Aggregation of Dynamic Risk Measures in Financial Management

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Abstract. This paper discusses aggregation of dynamic risks in financial management. The total risks in dynamic systems are usually estimated from risks at each time. This paper discusses what kind of aggregation methods are possible for dynamic risks. Coherent risk measures and their possible aggregation methods are investigated. This paper presents aggregation of dynamic coherent risks by use of generalized deviations. A few examples are also given.

1 Introduction

In the classical economic theory, the variance and the standard deviation have been used as risk indexes. Recently quantile-based risk criteria are employed widely in financial management. The concept of risk is different in its application fields. In engineering risks are considered in the both upper and lower areas from a true value since the risk is usually represented as the errors of the data to the true value. On the other hand in economics the concept of risk is given in a different way from the risk in engineering. The risk in economics is discussed only in an area of low rewards since the risk is connected deeply to losses and bankruptcy in financial management.

In this paper, we focus on the estimation of dynamic risks in financial management. The total estimation of dynamic risks are important for the stability of financial systems. The total risks in dynamic systems are usually estimated from risks at each time. The most popular methods for the total risks are the weighted arithmetic mean and the maximum of the risks over all periods. The method with the weighted arithmetic mean is sometimes insensitive to find the serious risks in dangerous situations ([10]). On the other hand regarding the method with the maximum it may happen to lose the chance to find out the other potential risks regarding the dynamic system since we observe only the largest risk through all periods. We can give ad hoc methods to construct a total risk from risks at each time. However is the total risk consistent as a risk measure? We need to investigate whether the total risk inherits propertires as a risk mesure from risks at each time. From the view point of aggregation operators ([1] and [9, Section 4.1]), this paper discusses what kind of aggregation methods are possible for dynamic risks.

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In Section 2 we investigate coherent risks and their possible direct aggregation methods. In Section 3 we discuss generalized deviations and their aggregation methods. In Section 4 we present aggregated dynamic coherent risks by use of generalized deviations. A few examples are also given.

2 Coherent Risk Measures

In recent financial management, the risk indexes derived from percentiles are used widely to estimate risks regarding losses and bankruptcy. Let (Ω, P) be a probability space, where P is a non-atomic probability measure. Let $\mathcal{X}$ be a set of integrable real random variables on Ω. The expectation of a random variable $X(\in \mathcal{X})$ is written by $E(X) := \int_\Omega X\, dP$.

Example 2.1 (Risk indexes defined by percentiles, Jorion [4], Tasche [7]).

(i) Value-at-risk (VaR): Let $X(\in \mathcal{X})$ be a real random variable on Ω for which there exist a non-empty open interval I and a strictly increasing and onto continuous distribution function $x(\in I) \mapsto F_X(x) := P(X < x)$. Then, the *value-at-risk (VaR)* at a risk-level probability p is given by the p-percentile of the distribution function F_X as follows:

$$\mathrm{VaR}_p(X) := \begin{cases} \inf I & \text{if } p = 0 \\ \sup\{x \in I \mid F_X(x) \le p\} & \text{if } 0 < p < 1 \\ \sup I & \text{if } p = 1. \end{cases} \tag{2.1}$$

(ii) Average value-at-risk (AVaR): Take $\mathcal{X}$ in the same way as (i). The *average value-at-risk (AVaR)* at a risk-level probability p is given by

$$\mathrm{AVaR}_p(X) := \begin{cases} \inf I & \text{if } p = 0 \\ \dfrac{1}{p} \displaystyle\int_0^p \mathrm{VaR}_q(X)\, dq & \text{if } 0 < p \le 1. \end{cases} \tag{2.2}$$

Let $\mathbb{R}$ be the set of all real numbers. Rockafellar and Uryasev [5] and Artzner et al. [2,3] introduce the following concept regarding risk measures.

Definition 2.1. A map $R : \mathcal{X} \mapsto \mathbb{R}$ is called a *(coherent) risk measure* on $\mathcal{X}$ if it satisfies the following conditions (R.a) – (R.e):

(R.a) $R(X) \le R(Y)$ for $X, Y \in \mathcal{X}$ satisfying $X \ge Y$. (*monotonicity*)
(R.b) $R(X + \theta) = R(X) - \theta$ for $X \in \mathcal{X}$ and real numbers θ.
(R.c) $R(\lambda X) = \lambda R(X)$ for $X \in \mathcal{X}$ and nonnegative real numbers λ. (*positive homogeneity*)
(R.d) $R(X + Y) \le R(X) + R(Y)$ for $X, Y \in \mathcal{X}$. (*sub-additivity*)
(R.e) $\lim_{k\to\infty} R(X_k) = R(X)$ for $\{X_k\} \subset \mathcal{X}$ and $X \in \mathcal{X}$ such that $\lim_{k\to\infty} X_k = X$ almost surely. (*continuity*)

The property (R.b) in Definition 2.1 is called *translation invariance* in financial management. We can easily check the following lemma for Example 2.1.

Lemma 2.1. *An index* $R = -\mathrm{VaR}$ *given by the value-at-risk satisfies the conditions of Definition 2.1 except for the sub-additivity (R.d). However an index* $R = -\mathrm{AVaR}_p$ *given by the average value-at-risk is a risk measure in the sense of Definition 2.1.*

Let T be a positive integer. Now we introduce risk measures for a *stochastic sequence*, where a random event at time $t(= 1, 2, \cdots, T)$ is denoted by a real random variable $X_t (\in \mathcal{X})$. In this paper, we represent the stochastic sequence simply as a *random vector* $\boldsymbol{X} = (X_1, X_2, \cdots, X_T)$. We discuss aggregation of risk measures $R_1(X_1)$, $R_2(X_2), \cdots, R_T(X_T)$ for a stochastic sequence of random variables $X_1, X_2, \cdots, X_T$. Denote a vector space of random variables in $\mathcal{X}$ by the product space $\mathcal{X}^T$. For random variables $\boldsymbol{X} = (X_1, X_2, \cdots, X_T) \in \mathcal{X}^T$ and $\boldsymbol{Y} = (Y_1, Y_2, \cdots, Y_T) \in \mathcal{X}^T$, a partial order $\boldsymbol{X} \geq \boldsymbol{Y}$ implies $X_t \geq Y_t$ for all $t = 1, 2, \cdots, T$. We introduce the following definition from Definition 2.1.

Definition 2.2. A map $\boldsymbol{R} : \mathcal{X}^T \mapsto \mathbb{R}$ is called a *(coherent) risk measure* on $\mathcal{X}^T$ if it satisfies the following conditions (R.a) – (R.e):

(R.a) $\boldsymbol{R}(\boldsymbol{X}) \leq \boldsymbol{R}(\boldsymbol{Y})$ for $\boldsymbol{X}, \boldsymbol{Y} \in \mathcal{X}^T$ satisfying $\boldsymbol{X} \geq \boldsymbol{Y}$. (*monotonicity*)
(R.b) $\boldsymbol{R}(\boldsymbol{X}+\boldsymbol{\theta}) = \boldsymbol{R}(\boldsymbol{X}) - \theta$ for $\boldsymbol{X} \in \mathcal{X}^T$ and real vectors $\boldsymbol{\theta} = (\theta, \theta, \cdots, \theta) \in \mathbb{R}^T$. (*translation invariance*)
(R.c) $\boldsymbol{R}(\lambda \boldsymbol{X}) = \lambda \boldsymbol{R}(\boldsymbol{X})$ for $\boldsymbol{X} \in \mathcal{X}^T$ and nonnegative real numbers λ. (*positive homogeneity*)
(R.d) $\boldsymbol{R}(\boldsymbol{X} + \boldsymbol{Y}) \leq \boldsymbol{R}(\boldsymbol{X}) + \boldsymbol{R}(\boldsymbol{Y})$ for $\boldsymbol{X}, \boldsymbol{Y} \in \mathcal{X}^T$. (*sub-additivity*)
(R.e) $\lim_{k\to\infty} \boldsymbol{R}(\boldsymbol{X}_k) = \boldsymbol{R}(\boldsymbol{X})$ for $\{\boldsymbol{X}_k\} \subset \mathcal{X}^T$ and $\boldsymbol{X} \in \mathcal{X}^T$ such that $\lim_{k\to\infty} \boldsymbol{X}_k = \boldsymbol{X}$ almost surely. (*continuity*)

We note that $\boldsymbol{R}(\mathbf{0}) = 0$ and $\boldsymbol{R}(\boldsymbol{\theta}) = -\theta$ for real vectors $\mathbf{0} = (0, 0, \cdots, 0) \in \mathbb{R}^T$ and $\boldsymbol{\theta} = (\theta, \theta, \cdots, \theta) \in \mathbb{R}^T$. The risk criterion $\boldsymbol{R}$ of a random variable $\boldsymbol{X} = (X_1, X_2, \cdots, X_T) \in \mathcal{X}^T$ is given by aggregation of risk indexes $R_1(X_1)$, $R_2(X_2), \cdots, R_T(X_T)$. Let a set of weighting vectors $\mathcal{W}^T := \{(w_1, w_2, \cdots, w_T) \mid w_t \geq 0\,(t = 1, 2, \cdots, T) \text{ and } \sum_{t=1}^T w_t = 1\}$. The following proposition can be checked easily.

Proposition 2.1. *Let* R_t *be a risk measure on* $\mathcal{X}$ *at time* $t = 1, 2, \cdots, T$. *The following (i) – (iii) hold.*

(i) The weighted average: *Let a weighting vector* $(w_1, w_2, \cdots, w_T) \in \mathcal{W}^T$. *Define a map* $\boldsymbol{R} : \mathcal{X}^T \mapsto \mathbb{R}$ *by*

$$\boldsymbol{R}(\boldsymbol{X}) := \sum_{t=1}^{T} w_t R_t(X_t) \tag{2.3}$$

for $\boldsymbol{X} = (X_1, X_2, \cdots, X_T) \in \mathcal{X}^T$. *Then* $\boldsymbol{R}$ *is a risk measure on* $\mathcal{X}^T$.

(ii) The order weighted average (Torra [8]): *Let* $(w_1, w_2, \cdots, w_T) \in \mathcal{W}^T$ *be a weighting vector satisfying* $w_1 \geq w_2 \geq \cdots \geq w_T \geq 0$. *Define a map* $\boldsymbol{R} : \mathcal{X}^T \mapsto \mathbb{R}$ *by*

$$\boldsymbol{R}(\boldsymbol{X}) := \sum_{t=1}^{T} w_t R_{(t)}(X_{(t)}) \tag{2.4}$$

for $\boldsymbol{X} = (X_1, X_2, \cdots, X_T) \in \mathcal{X}^T$, *where* $R_{(t)}(X_{(t)})$ *is the t-th largest risk values in* $\{R_1(X_1), R_2(X_2), \cdots, R_T(X_T)\}$. *Then* $\boldsymbol{R}$ *is a risk measure on* $\mathcal{X}^T$.

(iii) The maximum: *Define a map* $\boldsymbol{R} : \mathcal{X}^T \mapsto \mathbb{R}$ *by*

$$\boldsymbol{R}(\boldsymbol{X}) := \max\{R_1(X_1), R_2(X_2), \cdots, R_T(X_T)\} \tag{2.5}$$

for $\boldsymbol{X} = (X_1, X_2, \cdots, X_T) \in \mathcal{X}^T$. *Then* $\boldsymbol{R}$ *is a risk measure on* $\mathcal{X}^T$.

When we construct aggregation $\boldsymbol{R}$ directly from of risk indexes $R_1(X_1)$, $R_2(X_2), \cdots, R_T(X_T)$, it is difficult to find other methods except for the methods (i) – (iii) in Proposition 2.1 from the view point of aggregation operators ([1] and [9, Section 4.1]).

Example 2.2 (Average value-at-risks). By Proposition 2.1, the following (2.6) – (2.8) are risk measures induced from Example 2.1:

$$\boldsymbol{R}(\boldsymbol{X}) = \sum_{t=1}^{T} w_t(-\mathrm{AVaR}_{p_t}(X_t)) = -\sum_{t=1}^{T} w_t \mathrm{AVaR}_{p_t}(X_t), \tag{2.6}$$

$$\boldsymbol{R}(\boldsymbol{X}) = \sum_{t=1}^{T} w_t(-\mathrm{AVaR}_{p_{(t)}}(X_{(t)})) = -\sum_{t=1}^{T} w_t \mathrm{AVaR}_{p_{(t)}}(X_{(t)}), \tag{2.7}$$

$$\boldsymbol{R}(\boldsymbol{X}) = \max\{-\mathrm{AVaR}_{p_1}(X_1), -\mathrm{AVaR}_{p_2}(X_2), \cdots, -\mathrm{AVaR}_{p_T}(X_T)\} \tag{2.8}$$

for random variables $\boldsymbol{X} = (X_1, X_2, \cdots, X_T) \in \mathcal{X}^T$, where p_t $(0 < p_t < 1)$ is a given risk-level probability at time $t = 1, 2, \cdots, T$ and $-\mathrm{AVaR}_{p_{(t)}}(X_{(t)})$ is the t-th largest risk values in $\{-\mathrm{AVaR}_{p_1}(X_1), -\mathrm{AVaR}_{p_2}(X_2), \cdots, -\mathrm{AVaR}_{p_T}(X_T)\}$.

Let n be a positive integer. When we aggregate n risk indexes for a random variable X, we can use the following corollary derived from Proposition 2.1.

Corollary 2.1. *Let* R_i *be a risk measure on* $\mathcal{X}$ *for item* $i = 1, 2, \cdots, n$. *The following (i) – (iii) hold.*

(i) The weighted average: *Let a weighting vector* $(w_1, w_2, \cdots, w_n) \in \mathcal{W}^n$. *Define a map* $R : \mathcal{X} \mapsto \mathbb{R}$ *by*

$$R(X) := \sum_{i=1}^{n} w_i R_i(X) \tag{2.9}$$

for $X \in \mathcal{X}$. *Then* R *is a risk measure on* $\mathcal{X}$.

(ii) The order weighted average: *Let* $(w_1, w_2, \cdots, w_n) \in \mathcal{W}^n$ *be a weighting vector satisfying* $w_1 \geq w_2 \geq \cdots \geq w_n \geq 0$. *Define a map* $R : \mathcal{X} \mapsto \mathbb{R}$ *by*

$$R(X) := \sum_{i=1}^{n} w_i R_{(i)}(X) \tag{2.10}$$

for $X \in \mathcal{X}$, *where* $R_{(i)}(X)$ *is the* i*-th largest risk values in* $\{R_1(X), R_2(X), \cdots, R_n(X)\}$. *Then* R *is a risk measure on* $\mathcal{X}$.

(iii) The maximum: *Define a map* $R : \mathcal{X} \mapsto \mathbb{R}$ *by*

$$R(X) := \max\{R_1(X), R_2(X), \cdots, R_n(X)\} \tag{2.11}$$

for $X \in \mathcal{X}$. *Then* R *is a risk measure on* $\mathcal{X}$.

In the next section we discuss relations between risk measures and deviations to introduce other kinds of aggregation of risk measures.

3 Deviation Measures

Risk measure is related to deviation measures ([6]). In this section we introduce deviation measures to investigate indirect approaches which are different from direct methods in the previous section. Denote $L^2(\Omega)$ and $L^1(\Omega)$ the space of square integrable real random variables on Ω and the space of integrable real random variables on Ω respectively. We use a notation $a_- := \max\{-a, 0\}$ for real numbers a.

Example 3.1 (Classical deviations). The following criteria are classical deviations in financial management, engineering and so on.

(i) Let the space $\mathcal{X} = L^2(\Omega)$. The *standard deviation* of a random variable $X (\in \mathcal{X})$ is defined by $\sigma(X) := E((X - E(X))^2)^{1/2}$.

(ii) Let the space $\mathcal{X} = L^1(\Omega)$. The *absolute deviation* of a random variable $X (\in \mathcal{X})$ is defined by $W(X) := E(|X - E(X)|)$.

(iii) Let the space $\mathcal{X} = L^2(\Omega)$. The *lower standard semi-deviation* of a random variable $X (\in \mathcal{X})$ is defined by $\sigma_-(X) := E(((X - E(X))_-)^2)^{1/2}$.

(iv) Let the space $\mathcal{X} = L^1(\Omega)$. The *lower absolute semi-deviation* of a random variable $X (\in \mathcal{X})$ is defined by $W_-(X) := E((X - E(X))_-)$.

Recently Rockafellar et al. [6] has studied the following concept regarding deviations.

Definition 3.1. Let $\mathcal{X}$ be a set of real random variables on Ω. A map $D : \mathcal{X} \mapsto [0, \infty)$ is called a *deviation measure* on $\mathcal{X}$ if it satisfies the following conditions (D.a) – (D.e):

(D.a) $D(X) \geq 0$ and $D(\theta) = 0$ for $X \in \mathcal{X}$ and real numbers θ. (*positivity*)

(D.b) $D(X+\theta) = D(X)$ for $X \in \mathcal{X}$ and real number θ. (*translation invariance*)
(D.c) $D(\lambda X) = \lambda\, D(X)$ for $X \in \mathcal{X}$ and nonnegative real numbers λ. (*positive homogeneity*)
(D.d) $D(X+Y) \le D(X) + D(Y)$ for $X, Y \in \mathcal{X}$. (*sub-additivity*)
(D.e) $\lim_{k\to\infty} D(X_k) = D(X)$ for $\{X_k\} \subset \mathcal{X}$ and $X \in \mathcal{X}$ such that $\lim_{k\to\infty} X_k = X$ almost surely. (*continuity*)

Hence, we have the following lemma for Example 3.1.

Lemma 3.1. *The standard deviation σ, the absolute deviation W, the lower standard semi-deviation σ_- and the lower absolute semi-deviation W_- are deviation measures in the sense of Definition 3.1.*

Proof. We have $|a+b| \le |a| + |b|$ and $(a+b)_- \le a_- + b_-$ for $a, b \in \mathbb{R}$. We can easily check this lemma with these inequalities and Schwartz's inequality. □

For a deviation measure D, we put

$$N(X) := \frac{D(X) + D(-X)}{2} \tag{3.1}$$

for $X \in \mathcal{X}$. Then N is a semi-norm on $\mathcal{X}$, i.e, it satisfies the following conditions (N.a) – (N.c):

(N.a) $N(X) \ge 0$ and $N(0) = 0$ for $X \in \mathcal{X}$. (*positivity*)
(N.b) $N(\lambda X) = |\lambda|\, N(X)$ for $X \in \mathcal{X}$ and real numbers λ. (*homogeneity*)
(N.c) $N(X+Y) \le N(X) + N(Y)$ for $X, Y \in \mathcal{X}$. (*sub-additivity*)

We find from (3.1) that we can aggregate deviation measures in a similar way to norms on the space $\mathcal{X}$. Let $\mathcal{D}(\mathcal{X})$ denote the family of all deviation measures on $\mathcal{X}$. Then the following proposition shows $\mathcal{D}(\mathcal{X})$ becomes a *convex cone*, and it indicates a hint to construct a deviation criterion D of a random variable X from deviations $D_1(X)$ and $D_2(X)$ estimated by two viewpoints $D_1(\cdot)$ and $D_2(\cdot)$.

Proposition 3.1

(i) *Let $D \in \mathcal{D}(\mathcal{X})$ and a nonnegative real number λ. Then $\lambda D \in \mathcal{D}(\mathcal{X})$.*
(ii) *Let $D_1, D_2 \in \mathcal{D}(\mathcal{X})$. Then $D_1 + D_2 \in \mathcal{D}(\mathcal{X})$.*

The sum, the scalar multiplication and the shift on the vector space $\mathcal{X}^T$ are defined as follows: We put $\boldsymbol{X} + \boldsymbol{Y} = (X_1 + Y_1, X_2 + Y_2, \cdots, X_T + Y_T), \lambda\boldsymbol{X} = (\lambda X_1, \lambda X_2, \cdots, \lambda X_T)$ and $\boldsymbol{X} + \boldsymbol{\theta} = (X_1 + \theta, X_2 + \theta, \cdots, X_T + \theta)$ for $\boldsymbol{X} = (X_1, X_2, \cdots, X_T) \in \mathcal{X}^T, \boldsymbol{Y} = (Y_1, Y_2, \cdots, Y_T) \in \mathcal{X}^T$ and real numbers λ and real vectors $\boldsymbol{\theta} = (\theta, \theta, \cdots, \theta) \in \mathbb{R}^T$. We introduce the following definition for random vectors from Definition 3.1.

Definition 3.2. A map $\boldsymbol{D} : \mathcal{X}^T \mapsto \mathbb{R}$ is called a *deviation measure* on $\mathcal{X}^T$ if it satisfies the following conditions (D.a) – (D.e):

(D.a) $\boldsymbol{D}(\boldsymbol{X}) \geq 0$ and $\boldsymbol{D}(\boldsymbol{\theta}) = 0$ for $\boldsymbol{X} \in \mathcal{X}^T$ and real vectors $\boldsymbol{\theta} = (\theta, \theta, \cdots, \theta) \in \mathbb{R}^T$. (*positivity*)

(D.b) $\boldsymbol{D}(\boldsymbol{X} + \boldsymbol{\theta}) = \boldsymbol{D}(\boldsymbol{X})$ for $\boldsymbol{X} \in \mathcal{X}^T$ and real vectors $\boldsymbol{\theta} = (\theta, \theta, \cdots, \theta) \in \mathbb{R}^T$. (*translation invariance*)

(D.c) $\boldsymbol{D}(\lambda \boldsymbol{X}) = \lambda \boldsymbol{D}(\boldsymbol{X})$ for $\boldsymbol{X} \in \mathcal{X}^T$ and nonnegative real numbers λ. (*positive homogeneity*)

(D.d) $\boldsymbol{D}(\boldsymbol{X} + \boldsymbol{Y}) \leq \boldsymbol{D}(\boldsymbol{X}) + \boldsymbol{D}(\boldsymbol{Y})$ for $\boldsymbol{X}, \boldsymbol{Y} \in \mathcal{X}^T$. (*sub-additivity*)

(D.e) $\lim_{k\to\infty} \boldsymbol{D}(\boldsymbol{X}_k) = \boldsymbol{D}(\boldsymbol{X})$ for $\{\boldsymbol{X}_k\} \subset \mathcal{X}^T$ and $\boldsymbol{X} \in \mathcal{X}^T$ such that $\lim_{k\to\infty} \boldsymbol{X}_k = \boldsymbol{X}$ almost surely. (*continuity*)

The following proposition shows methods to construct a deviation $\boldsymbol{D}$ on $\mathcal{X}^T$ from deviations $D_1(X_1), D_2(X_2), \cdots, D_T(X_T)$ for a random vector $\boldsymbol{X} = (X_1, X_2, \cdots, X_T) \in \mathcal{X}^T$.

Theorem 3.1. *Let D_t be a deviation measure on $\mathcal{X}$ at time $t = 1, 2, \cdots, T$. Let d be a real number satisfying $1 \leq d < \infty$. The following (i) – (iii) hold.*

(i) The generalized weighted average: *Let a weighting vector $(w_1, w_2, \cdots, w_T) \in \mathcal{W}^T$. Define a map $\boldsymbol{D} : \mathcal{X}^T \mapsto [0, \infty)$ by*

$$\boldsymbol{D}(\boldsymbol{X}) := \left(\sum_{t=1}^{T} w_t D_t(X_t)^d \right)^{1/d} \tag{3.2}$$

for $\boldsymbol{X} = (X_1, X_2, \cdots, X_T) \in \mathcal{X}^T$. Then $\boldsymbol{D}$ is a deviation measure on $\mathcal{X}^T$.

(ii) The generalized order weighted average: *Let $(w_1, w_2, \cdots, w_T) \in \mathcal{W}^T$ be a weighting vector satisfying $w_1 \geq w_2 \geq \cdots \geq w_T \geq 0$. Define a map $\boldsymbol{D} : \mathcal{X}^T \mapsto [0, \infty)$ by*

$$\boldsymbol{D}(\boldsymbol{X}) := \left(\sum_{t=1}^{T} w_t D_{(t)}(X_{(t)})^d \right)^{1/d} \tag{3.3}$$

for $\boldsymbol{X} = (X_1, X_2, \cdots, X_T) \in \mathcal{X}^T$, where $D_{(t)}(X_{(t)})$ is the t-th largest deviation values in $\{D_1(X_1), D_2(X_2), \cdots, D_T(X_T)\}$. Then $\boldsymbol{D}$ is a deviation measure on $\mathcal{X}^T$.

(iii) The maximum: *Define a map $\boldsymbol{D} : \mathcal{X}^T \mapsto [0, \infty)$ by*

$$\boldsymbol{D}(\boldsymbol{X}) := \max\{D_1(X_1), D_2(X_2), \cdots, D_T(X_T)\} \tag{3.4}$$

for $\boldsymbol{X} = (X_1, X_2, \cdots, X_T) \in \mathcal{X}^T$. Then $\boldsymbol{D}$ is a deviation measure on $\mathcal{X}^T$.

Proof. (i) We can check this proposition easily with Minkowski's inequality. (ii) Let (t) denote indexes for the t-th largest deviation values in $\{D_1(X_1 + Y_1), D_2(X_2 + Y_2), \cdots, D_T(X_T + Y_T)\}$. By Minkowski's inequality, we get

$$
\begin{aligned}
\boldsymbol{D}(\boldsymbol{X}+\boldsymbol{Y}) &\leq \left(\sum_{t=1}^{T} w_t \left(D_{(t)}(X_{(t)}) + D_{(t)}(Y_{(t)})\right)^d\right)^{1/d} \\
&\leq \left(\sum_{t=1}^{T} w_t D_{(t)}(X_{(t)})^d\right)^{1/d} + \left(\sum_{t=1}^{T} w_t D_{(t)}(Y_{(t)})^d\right)^{1/d} \\
&\leq \boldsymbol{D}(\boldsymbol{X}) + \boldsymbol{D}(\boldsymbol{Y}).
\end{aligned}
$$

We can easily check the other conditions. □

Let n be a positive integer. When we aggregate n deviation indexes for a random variable X, we can use the following corollary derived from Proposition 3.1.

Corollary 3.1. *Let D_i be a deviation measure on $\mathcal{X}$ for item $i = 1, 2, \cdots, n$. Let d be a real number satisfying $1 \leq d < \infty$. The following (i) – (iii) hold.*

(i) The generalized weighted average: *Let a weighting vector $(w_1, w_2, \cdots, w_n) \in \mathcal{W}^n$. Define a map $D : \mathcal{X} \mapsto [0, \infty)$ by*

$$
D(X) := \left(\sum_{i=1}^{n} w_i D_i(X)^d\right)^{1/d} \tag{3.5}
$$

for $X \in \mathcal{X}$. Then D is a deviation measure on $\mathcal{X}$.

(ii) The generalized order weighted average: *Let $(w_1, w_2, \cdots, w_n) \in \mathcal{W}^n$ be a weighting vector satisfying $w_1 \geq w_2 \geq \cdots \geq w_n \geq 0$. Define a map $D : \mathcal{X} \mapsto [0, \infty)$ by*

$$
D(X) := \left(\sum_{i=1}^{n} w_i D_{(i)}(X)^d\right)^{1/d} \tag{3.6}
$$

for $X \in \mathcal{X}$, where $D_{(i)}(X)$ is the i-th largest deviation values in $\{D_1(X), D_2(X), \cdots, D_n(X)\}$. Then D is a deviation measure on $\mathcal{X}$.

(iii) The maximum: *Define a map $D : \mathcal{X} \mapsto [0, \infty)$ by*

$$
D(X) := \max\{D_1(X), D_2(X), \cdots, D_n(X)\} \tag{3.7}
$$

for $X \in \mathcal{X}$. Then D is a deviation measure on $\mathcal{X}$.

4 Construction of Risk Measures by Use of Deviation Measures

In this section we construct coherent risk measures for random vectors by use of deviation measures. Now we introduce the following definition for random vectors.

Definition 4.1. A map $\boldsymbol{E} : \mathcal{X}^T \mapsto \mathbb{R}$ is called an *expectation measure* on $\mathcal{X}^T$ if it satisfies the following conditions (E.a) – (E.d):

(E.a) $\boldsymbol{E}(\boldsymbol{\theta}) = \theta$ for real vectors $\boldsymbol{\theta} = (\theta, \theta, \cdots, \theta) \in \mathbb{R}^T$.
(E.b) $\boldsymbol{E}(\lambda \boldsymbol{X}) = \lambda \boldsymbol{E}(\boldsymbol{X})$ for $\boldsymbol{X} \in \mathcal{X}^T$ and real numbers λ. (*homogeneity*)
(E.c) $\boldsymbol{E}(\boldsymbol{X} + \boldsymbol{Y}) = \boldsymbol{E}(\boldsymbol{X}) + \boldsymbol{E}(\boldsymbol{Y})$ for $\boldsymbol{X}, \boldsymbol{Y} \in \mathcal{X}^T$. (*additivity*)
(E.d) $\lim_{k\to\infty} \boldsymbol{E}(\boldsymbol{X}_k) = \boldsymbol{E}(\boldsymbol{X})$ for $\{\boldsymbol{X}_k\} \subset \mathcal{X}^T$ and $\boldsymbol{X} \in \mathcal{X}^T$ such that $\lim_{k\to\infty} \boldsymbol{X}_k = \boldsymbol{X}$ almost surely. (*continuity*)

The following lemma shows the relation between deviation measures D on $\mathcal{X}$ and risk measures R on $\mathcal{X}$.

Lemma 4.1

(i) *Let D be a deviation measure on $\mathcal{X}$. Suppose*

$$D(X) \le E(X) - \operatorname*{ess\,inf}_{\omega} X(\omega) \quad \text{for } X \in \mathcal{X}. \tag{4.1}$$

Define

$$R(X) := D(X) - E(X)$$

for $X \in \mathcal{X}$. Then R is a risk measure on $\mathcal{X}$.
(ii) *Let R be a risk measure on $\mathcal{X}$. Suppose*

$$R(X) + E(X) \ge 0 \quad \text{for } X \in \mathcal{X}. \tag{4.2}$$

Define

$$D(X) := R(X) + E(X)$$

for $X \in \mathcal{X}$. Then D is a deviation measure on $\mathcal{X}$.

Proof. (i) From (D.b) – (D.d), we can easily check (R.b) – (R.d). (R.a) Let $X, Y \in \mathcal{X}$ satisfying $X \ge Y$. Let $Z := X - Y \ge 0$. Then from the assumption, we have $D(Z) \le D(Z) + \operatorname{ess\,inf}_\omega Z(\omega) \le E(Z)$. Then $R(Z) \le 0$. Then from (R.d) we obtain $R(X) = R(Y + Z) \le R(Y) + R(Z) \le R(Y)$. Thus we also get (R.a).

(ii) From (R.b) – (R.d), we can easily check (D.b) – (D.d). (D.a) Let $X \in \mathcal{X}$. From the assumption we have $D(X) = R(X) + E(X) \ge 0$. Let θ be a real number. From (R.c) we have $R(0) = 0$ and from (R.b) we also have $R(\theta) = R(0 + \theta) = R(0) - \theta = -\theta$. Therefore we obtain $D(\theta) = R(\theta) + E(\theta) = -\theta + \theta = 0$. Thus this lemma holds. □

Remark. The lower standard semi-deviation σ_- and the lower absolute semi-deviation W_- satisfy the condition (4.1) in Lemma 4.1(i). On the other hand, $-\mathrm{AVaR}_p$ is a risk measure which satisfies the condition (4.2) in Lemma 4.1(i) if $\lim_{x \downarrow \inf I} x F_X(x) = 0$.

Extending Lemma 4.1, the following lemma shows the relation between deviation measures $\boldsymbol{D}$ on $\mathcal{X}^T$ and risk measures $\boldsymbol{R}$ on $\mathcal{X}^T$.

Lemma 4.2

(i) *Let* $\boldsymbol{D}$ *be a deviation measure on* $\mathcal{X}^T$*. Suppose*

$$\boldsymbol{D}(\boldsymbol{X}) \le \boldsymbol{E}(\boldsymbol{X}) - \operatorname*{ess\,inf}_{\omega} \min_{1\le t\le T} X_t(\omega) \quad \text{for } \boldsymbol{X} \in \mathcal{X}^T. \tag{4.3}$$

Define

$$\boldsymbol{R}(\boldsymbol{X}) := \boldsymbol{D}(\boldsymbol{X}) - \boldsymbol{E}(\boldsymbol{X})$$

for $\boldsymbol{X} \in \mathcal{X}^T$*. Then* $\boldsymbol{R}$ *is a risk measure on* $\mathcal{X}^T$*.*

(ii) *Let* $\boldsymbol{R}$ *be a risk measure on* $\mathcal{X}^T$*. Suppose*

$$\boldsymbol{R}(\boldsymbol{X}) + \boldsymbol{E}(\boldsymbol{X}) \ge 0 \quad \text{for } \boldsymbol{X} \in \mathcal{X}^T. \tag{4.4}$$

Define

$$\boldsymbol{D}(\boldsymbol{X}) := \boldsymbol{R}(\boldsymbol{X}) + \boldsymbol{E}(\boldsymbol{X})$$

for $\boldsymbol{X} \in \mathcal{X}^T$*. Then* $\boldsymbol{D}$ *be a deviation measure on* $\mathcal{X}^T$*.*

Proof. The proof is in the same way as Lemma 4.1. □

From this lemma, we can derive indirect construction methods for risk measures for stochastic sequences.

Theorem 4.1. *Let a weighting vector* $(v_1, v_2, \cdots, v_T) \in \mathcal{W}^T$ *and let an expectation measure*

$$\boldsymbol{E}(\boldsymbol{X}) = \sum_{t=1}^{T} v_t E(X_t)$$

for $\boldsymbol{X} = (X_1, X_2, \cdots, X_T) \in \mathcal{X}^T$*, Assume* $R_t(X_t) + \boldsymbol{E}(\boldsymbol{X}) \ge 0$ *for* $\boldsymbol{X} = (X_1, X_2, \cdots, X_T) \in \mathcal{X}^T$ *and* $t = 1, 2, \cdots, T$*. Let* d *be a real number satisfying* $1 \le d < \infty$*. The following (i) and (ii) hold.*

(i) The weighted average: *Let a weighting vector* $(w_1, w_2, \cdots, w_T) \in \mathcal{W}^T$*. Define a map* $\boldsymbol{R} : \mathcal{X}^T \mapsto \mathbb{R}$ *by*

$$\boldsymbol{R}(\boldsymbol{X}) := \left(\sum_{t=1}^{T} w_t (R_t(X_t) + \boldsymbol{E}(\boldsymbol{X}))^d \right)^{1/d} - \boldsymbol{E}(\boldsymbol{X}) \tag{4.5}$$

for $\boldsymbol{X} = (X_1, X_2, \cdots, X_T) \in \mathcal{X}^T$*. Then* $\boldsymbol{R}$ *is a risk measure on* $\mathcal{X}^T$*.*

(ii) The order weighted average: *Let* $(w_1, w_2, \cdots, w_T) \in \mathcal{W}^T$ *be a weighting vector satisfying* $w_1 \ge w_2 \ge \cdots \ge w_T \ge 0$*. Define a map* $\boldsymbol{R} : \mathcal{X}^T \mapsto \mathbb{R}$ *by*

$$\boldsymbol{R}(\boldsymbol{X}) := \left(\sum_{t=1}^{T} w_t (R_{(t)}(X_{(t)}) + \boldsymbol{E}(\boldsymbol{X}))^d \right)^{1/d} - \boldsymbol{E}(\boldsymbol{X}) \tag{4.6}$$

for $\boldsymbol{X} = (X_1, X_2, \cdots, X_T) \in \mathcal{X}^T$*, where* $R_{(t)}(X_{(t)}) + E(X_{(t)})$ *is the* t*-th largest risk values in* $\{R_1(X_1) + E(X_1), R_2(X_2) + E(X_2), \cdots, R_T(X_T) + E(X_T)\}$*. Then* $\boldsymbol{R}$ *is a risk measure on* $\mathcal{X}^T$*.*

Proof. (i) First we have $R_t(X_t) + \boldsymbol{E}(\boldsymbol{X}) \geq 0$ for $\boldsymbol{X} = (X_1, X_2, \cdots, X_T) \in \mathcal{X}^T$. Let

$$\boldsymbol{D}(\boldsymbol{X}) = \left(\sum_{t=1}^{T} w_t (R_t(X_t) + \boldsymbol{E}(\boldsymbol{X}))^d \right)^{1/d} \tag{4.7}$$

for $\boldsymbol{X} = (X_1, X_2, \cdots, X_T) \in \mathcal{X}^T$. We can easily check $\boldsymbol{D}$ satisfies (D.a) – (D.c) in Definition 3.2 since $R_t(\theta) = -\theta$, $\boldsymbol{E}(\boldsymbol{\theta}) = \theta$ and $R_t(X_t + \theta) = R_t(X_t) - \theta$ for $\boldsymbol{X} \in \mathcal{X}^T$ and real vectors $\boldsymbol{\theta} = (\theta, \theta, \cdots, \theta) \in \mathbb{R}^T$. Then by Minkowski's inequality we obtain that $\boldsymbol{D}$ is a deviation measure on $\mathcal{X}^T$.

Next we fix any random vector $\boldsymbol{X} = (X_1, X_2, \cdots, X_T) \in \mathcal{X}^T$. Put a constant $c = \operatorname{ess\,inf}_\omega \min_t X_t(\omega)$. Then we have $X_t - c \geq 0$ for $t = 1, 2, \cdots, T$. Since R_t is a risk measure, from (R.a) – (R.c) in Definition 2.1 we get $R_t(X_t) + c = R_t(X_t - c) \leq R_t(0) = 0$ for $t = 1, 2, \cdots, T$. Thus it holds that $R_t(X_t) \leq -c$ for $t = 1, 2, \cdots, T$ and $X_t \in \mathcal{X}$. Hence we have

$$\begin{aligned} \boldsymbol{D}(\boldsymbol{X}) - \boldsymbol{E}(\boldsymbol{X}) &= \left(\sum_{t=1}^{T} w_t (R_t(X_t) + \boldsymbol{E}(\boldsymbol{X}))^d \right)^{1/d} - \boldsymbol{E}(\boldsymbol{X}) \\ &\leq \left(\sum_{t=1}^{T} w_t (-c + \boldsymbol{E}(\boldsymbol{X}))^d \right)^{1/d} - \boldsymbol{E}(\boldsymbol{X}) \\ &= -c = -\operatorname*{ess\,inf}_{\omega} \min_{t} X_t(\omega). \end{aligned}$$

Thus by Lemma 4.1(i) we obtain that $\boldsymbol{R} = \boldsymbol{D} - \boldsymbol{E}$ is a risk measure on $\mathcal{X}^T$. We can check (ii) in the same way. □

Let n be a positive integer. When we have n risk indexes for a random variable X, we can apply Theorem 4.1 to aggregation of these risk indexes.

Corollary 4.1. *Let R_i be a risk measure on $\mathcal{X}$ satisfying $R_i(\cdot) + E(\cdot) \geq 0$ on $\mathcal{X}$ for item $i = 1, 2, \cdots, n$. Let d be a real number satisfying $1 \leq d < \infty$. The following (i) and (ii) hold.*

(i) The weighted average: *Let a weighting vector $(w_1, w_2, \cdots, w_n) \in \mathcal{W}^n$. Define a map $R : \mathcal{X} \mapsto \mathbb{R}$ by*

$$R(X) := \left(\sum_{i=1}^{n} w_i (R_i(X) + E(X))^d \right)^{1/d} - E(X) \tag{4.8}$$

for $X \in \mathcal{X}$. Then R is a risk measure on $\mathcal{X}$.

(ii) The order weighted average: *Let $(w_1, w_2, \cdots, w_n) \in \mathcal{W}^n$ be a weighting vector satisfying $w_1 \geq w_2 \geq \cdots \geq w_n \geq 0$. Define a map $R : \mathcal{X} \mapsto \mathbb{R}$ by*

$$R(X) := \left(\sum_{i=1}^{n} w_i (R_{(i)}(X) + E(X))^d \right)^{1/d} - E(X) \tag{4.9}$$

for $X \in \mathcal{X}$, where $R_{(i)}(X)$ is the i-th largest risk values in $\{R_1(X), R_2(X), \cdots, R_n(X)\}$. Then R is a risk measure on $\mathcal{X}$.

We obtain the following example since $\mathrm{AVaR}_p(\cdot) \le E(\cdot)$ holds on $\mathcal{X}$ for probabilities p $(0 < p < 1)$.

Example 4.1 (Dynamic average value-at-risks). Let d be a real number satisfying $1 \le d < \infty$. Let p_t $(0 < p_t < 1)$ is a risk-level probability at time $t = 1, 2, \cdots, T$. The following (i) and (ii) hold.

(i) The weighted average: Let a weighting vector $(w_1, w_2, \cdots, w_T) \in \mathcal{X}$. Define a map $\boldsymbol{R} : \mathcal{X}^T \mapsto \mathbb{R}$ by

$$\boldsymbol{R}(\boldsymbol{X}) := \left(\sum_{t=1}^{T} w_t (-\mathrm{AVaR}_{p_t}(X_t) + \boldsymbol{E}(\boldsymbol{X}))^d \right)^{1/d} - \boldsymbol{E}(\boldsymbol{X}) \tag{4.10}$$

for $\boldsymbol{X} = (X_1, X_2, \cdots, X_T) \in \mathcal{X}^T$. Then $\boldsymbol{R}$ is a risk measure on $\mathcal{X}^T$.

(ii) The order weighted average: Let $(w_1, w_2, \cdots, w_T) \in \mathcal{X}$ be a weighting vector satisfying $w_1 \ge w_2 \ge \cdots \ge w_T \ge 0$. Define a map $\boldsymbol{R} : \mathcal{X}^T \mapsto \mathbb{R}$ by

$$\boldsymbol{R}(\boldsymbol{X}) := \left(\sum_{t=1}^{T} w_t (-\mathrm{AVaR}_{p_{(t)}}(X_{(t)}) + \boldsymbol{E}(\boldsymbol{X}))^d \right)^{1/d} - \boldsymbol{E}(\boldsymbol{X}) \tag{4.11}$$

for $\boldsymbol{X} = (X_1, X_2, \cdots, X_T) \in \mathcal{X}^T$, where $-\mathrm{AVaR}_{p_t}(X_t) + E(X_t)$ is the t-th largest risk values in $\{-\mathrm{AVaR}_{p_1}(X_1) + E(X_1), -\mathrm{AVaR}_{p_2}(X_2) + E(X_2), \cdots, -\mathrm{AVaR}_{p_T}(X_T) + E(X_T)\}$. Then $\boldsymbol{R}$ is a risk measure on $\mathcal{X}^T$.

References

1. Aczél, J.: On weighted mean values. Bulletin of the American Math. Society 54, 392–400 (1948)
2. Artzner, P., Delbaen, F., Eber, J.-M., Heath, D.: Thinking coherently. Risk 10, 68–71 (1997)
3. Artzner, P., Delbaen, F., Eber, J.-M., Heath, D.: Coherent measures of risk. Mathematical Finance 9, 203–228 (1999)
4. Jorion, P.: Value-a- Risk: The New Benchmark for Managing Financial Risk, 3rd edn. McGraw-Hill, New York (2006)
5. Rockafellar, R.T., Uryasev, S.P.: Optimization of conditional value-at-risk. Journal of Risk 2, 21–42 (2000)
6. Rockafellar, R.T., Uryasev, S.P., Zabarankin, M.: Generalized deviations in risk analysis. J. Finance and Stochastics 10, 51–74 (2006)
7. Tasche, D.: Expected shortfall and beyond. Journal of Banking Finance 26, 1519–1533 (2002)
8. Torra, V.: The weighted OWA operator. Int. J. of Intel. Syst. 12, 153–166 (1997)
9. Torra, V., Narukawa, Y.: Modeling Decisions. Springer, Heidelberg (2007)
10. Yoshida, Y.: A dynamic risk allocation of value-at-risks with portfolioss. Journal of Advanced Computational Intelligence and Intelligent Informatics 16, 800–806 (2012)