Kant on Geometry and Experience

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Towards the end of the eighteenth century, at the height of the German Enlightenment, Immanuel Kant developed a revolutionary theory of space and geometry that aimed to explain the distinctive relation of the mathematical science of geometry to our experience of the world around us—both our ordinary perceptual experience of the world in space and the more refined empirical knowledge of this same world afforded by the new mathematical science of nature. From the perspective of our contemporary conception of space and geometry, as it was first developed in the late nineteenth century by such thinkers as Helmholtz, Mach, and Poincaré, Kant's earlier conception thereby involves a conflation of what we now distinguish as mathematical, perceptual, and physical space. According to this contemporary conception, mathematical space is the object of pure geometry, perceptual space is that within which empirical objects are first given to our senses, and physical space results from applying the propositions of pure geometry to the objects of the (empirical) science of physics—which, first and foremost, studies the motions of such objects in (physical) space. Yet it is essential to Kant's conception that the three types of space (mathematical, perceptual, and physical) among which we now sharply distinguish are necessarily identical with one another, for it is in precisely this way, for Kant, that a priori knowledge of the empirical world around us is possible.

By contrast, it is precisely by insisting on the sharp distinction in question that we now find a fundamental error in Kant's theory, and we are thereby inclined to deny the possibility of what he calls *synthetic* a priori knowledge of the empirical around us—the possibility of any substantive a priori knowledge of its factual

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structure. From an historical point of view, however, it is important to acknowledge that our contemporary conception evolved in explicit reaction to Kant's and, as a result, that we are still very much in his debt. It is also important to appreciate the extent to which Kant's theory represents the culmination of the new view of space and geometry characteristic of the early modern period, according to which space—the very same space in which we live and move and perceive—is essentially geometrical, so that it can then serve as the basis for the new mathematical science of physical nature.[1] Kant's conception, in this sense, represents the culmination of such attempts to reconfigure our understanding of space and geometry as the Cartesian conception that the nature or essence of matter is pure (three-dimensional) extension, the Newtonian conception of absolute space developed in the famous Scholium to the Definitions in the *Principia*, and the Leibnizean conception of space as governing the "well-founded phenomena" described by the new mathematical science but not the underlying metaphysical reality knowable by the pure intellect alone.[2]

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Kant's explains his distinctive conception of space and geometry in the *Critique of Pure Reason* (1st ed. 1781, 2nd ed. 1787) in terms of their roles in what he takes to be the a priori necessary conditions underlying all human experience. These conditions, for Kant, are two-fold. On the one side are the a priori *sensible* conditions of such experience: space and time, as what Kant calls our pure forms of outer and inner intuition. On the other are the a priori *intellectual* conditions: the pure concepts or categories of the understanding, under the four headings of quantity, quality, relation, and modality. Under the heading of quantity, for example, are the three categories of unity, plurality, and totality; under the heading of relation are the three categories of substance, causality, and community; and so on. Kant takes such pure concepts of the understanding to be derived from the forms of judgement of traditional logic: unity, plurality, and totality from the forms of universal, particular, and singular judgements; substance, causality, and community, from the forms of categorical, hypothetical, and disjunctive judgements; and so on.

The faculty of sensibility, for Kant, is our passive or receptive faculty for receiving sensory impressions. In sharp contrast with all forms of traditional rationalism from Plato through Leibniz, however, Kant takes this receptive faculty to be itself a source of significant a priori knowledge, notably, the science of

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[1] Aside from the development of the new mathematical science of physical nature, the early modern conception of space as essentially geometrical has other important sources as well—notably, the development of linear perspective in the painting of the Italian Renaissance. For the latter see, for example, Edgerton (1991).

[2] I shall return in the final section of this essay to the sense in which Kant's conception represents the culmination these early modern attempts—and, at the same time, is also quite essential for understanding the later development of our contemporary (explicitly anti-Kantian) conception.

geometry as grounded in our outer (spatial) pure intuition. The Transcendental Aesthetic that begins the *Critique of Pure Reason* is devoted to our sensible faculty, and Kant there describes sensibility as having a *form*—"in which the manifold of all appearances can be ordered in certain relations" (A20/B34)—where this form (e.g., the spatial form of all outer appearances) is invariant under all changes in the *matter* (corresponding to sensation) that is taken up or received within it.[3] The faculty of understanding, by contrast, is our active or spontaneous faculty of thought, which, considered by itself, has no intrinsic relation to our spatio-temporal sensibility. In particular, the formal structure of the understanding is quite distinct from that of sensibility, and it comprises, in the first instance, just the structure of the traditional logic of concepts and judgements: the tree of Porphyry for concepts, for example, and the originally Aristotelian classification of the forms of judgement.[4]

Yet this purely logical structure, for Kant, is empty of all content, and it can only acquire objective meaning or what Kant calls "relation to an object" by being brought into necessary connection with our spatio-temporal faculty of sensibility. Despite the fact that the categories originate in the formal structure of traditional logic, and, in this respect, are entirely independent of all sensory experience, they nevertheless require what Kant calls a *schematism* in terms of our spatio-temporal sensibility in order to have objective meaning for us: substance in terms of the representation of spatio-temporal permanence, causality in terms of the represen-

[3] I cite the *Critique of Pure Reason* by the standard A/B pagination of the first and second editions. All translations from Kant's German are my own. The passage just quoted reads in full (ibid.): "I call that in the appearance which corresponds to sensation its *matter*, but that which brings it about that the manifold of appearances can be ordered in certain relations I call the *form* of appearance. Since that within which sensations can alone be ordered and arranged in a certain form cannot itself be sensation in turn, the matter of all appearance, to be sure, is only given to us a posteriori, but its form must already lie ready for it in the mind a priori and can therefore be considered separately from all sensation." (In B the words "can be" in the first sentence replace "are" in A).

[4] In the tree of Porphyry the highest genus Being is divided into the species Created and Uncreated, Created is divided into the lower species Material and Immaterial, Material into the lower species Animate and Inanimate, and Animate into the still lower species Rational and Irrational: Human Being is thus defined as Rational Animate Material Created Being. In the traditional Aristotelian classification of the logical forms of judgement the square of opposition depicts the logical relationships among judgements with respect to logical quantity (universal or particular) and quality (affirmative or negative), resulting in the four forms A, E, I, and O: A = Every S is P, E = No S is P, I = Some S is P, O = Some S is not P. Kant himself goes beyond the traditional Aristotelian classification, not only by adding the triads of categorical, hypothetical, and disjunctive judgements and problematic, assertoric, and apodictic judgements, but also by adding singular judgements to the two traditional forms of logical quantity (universal or particular) and "infinite" judgements to the two traditional forms of logical quality (affirmative or negative): see A70–76/B95–101. The resulting table of exactly twelve logical forms of judgement, together with the corresponding table of categories, has generated considerable scholarly controversy. I shall briefly touch on one such controversy below.

tation of temporal succession, and so on.[5] The categories thereby acquire a more than purely logical meaning and result in substantive a priori claims about the most fundamental structure of our (sensible) experience: that substance is permanent in space and time, for example, or that every alteration in the state of a substance must have a cause.[6] It is in precisely this way, for Kant, that all of our a priori knowledge —both sensible and intellectual—can only be ultimately understood in terms of the formal conditions of the possibility of experience, that is, of empirical knowledge.

These peculiarities of Kant's conception of the a priori and its necessary relation to empirical knowledge leads to the main problem addressed in the Transcendental Analytic of the *Critique of Pure Reason* (which is concerned with what Kant calls "transcendental" as opposed to "formal" logic). If the a priori concepts or categories of the understanding originate in the logical forms of judgement, entirely independently of all sensible experience, how can we show that they have more than a purely logical meaning—that they do relate a priori to objects and, in fact, to all possible objects of our (human) sense experience? The pure forms of sensibility are not subject to this difficulty, since, assuming that there are such forms, they are precisely the forms of what is sensibly received or given. They therefore relate, necessarily, to all possible objects of our senses, that is, to all possible objects in space and time. But the categories are pure forms of *thought*, not forms of *sensory perception*, and so, in this case, an additional step is needed. We need to show that the spatio-temporal schematization that they require in order to acquire objective meaning and ground empirical knowledge is indeed forthcoming.

The key step in solving this problem is taken in the Transcendental Deduction of the Categories, as becomes especially clear in the notoriously difficult § 26 of the (completely rewritten) version in the second edition of the *Critique*. The title of this section is "Transcendental Deduction of the Universally Possible use in Experience of the Pure Concepts of the Understanding," and Kant begins by describing the problem as one of explaining "the possibility of knowing a priori, *by means of categories*, whatever objects *may present themselves to our senses*—not, indeed, with respect to the form of their intuition, but with respect to the laws of their combination … [, f]or if they were not serviceable in this way, it would not become clear how everything that may merely be presented to our senses must stand under laws that arise a priori from the understanding alone" (B159–160). Kant then notes that, "under *the synthesis of apprehension*," he "understand[s] the composition [*Zusammensetzung*] of the manifold in an empirical intuition, whereby perception, i.e., empirical consciousness of [the empirical intuition] (as appearance), becomes

[5] See the discussion of substance and causality in the Schematism chapter (A144/B183): "The schema of substance is the permanence of the real in time, i.e., the representation of it as a substratum of empirical time determination in general—which therefore remains while everything else changes. The schema of cause and the causality of a thing in general is the real, upon which, whenever it is posited, something else always follows. It therefore consists of the succession of the manifold, in so far as it is subject to a rule".

[6] These are the first two Analogies of Experience. I shall return to the Analogies, together with other principles of the understanding, below.

possible" (B160). The synthesis of apprehension, therefore, is just the process of taking up the matter of sensible and empirical intuition (corresponding to sensation) into conscious awareness, and Kant calls this process "perception"—of empirical objects that may appear before our senses.

We now come to the argument proper. The synthesis of apprehension, Kant says, "must always be in accordance with" our "a priori *forms* of outer and inner sensible intuition in the representations of space and time" (B160). This is straightforward, because it merely reiterates that the representations in question *are* our two forms of outer and inner intuition. But the crucial (and extraordinarily difficult) point, around which the argument turns, immediately follows. Kant first reminds us that space and time are unified or unitary representations in a sense already articulated in the Aesthetic (ibid.): "[S]pace and time are represented a priori, not merely as *forms* of sensible intuition, but as *intuitions* themselves (which contain a manifold), and thus with the determination of the *unity* of this manifold (see the transcendental aesthetic*)." He then infers that precisely this unity must therefore govern the synthesis of perception (B160–161): "Therefore, *unity of the synthesis* of the manifold, outside us or in us, and thus a *combination* with which everything that is to be represented in space or time as determined must accord, is itself already given a priori, as condition of the synthesis of all *apprehension*, simultaneously with (not in) these intuitions." And it finally becomes clear, in the immediately following sentence, that this same unity is actually due to the understanding rather than sensibility (B161): "But this synthetic unity can be no other than that of the combination of the manifold of a given *intuition in general* in an original consciousness, in accordance with the categories, only applied to our *sensible intuition*."

This is why, in the second sentence, Kant insists that the synthetic unity in question is given "with" rather than "in" the intuitions of space and time themselves. Indeed, Kant has already suggested his doctrine of the *transcendental unity of apperception*—the highest and most general form of unity of which the understanding is capable—by using the term "combination [*Verbindung*]" here (which is then repeated in the third sentence). For "combination" is introduced as a technical term at the very beginning of the Deduction (§ 15) to designate the activity most characteristic of the understanding (B129–130): "[T]he *combination* (*conjunctio*) of a manifold in general can never come into us through the senses, and can thus not be simultaneously contained in the pure form of sensible intuition; for it is an act of the spontaneity of the power of representation, and since one must call this, in distinction from sensibility, understanding, all combination is an action of the understanding, which we would designate with the title *synthesis* in order thereby to call attention, at the same time, to the fact that we can represent nothing as combined in the object without ourselves having previously combined it."

Kant continues, in the following paragraph, by asserting that "[c]ombination is the representation of the *synthetic* unity of the manifold" (B130), and this unity, he says, "precedes all concepts of combination" and thus "is not, for example, the category of unity" (B131). Indeed, it cannot be the product of any of the categories, "for all categories are based on logical functions in judging, but in these combination, and thus unity of given concepts, is already thought" (ibid). Therefore, Kant concludes, "we must seek this unity … still higher, namely, in that which contains the ground of the unity of different concepts in judging, and thus of the possibility of the understanding, even in its logical use" (ibid). The required ground, according to the following section (§ 16), is "the original-synthetic unity of apperception"—namely, the representation "*I think*, which must be *able* to accompany all my representations" (ibid). And, according to the next section (§ 17), "[t]he principle of the synthetic unity of apperception is the highest principle of all use of the understanding" (B136).

There can be very little doubt, therefore, that Kant, in § 26, is asserting that the very same unity that was first introduced in the Aesthetic as characteristic of space and time themselves can—surprisingly—now be seen to be due to the understanding after all. Indeed, if he were not asserting this it would be extremely hard to see how Kant could conclude his argument with the claim that the *categories* are conditions of the possibility of experience (B161): "Consequently all synthesis, even that whereby perception becomes possible, stands under the categories, and, since experience is knowledge through connected perceptions, the categories are conditions of the possibility of experience, and thus are a priori valid for all objects of experience." The unity of apperception—"the highest principle of all use of the understanding"—is the ultimate ground of both the categories and the characteristic unity of space and time.

This conclusion, however, is puzzling in the extreme.[7] The crucial difficulty arises in the first sentence of the main argument of § 26 (B160), which contains the justificatory reference back to the Aesthetic. For the primary claim of the Aesthetic, in this connection, is that the characteristic unity of space and time is *intuitive* rather than *conceptual*. So how can we possibly begin with a unity that was earlier explicitly introduced as *non*-conceptual and conclude that this same unity is due to the understanding after all? Indeed, when we examine the footnote attached to Kant's reference back to the Aesthetic, we can appreciate even more how deeply problematic the situation appears to be, and I shall consider this in detail in what follows. For now, however, I shall simply observe that the first sentence of the footnote illustrates Kant's point by the example of "[s]pace, represented as *object* (as is actually required in geometry)" (B160n). Careful attention to this example,

[7] It is puzzling, in particular, because Kant here appears to take back his insistence that sensibility and understanding are two quite different faculties with two quite different a priori formal structures. Indeed, an important line of thought in post-Kantian German philosophy, including both the post-Kantian German idealists and the Marburg neo-Kantians, explicitly appeals to what Kant says in § 26 to motivate an "intellectualist" reading according to which the forms of intuition become absorbed into the more fundamental unity of the understanding. And another important line of thought, culminating in the notorious "common root" interpretation of Martin Heidegger, insists on the radical independence of sensibility—leaving us, in the end, with no plausible reading of § 26. (For further discussion and references relevant to these two lines of thought see Friedman (2015)). The interpretation I am developing here aims fully to incorporate the ineliminable role of the understanding in the characteristic unity of space and time appealed to in § 26 while simultaneously preserving the independent contribution of sensibility.

I shall argue, illuminates the precise character of the difficulty in question—and also points the way to its proper resolution.

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Before focussing on the footnote and the example of geometry, let us first consider the earlier passage from the Aesthetic to which Kant apparently refers in the main text. The passage in question is the third paragraph of the Metaphysical Exposition of Space, where Kant appeals to the characteristic unity and singularity of our representation of space to argue that it must be an intuition rather than a concept—and, indeed, it must therefore constitute the a priori form of all outer intuition. The characteristic properties of this representation to which Kant appeals are, first, that "one can only represent to oneself a single [*einigen*] space, and if one speaks of many spaces, one understands by this only parts of one and the same unique [*alleinigen*] space" (A24/B39), and, second, that "these parts cannot precede the single all-encompassing [*einigen allbefassenden*] space, as it were as its constituents (out of which a composition [*Zusammensetzung*] would be possible); rather, they can only be thought *within it*" (ibid.). It is for this reason, in fact, that "the general concept of *spaces* (emphasis added)"—in the plural, that is, the finite spatial regions that are parts of the "single all-encompassing space"—"rests solely on limitations," which carve out such regions from the single infinite space that contains them all.[8]

The crucial point, therefore, is that space is a singular individual representation, whose whole-part structure is completely different from that of any general concept. In the former case the whole "all-encompassing" space precedes and makes possible all of its limited parts (finite spatial regions), whereas, in the latter, the "parts" of any concept—that is, the "partial concepts [*Teilbegriffe*]" which together constitute its definition (as Rational and Animal are parts of the definition of Human Being: compare note 4 above)—precede and make possible the whole. The unity of a general concept, in this sense, is essentially different from that of our representation of space (and similarly for our representation of time), and this is the primary

[8] The paragraph reads in full (A24–25/B39): "Space is not a discursive, or, as one says, general concept of relations of things in general, but a pure intuition. For, first, one can only represent to oneself a single space, and if one speaks of many spaces, one understands by this only parts of one and the same unique space. These parts cannot precede the single all-encompassing space, as it were as its constituents (out of which a composition would be possible); rather, they can only be thought *within it*. It is essentially single; the manifold in it, and the general concept of spaces as such, rests solely on limitations. From this it follows that an a priori intuition (that is not empirical) underlies all concepts of space. Thus all geometrical principles, e.g., that in a triangle two sides together are greater than the third, are never derived from general concepts of line and triangle, but rather from intuition, and, in fact, with apodictic certainty." The final sentence makes it clear that the science of geometry is implicated in the distinctive whole-part structure that Kant is attempting to delineate, a point to which I shall return below.

reason, in the Aesthetic, that space and time count as intuitive rather than conceptual representations for Kant.[9]

Yet the difficult passage from § 26 of the Deduction appears to appeal to precisely the characteristically non-conceptual unity of space and time to argue that this same unity is actually due to the "unity of synthesis" that is most characteristic of the understanding—"even in its logical use" (B131). And the appended footnote only compounds the appearance of paradox. The first sentence says that "[s]pace, represented as object (as is actually required in geometry)," contains a "*grasping together* [*Zusammenfassung*] of the manifold, given in accordance with the form of sensibility, in an *intuitive* representation, so that the *form of intuition* gives merely a manifold, but the *formal intuition* gives unity of representation" (B160n). So the "unity of representation" here appears to be the "all-encompassing [*allbefassenden*]" unity of space emphasized in our passage from the Aesthetic (A24–25/B39).

The second sentence confirms this idea, and also explains why Kant had previously, in sharp contrast with his present point, taken the unity in question to be sensible as opposed to intellectual (B160–161n): "In the Aesthetic I counted this unity [as belonging] to sensibility, only in order to remark that it precedes all concepts, although it in fact presupposes a synthesis that does not belong to the senses but through which all concepts of space and time first become possible." The third sentence, however, appears to take this back, and even to contradict itself (B161n): "For, since through it (in that the understanding determines sensibility) space or time are first *given*, the unity of this a priori intuition belongs to space and time, and not to the concept of the understanding (§ 24)." Thus, after reiterating that the synthetic unity in question is a product of the understanding, Kant appears explicitly to deny that it is due to the understanding after all.[10]

I have recently proposed a solution to these apparent paradoxes that emerged out of my evolving work on Kant's theory of geometry. My original interpretation of this theory in Friedman (1985) emphasized the importance of Euclidean constructive reasoning for Kant and, in particular, appealed to Kant's understanding of such reasoning to explain the sense in which geometry, for him, is synthetic rather than analytic—an essentially intuitive rather than purely logical science. Yet I did not there explain the necessary relation between the science of geometry and what Kant calls our pure form of outer intuition: the (three-dimensional) space of perception within which all objects of outer sense necessarily appear to us. I first proposed such an explanation, which establishes a link between geometry and our

[9] This crucial difference in whole-part structure is emphasized especially clearly in the immediately following fourth paragraph of the Metaphysical Exposition of Space in the second edition (B39–40): "Space is represented as an infinite *given* magnitude. Now one must certainly think every concept as a representation that is contained in an infinite aggregate of different possible representations (as their common mark), and it therefore contains these *under itself*. But no concept, as such, can be so thought as if it were to contain an infinite aggregate of representations *within itself*. However space is thought in precisely this way (for all parts of space *in infinitum* exist simultaneously). Therefore, the original representation of space is an a priori *intuition*, and not a *concept*."

[10] Compare note 7 above, together with the paragraph to which it is appended.

passage from the Aesthetic (A24–25/B39), in Friedman (2000a).[11] And I found the missing link, in turn, in Kant's discussion of the relationship between what he calls "metaphysical" and "geometrical" space in his comments on essays by the mathematician Abraham Kästner in 1790.[12]

Kant's comments first describes the relationship between the two types of space as follows:

> Metaphysics must show how one *has* the representation of space, but geometry teaches how one can *describe* a space, i.e., can present it in intuition a priori (not by drawing). In the former space is considered as it is *given*, prior to all determination of it in accordance with a certain concept of the object, in the latter a [space] is *made*. In the former it is *original* and only a (single [*einiger*]) *space*, in the latter it is *derivative* and here there are (many) *spaces* —concerning which, however, the geometer, in agreement with the metaphysician, must admit, as a consequence of the fundamental representation of space, that they can all be thought only as parts of the single [*einigen*] original space. (20, 419)[13]

So it appears, in particular, that the (plural) *spaces* of the geometer—i.e., the figures or finite spatial regions that are iteratively constructed in Euclidean proofs—are prominent examples of the *parts* of the "single all-encompassing space" according to our passage from the Aesthetic (A24–25/B39).[14]

Kant's comments go on to discuss the different types of infinity belonging to geometrical and metaphysical space:

> [A]nd so the geometer grounds the possibility of his problem—to increase a given space (of which there are many) to infinity—on the original representation of a single [*einigen*], infinite, *subjectively given* space. This accords very well with [the fact] that geometrical and objectively given space is always finite, for it is only given in so far as it is *made*. That, however, metaphysical, i.e., original, but merely subjectively given space—which (because there are not many of them) can be brought under no concept that would be capable of a construction, but still contains the ground of construction of all possible geometrical concepts—is *infinite*, is only to say that it consists in the pure form of the mode of sensible

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[11] I thereby attempted to build a bridge between the "logical" interpretation of Kant's theory of geometry developed in my earlier paper (an approach that was first articulated by Jaakko Hintikka) and the "phenomenological" interpretation articulated and defended by Charles Parsons and Emily Carson. For further discussion of this issue see also Parsons (1992).

[12] Kästner's three essays on space and geometry were first published in J.A. Eberhard's *Philosophisches Magazin* in 1790. Eberhard's intention was to attack the *Critique of Pure Reason* on behalf of the Leibnizean philosophy, and Kästner's essays were included as part of this attack. Kant's comments on Kästner, sent to J. Schultz on behalf of the latter's defense of the Kantian philosophy in his reviews of Eberhard's *Magazin*, were first published by Wilhelm Dilthey in the *Archiv für Geschichte der Philosophie* in 1890. They are partially translated in Appendix B to Allison (1973), which also discusses the historical background in Chapter I of Part One. Kant's comments have played a not inconsiderable role in the subsequent discussion of space and geometry in § 26, and, after presenting my own interpretation, I shall touch on some of this discussion below.

[13] All references to Kant's works other than the first *Critique* are to volume and page numbers in Kant (1900-).

[14] This point becomes clearer in light of the final sentence of our passage from the Aesthetic—which brings Euclid's geometry explicitly into the picture (see note 8 above: the example there is Proposition I.20 of the *Elements*).

> representation of the subject as a priori intuition; and thus in this form of sensible intuition, as singular [*einzelnen*] representation, the possibility of all spaces, which proceeds to infinity, is *given*. (20, 420–421)

Thus, whereas geometrical space is only *potentially* infinite (as it emerges step-by-step in an iterative procedure), metaphysical space, in a sense, is *actually* infinite —in so far as the former presupposes the latter as an already given infinite whole. Geometrical construction presupposes a single "*subjectively given*" metaphysical space within which all such construction takes place.[15]

In Friedman (2000a) I interpreted the relationship between these two kinds of space as follows. Metaphysical space is the manifold of all oriented perspectives that an idealized perceiving subject can possibly take up. The subject can take up these perspectives successively by operations of translation and rotation—by translating its perspective from any point to any other point and changing its orientation by a rotation around any such point. In this way, in particular, any spatial object located anywhere in space is perceivable, in principle, by the same perceiving subject. The crucial idea is then that the transcendental unity of apperception—the highest principle of the pure understanding—thereby unifies the manifold of possible perspectives into a single "all-encompassing" unitary *space* by requiring that the perceiving subject, now considered as also a thinking subject, is able, in principle, to move everywhere throughout the manifold by such translations and rotations. But this then implies that Euclidean geometry is applicable to all such objects of perception as well, since Euclidean constructions, in turn, are precisely those generated by the two operations of translation (in drawing a straight line from point to point) and rotation (of such a line around a point in a given plane yielding a circle).[16]

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[15] Immediately preceding this passage Kant illustrates the distinction by contrasting geometry with arithmetic (20, 419–420): "Now when the geometer says that a line, no matter how far it has been continually drawn, can always be extended still further, this does not signify what is said of number in arithmetic, that one can always increase it by addition of other units or numbers without end (for the added numbers and magnitudes, which are thereby expressed, are possible for themselves, without needing to belong with the preceding as parts to a [whole] magnitude). Rather [to say] that a line can be continually drawn to infinity is to say as much as that the space in which I describe the line is greater than any line that I may describe within it." Thus, while the figures iteratively constructed in geometry are only potentially infinite, like the numbers, the former, but not the latter, presuppose a single "all-encompassing" magnitude within which all are contained as parts: i.e., the space "represented as an infinite *given* magnitude" of note 9 above (B39).

[16] This connection between Euclidean constructions and the operations in question is suggested by Kant himself (20, 410–411): "[I]t is very correctly said [by Kästner] that 'Euclid assumes the possibility of drawing a straight line and describing a circle without proving it'—which means without proving this possibility *through inferences*. For *description*, which takes place a priori through the imagination in accordance with a rule and is called construction, is itself the proof of the possibility of the object. However, that the possibility of a straight line and a circle can be proved, not *mediately* through inferences, but only immediately through the construction of these concepts (which is in no way empirical), is due to the circumstance that among all constructions (presentations determined in accordance with a rule in a priori intuition) some must still be *the first* —namely, the *drawing* or describing (in thought) of a straight line and the *rotating* of such a line

I appealed to these ideas in proposing an interpretation of the problematic footnote to § 26 in Friedman (2012a). The "unity of representation" mentioned in the second sentence of this footnote is indeed that considered in our passage from the Aesthetic (A24–25/B39), and Kant is indeed saying that this unity is a product of the understanding. It does not follow, however, that it is a *conceptual* unity—that it depends on the unity of any particular concept. It does not depend on the unity of any geometrical concept, for example, for the schemata of all geometrical concepts are generated by Euclidean (straight-edge and compass) constructions, and these presuppose, according to Kant, the prior unity of (metaphysical) space as a single whole. Nor does it depend on the unity of any category or pure concept of the understanding. For, by enumeration, we can see that none of their schemata result in any such object, i.e., space as a singular given object of intuition.

Rather, the unity of space as a singular given whole results directly from the transcendental unity of apperception, prior to any particular category, in virtue of the circumstance that the former unity, as suggested, results from requiring that the perceiving subject (which has available to it the manifold of all possible perspectives) is also a *thinking* subject. For the latter, as Kant says in § 16, must be "one and the same" in all of its conscious representations (B132).[17] The unity of apperception, as Kant says in § 15, is not that of any particular category but something "still higher"—namely, "that which itself contains the ground of the unity of different concepts in judging, and hence of the possibility of the understanding, even in its logical use" (B131). This is why Kant can correctly say, in the last sentence of the footnote to § 26 (B161n; emphasis added), that "the unity of this a priori intuition belongs to space and time, and not to the *concept* [i.e., *category*—MF] of the understanding (§ 24)."

If we follow the reference of this sentence back to § 24, moreover, we find that Kant there describes the figurative synthesis or transcendental synthesis of the imagination as "an action of the understanding on sensibility and its *first* application (at the same time the ground of all the rest) to objects of the intuition possible for us" (B152; emphasis added). He then proceeds to illustrates this synthesis by Euclidean constructions and explains that it also involves motion "as action of the subject":

(Footnote 16 continued)
around a fixed point—where the latter cannot be derived from the former, nor can it be derived from any other construction of the concept of a magnitude."

[17] More fully (B132): "[A]ll the manifold of intuition has a necessary relation to the *I think* in the same subject in which this manifold is encountered. But this representation is an act of *spontaneity*, i.e., it cannot be viewed as belonging to sensibility. I call it *pure apperception*, in order to distinguish it from the *empirical*, or also *original apperception*, because it is that self-consciousness, which—in so far as it brings forth the representation *I think* that must be able to accompany all others, and in all consciousness is one and the same—can be accompanied in turn by no other." Thus, the *I think* is the subject of which all other representations are predicated, whereas it can be predicated of no other representation in turn, and it is in precisely this sense that the *I think* cannot itself be a concept.

> We also always observe this [the transcendental synthesis of the imagination] in ourselves. We can think no line without *drawing* it in thought, no circle without *describing* it. We can in no way represent the three dimensions of space without *setting* three lines at right angles to one another from the same point. And we cannot represent time itself without attending, in the *drawing* of a straight line (which is to be the outer figurative representation of time), merely to the action of synthesis of the manifold, through which we successively determine inner sense, and thereby attend to the succession of this determination in it. Motion, as action of the subject (not as determination of an object*), and thus the synthesis of the manifold in space—if we abstract from the latter and attend merely to the action by which we determine *inner* sense in accordance with its form—[such motion] even first produces the concept of succession. (B154–155)

Thus, Kant begins with the two fundamental geometrical constructions (of lines and circles), and, after referring to a further construction (of three perpendicular lines), he emphasizes the motion ("as action of the subject") involved in drawing a straight line (and therefore involved in any further geometrical construction as well).

In the appended footnote, finally, Kant says that the relevant kind of motion (as an action of the subject rather than a determination of an object), "is a pure act of successive synthesis of the manifold in outer intuition in general through the productive imagination, and it belongs not only to geometry [viz., in the construction of geometrical concepts—MF], but *even to transcendental philosophy* [presumably, in the unification of the whole of space, and time, as *formal intuitions*—MF]" (B155n; emphasis added).[18] And one should especially observe how the representation of time necessarily enters here along with that of space. For the motion involved "in the *drawing* of a straight line" is what Kant calls "the outer figurative representation of **time**" (B154; bold emphasis added).[19]

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I shall return to our representation of time in the penultimate section of this essay. For now, however, I shall add some further reflections on what we have already learned about space. This will help to clarify the special role of space and geometry, for Kant, among the mathematical sciences. It will thereby clarify, as well, the distinctive contribution of space and geometry within his conception of the necessary a priori conditions underlying all human experience.

I have argued that an adequate understanding of the problematic footnote to § 26 involves the distinction Kant makes explicit in his comments on Kästner between

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[18] The footnote reads in full (B155n): "*Motion of an *object* in space does not belong in a pure science and thus not in geometry. For, that something is movable cannot be cognized a priori but only through experience. But motion, as the *describing* of a space, is a pure act of successive synthesis of the manifold in outer intuition in general through the productive imagination, and it belongs not only to geometry, but even to transcendental philosophy."

[19] The precise relationship between the representation of *motion* ("as action of the subject") in § 24 and the representation of time as a *formal intuition* suggested in § 26 is a delicate and subtle matter into which I cannot delve more deeply here; I provide some further discussion in Friedman (2015).

metaphysical and geometrical space—where the latter is generated step by step via Euclidean constructions of particular figures (lines, circles, triangles, and so on), and the former is given all at once, as it were, as *actually* rather than merely *potentially* infinite. Metaphysical space is thus the single "all-encompassing" whole within which all Euclidean constructions—along with the schemata of all geometrical concepts—are thereby made possible. It is in precisely this way that its characteristic unity precedes and makes possible "all *concepts* of space" (B161n; emphasis added), that is, all concepts of determinate regions of space ("spaces" in the plural) constituting particular geometrical figures [*Gestalten*].

Kant first discusses the characteristic unity of concepts in relation to our cognition of their corresponding objects in § 17. The *understanding*, he says, is "the faculty of *cognitions*," where these "consist is the determinate relation of given representations to an object" (B137). But an *object*, Kant continues, "is that in whose concept a given intuition is *united*," and "all unification of representations requires the unity of consciousness in their synthesis" (ibid.). He illustrates these claims by the unification of a given spatial manifold under the concept of a line (segment), whose object is just the determinate spatial figure (the determinate line segment) thus generated (B137–138): "[I]n order to cognize anything in space, e.g., a line, I must *draw* it, and therefore bring into being synthetically a determinate combination of the manifold, in such a way that the unity of this action is at the same time the unity of consciousness (in the concept of a line), and only thereby is an object (a determinate space) first cognized."

Yet when Kant discusses "[s]pace, represented as *object* (as is actually required in geometry)" in the problematic footnote to § 26 (B160n), he does not mean an object in *this* sense: he does not mean the object of any particular geometrical concept (or, indeed, of any other concept). The single unitary space discussed in the first sentence of the footnote is not geometrical space but rather the metaphysical space that precedes and makes possible all (geometrical) "*concepts* of space" (B161n; emphasis added). Kant's philosophical (or "metaphysical") claim is then that these (geometrical) concepts, together with their finite bounded objects (particular spatial figures), are themselves only possible in virtue of the prior "all-encompassing" metaphysical space in which all such bounded objects appear as parts. This prior metaphysical space—the whole of space as a formal intuition—is not an object *of* the science of geometry but rather an object considered at an entirely different level of abstraction (peculiar to what Kant calls "transcendental philosophy"), which, from a philosophical as opposed to a purely geometrical point of view, can nevertheless be seen as *presupposed* by the science of geometry.[20]

[20] I observed that interpreters have appealed to Kant's comments on Kästner while discussing space and geometry in § 26 (see note 12 above): notably, Martin Heidegger, in his lecture course on *Phenomenological Interpretation of Kant's* ***Critique of Pure Reason*** in the winter semester of 1927–1928 (1977, § 9), and Michel Fichant (1997), published along with his French translation of Kant's comments. Both Heidegger and Fichant, however, interpret space as a "formal intuition" in the footnote to § 26 as *geometrical* space in the terminology of the comments on Kästner—so that, according to them, the formal intuition of space is derivative from the more original "form of intuition" within which geometrical construction takes place. But this reading is incompatible with

Kant's more general philosophical claim concerns the role of space as a condition of the possibility of experience (empirical cognition)—and therefore its relationship, more specifically, to the pure concepts or categories of the understanding. The relevant concepts here are the categories of *quantity* or *magnitude* [*Größe*], and Kant emphasizes their role in his first illustration following the main argument of § 26:

> Thus, e.g., if I make the empirical intuition of a house into perception through apprehension of the manifold [of this intuition], the *necessary unity* of space and of outer sensible intuition in general lies at the basis, and I draw, as it were, its figure [*Gestalt*] in accordance with this synthetic unity of the manifold in space. Precisely the same synthetic unity, however, if I abstract from the form of space, has its seat in the understanding, and is the category of the synthetic unity of the homogeneous in an intuition in general, i.e., the category of *magnitude* [*Größe*], with which this synthesis of apprehension, i.e., the perception, must therefore completely conform. (B162)

All objects of outer sense, in other words, occupy determinate regions of space, and are therefore conceptualizable as measurable geometrical magnitudes (in determining, for example, how many square meters of floor space there are in a particular house).

Yet the pure intellectual concept of magnitudes as such, in contrast to the subspecies of specifically spatial (geometrical) magnitudes, "abstracts" from the form of space and considers only "the synthetic unity of the homogenous in an intuition in general"—or, as Kant puts it in the Axioms of Intuition, it involves "the composition [*Zusammensetzung*] of the homogeneous and the consciousness of the synthetic unity of this (homogeneous) manifold" (B202–203).[21] By "the composition of the homogeneous" Kant has primarily in mind the addition operation definitive of a certain magnitude kind (such as lengths, areas, and volumes), in virtue of which magnitudes within a single kind (but not, in general, magnitudes from different kinds) can be composed or added together so as to yield a magnitude equal to the sum of the two. Kant has primarily in mind, in other words, the Ancient

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(Footnote 20 continued)
Kant's claim in the footnote that space as a formal intuition is both unified and singular in the sense of the Aesthetic—and, most importantly, that it precedes and makes possible all *concepts* of space. Here I am in agreement with Béatrice Longuenesse: for her comments on Heidegger in this connection see Longuenesse (1998a, pp. 224–225); for her parallel comments on Fichant see Longuenesse (1998b/2005, pp. 67–69). I shall return to the relationship between my reading and Longuenesse's below. (I am indebted to Vincenzo De Risi for first calling the exchange between Fichant and Longuenesse to my attention and for prompting me to consider more carefully the relationship between my reading of § 26 and Longuenesse's).

[21] More fully (B202–203): "All appearances contain, in accordance with their form, an intuition in space and time, which lies at the basis of all of them a priori. They can therefore be apprehended in no other way—i.e., be taken up in empirical consciousness—except through the synthesis of the manifold whereby a determinate space or time is generated, i.e., through the composition [*Zusammensetzung*] of the homogeneous and the consciousness of the synthetic unity of this (homogeneous) manifold. But the consciousness of the homogeneous manifold in intuition in general, in so far as the representation of an object first becomes possible, is the concept of a magnitude (*quanti*)."

Greek theory of ratios and proportion (rigorously formulated in Book V of the *Elements*), but now extended well beyond the realm of geometry proper to encompass a wide variety of physical magnitudes (including masses, velocities, accelerations, and forces) in the new science of the modern era.[22]

Nevertheless, despite this envisioned extension, Kant takes specifically geometrical magnitudes to be primary. In the Axioms of Intuition he again appeals, in the first place, to the successive synthesis involved in drawing a line (A162–163/B203): "I can generate no line, no matter how small, without drawing it in thought, i.e., by generating all its parts successively from a point, and thereby first delineating this intuition." He then refers to the axioms of geometry (B163/B204): "On this successive synthesis of the productive imagination in the generation of figures is grounded the mathematics of extension (geometry), together with its axioms, which express the conditions of a priori sensible intuition under which alone the schema of a pure concept of outer intuition can arise." And he finally asserts that the axioms of geometry, in this respect, are uniquely privileged (ibid.): "These are the axioms which properly concern only magnitudes (*quanta*) as such."

The sense in which geometry is thereby privileged becomes clearer in the immediately following contrast with *quantity* (*quantitas*) and the science of arithmetic (A163–164/B204): "But in what concerns quantity (*quantitas*), i.e., the answer to the question how large something is, there are in the proper sense no axioms, although various of these propositions are synthetic and immediately certain (*indemonstrabilia*)." Kant illustrates the latter with "evident propositions of numerical relations," such as "7 + 5 = 12," which are "singular" and "not general, like those of geometry" (A164/B205).[23] The import of this last distinction, in turn, becomes clearer in Kant's important letter of November 25, 1788 (to his student Johann Schultz) concerning the science of arithmetic (10, 555): "Arithmetic certainly has no *axioms*, because it properly has no *quantum*, i.e., no object [*Gegenstand*] of intuition as magnitude as object [*Obiecte*], but merely *quantity* [*Quantität*], i.e., the concept of a thing in general through determination of magnitude." Instead, Kant continues, arithmetic has only "*postulates*, i.e., immediately certain practical judgements," and he illustrates the latter by the singular judgement "3 + 4 = 7" (10, 555–556).

Kant's claim, therefore, is that arithmetic, unlike geometry, has no proper domain of objects of its own—no *quanta* or objects of intuition as magnitudes. Arithmetic is rather employed in calculating the magnitudes of any such *quanta* there happen to be, but the latter, for Kant, must be given from outside of arithmetic

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[22] For discussion of the Ancient Greek theory of ratios and proportion see Stein (1990). For further discussion of this theory in relation to Kant see Friedman (1990), and Sutherland (2004a, b); 2006).

[23] Kant here illustrates the generality of geometry by the Euclidean construction of a triangle in general (A164–165/B205): "If I say that through three lines, of which two taken together are greater than the third, a triangle can be drawn, I have here the mere function of the productive imagination, which can draw the lines greater or smaller, and thereby allow them to meet at any and all arbitrary angles." (This is Proposition I.22 of the *Elements*; compare note 14 above).

itself. Kant thus does not understand arithmetic as we do: as an axiomatic science formulating universal truths about the (potentially) infinite domain of natural numbers. Nor, in Kant's own terms, is arithmetic an axiomatic science like geometry, which formulates universal truths about the (potentially) infinite domain of geometrical figures generated by Euclidean constructions—which, as we have seen, can be given or constructed in *pure* (rather than *empirical*) intuition. In particular, the potential infinity of this domain is guaranteed by the single, all-encompassing, and actually infinite formal intuition of space, which, in the end, constitutes the pure form of all outer (spatial) perception.

My reading of how the transcendental unity of apperception originally unifies our pure form of spatial intuition into a corresponding "all-encompassing" formal intuition is thus essentially connected with the science of geometry—the most fundamental science of mathematical magnitude. For I understand the pure form of intuition of space as a mere (not yet synthesized) manifold of possible spatial perspectives on possible objects of outer sense, where each such perspective comprises a point of view and an orientation with respect to a local spatial region in the vicinity of a perceiving subject. The unity of apperception then transforms such a not yet unified manifold into a single unitary *space* by the requirement that any such local perspective must be accessible to the same perceiving subject via (continuous) motion—via a (continuous) sequence of translations and rotations. And this implies, as we have seen, that the science of geometry must be applicable to all outer objects of perception. Space is thereby necessarily represented as comprising all specifically *geometrical* mathematical magnitudes.

It does not follow, however, that the transcendental unity of apperception and the pure concepts of the understanding take over the role of our pure forms of intuition, that there is no independent contribution of sensibility as in "intellectualist" readings like that of the Marburg neo-Kantians (see note 7 above). Rather, the representation of space as a formal intuition—as a single unitary (metaphysical) space within which all geometrical constructions take place—is a direct realization, as it were, of the transcendental unity of apperception within our particular pure form of outer intuition. For this form of intuition originally consists of an aggregate or manifold of possible local spatial perspectives, which the transcendental unity of apperception then transforms into a single, unitary, geometrical (Euclidean) space in the way that I have sketched above. Whereas our original form of outer intuition does not have the (geometrical) structure in question independently of transcendental apperception, it is equally true that no such realization of the latter can arise independently of our original form of outer intuition: this *particular* realization of the unity of apperception can by no means be derived in what Kant calls a manifold of intuition in general.[24]

[24] As we have seen, the pure intellectual concept of magnitudes in general abstracts from the structure of specifically spatial (geometrical) magnitudes and involves only "the synthetic unity of the homogenous in an intuition in general" (B162; compare note 21 above, together with the paragraph to which it is appended).

That there is a uniquely privileged mathematical science, the science of geometry, which establishes universal truths about a special domain of magnitudes (spatial regions as *quanta*) constructible in pure intuition, therefore depends on the existence of *our* pure form of outer intuition. Yet it also depends—mutually and equally—on the action of the transcendental unity of apperception (understood, in the first instance, in terms of a manifold of intuition in general) on this particular form of sensibility. Sensibility does make an independent contribution to the synthetic determination of appearances by the understanding, but it cannot make this contribution, of course, independently of the understanding. In particular, the distinctively geometrical structure realized in our pure form of outer intuition, on my reading, is the one and only realization of the unity of apperception in a domain of objects or magnitudes constructible in pure intuition.[25] And the understanding, on my reading, can only subsequently operate on empirical intuition through the mediation of the resulting formal intuition of space. The fundamental aim of the understanding, in this context, is to secure the possibility of the modern mathematical science of nature—which, as suggested, essentially involves a greatly expanded domain of physical magnitudes extending far beyond those traditionally considered in geometry.[26] It is in this way, as I shall argue in the penultimate section of this essay, that we can finally secure the possibility of what Kant calls experience.

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It is illuminating, at this point, to compare my reading of § 26 of the Deduction with that developed by Béatrice Longuenesse. For, as suggested, I am in agreement with Longuenesse concerning fundamental issues surrounding the interpretation of this crucial section (see note 20 above). In particular, I agree with her that that the synthetic unity in question proceeds from the transcendental unity of apperception itself, prior to the synthetic unity of any particular concept or category, and we also agree that the unity characteristic of space as a formal intuition is precisely the same as that which was earlier characterized as intuitive as opposed to conceptual unity in discussing our form of outer intuition in the Transcendental Aesthetic. Yet the understanding originally affects sensibility, for Longuenesse, in *empirical* rather than *pure* intuition: in the process of "comparison, reflection, and abstraction" by which we ascend from what is sensibly given in perception to form ever more general empirical concepts. Moreover, it turns out, for Longuenesse, that arithmetic is prior to geometry in our application of the categories of quantity to objects of outer intuition—while, as we have just seen, the priority on my reading is exactly the reverse.[27]

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[25] I am here indebted to a very helpful conversation with Graciela De Pierris concerning the precise connection between the transcendental unity of apperception and geometry in my reading.

[26] Compare the paragraph to which note 22 above is appended.

[27] As I have said, I first arrived at my reading of the role of the transcendental unity of apperception in § 26 in the course of the ongoing development of my work on Kant's theory of geometry.

Arithmetic becomes prior, for Longuenesse, in the following way. In her general conception of how the understanding originally affects sensibility the logical forms of judgement on which the categories are based are fundamentally forms of empirical concept formation by which we ascend from the sensible given in forming ever more general empirical concepts that can then figure in full-fledged acts of judgement. The understanding as the mere "capacity to judge [*Vermögen zu urteilen*]," however, can originally affect the sensible given in empirical intuition prior to all actual conceptualization and judgement. And, in this way, the understanding can "generate" our pure forms of outer and inner intuition (the formal intuitions of space and time) "as the necessary *intuitive* counterpart of our discursive capacity to reflect *universal* concepts, concepts whose extension (the multiplicities of singular objects thought under them) is potentially unlimited."[28] Space and time thus become a priori "prepared," as it were, for a corresponding potentially unlimited process of forming ever more general empirical concepts, and the application of the categories, as derived from the logical forms of judgement, to any sensible object given in space and time is thereby secured.

As we have seen, the logical forms associated with the categories of quantity are universal, particular, and singular.[29] For Longuenesse, subsumption of empirical intuitions under these forms involves a movement of thought from singular sensible items thought under some concepts ('This body is heavy', 'This other body is heavy', etc.) to the particular judgement involving the same concepts ('Some [a number of] bodies are heavy'), to the corresponding universal judgement ('All bodies are heavy').[30] The resulting role for the categories of quantity is then to think

(Footnote 27 continued)
In my critical study of Longuenesse (1998a), Friedman (2000b), I discussed her interpretation of the categories of quantity, but I did not consider her views on the "pre-conceptual" synthesis of space and time (which views were then opaque to me). Only after I arrived at my own interpretation of § 26, in Friedman (2012a), did I appreciate what I now see as her important insight. My differences with Longuenesse concerning the formulation and articulation of this insight will emerge in what follows.

[28] See Longuenesse (1998a, p. 224): "[I]f one reads 'the understanding' as *das Vermögen zu urteilen*, the capacity to judge, then one can understand, as I suggested earlier, that the capacity to form judgments, 'affecting sensibility,' generates the pure intuitions of space and time as the necessary *intuitive* counterpart to our discursive capacity to reflect *universal* concepts, concepts whose extension (the multiplicities of singular objects thought under them) is potentially unlimited. When this original intuition is produced, no concept is thereby *yet* generated. Everything, as it were, remains to be done. But part of the minimal equipment that a human being, capable of discursive thought, has at his disposal, is the capacity to generate the 'pure' intuitions of space and time as that in which empirical objects are instances of *concepts* (i.e., universal representations, representations whose logical extension is unlimited)."

[29] See note 4 above, together with the paragraph to which it is appended and the preceding paragraph.

[30] Longuenesse thus follows a well-known paper by Frede and Krüger (1970) in taking the correspondence between categories and forms of judgement to be reversed in Kant's published tables. This correspondence, according to Frede and Krüger, should align singular judgements with unity and universal judgements with totality rather than the other way around. (This is the controversy to which I allude at the end of note 4 above).

the extension of some general concept—the aggregate of singular items thought under this concept—in such a way that it can now be counted or enumerated. For Longuenesse, therefore, Kant's conception of quantity turns out to be very similar to Frege's later conception of number, as a notion that attaches primarily to the extension of a given concept and assigns a (cardinal) number to such a collection of objects so conceptualized. Finally, the figurative synthesis or transcendental synthesis of the imagination corresponding to these categories—the *schema* of the categories of quantity—is that spatio-temporal synthesis by which extensions of concepts in general are placed in space and time as "homogeneous multiplicities" countable or enumerable by successively iterated synthesis ('Here-now is one body', 'Here-now is a second body', 'Here-now is a third body', etc.). The pure intuitions of space and time, as synthesized in accordance with the logical forms of quantity, thereby provide something like a primitive mathematical theory of sets or aggregates: a general theory of the possible extensions of concepts, which, in particular, guarantees that such extensions can be potentially infinite in principle.[31]

Longuenesse is explicit that by "homogeneous multiplicity" she always means a number of distinct (sensible) particulars falling under a shared general concept.[32] The point, it appears, is that space and time serve to guarantee that, no matter how many concepts two sensible particulars share, it is always possible for them to be distinct: it is always possible for there to be *numerical* diversity together with *generic* identity, no matter how extensive the latter may be.[33] Thus the arithmetical units that are now to be successively enumerated and thereby added together are homogeneous in the purely logical sense: they are simply objects falling under the same general concept. The specifically mathematical notion of homogeneity figuring in the traditional theory of ratios or proportion—the property of being "composable" elements of a single magnitude kind or "dimension" (such as lengths, areas, or volumes)—is not yet at issue.[34] Indeed, there is no need at all for this specifically mathematical notion if the point (at least so far) is solely to guarantee that the extensions of empirical concepts, in general, can always be potentially infinite.

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[31] See Longuenesse (1998a, p. 276): "For this to be fully clear, Kant should have said that the concept of number is not an ordinary concept, that is, not a 'common concept' that can be predicated, as mark or a combination of marks, of another concept. It is different from 'common concepts,' since it reflects as such (as multiplicities or, as Cantor will say, as sets having a determinate 'power') sets of objects defined by a concept."

[32] See Longuenesse (1998a, p. 252; bold emphasis added): "I maintain that according to Kant, even the category of quantity is originally acquired insofar as the power of judgment, reflecting on the sensible given in order to subordinate representations to empirical concepts combined in judgments, generates the *schema* of quantity—that is, a successive synthesis of homogeneous elements (**where 'homogeneous' means 'reflected under the same concept'**)."

[33] Kant discusses this situation in the Amphiboly in the course of criticizing Leibniz's doctrine of the identity of indiscernibles (A263–264/B319–320, A271–272/B327–328). Compare Longuenesse's discussion of the Amphiboly in her chapter on "Concepts of Comparison, Forms of Judgment, Concept Formation" (1998a, pp. 132–135).

[34] See note 22 above, together with the paragraph to which it is appended. For discussion of a variety of different notions of homogeneity—both logical and mathematical—see especially Sutherland (2004b).

Now, as Longuenesse emphasizes, arithmetical enumeration is certainly applicable to quantities that are mathematically (dimensionally) homogeneous in terms of the traditional theory of continuous magnitudes. For, in order to measure such a magnitude and assign it a numerical value, we can arbitrarily choose a unit (meters or feet in the case of length, for example) and then count or enumerate the number of such units composing the given magnitude in question. In particular, we may (exhaustively) decompose a given length into a multiplicity of (non-overlapping) line segments and take the measure of this length to be simply the number (in the Fregean sense) of the objects falling under the concept ‘segments in the decomposition equal to the chosen unit’.[35] Yet Longuenesse’s prioritizing of arithmetic over geometry may lead to misunderstandings concerning the fundamental mathematical differences between the two cases—between arithmetic as the science of discrete magnitude and geometry as the science of continuous magnitude. As a result, Kant’s conception of the relationship between the categories of quantity and the science(s) of continuous magnitude may also become obscured.

There are two mathematical points worth noting here. In the first place, arithmetical units in the Fregean sense are merely logically homogenous or “equal” to one another: all they need have in common is that they fall under the same general concept, and, in particular, they need not have the same “size” or “magnitude” in any specifically mathematical sense. By contrast, the geometrical units applicable to continuous magnitudes (meters, square meters, cubic meters, and so on) necessarily have the same size or magnitude themselves, and they must all be congruent or equal to one another in precisely this sense—which, as such, goes far beyond the purely logical notion of conceptual homogeneity. In other words, although there is no doubt that such geometrical units *can* be taken to be homogeneous in the logical sense, the notions of equality and proportion relevant to continuous magnitudes have additional specifically mathematical structure. And it is precisely this additional structure that is required for their quantitative comparison.

A second and closely related point makes the fundamental difference between arithmetic and geometry—discrete and continuous magnitude—even more evident. One of the most significant discoveries of Ancient Greek mathematics was the phenomenon of *incommensurability* for continuous magnitudes: the existence of pairs of such magnitudes (like the side and diagonal of a square) that are not jointly measurable by any common unit, no matter how small a unit we choose. So the ratio or proportion between such magnitudes is not representable as a ratio of two whole numbers (as a rational fraction), but only, as we would now put it, by an irrational number (in our example $\sqrt{2}$). In this precise sense, therefore, it is possible

[35] See Longuenesse (1998a, p. 265): “[T]he same capacity to judge that makes us capable of reflecting our intuitions according to the logical form of quantity also makes us capable of recognizing in the line a plurality of homogeneous segments, thought under the concept ‘equal to the segment *s*, the unit of measurement.’ To ‘subsume under the concept of quantity’ is to count these segments, that is, to reflect the unity of this plurality of homogeneous elements.” By contrast, the notion of homogeneity used in the traditional theory of proportion, as noted, has an essentially dimensional significance: lengths may be composed with lengths but not areas, areas with areas but not volumes.

to demonstrate that equalities and proportions between continuous magnitudes are incapable of purely arithmetical representation: geometry, in this sense, is demonstrably irreducible to arithmetic. Thus, while arithmetic can be subsumed under the general theory of proportion (now restricted to commensurable magnitudes), the latter can by no means be subsumed under the former. And it was just this mathematical discovery that initiated the first rigorous formulation of the general theory of proportion (including specifically incommensurable magnitudes) in Book V of the *Elements*.[36]

I am not claiming that Longuenesse ignores these mathematical differences between discrete and continuous magnitudes. Indeed, she provides an interesting discussion of specifically continuous magnitudes (*quanta continua*), and she also considers Kant's own treatment of irrational magnitudes (like $\sqrt{2}$) in his correspondence with August Rehberg.[37] The problem, rather, is that Longuenesse's treatment of the categories of quantity may easily suggest that the transition from the discrete to the continuous case is smoother and more natural than it actually is. It may suggest, in particular, that one can begin with a conception of these categories tailored to the enumeration of discrete aggregates of individuals, which are homogeneous only in the sense of falling under the same general concept, and then move to their application to truly continuous magnitudes without sufficiently attending to the quite different notion of (mathematical) homogeneity characteristic of the traditional theory of proportion. I want also to insist, accordingly, that Kant's own conception of the categories of quantity is not originally tailored to discrete aggregates of individuals at all, but rather to precisely the traditional general theory of proportion for continuous magnitudes.

What, for Kant, is the relationship between the categories of quantity, on the one side, and the representations of space and time as continuous magnitudes, on the other? For Longuenesse, as observed, space and time function in this context as principles of individuation for discrete sensible particulars falling under a common general concept. They secure the possibility of numerical diversity together with generic identity and thereby secure the possibility that the extension of any given empirical concept is potentially infinite. Moreover, they then make it possible to count or enumerate the individuals in any such extension via successively iterated synthesis. Note, however, that we have no reason so far to take space and time to be *continuous* magnitudes. We have no reason to require that there exists a congruence relation between spatial and temporal parts or regions (lengths, areas, and volumes), that the relations between such parts are thereby representable in the traditional theory of proportion, or, in modern terms, that the parts in question have any metrical properties whatsoever. It is only necessary, rather, that the parts of both space and time be sufficiently distinct from one another and sufficiently numerous

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[36] A specifically arithmetical development of the theory of proportion, devoted to *commensurable* magnitudes (numbers), follows in Book VII. See the "Introductory Note" to Book V in Heath's edition of the *Elements* (1926, vol. 3, pp. 112–113) for discussion of how the discovery of incommensurables is reflected in its structure.

[37] See Longuenesse (1998a, pp. 263–271 and 262–263, respectively).

in a purely mereological sense: they may even be wholes consisting of discrete aggregates of parts.

For Longuenesse, that space and time are continuous magnitudes appears to be a brute fact about our two forms of sensible intuition. In particular, it has no essential connection with either our original conception of how the categories of quantity operate or our original conception of how the transcendental unity of apperception first gives unity to our forms of sensible intuition. For, if we begin by conceiving the operation of the categories of quantity in terms of the possibility of enumerating discrete individuals falling under the same general concept, then, as we have just seen, there is no need for our original conception of space and time to involve anything more than the possibility of potentially indefinite numerical diversity together with generic identity. Moreover, as we have also seen, Longuenesse conceives the original act of the understanding by which it first unifies sensibility to be that action of the "capacity to judge" by which, in empirical intuition, it "generates" our pure forms of outer and inner intuition (the formal intuitions of space and time) "as the necessary *intuitive* counterpart of our discursive capacity to reflect *universal* concepts" (see note 28 above). And so, once again, no further properties of space and time beyond their purely mereological (discretely individuative) properties are involved.

On my reading, by contrast, there is an essential connection between our forms of intuition and the relevant activities of the understanding. For I have suggested that, when Kant explains the category of magnitude in terms of "the composition [*Zusammensetzung*] of the homogeneous and the consciousness of the synthetic unity of this (homogeneous) manifold [in intuition in general]" (B202–203), he has specifically in mind the traditional theory of proportion articulated for continuous magnitudes.[38] To be sure, he is considering this theory in "abstraction" from our particular forms of spatio-temporal intuition and, in this sense, only as applied to a manifold of intuition in general. Yet Kant is not, on my view, beginning from a Fregean conception of number as enumerating an aggregate of discrete individuals falling under a common general concept, and he does not need to make a transition from discrete magnitude (arithmetic) to continuous magnitude (geometry and the theory of proportion). On the contrary, specifically continuous magnitudes are his focus from the very beginning, and so it is no wonder, in particular, that all of Kant's examples of magnitudes (*quanta*) in both the Transcendental Deduction and the Axioms of Intuition are continuous rather than discrete.[39]

[38] See note 21 above, together with paragraph to which it is appended. If, however, we take the traditional theory of proportion as Kant's model instead of arithmetic, we need to develop an alternative account to Longuenesse's of the correspondence between the categories of quantity and the (quantitative) logical forms of judgement (see note 30 above). Thompson (1989) has developed such an account, although I believe that it needs more work. Compare the discussion of Thompson in Longuenesse (2005, pp. 45–46)—which, in particular, suggests that she may be willing to revise her account of this correspondence accordingly.

[39] Longuenesse acknowledges—and even emphasizes—that the application of the categories of quantity to continuous magnitudes is most important to Kant. For example, in her section on continuous magnitudes (see note 37 above) she says that "the most important aspect of the

It is even more significant, however, that my conception of how the transcendental unity of apperception originally unifies our pure form of spatial intuition into a corresponding singular formal intuition is also essentially connected with geometry—the most fundamental science of continuous magnitude. For, as explained at the end of the previous section, it follows from my reading of § 26 that the representation of space as a formal intuition—as a single unitary (metaphysical) space within which all geometrical constructions take place—is a direct realization of the transcendental unity of apperception within our pure form of outer intuition. Space, in this sense, is necessarily represented as a specifically geometrical—and therefore continuous—mathematical magnitude, and it is so represented, in particular, on behalf of the transcendental unity of apperception. Nevertheless, as also explained at the end of the previous section, my reading preserves the independent contribution of sensibility, for this particular realization of the unity of apperception can by no means be derived in what Kant calls a manifold of intuition in general.[40]

In the end, neither my view nor Longuenesse's is subject to the standard criticism of "intellectualist" readings of § 26, according to which Kant's original conception of the faculties of sensibility and understanding as having distinct and independent a priori formal structures is thereby necessarily subverted (compare again note 7 above). Indeed, Longuenesse's view that the application of the categories of quantity to specifically continuous magnitudes is a result of the entirely contingent fact—so far as the understanding is concerned—that space and time happen to be continuous (rather than discrete) magnitudes makes the independence of the two faculties, on her view, especially clear. On my view, by contrast, we do not have radical independence in *this* sense because the categories of quantity have specifically continuous magnitudes in view from the very beginning, and sensibility

(Footnote 39 continued)
category of quantity" is "the role it plays in the determination of a *quantum*" (1998a, p. 265), and she goes on to single out continuous *quanta* in particular (p. 266): "[T]he category of quantity (*Quantität*) finds its most fruitful use when it serves to determine the *quantitas* of a *quantum*, that is, the *Größe*, the *magnitude* of an objects itself given as a continuous magnitude, a *quantum continuum* in space and time." Yet, because of her overriding emphasis on arithmetic and discrete magnitude in the original application of the categories of quantity, she is also led to the surprising claim that in the Axioms of Intuition "appearances are treated essentially as aggregates, namely discrete magnitudes" (2005, p. 50). One can see what she means by this from her earlier discussion of continuous magnitude. In particular, she there (1998a, p. 264) focusses on the notion of a "*quantum* 'in itself' *continuum*, but which I can represent as *discretum* by choosing a unit of measurement to determine the *quantitas* of this *quantum*—that is, *quoties in eo unum sit positum*, how many times a unit is posited in it." However, by failing to emphasize here that this arithmetical "representation" of a continuous magnitude (as discrete) is limited, and is in fact impossible in the comparison of *incommensurable* magnitudes, her discussion may easily give the impression that continuous magnitude can be considered as a species of discrete magnitude.

[40] See again note 24 above, together with the paragraph to which it is appended and the preceding paragraph.

makes an independent contribution by providing objective reality for these categories so understood.[41]

The crucial difference between Longuenesse's and my reading, therefore, is that I do not conceive the original action of the understanding on sensibility as aiming, in the first instance, to secure the possibility of "reflecting on the sensible given in order to subordinate representations to empirical concepts."[42] Rather, this action of the understanding on sensibility originally operates on the *pure* form of our outer intuition, by requiring that the perceiving subject—considered also as a thinking subject—is necessarily "one and the same" in all of its conscious representations (B132).[43] This action of the understanding, once again, results in the representation of space as a single, unified, *geometrical* (and therefore continuous) mathematical magnitude.[44] And it is only through the mediation of this representation, in the end,

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[41] As observed (note 27 above), I did not sufficiently appreciate the importance of Longuenesse's reading of § 26 of the Deduction in Friedman (2000b). In particular, I did not then sufficiently appreciate the way in which her account preserves the independent contribution of sensibility by viewing continuity as a de facto property of space and time as our two forms of pure intuition—and, as a result, I did not sufficiently appreciate the way in which Longuenesse can and does make a transition from considering what she takes to be the original application of the categories of quantity in the enumeration of discrete particulars to the measurement of continuous magnitudes in space and time (see notes 37 and 39 above, together with the paragraphs to which they are appended). So Longuenesse (2001/2005) is perfectly correct to rectify these oversights and to emphasize, accordingly, that a proper appreciation of her reading requires one "to pay attention to the distinct and complementary roles Kant assigns to the logical forms of judgement, on the one hand, and to the pure forms of intuition and synthesis of the imagination, on the other" (2005, p. 53). I now agree with Longuenesse in emphasizing the "distinct and complementary roles" of the understanding and sensibility, but, at the same time, I differ with her on two remaining issues: (i) the structure of the categories of quantity, (ii) the relation of the transcendental unity of apperception to space (and time) in the transcendental synthesis of imagination. With respect to the first issue, the main point is that I take the application of these categories, first and foremost, to be to continuous rather than discrete magnitudes, and so, on my reading, there is no transition from the discrete to the continuous case at all. For the second issue see note 44 below.

[42] Compare, once again, note 28 above, together with the paragraph to which it is appended.

[43] See note 17 above, together with the paragraph to which it is appended.

[44] I am not suggesting that Kant appeals to the understanding to explain the continuity of space and time. In particular, it makes perfect sense, from the point of view of contemporary mathematics, to assign the property of continuity to the original form of spatial intuition (as a manifold of perspectives) and to appeal to the possibility of continuous motions (translations and rotations) only to explain the resulting *metrical* structure. The point, on my reading, is rather, from a contemporary point of view, that Kant assumes a full Euclidean (metrical) structure for space from the beginning—in virtue of which all its parts (lengths, areas, and volumes) then counts as three-dimensional continuous magnitudes in the sense of the traditional theory of proportion. He appeals to the understanding, however, as that which is originally responsible for the status of space as a unified and unitary formal intuition (metaphysical space) within which all (Euclidean) geometrical constructions take place. The fundamental difference between myself and Longuenesse here is that it is precisely this pure geometrical structure, on my view, that most directly realizes the transcendental unity of apperception in sensibility. (I am indebted to Vincenzo De Risi for raising the particular question of continuity in this connection).

that the application of the transcendental unity of apperception—and therefore the categories—to *empirical* intuition is then possible.[45]

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When discussing the question "How is pure mathematics possible?" in the *Prolegomena to any Future Metaphysics* (1783), Kant isolates three principal mathematical sciences, namely, geometry, arithmetic, and "pure mechanics" (4, 283): "Geometry takes as basis the pure intuition of space. Even arithmetic brings its concepts of numbers into being through the successive addition of units in time; above all, however, pure mechanics can bring its concept of motion into being only by means of the representation of time." This passage suggests that it is pure mechanics, rather than arithmetic, which relates most directly to time. The reason, as Kant explains in the letter to Schultz, is that numbers are not themselves temporal objects (10, 556–557). Numbers are only "pure determinations of quantity," and not, like "every alteration (as a *quantum*)," properly temporal objects.[46] Whereas all calculation with numbers takes place *within* our pure intuition of time, the numbers themselves do not relate to parts of time or temporal intervals in the way in which the science of geometry—through the construction of figures—relates to parts of space or spatial regions corresponding to these figures.[47]

In the second edition of the *Critique* Kant added two new sections to the Transcendental Aesthetic: a "transcendental exposition of the concept of space" (§ 3) and a "transcendental exposition of the concept of time" (§ 5). The first argues

[45] See again the paragraph to which notes 25 and 26 above are appended. As I shall argue in the next section, it turns out that the mediation in question essentially involves the Newtonian mathematical theory of motion. In Longuenesse's reading, by contrast, there are two distinct aspects to Kant's grounding of experience—one involving the medium-sized sensible particulars of our ordinary perceptual experience, the other involving the objects of Newtonian mathematical natural science (2005, p. 54): "[I]f my reading is correct, Kant's argument is an attempt to account both for the pull of Aristotelianism in our ordinary perceptual world and for the truth of Newtonianism." I, for one, find much less of a contrast in Kant between ordinary and scientific experience, and much less room, accordingly, for an independent grounding of the former. This does not mean, however, that the argument of the first *Critique*, on my view, simply collapses into that of the *Metaphysical Foundations of Natural Science*: see the Conclusion to Friedman (2013) for my account of the fundamentally different perspectives (on the same phenomenal world) represented in these two works.

[46] More fully (ibid.): "Time, as you correctly remark, has no influence on the properties of numbers (as pure determinations of magnitude), as [it does], e.g., on the properties of every alteration (as a *quantum*), which is itself only possible relative to a specific constitution of inner sense and its form (time), and the science of number, regardless of the succession that every construction of magnitude requires, is a pure intellectual synthesis, which we represent to ourselves in thought."

[47] The crucial difference is that, whereas the science of number or arithmetic, for Kant, certainly presupposes the possibility of indefinite iteration (succession) in time, it does not yet constitute the determination of parts of time as mathematical magnitudes or *quanta*. As explained below, time only acquires what we would now call a metrical structure by means of precisely the mathematical theory of motion, whereby parts of time in particular can now be considered as *quanta*.

that space is indeed a pure or a priori intuition by appealing to the synthetic a priori science of geometry. The second, however, focusses on what Kant calls the "general doctrine of motion [*allgemeine Bewegungslehre*]." "[T]he concept of alteration," Kant says, "and, along with it, the concept of motion (as alteration of place) is possible only in and through the representation of time" (B48). The reason, he continues, is that "[o]nly in time can two contradictorily opposed determinations in one thing be met with, namely, *successively*" (B48–49). Therefore, he concludes, "our concept of time explains as much synthetic a priori knowledge as is set forth in the general doctrine of motion, which is by no means unfruitful" (ibid.). This strongly confirms the idea that it is the mathematical science of motion ("pure mechanics"), not arithmetic, which relates to time as geometry does to space—as the latter science, in particular, relates to the parts of space or spatial regions corresponding to geometrical figures (see again note 47 above).

It is in the *Metaphysical Foundations of Natural Science* (1786) that Kant develops in detail what he takes to be the synthetic a priori principles contained in the general doctrine of motion. He explains in the Preface that the natural science for which he is providing a metaphysical foundation is "either a pure or an applied *doctrine of motion* [*reine oder angewandte Bewegungslehre*]" (4, 476). Moreover, he concludes the Preface by saying that he wants to bring his enterprise "into union with the mathematical doctrine of motion [*der mathematischen Bewegungslehre*]" (478) and suggesting that he has Newton's *Principia* most centrally in mind.[48] Then, in the first chapter or Phoronomy, Kant characterizes the object of this synthetic a priori science as "the movable in space" (480) and remarks that "this concept, as empirical, could only find a place in a natural science, as applied metaphysics, which concerns itself with a concept given through experience, although in accordance with a priori principles" (482). Nevertheless, he also suggests that he is envisioning a *transition* from what § 24 of the B Deduction will call the pure act of motion of the subject—"as the *describing* of a space" (B155n)—to the motion of an empirically given object (a perceptible body) considered in the *Metaphysical Foundations*. For he begins the Phoronomy by considering moving matter as an abstract mathematical point—whereby "motion can only be considered as the *describing of space*" (489)—and reserves its subsumption under the more empirical concept of an extended (massive) body for later.[49]

The *Metaphysical Foundations* is organized into four main chapters—the Phoronomy, Dynamics, Mechanics, and Phenomenology—in accordance with the four headings of the table of categories (quantity, quality, relation, and modality).

[48] For further discussion of the relationship between Kant's "mathematical doctrine of motion" and Newton's *Principia* see Friedman (2012b) and (more fully) Friedman (2013).

[49] The quoted passage reads more fully (ibid.): "In phoronomy, since I am acquainted with matter through no other property but its movability, and thus consider it only as a point, motion can only be considered as the *describing of a space*—in such a way, however, that I attend not solely, as in geometry, to the space described, but also to the time in which, and thus to the velocity with which, a point describes a space. Phoronomy is thus just the pure doctrine of magnitude (*Mathesis*) of motion." For further discussion of the transition from pure to empirical motion see again Friedman (2012b) and (more fully) Friedman (2013).

In the third chapter Kant formulates his own three "Laws of Mechanics," which he employs in the fourth chapter to determine the true or actual motions in the cosmos from the merely apparent motions that we observe from our parochial position here on the surface of the earth.[50] He thereby shows how we can move from the mere "appearance [*Erscheinung*]" of motion to a determinate "experience [*Erfahrung*]" thereof (554–555). Moreover, whereas Kant's three Laws of Mechanics are derived as more specific realizations or instantiations of the three Analogies of Experience, his procedure for determining true from merely apparent motions involves a more specific realization or instantiation of the three Postulates of Empirical Thought. He determines the true from the merely apparent motions, in other words, by successively applying the three modal categories of possibility, actuality, and necessity. I have argued elsewhere, and in great detail, that Kant's model for this procedure is precisely Book 3 of the *Principia*, where Newton determines the true motions in the solar system from the initial "Phenomena" encapsulated in Kepler's laws of planetary motion and, at the same time, thereby establishes the law of universal gravitation.[51]

It is especially significant that Kant's Laws of Mechanics are more specific realizations of the Analogies of Experience. For the latter are characterized in the first *Critique* as the fundamental principles for the determination of time:

> These, then, are the three analogies of experience. They are nothing else but the principles for the determination of the existence of appearances in time with respect to all of its three modes, the relation to time itself as a magnitude (the magnitude of existence, i.e., duration), the relation in time as a series (successively), and finally [the relation] in time as a totality of all existence (simultaneously). This unity of time determination is thoroughly dynamical; that is, time is not viewed as that in which experience immediately determines the place of an existent, which is impossible, because absolute time is no object of perception by means of which appearances could be bound together; rather, the rule of the understanding, by means of which alone the existence of the appearances can acquire synthetic unity with respect to temporal relations, determines for each [appearance] its position in time, and thus [determines this] a priori and valid for each and every time. (A215/B262)

Just as we need the transcendental unity of apperception, in connection with the categories of quantity, to secure the application of the mathematical science of

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[50] Kant's three Laws of Mechanics are the conservation of the total quantity of matter, the law of inertia, and the equality of action and reaction; compare the discussion (and illustration) of the synthetic a priori propositions of pure natural science in the Introduction to the second edition of the *Critique* (B20–21). I (briefly) comment on the relationship between these laws and the Newtonian Laws of Motion in Friedman (2012b) and (more fully) in Friedman (2013).

[51] See, e.g., Friedman (2012c) and (more fully) Friedman (2013). The successive determination of true from merely apparent motions, for Kant, involves a nested sequence of ever more comprehensive rotating systems—as we proceed from our parochial perspective here on earth, to the more comprehensive perspective of the center of mass of the solar system, to the even more comprehensive perspective of the center of mass of the Milky Way galaxy, and so on ad infinitum. Kant thereby reinterprets Newtonian absolute space (and extends the Newtonian determination of true from merely apparent motions far beyond the solar system) as the regulative idea (which can never be actually attained) of the ideal limit of this procedure: the perspective (which can never be actually attained) of the center of gravity of all matter.

geometry to all objects that may be presented within this form, we need the same transcendental unity of the understanding, in connection with the categories of relation, to generate a parallel mathematical structure (for duration, succession, and simultaneity) governing all objects that may be presented to us in time—that is, all objects of the senses whatsoever. And it is only at this point, in particular, that parts of time (temporal intervals) are themselves determined as mathematical magnitudes (*quanta*).[52]

But there is a crucial disanalogy between the two cases. The objects or magnitudes (*quanta*) considered in geometry, as explained, can be given or constructed in pure intuition—which, in turn, is the necessary form of all empirical intuition of outer objects. The Axioms of Intuition, therefore, are constitutive of such objects as appearances. The Analogies of Experience, however, as what Kant calls "dynamical" rather than "mathematical" principles, are concerned with "*existence* [*Dasein*] and the *relation* among [the appearances] with respect to [their] existence" (A178/B220). Further, because "the existence of appearances cannot be cognized a priori" (A178/B221), because "[existence] cannot be constructed" (A179/B221), the latter principles, unlike the former, cannot be constitutive of appearances (A180/B222–223): "An Analogy of Experience will thus only be a rule in accordance with which from perceptions unity of experience may arise (not, like perception itself, as empirical intuition in general), and it is valid as [a] principle of the objects (the appearances) not *constitutively* but merely *regulatively*."[53]

I can now delineate more exactly the uniquely privileged role of the mathematical science of geometry in Kant's conception of the possibility of experience. Geometry, for Kant, involves a procedure whereby all the objects of this science—all the figures considered in Euclid's geometry—are constructed step-by-step in pure intuition within space as a singular and unitary formal intuition (metaphysical space). Since all (outer) appearances as empirical intuitions are also given within this space, geometry necessarily applies to all objects of outer sense merely considered as objects of perception or appearance. The mathematical structure of time resulting from the general doctrine of motion, by contrast, can by no means be constructed in pure intuition. It can only arise within the context of the relational

[52] Compare again note 47 above. In applying the Analogies of Experience to the mathematical science of motion, in particular, we determine the magnitudes of temporal intervals by reference to idealized perfectly uniform motions, which then set the standard for correcting the actually non-uniform motions found in nature. In Newton's famous remarks concerning "absolute, true, and mathematical time" in the *Principia* (1999, p. 408), for example, we thereby correct the common "sensible measures" of time such as "an hour, a day, a month, a year" (ibid.), and I argue in Friedman (2013) that Kant takes this procedure as his model for time determination in the above passage from the Analogies (A215/B262). I also argue that Kant has the same procedure in mind in his Second Remark to the Refutation of Idealism, according to which, for example, we "undertake [*vornehmen*]" such time determination from the observed "motion of the sun with respect to objects on the earth" (B277–278).

[53] Although the Analogies of Experience are thus not constitutive of *appearances*, they are (of course) constitutive of what Kant calls "experience." Compare Kant's discussion of this distinction in the Appendix to the Transcendental Dialectic (A664/B692).

categories, and it thus involves a crucial transition from objects of perception or appearance to objects of what Kant calls experience. So we can only determine objects within this structure as objects of experience by beginning our determination in empirical rather than pure intuition.

This crucial asymmetry, for Kant, between the mathematical structure of time and that of space sheds further light on the independent contribution of the faculty of sensibility to the determination of the objects of experience by the understanding. I have explained the independent contribution of space as the form of outer sense in terms of the circumstance that geometry is the only mathematical science whose objects (as magnitudes) are determinable in pure intuition. It is now clear, however, that it is only by taking account of the characteristic structure of both space and time —the structure of our *spatio-temporal* sensibility—that we can fully appreciate the way in which our understanding can similarly determine the objects of experience. For the latter objects can only be so determined in empirical rather than pure intuition, and, for this purpose, we need to make a transition from *perception* (in accordance with the mathematical principles) to *experience* (in accordance with the dynamical principles).

The *Metaphysical Foundations*, I have suggested, takes the argument of Book 3 of the *Principia* as its model for determining true from merely apparent motions, and thus for determining "experience" from "appearance." In this procedure Kant substitutes his own Laws of Mechanics for Newton's Laws of Motion, where these Laws of Mechanics, in turn, are more specific realizations or instantiations of the Analogies of Experience. The determination in question, moreover, proceeds in accordance with the modal categories of possibility, actuality, and necessity, and thus by a more specific realization or instantiation of the Postulates of Empirical Thought.[54] So at the end of Kant's procedure, in particular, we have determined the resulting causal interactions between each body and every other body subject to the law of universal gravitation as *necessary* in the sense of the third Postulate (A218/B266): "That whose coherence [*Zusammenhang*] with the actual is determined in accordance with the universal conditions of experience, is (exists as) *necessary*." Indeed, as I have argued in detail elsewhere, it turns out that the law of universal gravitation itself (in sharp contrast with the Keplerian Phenomena from which it is inferred) is thereby determined, at the same time, as a universally valid and necessary law—as opposed to a merely inductive regularity or general rule.[55]

It follows, more generally, that the transition from what Kant calls "perception" to what he calls "experience" is also a transition from that which is merely *actual* (in the sense of the Postulates) to that which is *necessary* (in the same sense). For Kant says of the Postulates as a whole that they "together concern the synthesis of mere intuition (the form of appearance), of perception (the matter of appearance), and of experience (the relation of these perceptions)" (A180/B223). Indeed, in the second edition Kant reformulates the general principle governing all three

[54] See the paragraph to which note 51 above is appended.

[55] Friedman (2012c) is my most recent detailed discussion of this point.

Analogies so as, in effect, to explain “experience” in terms of such necessity (B218): “*Experience is only possible through the representation of a necessary connection* [*Verknüpfung*] *of perceptions*.” And this explanation, in turn, reflects the intervening discussion in the *Prolegomena* of how “judgements of experience” differ from “judgements of perception” by the transformation of a merely inductive general rule (“If a body is illuminated long enough by the sun it becomes warm”) into a genuine causal law (“The sun through its light is the cause of the warmth”).[56]

The discussion in the *Prolegomena* concludes by illustrating Kant’s conception of universally valid and necessary laws of nature more precisely. Kant says that he will illustrate his fundamental claim, that “*the understanding does not extract its laws* (a priori) *from, but prescribes them to, nature*” (§ 36; 4, 320), with “an example, which is supposed to show that laws which we discover in objects of sensible intuition, especially if these laws have been cognized as necessary, are already held by us to be such as have been put there by the understanding, although they are otherwise in all respects like the laws of nature that we attribute to experience” (§ 37; ibid.). And the example of such a law considered in the immediately following section is none other than the law of universal gravitation (§ 38; 4, 321): “a physical law of reciprocal attraction, extending to all material nature, the rule of which is that these attractions decrease inversely with the square of the distance from each attracting point.”

It is striking, therefore, that this part of the *Prolegomena* is also echoed in § 26 of the Deduction. In particular, the conclusion of § 36 of the *Prolegomena* is echoed by the introductory remarks of § 26 of the Deduction where Kant announces the goal of the argument to follow: namely, to explain “the possibility of knowing a priori, *by means of categories*, whatever objects *may present themselves to our senses*, not, indeed, with respect to the form of their intuition, but with respect to the laws of their combination—**and thus to prescribe the law to nature and even make nature possible**” (B159; bold emphasis added). Moreover, after the conclusion of the main argument in § 26—“Consequently all synthesis, even that whereby perception becomes possible, stands under the categories, and, since experience is knowledge through connected [*verknüpfte*] perceptions, the categories are conditions of the possibility of experience, and thus are a priori valid for all objects of experience” (B161)—Kant finally arrives at the claim that the understanding is thus “the original ground of [nature’s] necessary lawfulness (as *natura formaliter spectata*)” (B165). And this, in turn, echoes the second introductory question posed in § 36 of the *Prolegomena* (4, 318): “How is nature possible in the *formal* sense, as the sum total of the rules to which appearances must be subject if they are to be thought as connected [*verknüpft*] in one experience?”

It is not unreasonable to suppose, therefore, that Newtonian natural science in general and the law of universal gravitation in particular are just as relevant to the conception of experience articulated in the second edition Deduction as they are

[56] This is the famous response to Hume’s “*crux metaphysicorum*” in the *Prolegomena* (§ 29; 4, 312). For a detailed discussion see again Friedman (2012c).

(explicitly) in the corresponding sections of the *Prolegomena*.[57] And it is quite clear, in any case, that Kant's treatment of the possibility of experience in the Deduction is just as involved with the question of how pure natural science is possible. The formal intuition of space as a whole highlighted in the footnote to § 26—"[s]pace represented as *object* (as is actually required in geometry)" (B160n)—is the three-dimensional, infinite, essentially geometrical space central to the new science of nature. It is that space in which all of nature is contained so as thereby to subject it to a unified system of mathematically formulated universally valid laws.[58] This modern conception of the laws of nature, Kant sees, has been finally successfully realized by Newton, who shows, for the first time, how we can thereby rigorously treat temporal duration as a mathematical magnitude as well. Kant incorporates this insight into his own revolutionary conception of transcendental time determination in accordance with the Analogies of Experience, whereby the universally valid and necessary laws of nature turn out to be prescribed to nature by us. Nature, on this conception, is nothing more nor less than the sum total of sensible objects in space and time, as necessarily subject to the lawgiving activity of the understanding. And it is in precisely this way that nature itself, for Kant, becomes the necessarily correlative object of our (human) experience.

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At the beginning of this essay I suggested that Kant's revolutionary conception of the relationship between geometry and experience is centrally situated between the early modern conception of space—the very same space in which we live and move and perceive—as essentially geometrical and our contemporary conception, where the latter, by contrast, is based on a clear and sharp distinction between mathematical, perceptual, and physical space. I shall conclude, accordingly, by providing a bit more detail concerning the way in which Kant's conception is so situated.

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[57] The relevance of the law of universal gravitation in particular is suggested in § 19 of the Deduction, which develops an account of the "*necessary unity*" belonging to the representations combined in any judgement as such—"i.e., a relation that is *objectively valid*, and is sufficiently distinguished from the relation of precisely the same representations in which there would be only subjective validity, e.g., in accordance with laws of association" (B142). Kant then illustrates his point by the relation between subject and predicate in the judgement "Bodies are heavy" (ibid.). This discussion continues the discussion of "judgements of experience" developed in the *Prolegomena*, and the example Kant chooses invokes universal gravitation as discussed in both § 38 of the *Prolegomena* and the *Metaphysical Foundations of Natural Science*. See Friedman (2012c) and Friedman (2015) for further relevant details concerning the relationship between the second edition Deduction and the *Prolegomena*.

[58] That the space in question functions as a foundation for both the mathematical science of (Euclidean) geometry and the modern conception of universally valid mathematical laws of nature sheds light on the sense in which the original act of the understanding responsible for the necessary unity of this space is more general than the unifying activity expressed in any *particular* category. We are thereby not confined, for example, to either the categories of quantity (as realized in the science of geometry) or the categories of relation (as realized in the universal laws of nature); rather, the original act in question insures the application of all of the categories together.

The early modern conception begins, as I suggested, with the Cartesian view that the essence or nature of matter, and thus of all of physical nature, is pure (three-dimensional) extension, so that physical nature itself consists of the objects of pure geometry actualized or made real. But this elegant and austere view was rejected by virtually all post-Cartesian thinkers, who took some additional, characteristically physical property of matter—such as impenetrability, mass, or force—to be unavoidably required. Thus Newton explicitly opposed the Cartesian metaphysics of space and matter with his own metaphysical conception of (absolute) space as a divine emanation, where properties such as impenetrability, mass, and force are then bestowed on particular regions of space by a separate act of divine creation.[59] But Leibniz, who of course entirely rejected Newtonian absolute space, took the space of the new mathematical science of physical nature to govern only the "well-founded phenomena" described by this science but not the more fundamental metaphysical reality underlying the phenomena.[60] And, as I have argued in detail elsewhere, Kant himself should be read as articulating a kind of synthesis of Leibniz and Newton—where, on the one side, Kant agrees with Leibniz that the new mathematical physics requires a metaphysical foundation in the purely intellectual (and so far non-spatio-temporal) concepts of substance, action, and force, and, on the other side, he agrees with Newton that such concepts only have sense and meaning when realized or instantiated (in Kantian terminology "schematized") within geometrical space.[61]

The primary aim of the present essay has been to explain in detail exactly how, for Kant, purely intellectual concepts—and, indeed, the pure intellect itself—are realized or instantiated in geometrical space, and exactly how, as a result, they then make our experience of nature possible. So what remains is briefly to consider exactly how Kant's late eighteenth-century account of these matters led from the early modern conception of space and geometry that was his starting point to our contemporary conception. The basic point I want to emphasize is that, although the latter conception, as explained, is indeed diametrically opposed to his, there is nonetheless a continuous conceptual evolution from Kant's late eighteenth-century account to the early twentieth-century developments in both geometry and physics that had our contemporary conception as a central result.

For, as I have argued in detail elsewhere, Kant's conception of space as both the pure form of our outer sensible experience and the basis for the mathematical science of geometry was taken up and generalized in the nineteenth century by first

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[59] This conception is explicitly developed in Newton's unpublished *De Gravitatione*, but it also surfaces in some of his best-known published works, such as the General Scholium to the *Principia* and the Queries to the *Opticks*. See Stein (2002) for a detailed discussion, and compare Janiak (2008) for a somewhat different perspective.

[60] This, in any case, is the traditional understanding of Leibniz, which was certainly shared by Kant. Two recent more sophisticated interpretations—which argue for greater continuity between Leibniz's "phenomenalism" and Kant's—are Adams (1994) and De Risi (2007).

[61] See again Friedman (2013) for a detailed development of this reading, and compare Friedman (2009) for Kant's relationship, in particular, to Newton's metaphysics of space.

Hermann von Helmholtz and then Henri Poincaré, in an effort to accommodate what they took to be correct in Kant's conception to the new mathematical discoveries in non-Euclidean geometry.[62] During approximately the same time, moreover, Ernst Mach (and others) reconsidered the conceptual foundations of Newtonian physics, with the result that Newtonian absolute space was replaced by what we now call inertial frames of reference.[63] And Albert Einstein was explicitly indebted to both of these sets of developments (in geometry and physics respectively) in his revolutionary early twentieth-century theories of relativity.[64] In particular, as Einstein himself tells us in his aptly titled lecture, "Geometry and Experience," he could never have developed his general theory of relativity—which replaced the Newtonian theory of universal gravitation with a theory of dynamical variably-curved space-time—without the intervening work of Helmholtz and Poincaré.[65] It was in just this lecture, finally, that Einstein canonically articulated our contemporary (explicitly anti-Kantian) conception of the relationship between geometry and experience for us.[66]

References

Adams, R. M. (1994). *Leibniz: Determinist, theist, idealist*. Oxford: Oxford University Press.

Allison, H. E. (1973). *The Kant-Eberhard controversy*. Baltimore, London: Johns Hopkins University Press.

De Risi, V. (2007). *Geometry and monadology: Leibniz's analysis situs and philosophy of space*. Basel, Boston, Berlin: Birkhäuser Verlag.

[62] I discuss the relationship between Helmholtz's and Poincaré's conception of geometry as based on the principle of free mobility and Kant's "perspectival" conception of space as our pure form of outer sensible intuition in Friedman (2000a).

[63] For the relationship between Kant's reinterpretation of the Newtonian conception of absolute space (compare note 51 above) and the concept of what we now call an inertial frame of reference see Friedman (2013, pp. 503–509), where I also refer to DiSalle (2006) in the same connection. For Mach and the concept of inertial frame see DiSalle (2002).

[64] My most detailed discussion of the conceptual development from Kant through Helmholtz, Mach, and Poincaré to Einstein is Friedman (2010, pp. 621–664).

[65] See Einstein (1921); for further discussion see Friedman (2002).

[66] See Einstein (1921, pp. 3–4, 1923, pp. 28–29; my translation): "In so far as the propositions of mathematics refer to reality they are not certain; and in so far as they are certain they do not refer to reality. Full clarity about the situation appears to me to have been first obtained in general by that tendency in mathematics known under the name of 'axiomatics'. The advance achieved by axiomatics consists in having cleanly separated the formal-logical element from the material or intuitive content. According to axiomatics only the formal-logical element constitutes the object of mathematics, but not the intuitive or other content connected with the formal-logical elements." Although Einstein does not explicitly mention Kant here, these famous words were clearly intended and standardly taken as a rebuttal of the Kantian conception that mathematics (especially geometry) is paradigmatic of synthetic a priori knowledge. They were so standardly taken, in particular, by the logical empiricists beginning with Moritz Schlick—who is in turn favorably cited in precisely this connection by Einstein. For further discussion see again Friedman (2002).

DiSalle, R. (2002). Reconsidering Ernst Mach on space, time, and motion. In D. B. Malament (Ed.), *Reading natural philosophy: Essays in the history and philosophy of science and mathematics* (pp. 167–191). Chicago and La Salle, Ill.: Open Court Publishing Company.

DiSalle, R. (2006). *Understanding space-time: The philosophical development of physics from Newton to Einstein*. Cambridge: Cambridge University Press.

Edgerton, S. Y. (1991). *The heritage of Giotto's geometry: Art and science on the eve of the scientific revolution*. Ithaca, London: Cornell University Press.

Einstein, A. (1921). *Geometrie und Erfahrung. Erweiterte Fassung des Festvortrages gehalten an der Preussischen Akademie der Wissenschaft zu Berlin am 27. Januar 1921* (G. Jeffrey & W. Perrett (Eds.), Trans.). Berlin: Springer. (*Sidelights on Relativity,* pp. 27–55. London: Methuen, 1923).

Fichant, M. (1997). 'L'Espace est représenté comme une grandeur infinie donnée'. La radicalité de l'Esthétique. *Philosophie, 50*, 20–48.

Frede, M., & Krüger, L. (1970). Über die Zuordnung der Quantitäten des Urteils und der Kategorien der Größe bei Kant. *Kant-Studien, 61*, 28–49.

Friedman, M. (1985). Kant's theory of geometry. *Philosophical Review* 94 (pp. 455–506).

Friedman, M. (1990). Kant on concepts and intuitions in the mathematical sciences. *Synthese* 84 (pp. 213–257).

Friedman, M. (2000a). Geometry, construction, and intuition in Kant and his successors. In G. Scher & R. Tieszen (Eds.), *Between logic and intuition: Essays in honor of Charles Parsons* (pp. 186–218). Cambridge: Cambridge University Press.

Friedman, M. (2000b). Logical form and the order of nature: Comments on Béatrice Longuenesse's Kant and the capacity to judge. *Archiv für Geschichte der Philosophie, 82*, 202–215.

Friedman, M. (2002). Geometry as a branch of physics: Background and context for Einstein's 'geometry and experience'. In Malament, *op. cit.,* (pp. 193–229).

Friedman, M. (2009). Newton and Kant on absolute space: From theology to transcendental philosophy. In M. Bitbol, P. Kerszberg, & J. Petitot (Eds.), *Constituting objectivity: Transcendental perspectives on modern physics* (pp. 35–50). Berlin: Springer.

Friedman, M. (2010). Synthetic history reconsidered. In M. Domski & M. Dickson (Eds.), *Discourse on a new method: Reinvigorating the marriage of history and philosophy of science* (pp. 571–813). Chicago, La Salle, Ill.: Open Court Publishing Company.

Friedman, M. (2012a). Kant on geometry and spatial intuition. *Synthese, 186*, 231–255.

Friedman, M. (2012b). Newton and Kant: Quantity of matter in the metaphysical foundations of natural science. *Southern Journal of Philosophy, 50*, 482–503.

Friedman, M. (2012c). The *Prolegomena* and natural science. In H. Lyre & O. Schliemann (Eds.), *Kants Prolegomena. Ein kooperativer Kommentar* (pp. 299–326). Frankfurt am Main: Klostermann.

Friedman, M. (2013). *Kant's construction of nature: A reading of the metaphysical foundations of natural science*. Cambridge: Cambridge University Press.

Friedman, M. (2015). Space and Geometry in the B Deduction. In C. Posy & O. Rechter (Eds.), *Kant's philosophy of mathematics, vol. I: The critical philosophy and its background.* Cambridge: Cambridge University Press.

Heath, T. L. (Ed.). (1926). *The thirteen books of Euclid's elements*. Cambridge: Cambridge University Press.

Heidegger, M. (1977). *Phänomenologische Interpretation von Kant's Kritik der reinen Vernunft.* Frankfurt am Main: Klostermann). English edition: (1997). *Phenomenological Interpretation of Kant's Critique of Pure Reason.* Bloomington and Indianapolis: Indiana University Press.

Janiak, A. (2008). *Newton as philosopher*. Cambridge: Cambridge University Press.

Kant, I. (Ed.). (1900-). *Kant's gesammelte Schriften* (Vol. 29). Berlin: Royal Prussian (later German and Berlin-Brandenburg) Academy of Sciences, Berlin: Georg Reimer (later Walter de Gruyter).

Longuenesse, B. (1998a). *Kant and the capacity to judge: Sensibility and discursivity in the transcendental analytic of the critique of pure reason*. Princeton: Princeton University Press.

Longuenesse, B. (1998b). Synthèse et donation (Réponse à M. Fichant), *Philosophie* 60 (pp. 79–91). (Translated in B. Longuenesse 2005, pp. 64–78).

Longuenesse, B. (2001). Synthesis, logical forms, and the objects of our ordinary experience: Response to Michael Friedman. In *Archiv für Geschichte der Philosophie* 83 (pp. 199–212). (Reprinted in B. Longuenesse 2005, pp. 39–63).

Longuenesse, B. (2005). *Kant on the human standpoint*. Cambridge: Cambridge University Press.

Newton, I. (1999). *The principia*: *Mathematical principles of natural philosophy* (I. B. Cohen & A. Whitman, Ed., & Trans., assisted by J. Budenz). Berkeley and Los Angeles: University of California Press.

Parsons, C. (1992). The transcendental aesthetic. In P. Guyer (Ed.), *The Cambridge companion to Kant* (pp. 62–100). Cambridge: Cambridge University Press.

Stein, H. (1990). Eudoxus and Dedekind: On the ancient Greek theory of ratios, and its relation to modern mathematics. *Synthese, 84*, 163–211.

Stein, H. (2002). Newton's metaphysics. In I. B. Cohen & G. E. Smith (Eds.), *The Cambridge companion to Newton*. Cambridge: Cambridge University Press.

Sutherland, D. (2004a). The role of magnitude in Kant's critical philosophy. *Canadian Journal of Philosophy, 34*, 411–442.

Sutherland, D. (2004b). Kant's philosophy of mathematics and the Greek mathematical tradition. *Philosophical Review, 113*, 157–201.

Sutherland, D. (2006). Kant on arithmetic, algebra, and the theory of proportion. *Journal of the History of Philosophy, 44*, 533–558.

Thompson, M. (1989). Unity, plurality, and totality as Kantian categories. *The Monist, 72*, 168–189.