Introduction

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Today the definition of geometry as the *science of space* is generally accepted by both epistemologists and mathematicians. The history of modern geometry is entirely built around the mathematical notion of space, and different approaches to this science, from Gauss' studies of intrinsic curvature to the Erlangen Program, from the discovery of General Relativity to the most recent developments in topology (take, for instance, Thurston's geometrization conjecture and its proof) rely on a general understanding of mathematical space that remains constant through different perspectives and offers a common ground for regarding all these developments as parts of a single enterprise. Modern geometry is simply inconceivable without the notion of space.

Nonetheless, the definition of geometry as the science of space, however standard, is properly speaking *modern*. Should we go through the thirteen books of Euclid's *Elements*, or in fact the entire corpus of ancient mathematics, we would find almost no occurrence of spatial concepts or terms. Were we to follow the millennial development of Classical geometry in the Middle Ages or the Renaissance, we would still not find any reference to space. The first (and quite rhetorical) mention of *spatium* in a geometrical essay does not predate the last decades of the sixteenth century. To see spatial notions effectively *employed* in geometrical reasoning, we have to wait for another one hundred years. Leibniz' work on geometry (the *analysis situs*) is probably the first attempt in this direction, and in any case it ushered in a general discussion about the object of geometrical investigation. The eighteenth century debated whether geometry had to be regarded as a science of space, and this new idea initially attracted more opponents than supporters; by the end of the century the spatial backers won their battle, and the nineteenth century

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declared that space was indeed the object of geometry. The *mathematization of space* was then complete: classical geometry came to an end and modern geometry was born.

In fact, even though the divide between ancient and modern geometry may be arbitrarily demarcated into several historically distinct episodes (such as the birth of algebraic geometry or the discovery of the infinitesimal calculus), the transformation of geometry into a science of space is probably the most important change the discipline underwent during the course of its development. In accepting space as its object of investigation, geometry began to study relational *structures* instead of single figures or magnitudes (like triangles or conic sections). In this sense, the entire structuralist approach of modern mathematics is grounded in this important shift of perspective of eighteenth-century geometry, which (to use Cassirer's words) turned a classical geometry of substances (i.e. figures) into a geometry of functions (structures). As soon as space came to be regarded as a mathematical structure, in fact, it was declined in the plural: non-Euclidean geometries, projective geometry, and then the theory of manifolds, were all disclosed by this new approach to space, whereas a geometry of figures could not possibly produce the idea of different global structures. Moreover, the very *spatiality* of these structures was of primary importance. While the primary aim of classical geometry was the calculation of lengths, areas and volumes of given figures, in a geometry of space the notions of position, incidence or direction may play a central role. Once again, non-metric geometry, projective geometry or topology were inconceivable without such a transition from a science of figures to one of space. The transformation of the object of geometry, moreover, had several consequences outside the strictest mathematical domain, and the entire philosophical development of the last two centuries is scarcely understandable without the notion of space conceived as a mathematical structure.

This new picture of mathematics, which in many respects is still the one we have of this science, found its beginnings in the assessment of geometry as the science of mathematical space, and it is grounded in the long process that transformed Greek geometry into a spatial endeavor. This evolution occurred for the most part outside the realm of mathematics, and involved a gradual evolution of the metaphysical conceptions of space, the epistemology of modern science, and several other advances in various disciplines, from geography to the theory of perspective, from mechanics to cosmology. I will now try to sketch a very brief history of the transformation of geometry from the Greek science of figures to the nineteenth-century science of space, with special emphasis on the philosophical aspects of the development. This is obviously a very limited perspective on the whole matter, and it seriously risks several oversimplifications; yet, I think that it may still be useful to have a guiding thread to lead us through the complex history of the mathematization of space before engaging in further and more detailed investigations. To this effect, I would distinguish four stages in the relations between geometry and space: a geometry without space, a geometry in material extension, a geometry in space, and finally a geometry of space.

(1) *A geometry without space*. The first stage encompasses Greek geometry in the Classical and Hellenistic Ages. Here Geometry is completely devoid of any

spatial content or reference. The objects of geometrical investigations are individual figures (triangles, circles, and so on), which are conceived as reciprocally unrelated and are not embedded in any spatial background. They may be regarded as Platonic ideas or Aristotelian abstractions from material substances (or in several other ways), but in any case these singular figures are the sole objects of geometrical enquiry. Accordingly, their definitions are considered to be the true principles of demonstration and theorems only deal with the properties of these figures.

In the epistemology of Plato we do not find any general characterization of the object of geometry since mathematics is for him defined through its method (hypothetical and diagrammatic) much more than through its subject matter. Still, one can easily understand that he regarded *form* and *shape* (εἶδος and σχήμα) as the main features of geometrical objects. In the age of Plato or shortly thereafter, however, a first general theory of magnitudes and proportions was developed. This theory, which is often attributed to Eudoxus and was later embedded (as Book Five) in the Euclidean *Elements*, was general enough to state and prove theorems that uniformly apply to any kind of continuous extension (be it a line, a plane figure or a solid). In this way, the manifold objects of geometrical (or stereometrical) investigation were reduced to a common *genus*, and it comes as no surprise that Aristotle was able to define geometry through its subject matter (rather than its method) stating that the object of geometrical investigation is continuous quantity, or magnitude (μέγεθος). This definition immediately acquired paramount importance, and was unfailingly repeated by mathematicians and philosophers for more than two thousand years. Even though Aristotle and his successors clearly conceived magnitudes as shaped extensions, and thus as classical geometrical figures (since no abstract or numerical notion of a continuous magnitude was available), shape, form or position were simply regarded as accidents and properties of geometrical objects essentially conceived as concrete magnitudes. The Aristotelian definition bent geometry in the direction of a science of measure, whose principal aims were to compute length, areas and volumes of given magnitudes. In this respect, it concealed the possibility of a mathematical development of the notions of geometrical form or shape, or the (purely spatial) concept of reciprocal position (θέσις) of geometrical objects. Even though we actually find several uses of the concepts of shape and position in most ancient Greek geometry (for instance: a treatment of similarity in Thales, a wide employment of the concept of position in Pythagoreanism, the notion of form in Plato, and all of these in the Euclidean treatise on *Data*), we are not allowed to suppose, of course, that without Aristotle's influence this science actually had the possibility of developing toward non-metrical (and possibly spatial) outcomes. The advancements of Classical geometry as a theory of measurement are largely independent of any epistemological claim, and seem to be essential to this science, which was born (at least in words) to measure the Earth. Yet, it is certainly true that the Aristotelian definition of geometry as a science of magnitudes, so widely accepted in the following centuries, acted as a powerful constraint to later attempts (in the Renaissance or the Early Modern Age) to transform geometry into a science of space, positions and configurations, that could supersede a metric theory of measure; and even Kant, at the very end of our story, could not but define geometry through the category of quantity—as Aristotle had done.

In the Classical Age, spatial notions remained mostly extraneous to geometry. While Plato had understood geometrical objects as pure ideas, the later Greek philosophers, who regarded them as magnitudes (something that in principle might be located somewhere), usually strongly denied that they are in space and claimed that mathematical objects are nowhere (τὰ μαθηματικὰ οὐ πού).

On the other hand, the concept of space itself is altogether missing from the cultural background of Antiquity. The Platonic notion of χώρα seems to refer to a material extension rather than to a proper spatial container (although the interpretation is controversial). The void of pre-Socratic atomism seems to mean much more a mere nothingness than a positive spatial extension, while the void of Epicurean atomism may be extended, but has a few features in common with matter (it moves away when atoms arrive and it is not properly filled by them); in sum, the vacuum of Greek atomism is a spaceless void rather than a prefiguration of space. Both Stoics and Epicureans, apparently, also had a notion of space as something extended that may be filled or not, but they did not even have a name for such an entity which they labeled "the intangible nature" or "what can be occupied by a body"; no properties of such a thing were ever spelled out, nor does it seem they were in any way connected to quantity or mathematics. We do have, of course, several important philosophical treatments of the concept of place (τόπος), which is however an ecological notion rather than a geometrical one and is related to orientation in the environment, geographical position, or cosmic localization (to be in a place is to be in the market, on a boat, in Athens, in a vessel, in the outermost heaven). In all these cases, the place of a body is just another body which is regarded as having the role of a place. Accordingly, there is no space as an independent being that may be quantified, or measured, or otherwise mathematized. Even geography, which developed several mathematical techniques to represent the world, was mostly concerned with the measurement of an object, the Earth or perhaps the Ecumene, rather than its place. It cannot be denied, however, that the plurality of the representational techniques employed in geography and astronomy already in Antiquity (produced by the impossibility of isometrically projecting a sphere on a map) trespassed on the simple conception of measuring given magnitudes, and paved the way for understanding how different geometrical structures may be employed to capture the world. The lack of a veritable concept of space, however, still prevented the possibility of regarding these different geometrical representations as realizations of space structures, or of conceiving them as anything more than an application of the geometry of magnitudes to a specific cosmic object.

(2) *A geometry in material extension*. This very general metaphysical picture, however, began to change already in the Neoplatonic philosophy of Late Antiquity. Aristotelian ontology regarded quantity (and thus magnitude) as an accident of a given individual substance; and this was in perfect agreement with the idea of a geometry of individual figures. Neoplatonic metaphysics, on the other hand, admitted different ontological stratifications, and might conceive (in a few authors)

that quantity is not an accident of the substance, but an accident of matter itself. Matter was thus regarded as quantified and extended, or at least endowed with magnitude as its primary property. This notion of an extended matter, that came from Plato's conception of the Receptacle, had nonetheless acquired an Aristotelian bent, as magnitude was now conceived by Neoplatonists as a positive property that may be geometrically investigated rather than (as in Plato) the unknowable material substrate in which Forms appear. The immediate consequence is that extended matter is in fact the substrate of geometrical objects and their ontological condition, and that this matter exists (in some sense) before being formed in the particular shapes of the individual geometrical figures. This insubstantial extension is thus a kind of background in which geometrical objects have their seat. It is not yet space, as it is a material extension and it is not explicitly connected with any local or positional property; moreover, the material extension is itself devoid of any geometrical property, and geometry remains the science of those definite and individual magnitudes that are made up by this extension: the latter is the condition, rather than the object, of geometry. Yet, this Neoplatonic quantified matter in which geometrical magnitudes are first delineated is the ancestor of the modern concept of space.

This conception of an extended matter as the substrate of geometrical magnitudes is quite general, and it received very different treatments by different metaphysicians and schools. There is no need here to go into the many different perspectives that very different authors held on this topic, as we are only interested in this conception of matter as a prehistory of the notion of space. It needs to be said, however, that while several philosophers intended quantified matter as corporeal matter (the matter of the world) and thus connected the ontology of mathematical objects to that of cosmic extension (possibly through an abstractionist theory of mathematical entities inspired by Aristotle), a few authors from Late Antiquity on regarded the matter of extended mathematical objects as "intelligible matter" (another term taken from Aristotle), or imaginative matter (ὕλη φανταστική). In this way, they advanced an ontology of geometrical objects as ideal entities that are constructed by the mathematician in the imagination. Imagination is conceived as a sort of blackboard on which figures are drawn. Even though, once again, the blackboard itself is not thematized as an object of geometrical investigation, its relevance for the history of the concept of mathematical space is enormous. The main supporter of this view in Late Antiquity was Proclus, who developed an important theory of mathematical "projective" imagination in his philosophical commentary to the First Book of Euclid's *Elements*, but the idea that imagination is the proper faculty of geometry (a conception that was absent from the Greek thought of the Classical Age) had a widespread diffusion in the centuries to come.

In any case, the significance of the ontology of quantified matter (be it corporeal or imaginary) went well beyond the strictest Neoplatonic doctrines, and was generally accepted in the Middle Ages. Especially through the Averroistic tradition, the doctrine of the *dimensiones indeterminatae* of matter (a matter endowed with extension and indeterminate magnitude but without a determinate shape) enjoyed diffusion and acquired relevance in the West (think of Aquinas' *materia signata quantitate*) and formed the basic geometric ontology for several centuries. In the Renaissance, most Late Scholastic authors still conceived matter as essentially extended and regarded it as the ontological condition for the existence of the proper mathematical objects (the shaped magnitudes). The connection of geometry with the imagination remained strong during the Middle Ages, even though the original writings of Proclus were unknown and the Scholastics had to rely on secondary sources. The debate on quantified matter in imagination however reached its highpoint in the Renaissance, when Alessandro Piccolomini (in 1547) interpreted the recently rediscovered Proclus and engendered the so-called *quaestio de certitudine mathematicarum*, a quarrel among metaphysicians and epistemologists that continued for more than a century.

A few philosophers were ready to endorse the Averroistic tradition of quantified corporeal matter as the substrate of geometrical entities to the point of rejecting several Euclidean principles and theorems that cannot be accommodated to bodily magnitudes, since the latter are (for instance) necessarily tridimensional (leaving no space for a pure plane geometry), or since they may not be infinitely divisible; a vast number of indivisibilist mathematicians, in fact, ranging from the Middle Ages (Gerard of Odo or Nicholas d'Autrecourt) to the Early Modern Age (Bruno, Arriaga), grounded their rejection of infinite divisibility and (in some cases) a new finitistic mathematics on their views about the features of bodily extension.

Many others, however, followed Proclus and appealed to the imagination in order to ground the possibility of ideal geometrical constructions. This conception was especially widespread among professional mathematicians, from Peletier to Clavius, Barrow or Borelli, but was shared by several philosophers as well.

Descartes was able to encompass both traditions, with a complex mathematical ontology that conceived geometrical objects as parts of the *res extensa*, but which also stressed their connection with the faculty of imagination. It is remarkable how, in the hands of a great mathematician, the metaphysics of geometrical extension yielded important mathematical outcomes. The algebraization of geometry, in fact, required geometrical figures to be conceived as embedded in a larger (indefinite) extension, that can be captured in a system of coordinates. Such a fundamental mathematical development was not possible in the classical ontology of individual figures, and was in need of an ontology of quantified matter. A few further issues and techniques in geometry, like the theory of *loci* or the use of motion in a proof, acquired new meanings in the wake of this ontological transformation.

Yet, in the innumerable disputes that divided both Scholastics and new philosophers at the end of the Renaissance, and produced such different metaphysical systems, the conception of geometry remained firmly chained to these two pillars: it was the science of magnitudes, as Aristotle had said; and these magnitudes were to be conceived as parts and chunks of an originally extended and quantified matter.

Another development, however, was maturing in the Neoplatonic tradition: the ancient conception of place as an ecological notion was substituted by a concept of space as a three-dimensional extension. This conception is generally ascribed to the Aristotelian commentators Philoponus and Simplicius in Late Antiquity, who first worked it out at some length (with a few differences between them) disavowing their master's opinion. During the Middle Ages the notion was further developed, even though its acceptance always remained controversial, given the authority of Aristotle who had opposed it; but in the Renaissance it gained numerous followers among new philosophers and Aristotelians as well. From this perspective, the three-dimensional extension of place was still conceived as an accident dependent on the existence of the (individual) located substance, rather than an all-encompassing space ontologically independent of the bodies in it. This notwithstanding, the extension of place was quantified and could be measured like any other magnitude (which was not the case of the classical "ecological" place), and this was an important step forward toward the mathematization of space. Nevertheless, three-dimensional place had no special link with quantity or geometry, and it remained one magnitude among others. It was considered measurable, in fact, only by virtue of its capacity to receive matter, since matter only (as we have seen) is the ontological substrate of magnitude.

A further development coming from the Middle Ages was the introduction of the concept of an imaginary space (*spatium imaginarium*). This notion also came from the Averroistic tradition, and hinted at the fact that the imagination cannot help but conceive further extension beyond the boundaries of heaven. The debate on the nature of imaginary space was wide ranging, but we need only observe that it had several advantages over the three-dimensional concept of place: it was a veritable *space*, in the sense that it was not limited to the extension of the individual located body, but rather it encompassed an infinite (or, shall we say, indefinite) extension; and it was a creation of the imagination, which again hinted toward a connection of it with the ontology of mathematical objects as proper products of the *phantasia*. However, this possible connection remained unexplored in the Middle Ages and the Renaissance since the imaginary spaces were conceived as *purely local* constructions, which had nothing to do with quantity in the proper sense; indeed, several authors insisted that being devoid of matter, imaginary spaces have no quantity whatsoever. Even Hobbes, well into the seventeenth century, was still claiming that while geometry may have a substrate in imaginary space, this only comes from the corporeal origin of imaginary space itself—since extension and quantity are nothing but bodily features.

(3) *A geometry in space*. By the end of the Renaissance, however, these manifold traditions were ripe enough to merge into something new. In particular, the Philoponian conception of three-dimensional place and the notion of an infinite imaginary space finally met the ontology of quantified matter and geometrical objects. The hint may have come from Ficino, who translating Plotinus' *Enneads* and dealing with the Neoplatonic treatment of quantified matter (ὄγκος, mass) realized that this was just pure extension (primary matter plus quantity) rather than bodily extension, and thus ventured to translate the term as *spatium*. Ficino did not draw any consequence from his own translation, and continued to profess a standard Neoplatonic philosophy of mathematics and an Aristotelian conception of place. Several years later, however, Francesco Patrizi, a Neoplatonic disciple of Ficino and a vigorous renewer of metaphysics, followed the hint and superimposed

the metaphysics of quantified matter onto that of three-dimensional space. The result was a new conception of space as a three-dimensional infinite extension which is independent of matter and bodies, and is pre-exististent to them as a condition of their own existence. Patrizi's new metaphysics was probably more suggestive than well argued, and was developed in the same years that others (Bruno, in particular) were also elaborating similar views. However, Patrizi was alone in stressing a new conception of mathematics. He claimed that space is originally and essentially quantified and extended, whereas matter and bodies are quantified only because they are embedded in space. He was thus turning the preceding conception upside down by claiming that the possibility of mathematizing the natural world depends on the possibility of applying geometry to space itself. To his essay *On Physical Space* Patrizi added a treatise *On Mathematical Space*, and an entire book *On a New Geometry* (*Della nuova geometria*, 1586), in which he explicitly asserted that "the general subject of mathematics is space" and he tried to rewrite a portion of Euclid's *Elements* in such a way as to reflect the change in the object of study. To my knowledge, this is the first occurrence of such a claim in a book of geometry, and almost the birth of the idea of a geometry of space.

We should add, however, that there is not much more here than an idea and a project. Patrizi's skills as a mathematician were very scarce, and in any case he was not able to foster any real improvement in Classical geometry. Patrizi was still discussing straight lines and circles, not space, and he was not conceiving space as a structure that could be investigated. His claim that space is essentially quantity had allowed him to think of it as the substrate of geometry (a big step forward), but had also imprisoned his "new geometry" in the usual cage of a science of magnitudes (rather than spatial relations), and Patrizi did not really question the Aristotelian tenet that magnitudes are the objects of geometrical investigations. He only changed the ontology, stating that these magnitudes are not just chunks of matter, but rather regions of the all-encompassing space. In sum, his is not a geometry *of* space, but a geometry of figures and magnitudes *in* space.

Patrizi's metaphysics of space enjoyed a wide diffusion, and informed the new atomistic philosophy of the seventeenth century (Gassendi was very familiar with his work), as well as further Neoplatonic developments (such as those of the Cambridge Platonists), and directly or indirectly the new metaphysics spread during the course of the century. Newton himself may be regarded as an heir of these Renaissance metaphysical conceptions of space, and in the *De gravitatione*, in particular, he gives a description of space and the figures existing in it which are reminiscent of several passages of Patrizi's. The idea that the objects of geometry are spatial figures gradually spread, and in a few decades it became commonplace in several circles. The two opposing views that geometrical objects are material or rather spatial entities battled with each other over the next hundred years, and if at the beginning of the seventeenth century nearly everyone had advocated the former position, by the end of the century only a few indomitable Cartesians (or very late Aristotelians) were opposing the latter. The battle was purely metaphysical, though, as all of the participants were still working within the framework of the classical geometry of figures, and (almost) no mathematical consequences were drawn from their different geometric ontologies. Newton, for one, considered himself a classical geometer, and his innovations in the field do not at all concern the subject matter of geometrical investigations (even though his conception of space may have influenced his geometrical endeavors).

Among the several developments that contributed to the discovery of space as a structure, we may at least mention here the development of a mathematical geography and cosmography, as well as the tradition of perspectival studies.

In his celebrated essay on *Perspektive als symbolischer Form*, Panofsky claimed that the modern concept of space was born in the Renaissance treatises on perspective. At least from the time of Giotto or Duccio, to be sure, a new form of spatial organization may be found in pictorial representations. But this new way of representing space was only codified much later, in the late fifteenth century, when a *mathematical* reflection on those painting techniques began to be available in the works of Alberti, and then Piero della Francesca, Albrecht Dürer, and many others. In the sixteenth century, we find an explosion of geometrical essays on the theory of perspective that explicitly discuss the mathematical structure of place and space. The humanist Pomponio Gaurico, in his book *De sculptura* (1504), claimed that the painter should first depict the *locus* of the represented things, and only then the things themselves, since place is prior to located objects. He simply meant that the perspectival technique (the *locus*) should be selected in advance; but his choice of words shows a conception of place that is not an object but rather a mathematical structure which is a condition and an order of objects. Different structures (different spaces) with different mathematical properties may be employed to represent the very same things. While ancient paintings may have depicted single objects in perspective, juxtaposing certain objects with others (each object, I mean, with its own perspective and vanishing point), a modern painter might offer what Gaurico calls a perspective which *ad totum opus pertinet* encompassing the whole scene in a single space. In this respect we moved from a perspective of substances (figures) to a perspective of structures and relational systems.

It is true that this use of the word “*locus*” to mean a perspectival technique may seem quite eccentric, and it was at first confined to the treatises on this discipline without acquiring any more general sense. The essays on perspective, however, became more and more mathematical over the years, and by the end of the sixteenth century they had already transformed from painters’ handbooks into studies for mathematicians (think of Guidobaldo del Monte’s *Perspectivae libri sex* in 1600). When Desargues and then Pascal or La Hire in the seventeenth century applied perspectival techniques to prove classical geometrical theorems, the development was complete: and the perspectivists’ conception of place or space as a structure and an order had became a tool for “real” mathematics. For the first time a few purely spatial notions, like those of *position* or *situation* of points and lines, and the study of spatial incidence and configurations, became an object of geometrical enquiry.

(4) *A geometry of space*. The synthesis of all these different trends toward a geometry of space was first realized in Leibniz’ work on *analysis situs*. Leibniz had

the opportunity to read Pascal's lost treatise on conic sections in manuscript (where Pascal had probably employed the notions of *space* and *situation* to ground projective techniques in geometry) and devoted himself to the task of developing a new geometry entirely based on the spatial notion of situation (*situs*).

Leibniz' conception of space, which he famously opposed to Newton's absolute space, was that of a system of situational relations (an *ordre de situations*), and thus a structure. The new *analysis situs*, in developing a geometry of situations, had to study the structure of space itself. This was the first time in history that space was conceived as a structure and geometry as its science. In fact, Leibniz' work on *analysis situs* revolves around the definition and interrelation of a few structural properties of space, like uniformity (our isotropy), homogeneity (our manifold structure), dimensionality, continuity, connectedness, flatness, and so forth, that had been never studied before. The possibility and properties of every figure should be reduced to those of the ambient space, which is the only real object of geometrical investigation. Leibniz' aim with his *analysis situs* is in fact to show that only one all-encompassing spatial structure is possible, thus to ground on absolute foundations the existence of a three-dimensional, uniform, homogeneous, continuous, connected, Euclidean space. And if his goals were unreachable, his endeavors were nevertheless quite interesting as they show how a geometer could forsake the classical geometry of figures and ruler-and-compass constructions and attempt a much more abstract study of the structural properties of a system of relations.

The concept of situation is generally reduced to that of distance (a situation between two points being expressed by their distance), and this fact prevented Leibniz from developing a truly non-metrical geometry (a projective geometry, for instance). This notwithstanding, geometry is for Leibniz first a science of position and place, and only derivatively a science of quantity. He attempted, in fact, to develop a wide-ranging theory of similarity that should have been (partially) free from metrical considerations. Leibniz' relational conception of space also offered a new solution to the problem of the composition of the continuum, which stated that whereas a set of unextended points (however numerous) could never build up a continuous extension, a set of points endowed with situational relations (that is, points considered as terms of relations of distance) may nonetheless be extended, since extension and continuity themselves are nothing but structural properties of such a set. This approach, which was very modern and had never before been attempted, opened the way for a general set-theoretical and structural view of space.

However promising, Leibniz' attempts at a geometry of space remained unpublished, and the eighteenth century only caught a few glimpses of this new science. The Leibnizian program in geometry, however, was well known, and it sparked a lasting debate (especially in Germany) on the objects and methods of geometry. It opposed traditional geometers, who still claimed that figures in space are the object of geometry and appealed to the authority of Euclid, to Leibnizians who advocated the advantages of a geometry of space but could not show any significant consequence of this new science (Wolff, for one, stated that geometry is the science of space but was unable to produce any result that went beyond Euclid).

The same seeds that had engendered Leibniz' *analysis situs* in the seventeenth century, however, continued to bear fruit in the eighteenth century. The developments of a mathematical theory of perspective ended up with Monge's and Poncelet's first attempts at a veritable descriptive and projective geometry, which required an array of spatial and local concepts and tools, and could not help but rely on the idea of a science of space. Closely related to these mathematical developments, the theory of visual perception also pushed a few philosophers on the same shore. The geometrization of visual space had already begun (I mean, with full philosophical awareness) in Descartes' *Dioptrique* and theory of perception, which represented a fundamental step forward toward a mathematization of the world accomplished through a (kind of) spatial structure. But this *géométrisation du regard* continued in the eighteenth century in the works of Berkeley, Hume and Reid (among others), who discussed the manifold properties (and alleged differences) of visual and tactile spaces; and while Berkeley seems to aim at a projective conception of visual space that may not be captured by a geometrical theory, Hume's conceptions of perceptual *minima* (also stemming from Berkeley) may tend toward a finitistic geometry, and Thomas Reid's (quasi) axiomatization of spherical geometry as the geometry of sight professes that space is nothing but a perceptual structure, that a plurality of such structures is possible, and that they can be differently treated from a mathematical point of view.

In the same years, other mathematicians were developing their first essays on non-Euclidean geometry. Saccheri in 1733 was still unable to recognize his "obtuse angle hypothesis" as an instance of spherical geometry (the same that was discussed by Reid), since he was a hyper-classicist mathematician who considered his own studies on the Parallel Postulate to be a Euclidean exercise in a geometry of figures. As early as 1766, however, Lambert realized that the Parallel Postulate does not concern parallel or incident lines, but the deeper structure of space itself. Wallis had shown that the Postulate is in fact equivalent to the possibility of transformations in space through similarity, and the latter is clearly a "second order" property about figures rather than the property of a single figure (or a couple of lines). This allowed Lambert (who had also worked on perspective) to recognize spherical geometry as a description of a non-Euclidean space, thus opening the way to conceive an abstract model for hyperbolic space as well. And while Lambert had written a *Theorie der Parallellinien*, János Bolyai, at the culmination of the non-Euclidean revolution, would later write a *Scientia spatii* (1832), or *Raumlehre*, fully realizing that his researches disclosed the science of a different space rather than a new theory of parallel lines.

Yet another line of enquiry and discussion concerned the role of principles in geometry. Definitions had been considered as the true principles of demonstration from Antiquity to the Eighteenth century, reflecting the idea that the objects of geometry are individually defined figures. From this perspective, axioms were generally regarded as immediate consequences of the definitions. In the eighteenth century, a few mathematicians began to claim that axioms should be the true principles of geometry, and definitions should follow from them (an instance of this attitude is once again in Lambert). This new epistemological claim mirrored the growing awareness that only the complete system of figures and objects, that is the space structure as such, is the true subject of investigation; as well as the awareness that a plurality of such structures (defined by different systems of axioms) are in fact possible.

In sum, just as the seventeenth century had witnessed the battle between supporters of a science of material figures against supporters of spatial figures, so the eighteenth century was divided among those who still professed a geometry of figures (in space) and those who looked for a new geometry *of* space.

The century's richest and most developed epistemology of mathematics, that of Immanuel Kant, still reflected these quarrels. Kant's philosophy of mathematics may be easily seen as the culminating point of several philosophical traditions that we have mentioned so far. His theory of productive imagination and its application to geometry, in particular, clearly draws from the Proclean tradition of a projective *phantasia*, while the idea of the applicability of mathematics to the phenomenal world through the mathematization of space comes from the Neoplatonic tradition of the Early Modern Age. And yet all these elements (and many more) are merged together into a new synthesis that seems to push them to their maximum conceptual strength. In many respects, in the epistemology of mathematics Kant appears to be the last philosopher of the classical age, and to be advancing a complex and consistent theory of geometry as a Euclidean science of figures, rather than a modern science of space. Not only did he strongly attack Leibniz' point of view on the essence of space (now conceived as a pure intuition) and the nature of a mathematical proof (based on synthetic *a priori* judgments rather than analytical statements), but his entire positive theory of mathematics is grounded on a constructivist stance that makes use of the synthesis in imagination for the composition (*Zusammensetzung*) of individual finite figures in space. These figures are the objects of investigation in geometry, as in the classical tradition; more than that, they fall under the category of quantity and are called *magnitudes*, as in the Aristotelian tradition. Space itself is regarded as the condition of geometry much more than its proper object: a formal intuition which is needed as the background of the productive activity of the imagination. This space has a unity and may be regarded as the product of a *sui generis* synthetic act, which is an intellectual but pre-categorial "putting together" (*Zusammenfassung*) of the spatial manifold. This act, which gives a unitary structure to geometrical space, is not a synthesis of imagination ruled by the category of quantity (a "composition" in the proper sense), and thus the structure of space as a whole does not fall under the scrutiny of geometry. Yet, the very concept of this act, which is explicitly (albeit peripherally) thematized in Critical philosophy, distances it from the classical Neoplatonic tradition, and reflects the idea that space as a whole may have a unitary logical structure.

In any event, the first generation of Kantian followers no longer had any doubt that space has a given geometrical structure, and in this respect Kantianism (often blended with Leibnizianism) became an important cultural force (perhaps against the will of Kant himself) to foster the idea of a new science of space. It was through these transformations and philosophical quarrels that the modern party eventually won out against old Euclid, and by the dawn of the nineteenth century it was so common to state that geometry is the *science of space* that almost no one realized that a different opinion had been possible in the past.

The history of the mathematization of space was in fact much more complex (and less linear) than the idealized picture given above. I hope that the essays presented in this book, which illustrate a few episodes of this history in great detail, will help the reader to understand the long development that transformed Greek geometry into Modern mathematics. They also show the many and various epistemological contributions that went into this conceptual development, and point to a few of the several disciplines and figures involved in the "spatial turn" of geometry.