Sub-propositional Fragments of the Interval Temporal Logic of Allen's Relations*

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Abstract. Interval temporal logics provide a natural framework for temporal reasoning about interval structures over linearly ordered domains, where intervals are taken as the primitive ontological entities. The most influential propositional interval-based logic is probably Halpern and Shoham's Modal Logic of Time Intervals, a.k.a. HS. While most studies focused on the computational properties of the syntactic fragments that arise by considering only a subset of the set of modalities, the fragments that are obtained by weakening the propositional side have received very scarce attention. Here, we approach this problem by considering various sub-propositional fragments of HS, such as the so-called Horn, Krom, and core fragment. We prove that the Horn fragment of HS is undecidable on every interesting class of linearly ordered sets, and we briefly discuss the difficulties that arise when considering the other fragments.

1 Introduction

Most temporal logics proposed in the literature assume a point-based model of time, and they have been successfully applied in a variety of fields. However, a number of relevant application domains, such as planning and synthesis of controllers, are characterized by advanced features that are neglected or dealt with in an unsatisfactory way by point-based formalisms. Interval temporal logics provide a natural framework for temporal reasoning about interval structures over linearly (or partially) ordered domains. They take time intervals as the primitive ontological entities and define truth of formulas relative to time intervals, rather than time points; their modalities correspond to various relations between

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pairs of intervals. Applications of interval-based reasoning systems range from hardware and real-time system verification to natural language processing, from constraint satisfaction to planning [1, 11, 20, 23].

The well-known logic HS [16] features a set of modalities that make it possible to express all Allen's interval relations [1]. HS is highly undecidable over most classes of linear orders, and this result motivated the search for (syntactic) HS fragments offering a good balance between expressiveness and decidability/complexity. The few decidable fragments that have been found present complexities that range from NP-complete (in very simple cases) to NExpTime-complete, to ExpSpace-complete, to non-primitive recursive [5, 6, 8, 10, 17–19]. While the classification of fragments of HS in terms of the allowed modal operators can be considered almost completed, sub-propositional fragments of HS have received very scarce attention in the literature. Three propositional restrictions are often mentioned in the context of propositional, first-order, and modal logics, namely the *Horn*, *Krom*, and *core* fragments. They are all based on the *clausal* form of formulas, i.e., implications of the type $(\lambda_1 \wedge \ldots \wedge \lambda_n) \rightarrow (\lambda_{n+1} \vee \ldots \vee \lambda_{n+m})$ and define a particular fragment by limiting the applicability of Boolean operators and the number of literals in the clauses. In the case of modal logics, the restriction to Horn and core clauses can be separated into two cases, that basically differ from each other on the role played by existential modalities (diamonds). In the classical version, one may freely use both existential (diamond) and universal (box) modalities in *positive literals* [13, 14, 22], while in Artale's et. al. version [3] the use of existential modalities is restricted to obtain better computational properties. This duality does not affect the Krom fragment, since the existential modalities can be recovered using only boxes (preserving the satisfiability).

In this paper, we consider the five expressively different sub-propositional fragments of HS that emerge from the above discussion, and we prove that the Horn fragment of HS is undecidable under very weak assumptions of the underlying linear order (in fact, it is undecidable in any class of linear orders where full HS is). While inspired by existing work, our proof, which is the main contribution of this paper, necessarily differs from previous ones due to the limited expressive power of the Horn fragment. We conclude the paper by briefly discussing the reasons that make the Krom and core fragments more difficult to deal with.

2 HS: Syntax and Semantics

Let $\mathbb{D} = \langle D, < \rangle$ be a linearly ordered set. An *interval* over $\mathbb{D}$ is an ordered pair $[x, y]$, where $a, b \in D$ and $a < b$. In this paper, we assume the *strict semantics*, that is, we exclude point intervals and only consider strict intervals. The adoption of the strict semantics instead of the *non-strict semantics*, which includes point intervals, conforms to the definition of interval adopted by Allen in [1], but differs from the one given by Halpern and Shoham in [16]. It has at least two strong motivations: first, a number of representation paradoxes

$\langle A \rangle$	$[x,y]R_A[x',y'] \Leftrightarrow y = x'$
$\langle L \rangle$	$[x,y]R_L[x',y'] \Leftrightarrow y < x'$
$\langle B \rangle$	$[x,y]R_B[x',y'] \Leftrightarrow x = x', y' < y$
$\langle E \rangle$	$[x,y]R_E[x',y'] \Leftrightarrow y = y', x < x'$
$\langle D \rangle$	$[x,y]R_D[x',y'] \Leftrightarrow x < x', y' < y$
$\langle O \rangle$	$[x,y]R_O[x',y'] \Leftrightarrow x < x' < y < y'$

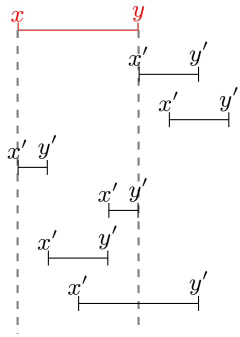

Fig. 1. Allen's interval relations and the corresponding HS modalities

arise when the non-strict semantics is adopted, due to the presence of point intervals, as pointed out in [1]; second, when point intervals are included there seems to be no intuitive semantics for interval relations that makes them both pairwise disjoint and jointly exhaustive. It should be observed that, from the decidability/undecidability/complexity point of view, no differences have ever been found between the two semantic choices; there are no reasons to suspect that sub-propositional strict and non-strict HS restrictions might behave in a different way. If we exclude the identity relation, there are 12 different relations between two strict intervals in a linear order, often called *Allen's relations* [1]: the six relations R_A (adjacent to), R_L (later than), R_B (begins), R_E (ends), R_D (during), and R_O (overlaps), depicted in Fig. 1, and their inverses, that is, $R_{\overline{X}} = (R_X)^{-1}$, for each $X \in \{A, L, B, E, D, O\}$.

We interpret interval structures as Kripke structures, with Allen's relations playing the role of the accessibility relations. Thus, we associate a universal modality $[X]$ and an existential modality $\langle X \rangle$ with each Allen relation R_X. For each $X \in \{A, L, B, E, D, O\}$, the *transposes* of the modalities $[X]$ and $\langle X \rangle$ are the modalities $[\overline{X}]$ and $\langle \overline{X} \rangle$, corresponding to the inverse relation $R_{\overline{X}}$ of R_X. Halpern and Shoham's logic HS [16] is a multi-modal logic with formulas built from a finite, non-empty set $\mathcal{AP}$ of atomic propositions (also referred to as proposition letters), the classical propositional connectives, and a pair of modalities for each Allen relation:

$$\varphi ::= \bot \mid p \mid \neg\varphi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \langle X \rangle\varphi \mid [X]\varphi \mid \langle \overline{X} \rangle\varphi \mid [\overline{X}]\varphi, \tag{1}$$

where $p \in \mathcal{AP}$ and $X \in \{A, L, B, E, D, O\}$. The (strict) semantics of HS is given in terms of *interval models* $M = \langle \mathbb{I}(\mathbb{D}), V \rangle$, where $\mathbb{D}$ is a linear order, $\mathbb{I}(\mathbb{D})$ is the set of all (strict) intervals over $\mathbb{D}$, and V is a *valuation function* $V : \mathcal{AP} \mapsto 2^{\mathbb{I}(\mathbb{D})}$, which assigns to each atomic proposition $p \in \mathcal{AP}$ the set of intervals $V(p)$ on which p holds. The *truth* of a formula on a given interval $[x, y]$ in an interval model M is defined by structural induction on formulas as follows:

- $M, [x, y] \Vdash p$ if and only if $[x, y] \in V(p)$;
- Boolean connectives are dealt with in the standard way;

- $M, [x, y] \Vdash \langle X \rangle \psi$ if and only if there exists $[x', y']$ such that $[x, y] R_X [x', y']$ and $M, [x', y'] \Vdash \psi$;
- $M, [x, y] \Vdash [X] \psi$ if and only if for every $[x', y']$ such that $[x, y] R_X [x', y']$ we have that $M, [x', y'] \Vdash \psi$;
- $M, [x, y] \Vdash \langle \overline{X} \rangle \psi$ if and only if there exists $[x', y']$ such that $[x, y] R_{\overline{X}} [x', y']$ and $M, [x', y'] \Vdash \psi$;
- $M, [x, y] \Vdash [\overline{X}] \psi$ if and only if for every $[x', y']$ such that $[x, y] R_{\overline{X}} [x', y']$ we have that $M, [x', y'] \Vdash \psi$.

Formulas of HS can be interpreted over different classes of interval models, built from different classes of linear orders. Among others, we mention the following important classes of linear orders:

(i) the class of *all* linear orders Lin;
(ii) the class of *dense* linear orders Den, that is, those in which for every pair of distinct points there exists at least one point in between them (e.g., $\mathbb{Q}$ and $\mathbb{R}$);
(iii) the class of *strongly discrete* linear orders Dis, that is, those in which there is a finite number of elements between any two distinct elements;
(iv) the class of *weakly discrete* linear orders WDis, where every element, apart from the greatest element—if it exists—has an immediate successor, and every element, other than the least element—if it exists—has an immediate predecessor (this class includes, e.g., $\mathbb{Z} + \mathbb{Z}$);
(v) the class of *finite* linear orders Fin, that is, those having only finitely many points.

It is important to observe that all classes mentioned above, except Fin, share the common characteristic that possess at least one linear order with an infinitely ascending sequence of points (*infinite chain*).

3 Sub-propositional Fragments of HS

A *syntactical* fragment of HS can be defined by restricting the grammar (1) either by limiting the set of modalities that are included in the language, by limiting nesting of temporal modalities, or by restricting the application of boolean operators. While the first choice (limiting the set of modalities) has been extensively explored, the other two choices has received much scarcer attention. One of the very few examples is [9], where a NP-complete fragment of the temporal logic CDT (which includes HS) has been identified by limiting the nesting of temporal modalities. Here, we study restrictions of interval-based temporal logics along a different line: we limit the applicability of Boolean operators.

To enter into the details we need to start by defining the clausal form of HS-formulas. Clausal forms of modal logics, such as K, can be found, e.g., in [21]. In the context of temporal logics, such as Linear Temporal Logic (LTL), clausal forms [15] have been extensively explored for its applications in automated reasoning. No clausal forms for pure interval-based temporal logics have been proposed so far, to the best of our knowledge. We first introduce the notion of *positive temporal literals*, given by the following grammar:

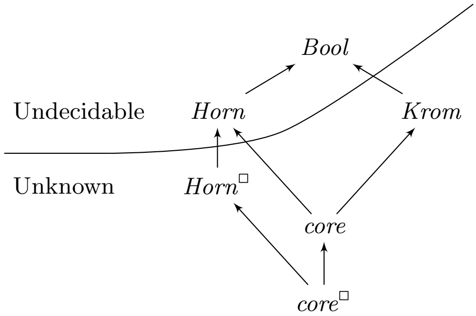

Fig. 2. Relative expressive power between sub-propositional restrictions and their decidability status for HS

$$\lambda ::= \bot \mid p \mid \langle X\rangle\bot \mid [X]\bot \mid \langle X\rangle p \mid [X]p \mid [U]p, \tag{2}$$

where $[X]$ and $\langle X\rangle$ HS modalities, and $[U]$ is the *universal* modality, that can be defined in HS in several ways, such as:

$$[U]\varphi = [\overline{A}][\overline{A}][A]\varphi \wedge [\overline{A}][A]\varphi \wedge [\overline{A}][A][A]\varphi. \tag{3}$$

An HS-formula is said to be in *clausal* form if and only if it can be written following the grammar:

$$\varphi ::= \lambda \mid \neg\lambda \mid [U](\neg\lambda_1 \vee \ldots \vee \neg\lambda_n \vee \lambda_{n+1} \vee \ldots \vee \lambda_{n+m}) \mid \varphi \wedge \varphi. \tag{4}$$

Every HS-formula can be transformed into an equi-satisfiable conjunction of HS-clauses; this transformation is rather standard.

Definition 1. *An* HS*-clause is said to be* Bool *if it can be obtained from (4); it is said to be* Horn *if* $m \leq 1$*; it is said to be* Krom *if* $n + m \leq 2$*; finally, it is said to be* core *if it is both* Bool *and* Horn*, that is, if* $n + m \leq 2$ *and* $m \leq 1$.

Here, we follow the classical definition of modal clauses [14, 15, 21, 22]. In [3], positive literals are defined restricting (2) by eliminating $\langle X\rangle\lambda$. As far as Bool and Krom clauses are concerned, this elimination does not weaken the expressive power; as a matter of fact, formulas of the type $\varphi = \langle X\rangle\psi$ can be recovered by introducing a new propositional letter p_φ, and by using the conjunction of clauses $\neg[X]p_\varphi \wedge [U](p_\varphi \vee \psi)$, which is clearly equi-satisfiable to φ. On the other hand, this is not necessarily true for Horn and core clauses. If we denote the latter fragments by $Horn^\Box$ and $core^\Box$, respectively, the relative expressive power for sub-propositional fragments of HS is as displayed in Fig. 2.

In this paper we prove that restricting to the Horn fragment of HS (HS_{Horn}) is not sufficient to recover decidability. The decidability/undecidability status of the Horn$^\Box$, Krom, and core$^\Box$ fragments is still an open problem. For the sake of comparison, we mention here that for LTL, whose satisfiability problem is PSPACE-complete [25], the complexity does not change neither when we restrict to the Horn fragment [12] nor to the Horn$^\Box$ fragment (Chen and Lin's proof the use of diamond positive literal is not essential). In [3] it is proved that the core$^{\Box}$ fragment of LTL is NP-hard and that the Krom fragment is NP, proving that all remaining restrictions of LTL are, in fact, NP-complete. Finally, only the Horn fragment of the modal logic K has been studied, and its complexity is the same as in the case of full K [22], that is, PSPACE-complete. Although the Krom and the core restrictions of modal and temporal logics have not received much attention in the literature, similar restrictions have been studied at least in the context of Description Logics, both in the atemporal case [2], and in the temporal one [4], justifying the interest in such sub-propositional limitations.

4 Undecidability of HS_{Horn} in the Infinite Case

In this section, we assume that HS_{Horn} is interpreted in any class of linearly ordered sets that possesses at least one linear order with an infinite chain, therefore solving the cases $\mathsf{Lin}, \mathsf{Den}, \mathsf{Dis}$, and WDis. In the next section, we show how to modify the proof to deal with the case of Fin. Our construction adapts to the restricted applicability of Boolean operators the ideas from both the original undecidability proof for full HS [16], as well as the more recent undecidability proofs for fragments of HS [7]. It is based on a reduction of the *non-halting problem* of a deterministic Turing Machine on empty input [24].

A *Turing Machine* is defined as a tuple $\mathcal{A} = (Q, \Sigma, \Gamma, \delta, q_0, q_f)$, where Q is the set of states, q_0 (resp., q_f) is the initial (resp., final) state, Σ is the machine's alphabet that does not contain $\sqcup$ (blank), $\Gamma = \Sigma \cup \{\sqcup\}$ is the tape alphabet, and $\delta : Q \times \Gamma \to Q \times \Gamma \times \{L, R\}$ is the transition function (L, R represent the possible moves on the machine's tape: left, right). Even under the assumption that $\Sigma = \{0, 1\}$ and that the input is empty, both the halting and the non-halting problem for a deterministic Turing Machine are undecidable [24] (as a matter of fact, the former is R.E.-complete, while the latter is CO-R.E.-complete).

Our reduction is based on the idea of representing the *computation history* of $\mathcal{A}$. A *configuration* represents the status of $\mathcal{A}$ at a given moment of the computation, and includes the content of the tape, the position of the reading head, and the current state. Elements of the tape will be placed over *unit* intervals (or, simply, *units*), which we shall denote by u. We shall use the propositional symbol $*$ to separate successive configurations, $0, 1, \sqcup$ to represent tape cells *not* under the machine's head, and the propositional symbols q^c, with $q \in Q \setminus \{q_f\}$ and $c \in \{0, 1, \sqcup\}$, to represent the tape cell under the head and the current (non-final) state of the machine. Let $\mathcal{L}$ be the set $\{0, 1, \sqcup, *\} \cup \{q^c \mid q \in Q \setminus \{q_f\} \wedge c \in \{0, 1, \sqcup\}\} \cup \{q_f\}$, and consider the following group of formulas.

$\phi_1 = \langle A \rangle u \wedge [U](u \to \langle A \rangle u)$ *u-chain exists*

$\phi_2 = \langle A \rangle Start \wedge [U](Start \to \neg\langle \overline{A} \rangle u) \wedge [U](Start \to \neg\langle \overline{L} \rangle u)$ *no u in the past*

$\phi_3 = [U](u \to \neg\langle B \rangle u) \wedge [U](u \to \neg\langle E \rangle u)$ *u-chain unique (1)*

$\phi_4 = [U](u \to \neg\langle D \rangle u) \wedge [U](u \to \neg\langle O \rangle u)$ *u-chain unique (2)*

$\phi_5 = \bigwedge_{l \in \mathcal{L}} [U](l \to u)$ *tape/state propositions and $*$ are units*

$\phi_6 = \bigwedge_{l, l' \in \mathcal{L}, l \neq l'} [U](l \to \neg l')$ *tape/state propositions and $*$ are unique*

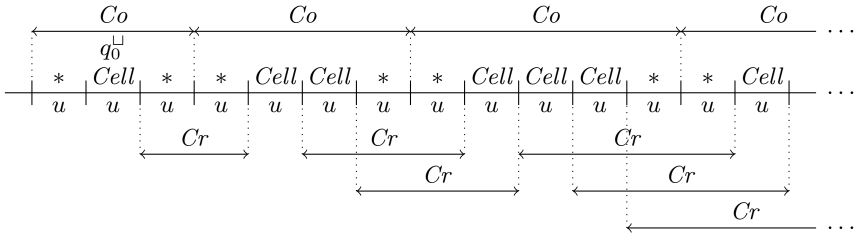

Fig. 3. Configurations

Lemma 1. *Suppose that* $M, [x, y] \Vdash \phi_1 \wedge \ldots \wedge \phi_6$*, then there exists an infinite sequence of points* $y = y_0 < y_1 < \ldots$ *such that:*

1. *for each* $i \geq 0$*,* $M, [y_i, y_{i+1}] \Vdash u$*;*
2. *no other interval* $[z, t]$ *satisfies* u*, unless* $z > y_i$ *for each* $i \geq 0$*;*
3. *for each interval* $[z, t]$*, if* $M, [z, t] \Vdash l$ *and* $l \in \mathcal{L}$*, then* $M, [z, t] \Vdash u$*;*
4. *for each* $l_1, l_2 \in \mathcal{L}$ *where* $l_1 \neq l_2$*,* $M, [z, t] \Vdash l_1$ *implies* $M, [z, t] \not\Vdash l_2$*.*

Proof. Since $M, [x, y] \Vdash \langle A \rangle u$, there exists $y' > y$ such that $[y, y']$ satisfies u; let us call $y_0 = y$ and $y_1 = y'$. From the fact that $M, [x, y] \Vdash [U](u \rightarrow \langle A \rangle u)$ we can easily conclude that the chain $y_0, y_1, \ldots$ exists (proving (1)). Consider now an interval $[z, t]$, such that $z \leq y_i$ for some y_i, $M, [z, t] \Vdash u$, but $[z, t] \neq [y_j, y_{j+1}]$ for each $j \geq 0$. We can assume w.l.o.g. that y_i is the smallest point of the chain such that $z \leq y_i$. Towards a contradiction, assume $z = y_i$; this means that $[z, t]$ is a u-interval that starts or is started by the u-interval $[y_i, y_{i+1}]$, which contradicts ϕ_3. Hence, $z < y_i$, and we can distinguish between the following cases. If $t > y_i$ then $[z, t]$ either contains, is finished by, or overlaps the u-interval $[y_j, y_{j+1}]$ in contradiction with ϕ_3 or ϕ_4. If $t \leq y_i$ and $y_i > y_0$ then $[z, t]$ is contained in the u-interval $[y_{j-1}, y_j]$, in contradiction with ϕ_4. Finally, if $y_i = y_0$ then $t \leq y_0$ and we have a contradiction with ϕ_2 (proving (2)). Thanks to ϕ_5 and ϕ_6, if some $l \in \mathcal{L}$ labels an interval, then it must be a u-interval (proving (3)), and such l is unique (proving (4)). □

Remark 1. Notice that Lemma 1 enables us to use a copy of $\mathbb{N}$, represented by the sequence $y_0, y_1, \ldots$, embedded into the (not necessarily discrete) linearly ordered set under consideration.

Configurations (denoted by *Co*) must be composed by unit intervals; there must be an infinite sequence of them; and each one must be started and finished by a unit labeled by $*$ (see Fig. 3). We use the proposition *Cell* to characterize unit intervals containing a tape symbol (and not an $*$). Consider the following formulas:

$\phi_7 = \langle A \rangle Co \wedge [U](Co \rightarrow \langle B \rangle *) \wedge [U](Co \rightarrow \langle E \rangle *)$ *configuration structure*
$\phi_8 = [U](Co \rightarrow \langle A \rangle Co)$ *configuration sequence*

$\phi_9 = [U](Co \rightarrow [B]\neg Co) \wedge [U](Co \rightarrow [E]\neg Co)$ *configurations relations*
$\phi_{10} = [U](* \rightarrow \neg Cell) \wedge [U](Cell \rightarrow u) \wedge [U]((\langle\overline{D}\rangle Co \wedge u) \rightarrow Cell)$ *Cell iff* $\neg *$

Now, we have to make sure that the initial configuration is exactly as requested by the problem, that is, empty tape with the machine in the initial state q_0. This implies that the first Co must be a sequence of three unit intervals labeled respectively with $*$, $q_0^{\sqcup}$, and $*$. In order to encode exactly this situation, we make use of three new propositions N_1, N_2, and N_3.

$\phi_{11} = \langle A\rangle N_1 \wedge [U](N_1 \rightarrow \langle A\rangle N_2) \wedge [U](N_2 \rightarrow \langle A\rangle N_3)$ *Ns' position*
$\phi_{12} = [U](N_1 \rightarrow *) \wedge [U](N_2 \rightarrow q_0^{\sqcup}) \wedge [U](N_3 \rightarrow *)$ *N_1, N_2, N_3's content*

The length of successive configurations is controlled by the proposition Cr:

$\phi_{13} = [U](Cell \rightarrow \langle A\rangle Cr) \wedge [U](Cr \rightarrow \langle A\rangle Cell)$ *all cells forward-corr to cell*
$\phi_{14} = [U](Cell \wedge \langle A\rangle Cell \rightarrow \langle\overline{A}\rangle Cr)$ *all cells, but the last, back-corr to cell*
$\phi_{15} = [U](Cr \rightarrow [B]\neg Cr) \wedge [U](Cr \rightarrow [E]\neg Cr)$ *correspondences relations (1)*
$\phi_{16} = [U](Cr \rightarrow [D]\neg Cr) \wedge [U](Cr \rightarrow \langle\overline{A}\rangle Cell)$ *correspondences relations (2)*
$\phi_{17} = [U](Co \rightarrow [D]\neg Cr) \wedge [U](Co \rightarrow [E]\neg Cr)$ *config./corr. (1)*
$\phi_{18} = [U](Co \rightarrow [\overline{D}]\neg Cr) \wedge [U](Co \rightarrow [\overline{E}]\neg Cr)$ *config./corr. (2)*

Lemma 2. *Suppose that $M, [x, y] \Vdash \phi_1 \wedge \ldots \wedge \phi_{18}$, and consider the infinite sequence $y_0, y_1, \ldots$, where $y = y_0$, whose existence is guaranteed by Lemma 1. Then, there exists an infinite sequence of indexes $k_0, k_1, \ldots$, such that $y_0 = y_{k_0}$ and:*

1. *$M, [y_0, y_1] \Vdash *$, $M, [y_1, y_2] \Vdash q_0^{\sqcup}$, and $M, [y_2, y_3] \Vdash *$;*
2. *for each $i \geq 0$, $M, [y_{k_i}, y_{k_{(i+1)}}] \Vdash Co$;*
3. *for each $i \geq 0$, $M, [y_{k_i}, y_{(k_i+1)}] \Vdash *$, $M, [y_{(k_{(i+1)}-1)}, y_{k_{(i+1)}}] \Vdash *$;*
4. *for each $i \geq 0$, $j \geq 1$, $M, [y_{(k_i+j)}, y_{(k_i+j+1)}] \Vdash Cell \wedge \neg *$;*
5. *for each $i \geq 0$, $j \geq 2$, $M, [y_{(k_i+j)}, y_{(k_{(i+1)}+j-1)}] \Vdash Cr$;*
6. *$k_1 - k_0 = 3$ and, for every $i > 1$, $0 \leq (k_i - k_{(i-1)}) - (k_{(i-1)} - k_{(i-2)}) \leq 1$;*
7. *no other interval $[z, t]$ satisfies Co nor Cr, unless $z > y_i$ for each $i \geq 0$.*

Proof. Since $M, [x, y] \Vdash \phi_{11} \wedge \phi_{12}$, the first three units of the chain $y_0, y_1, \ldots$ are determined, and are, in this order, $*, q_0^{\sqcup}$, and $*$ (proving (1)).

Now, let us call $y_{k_0} = y_0$. The fact that a chain of Co-intervals starts at y_{k_0} is guaranteed by ϕ_7 and ϕ_8. We prove (2)–(6) by induction on the index i. For the base case, we need to prove that the Co-interval $[y_{k_0}, t]$ is such that $t = y_3$. Suppose, for the sake of contradiction, that $t < y_3$; in this case, we have a contradiction with ϕ_7 and with Lemma 1. If, on the other hand, $t > y_3$, then the Co-interval $[y_{k_0}, t]$ strictly contains $*$ by (1), and this is in contradiction with ϕ_{10}. Therefore, $t = y_3$, and we can set $y_{k_1} = y_3$. It remains to be shown that (5) holds for the base case. We know that $[y_{k_1}, t] = [y_3, t]$ is a Co-interval for some t, that $[y_3, y_4]$ is a $*$-interval, and that $[y_4, y_5]$ is a $Cell$-interval (by ϕ_7, ϕ_{10}). We also know that $[y_2, z]$ is a Cr-interval for some z by ϕ_{13}. We want to prove $z = y_4$. By ϕ_{13} we deduce that $z \geq y_4$. Suppose, by the sake of contradiction, that $z > y_4$. Let us analyze the content of $[y_5, y_6]$. If it is $*$, then $z \geq y_6$, and by ϕ_7 and ϕ_{10}, $[y_3, y_6]$ is a Co-interval, which either ends or is strictly contained in the Cr-interval $[y_2, z]$, a contradiction with ϕ_{18}. If it is $Cell$, by ϕ_{14}, $[s, y_4]$ must be a Cr-interval for some s. It happens that $s \geq y_2$ contradicts ϕ_{15} or ϕ_{16}, and that $s < y_2$ contradicts ϕ_{16} or Lemma 1. Therefore, s cannot be placed anywhere, and $z = y_4$. Thus (5) holds in the base case.

For the inductive case, assume (2)–(6) hold up to $i-1$: we prove that (2)–(6) hold for i. By induction hypothesis, $[y_{k_{(i-1)}}, y_{k_i}]$ is a Co-interval for which (2)–(6) hold. By ϕ_8, $[y_{k_i}, t]$ is a Co-interval for some t. Assume that $(k_i - 1) - (k_{(i-1)} + 1) = n$, that is, assume that the Co-interval $[y_{k_{(i-1)}}, y_{k_i}]$ has precisely n (non-$*$) cells. Since $[y_{(k_i-1)}, y_{(k_i+n)}]$ is a Cr-interval by (5) applied on $i-1$, then $[y_{(k_i+n)}, y_{(k_i+n+1)}]$ is a $Cell$-interval, and $[y_{(k_i+n+1)}, y_{(k_i+n+2)}]$ is either $*$-interval or a $Cell$-interval. In the first case, we let $y_{k_{(i+1)}} = y_{(k_i+n+2)}$ (proving (6) on i); in the second case, $[y_{(k_i+n+2)}, y_{(k_i+n+3)}]$ must be a $*$-interval (otherwise, we apply the same argument as in the base case, showing that there would be a Cr-interval whose starting point cannot be placed anywhere), and therefore we let $y_{k_{(i+1)}} = y_{(k_i+n+3)}$ (again, proving (6) on i). This argument also proves (2)–(4) for i. It remains to be proved that (5) holds the inductive case. To this end, we proceed, again, by induction on j, starting with the base case $j = 2$. By ϕ_{13}, $[y_{(k_i+2)}, z]$ is a Cr-interval. From ϕ_{17} we know $z > y_{k_{(i+1)}}$. Observe that $[y_{k_{(i+1)}}, t]$ is a Co-interval for some t; from ϕ_{18} we know that $z < t$. Towards a contradiction, assume $z > y_{k_{(i+1)}+1}$. Thanks to ϕ_{13}, the point z must start a cell, so that we can assume w.l.g. that $[y_{k_{(i+1)}+2}, y_{k_{(i+1)}+3}]$ is a cell. Then, by ϕ_{14}, $[s, y_{k_{(i+1)}+1}]$ must be a Cr-interval for some s, and by the same argument that we used before, we can prove that s cannot be placed anywhere. Thus, $z = y_{k_{(i+1)}+1}$ (proving (5)) in the base case. Now, it is easy to see that the inductive case proceeds in the same way; we can then conclude that (5) holds for each $j > 2$.

Finally, suppose that $M, [z, t] \Vdash Co$, $z \leq y_i$ for some i, and $[z, t] \neq [y_{k_i}, y_{k_{i+1}}]$ for each i. If $z < y_0$, then, by ϕ_7, z must start some u-interval, which is in contradiction with Lemma 1. Otherwise, $[z, t]$ is a Co-interval either contained, or started by, or ended by another Co-interval, which is in contradiction with ϕ_9. A similar reasoning applies for Cr-intervals (proving (7)). □

The above two lemmas help us to set the underlying structure which we can now use to ensure the correct behaviour of $\mathcal{A}$. We are left with the problem of encoding the transition function δ. To this end, we enrich our language with a new set of propositional letters $\mathcal{L}^t = \{(l_1, l_2, l_3) \mid \forall i(1 \leq i \leq 3 \rightarrow l_i \in \mathcal{L})\}$ ($=\mathcal{L} \times \mathcal{L} \times \mathcal{L}$). Each proposition in $\mathcal{L}^t$ represents the content of three consecutive u-intervals; in this way, we have all information needed to encode δ at each step by reading only one proposition. We proceed as follows: first, the value of three successive cells is encoded in the correct triple from $\mathcal{L}^t$ and placed over a Cr-interval; second, this information is used to label the cells of the next configuration (see Fig 4) by taking into account the transition function δ. In the encoding of δ, we treat as special cases the situations in which: *(i)* the head is at the last cell of the segment of the tape currently shown and the head must be moved to the right and, *(ii)* the head is at the first cell of the tape and the head must be moved to the left. Consider the following formulas,

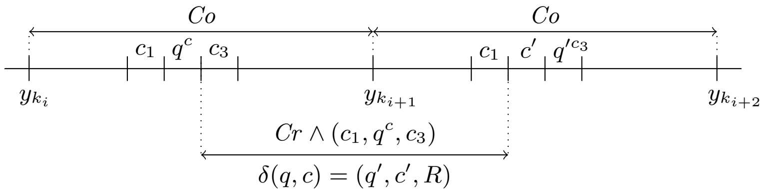

Fig. 4. An example of transition

where $c, c', c_1, c_2, c_3 \in \{0, 1, \sqcup, *\}$ and $q, q' \in Q$ (by a little abuse of notation, we assume that all symbols q_f^c are equal to q_f).

$\phi_{19} = \bigwedge_{l_1,l_2,l_3 \in \mathcal{L}, l_2 \neq *} [U]((\langle \overline{A} \rangle l_1 \wedge l_2 \wedge \langle A \rangle l_3) \rightarrow \langle A \rangle (l_1, l_2, l_3))$ *info transfer*

$\phi_{20} = \bigwedge_{l_1,l_2,l_3 \in \mathcal{L}} [U]((l_1, l_2, l_3) \rightarrow Cr)$ *triple structure*

$\phi_{21} = \bigwedge_{(c_1,c_2,c_3) \in \mathcal{L}^t} [U]((c_1, c_2, c_3) \rightarrow \langle A \rangle c_2)$ *far from the head*

$\phi_{22} = \bigwedge_{(c_1,q^c,c_3) \in \mathcal{L}^t}^{\delta(q,c)=(q',c',R)} [U]((c_1, q^c, c_3) \rightarrow \langle A \rangle c')$ *rightwards (1)*

$\phi_{23} = \bigwedge_{(q^c,c_2,c_3) \in \mathcal{L}^t, c_2 \neq *}^{\delta(q,c)=(q',c',R)} [U]((q^c, c_2, c_3) \rightarrow \langle A \rangle q'^{c_2})$ *rightwards (2)*

$\phi_{24} = \bigwedge_{(c_1,c_2,q^c) \in \mathcal{L}^t}^{\delta(q,c)=(q',c',R)} [U]((c_1, c_2, q^c) \rightarrow \langle A \rangle c_2))$ *rightwards (3)*

$\phi_{25} = \bigwedge_{(c_1,q^c,*) \in \mathcal{L}^t}^{\delta(q,c)=(q',c',R)} [U]((c_1, q^c, *) \rightarrow \langle A \rangle N^{q',c'})$ *last cell (1)*

$\phi_{26} = \bigwedge_{N^{q',c'}} ([U](N^{q',c'} \rightarrow c') \wedge [U](N^{q',c'} \rightarrow \langle A \rangle q'^{\sqcup}))$ *last cell (2)*

$\phi_{27} = \bigwedge_{(c_1,q^c,c_3) \in \mathcal{L}^t, c_1 \neq *}^{\delta(q,c)=(q',c',L)} [U]((c_1, q^c, c_3) \rightarrow \langle A \rangle c')$ *leftwards (1)*

$\phi_{28} = \bigwedge_{(q^c,c_2,c_3) \in \mathcal{L}^t}^{\delta(q,c)=(q',c',L)} [U]((q^c, c_2, c_3) \rightarrow \langle A \rangle c_2)$ *leftwards (2)*

$\phi_{29} = \bigwedge_{(c_1,c_2,q^c) \in \mathcal{L}^t, c_2 \neq *}^{\delta(q,c)=(q',c',L)} [U]((c_1, c_2, q^c) \rightarrow \langle A \rangle q'^{c_2})$ *leftwards (3)*

$\phi_{30} = \bigwedge_{(*,q^c,c_3) \in \mathcal{L}^t}^{\delta(q,c)=(q',c',L)} [U]((*, q^c, c_3) \rightarrow \langle A \rangle q'^{c'})$ *first cell (2)*

We can now prove that our construction works as designed. For a Turing Machine $\mathcal{A}$, we denote by C any $\mathcal{A}$-configuration, univocally determined by the content of the (interesting prefix of the) tape, the position of the reading head, and the state. An $\mathcal{A}$-configuration can be seen as the semantical counterpart of a *Co*-interval in our construction. An $\mathcal{A}$-configuration is said to be *initial* if its state is q_0, and *final* if its state is q_f and for any two $\mathcal{A}$-configurations C, C', we say that C' is the *successor* of C if and only if C' is obtained by C after exactly one application of δ. Finally, a *Co*-interval $[y_{k_i}, y_{k_{i+1}}]$ is said to be *coherent* if and only if the following two conditions apply: *(i)* there exists exactly one u-interval $[y_{(k_i+j)}, y_{(k_i+j+1)}]$ labeled by a symbol of the type q^d, where $q \in Q$ and $d \in \{0, 1, \sqcup\}$; *(ii)* every other interval $[y_{(k_i+h)}, y_{(k_i+h+1)}]$ labeled with *Cell* is also labeled by a symbol $d \in \{0, 1, \sqcup\}$. The following lemma allows us to determine the link between $\mathcal{A}$-configurations and *Co*-intervals.

Lemma 3. *Suppose that* $M, [x, y] \Vdash \phi_1 \wedge \ldots \wedge \phi_{30}$*, consider the infinite sequence* $y_0, y_1, \ldots$*, where* $y = y_0$*, whose existence is guaranteed by Lemma 1, and the sequence* $k_0, k_1, \ldots$ *of indexes whose existence is guaranteed by Lemma 2. Then:*

1. *the Co-interval* $[y_{k_0}, y_{k_1}]$ *represents the initial* $\mathcal{A}$*-configuration when the Turing Machine* $\mathcal{A}$ *has an empty input;*
2. *the Co-interval* $[y_{k_{i+1}}, y_{k_{i+2}}]$ *is coherent for each* $i \geq 0$*;*
3. *the* $\mathcal{A}$*-configuration represented by the Co-interval* $[y_{k_i}, y_{k_{i+1}}]$ *is the successor of the* $\mathcal{A}$*-configuration represented by the Co-interval* $[y_{k_{i-1}}, y_{k_i}]$*, for each* $i > 0$*.*

Proof. The content of the interval $[y_{k_0}, y_{k_1}]$ is set as in Lemma 2, proving its coherence and its status of initial configuration (proving (1)). Points (2) and (3) must be proved together, and by induction; the base case is, as a matter of fact, a consequence of (1) (notice that at the base case, (3) is trivially satisfied). Consider, now, an index $i > 0$ and the $\mathcal{A}$-configuration C represented by the *Co*-interval $[y_{k_{i-1}}, y_{k_i}]$. Assume that the state in C is q, and that the head is reading $c \neq *$. There are several cases to be considered, depending on the movement required by δ, the relative position of the j-th cell ($j \geq 1$) currently read by the head (labeled, by hypothesis, with q^c), and the content c of its adjacent cell.

(a) $\delta(q, c) = (q', c', R)$ and the $(j+1)$-th unit is not $*$. By ϕ_{19} and ϕ_{20}, the unique *Cr*-interval $[y_{(k_{i-1}+j+1)}, y_{(k_i+j)}]$ is also labeled by (c_1, q^c, c_2) for some c_1, c_2. As a consequence, by ϕ_{22}, the j-th unit of $[y_{k_i}, y_{k_{i+1}}]$ is labeled by c'. Now, if $j > 1$, then the $(j-1)$-th unit is a cell, and, by ϕ_{19} and ϕ_{20}, the *Cr*-interval $[y_{(k_{i-1}+j)}, y_{(k_i+j-1)}]$ is labeled by (c_3, c_1, q^c) (for some c_3); therefore ϕ_{24} applies, meaning that the $(j-1)$-th unit of $[y_{k_{i+1}}, y_{k_{i+2}}]$ is labeled by c_1. Similarly, the value of the $(j+1)$-th cell (which cannot be $*$), is set by ϕ_{23}. Now, by the coherence of $[y_{k_{i-1}}, y_{k_i}]$ (inductive hypothesis), every unit strictly before the $(j-1)$-th (excluding the 0-th unit) is a cell, as well as every unit strictly after the $(j+1)$-th (excluding the last one). In the case of $j = 1$, it is clear by Lemma 2, that the first unit of $[y_{k_{i+1}}, y_{k_{i+2}}]$ is labeled by $*$, and the rest of the proof is similar to the case $j > 1$. Therefore, by ϕ_{19} and ϕ_{20}, their corresponding *Cr*-intervals are labeled by triples that do not include q^c for any c, and thanks to ϕ_{21} their corresponding units in the *Co*-interval $[y_{k_i}, y_{k_{i+1}}]$ are cells (and their content, which is preserved, cannot be q^c for any c). Thus, $[y_{k_i}, y_{k_{(i+1)}}]$ is a coherent $\mathcal{A}$-configuration, and its content is obtained by exactly one application of δ (proving (2) and (3) in this case).

(b) $\delta(q, c) = (q', c', R)$ and the $(j+1)$-th unit is $*$. The content of the j-th unit of the i-th *Co*-interval is determined by ϕ_{25} and ϕ_{26}. In particular, the j-th unit of the i-th *Cr*-interval is labeled by $N^{q', c'}$, which implies that it is also labeled by c' (and therefore it is a cell), and that the $(j+1)$-th cell must exist and must be labeled by $q'^{\sqcup}$. The content of the remaining cells, and therefore the coherence of the the i-th *Co*-interval can be then deduced by applying the same argument as before (proving (2) and (3) in this case).

(c) $\delta(q, c) = (q', c', L)$ and the $(j-1)$-th unit is not $*$. In this case, one can proceed as in case *(a)*, only applying ϕ_{27}, ϕ_{28}, and ϕ_{29}.

(d) $\delta(q, c) = (q', c', L)$ and the $(j-1)$-th unit is $*$. In this case, one can proceed as in case *(a)*. The requirement ϕ_{30} plays a major role here: by definition, when δ demands a movement leftwards while the head is on the first cell, the head should not move.

□

The construction is now completed.

Theorem 1. *Let $\mathcal{A}$ be a deterministic Turing Machine. Then, $\mathcal{A}$ diverges on empty input if and only if the* HS_{Horn}*-formula*

$$NotHalts = \phi_1 \wedge \ldots \wedge \phi_{30} \wedge \neg\langle L\rangle q_f$$

is satisfiable on a model with an infinite chain.

Proof. If the formula $\phi_1 \wedge \ldots \wedge \phi_{30} \wedge \neg\langle L\rangle q_f$ is satisfiable, using Lemmas 1–3, we get the desired construction for proving that the Turing Machine $\mathcal{A}$ has an infinite computation on empty input. Conversely, if $\mathcal{A}$ does not halt on empty input, it is a straightforward exercise to prove the satisfiability of the formula $NotHalts$. □

Corollary 1. *The satisfiability problem for* HS_{Horn} *over* Lin*,* Dis*,* WDis*, and* Den *is undecidable.*

5 Undecidability of HS_{Horn} in the Finite Case

When we restrict our attention to the class Fin, the reduction of the non-halting problem for deterministic Turing machines can no longer be carried out, since we cannot represent an infinite computation on a structure with a finite number of points. Nevertheless, undecidability of HS_{Horn} can be proved by a reduction of the *halting* problem for deterministic Turing machines. In this case the formula must represent a finite computation reaching the final state q_f, and, thus, can be satisfied by a finite model. This can be achieved by very small changes in the formulas we used in the previous section, which we briefly summarize here.

First of all, the u-chain now becomes finite, and its encoding can be simplified by exploiting the strong discreteness of the model. Hence, formulas $\phi_1 - \phi_4$ must be replaced by the following two formulas:

$$\psi_1 = [\overline{A}]\bot \wedge \langle A\rangle u \wedge [U]((u \wedge \langle A\rangle\top) \rightarrow \langle A\rangle u) \qquad \textit{u-chain exists}$$
$$\psi_2 = [U](u \rightarrow [B]\bot) \qquad \textit{u is of length } 1$$

Similarly, the chain of Co-intervals must be finite, and hence ϕ_8 must be changed:

$$\psi_8 = [U]((Co \wedge \langle A\rangle\top) \rightarrow \langle A\rangle Co) \qquad \textit{configuration sequence}$$

The structure of Cr-intervals requires a little more attention: now, it is no longer true that every cell of each configuration starts a Cr-interval, but only those cells that are not in the last configuration. This can be achieved by adding a new proposition *Cont* and replacing ϕ_{13} with:

$\psi_{13a} = [U]((Co \wedge \langle A\rangle\top) \rightarrow Cont)$ *mark the non-last Co*
$\psi_{13b} = [U]((Cell \wedge \langle\overline{D}\rangle Cont) \rightarrow \langle A\rangle Cr)$ *forward-corr to cell*

All other formulas remains unchanged.

Theorem 2. *Let $\mathcal{A}$ be a deterministic Turing Machine. Then, $\mathcal{A}$ converges on empty input if and only if the* HS_{Horn}*-formula*

$$Halts = \psi_1 \wedge \psi_2 \wedge \phi_5 \wedge \ldots \wedge \phi_7 \wedge \psi_8 \wedge \phi_9 \wedge \ldots \wedge \phi_{12} \wedge \psi_{13a} \wedge \psi_{13b} \wedge \phi_{14} \wedge \ldots \wedge \phi_{30} \wedge \langle L\rangle q_f$$

is satisfiable on a finite model.

Corollary 2. *The satisfiability problem for* HS_{Horn} *over* Fin *is undecidable.*

6 Conclusions

Sub-propositional fragments of classical and modal logics, such as the Horn and Krom fragments, have been extensively studied. The generally high complexity of the (few) decidable interval-based temporal logics justifies a certain interest in exploring the sub-propositional fragments of HS in search of languages that present a better computational behaviour, and yet are, expressiveness-wise, suitable for some applications. In this paper we proved a first negative result in this sense, by showing that HS is still undecidable when its Horn fragment is considered. This result has been obtained under very weak assumptions on the class of models in which the logic is interpreted; as a matter of fact, we proved that HS_{Horn} is undecidable on every meaningful class of linearly ordered set (precisely as full HS is).

Despite this initial result, we believe that sub-propositional fragments of interval temporal logics deserves further study. On one hand, we plan to consider the Horn$^{\Box}$ fragments of decidable interval logics such as $\mathsf{A\overline{A}}$ and $\mathsf{B\overline{B}L\overline{L}}$, to understand whether or not their satisfiability problem present a better computational behaviour; initial analysis in this sense suggest that this could be the case. On the other hand, the decidability of the satisfiability problem is still an open issue for HS_{Krom}, HS_{core}, as well as for $\mathsf{HS}_{Horn^{\Box}}$ and $\mathsf{HS}_{core^{\Box}}$ (the weaker definitions of the Horn and core fragments considered in [3]). In this respect, it is worth to observe that in our construction of the formula *NotHalts* only three clauses, namely ϕ_{10}, ϕ_{14}, and ϕ_{19}, are not core. We are pretty confident that the first two formulas, ϕ_{10} and ϕ_{14}, can be rewritten in the core fragment. The last one, though, presents more difficulties. In addition, the construction makes an extensive use of diamond modalities, and hence seems not be applicable to the fragments $\mathsf{HS}_{Horn^{\Box}}$ and $\mathsf{HS}_{core^{\Box}}$, suggesting that they may even be decidable.

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