Chapter 5

Two-Timing, Geometric, and Multi-scale Methods

(a) Elementary Two-Timing

The Brooklyn native Julian Cole (1925–1999) got his Ph.D. in aeronautics at Caltech in 1949 with Hans Liepmann (a German émigre of 1939) as his advisor. He remained on the Caltech faculty until 1968 where he maintained active contact with Kaplun, Lagerstrom and others in aeronautics, applied mathematics and industry, attempting to understand singular perturbations more deeply and to apply its growing methodology. Then he moved to UCLA and, ultimately, Rensselaer. He and his Jerusalem-born student Jerry Kevorkian, who spent his academic career at the University of Washington, developed and applied asymptotic methods involving two- (i.e. multi-) time or multiple scales in the early 1960s (cf. the obituary of Cole by Bluman et al. [48]). Related approaches were made by the Soviet Kuzmak [273], the Australian Mahony [305], and the American Cochran [89], among others, but Cole and Kevorkian had the dominant long-term impact. The previously cited work of Lomov is also recommended reading, as is the paper by Levey and Mahony [286]. The monograph *Perturbation Methods in Applied Mathematics*, Cole [92], considered singular perturbations in a broad applied math setting, where both the development of the underlying techniques and significant and diverse applications were included. The book approaches matching using intermediate limits and presumes a corresponding overlap of inner and outer domains. The examples used are generally very instructive and quite nontrivial.

Two-variable expansions are naturally introduced, for example, in the presence of a small disturbance acting cumulatively for a long time. The simultaneous occurrence of a small parameter and a long time interval implies

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that we, indeed, encounter a *two-parameter* singular perturbation *problem* (cf. O'Malley [366]). Smith [466] aptly called such initial value problems singular perturbations with a nonuniformity at infinity. The two-timing approach generalizes and extends the classical Poincaré–Lindstedt method of *strained coordinates* (cf., e.g., Poincaré [394], Minorsky [319], and Murdock [335]) which is often applied to solve Duffing's equation and to describe related nonlinear oscillations. The procedure ultimately provides the same asymptotic expansion for the solution as the method of averaging (cf. Bogoliubov and Mitropolsky [51]), which is long known to be mathematically justified. Greatly expanded editions of Cole's book, coauthored with Kevorkian, appeared in 1981 and 1996 [247, 248]. The late Peter Chapman, a colleague of Mahony in Perth, developed a promising manuscript [77] that may never have been finished. It explained the ongoing work of Mahony and the paper of Kuzmak. Mahony's work, more generally, is reviewed in Fowkes and Silberstein [153]. The major recent generalization from two-timing to multiscale modeling is considered in E [130].

As a first example, we shall describe the application of two-timing to the nearly linear *Rayleigh equation*

$$\ddot{y} + y = \epsilon\left(\dot{y} - \frac{1}{3}\dot{y}^3\right) \quad \text{on } t \geq 0 \tag{5.1}$$

with initial values

$$y(0) = 0 \quad \text{and} \quad \dot{y}(0) = 1. \tag{5.2}$$

This, presumably, describes the oscillations of a clarinet reed (cf. Rayleigh [409]). Note that if y satisfies the Rayleigh equation, $\dot{y}$ will satisfy the van der Pol equation (which we will later study).

Two-timing anticipates that the solution of (5.1–5.2) will evolve, depending on both the given fast-time t and the introduced *slow time*

$$\tau \equiv \epsilon t \tag{5.3}$$

using a formal *two-time* power series *expansion*

$$y(t,\epsilon) = Y(t,\tau,\epsilon) \sim Y_0(t,\tau) + \epsilon Y_1(t,\tau) + \epsilon^2 Y_2(t,\tau) + \dots \tag{5.4}$$

when $t = O(1/\epsilon)$, i.e. $\tau = O(1)$. (Don't be confused because we used $\tau = \frac{t}{\epsilon}$ as a fast time in Chap. 3.) Sophisticates will realize that (5.4) is a generalized asymptotic expansion, since its coefficients Y_k depend on ϵ through τ. The chain rule requires that

$$\dot{y} = Y_t + \epsilon Y_\tau \quad \text{and} \quad \ddot{y} = Y_{tt} + 2\epsilon Y_{t\tau} + \epsilon^2 Y_{\tau\tau},$$

so equation (5.1) implies that the two-time expansion (5.4) must satisfy the partial differential equation

$$Y_{tt} + Y + \epsilon\left(2Y_{t\tau} - Y_t + \frac{1}{3}Y_t^3\right) + \epsilon^2(Y_{\tau\tau} - Y_\tau + Y_t^2 Y_\tau) + \epsilon^3 Y_t Y_\tau^2 + \frac{\epsilon^4}{3}Y_\tau^3 = 0. \tag{5.5}$$

Converting the ODE (5.1) to the PDE (5.5) may not, at first, seem like a step forward, but wait and experience its success. Equating coefficients of successive powers of ϵ as a regular perturbation expansion requires

$$Y_{0tt} + Y_0 = 0, \tag{5.6}$$

$$Y_{1tt} + Y_1 = -2Y_{0t\tau} + Y_{0t} - \frac{1}{3}Y_{0t}^3, \tag{5.7}$$

etc. From (5.6), it follows that Y_0 must be a linear combination of $\cos t$ and $\sin t$ with τ-dependent coefficients, so we set

$$Y_0(t,\tau) = A_0(\tau)\cos t + B_0(\tau)\sin t, \tag{5.8}$$

where A_0 and B_0 so far remain undetermined, except for their initial values since

$$\begin{cases} y(0) = Y_0(0,0) = A_0(0) = 0 \\ \text{and} \\ \dot{y}(0) \sim Y_{0t}(0,0) = B_0(0) = 1. \end{cases} \tag{5.9}$$

Thus, the representation (5.8) introduces *amplitudes* A_0 and B_0 that are slowly varying functions of t.

Next, using the partial differential equation (5.7) for Y_1, we require

$$\begin{aligned} Y_{1tt} + Y_1 = &-2\left(-\frac{dA_0}{d\tau}\sin t + \frac{dB_0}{d\tau}\cos t\right) \\ &+ (-A_0\sin t + B_0\cos t) \\ &- \frac{1}{3}(-A_0\sin t + B_0\cos t)^3. \end{aligned} \tag{5.10}$$

Recalling the trigonometric identities $\sin^3 t = \frac{3}{4}\sin t - \frac{1}{4}\sin 3t$, $\sin^2 t\cos t = \frac{1}{4}\cos t - \frac{1}{4}\cos 3t$, $\sin t\cos^2 t = \frac{1}{4}\sin t + \frac{1}{4}\sin 3t$, and $\cos^3 t = \frac{3}{4}\cos t + \frac{1}{4}\cos 3t$, we find that $Y_{1tt} + Y_1$ is a linear combination of $\sin t$, $\cos t$, $\sin 3t$, and $\cos 3t$, with τ-dependent coefficients. The general solution for Y_1 follows simply by the method of undetermined coefficients. Multiples of $\sin t$ and $\cos t$ in the forcing term yield unbounded responses like $t\sin t$ and $t\cos t$ in the particular solution. Since $\sin t$ and $\cos t$ are solutions of the homogeneous equation, the presence of such terms in the forcing is said to *resonate* with the complementary solutions for Y_1. Such *secular* response terms can't be allowed if the expansion $Y(t,\tau,\epsilon)$ is to remain asymptotic for large t. Specifically, since (5.10) implies that

$$\begin{aligned} Y_{1tt} + Y_1 = &\left(2\frac{dA_0}{d\tau} - A_0 + \frac{A_0^3}{4} + \frac{A_0B_0^2}{4}\right)\sin t \\ &+ \left(-2\frac{dB_0}{d\tau} + B_0 - \frac{A_0^2B_0}{4} - \frac{B_0^3}{4}\right)\cos t \\ &+ \left(-\frac{A_0^3}{12} + \frac{A_0B_0^2}{4}\right)\sin 3t + \left(\frac{A_0^2B_0}{4} - \frac{B_0^3}{12}\right)\cos 3t, \end{aligned} \tag{5.11}$$

to make the first harmonics disappear in the forcing thereby requires A_0 and B_0 to satisfy the coupled vector initial value problem

$$\begin{cases} 2\frac{dA_0}{d\tau} = A_0 - \frac{A_0}{4}(A_0^2 + B_0^2), \quad A_0(0) = 0 \\ \text{and} \\ 2\frac{dB_0}{d\tau} = B_0 - \frac{B_0}{4}(A_0^2 + B_0^2), \quad B_0(0) = 1. \end{cases} \tag{5.12}$$

Uniqueness implies that

$$A_0(\tau) = 0 \tag{5.13}$$

while the explicit solution of the remaining Bernoulli equation determines

$$B_0(\tau) = \frac{2}{\sqrt{1 + 3e^{-\tau}}}, \tag{5.14}$$

which remains defined for all $\tau \geq 0$.

Thus, Y_1 must be a bounded solution of

$$Y_{1tt} + Y_1 = -\frac{B_0^3}{12}\cos 3t.$$

A particular solution as a slowly varying multiple of $\cos 3t$ follows using undetermined coefficients and since its complementary solution will be a linear combination of $\cos t$ and $\sin t$, Y_1 will have the form

$$Y_1(t, \tau) = A_1(\tau)\cos t + B_1(\tau)\sin t + \frac{B_0^3(\tau)}{96}\cos 3t.$$

Moreover, A_1 and B_1 must satisfy the initial conditions

$$0 = Y_1(0, 0) = A_1(0) + \frac{B_0^3(0)}{96} \quad \text{and} \quad 0 = Y_{1t}(0, 0) + Y_{0\tau}(0, 0) = B_1(0) + A_0(0),$$

so

$$A_1(0) = -\frac{1}{96} \qquad \text{and} \qquad B_1(0) = 0.$$

We will completely specify A_1 and B_1 as an exercise. Equations (5.13–5.14) determines the limiting two-time approximation

$$y(t, \epsilon) = Y_0(t, \tau) + O(\epsilon) \tag{5.15}$$

on $0 \leq \tau < \infty$ with

$$Y_0(t, \tau) = \frac{2\sin t}{\sqrt{1 + 3e^{-\tau}}}. \tag{5.16}$$

Proofs justifying the two-time technique by proving the estimate (5.15) are given, e.g., in Smith [466]. They can be expected to hold with some tradeoffs both on longer time intervals and to higher-order. Note that Poincaré in the preface to the first volume of *Celestial Mechanics* [394] reported

> All efforts of geometers in the second half of this century have had as main objective the elimination of secular terms.

Exercise

Knowing Y_0, determine Y_1 completely by eliminating resonant terms in the differential equation

$$Y_{2tt} + Y_2 + (2Y_{1t\tau} - Y_1 + Y_{0t}^3 Y_{1t}) + Y_{0\tau\tau} - Y_{0\tau} + Y_{0t}^2 Y_{0\tau} = 0$$

for Y_2 by appropriately determining the first harmonic coefficients $A_1(\tau)$ and $B_1(\tau)$ of Y_1.

Paul Germain [169] introduces *multiple scales* more broadly:

> when a physical phenomenon is thought to be represented by the occurrence of steep gradients in one variable only .... Assume that the manifold across which the gradients are steep is $F(x,t) =$ constant. Then, the mathematical progressive wave structure is
>
> $$U(t, x, \frac{F}{\epsilon}, \epsilon)$$
>
> with $\xi \equiv \frac{F}{\epsilon}$ considered as a fifth variable.

See Germain [169] and Zeytounian [533] for more details.

Historical Comment

Poincaré won the (Swedish and Norwegian) *King Oscar II Prize* in 1889 (celebrating the king's 60th birthday) for his work on the three-body problem. The hastily prepared submission was, however, in error (cf. Barrow-Green [29]). (Distributed copies were collected and trashed, but Barrow-Green recently found one remaining in the Mittag-Leffler Institute library. Mittag-Leffler was a judge (together with Hermite and Weierstrass), organizer of the prize, and founding editor of *Acta Mathematica*.) A corrected version of Poincaré's paper was published in *Acta Mathematica* in 1890 (at Poincaré's expense). Much of the difference relates to whether the formal solutions found by the astronomer Lindstedt in 1883 were convergent or simply asymptotic (and to the nonintegrability of the three-body problem). This distinction was further highlighted in the introduction to the second volume of Poincaré's *Celestial Mechanics* [394] where a difference between "astronomers" and "mathematicians" may be understood if we realize that astronomers traditionally call asymptotic series convergent. Many years later, KAM theory (the research of Kolmogorov, Arnold, and Moser from 1954 to 1963 on the persistence of quasi-periodic motions under small perturbations) shows that some similar results actually converge (cf. Arnold et al. [12]). Curiously, Szpiro [478] suggests that the Swedes Lindstedt and Gyldén each claimed some of their work had precedence over Poincaré's, but this is not noted by the more scholarly Barrow-Green. More details can be found in Charpentier et al. [78], Verhulst [502], and Gray [182].

According to Stubhaug [477], Gyldén, head of the Stockholm Observatory, characterized the entire prize as

humbug.

Ziegler [537] reports that Mittag-Leffler tried to get a Nobel prize for Poincaré from their initiation in 1901 until Poincaré's death. Stubhaug emphasizes that Mittag-Leffler wanted to promote theoretical physics for the prize as well as Poincaré. (It's never been clear why there's no mathematics Nobel, but there's now an *Abel prize* and other, even bigger, new ones.)

Now consider the initial value problem

$$\ddot{y} + y + \epsilon y^3 = 0, \quad y(0) = 1, \quad \dot{y}(0) = 0 \tag{5.17}$$

for *Duffing's equation* on $t \geq 0$ (Kovacic and Brennan [261] provides a brief biography of Georg Duffing (1861–1944) as well as a partial translation of his 1918 book *Forced Oscillations with Variable Natural Frequency and their Technical Significance*, originally published by Sammlung Vieweg, Braunschweig, in German.). Using a phase-plane analysis, for example, one can readily convince oneself that the solution is periodic (cf. Mudavanhu et al. [332]). A regular perturbation expansion produces artificial secular terms that are clearly spurious, so we might naturally instead seek an asymptotic solution

$$y(t, \epsilon) = z(s, \epsilon) \sim z_0(s) + \epsilon z_1(s) + \dots \tag{5.18}$$

as a function of a so-called *strained* coordinate

$$s = (1 + \epsilon\Omega(\epsilon))t \tag{5.19}$$

where the frequency perturbation

$$\Omega(\epsilon) \sim \Omega_0 + \epsilon\Omega_1 + \dots \tag{5.20}$$

for constants Ω_j is to be determined termwise to achieve periodicity of the terms of the expansion for z with respect to the strained s. Clearly, z will need to satisfy the initial value problem

$$(1 + \epsilon\Omega(\epsilon))^2 \frac{d^2 z}{ds^2} + z + \epsilon z^3 = 0, \quad z(0, \epsilon) = 1, \quad \frac{dz}{ds}(0, \epsilon) = 0. \tag{5.21}$$

Proceeding termwise, we obtain successive initial value problems

$$\frac{d^2 z_0}{ds^2} + z_0 = 0, \quad z_0(0) = 1, \quad \frac{dz_0}{ds}(0) = 0,$$

$$\frac{d^2 z_1}{ds^2} + z_1 + 2\Omega_0 \frac{d^2 z_0}{ds^2} + z_0^3 = 0, \quad z_1(0) = 0, \quad \frac{dz_1}{ds}(0) = 0,$$

etc. Since

$$z_0(s) = \cos s, \tag{5.22}$$

z_1 must satisfy

$$\frac{d^2 z_1}{ds^2} + z_1 = \left(2\Omega_0 - \frac{3}{4}\right)\cos s - \frac{1}{4}\cos 3s.$$

To avoid secular terms in z_1, we must pick

$$\Omega_0 = 3/8 \tag{5.23}$$

to obtain

$$z_1(s) = -\frac{1}{32}\cos s + \frac{1}{32}\cos 3s. \tag{5.24}$$

At the next stage, we find

$$s = \left(1 + \frac{3\epsilon}{8} - \frac{21}{256}\epsilon^2 + \dots\right) t \tag{5.25}$$

and

$$\begin{aligned} y(t,\epsilon) = z(s,\epsilon) = &\left(1 - \frac{\epsilon}{32} + \frac{23}{1024}\epsilon^2 + \dots\right)\cos s \\ &+ \epsilon\left(\frac{1}{32} - \frac{\epsilon}{64} + \dots\right)\cos 3s \\ &+ \epsilon^2\left(\frac{1}{1024} + \dots\right)\cos 5s + \dots . \end{aligned} \tag{5.26}$$

The series for the frequency actually converges. Andersen and Geer [7] calculated series for the corresponding periodic solution of the van der Pol equation to $O(\epsilon^{24})$ terms using Macsyma and to $O(\epsilon^{164})$ terms via Taylor series.

Note that use of the coordinate s (which is only ever approximated as a polynomial of increasing order in ϵ) corresponds to a *multitime* expansion using the times t, ϵt, $\epsilon^2 t, \dots$. (See below.) Later scales are not needed when we bound $\epsilon^k t$ for any fixed k.

More generally, nonlinear *clock functions*

$$\tau_i(t,\epsilon), \quad i = 0, 1, \dots, N$$

are sometimes used to determine multitime expansions

$$x(t,\epsilon) = X(\tau_0, \tau_1, \dots, \tau_N, \epsilon).$$

Ablowitz [1] points out that this *frequency-shift* method, that he calls the Stokes-Poincaré approach, is limited to equations in conservation form (see Sect. (c)). (It does not apply, e.g., to the van der Pol equation, though it will provide its periodic limit cycle.)

More generally, when one considers the nearly linear equation

$$\ddot{y} + y + \epsilon f(y, \dot{y}) = 0, \tag{5.27}$$

two-timing produces a first term approximation

$$Y_0(t, \tau) = A_0(\tau) \cos t + B_0(\tau) \sin t \tag{5.28}$$

for $\tau = \epsilon t$ and requires the second term to satisfy a resulting nonhomogeneous equation

$$\frac{\partial^2 Y_1}{\partial t^2} + Y_1 = P(t, \tau) \tag{5.29}$$

determined in terms of A_0 and B_0. The *Fredholm alternative* requires the two orthogonality conditions,

$$\int_0^{2\pi} P(t, \tau) \cos t \, dt = 0 \quad \text{and} \quad \int_0^{2\pi} P(t, \tau) \sin t \, dt = 0, \tag{5.30}$$

which coincide with the differential equations for A_0 and B_0 needed to eliminate secular terms in Y_1. An alternative representation

$$Y_0(t, \tau) = C_0(\tau) \cos(t + D_0(\tau))$$

with slowly varying amplitude C_0 and phase D_0 (instead of (5.28)) would also be effective. Kevorkian and Cole [249] use multiple scale methods for a variety of problems.

Exercise (cf. Hinch [206])

Show that Duffing's equation could also be solved by directly seeking a solution of the form

$$y(t, \epsilon) = z(s, \epsilon) \tag{5.31}$$

where s is determined by inverting a *near-identity transformation*

$$t = s + \epsilon t_1(s) + \epsilon^2 t_2(s) + \dots \tag{5.32}$$

for functions $t_j(s)$ that provide a periodic solution $z(s, \epsilon)$ termwise.

One can obtain Mathieu's equation

$$\frac{d^2 x}{dt^2} + (\delta + \epsilon \cos t) x = 0 \tag{5.33}$$

as a linearization of Duffing's equation (cf. Jordan and Smith [230]). Moreover, one can study what parameter values δ and ϵ provide bounded or unbounded solutions. Transitions of solutions $x(t, \epsilon)$ from stability to instability occur along curves $\delta(\epsilon)$ called *tongues* that can be obtained using perturbation methods (cf. Nayfeh and Mook [345]). Bifurcations may involve *hidden* time scales, like $\epsilon^{3/2}t$ (cf. Chen et al. [81] and Verhulst [504]). The traditional two-time expansion

$$Y(t, \tau, \epsilon) \sim Y_0(t, \tau) + \epsilon Y_1(t, \tau) + \dots \tag{5.34}$$

is effective for equation (5.27). We can expect such results to hold for bounded τ values, though Greenlee and Snow [184] showed that with appropriate damping, two-timing is valid on the whole half-line $t \geq 0$.

We can also apply multiple scales to equations with boundary layer behavior, even though Lagerstrom [276] sought a dichotomous distinction between *layer type* and *secular* problems. Consider, for example, the nonlinear two-point boundary value problem

$$\epsilon y'' + a(x)y' + g(x, y) = 0 \tag{5.35}$$

on $0 \leq x \leq 1$ where

$$a(x) > 0, \tag{5.36}$$

a and g are smooth, and bounded end values

$$y(0) \quad \text{and} \quad y(1) \tag{5.37}$$

are prescribed. We anticipate having an initial boundary layer due to the sign of a, so we introduce the *stretched* (fast) *variable*

$$\eta = \frac{1}{\epsilon} \int_0^x a(s)\, ds \tag{5.38}$$

(known to be appropriate for the corresponding linear equations) and seek an asymptotic solution to the two-point problem for (5.35) in the two-variable (or multiscale) form

$$y(x, \epsilon) = Y(x, \eta, \epsilon) \sim Y_0(x, \eta) + \epsilon Y_1(x, \eta) + \dots. \tag{5.39}$$

Since

$$y' = Y_x + \frac{a(x)}{\epsilon} Y_\eta$$

and

$$\epsilon y'' = \epsilon Y_{xx} + 2a(x)Y_{x\eta} + a'(x)Y_\eta + \frac{a^2(x)}{\epsilon} Y_{\eta\eta},$$

the ordinary differential equation (5.35) for y requires Y to satisfy the partial differential equation

$$\begin{aligned} \frac{a^2(x)}{\epsilon}(Y_{\eta\eta} + Y_\eta) + \big(2a(x)Y_{x\eta} + a'(x)Y_\eta \\ + a(x)Y_x + g(x, Y)\big) + \epsilon Y_{xx} = 0 \end{aligned} \tag{5.40}$$

as a regular perturbation series, with the coefficients Y_k in (5.39) depending on x and η simultaneously. Equating successive coefficients in (5.40) to zero then requires that

$$Y_{0\eta\eta} + Y_{0\eta} = 0, \tag{5.41}$$

$$Y_{1\eta\eta} + Y_{1\eta} + \frac{1}{a^2(x)}\left(2a(x)Y_{0x\eta} + a'(x)Y_{0\eta} + a(x)Y_{0x} + g(x, Y_0)\right) = 0, \tag{5.42}$$

etc. Equation (5.41) implies that Y_0 is a linear combination of 1 and $e^{-\eta}$, with undetermined coefficients depending on the slow variable x. Thus, we set

$$Y_0(x, \eta) = A_0(x) + B_0(x)e^{-\eta}. \tag{5.43}$$

The boundary values then require that

$$A_0(0) + B_0(0) = y(0) \quad \text{and} \quad A_0(1) \sim y(1) \tag{5.44}$$

since $e^{-\eta}$ is asymptotically negligible outside the initial layer. The equation (5.42) then requires that

$$\begin{aligned} Y_{1\eta\eta} + Y_{1\eta} + \frac{1}{a^2(x)} &\big(-2a(x)B_0'(x)e^{-\eta} - a'(x)B_0(x)e^{-\eta} \\ &+ a(x)\left(A_0' + B_0'e^{-\eta}\right) + g(x, A_0 + B_0e^{-\eta})\big) = 0. \end{aligned} \tag{5.45}$$

We expand

$$g(x, A_0 + B_0e^{-\eta}) = g(x, A_0) + g_y(x, A_0)B_0e^{-\eta} + \frac{1}{2}g_{yy}(x, A_0)(B_0^2e^{-2\eta}) + \dots$$

about $y = A_0$. To prevent secular terms in Y_1, we must make the coefficients of 1 and $e^{-\eta}$ in the forcing term of (5.45) be zero. Thus, we will need A_0 to satisfy the limiting nonlinear equation

$$a(x)A_0' + g(x, A_0) = 0$$

while B_0 must satisfy the coupled linear equation

$$-a(x)B_0' + (-a'(x) + g_y(x, A_0))B_0 = 0.$$

Using the terminal condition for A_0, we shall assume that a unique solution to the reduced problem

$$A_0' = -\frac{g(x, A_0)}{a(x)}, \quad A_0(1) = y(1) \tag{5.46}$$

is defined throughout $0 \le x \le 1$. We may have to obtain A_0 numerically. Then, $B_0(x)$ is uniquely determined from the linear problem

$$(aB_0)' = g_y(x, A_0)B_0, \quad B_0(0) = y(0) - A_0(0),$$

i.e.

$$B_0(x) = e^{\int_0^x \frac{g_y(s, A_0(s))}{a(s)} ds} \frac{a(0)}{a(x)} (y(0) - A_0(0)). \tag{5.47}$$

This completely specifies Y_0. Next, we will need to integrate

$$Y_{1\eta\eta} + Y_{1\eta} + \frac{1}{a^2(x)}(g(x, A_0 + B_0 e^{-\eta}) - g(x, A_0) - g_y(x, A_0) B_0 e^{-\eta}) = 0$$

using variation of parameters to determine its complementary solution by applying the initial conditions and then eliminating the resulting resonant terms in the differential equation for Y_2. Generalizations of these methods are found in O'Malley [362, 363], Smith [465], and elsewhere.

To illustrate how our earlier results on turning point problems fit the two-time ansatz, consider the two examples that follow.

Example 1

Recall that solutions of the two-point problem

$$\epsilon y'' + xy' = 0, \quad -1 \le x \le 1, \quad y(\pm 1) = \pm 1 \tag{5.48}$$

have the form

$$y(x, \epsilon) = A + B \int_0^x e^{-s^2/2\epsilon} ds$$

for constants $A(\epsilon)$ and $B(\epsilon)$. Applying the boundary conditions, we get $A = 0$ and $B = (\int_0^1 e^{-s^2/2\epsilon} ds)^{-1}$, so the asymptotic solution

$$y(x, \epsilon) \sim \frac{\int_0^{x/\sqrt{\epsilon}} e^{-t^2/2}\, dt}{\int_0^{\infty} e^{-t^2/2}\, dt} \tag{5.49}$$

is an odd function of the stretched variable

$$\xi = x/\sqrt{\epsilon}. \tag{5.50}$$

If we directly sought the solution as

$$y(x, \epsilon) = C(\xi), \tag{5.51}$$

C would satisfy the boundary value problem

$$\frac{d^2 C}{d\xi^2} + \xi \frac{dC}{d\xi} = 0, \quad C(\pm\infty) = \pm 1,$$

as found. The symmetric shock layer $C(\xi)$ clearly connects the outer solutions ∓ 1 on opposite sides of the turning point.

Example 2

Recall that the differential equation

$$\epsilon y'' + 2xy' - 2y = 0, \quad -1 \le x \le 1, \;\; y(-1) = -1, \;\; y(1) = 2 \tag{5.52}$$

has the exact solution

$$y(x, \epsilon) = x + x \frac{\int_{-1}^{x} e^{-s^2/\epsilon}\, ds}{\int_{-1}^{1} e^{-s^2/\epsilon}\, ds} + \frac{\epsilon e^{-x^2/\epsilon}}{2 \int_{-1}^{1} e^{-s^2/\epsilon}\, ds}$$

depending on the variables x and $\xi \equiv x/\sqrt{\epsilon}$. Indeed, it has the simple form

$$y(x, \epsilon) = Y(x, \xi, \sqrt{\epsilon}) \sim x + xC_0(\xi) + \epsilon C_2(\xi) \tag{5.53}$$

where $C_0(\xi) \equiv \frac{\int_{-\infty}^{\xi} e^{-t^2} dt}{\int_{-\infty}^{\infty} e^{-t^2} dt}$ and $C_2(\xi) \equiv \frac{e^{-\xi^2}}{2\int_{-\infty}^{\infty} e^{-t^2} dt}$. As expected,

$$Y(x, \xi, 0) \to \begin{cases} 2x & \text{as } \xi \to \infty \\ x & \text{as } \xi \to -\infty \end{cases}$$

while $C_2(\xi) \to 0 \quad \text{as} \quad \xi \to \pm\infty$.

Murdock [335] considered the so-called *harmonic resonance* problem of finding solutions of period $\frac{2\pi}{\omega(\epsilon)}$ for the equation

$$\ddot{y} + y = \epsilon f(y, \dot{y}, \omega(\epsilon)t) \tag{5.54}$$

where

$$\omega(\epsilon) \sim 1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \ldots \tag{5.55}$$

is specified. He lets the initial values take the form

$$\begin{cases} y(0) & = \alpha_0 + \epsilon\alpha_1 + \ldots \\ \text{and} & \\ \dot{y}(0) & = \beta_0 + \epsilon\beta_1 + \ldots \end{cases} \tag{5.56}$$

(to be determined termwise) with

$$y \sim y_0(\omega t) + \epsilon y_1(\omega t) + \ldots . \tag{5.57}$$

Exercises

1. To further motivate two-scale expansions, consider the scalar linear initial value problem

   $$\epsilon u' + a(x)u = b(x), \quad x \ge 0, \;\; u(0) \quad \text{prescribed}$$

   on a finite interval where $a(x) > 0$ and a and b are smooth.

(a) Obtain a formal asymptotic solution in the form

$$u(x,\epsilon) = U(x,\epsilon) + e^{-\frac{1}{\epsilon}\int_0^x a(s)\,ds}\big(u(0) - U(0,\epsilon)\big)$$

where U is an outer expansion

$$U(x,\epsilon) \sim U_0(x) + \epsilon U_1(x) + \epsilon^2 U_2(x) + \dots$$

(b) Integrate the exact solution

$$u(x,\epsilon) = e^{-\frac{1}{\epsilon}\int_0^x a(s)\,ds}\,u(0) + \frac{1}{\epsilon}\int_0^x e^{-\frac{1}{\epsilon}\int_t^x a(s)\,ds}\,b(t)\,dt$$

by parts to show that

$$u(x,\epsilon) = \frac{b(x)}{a(x)} + e^{-\frac{1}{\epsilon}\int_0^x a(s)\,ds}\left(u(0) - \frac{b(0)}{a(0)}\right) + O(\epsilon).$$

(c) Integrate the exact solution again by parts to show that

$$u(x,\epsilon) = U_0(x) + \epsilon U_1(x) + e^{-\frac{1}{\epsilon}\int_0^x a(s)\,ds}\big(u(0) - U_0(0) - \epsilon U_1(0)\big) + O(\epsilon^2).$$

(d) Find the exact solution to the two-point boundary value problem

$$\epsilon y'' + a(x)y' = f(x), \quad 0 \le x \le 1$$

with

$$y(0) \quad \text{and} \quad y(1) \text{ prescribed.}$$

For $a(x) > 0$ and a and f smooth, use integration by parts to show that the asymptotic solution has the two-variable form

$$y(x,\epsilon) = Y(x,\epsilon) + e^{-\frac{1}{\epsilon}\int_0^x a(s)\,ds}(y(0) - Y(0,\epsilon))$$

where the outer solution Y has an asymptotic series expansion in ϵ.

2. Consider the scalar two-point problem

$$\epsilon y'' + f(x,y,y',\epsilon) = 0, \quad 0 \le x \le 1$$

with $y(0)$ and $y(1)$ prescribed in cases when the reduced problem

$$f(x,Y,Y',0) = 0, \quad Y(1) = y(1)$$

has a solution $Y_0(x)$ on $0 \le x \le 1$ with

$$f_{y'}(x,Y_0,Y_0',0) \ge \sigma$$

for a positive constant σ. Provide examples for which one can use the fast variable

$$\frac{1}{\epsilon}\int_0^t f_{y'}(s,Y_0(s),Y_0'(s),0)\,ds$$

(cf. Willett [524], O'Malley [362], Searl [443], and Rosenblat [420]).

3. (cf. Searl [443]) Consider the *Cole-Lagerstrom problem*

$$\epsilon\ddot{x} + x\dot{x} - x = 0, \qquad x(0) = \alpha, \quad x(1) = \beta.$$

   (a) Try solving the problem via two-timing by setting

$$x(t, \tau, \epsilon) = x_0(t, \tau) + \epsilon x_1(t, \tau) + \cdots$$

   with the slow time $\tau = \epsilon t$. Show that x_0 must satisfy

$$x_{0\tau\tau} + x_0 x_{0\tau} = 0$$

   while

$$x_{1\tau\tau} + x_0 x_{1\tau} + x_1 x_{0\tau} + 2x_{0t\tau} + x_0 x_{0\tau} - x_0 = 0.$$

   Then take

$$x_0(t, \tau) = u_0(t) \tanh\left(\frac{u_0(t)}{2}\tau + v_0(t)\right)$$

   with $u_0(1) = \beta$ and $v_0(0) = \tanh^{-1}\left(\frac{\alpha}{u_0(0)}\right)$.

   (b) Determine the functions u_0 and v_0 to eliminate secular terms in x_1. Note that $\operatorname{sech}^2\left(\frac{u_0}{2}\tau + v_0\right)$ is a solution of the homogeneous linearized equation

$$x_{\tau\tau} + x_0(t, \tau)x_\tau + x_{0\tau}(t, \tau)x = 0,$$

   with t as a parameter. Searl determines $u_0(t) = t + \beta - 1$ and $v_0(t) = v_0(0) = \tanh^{-1}\left(\frac{\alpha}{\beta-1}\right)$ when $|\alpha| < |\beta - 1|$.

   (c) Can you find solutions for all αs and βs?

4. (a) Solve the linear initial value problem

$$\ddot{y} + y + \epsilon e^{-t} y = 0, \quad y(0) = 1, \quad \dot{y}(0) = 0$$

   on $t \leq 0$ in terms of Bessel functions.

   (b) Show that the regular perturbation expansion provides the asymptotic solution with no secular terms.

The Palestinian-American Ali Nayfeh (1933–) got his Ph.D. in aeronautics at Stanford in 1964, with Milton Van Dyke as his advisor. He's been in the department of engineering science and mechanics at Virginia Tech since 1971. His 1973 text, *Perturbation Methods* [341], now reissued as a Wiley Classic, surveyed the growing literature and provided detailed solutions to numerous examples. This and several related books by him have been very successful pedagogically for two generations of engineering and science students. Nayfeh [342] again provides solutions to many perturbation problems, while Nayfeh [343] updates his discussion of two-timing.

(b) Lighthill's Method

Sir M. James Lighthill (1924–1988) was a British applied mathematician and administrator who held important positions at the University of Manchester, the Royal Aircraft Establishment, Imperial College London, Trinity College Cambridge, and University College London (see Pedley [390] and the biography Debnath [115]).

One of the topics presented by Nayfeh [341] is *coordinate stretching*. It generalizes the Poincaré-Lindstedt method and was called the *PLK method* by von Kármán's student H.-S. Tsien (after Poincaré, Lighthill, and Kuo) (cf. Tsien [486]). (Tsien lost his security clearance in 1950 and spent 5 years under house arrest in California before returning to China to lead its rocket program. He is the subject of a biography (Chang [75]). Lighthill gave the *Ludwig Prandtl Memorial Lecture* to GAMM in 1961. He commented

> Indeed, his revolutionary discovery of the boundary layer in 1904 had the same transforming effect on fluid mechanics as Einstein's 1905 discoveries on other parts of physics.

A simple example of Lighthill's method (cf. Lighthill [290], Nayfeh [341], de Jager and Jiang [224], and Johnson [226]) is provided by the nonlinear initial value problem

$$(x+\epsilon u)\frac{du}{dx}+u=0, \quad u(1)=1. \tag{5.58}$$

A regular perturbation expansion

$$u(x,\epsilon)=u_0(x)+\epsilon u_1(x)+\dots$$

breaks down at $x=0$, though the given equation only becomes singular when $x+\epsilon u=0$. Proceeding termwise, we'd need

$$x\frac{du_0}{dx}+u_0=0, \quad u_0(1)=1,$$

so $u_0(x)=1/x$. Then $x\frac{du_1}{dx}+u_1+u_0\frac{du_0}{dx}=0$ and $u_1(1)=0$ imply that

$$u_1(x)=\frac{x^2-1}{2x^3}.$$

The increasing singularity of the terms $u_k(x)$ at $x=0$ is a difficulty which we might compensate for by introducing a *near-identity transformation*

$$x=\xi+\epsilon f(\xi,\epsilon)\sim\xi+\epsilon f_0(\xi)+\epsilon^2 f_1(\xi)+\dots \tag{5.59}$$

with a yet unspecified function f to define the new coordinate $\xi(x,\epsilon)$ by inversion and a corresponding regular perturbation expansion

$$U(\xi,\epsilon)=U_0(\xi)+\epsilon U_1(\xi)+\dots \tag{5.60}$$

for the solution of (5.58) as a function of ξ, which we hope will be defined at $x = 0$. (Bush [62] makes the helpful suggestion that subsequent coefficients U_k should be no more singular than previous ones.) Since $\frac{d\xi}{dx} = \frac{1}{1+\epsilon\frac{df}{d\xi}}$, the given differential equation (5.58) transforms to

$$(\xi + \epsilon f(\xi,\epsilon) + \epsilon U)\frac{dU}{d\xi} + \left(1 + \epsilon\frac{df}{d\xi}(\xi,\epsilon)\right)U = 0. \tag{5.61}$$

The regular perturbation process now implies the sequence of equations

$$\xi\frac{dU_0}{d\xi} + U_0 = 0, \tag{5.62}$$

$$\xi\frac{dU_1}{d\xi} + U_1 + f_0(\xi)\frac{dU_0}{d\xi} + \frac{df_0}{d\xi}U_0 + U_0\frac{dU_0}{d\xi} = 0, \tag{5.63}$$

etc., for the coefficients U_k in (5.60). The boundary values need to be determined from the terminal condition

$$U(\xi^*,\epsilon) = 1 \quad \text{where} \quad 1 = \xi^* + \epsilon f(\xi^*,\epsilon). \tag{5.64}$$

Taking

$$\xi^* \sim 1 + b_0\epsilon + b_1\epsilon^2 + \ldots, \tag{5.65}$$

the ϵ coefficient in (5.64), $1 = (1 + b_0\epsilon + b_1\epsilon^2 + \ldots) + \epsilon f_0(1 + \epsilon b_0 + \ldots) + \epsilon^2 f_1(1+\ldots) + \ldots$, implies that

$$b_0 = -f_0(1),$$

so $U(\xi^*,\epsilon) = U_0(1 - f_0(1)\epsilon + \ldots) + \epsilon U_1(1+\ldots) + \ldots = 1$ determines the end values

$$U_0(1) = 1, \tag{5.66}$$

and

$$U_1(1) = U_0'(1)f_0(1) \tag{5.67}$$

etc. Returning to (5.62), $\frac{d}{d\xi}(\xi U_0) = 0, \quad U_0(1) = 1$ implies that

$$U_0(\xi) = \frac{1}{\xi} \tag{5.68}$$

(compared to $1/x$ for u_0). Now there still remains much flexibility in picking f in the near-identity transformation (5.59). We will compensate the singular term $U_0\frac{dU_0}{d\xi}$ in (5.63) by asking that f_0 satisfies

$$\frac{df_0}{d\xi}U_0 + f_0\frac{dU_0}{d\xi} + U_0\frac{dU_0}{d\xi} = 0, \tag{5.69}$$

leaving

$$\xi \frac{dU_1}{d\xi} + U_1 = 0, \quad U_1(1) = -f_0(1). \tag{5.70}$$

from (5.63). If we now take $f_0(1) = 0$, we get

$$U_1(\xi) = 0, \tag{5.71}$$

leaving (5.63) as the initial value problem $\frac{1}{\xi}\frac{df_0}{d\xi} - \frac{f_0}{\xi^2} - \frac{1}{\xi^3} = 0$, $f_0(1) = 0$. Integration yields

$$f_0(\xi) = \frac{1}{2}\left(\xi - \frac{1}{\xi}\right). \tag{5.72}$$

Taking all later f_ks in (5.59) to be zero, as well as all later U_ks in (5.60), we simply obtain the quadratic near-identity transformation

$$x = \xi + \frac{\epsilon}{2}\left(\xi - \frac{1}{\xi}\right) \tag{5.73}$$

and the one-term solution

$$U = U_0(\xi) = \frac{1}{\xi}. \tag{5.74}$$

Since (5.73) has the inverse $\xi = \frac{x+\sqrt{x^2+2\epsilon+\epsilon^2}}{2+\epsilon}$, the solution (5.74) of (5.58) is

$$u(x,\epsilon) = \frac{2+\epsilon}{x+\sqrt{x^2+2\epsilon+\epsilon^2}} = \frac{-x+\sqrt{x^2+2\epsilon+\epsilon^2}}{\epsilon}. \tag{5.75}$$

Amazingly, this is the *exact* solution, as can be checked by integrating

$$\left(xu + \frac{\epsilon}{2}u^2\right)' = 0, \quad u(1) = 1.$$

We note a closely related method of George Temple [482]. Recall, too, that Kaplun [235] called a coordinate ξ *optimal* when it leads to a uniformly valid solution $U(\xi,\epsilon)$. Readers should consult Comstock [95] regarding Lighthill's method and the controversy that once surrounded it. Johnson [226] considers more general initial value problems for

$$(x + \epsilon u)u' + (\alpha + \beta x)u = 0$$

for $\alpha > 0$ and $u(1)$ prescribed, while Sibuya and Takahasi [459] provide a proof for equations

$$(x + \epsilon u)u' + q(x)u = r(x).$$

Awrejcewicz and Krysko [18] more generally suppose one begins with a *naive* expansion

$$f(x,\epsilon) \sim f_0(x) + \epsilon f_1(x) + \epsilon^2 f_2(x) + \dots \tag{5.76}$$

that is not uniformly valid. They next introduce the deformed variable X via

$$x = X + \epsilon\nu_1(X) + \epsilon^2\nu_2(X) + \dots \tag{5.77}$$

to obtain

$$f(x,\epsilon) = F(X,\epsilon) \sim F_0(X) + \epsilon F_1(X) + \epsilon^2 F_2(X) + \dots . \tag{5.78}$$

Then, they pick the deformation coefficients ν_1, ν_2, ... in (5.77) to achieve a uniformly suitable series for F.

(c) Phase-Plane Methods and Relaxation Oscillations

Scalar boundary value problems for singularly perturbed equations in *conservation form*,

$$\begin{cases} \epsilon^2\ddot{x} + f(x) = 0, \quad 0 \le t \le 1 \\ \text{with} \\ x(0) \text{ and } x(1) \text{ prescribed,} \end{cases} \tag{5.79}$$

can be integrated by introducing the potential energy

$$V(x) = \int^x f(s)\, ds \tag{5.80}$$

and invoking the resulting *conservation of energy* principle

$$\frac{1}{2}\epsilon^2\dot{x}^2 + V(x) = E \tag{5.81}$$

for a constant total energy E fixed on each solution trajectory (cf. O'Malley [367], Lutz and Goze [300], and Ou and Wong [384]). (The classical graphical treatment of conservation equations with $\epsilon = 1$ is given, e.g., in Jordan and Smith [230].) This follows immediately by direct integration, after multiplying the differential equation of (5.79) by $\dot{x}$. If we set

$$y = \epsilon\dot{x}, \tag{5.82}$$

we can describe the motion in the x-y *phase plane* by considering the singularly perturbed system

$$\begin{cases} \epsilon\dot{x} = y \\ \epsilon\dot{y} = -f(x). \end{cases} \tag{5.83}$$

To get a real trajectory, the constant E must always exceed $V(x)$ since $E - V = y^2/2 \ge 0$.

Indeed, since $dt = \pm\frac{\epsilon\, dx}{\sqrt{2(E-V(x))}}$, E must be only just slightly greater than the maximum of V on any bounded trajectory in order to use up one unit of transit time in going from one prescribed endvalue to the other, i.e., so that

$$\int_0^1 dt = \epsilon \int_{x(0)}^{x(1)} \frac{dx}{\sqrt{2(E - V(x))}} = 1 \tag{5.84}$$

for integration along the path $x(t)$ traversed. Moreover, most time must be spent near rest points of (5.83) corresponding to such maxima, because $dt = O(\epsilon)$ elsewhere. Be aware that such two-point problems generally have more than one solution that follow related, but different, phase-plane trajectories (with slightly different E levels), as we shall demonstrate with Example 2 below.

Example 1

Consider the simple linear problem

$$\begin{cases} \epsilon^2 \ddot{x} - x = 0, \quad 0 \le t \le 1 \\ \text{with} \\ x(0) = 1 \text{ and } x(1) = 2. \end{cases} \tag{5.85}$$

Here, we can take

$$V = \frac{1}{2} x^2.$$

Note that solutions $x(t)$ satisfy a maximum principle (cf. Dorr et al. [124]). Since solutions are linear combinations of $e^{\pm t/\epsilon}$, we write the unique solution of (5.85) as

$$x(t, \epsilon) = e^{-t/\epsilon} c + e^{-(1-t)/\epsilon} k$$

for constants $c(\epsilon)$ and $k(\epsilon)$ that satisfy the linear system

$$c + e^{-1/\epsilon} k = 1 \quad \text{and} \quad e^{-1/\epsilon} c + k = 2.$$

Up to asymptotically negligible quantities, we get $c \sim 1$ and $k \sim 2$, so

$$x(t, \epsilon) \sim e^{-t/\epsilon} + 2e^{-(1-t)/\epsilon}. \tag{5.86}$$

Then

$$y = \epsilon \dot{x} \sim -e^{-t/\epsilon} + 2e^{-(1-t)/\epsilon} \tag{5.87}$$

and the total energy on the corresponding trajectory is

$$E = \frac{1}{2}(y^2 - x^2) \sim -4e^{-1/\epsilon} \tag{5.88}$$

(negative, but asymptotically negligible). Graphically, see Figs. 5.1 and 5.2.

We plot $V(x)$ with a small negative E value in Fig. 5.3. This determines the allowed range of x values (omitting a neighborhood of $x = 0$) and graphically determines the corresponding real-valued $y = \pm\sqrt{2(E - V(x)}$.

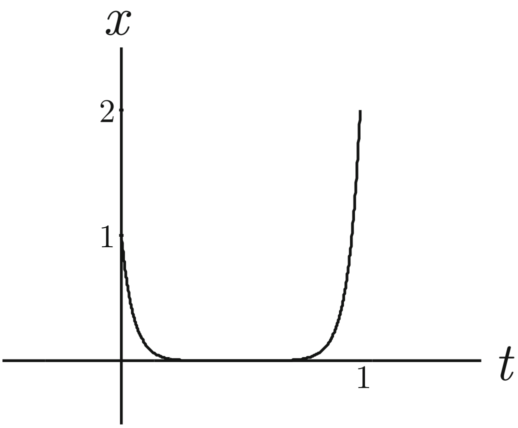

Figure 5.1: The asymptotic solution $x(t, \epsilon)$ of $\epsilon^2 \ddot{x} = x$

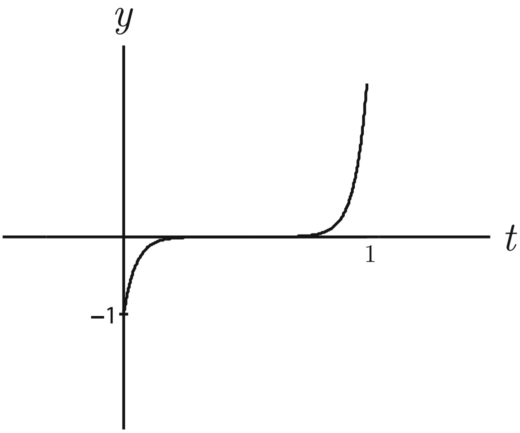

Figure 5.2: The solution $y(t, \epsilon) = \epsilon \dot{x}(t, \epsilon)$ of $\epsilon^2 \ddot{x} = x$

To obtain a trajectory joining $x(0) = 1$ and $x(1) = 2$, we must follow the dashed right orbit in the phase-plane (Fig. 5.4) where $x > 0$ and y is monotonically increasing. Let α' and α be the points where the orbit cuts the vertical line $x = 1$ (with α' below α) and let β' and β be the corresponding points where it cuts $x = 2$. Because motion only slows down near the rest point $(0, 0)$, $\epsilon \int_1^2 \frac{dx}{y} = 1$ requires y to be small most of the time. This rules out the trajectory $\alpha\beta$ as being too fast, so the only possible orbit is $\alpha'\alpha\beta$, which moves slowly near the rest point, but fast in the endpoint layers, as pictured in Figs. 5.1 and 5.2.

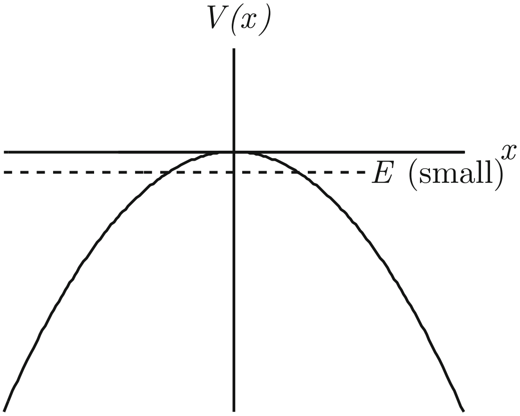

Figure 5.3: The potential $V(x) = -\frac{1}{2}x^2$

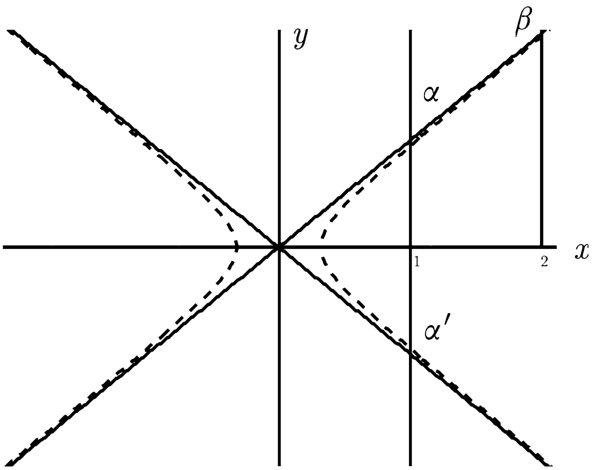

Figure 5.4: The dotted phase-plane orbit for the solution $\alpha'\alpha\beta$ of $\epsilon^2\ddot{x} = x$ with ϵ small

Example 2

Consider the nonlinear example

$$\begin{cases} \epsilon^2\ddot{x} + x^2 = 1, & -1 \le t \le 1 \\ \text{with} \quad x(\pm 1) = 0. \end{cases} \tag{5.89}$$

The example is important because it suggests that the method of matched expansions can mislead by suggesting the existence of *spurious solutions*. Actual solutions can be obtained in terms of elliptic integrals (cf. Byrd and

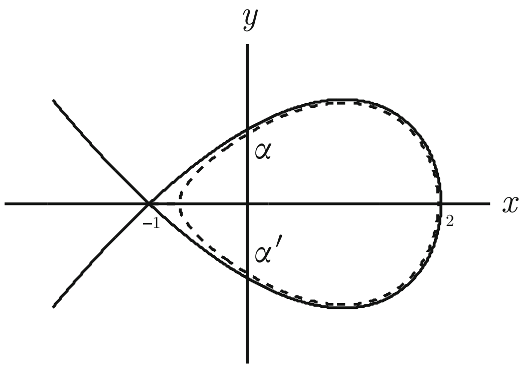

Figure 5.5: The x-y phase plane for $\epsilon^2\ddot{x} = 1 - x^2$

Friedman [66] and Kevorkian and Cole [249]). We take the potential energy to be

$$V(x) = \int_0^x (s^2 - 1)\, ds = \frac{x^3}{3} - x. \tag{5.90}$$

Since $V'(-1) = 0$ and $V''(-1) < 0$, V has a local maximum $2/3$ at $x = -1$. Significantly, $V(2) = 2/3$, too, though 2 is not a maximum of V. The phase-plane portrait for an E slightly less than $2/3$ is shown in Fig. 5.5.

To obtain a trajectory joining $x(-1) = 0$ and $x(1) = 0$, we will need the (dashed) orbit in Fig. 5.5 within the *separatrix* that passes through the rest point in the phase-plane. (A trajectory through the rest point cannot take finite time.) Let α' and α be the points where the trajectory hits the prescribed boundary value $x = 0$, with α' below α. Note that the orbit $\alpha\alpha'$ is too fast since it doesn't go near the rest point $(-1, 0)$, but repeated passages past both points $(-1, 0)$ and $(2, 0)$ are allowed (on trajectories with somewhat less than the upper bound $\frac{2}{3}$ for the energy E). To use up one unit of time, the orbit spends most time near $(-1, 0)$. By contrast, motion toward $(2, 0)$ and back is rapid, providing a thin *spike* in the x-y trajectory. The limit $X_0(t) = -1$ is a root of the reduced equation. The other root, $X_0(t) = 1$, a minimum of the potential energy V, is quite irrelevant asymptotically. The simplest (and shortest) solution $\alpha'\alpha$ is shown in Fig. 5.6. It has the form

$$x(t, \epsilon) \sim -1 + \frac{12e^{p_L}}{(1 + e^{p_L})^2} + \frac{12e^{p_R}}{(1 + e^{p_R})^2} \tag{5.91}$$

(cf. Carrier and Pearson [71] and Lange [280]) where $p_{L/R} = \frac{\sqrt{2}}{\epsilon}(1 \pm t) + 2\ln(\sqrt{3} + \sqrt{2})$ or, equivalently,

$$\begin{aligned} x(t, \epsilon) \sim\ & -1 + 3\,\mathrm{sech}^2\left(\frac{1 + t}{\sqrt{2}\epsilon} + \ln(\sqrt{3} + \sqrt{2})\right) \\ & + 3\,\mathrm{sech}^2\left(\frac{1 - t}{\sqrt{2}\epsilon} + \ln(\sqrt{3} + \sqrt{2})\right). \end{aligned} \tag{5.92}$$

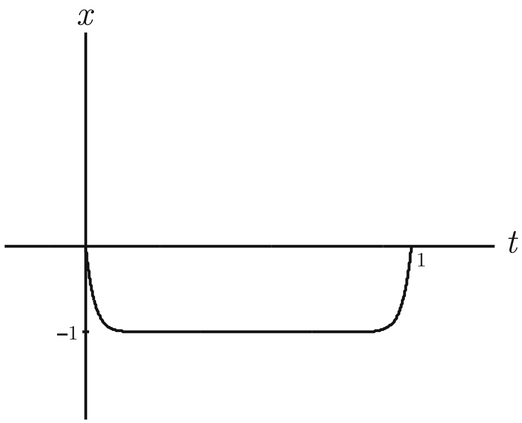

Figure 5.6: The x trajectory following $\alpha'\alpha$

The corresponding solution for $y = \epsilon\dot{x}$ follows by differentiation. Readers are urged to plot these functions to check the complicated formulas. Other endpoint layers, with spikes to 2, are possible for trajectories with lower energies E. For example, the solution $\alpha\alpha'\alpha$ has such an initial spike while $\alpha'\alpha\alpha'$ has a terminal spike, and $\alpha\alpha'\alpha\alpha'$ has a spike near both endpoints. Two of these solutions are shown in Figs. 5.7 and 5.8. Note the differences in the signs and sizes of the endpoint slopes.

According to Ou and Wong [384], the asymptotic solutions are, respectively, given by

$$\begin{aligned} x(t,\epsilon) \sim\ & -1 + 3\,\mathrm{sech}^2\left(-\frac{t+1}{\sqrt{2}\epsilon} + \ln(\sqrt{3}+\sqrt{2})\right) \\ & + 3\,\mathrm{sech}^2\left(\frac{1-t}{\sqrt{2}\epsilon} + \ln(\sqrt{3}+\sqrt{2})\right) \end{aligned} \tag{5.93}$$

$$\begin{aligned} x(t,\epsilon) \sim & -1 + 3\,\mathrm{sech}^2\left(\frac{t+1}{\sqrt{2}\epsilon} + \ln(\sqrt{3}+\sqrt{2})\right) \\ & + 3\,\mathrm{sech}^2\left(-\frac{(1-t)}{\sqrt{2}\epsilon} + \ln(\sqrt{3}+\sqrt{2})\right), \end{aligned} \tag{5.94}$$

and

$$\begin{aligned} x(t,\epsilon) \sim & -1 + 3\,\mathrm{sech}^2\left(-\frac{(t+1)}{\sqrt{2}\epsilon} + \ln(\sqrt{3}+\sqrt{2})\right) \\ & + 3\,\mathrm{sech}^2\left(-\frac{(1-t)}{\sqrt{2}\epsilon} + \ln(\sqrt{3}+\sqrt{2})\right). \end{aligned} \tag{5.95}$$

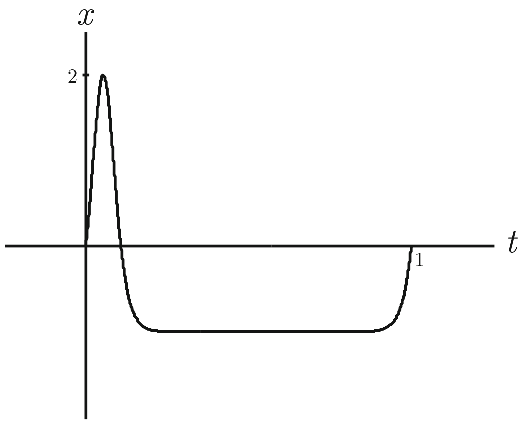

Figure 5.7: The x trajectory following $\alpha\alpha'\alpha$

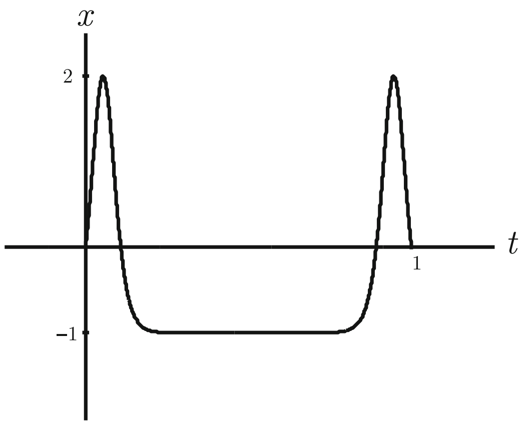

Figure 5.8: The x trajectory following $\alpha\alpha'\alpha\alpha'$

Interior spikes are another possibility. Carrier and Pearson [71] warned that classical matching allows one to formally add a spike

$$g(t) = 3\text{sech}^2\left(\frac{t - t_0}{\sqrt{2}\epsilon}\right) \tag{5.96}$$

about *any* interior point t_0 to obtain a possibly new asymptotic solution x. Indeed, one could seem to add several isolated spikes. However, Carrier and Pearson realized by a phase-plane analysis (such as ours) that these "solutions" are generally *spurious*. It would be allowable to add a single spike at the midpoint $t_0 = 0$ or two such spikes simultaneously at $t_0 = \pm 1/3$ (corresponding to one- and two-thirds of the interval length). Solutions with legitimate interior spikes can be obtained that pass near the rest point more than once, on a longer trajectory with somewhat less energy E. Such cycles take the same time of passage for each revolution in the phase- plane, so the resulting periodic motion in the x-t plane must feature nearly regularly spaced spikes, in addition to the endpoint layers already considered. Carrier and Pearson reassured readers

> The authors have never seen this occur with any problem which arose in a scientific context.

Nonetheless, our confidence in formal matching is diminished. Kath [242] provides extensions to slowly varying phase-planes.

Limiting attention to N *regularly spaced* $O(\epsilon)$-thick interior spikes will, as N increases, ultimately fill the t interval $(-1, 1)$, leaving scant space for the attractive outer limit -1 to apply. Thus, it's not surprising that Ou and Wong [384], using a natural *shooting* argument for initial values $x(-1, 0) = 0$ and a varying $\dot{x}(-1, 0) = k$, were able to show that actual solutions for appropriate k have at most $O(1/\epsilon)$ internal spikes. They also provided details regarding the asymptotic locations of those spikes as functions of ϵ. Also, note Ward [510].

Carrier [72] considered the nonautonomous problem

$$\epsilon^2\ddot{x} + 2(1 - t^2)x + x^2 - 1 = 0, \quad x(\pm 1) = 0. \tag{5.97}$$

No phase-plane argument applies. As you'd expect, however, he finds (cf. Bender and Orszag [36]) that the asymptotic solutions have the outer limit

$$-(1 - t^2) - \sqrt{(1 - t^2)^2 + 1}$$

with both boundary layers and various additional spikes possible. MacGillivray et al. [303] show that the interior spikes coalesce as $\epsilon \to 0$. Carrier again raised the specter of spurious solutions and considered the possibility of multiple solutions, differing in their endpoint slopes by only negligible amounts. Bender and Orszag display a number of numerical solutions for $\epsilon = 10^{-4}$, again suggesting a breakdown in the asymptotics as the number of spikes increases. The more recent results of Ai [4] suggest a pileup of spikes near the midpoint, but that isn't clear from Wong and Zhao [527], who also used shooting arguments. Hastings and McLeod [198] transform the problem to a more tractable equation of the Riccati form

$$\epsilon^2\ddot{u} = u(q(t, \epsilon) - u)$$

by setting $x = X + u$ for the outer solution X (i.e., by using the subtraction trick). They also develop extensive material regarding spikes and layers for more general reaction-diffusion models.

Carrier and Pearson [71] use singular perturbations as the penultimate topic in their ODE text. George Carrier (1918–2002) was a superb math modeller who repeatedly used asymptotic techniques to comprehend a wide variety of physical applications. He was active as a consultant to industry and government, winning the National Medal of Science in 1990. He also had great success and influence as a teacher, throughout his long career at Harvard (from 1952). Even earlier, he introduced Julian Cole to perturbation methods as an undergraduate at Cornell (where Carrier got his Ph.D.). Carl Pearson co-authored several books with Carrier, worked for Boeing, and was a professor of aeronautics and applied mathematics at the University of Washington.

Let's next consider an autonomous slow-fast planar system

$$\begin{cases} \dot{x} = f(x,y) \\ \epsilon\dot{y} = g(x,y) \end{cases} \tag{5.98}$$

for times $t \geq 0$. We can expect slow motion to follow the reduced system

$$\begin{cases} \frac{dX}{dt} = f(X,Y) \\ 0 = g(X,Y), \end{cases} \tag{5.99}$$

while fast motion should follow the stretched system

$$\frac{dy}{d\tau} = g(x,y) \tag{5.100}$$

with x as a parameter and fast time

$$\tau = \frac{t - t_0}{\epsilon} \quad \text{for some } t_0. \tag{5.101}$$

Thus, slow motion will lie on the *manifold*

$$\Gamma : g(x,y) = 0 \tag{5.102}$$

and fast motion will be off it. If the prescribed initial point

$$(x(0), y(0))$$

is not on Γ, we can immediately expect nearly vertical fast motion toward (or away from) Γ since g/ϵ will be large. If we next reach a stable (i.e., attractive) point

$$(x(0), y_1)$$

on Γ where $g_y < 0$, we then expect slow motion along Γ to follow until, say, g_y loses stability at a *junction point*

$$(x_2, y_2).$$

Then, we can again expect rapid motion away from the manifold until, perhaps, a stable *drop point*

$$(x_2, y_3)$$

on Γ is reached, when slow motion may again begin. When successive alternations between slow and fast motions produce a limiting *closed trajectory* with jerky, almost instantaneous, jumps in y, we will say we have a *relaxation oscillation*. The limiting period will be determined by integrating

$$dt = \frac{dx}{f(x,y)}$$

on the slow manifold Γ. Detailed asymptotics, especially near junction and drop points, is called for. We leave the connection to *hysteresis* open, but interested readers might note Mortell et al. [328].

Many times, oscillators like (5.98) arise from scalar second-order differential equations

$$\epsilon\ddot{y} - F'(y)\dot{y} + y = 0. \tag{5.103}$$

The van der Pol equation occurs when

$$F(y) = y - \frac{1}{3}y^3, \tag{5.104}$$

i.e. from

$$\frac{d^2y}{d\tau^2} - \lambda(1-y^2)\frac{dy}{d\tau} + y = 0 \tag{5.105}$$

when $\lambda = \frac{1}{\sqrt{\epsilon}}$ is large and $\tau = \lambda t$. For it, we let

$$\dot{x} = f(x,y) \equiv y \tag{5.106}$$

and integrate the resulting y equation (5.103) to get

$$\epsilon\dot{y} = g(x,y) \equiv F(y) - x. \tag{5.107}$$

Thus,

$$\Gamma : x = F(y) \tag{5.108}$$

is then an S-shaped curve, stable for $|y| > 1$, so the relaxation oscillation for the van der Pol equation jumps between arcs of Γ at $y = \pm 1$ (cf. Stoker [476]). See Figs. 5.9 and 5.10.

As anticipated, the limiting period of the corresponding trajectory will be

$$T = 2\int_{-2/3}^{2/3} \frac{dx}{y(x)} = 2\int_2^1 \frac{F'(y)}{y}dy = 3 - 2\ln 2, \tag{5.109}$$

integrating along one arc of Γ. Obtaining higher-order terms in the asymptotic expansion for the period or the amplitude requires much effort (cf. Stoker [476], Levinson [287], Mischenko and Rozov [321], and Grasman [179]). In particular, the asymptotic sequences that arise are far from obvious *a priori*.

The first to study relaxation oscillations seems to be Balthasar van der Pol (1889–1959) in 1926 (cf. Israel [222]). He was a scholarly Dutch engineer, with a Ph.D. from Utretcht based on work done in Cambridge, and was head physicist at Philips Physical Laboratory. He became particularly interested in modeling the human heart and its arrhythmias. (From 1945 to 1946, he was president of the Temporary University at Eindhoven, founded to replace other Dutch universities in occupied territory.) Currently, applications to neuroscience, featuring many coupled oscillations, are of great interest and quite complicated (cf. Ermentrout and Terman [142]). Earlier neural networks are modeled in Cronin [105, 106]. The simplest examples may be see-saws with water reservoirs, pictured in Grasman [179]. For other applications, see Sastry and Desoer [432].

One of the most interesting recent developments relates to the occurrence of *canards*. The topic was introduced by a group of French mathematicians in the 1970s, primarily concerned with applying non-standard analysis (cf. Diener and Diener [120]). (Some were working in Algeria.) They considered the forced van der Pol equation

$$\epsilon\ddot{y} + (y^2 - 1)\dot{y} + y = a \tag{5.110}$$

or, equivalently, the system

$$\begin{cases} \epsilon\dot{y} &= z - F(y) \\ \dot{z} &= a - y \end{cases} \tag{5.111}$$

in the Liénard plane where $F(y) = y - \frac{y^3}{3}$ and a is a constant. For $a = 0$ or 1, for example, we get periodic solutions that follow the limit cycle consisting of slow motion on the attractive arcs of the characteristic curve and fast

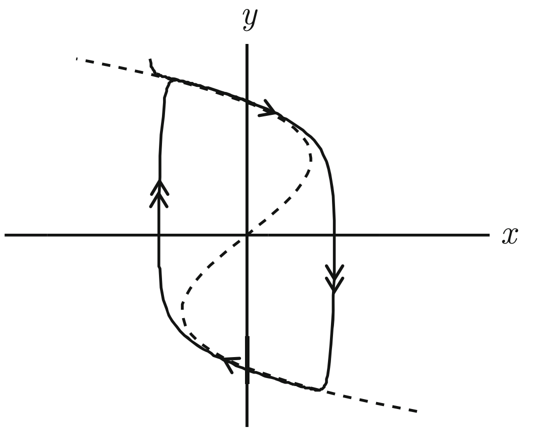

Figure 5.9: Approaching the limit cycle for the van der Pol equation

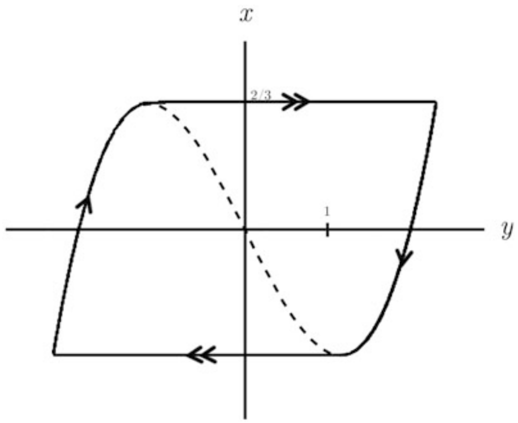

Figure 5.10: Limit cycle

horizontal trajectories. They found that canards occur for special values

$$a(\epsilon) \sim a_0 + \epsilon a_1 + \epsilon^2 a_2 + \dots \tag{5.112}$$

near $a_0 = 1$, resulting in solutions of the form

$$z \sim F(y) + \epsilon f_1(y) + \epsilon^2 f_2(y) + \dots \tag{5.113}$$

that travel up the unstable branch of the characteristic curve to produce a *canard sans tête* (Fig. 5.11) or a *canard avec tête* (Fig. 5.12). (Pardon my French!) Note the connection to Sect. (c) of Chap. 4. Motion up the dashed curve is unstable (i.e., repulsive).

Finding the power series is straightforward since $\dot{z} = z'\dot{y}$ and $\epsilon\dot{y} = z - F(y)$ imply that

$$\dot{z} = a_0 - y + \epsilon a_1 + \dots = (F'(y) + \epsilon f_1'(y) + \dots)(f_1(y) + \epsilon f_2(y) + \dots).$$

Thus, $F'(1) = 0$ shows that we need

$$a_0 = 1. \tag{5.114}$$

More generally, at $\epsilon = 0$, $a_0 - y = F'(y) f_1(y)$ determines

$$f_1(y) = \frac{a_0 - y}{F'(y)} = \frac{1}{1+y} \tag{5.115}$$

while the ϵ terms imply that

$$\begin{cases} a_1 = f_1'(1) f_1(1) = -\frac{1}{8} \\ \text{and} \\ f_2(y) = \frac{a_1 - f_1'(y) f_1(y)}{F'(y)} = \frac{(1+y)^3 - 8}{8(1-y)(1+y)^4}. \end{cases} \tag{5.116}$$

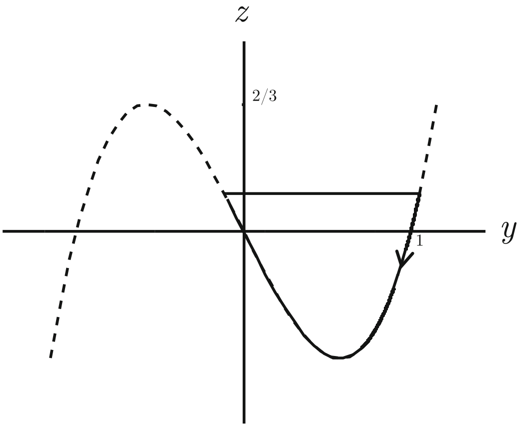

Figure 5.11: Canard sans tête

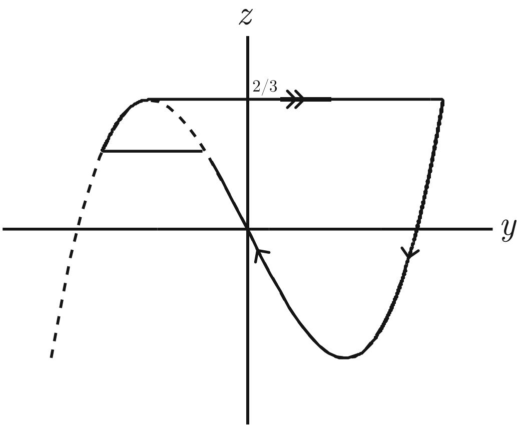

Figure 5.12: Canard avec tête

The French mathematicians used MACSYMA to get the expansions for a general f up to fifty terms while Zvonkin and Shubin [538] later provided recursion relations for them. For the van der Pol equation,

$$a = 1 - \frac{\epsilon}{8} - \frac{3}{32}\epsilon^2 + \dots \tag{5.117}$$

and

$$z = \frac{y^3}{3} - y - \frac{\epsilon}{y+1} - \frac{\epsilon^2}{8(y+1)^2}(y^4 + 4y + 7) + \dots . \tag{5.118}$$

These results were confirmed by Eckhaus [134] using classical (standard) analysis. Canalis-Durand [67] showed that the divergent series are of class Gevrey -1. Zvonkin and Shubin conclude

> Ducks and all phenomena connected with them can be effectively discovered by numerical computations for such *moderately infinitely small* values of ϵ as $\frac{1}{10}$ or $\frac{1}{20}$.

Exercise

Find canard solutions for the van der Pol equation with $\epsilon = 1/10$ and a nearly $0.9863132\ldots$. Picture them. Imagine doing so in North Africa 35 years ago!

Beware:

A canard's life is short.

If $\overline{a}$ is a duck value, any other duck value a satisfies

$$|a - \overline{a}| = e^{-\frac{1}{k\epsilon}}$$

for some $k > 0$. Once more, we're involved with exponential asymptotics. Some, though not the originators, say the hard-to-detect phenomenon was called a canard after the French newspaper slang for a hoax. For more details, see the Scholarpedia article by Wechselberger. (Also, note Braaksma [58].)

(d) Averaging and Renormalization Group Methods

The first appendix to Sanders et al. [430] presents a brief history of the method of *averaging*. Important contributions to that approach were made by Lagrange, van der Pol, Krylov, Bogoliubov, and Mitropolsky. See Samoilenko [429] for a survey of Soviet work.

A basic underlying idea concerns *variation of parameters* (i.e., variation of constants). Its linear version is elementary and well known, while its nonlinear version has recently been attributed to Alekseev [6], rather than Lagrange or Poisson, though it seems present in the earlier celestial mechanics literature (cf. Pollard [397] and Verhulst [499]). Consider the vector initial value problem

$$\dot{z} = f(z, t, \epsilon), \quad z(t_0) = a, \tag{5.119}$$

with f depending smoothly on ϵ, and suppose the unperturbed problem

$$\dot{x} = f(x, t, 0), \quad x(t_0) = a \tag{5.120}$$

has a known solution which we will denote (as in dynamical systems) by

$$x = \phi(a, t). \tag{5.121}$$

If we seek the solution of the given problem (5.119) using the ansatz

$$z = \phi(p, t) \tag{5.122}$$

for a *variable* function $p(t)$ subject to the initial condition $p(t_0) = a$, differentiation implies that

$$\dot{z} = \frac{\partial \phi}{\partial p}\frac{dp}{dt} + \frac{\partial \phi}{\partial t} = f(\phi, t, \epsilon).$$

But $\frac{\partial \phi}{\partial t} = f(\phi, t, 0)$ and $\frac{\partial \phi}{\partial p}$ is (at least locally) invertible by the existence-uniqueness theorem. Thus, p must satisfy the initial value problem

$$\frac{dp}{dt} = \left(\frac{\partial \phi}{\partial p}\right)^{-1} (f(\phi(p,t), t, \epsilon) - f(\phi(p,t), t, 0)), \quad p(t_0) = a. \tag{5.123}$$

Because $\frac{dp}{dt} = O(\epsilon)$, p will be *slowly varying*, so it can only change substantially on a long $O(1/\epsilon)$ time interval. Solving the nonlinear problem (5.123) for $p(t)$, numerically or otherwise, determines the desired solution $z = \phi(p, t)$ of (5.119). The constant a in (5.121) could be any parameter, not just the initial value.

Adrianov et al. [9] also consider the system (5.119). They reduce it to the *standard form*

$$\dot{z} = \epsilon Z(t, z, \epsilon) \tag{5.124}$$

by simply making a change of variables

$$z = G(t, x) \tag{5.125}$$

where

$$\frac{\partial G}{\partial t} = f + O(\epsilon), \tag{5.126}$$

$\frac{\partial G}{\partial x}$ is invertible, and

$$\dot{z} = \left(\frac{\partial G}{\partial x}\right)^{-1} \left(f - \frac{\partial G}{\partial t}\right) \equiv \epsilon Z(t, z, \epsilon). \tag{5.127}$$

Examples

1. For the nearly linear vector problem

$$\dot{z} = A(t)z + \epsilon g(z, t), \quad z(0) = a, \tag{5.128}$$

the limiting problem $\dot{x} = A(t)x$, $x(0) = a$ has the unique solution

$$x = \Phi(t)a$$

where the *fundamental matrix* $\Phi(t)$ (cf., e.g., Bellman [35]) satisfies the linear homogeneous matrix initial value problem

$$\dot{\Phi} = A(t)\Phi, \quad \Phi(0) = I.$$

(In particular, $\Phi(t)$ is the matrix exponential e^{At} when $A(t)$ is constant). Its n columns provide a full set of linearly independent solutions to (5.128) with $\epsilon = 0$, spanning all solutions. Assuming $\Phi(t)$ is available, variation of parameters determines the solution of the given problem (5.128) in the form

$$z(t) = \Phi(t)p(t). \tag{5.129}$$

Thus $\dot{z} = \dot{\Phi}p + \Phi\dot{p} = A\Phi p + \epsilon g(\Phi p, t)$ implies that the slowly varying amplitude $p(t)$ must satisfy the nonlinear vector initial value problem

$$\dot{p} = \epsilon\Phi^{-1}(t)g(\Phi(t)p, t), \quad p(0) = a. \tag{5.130}$$

Existence of $p(t)$ is guaranteed locally and its numerical solution is straightforward.

2. In the special case of nearly linear autonomous scalar oscillators

$$\ddot{x} + x = \epsilon h(x, \dot{x}) \tag{5.131}$$

with prescribed initial values $x(0)$ and $\dot{x}(0)$, we naturally write the solution for $\epsilon = 0$ in *polar coordinates* as

$$x = r\sin(t+\phi) \qquad \text{and} \qquad \dot{x} = r\cos(t+\phi)$$

for constants r and ϕ determined directly by the initial values. For $\epsilon \neq 0$, we use the traditional variation of parameters approach to instead introduce variable functions $R(t)$ and $\Psi(t)$ so that

$$x = R\sin(t+\Psi) \quad \text{and} \quad \dot{x} = R\cos(t+\Psi). \tag{5.132}$$

Differentiating the first expression with respect to t and comparing that result to the second requires R and Ψ to satisfy

$$\dot{R}\sin(t+\Psi) + R\cos(t+\Psi)\dot{\Psi} = 0.$$

Likewise, differentiating the second expression implies that

$$\ddot{x} + x = \dot{R}\cos(t+\Psi) - R\sin(t+\Psi)\dot{\Psi} = \epsilon h(x, \dot{x}).$$

Using Cramer's rule, we solve these two linear equations to obtain

$$\begin{cases} \dot{R} = \epsilon\cos(t+\Psi)\, h(R\sin(t+\Psi), R\cos(t+\Psi)) \\ \text{and} \\ R\dot{\Psi} = \epsilon\sin(t+\Psi)\, h(R\sin(t+\Psi), R\cos(t+\Psi)). \end{cases} \tag{5.133}$$

The resulting nonlinear initial value problem with

$$R(0) = r \quad \text{and} \quad \Psi(0) = \phi \tag{5.134}$$

will have a unique slowly varying solution $\begin{pmatrix} R \\ \Psi \end{pmatrix}$ for bounded τ values. This variation of parameters approach is the basis of a variety of averaging procedures. For Duffing's equation, for example, $h(x, \dot{x}) = -x^3$ so

$$\begin{cases} \dot{R} = -\epsilon R^3 \cos(t+\Psi)\sin^3(t+\Psi) \\ \text{and} \\ \dot{\Psi} = -\epsilon R^2 \sin^4(t+\Psi), \end{cases}$$

a problem we could solve numerically. Several different approximate methods will be found subsequently. For more examples, see O'Malley and Kirkinis [378].

The fundamental idea to vary arbitrary constants was contained in J.-L. Lagrange's *Analytical Mechanics* [277] in 1788. A 1997 translation of the 1811 edition states:

> For problems of mechanics which can only be resolved by approximate methods, the first solution is ordinarily found by considering only the primary forces acting on the bodies. In order to extend this solution to the secondary forces which are called perturbing forces, the simplest approach is to keep the form of the first solution by considering as variables the arbitrary constants contained in it. If the quantities which were neglected and which we want to take into account, are very small, the new variables will be nearly constant and the ordinary methods of approximation could be applied. Thus the difficulty is reduced to finding the equations between these variables.

If we naively seek a solution $\begin{pmatrix} R(t,\epsilon) \\ \Psi(t,\epsilon) \end{pmatrix}$ of (5.133–5.134) as a regular power series in ϵ, the first term will be a constant, which will cause the second term to grow like t as $t \to \infty$. This is equivalent to expanding the right-hand side as a Fourier series and realizing that a nonzero constant term would result in secular behavior for later terms because that part of the forcing resonates with the complementary solution of the leading-order homogeneous system. It might again be advisable to introduce a near-identity transformation to eliminate such secularities in accordance with the Fredholm alternative. The end result is to replace the system (5.133) for R and Ψ by its *average* over its 2π period, i.e. we use the autonomous averaged equation

$$\dot{\overline{R}} = \epsilon f_1(\overline{R}) \equiv \frac{\epsilon}{2\pi}\int_0^{2\pi} \cos s \, h(\overline{R}\sin s, \overline{R}\cos s)\, ds \quad \text{with} \quad \overline{R}(0) = r. \quad (5.135)$$

This nonlinear, but separable, initial value problem provides $\overline{R}$ uniquely as a function of the slow time

$$\tau = \epsilon t,$$

i.e.

$$\tau = \int_r^{\overline{R}} \frac{ds}{f_1(s)}. \tag{5.136}$$

(Finding $\overline{R}(\tau)$ explicitly won't always be possible, though we know $\overline{R}$ will move monotonically to a zero of f_1.) Using $\overline{R}$ in place of R makes approximate sense since R should only change by a small $O(\epsilon)$ amount over a period. Then, we can directly integrate the corresponding averaged equation

$$\dot{\overline{\Psi}} = \frac{\epsilon}{\overline{R}} f_2(\overline{R}) \equiv \frac{\epsilon}{2\pi \overline{R}} \int_0^{2\pi} \sin s\, h(\overline{R} \sin s, \overline{R} \cos s)\, ds$$

to provide the approximate phase

$$\overline{\Psi}(\tau) = \phi + \int_0^{\tau} \frac{f_2(\overline{R}(\tau))}{\overline{R}(\tau)}\, d\tau \tag{5.137}$$

in terms of $\overline{R}$. (A more complete argument, using near-identity transformations, is given, e.g., in Rand [408].) The resulting approximation (5.132) for x and $\dot{x}$ can be shown, by a *Gronwall inequality* argument (cf. Cesari [74]), to have an $O(\epsilon)$ error on any $0 \leq t \leq O(1/\epsilon)$ time interval. As a final caveat, we point out that blowup later is always an *ultimate* possibility. Verhulst [500] uses the separable equation

$$\dot{x} = 2\epsilon x^2 \sin^2 t$$

as an example. With $x(0) = 1$, the exact solution is $x(t, \epsilon) = \frac{1}{1 - \epsilon t + \frac{\epsilon}{2} \sin 2t}$. Moreover, when the forcing in $\dot{x} = \epsilon f(x, t, \epsilon)$ isn't periodic, one can conveniently use the *long-time average*

$$f_0(x) \equiv \lim_{T \to \infty} \frac{1}{T} \int_0^T f(x, s, 0)\, ds.$$

Higher-order averaging is also well studied and important in applications, as are extensions to even longer time intervals. An unusual variation is given in Coppola and Rand [97]. E [130] describes how the corresponding *homogenization* technique can be applied to certain elliptic and Hamilton–Jacobi equations. (See Bensoussan et al. [39], Bakhvalov and Panasenko [21], Bakhvalov et al. [22], and Holmes [209] as well.)

For the van der Pol equation, $h = (1 - x^2)\dot{x}$, so the averaged equations are easy to solve, viz.

$$\dot{\overline{R}} = \frac{\epsilon}{2} \overline{R} \left(1 - \frac{\overline{R}^2}{4} \right) \quad \text{and} \quad \dot{\overline{\Psi}} = 0.$$

This indicates that a steady-state limit cycle will occur (when $r > 0$) with $\overline{R}(\infty) = 2$.

The intimate relationship between two-timing and averaging, in general, becomes clear when one examines the asymptotic solution of the initial value problem for the vector system

$$\dot{x} = \epsilon f(x, t, \epsilon)$$

in the so-called *periodic standard form* (cf. Murdock [335], Sarlet [431], and Mudavanhu et al. [332]).

See de Jager and Jiang [224] for a variety of worked out examples of averaging.

3. The simple linear initial value problem

$$\ddot{y} + 2\epsilon\dot{y} + y = 0, \quad t \geq 0, \quad y(0) = 0, \quad \dot{y}(0) = 1 \tag{5.138}$$

with small damping has the exact solution

$$y(t, \epsilon) = \frac{e^{-\epsilon t}}{\sqrt{1-\epsilon^2}} \sin\left(\sqrt{1-\epsilon^2}\ t\right). \tag{5.139}$$

Although this solution is bounded, its regular perturbation expansion about $\epsilon = 0$ breaks down as t becomes unbounded, since secular terms arise. If we had naively set

$$y(t, \epsilon) = y_0(t) + \epsilon y_1(t) + \ldots, \tag{5.140}$$

we would need $\ddot{y}_0 + y_0 = 0$, $\ddot{y}_1 + y_1 = -2\dot{y}_0$, etc., so

$$y_0(t) = \alpha_0 \cos t + \beta_0 \sin t \tag{5.141}$$

for constants α_0 and β_0 and $\ddot{y}_1 + y_1 = 2\alpha_0 \sin t - 2\beta_0 \cos t$. This implies the secular term

$$y_1(t) = (\alpha_1 - \alpha_0 t) \cos t + (\beta_1 - \beta_0 t) \sin t \tag{5.142}$$

Renormalization group (or *RG*) *methods* (cf. Chen et al. [80] and [81] and Ziane [536]) proceed by eliminating the unbounded terms in a naive expansion (5.140) by replacing the ϵ-dependent initial amplitudes

$$\alpha_0 + \epsilon\alpha_1 + \ldots \quad \text{and} \quad \beta_0 + \epsilon\beta_1 + \ldots$$

that arise by instead using *slowly varying* amplitudes

$$A(\tau, \epsilon) \quad \text{and} \quad B(\tau, \epsilon)$$

depending on the slow time

$$\tau = \epsilon t \tag{5.143}$$

For bounded τ, then, they could simply seek an asymptotic solution for (5.138) in the form

$$y(t, \tau, \epsilon) = A(\tau, \epsilon) \cos t + B(\tau, \epsilon) \sin t \tag{5.144}$$

Then

$$\dot{y} = \left(B + \epsilon\frac{dA}{d\tau}\right)\cos t + \left(-A + \epsilon\frac{dB}{d\tau}\right)\sin t$$

while

$$\ddot{y} = \left(-A + 2\epsilon\frac{dB}{d\tau} + \epsilon^2\frac{d^2A}{d\tau^2}\right)\cos t + \left(-B - 2\epsilon\frac{dA}{d\tau} + \epsilon^2\frac{d^2B}{d\tau^2}\right)\sin t,$$

so (5.138) requires

$$\begin{aligned}\ddot{y} + 2\epsilon\dot{y} + y =& \epsilon\left[2\frac{dB}{d\tau} + B + \epsilon\left(\frac{d^2A}{d\tau^2} + \frac{dA}{d\tau}\right)\right]\cos t \\ &+ \epsilon\left[-2\frac{dA}{d\tau} - A + \epsilon\left(\frac{d^2B}{d\tau^2} + \frac{dB}{d\tau}\right)\right]\sin t = 0.\end{aligned} \tag{5.145}$$

The linear independence of the trig functions implies that the amplitudes A and B must satisfy the initial value problem

$$\begin{aligned}2\frac{dA}{d\tau} + 2A &= \epsilon\left(\frac{d^2B}{d\tau^2} + 2\frac{dB}{d\tau}\right), \quad A(0,\epsilon) = 0 \\ 2\frac{dB}{d\tau} + 2B &= -\epsilon\left(\frac{d^2A}{d\tau^2} + 2\frac{dA}{d\tau}\right), \quad B(0,\epsilon) = 1.\end{aligned} \tag{5.146}$$

Using series expansions

$$A(\tau,\epsilon) \sim \sum_{j\geq 0} A_j(\tau)\epsilon^j \quad \text{and} \quad B(\tau,\epsilon) \sim \sum_{j\geq 0} B_j(\tau)\epsilon^j, \tag{5.147}$$

we will need $\frac{dA_0}{d\tau} + A_0 = 0,\ A_0(0) = 0$ and $\frac{dB_0}{d\tau} + B_0 = 0,\ B_0(0) = 1$, so

$$A_0(\tau) = 0 \quad \text{and} \quad B_0(\tau) = e^{-\tau} \tag{5.148}$$

completely specify the limiting solution $y_0 = e^{-\tau}\sin t$. Next, we will need

$$2\frac{dA_1}{d\tau} + 2A_1 = \frac{d^2B_0}{d\tau^2} + 2\frac{dB_0}{d\tau} = -e^{-\tau}, \quad A_1(0) = 0$$

and

$$2\frac{dB_1}{d\tau} + 2B_1 = -\frac{d^2A_0}{d\tau^2} - 2\frac{dA_0}{d\tau} = 0, \quad B_1(0) = 0,$$

so

$$A_1(\tau) = -\frac{\tau}{2}e^{-\tau} \quad \text{and} \quad B_1(\tau) = 0. \tag{5.149}$$

Thus, we have the renormalized solution

$$y(t,\tau,\epsilon) = e^{-\tau}\sin t - \frac{\epsilon\tau}{2}e^{-\tau}\cos t + O(\epsilon^2) \tag{5.150}$$

for τ finite. For improved approximations, we must let the frequency of the trig functions vary with ϵ^2. For details, see O'Malley and Kirkinis [377]. (Recall that the exact solution (5.139) is a function of the fast time $\sqrt{1-\epsilon^2}t$ and the slow time $\tau = \epsilon t$. Kirkinis [252] provides an alternative elimination technique using *cumulants* (cf. Small [464]) to describe renormalization that is somewhat closer to the original ideas of Goldenfeld, Oono, and coworkers.

4. The linear initial value problem

$$\ddot{x} + x = \epsilon \sin t, \quad t \geq 0, \quad x(0) = 1, \quad \dot{x}(0) = 0 \tag{5.151}$$

can be solved exactly using variation of parameters. Thus, $x(t,\epsilon) = \cos t + \frac{\epsilon}{2}(\sin t - t\cos t)$. A more insightful representation, however, is to write $x(t,\tau,\epsilon) = \left(1 - \frac{\tau}{2}\right)\cos t + \frac{\epsilon}{2}\sin t$ for $\tau = \epsilon t$. This suggests that we directly seek an asymptotic solution of the form

$$x(t,\tau,\epsilon) = \alpha(\tau,\epsilon)\cos t + \epsilon\beta(\tau,\epsilon)\sin t \tag{5.152}$$

for bounded τ values and undetermined slowly varying coefficients α and β. Then

$$\dot{x} = \epsilon\frac{d\alpha}{d\tau}\cos t - \alpha\sin t + \epsilon^2\frac{d\beta}{d\tau}\sin t + \epsilon\beta\cos t$$

and

$$\ddot{x} = \epsilon^2\frac{d^2\alpha}{d\tau^2}\cos t - 2\epsilon\frac{d\alpha}{d\tau}\sin t - \alpha\cos t + \epsilon^3\frac{d^2\beta}{d\tau^2}\sin t + 2\epsilon^2\frac{d\beta}{d\tau}\cos t - \epsilon\beta\sin t.$$

Substituting into (5.151), we obtain

$$\epsilon^2\left(\frac{d^2\alpha}{d\tau^2} + 2\frac{d\beta}{d\tau}\right)\cos t + \epsilon\left(-2\frac{d\alpha}{d\tau} + \epsilon^2\frac{d^2\beta}{d\tau^2} - 1\right)\sin t = 0$$

with

$$\alpha(0,\epsilon) = 1 \quad \text{and} \quad \frac{d\alpha}{d\tau}(0,\epsilon) + \beta(0,\epsilon) = 0.$$

The linear independence of $\sin t$ and $\cos t$ implies that α and β must satisfy the initial value problem

$$\begin{cases} 2\frac{d\alpha}{d\tau} + 1 = \epsilon^2\frac{d^2\beta}{d\tau^2}, \quad \alpha(0,\epsilon) = 1 \\ \text{and} \\ 2\frac{d\beta}{d\tau} + \frac{d^2\alpha}{d\tau^2} = 0, \quad \beta(0,\epsilon) + \frac{d\alpha}{d\tau}(0,\epsilon) = 0. \end{cases} \tag{5.153}$$

We will solve the problem asymptotically for finite τ using power series

$$\begin{pmatrix} \alpha(\tau,\epsilon) \\ \beta(\tau,\epsilon) \end{pmatrix} \sim \sum_{j\geq 0}\begin{pmatrix} \alpha_j(\tau) \\ \beta_j(\tau) \end{pmatrix}\epsilon^j. \tag{5.154}$$

The leading terms require

$$2\frac{d\alpha_0}{d\tau} + 1 = 0, \quad \alpha_0(0) = 1$$

and

$$2\frac{d\beta_0}{d\tau} + \frac{d^2\alpha_0}{d\tau^2} = 0, \quad \beta_0(0) + \frac{d\alpha_0}{d\tau}(0) = 0.$$

Thus

$$\alpha_0(\tau) = 1 - \frac{\tau}{2} \quad \text{and} \quad \beta_0(\tau) = \frac{1}{2} \tag{5.155}$$

No further corrections are needed!

Exercises

1. Use two-timing to show that the asymptotic solution to the initial value problem

$$\ddot{y} + \epsilon\dot{y}|\dot{y}| + y = 0, \quad y(0) = 1, \quad \dot{y}(0) = 0$$

satisfies

$$y(t,\tau) = \frac{\cos t}{1 + \frac{4\tau}{3\pi}} + O(\epsilon)$$

for $\tau = \epsilon t$ bounded.

Hint: The $\sin t$ coefficient in the Fourier expansion of $\sin t\,|\sin t|$ is $\frac{8}{3\pi}$ (cf. Mattheij et al. [309]).

2. (cf. Smith [466]). Consider the initial value problem for a weakly coupled electrical circuit

$$\begin{cases} (1-\epsilon^2)Q_1'' + Q_1 = \epsilon Q_2, & Q_1(0) = q, \quad Q_1'(0) = 0 \\ (1-\epsilon^2)Q_2'' + Q_2 = \epsilon Q_1, & Q_2(0) = 0 = Q_2'(0) \end{cases}$$

and determine a solution of the form

$$\begin{aligned} Q_1(t) &= \epsilon A(\tau,\epsilon)\sin t + B(\tau,\epsilon)\cos t \\ Q_2(t) &= C(\tau,\epsilon)\sin t + \epsilon D(\tau,\epsilon)\cos t \end{aligned}$$

for $0 \le t \le \frac{1}{\epsilon}$ using averaging. Note the *beat* phenomenon!

3. (cf. O'Malley and Kirkinis [378]). Seek a solution of the Riccati equation

$$\dot{x} = -x^2 + \epsilon x$$

of the form

$$x(t,\epsilon) = \frac{A}{1 + At}$$

and show that the slowly varying coefficient A is given by

$$A(t,\epsilon) = \frac{\epsilon x(0)}{x(0)\left(1 - \epsilon t - e^{-\epsilon t}\right) + \epsilon}.$$

Historical Remarks

The very important work of Krylov, Bogoliubov, and Mitropolsky on averaging and its applications was somewhat delayed in reaching the West from Kiev. The 1937 Russian monograph by Krylov and Bogoliubov didn't appear in English until an abridged translation by Solomon Lefschetz was published by Princeton University Press [267] in 1947. The 1955 monograph by Bogoliubov and Mitropolsky, *Asymptotic Methods in the Theory of Nonlinear Oscillations*, appeared in an English version [51] by Hindustan Publishing of New Delhi in 1961 (distributed in the West by Gordon and Breach). (Its first sixty pages are now available on Google Scholar.) For many years, two-timing was justified (cf. Morrison [327] and Perko [391]) because it gave the same results as averaging. Now, direct proofs are known (cf., e.g., Murdock [336] and Murdock and Wang [337]).

The physicists Chen, Goldenfeld, and Oono [81] presented their *renormalization group* method as a unified approach to global asymptotic analysis. They, indeed, succeeded in finding asymptotic approximations for solutions to a wide variety of challenging problems from the literature. Figuring out what their fundamental ideas are is not simple, but their aim to provide a universal technique is one all can subscribe to. Their underlying technique is to eliminate secular terms in a regular perturbation expansion by replacing constants by slowly varying amplitudes (or *envelopes*) that satisfy appropriate RG equations to be integrated as initial value problems (cf. also Kunihiro [270] and [271], Ei et al. [139], and Kirkinis [253]). See Goldenfeld [172] for an ambitious update. It is clear that the renormalization group approach is a *resummation* technique, closely related to averaging methods that suppress secular terms. The recent work of DeVille et al. [118] and Roberts [418] emphasizes that it often produces an asymptotic solution in the classical Poincaré-Birkhoff *normal form* (cf. Guckenheimer and Holmes [188], Sanders et al. [430], and Nayfeh [344]). Woodruff [528, 529] independently developed a related *invariance* method (which deserves more attention than it has received), applying it to systems of the form

$$\dot{x} = M(\epsilon^{\alpha}t)x + \epsilon N(\epsilon^{\alpha}t, t, x)$$

for $\alpha = 1$ or 2, where the matrix M has distinct, nonzero, purely imaginary eigenvalues. Cheng [82] presents a hybrid scheme combining renormalization and two-timing while Chiba [86, 87] provides a simplified RG method.

In section 3.7A of Oono [381], a proto-renormalization group approach is developed for autonomous differential equations

$$Ly = \epsilon N(y)$$

where L is a constant-coefficient linear differential operator and N is nonlinear. A somewhat analogous procedure is developed in Mudavanhu and O'Malley [331].

In the following chapter, we shall give the renormalization group method a new and somewhat simplified presentation, which we hope will be further developed as a unification of many methods found in the literature.