncelet, M. (1822). *Traité des propriétés projectives des figures*. Paris: Bachelier.

Russo, L. (1998). The definitions of fundamental geometric entities contained in book I of Euclid's. *Elements, Archive for History of Exact Sciences, 52*(3), 195–219.

Wallis, J. (1693). De postulato quinto et definitione lib. 6 Euclidis deceptatio geometrica. In: *Operum Mathematicorum*, *2*, (pp. 665–678). Oxford: Oxford University Press.