Chapter 8
Compromise Programming and Utility Functions

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Abstract Proposed in the last decades of the twentieth century, the Compromise Programming (CP) model assumes that the decision maker looks for a compromise between objectives of different character, financial, ethical or others. As described by CP, the decision maker has in mind an ideal point, which is a basket containing the best feasible level of each objective. This ideal is a utopian infeasible basket of reference because all the best objectives cannot be simultaneously reached. Given an efficient frontier of baskets, the CP satisfying solution is to choose the basket closer to the ideal. More precisely, the CP solution is obtained by minimizing the distance between a frontier basket and the ideal. Distances are not necessarily measured by the Euclidean quadratic metric but by a conventional metric between one and infinity. Moreover, the distance in CP is not a purely geometric notion but a composite measure in which the geometric components are multiplied by the decision maker's preference weights for each objective. Years later the CP proposal, a linkage between CP and utility theory was investigated. Finally, Linear–quadratic composite metric looks for a compromise between aggressive (large risky acnievements) and conservative (balanced solutions) objectives.

8.1 Introduction to CP Modelling

To deal with a variety of MCDM methods assures competitiveness and complementarity, so that the use of a broad range of methods should not be abruptly reduced to one or a few. Together with GP and other techniques, CP is appropriate to make decisions in many fields such as finance, engineering, management and so on. Given that no method can be presented as superior to others, each of them is useful depending on environments and circumstances. Take, for example, classic lexicographic and weighted goal programming, which are the most usual goal

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programming models (Tamiz et al. 1995). They are especially appropriate for management scenarios in which the decision maker (DM) seeks "satisficing" solutions of bounded rationality by subjectively introducing a profusion of targets. In contrast, CP and multi-objective programming are more appropriate for finance/engineering scenarios where the DM cannot afford to replace objective information by subjective views, although the principle of bounded rationality is still accepted in a moderate way. Briefly speaking, CP raises the following optimization problem: to find the efficient alternative closest to a referential infeasible alternative, named utopia, ideal or anchor value. In greater detail, the characteristics of CP are as follows:

(a) It requires specifying the efficient frontier, namely, an allocation set in which no variable can be made better off without making some other variable worse off.
(b) It considers the ideal as an analytic reference for optimization.
(c) This ideal is an infeasible point which generally derives from the efficient frontier, i.e., the CP ideal is a vector whose components are the best values (anchor values) of the criteria.
(d) Therefore, unlike goal programming, the CP ideal is not a target established by the DM from his own views and judgments.
(e) The CP solution is obtained by minimizing the weighted distance from each efficient point to the infeasible ideal, so that the DM chooses the efficient alternative closest to the utopia.
(f) Therefore, CP, although using preference weights, searches for an optimal solution rather than for a "satisficing" solution in the most literal sense of this word.

Assuring efficiency by finding the efficient frontier prior to selecting the optimal solution is the standard procedure used in economics (utility optimization, multi-objective programming models, etc.). In economics, the production possibility set (Pareto efficient frontier) is determined prior to optimizing utility by Lagrange maximization. However, the two-step model (efficient frontier first) can be reduced to a single step (direct optimization) in problems where the presence of inefficient solutions is directly discarded. Descriptively considered, CP embraces different meanings and representations, one of them being an arbiter who looks for a compromise between parties with conflicting interests or opposite standpoints (Ballestero 2000). To undertake the CP minimization, the analyst should previously specify the objective function as a distance equation depending on the chosen metric, which is not necessarily the usual Euclidean quadratic metric. Linear metric is appealing to DMs who seek large outcomes involving imbalanced solutions in exchange for balanced (non-corner) solutions. In contrast, higher metrics such as the quadratic one or even higher are more appealing to DMs who turn to the precautionary principle of avoiding corner solutions. An extreme metric for the balancing purpose is the infinity norm; however, its use might be inappropriate from the achievement perspective (Ballestero 1997). There is CP literature in which a method is provided to solve the metric selection problem. This method relies on a linkage between the CP metric and Arrow's-Pratt's risk theory (Arrow 1965; Pratt 1964). Nevertheless, such an approach cannot be properly used in our deterministic context (Ballestero and Romero 1998; Krcmar et al. 2005; Stokes and Tozer 2002; Xia et al. 2001).

A large amount of CP papers have been published in the academic literature. Currently, more than 18,000 articles can be found in ScienceDirect, which is one of the world's leading full-text scientific database, from which more than 1,300 are applications in finance and, in particular, more than 300 papers include applications of CP-based models to the portfolio selection problem. One of the pioneering applications of CP for portfolio selection are due to Ballestero and Romero (1996) and since then several interesting works can be found in the literature. Some recently published applications are: Bilbao-Terol et al. (2006a,b), Amiri et al. (2011), Abdelaziz et al. (2007a), Ballestero and Plà-Santamaría (2003, 2004), Ballestero et al. (2007) and Perez Gladish et al. (2007).

In this chapter, the problem takes a different turn. The approach is deterministic, which appears to be more appealing than to combine CP with risk and probability without an axiomatic basis.

8.2 Choice Problems and the Decision Maker's Utility

In economics, utility is the cornerstone of classical and modern theory. This concept derives from Bentham's thought, which is known as utilitarianism. Economists assume utility maximization, which is stated as follows:

$$\begin{aligned} &\max Z = Z(x_1, x_2) \\ &\text{subject to } T(x_1, x_2) = k \end{aligned} \tag{8.1}$$

where (x_1, x_2) represents a choice for the decision-maker (e.g. commodity–mix in a consumer's choice problem, vector of outputs in a joint production problem, composition of a portfolio of securities, etc.); $Z(x_1, x_2)$ is the utility function for the decision-maker, and $T(x_1, x_2) = k$ is the attainable or feasible set (budgetary boundary in consumer theory, transformation curve in joint production problems, efficient frontier in portfolio analysis, etc.).

The essence of microeconomic analysis lies within structure (8.1). Thus, economic rationality is usually defined in terms of maximizing a consistent and transitive function such as $Z(x_1, x_2)$ subject to the satisfaction of the feasible set. This approach has long been used because of its elegance, although its empirical value is doubtful for practical reasons. Implementation of traditional analysis requires one obtaining a reliable mathematical representation of $Z(x_1, x_2)$ which demands very precise information not available in many scenarios. In other words, $Z(x_1, x_2)$ is often unknown. For example, an economist can rarely deal with a consumer's empirically elicited utility function, and still less with an empirical social utility function.

Moreover, it might be useful to remember that the logical soundness of the utility function has been severely criticized in several decision contexts. Some of the assumptions necessary to the acceptance of the existence of a utility function (comparability, reflexivity, transitivity, and continuity of preferences) seem questionable; e.g. the continuity of preferences in many decision making problems within the field of natural resources planning. However, this controversial topics will not be considered in the present paper. We do not seek to modify the core of the traditional paradigm since it is commonly accepted in the literature and has proved its explanatory power for the economist's intellectual necessities. On the contrary, we are looking for a bridge between utility functions and operational research, improving the potentiality of the traditional paradigm in economic applications.

8.3 Reviewing the CP Model

A first task in CP is to define the ideal point, also called the point of anchor values. This ideal is an infeasible utopian target, in which each CP variable reaches its optimum. No decision maker can optimize all the variables simultaneously. Imagine your ideal is to drive your car as fast as possible and simultaneously to minimize road accident risk, but this utopian aspiration is quite impossible to achieve. Then you look for a compromise between speed and security. Consider the following example related to SRI policies: A country which can produce food of two different types of farming:

(a) Organic food by agricultural systems that do not use chemical fertilizers and pesticides. This farming involves an SRI objective.
(b) Conventional food from crops in which chemical fertilizers and pesticides are used. This farming does not involve an SRI objective.

By allocating all the agricultural resources to organic food, the country can attain x_1^* units of food, whereas by allocating all the agricultural resources to conventional food, the country can reach x_2^* units. Hence, the obviously unattainable utopian basket (x_1^*, x_2^*), would be the CP ideal point. The country's dream consists in simultaneously producing x_1^* organic food and x_2^* conventional food; however, this dream is impossible. Indeed, the country can produce a mix (x_1, x_2) such as $T(x_1, x_2) = k$, where T is an efficient frontier whose extreme points are $(x_1^*, 0)$ and $(0, x_2^*)$.

Under similar situations, the basic structure of a CP choice is not (8.1) but the following alternative, which is not devoid of realism:

$$\begin{aligned} &\max Z = Z(x_1, x_2) \\ &\text{subject to } T(x_1, x_2) = k \end{aligned} \tag{8.2}$$

where $C_O(x_1, x_2)$ means the search for an compromise point along the T frontier. There is not a single rigid criterion for solving (8.2). Among many others, a simple way of compromising is obtained by taking:

$$x_1/x_1^* = x_2/x_2^* \tag{8.3}$$

However, there is a general criterion which is widely accepted in the literature: the decision-maker seeks a compromise solution as close as possible to the ideal point, the so called Zeleny's axiom of choice (Zeleny 1982). To achieve this closeness, a family of distance functions is introduced into the analysis. In consequence, the structure of a CP problem under Zeleny's axiom can be summarized as follows:

$$\begin{aligned} &\min L_p = [w_1^p(x_1 - x_1^*)^p + w_2^p(x2 - x_2^*)^p]^{1/p} \\ &\text{subject to } T(x_1, x_2) = k \\ &0 \leq x_1 \leq x_1^*, \quad 0 \leq x_2 \leq x_2^* \end{aligned} \tag{8.4}$$

where (x_1^*, x_2^*) is the ideal point which is usually derived from $T(x_1^*, 0) = k$ and $T(0, x_2^*) = k$; (w_1, w_2) is the vector of weights attached to both magnitudes; and p is a parameter defining the family of distance functions $1 \leq p \leq \infty$.

In CP, weights w_1 and w_2 can play two different roles: (i) shadow prices for normalizing both x_1 and x_2 magnitudes in order to make their aggregation possible; (ii) preferential indexes, when utility functions are not considered in the analysis. In this paper weights will only be used for normalizing purposes, since utility functions involve the preferential scheme.

For several values of the parameter p different baskets which are closest to the ideal point are obtained. Yu (1973) demonstrated that for the bi-criteria case the distance function L_∞, is monotone nondecreasing for $1 \leq p \leq \infty$. Thus, L_1 and L_∞ metrics define a subset of the attainable frontier, known as the compromise set.

The other best-compromise solutions fall between those corresponding to L_1 and L_∞ metrics, i.e., $L_p \in [L_1, L_\infty]$. Baskets on the compromise set enjoy some useful economic properties, such as feasibility, Paretian efficiency, independence of irrelevant alternatives, etc. (Yu 1985, Ch.4)

It is worth pointing out that Eq. (8.3) is a particular case of Eq. (8.4) when $p = \infty$ and weights are inversely proportional to the values (i.e. $w_1/w_2 = x_2^*/x_1^*$) as can easily be proved (see Ballestero and Romero 1991).

8.3.1 An Example of CP Setting from Economic and Ethical Objectives

Political leaders in a country usually pursue economics growth policies together with ethical policies. Electors and media can then wonder if the political programs are able to combine economic and ethical objectives in a coherent way by looking for a compromise between goals. Those programs which promise an ideal achievement of maximizing incompatible goals are not earnest and can be judged as demagogic. Suppose that the Y party is preparing an electoral program from the following objectives.

(i) Economic policy. To increase Gross Domestic Product (variable x_1) as much as possible. Domestic product to be reached should not be less than x_{1*} expressed in real terms to assure a reasonable income guaranteeing a decent standard of living to people.
(ii) Ethical policy. To increase environmental protection (variable x_2) as much as possible. This is needed to meet targets such as sustainable growth, rational use of natural resources, health, and low pollution. For this purpose, the index of environmental protection should not be less than x_{2*} scalarized units.

This involves a trade-off between (i) and (ii), so that more x_2 can be only obtained in exchange for less x_1 and viceversa. Electoral programs should consider the moral impossibility of promising ideal paradises, which overlook the trade off. In mathematical terms the trade off is measured on the Paretian efficient frontier (8.1), namely:

$$T(x_1, x_2) = k$$

In Fig. 8.1, curve *ABCD* represents this trade-off in its general formulation. Since more x_1 implies less x_2, the curve is decreasing. Concerning concavity, the shape of the curve is highlighted as follows. If x_1 has a value close to 0, then x_1 can strongly increase in exchange for a slight loss of x_2. Therefore, we have an almost horizontal slope at point D. On the contrary, suppose that x_1 reaches a high value close to OA. Then, slight increments in x_1 involve abrupt losses of x_2, so that the slope at point A is almost vertical.

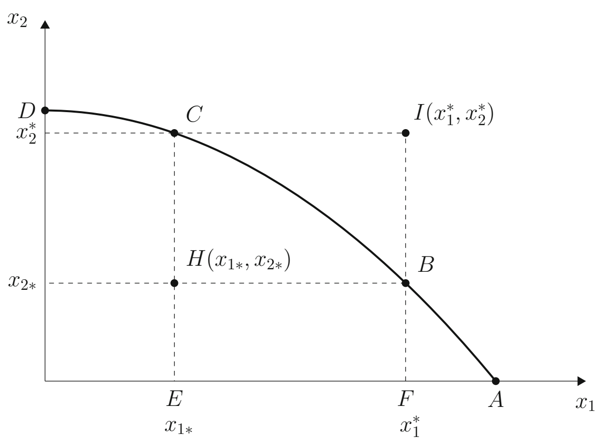

Fig. 8.1 Gross Domestic Product and environmental protection dilemma: CP setting

Figure 8.1 describes the CP setting. Ideal point $I(x_1^*, x_2^*)$ and anti-ideal point $H(x_{1*}, x_{2*})$ are graphed in connection with the efficient frontier.

OE is the minimum level of gross domestic product to assure acceptable lower limits for consumption, investment and employment. This is the x_{1*} anti-ideal value. Vertical line EC determines the ideal or anchor value $x_2^* = FI$.
OG is the minimum level of environmental protection to assure acceptable lower limits for critical environmental parameters. This is the x_{2*} anti-ideal value. Horizontal line GB determines the ideal or anchor value $x_1^* = FI$.

Given preference weights w_1 and w_2 for objectives (i) and (ii), the compromise solution is the frontier point which minimizes distance (8.4).

There is an ongoing issue that movements along the frontier curve can cause changes in this frontier. As shown in Fig. 8.2, the frontier could then shift upward to position $A'B'C'D'$ or downward to position $A''B''C''D''$. Consider a CP setting in which x$_1$ is company's income in aggregate terms while x_2 is an index of social protection including social security, subsidies, holidays and any other government initiative of social welfare. Conservative parties contend that the frontier curve will shift downward if x_2 is set high. This is because high levels of social protection discourage private investment. If so, social protection could finally turn out to be less than before due to downward shifts. Social democratic parties do not agree with this paradox. They contend that the frontier curve will keep unchanged or will shift upward because productivity increases with social welfare. To look into pros and cons of these political programs lies outside the limits of this book.

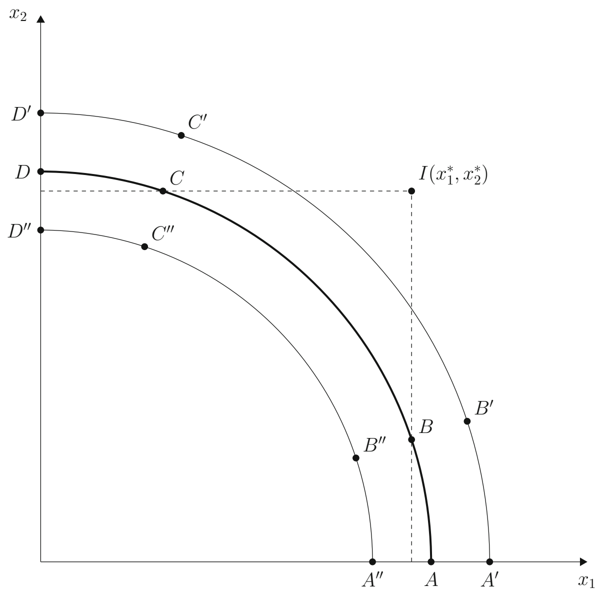

Fig. 8.2 Company's income and social protection dilemma: Frontier shifting

8.3.2 CP Proxy for the Decision Maker's Utility Function

Now, our purpose is to show how to use CP model (8.4) to help solve difficult problem (8.1). Before rushing into formal statements, we will highlight the issue in an intuitive way. To minimize CP distance (8.4) is equivalent to maximizing the following CP utility function:

$$\max((w_1 x_1)^P + (w_2 x_2)^p)^{1/p} \tag{8.5}$$

under satiation conditions $x_1 \leq x_1^*$ and $x_2 \leq x_2^*$

CP utility (8.5) is non-linear non-additive for p values other than 1. It is worth noting that additive utility does not satisfy important properties in economic analysis. Satiation at the ideal point is also meaningful. Given a utility map, satiation means that you will reach a utility top. To assume the existence of this utility top is more realistic that assuming non-satiation, which would involve that you will never reach the top.

As proven in the MCDM literature, the CP maximum (8.5) lies on the Yu compromise set on the T efficient frontier. This property is extended by the following theorem:

Theorem 1 *Under plausible assumptions, the Lagrangean maximum of utility Z with two attributes lies on the Yu compromise set on the T efficient frontier.*

A proof can be found elsewhere. See e.g. Ballestero and Romero (1991).

8.3.3 SRI Example: Carbon Pollution from a Power Plant

Imagine a conventional thermal power plant which uses coal as energy source. Pollution from this plant is very high. This specially affects tourists in the summer and people living in the area who spend leisure time outside their homes. Faced with this problem, the manager of the power company looks for a compromise between environmental and profitability goals, which are defined as follows.

(i) Environmental objective. To stop the activity of the plant for some weeks in the summer.
(ii) Profitability objective. To work the remaining weeks of the year.

Hereafter, we denote by h_1 and h_2 the yearly hours of activity and temporary closure time, respectively.

According to the European Emission Trading Scheme (EU ETS), a CO_2 emission limitation target C_0 tonnes per year is established for this kind of power plants. Let

C be the yearly level of CO_2 pollution from the plant. In this legal framework, the following cases can occur:

(a) $C = C_0$. As the pollution level from the plant is equal to the target, the company is authorised to work during the year without incurring any penalty. The company does not receive any premium either.
(b) $C > C_0$. Then, the company's activity is authorised if and only if the company purchases Certified Emission Reduction credits (CERs) for the gap $(C - C_0)$ from the primary market. These purchases are made at price P established by the competitive market, which involves an extra cost of $P(C - C_0)$ monetary units for the company.
(c) $C < C_0$. Then, the company can sell CERs amounting to $(C_0 - C)$ in the primary market at price P, which means an extra earning of $P(C_0 - C)$ monetary units for the company.

Annual earnings after interest, taxes, depreciation and amortization are:

$$Y = y(1-t)h_1 + P(C_0 - C) = y(1-t)h_1 + P(C_0 - ch_1) \quad (8.6)$$

where y denotes earnings per hour after interest, amortization and depreciation but before taxes; t denotes corporate tax rate; and c is CO_2 pollution per hour from the plant.

To look for a compromise between objectives (i) and (ii), the following CP model is formulated. See setting in Fig. 8.3.

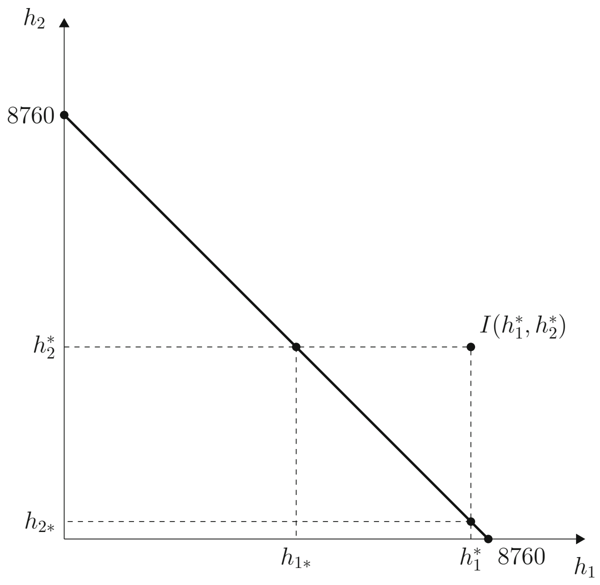

Fig. 8.3 CP setting and anchor values for environmental and profitability goals

$$h_1 + h_2 = 365 \times 24 = 8{,}760 \text{ hours per year} \tag{8.7}$$

Ideal point (h_1^*, h_2^*) and anti-ideal point (h_{1*}, h_{2*}) are stated as follows.

h_1^* is the maximum number of yearly hours to work by the plant. In our case, $h_1^* = (365 - 15) \times 24 = 8{,}400$ h, as the plant should stop for around 2 weeks for maintenance and control, regardless of the temporary closure time to meet environmental objective (i).

h_{1*} is the minimum number of yearly hours that the plant can work. To estimate it, we ask the company's manager on the minimum level of earnings that the company is willing to accept. Let Y_0 be this level. From Eq. (8.6), we have:

$$y(1-t)h_1 + P(C_0 - ch_1) \geq Y_0 \tag{8.8}$$

From Eq. (8.8) we obtain:

$$h_{1*} = \min h_1 = \frac{Y_0 - PC_0}{y(1-t) - cP} \tag{8.9}$$

where the variables took in year 2013 the following numerical values: $y = 4{,}500$ monetary units per hour; $t = 0.19$; $P = 4$ monetary units per CO_2 tonne, which was the market price for CERs; $C_0 = 10{,}000{,}000$ tonnes per year, which was the CO_2 emission limitation target for the plant; $c = 833$ tonnes per hour, which was CO_2 pollution from the plant; $Y_0 = 41{,}500{,}000$ monetary units a year. This amount was elicited by a dialogue between the analyst and the power plant manager who discloses that 41,500,000 monetary units was the minimum earning acceptable by the company. By specifying Eq. (8.9) with these numerical values, we get:

$$h_{1*} = \frac{41{,}500{,}000 - 4 \times 10{,}000{,}000}{4{,}500(1 - 0.19) - 833 \times 4} = 4{,}792$$

Moreover, we have:

$$h_1^* + h_{2*} = 8{,}760 \; h_{1*} + h_2^* = 8{,}760$$

These equations yield:

$$h_{2*} = 8{,}760 - 8{,}400 = 360 \quad \text{hours per year}$$

$$h_2^* = 8{,}760 - 4{,}792 = 3{,}968 \quad \text{hours per year}$$

From this setting, the CP model is defined as follows:

$$\begin{aligned}
&\min = [w_1^p(8{,}400 - h_1^*)^p + w_2^p(3{,}968 - h_2^*)^p]^{1/p} \\
&\text{subject to } h_1 + h_2 = 8{,}760 \\
&360 \leq h_2 \leq 3{,}968
\end{aligned} \tag{8.10}$$

From preference weights $\mathrm{w}_1 = 0.6$ and $\mathrm{w}_2 = 0.4$ for the objectives, together with the Euclidean metric $p = 2$, this model is solved by Lingo 11.0, which yields $h_1 = 7{,}290$ h for the activity time and $h_2 = 1{,}470$ h for the temporary closure time.

A sensitivity analysis can highlight robustness of the model with respect to metric p. If the decision maker's risk aversion for random changes in the variables is very strong, then a higher metric should be used. Readers can check the results.

8.4 Linear–Quadratic Composite Metric: Advanced Approaches

We here minimize the distance between utility at the CP ideal point and the utility at a frontier point on the criteria setting. This meaningful distance is treated by Taylor expansion around the ideal point, thus obtaining the linear–quadratic composite metric. Aggressive decision makers prefer large risky achievements but the conservative ones prefer prudent balanced solutions, which are far away from aggressive corner points. Linear–quadratic composite metric looks for a compromise between these aggressive and conservative objectives.

The manufactures are often interested in blending materials to achieve industrial products able to satisfy marketing criteria. Suppose a manufacturer who wants to obtain blends of materials by considering a set of marketing and SRI criteria such as quality standards, obsolescence, special necessities of customer segments, environmental requirements, and others. In this problem, every criterion can be associated with a decision variable. For example, a criterion such as environmental requirements is associated with a decision variable such as the amount of a given polluting material. From this correspondence, the level of the jth criterion can be measured by the x_j decision variable. The space of decision variables and the space of criteria coincide.

Let $(x_1, x_2, \ldots, x_j, \ldots, x_n)$ be a CP setting of criteria/decision variables, where every criterion behaves as "the more the better". Every x_j is greater than (or equal to) zero. In this setting, the ideal point is $I(x_1^*, x_2^*, \ldots, x_j^*, \ldots, x_n^*)$, where x_j^* is the highest feasible value of the jth criterion. As well known, the CP objective function is given by the distance function of metric p (between 1 and ∞) as follows:

$$Z = \left[\sum_{j=1}^{n} w_j{}^p (x_j^* - x_j)^p\right]^{1/p} \tag{8.11}$$

to be minimized subject to an efficient frontier and the non-negativity conditions, which is equivalent to:

$$\max U_Z = K - Z \tag{8.12}$$

subject to the efficient frontier, where K is a constant sufficiently large to assure that the difference (8.12) is positive. Function (8.12) has the meaning of a special utility function that will be called the Zeleny–Yu utility.

8.4.1 Utility Function: An Extended Approach

A question arises whether CP can be stated from more general utility functions than (8.12). Let

$$U(x_1, x_2, \ldots, x_j, \ldots, x_n) = \sum_{j=1}^{n} U_j(x_j) \tag{8.13}$$

be a general additive utility function of criteria on the CP map. From Eq. (8.13), consider the following constrained minimization:

$$\min \Delta = \sum_{j=1}^{n} U_j(x_j^*) - \sum_{j=1}^{n} U_j(x_j) \tag{8.14}$$

subject to the efficient frontier and the non–negativity conditions, where Δ is the deviation between the utility value at the ideal point and the utility value at a generic point on the efficient frontier.

Indeed, minimizing the Δ deviation can be viewed as the core of an extended compromise programming. A Taylor expansion around the ideal point with the Lagrange form of the remainder term converts Eq. (8.14) into:

$$\begin{aligned}\min \Delta = {} & \sum_{j=1}^{n} U_j(x_j^*) - \\ & \left[\sum_{j=1}^{n} U_j(x_j^*) + \sum_{j=1}^{n} U_j{}^{(1)}(x_j^*)(x_j - x_j^*) + 0.5 \sum_{j=1}^{n} U_j{}^{(2)}(\varepsilon_j)(x_j - x_j^*)^2\right] \\ = {} & \sum_{j=1}^{n} U_j{}^{(1)}(x_j^*)(x_j - x_j^*) - 0.5 \sum_{j=1}^{n} U_j{}^{(2)}(\varepsilon_j)(x_j - x_j^*)^2\end{aligned} \tag{8.15}$$

where $U_j^{(1)}$ and $U_j^{(2)}$ are the first and second partial derivatives of the utility function with respect to the j th variable, namely, the first and second derivatives of the utility term $U_j(x_j)$ of the additive function. Notice that expansion (8.15) does not state a mere approximate value but represents the exact value according to the following Taylor's theorem: the Lagrange form of the remainder term states that a numberε between x_j and x_j^* does exist if U_j is a function which is continuously differentiable on the closed interval $[x_j, x_j^*]$ and twice differentiable on the open interval (x_j, x_j^*).

Since the ε_j terms are unknown variables, we use a proxy for *min* Δ, which consists in replacing every ε_j by the respective x_j^* ideal value. Then, Eq. (8.15) becomes:

$$\min \Delta = \sum_{j=1}^{n} U_j(x_j^*) -$$
$$\left[\sum_{j=1}^{n} U_j(x_j^*) + \sum_{j=1}^{n} U_j{}^{(1)}(x_j^*)(x_j - x_j^*) + 0.5 \sum_{j=1}^{n} U_j{}^{(2)}(x_j^*)(x_j - x_j^*)^2 \right]$$
$$= \sum_{j=1}^{n} (-1) U_j{}^{(1)}(x_j^*)(x_j - x_j^*) - 0.5 \sum_{j=1}^{n} U_j{}^{(2)}(x_j^*)(x_j - x_j^*)^2 \tag{8.16}$$

8.4.2 Normalizing the x_j Criteria

For practical convenience, each x_j criterion is normalized by the following equation:

$$y_j = \frac{x_j - x_{j*}}{x_j^* - x_{j*}} \tag{8.17}$$

where x_j^* and x_{j*} are the ideal and anti-ideal values, respectively, while the normalized y_j ranges between 0 and 1. Therefore, the normalized ideal is $y_j^* = 1$ while the normalized anti-ideal is $y_{j*} = 0$ for all j. In the special and frequent case of zero anti-ideal, Eq. (8.17) becomes $y_j = x_j / x_{j*}$. Later, this normalization will be use to transform Eq. (8.16). Our next task is to specify the partial derivatives in an understandable CP language.

8.4.3 Normalizing the Objective Function

By normalizing the x_j variables according to the previous section, objective function (8.16) becomes:

$$\min \Delta = \sum_{j=1}^{n} U_j^{(1)}(1)(1-y_j) - 0.5\sum_{j=1}^{n} U_j^{(2)}(1)(1-y_j)^2 \qquad (8.18)$$

8.4.4 Linear–Quadratic CP Achievement Function

The statement in Sects. 8.4.1–8.4.3 leads to a particular utility–based compromise objective function, which is called the linear-quadratic CP achievement (Ballestero 2007). This is interesting, not only for blend design but also to straightforwardly solve a wide range of compromise programs of management. Linear-quadratic CP achievement can be stated with any number of criteria. Hereafter, the analysis will be limited by considering only two criteria, as this special case often appear in managerial and finance applications. In the previous subsections, the CP approach has been entirely developed in a rather general utility framework. No particular type of utility, such as exponential, logarithmic, power or any other, has been used. However, to derive the linear-quadratic CP achievement in our context we use the classic Cobb–Douglas utility function U_{CD} with two criteria as an operational tool, namely:

$$U_{CD} = y_1^{V_1} y_2^{V_2}; \quad 0 \leq V_1,\ V_2 \leq 1; \quad V_1 + V_2 = 1 \qquad (8.19)$$

whose first and second partial derivatives specified at the ideal point $y_j^* = 1$ $(j = 1, 2)$ are:

$$\begin{aligned} U_{CD_j}^{(1)}(1) &= V_j \\ U_{CD_j}^{(2)}(1) &= V_j(V_j - 1); \quad j = 1, 2 \end{aligned} \qquad (8.20)$$

By introducing partial derivatives (8.20) into CP objective function (8.18), we obtain the linear-quadratic CP achievement function:

$$\begin{aligned} \min \Delta = {} & (V_1(1-y_1) + V_2(1-y_2)) + \\ & 0.5\left(V_1(1-V_1)(1-y_1)^2 + V_2(1-V_2)(1-y_2)^2\right) \end{aligned} \qquad (8.21)$$

which should be optimized subject to the normalized efficient frontier.

Parameters V_1 and V_2 have a clear meaning of preference weights for the respective criteria. This can be checked by rating utility (8.19) in its logarithmic form:

$$\log U_{CD} = V_1 \log y_1 + V_2 \log y_2 \tag{8.22}$$

Therefore, V_1 and V_2 can be elicited through a dialogue about preferences between the analyst and the decision maker. An example of this dialogue is as follows.

Analyst. Do you prefer the $j = 1$ criterion to the $j = 2$ criterion, or viceversa?
Decision Maker. I prefer $j = 1$.
Analyst. How much?
Decision Maker. I give 3 points to $j = 1$ and 2 points to $j = 2$.

From this dialogue, we get $V_1 = 3/5$ and $V_2 = 2/5$.

8.4.5 A Case of Polymer Industry

This section describes an example of industrial blending in which the manufacturer's decisions are made from marketing criteria rather than from SRI criteria. Later in this chapter, an example involving SRI objectives will be developed. In both examples, the linear-quadratic CP achievement (8.21) will be used.

Suppose a manufacturer who faces with the problem of blending three types of polymer fibers. The product should have two desirable properties, tenacity and elongation at break, which are the CP criteria. Let q_i $(i = 1, 2, 3)$ be the percentage of the ith fiber in the blend, these percentages being the decision variables. Laboratory experiments to evaluate and enhance the product design show that tenacity in the blend is governed by the following equation:

$$x_1 = 1{,}132 \left(\sum_{i=1}^{3} t_i q_i \right) - 0.012 \left(\sum_{i=1}^{3} t_i q_i \right)^2 \tag{8.23}$$

where t_i is tensile strength per unit of the ith fiber. In Eq. (8.23), the negative quadratic term is due to a synergy effect which negatively influences tenacity in the blend. Elongation at break is roughly evaluated by the equation:

$$x_2 = \sum_{i=1}^{3} e_i q_i \tag{8.24}$$

where e_i is elongation at break per unit of the ith fiber. In Table 8.1, both t_i and e_i values $(i = 1, 2, 3)$ are recorded. Note the high inverse correlation between

Table 8.1 Polymer fiber blends: Basic data and the efficient frontier

Fiber code	Efficient frontier																
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)
1	20	92	0.41	0.44	0.48	0.51	0.55	0.58	0.62	0.65	0.69	0.73	0.76	0.8	0.83	0.87	0.9
2	29	72	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	64	21	0.59	0.56	0.52	0.49	0.45	0.42	0.38	0.35	0.31	0.27	0.24	0.2	0.17	0.13	0.1
x1			26.68	26.61	26.48	26.29	26.05	25.75	25.39	24.97	24.5	23.97	23.38	22.73	22.03	21.26	20.44
x2			50	52.5	55	57.5	60	62.5	65	67.5	70	72.5	75	77.5	80	82.5	85
y1			1	0.99	0.97	0.94	0.9	0.85	0.79	0.73	0.65	0.57	0.47	0.37	0.25	0.13	0
y2			0	0.07	0.14	0.21	0.29	0.36	0.43	0.5	0.57	0.64	0.71	0.79	0.86	0.93	1

Column description: (1) tensile strength, measured in Newton per square millimeter or Megapascal; (2) Elongation at break, measured in percentage; (3)–(17) 15 efficient blends with their corresponding tenacity (x_1) and elongation at break (x_2) values, as well as the respective normalized tenacity (y_1) and normalized elongation at break (y_2) values.

tenacity (ability of the product to withstand tension, measured in Newton per square millimetre) and elongation at break (ability to stretch, measured in percentage).

First step. Determine the efficient frontier by maximizing tenacity subject to parametric levels of elongation at break from Eq. (8.24) and the constraint:

$$\sum_{i=1}^{3} q_i = 1 \tag{8.25}$$

saying that the sum of percentages is equal to unity. The parametric values of elongation x_2 range between 50 and 85 with intervals of 2.5. If elongation decreases below 50, then tenacity decreases below 26.68, and therefore, elongation values lower than 50 should be discarded as they lead to results worse than the combination ($x_1 = 50; x_2 = 26.68$) (see Table 8.1). Moreover, if elongation increases above 85, then tenacity decreases below 20, and consequently, using fiber number 1 alone is better than using a blend (check this in Table 8.1, upper half). In sum, the trade-off between elongation and tenacity appears only over the range (50, 85).

Second step. Normalize (standardize) both x_1 and x_2 criteria by Eq. (8.17), where the ideal and anti-ideal values are 26.68 and 20.44 for tenacity, while they are 85 and 50 for elongation at break. In Table 8.1, bottom half, the normalized values y_1 and y_2 are displayed.

Third step. Elicit preferences and attitudes to imbalance by the dialogue stated in Sect. 8.4.4 for the special case $n = 2$, thus obtaining Y_1 % and $(100 - Y_1)$ % from the decision maker's answer. In Table 8.2, several possible answers are considered, and therefore, their corresponding Y_1 percentages are displayed as parametric values.

Fourth step. For each possible answer, minimize Eq. (8.21) once specified numerically with the respective Y_1 percentage, this minimization being subject to the normalized efficient frontier given in Table 8.1.

Results are shown in Table 8.2 for a scale of parametric values Y_1 from 0.1 to 99.9 %. As a robustness analysis, this table visualizes each interval of parameter Y_1 for which the solutions given by the composite metric do not change, the intervals being separated by horizontal lines. For comparison, the solutions with metrics $p = 1$ and $p = \infty$ are also recorded in the same table. As these results come from a mere example of two criteria, their validity is very limited. They are summarized as follows:

(a) Metric $p = 1$ gives a wide range of corner solutions with larger achievements. Therefore, this is not a fitting metric for decision makers with significant aversion to imbalance

Table 8.2 Solutions with the composite metric, metric 1 and the infinity norm for different Y_1 percentages

	Composite metric		$h = 1$		Infinity norm		
Y_1	y_1	y_2	y_1	y_2	y_1	y_2	*Error*
0.1	0	1	0	1	0	1	0.1
1	0	1	0	1	0	1	1
2	0	1	0	1	0	1	2
3	0	1	0	1	0	1	3
5	0	1	0	1	0.13	0.93	−2.3
10	0	1	0	1	0.13	0.93	2.4
15	0	1	0	1	0.25	0.86	−0.65
20	0	1	0	1	0.25	0.86	3.8
24	0	1	0	1	0.37	0.79	−0.84
25	0.13	0.93	0	1	0.37	0.79	0
27	0.13	0.93	0	1	0.37	0.79	1.68
28	0.25	0.86	0	1	0.37	0.79	2.52
29	0.25	0.86	0	1	0.37	0.79	3.36
30	0.37	0.79	0	1	0.37	0.79	4.2
35	0.37	0.79	0	1	0.47	0.71	−0.3
38	0.37	0.79	0.37	0.79	0.47	0.71	2.16
39	0.57	0.64	0.37	0.79	0.47	0.71	2.98
40	0.57	0.64	0.37	0.79	0.47	0.71	3.8
45	0.57	0.64	0.57	0.64	0.57	0.64	−0.45
46	0.57	0.64	0.57	0.64	0.57	0.64	0.34
47	0.65	0.57	0.73	0.5	0.57	0.64	1.13
48	0.73	0.5	0.73	0.5	0.57	0.64	1.92
49	0.73	0.5	0.73	0.5	0.57	0.64	2.71
50	0.73	0.5	0.73	0.5	0.57	0.64	3.5
57	0.73	0.5	0.85	0.36	0.65	0.57	1.46
58	0.73	0.5	0.85	0.36	0.65	0.57	2.24
59	0.79	0.43	0.9	0.29	0.65	0.57	3.02
60	0.85	0.36	0.9	0.29	0.73	0.5	−3.8
64	0.85	0.36	0.9	0.29	0.73	0.5	−0.72
65	0.85	0.36	0.9	0.29	0.73	0.5	0.05
66	0.85	0.36	0.9	0.29	0.73	0.5	0.82
67	0.9	0.29	0.94	0.21	0.73	0.5	1.59
70	0.9	0.29	0.97	0.14	0.79	0.43	−2.4
75	0.9	0.29	0.97	0.14	0.79	0.43	1.5
76	0.94	0.21	0.97	0.14	0.79	0.43	2.28
79	0.94	0.21	0.99	0.07	0.85	0.36	−1.59
80	0.97	0.14	0.99	0.07	0.85	0.36	−0.8
85	0.97	0.14	0.99	0.07	0.9	0.29	−2.15
86	0.97	0.14	0.99	0.07	0.9	0.29	−1.34
87	0.99	0.07	0.99	0.07	0.9	0.29	−0.53

(continued)

Table 8.2 (continued)

	Composite metric		$h = 1$		Infinity norm		
Y_1	y_1	y_2	y_1	y_2	y_1	y_2	*Error*
90	0.99	0.07	1	0	0.9	0.29	1.9
92	0.99	0.07	1	0	0.94	0.21	−0.8
93	1	0	1	0	0.94	0.21	0.05
99	1	0	1	0	0.99	0.07	0.06
99.9	1	0	1	0	1	0	−0.1

(b) Given the discrete frontier in this example, the infinity norm only provides rough solutions affected by errors of considerable magnitude. From the basic equation of the infinity norm:

$$Y_1 = (1 - y_1) = (100 - Y_1)(1 - y_2) \tag{8.26}$$

the error corresponding to each frontier point (y_1, y_2) is computed as the difference between both sides of this equation, namely:

$$e(Y_1) = Y_1(1 - y_1) - (100 - Y_1)(1 - y_2) \tag{8.27}$$

In Table 8.2, the rough solution given by the infinity norm for each Y_1 preference weight is the frontier point minimizing error (8.27) in the set of the 15 frontier points recorded in Table 8.1, last two rows. Indeed, the errors shown in the last column of Table 8.2 do not allow us to draw conclusions on the accuracy of results from the infinity norm, which appears to be rather inapplicable. In particular, the rough solution $y_1 = 0$ and $y_2 = 1$ for the first four rows in Table 8.2 is affected by a percentage error of 200 %, and therefore, using here the infinity norm is unacceptable. The same occurs with the last row of the table. Only for $Y_1 = 25$, zero error is obtained.

Conclusions

We describe Compromise Programming (CP) as a multicriteria technique related to utility. Because optimizing utility is quite difficult in practice, the existence of a linkage between utility U and CP is appealing to construct a CP proxy for utility optimization. Analytically, CP can be viewed as a method to maximize the decision maker's utility function subject to an efficient frontier of criteria an the non-negativity constraints in a deterministic context. The lack of information necessary to build a reliable utility function is mitigated by resorting to the technical information derived from the efficient frontier. Regarding MCDM literature we explain how the CP solution lies on the Yu compromise set on the T efficient frontier.

(continued)

A first application of CP considering economic and ethical objectives is developed. The CP setting represents the Gross Domestic Product and environmental protection dilemma. Indeed, to reach the maximum Gross Domestic Product together with the maximum environmental protection is a utopian infeasible basket of reference because these objectives cannot be simultaneously reached.

An SRI numerical example is presented in order to describe how to apply the CP technique. The manager in a thermal power plant looks for a compromise between environmental and profitability goals facing with the problem of defining time for activity and time for temporary closure. Ideal and antiideal points are established by the manager. Then, the CP model is formulated taking into account preferences for the objectives. The selected CP metric is the Euclidean metric $p = 2$.

Compromise programming (CP) is viewed as the maximization of the decision maker's additive utility function (whose arguments are the criteria under consideration) subject to an efficient frontier of criteria and the non-negativity constraints in a deterministic context. This is equivalent to minimizing the difference between utility at the ideal point and utility at a frontier point on the criteria map, a meaningful statement as minimizing distances to the utopia is the ethos of compromise programming. By Taylor expansion of utility around the ideal point, the distance to the utopia becomes the weighted sum of linear and quadratic CP distances, which gives us the composite metric. While the linear terms pursue achievement, the quadratic ones pursue balanced (non-corner) solutions. Because some decision makers fear imbalance while others prefer large achievements even to the detriment of balance. Section 8.4 defines an aversion to imbalance ratio, so that the composite linear-quadratic metric should conform to this ratio depending on the decision maker's preferences and attitudes.

This composite metric seems to be appealing to analysts and users, not only because of its utility foundation but also because practitioners can easily specify the objective function without undertaking the unsolved problem of determining the best metric.

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