# Intuitionistic Fuzzy Rough Approximation Operators Determined by Intuitionistic Fuzzy Triangular Norms

Wei-Zhi Wu[1,2], Shen-Ming Gu[1,2], Tong-Jun Li[1], and You-Hong Xu[1,2]

[1] School of Mathematics, Physics and Information Science,
Zhejiang Ocean University, Zhoushan, Zhejiang, 316022, P.R. China
[2] Chongqing Key Laboratory of Computational Intelligence,
Chongqing University of Posts and Telecommunications, Chongqing, 400065, China
{wuwz,gsm,litj,xyh}@zjou.edu.cn

**Abstract.** In this paper, relation-based intuitionistic fuzzy rough approximation operators determined by an intuitionistic fuzzy triangular norm $T$ are investigated. By employing an intuitionistic fuzzy triangular norm $T$ and its dual intuitionistic fuzzy triangular conorm, lower and upper approximations of intuitionistic fuzzy sets with respect to an intuitionistic fuzzy approximation space are first introduced. Properties of $T$-intuitionistic fuzzy rough approximation operators are then examined. Relationships between special types of intuitionistic fuzzy relations and properties of $T$-intuitionistic fuzzy rough approximation operators are further explored.

**Keywords:** Approximation operators, Intuitionistic fuzzy rough sets, Intuitionistic fuzzy sets, Intuitionistic fuzzy triangular norms, Rough sets.

## 1 Introduction

Rough set theory [8] is a new mathematical approach to deal with insufficient and incomplete information. The basic structure of rough set theory is an approximation space consisting of a universe of discourse and a binary relation imposed on it. Based on the approximation space, the notions of lower and upper approximation operators can be constructed. Using the concepts of lower and upper approximations in rough set theory, knowledge hidden in information tables may be unravelled and expressed in the form of decision rules.

One of the main directions in the research of rough set theory is naturally the generalization of concepts of Pawlak rough set approximation operators. Many authors have generalized the notion of rough set approximations by using non-equivalence binary relations. Other authors have also generalized the notion of rough set approximations into the fuzzy environment, and the results are called rough fuzzy sets (fuzzy sets approximated by a crisp approximation space) and fuzzy rough sets (fuzzy or crisp sets approximated by a fuzzy approximation space). As a more general case of fuzzy sets, the concept of intuitionistic fuzzy

D. Miao et al. (Eds.): RSKT 2014, LNAI 8818, pp. 653–662, 2014.
DOI: 10.1007/978-3-319-11740-9_60 

(IF for short) sets, which was originated by Atanassov [1], has played a useful role in the research of uncertainty theories. Unlike a fuzzy set, which gives a degree of which element belongs to a set, an IF set gives both a membership degree and a nonmembership degree. Obviously, an IF set is more objective than a fuzzy set to describe the vagueness of data or information. The combination of IF set theory and rough set theory is a new hybrid model to describe the uncertain information and has become an interesting research issue over the years (see e.g. [2, 3, 5, 6, 9–12, 14–17]).

It is well-known that the dual properties of lower and upper approximation operators are of particular importance in the analysis of mathematical structures in rough set theory. The dual pairs of lower and upper approximation operators in the rough set theory are strongly related to the interior and closure operators in topological space, the necessity (box) and possibility (diamond) operators in modal logic, and the belief and plausibility functions in the Dempster-Shafer theory of evidence. On the other hand, we know that there are a lot of triangular norms which have been widely used in fuzzy set research. It should be noted that fuzzy inference results often depend upon the choice of the triangular norm. For analyzing uncertainty in complicated fuzzy systems, dual pairs of lower and upper fuzzy rough approximations defined by arbitrary triangular norms in rough set theory have been developed [7, 13]. According to this research line, the main objective of this paper is to present the study of IF rough sets determined by IF triangular norms. We will define a dual pair of lower and upper $T$-IF rough approximation operators and examine their essential properties.

## 2 Basic Notions Related to Intuitionistic Fuzzy Sets

In this section we recall some basic notions and previous results about intuitionistic fuzzy sets which will be used in the later parts of this paper.

Throughout this paper, $U$ will be a nonempty set called the universe of discourse. The class of all subsets (respectively, fuzzy subsets) of $U$ will be denoted by $\mathcal{P}(U)$ (respectively, by $\mathcal{F}(U)$). In what follows, $1_y$ will denote the fuzzy singleton with value 1 at $y$ and 0 elsewhere; $1_M$ will denote the characteristic function of a crisp set $M \in \mathcal{P}(U)$. For any $A \in \mathcal{F}(U)$, the complement of $A$ will be denoted by $\sim A$, i.e. $(\sim A)(x) = 1 - A(x)$ for all $x \in U$.

We first review a lattice on $[0,1] \times [0,1]$ originated by Cornelis *et al.* [4].

**Definition 1.** *Denote*

$$L^* = \{(x_1, x_2) \in [0,1] \times [0,1] \mid x_1 + x_2 \leq 1\}. \tag{1}$$

*A relation* $\leq_{L^*}$ *on* $L^*$ *is defined as follows:* $\forall (x_1, x_2), (y_1, y_2) \in L^*$,

$$(x_1, x_2) \leq_{L^*} (y_1, y_2) \iff x_1 \leq y_1 \text{ and } x_2 \geq y_2. \tag{2}$$

The relation $\leq_{L^*}$ is a partial ordering on $L^*$ and the pair $(L^*, \leq_{L^*})$ is a complete lattice with the smallest element $0_{L^*} = (0, 1)$ and the greatest element $1_{L^*} = (1, 0)$ . The meet operator $\wedge$ and the join operator $\vee$ on $(L^*, \leq_{L^*})$ linked to the ordering $\leq_{L^*}$ are, respectively, defined as follows: $\forall (x_1, x_2), (y_1, y_2) \in L^*$,

$$\begin{array}{l}(x_1, x_2) \wedge (y_1, y_2) = (\min(x_1, y_1), \max(x_2, y_2)),\\(x_1, x_2) \vee (y_1, y_2) = (\max(x_1, y_1), \min(x_2, y_2)).\end{array} \tag{3}$$

And for any index set $J$ and $a_j = (x_j, y_j) \in L^*$, $j \in J$, we define

$$\begin{array}{l}\bigwedge_{j\in J} a_j = \bigwedge_{j\in J} (x_j, y_j) = (\bigwedge_{j\in J} x_j, \bigvee_{j\in J} y_j),\\\bigvee_{j\in J} a_j = \bigvee_{j\in J} (x_j, y_j) = (\bigvee_{j\in J} x_j, \bigwedge_{j\in J} y_j).\end{array} \tag{4}$$

Meanwhile, an order relation $\geq_{L^*}$ on $L^*$ is defined as follows: $\forall x = (x_1, x_2), y = (y_1, y_2) \in L^*$,

$$(y_1, y_2) \geq_{L^*} (x_1, x_2) \iff (x_1, x_2) \leq_{L^*} (y_1, y_2), \tag{5}$$

and

$$x = y \iff x \leq_{L^*} y \text{ and } y \leq_{L^*} x. \tag{6}$$

For $(x_1, x_2) \in L^*$, we define the complement element of $(x_1, x_2)$ in $L^*$ as follows:

$$1_{L^*} - (x_1, x_2) = (x_2, x_1). \tag{7}$$

Since $\leq_{L^*}$ is a partial ordering, the order-theoretic definitions of conjunction and disjunction on $L^*$ called *IF triangular norm* (IF $t$-norm for short) and *IF triangular conorm* (IF $t$-conorm for short) are introduced as follows:

**Definition 2.** *An IF triangular norm (IF t-norm for short) on $L^*$ is an increasing, commutative, associative mapping $T : L^* \times L^* \to L^*$ satisfying $T(1_{L^*}, x) = x$ for all $x \in L^*$.*

**Definition 3.** *An IF triangular conorm (IF t-conorm for short) on $L^*$ is an increasing, commutative, associative mapping $S : L^* \times L^* \to L^*$ satisfying $S(0_{L^*}, x) = x$ for all $x \in L^*$.*

Obviously, the greatest IF $t$-norm (respectively, the smallest IF $t$-conorm) with respect to (w.r.t.) the ordering $\leq_{L^*}$ is min (respectively, max), defined by $\min(x, y) = x \wedge y$ (respectively, $\max(x, y) = x \vee y$) for all $x, y \in L^*$.

An IF $t$-norm $T$ and an IF $t$-conorm $S$ on $L^*$ are said to be *dual* if

$$\begin{array}{l}T(x, y) = 1_{L^*} - S(1_{L^*} - x, 1_{L^*} - y), \forall x, y \in L^*,\\S(x, y) = 1_{L^*} - T(1_{L^*} - x, 1_{L^*} - y), \forall x, y \in L^*.\end{array} \tag{8}$$

Each IF $t$-norm $T$ can be associated two functions $T_1$, $T_2 : L^* \times L^* \to [0, 1]$ which are defined as follows:

$$T(a, b) = (T_1(a, b), T_2(a, b)), \forall a, b \in L^*. \tag{9}$$

Likewise, from an IF $t$-conorm $S$ on $L^*$, we can derive two functions $S_1$, $S_2$ : $L^* \times L^* \to [0,1]$ which satisfy the following equation.

$$S(a,b) = (S_1(a,b), S_2(a,b)), \forall a,b \in L^*. \tag{10}$$

Since $T$ and $S$ are increasing, by Eq. (2) we can conclude

**Proposition 1.** *If $T$ is an IF $t$-norm on $L^*$ and $S$ the IF $t$-conorm on $L^*$ dual to $T$, then $T_1$ and $S_1$ are increasing and $T_2$ and $S_2$ are decreasing for both arguments.*

**Proposition 2.** *If $T$ is an IF $t$-norm on $L^*$, and $S$ is the IF $t$-conorm on $L^*$ dual to $T$. Then*
(1) $S_1(a,b) = T_2(1_{L^*} - a, 1_{L^*} - b)$, *for all* $a,b \in L^*$.
(2) $S_2(a,b) = T_1(1_{L^*} - a, 1_{L^*} - b)$, *for all* $a,b \in L^*$.
(3) $T_1(a,b) = S_2(1_{L^*} - a, 1_{L^*} - b)$, *for all* $a,b \in L^*$.
(4) $T_2(a,b) = S_1(1_{L^*} - a, 1_{L^*} - b)$, *for all* $a,b \in L^*$.

*Proof.* For $a = (a_1,a_2), b = (b_1,b_2) \in L^*$, by Eqs. (7) and (8), we have

$$\begin{aligned} S(a,b) &= (S_1(a,b), S_2(a,b)) = 1_{L^*} - T(1_{L^*} - a, 1_{L^*} - b) \\ &= 1_{L^*} - (T_1(1_{L^*} - a, 1_{L^*} - b), T_2(1_{L^*} - a, 1_{L^*} - b)) \\ &= 1_{L^*} - (T_1((a_2,a_1),(b_2,b_1)), T_2((a_2,a_1),(b_2,b_1))) \\ &= (T_2((a_2,a_1),(b_2,b_1)), T_1((a_2,a_1),(b_2,b_1))) \\ &= (T_2(1_{L^*} - a, 1_{L^*} - b), T_1(1_{L^*} - a, 1_{L^*} - b)). \end{aligned}$$

Thus (1) and (2) hold. Similarly, we can conclude (3) and (4).

**Definition 4.** [1] *Let a set $U$ be fixed. An IF set $A$ in $U$ is an object having the form*

$$A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle \mid x \in U\},$$

*where $\mu_A : U \to [0,1]$ and $\gamma_A : U \to [0,1]$ satisfy $0 \le \mu_A(x) + \gamma_A(x) \le 1$ for all $x \in U$, and $\mu_A(x)$ and $\gamma_A(x)$ are, respectively, called the degree of membership and the degree of non-membership of the element $x \in U$ to $A$. The family of all IF subsets in $U$ is denoted by $\mathcal{IF}(U)$. The complement of an IF set $A$ is defined by $\sim A = \{\langle x, \gamma_A(x), \mu_A(x)\rangle \mid x \in U\}$.*

It can be observed that an IF set $A$ is associated with two fuzzy sets $\mu_A$ and $\gamma_A$. Here, we denote $A(x) = (\mu_A(x), \gamma_A(x))$, then it is clear that $A \in \mathcal{IF}(U)$ iff $A(x) \in L^*$ for all $x \in U$. Obviously, a fuzzy set $A = \{\langle x, \mu_A(x)\rangle \mid x \in U\}$ can be identified with the IF set of the form $\{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle \mid x \in U\}$. Thus an IF set is indeed an extension of a fuzzy set.

Some basic operations on $\mathcal{IF}(U)$ are introduced as follows [1]: for $A, B, A_i \in \mathcal{IF}(U)$, $i \in J$, $J$ is an index set,

- $A \subseteq B$ iff $\mu_A(x) \le \mu_B(x)$ and $\gamma_A(x) \ge \gamma_B(x)$ for all $x \in U$,
- $A \supseteq B$ iff $B \subseteq A$,
- $A = B$ iff $A \subseteq B$ and $B \subseteq A$,
- $A \cap B = \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\gamma_A(x), \gamma_B(x))\rangle \mid x \in U\}$,

- $A \cup B = \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\gamma_A(x), \gamma_B(x))\rangle \mid x \in U\}$,
- $\bigcap_{i\in J} A_i = \{\langle x, \bigwedge_{i\in J} \mu_{A_i}(x), \bigvee_{i\in J} \gamma_{A_i}(x)\rangle \mid x \in U\}$,
- $\bigcup_{i\in J} A_i = \{\langle x, \bigvee_{i\in J} \mu_{A_i}(x), \bigwedge_{i\in J} \gamma_{A_i}(x)\rangle \mid x \in U\}$.

For $(\alpha, \beta) \in L^*$, $\widehat{(\alpha,\beta)}$ will be denoted by the constant IF set: $\widehat{(\alpha,\beta)}(x) = (\alpha, \beta)$, for all $x \in U$. For any $y \in U$ and $M \in \mathcal{P}(U)$, IF sets $1_y$, $1_{U-\{y\}}$, and $1_M$ are, respectively, defined as follows: for $x \in U$,

$$\mu_{1_y}(x) = \begin{cases} 1, & \text{if} \quad x = y, \\ 0, & \text{if} \quad x \neq y. \end{cases} \quad \gamma_{1_y}(x) = \begin{cases} 0, & \text{if} \quad x = y, \\ 1, & \text{if} \quad x \neq y. \end{cases}$$

$$\mu_{1_{U-\{y\}}}(x) = \begin{cases} 0, & \text{if} \quad x = y, \\ 1, & \text{if} \quad x \neq y. \end{cases} \quad \gamma_{1_{U-\{y\}}}(x) = \begin{cases} 1, & \text{if} \quad x = y, \\ 0, & \text{if} \quad x \neq y. \end{cases}$$

$$\mu_{1_M}(x) = \begin{cases} 1, & \text{if} \quad x \in M, \\ 0, & \text{if} \quad x \notin M. \end{cases} \quad \gamma_{1_M}(x) = \begin{cases} 0, & \text{if} \quad x \in M, \\ 1, & \text{if} \quad x \notin M. \end{cases}$$

The IF universe set is $U = 1_U = \widehat{(1,0)} = \widehat{1_{L^*}} = \{\langle x, 1, 0\rangle \mid x \in U\}$ and the IF empty set is $\emptyset = \widehat{(0,1)} = \widehat{0_{L^*}} = \{\langle x, 0, 1\rangle \mid x \in U\}$.

By using $L^*$, IF sets on $U$ can be represented as follows: for $A, B, A_j \in \mathcal{IF}(U)(j \in J, J$ is an index set), $x, y \in U$, and $M \in \mathcal{P}(U)$

- $A(x) = (\mu_A(x), \gamma_A(x)) \in L^*$,
- $U(x) = (1, 0) = 1_{L^*}$,
- $\emptyset(x) = (0, 1) = 0_{L^*}$,
- $x = y \Longrightarrow 1_y(x) = 1_{L^*}$ and $1_{U-\{y\}}(x) = 0_{L^*}$,
- $x \neq y \Longrightarrow 1_y(x) = 0_{L^*}$ and $1_{U-\{y\}}(x) = 1_{L^*}$,
- $x \in M \Longrightarrow 1_M(x) = 1_{L^*}$,
- $x \notin M \Longrightarrow 1_M(x) = 0_{L^*}$,
- $A \subseteq B \Longleftrightarrow A(x) \leq_{L^*} B(x), \forall x \in U \Longleftrightarrow B(x) \geq_{L^*} A(x), \forall x \in U$,
- $\big(\bigcap_{j\in J} A_j\big)(x) = \bigwedge_{j\in J} A_j(x) = \big(\bigwedge_{j\in J} \mu_{A_j}(x), \bigvee_{j\in J} \gamma_{A_j}(x)\big) \in L^*$,
- $\big(\bigcup_{j\in J} A_j\big)(x) = \bigvee_{j\in J} A_j(x) = \big(\bigvee_{j\in J} \mu_{A_j}(x), \bigwedge_{j\in J} \gamma_{A_j}(x)\big) \in L^*$.

For two IF sets $A$, $B \in \mathcal{IF}(U)$, we define two IF sets $A \cap_T B$ and $A \cup_{\mathcal{S}} B$ as follows:

$$\begin{aligned} (A \cap_T B)(x) &= T(A(x), B(x)), \ x \in U, \\ (A \cup_{\mathcal{S}} B)(x) &= \mathcal{S}(A(x), B(x)), \ x \in U. \end{aligned} \tag{11}$$

It can easily be verified that

$$A \cup_{\mathcal{S}} B = \sim ((\sim A) \cap_T (\sim B)). \tag{12}$$

## 3 *T*-Intuitionistic Fuzzy Rough Approximation Operators

In this section, by employing an IF $t$-norm $T$ and its dual IF $t$-conorm $S$ on $L^*$, we will define the lower and upper approximations of IF sets w.r.t. an arbitrary IF approximation space and discuss properties of $T$-IF rough approximation operators.

**Definition 5.** *Let $U$ and $W$ be two nonempty universes of discourse. A subset $R \in \mathcal{IF}(U \times W)$ is referred to as an IF binary relation from $U$ to $W$, namely, $R$ is given by*

$$R = \{\langle (x, y), \mu_R(x, y), \gamma_R(x, y)\rangle \mid (x, y) \in U \times W\}, \tag{13}$$

*where $\mu_R : U \times W \to [0, 1]$ and $\gamma_R : U \times W \to [0, 1]$ satisfy $0 \leq \mu_R(x, y) + \gamma_R(x, y) \leq 1$ for all $(x, y) \in U \times W$. We denote the family of all IF relations from $U$ to $W$ by $\mathcal{IFR}(U \times W)$. An IF relation $R \in \mathcal{IFR}(U \times W)$ is said to be serial if $\bigvee_{y \in W} R(x, y) = 1_{L^*}$ for all $x \in U$. If $U = W$, $R \in \mathcal{IFR}(U \times U)$ is called an IF binary relation on $U$. $R \in \mathcal{IFR}(U \times U)$ is said to be reflexive if $R(x, x) = 1_{L^*}$ for all $x \in U$. $R$ is said to be symmetric if $R(x, y) = R(y, x)$ for all $x, y \in U$. $R$ is said to be $T$-transitive if $\bigvee_{y \in U} T(R(x, y), R(y, z)) \leq_{L^*} R(x, z)$ for all $x, z \in U$, where $T$ is an IF $t$-norm.*

Throughout this section, we always assume that $T$ is an IF continuous $t$-norm on $L^*$ and $S$ the IF $t$-conorm dual to $T$.

**Definition 6.** *Let $U$ and $W$ be two non-empty universes of discourse and $R$ an IF relation from $U$ to $W$, then the triple $(U, W, R)$ is called a generalized IF approximation space. For $A \in \mathcal{IF}(W)$, the $T$-lower and $T$-upper approximations of $A$, denoted as $\underline{R}(A)$ and $\overline{R}(A)$, respectively, w.r.t. the approximation space $(U, W, R)$ are IF sets of $U$ and are, respectively, defined as follows:*

$$\underline{R}(A)(x) = \bigwedge_{y \in W} S(1_{L^*} - R(x, y), A(y)), \quad x \in U. \tag{14}$$

$$\overline{R}(A)(x) = \bigvee_{y \in W} T(R(x, y), A(y)), \quad x \in U. \tag{15}$$

*The operators $\underline{R}$, $\overline{R} : \mathcal{IF}(W) \to \mathcal{IF}(U)$ are, respectively, referred to as $T$-lower and $T$-upper IF rough approximation operators of $(U, W, R)$, and the pair $(\underline{R}(A), \overline{R}(A))$ is called the $T$-IF rough set of $A$ w.r.t. $(U, W, R)$.*

**Theorem 1.** *Let $(U, W, R_1)$ and $(U, W, R_2)$ be two IF approximation spaces, if $R_1 \subseteq R_2$, then*
(1) *$\overline{R_1}(A) \subseteq \overline{R_2}(A)$ for all $A \in \mathcal{IF}(W)$.*
(2) *$\underline{R_2}(A) \subseteq \underline{R_1}(A)$ for all $A \in \mathcal{IF}(W)$.*

*Proof.* It can be deduced directly from Definition 6.

**Definition 7.** *If $U, V, W$ are three nonempty sets, $R_1$ is an IF relation from $U$ to $V$, and $R_2$ is an IF relation from $V$ to $W$, we define an IF relation from $U$ to $W$, denoted $R_1 \circ R_2$, called the $T$-composition of $R_1$ and $R_2$ as follows:*

$$R_1 \circ R_2(x, z) = \bigvee_{y \in V} T(R_1(x, y), R_2(y, z)), \forall (x, z) \in U \times W. \tag{16}$$

**Theorem 2.** *Let* $(U, V, R_1)$ *and* $(V, W, R_2)$ *be two IF approximation spaces, then*
(1) $\overline{R_1 \circ R_2}(A) = \overline{R_1}(\overline{R_2}(A))$ *for all* $A \in \mathcal{IF}(W)$.
(2) $\underline{R_1 \circ R_2}(A) = \underline{R_1}(\underline{R_2}(A))$ *for all* $A \in \mathcal{IF}(W)$.

*Proof.* (1) For any $u \in U$, we have

$$\begin{aligned}
\overline{R_1}(\overline{R_2}(A))(u) &= \bigvee_{v\in V} T(R_1(u,v), \overline{R_2}(A)(v)) \\
&= \bigvee_{v\in V} T(R_1(u,v), \bigvee_{w\in W} T(R_2(v,w), A(w))) \\
&= \bigvee_{v\in V} \bigvee_{w\in W} T(R_1(u,v), T(R_2(v,w), A(w))) \\
&= \bigvee_{w\in W} \bigvee_{v\in V} T(T(R_1(u,v), R_2(v,w)), A(w)) \\
&= \bigvee_{w\in W} T( \bigvee_{v\in V} T(R_1(u,v), R_2(v,w)), A(w)) \\
&= \bigvee_{w\in W} T(R_1 \circ R_2(u,w), A(w)) \\
&= \overline{R_1 \circ R_2}(A)(u).
\end{aligned}$$

Thus, $\overline{R_1 \circ R_2}(A) = \overline{R_1}(\overline{R_2}(A))$.
(2) It is similar to the proof of (1).

**Theorem 3.** *Let* $(U, W, R)$ *be an IF approximation space,* $T$ *an IF t-norm on* $L^*$*, and* $S$ *the IF t-conorm dual to* $T$*, then*
(IFL1) $\underline{R}(A) = \sim \overline{R}(\sim A)$ *for all* $A \in \mathcal{IF}(W)$.
(IFU1) $\overline{R}(A) = \sim \underline{R}(\sim A)$ *for all* $A \in \mathcal{IF}(W)$.

*Proof.* For any $A \in \mathcal{IF}(W)$ and $x \in U$, by Eq. (4) and Proposition 2 we have

$$\begin{aligned}
\overline{R}(\sim A)(x) &= \bigvee_{y\in W} T(R(x,y), (\sim A)(y)) \\
&= \bigvee_{y\in W} (T_1(R(x,y), 1_{L^*} - A(y)), T_2(R(x,y), 1_{L^*} - A(y))) \\
&= ( \bigvee_{y\in W} T_1(R(x,y), 1_{L^*} - A(y)), \bigwedge_{y\in W} T_2(R(x,y), 1_{L^*} - A(y))) \\
&= ( \bigvee_{y\in W} S_2(1_{L^*} - R(x,y), A(y)), \bigwedge_{y\in W} S_1(1_{L^*} - R(x,y), A(y))) \\
&= 1_{L^*} - ( \bigwedge_{y\in W} S_1(1_{L^*} - R(x,y), A(y)), \bigvee_{y\in W} S_2(1_{L^*} - R(x,y), A(y))).
\end{aligned}$$

Thus

$$\begin{aligned}
(\sim \overline{R}(\sim A))(x) &= 1_{L^*} - \overline{R}(\sim A)(x) \\
&= ( \bigwedge_{y\in W} S_1(1_{L^*} - R(x,y), A(y)), \bigvee_{y\in W} S_2(1_{L^*} - R(x,y), A(y))) \\
&= \bigwedge_{y\in W} (S_1(1_{L^*} - R(x,y), A(y)), S_2(1_{L^*} - R(x,y), A(y))) \\
&= \bigwedge_{y\in W} S(1_{L^*} - R(x,y), A(y)) \\
&= \underline{R}(A)(x).
\end{aligned}$$

Therefore, we conclude (IFL1). Similarly, we can prove that (IFU1) holds.

Properties (IFL1) and (IFU1) in Theorem 3 show that the $T$-IF rough approximation operators $\underline{R}$ and $\overline{R}$ are dual with each other. The following theorem presents some basic properties of $T$-IF rough approximation operators.

**Theorem 4.** *Let $(U, W, R)$ be an IF approximation space. Then the upper and lower $T$-fuzzy rough approximation operators defined in Definition 6 satisfy the following properties: For all $A, B \in \mathcal{IF}(W)$, $A_j \in \mathcal{IF}(W)(\forall j \in J, J$ is an index set), $M \subseteq W$, $(x, y) \in U \times W$ and all $(\alpha, \beta) \in L^*$,*

(IFL2) $\underline{R}(\widehat{(\alpha,\beta)} \cup_S (A) = \widehat{(\alpha,\beta)} \cup_S \underline{R}(A)$.
(IFU2) $\overline{R}\big(\widehat{(\alpha,\beta)} \cap_T A\big) = \widehat{(\alpha,\beta)} \cap_T \overline{R}(A)$.
(IFL3) $\underline{R}(\bigcap_{j\in J} A_j) = \bigcap_{j\in J} \underline{R}(A_j)$.
(IFU3) $\overline{R}(\bigcup_{j\in J} A_j) = \bigcup_{j\in J} \overline{R}(A_j)$.
(IFL4) $A \subseteq B \Longrightarrow \underline{R}(A) \subseteq \underline{R}(B)$.
(IFU4) $A \subseteq B \Longrightarrow \overline{R}(A) \subseteq \overline{R}(B)$.
(IFL5) $\underline{R}(\bigcup_{j\in J} A_j) \supseteq \bigcup_{j\in J} \underline{R}(A_j)$.
(IFU5) $\overline{R}(\bigcap_{j\in J} A_j) \subseteq \bigcap_{j\in J} \overline{R}(A_j)$.
(IFL6) $\underline{R}(W) = U$.
(IFU6) $\overline{R}(\emptyset_W) = \emptyset_U$.
(IFL7) $\underline{R}(1_{W-\{y\}})(x) = 1_{L^*} - R(x, y)$.
(FU7) $\overline{R}(1_y)(x) = R(x, y)$.
(IFL8) $\widehat{(\alpha,\beta)} \subseteq \underline{R}(\widehat{(\alpha,\beta)})$.
(IFU8) $\overline{R}(\widehat{(\alpha,\beta)}) \subseteq \widehat{(\alpha,\beta)}$.
(IFL9) $\underline{R}(1_M)(x) = \bigwedge_{y\notin M} (1_{L^*} - R(x, y))$.
(IFU9) $\overline{R}(1_M)(x) = \bigvee_{y\in M} R(x, y)$.
(IFL10) $\underline{R}(1_{W-\{y\}} \cup_S \widehat{(\alpha,\beta)})(x) = S(1_{L^*} - R(x, y), (\alpha, \beta))$.
(IFU10) $\overline{R}(1_y \cap_T \widehat{(\alpha,\beta)})(x) = T(R(x, y), (\alpha, \beta))$.

*Proof.* The proof for properties of the upper $T$-IF rough approximation operator can be found in [17], and properties of lower $T$-IF rough approximation operator can be deduced directly by employing the dual properties (IFL1) and (IFU1) in Theorem 3.

By using Theorem 5 in [17] and the dualities in Theorem 3, we can obtain following Theorems 5-8, which show that properties of some special IF relations, say serial IF relations, reflexive IF relations, symmetric IF relations, and $T$-transitive IF relations, can be equivalently characterized by properties of the $T$-IF rough approximation operators.

**Theorem 5.** *Let $(U, W, R)$ be an IF approximation space, then*

$$\begin{aligned} R \ \textit{is serial} &\Longleftrightarrow \text{(IFL0)}\ \underline{R}(\widehat{(\alpha,\beta)}) = \widehat{(\alpha,\beta)},\ \forall(\alpha,\beta) \in L^*. \\ &\Longleftrightarrow \text{(IFU0)}\ \overline{R}(\widehat{(\alpha,\beta)}) = \widehat{(\alpha,\beta)},\ \forall(\alpha,\beta) \in L^*. \\ &\Longleftrightarrow \text{(IFL0)}'\ \underline{R}(\emptyset_W) = \emptyset_U. \\ &\Longleftrightarrow \text{(IFU0)}'\ \overline{R}(W) = U. \end{aligned}$$

**Theorem 6.** *Let $(U, R)$ be an IF approximation space (i.e. $R$ is an IF relation on $U$), then*

$$\begin{aligned} R \ \textit{is reflexive} &\Longleftrightarrow \text{(IFLR)}\quad \underline{R}(A) \subseteq A, \quad \forall A \in \mathcal{IF}(U). \\ &\Longleftrightarrow \text{(IFUR)}\quad A \subseteq \overline{R}(A), \quad \forall A \in \mathcal{IF}(U). \end{aligned}$$

**Theorem 7.** *Let $(U, R)$ be an IF approximation space, then*

$$\begin{aligned} R \ \textit{is symmetric} &\Longleftrightarrow \text{(IFLS)}\ \underline{R}(1_{U-\{x\}})(y) = \underline{R}(1_{U-\{y\}})(x), \forall(x,y) \in U \times U. \\ &\Longleftrightarrow \text{(IFUS)}\ \overline{R}(1_x)(y) = \overline{R}(1_y)(x), \forall(x,y) \in U \times U. \end{aligned}$$

**Theorem 8.** *Let $(U, R)$ be an IF approximation space, then*

$$\begin{aligned} R \ \textit{is}\ T-\textit{transitive} &\Longleftrightarrow \text{(IFLT)}\ \underline{R}(A) \subseteq \underline{R}(\underline{R}(A)), \forall A \in \mathcal{IF}(U). \\ &\Longleftrightarrow \text{(IFUT)}\overline{R}(\overline{R}(A)) \subseteq \overline{R}(A), \forall A \in \mathcal{IF}(U). \end{aligned}$$

## 4 Conclusion

We have studied a general type of relation-based intuitionistic fuzzy rough sets determined by IF triangular norms with their dual IF triangular conorms. We have introduced a dual pair of $T$-lower and $T$-upper IF rough approximation operators induced from a generalized IF approximation space. We have presented some properties of $T$-lower and $T$-upper IF rough approximation operators and have also examined essential properties of $T$-IF rough approximation operators corresponding to some special types of IF binary relations. For further study, we will investigate more mathematical structures of the $T$-IF rough approximation operators.

**Acknowledgement.** This work was supported by grants from the National Natural Science Foundation of China (Nos. 61272021, 61075120, 11071284, and 61173181), the Zhejiang Provincial Natural Science Foundation of China (Nos. LZ12F03002 and LY14F030001), and Chongqing Key Laboratory of Computational Intelligence (No. CQ-LCI-2013-01).

## References

1. Atanassov, K.: Intuitionistic Fuzzy Sets: Theory and Applications. Physica-Verlag, Heidelberg (1999)
2. Chakrabarty, K., Gedeon, T., Koczy, L.: Intuitionistic fuzzy rough set. In: Proceedings of 4th Joint Conference on Information Sciences (JCIS), Durham, NC, pp. 211–214 (1998)

3. Cornelis, C., Cock, M.D., Kerre, E.E.: Intuitionistic fuzzy rough sets: at the crossroads of imperfect knowledge. Expert Systems 20, 260–270 (2003)
4. Cornelis, C., Deschrijver, G., Kerre, E.E.: Implication in intuitionistic fuzzy and interval-valued fuzzy set theory: construction, classification, application. International Journal of Approximate Reasoning 35, 55–95 (2004)
5. Huang, B., Li, H.-X., Wei, D.-K.: Dominance-based rough set model in intuitionistic fuzzy information systems. Knowledge-Based Systems 28, 115–123 (2012)
6. Jena, S.P., Ghosh, S.K.: Intuitionistic fuzzy rough sets. Notes on Intuitionistic Fuzzy Sets 8, 1–18 (2002)
7. Mi, J.-S., Leung, Y., Zhao, H.-Y., Feng, T.: Generalized fuzzy rough sets determined by a triangular norm. Information Sciences 178, 3203–3213 (2008)
8. Pawlak, Z.: Rough Sets: Theoretical Aspects of Reasoning about Data. Kluwer Academic Publishers, Boston (1991)
9. Radzikowska, A.M.: Rough approximation operations based on IF sets. In: Rutkowski, L., Tadeusiewicz, R., Zadeh, L.A., Żurada, J.M. (eds.) ICAISC 2006. LNCS (LNAI), vol. 4029, pp. 528–537. Springer, Heidelberg (2006)
10. Rizvi, S., Naqvi, H.J., Nadeem, D.: Rough intuitionistic fuzzy set. In: Proceedings of the 6th Joint Conference on Information Sciences (JCIS), Durham, NC, pp. 101–104 (2002)
11. Samanta, S.K., Mondal, T.K.: Intuitionistic fuzzy rough sets and rough intuitionistic fuzzy sets. Journal of Fuzzy Mathematics 9, 561–582 (2001)
12. Sun, B., Ma, W., Liu, Q.: An approach to decision making based on intuitionistic fuzzy rough sets over two universes. Journal of the Operational Research Society 64, 1079–1089 (2013)
13. Wu, W.-Z.: On some mathematical structures of $T$-fuzzy rough set algebras in infinite universes of discourse. Fundamenta Informaticae 108, 337–369 (2011)
14. Zhang, Z.M.: Generalized intuitionistic fuzzy rough sets based on intuitionistic fuzzy coverings. Information Sciences 198, 186–206 (2012)
15. Zhou, L., Wu, W.-Z.: On generalized intuitionistic fuzzy approximation operators. Information Sciences 178, 2448–2465 (2008)
16. Zhou, L., Wu, W.-Z.: Characterization of rough set approximations in Atanassov intuitionistic fuzzy set theory. Computers and Mathematics with Applications 62, 282–296 (2011)
17. Zhou, L., Wu, W.-Z., Zhang, W.-X.: On characterization of intuitionistic fuzzy rough sets based on intuitionistic fuzzy implicators. Information Sciences 179, 883–898 (2009)