A Modal Logic of Knowledge, Belief, and Estimation

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Abstract. We introduce $\mathbf{KBE}$, a modal epistemic logic for reasoning about *Knowledge*, *Belief* and *Estimation*, three attitudes involved in an agent's decision-making process. In our logic, *Knowledge* and *Belief* are captured by $\mathbf{S4.2}$, a modal logic holding a distinguished position among the epistemic logics investigated in AI and Philosophy. The *Estimation* operator of $\mathbf{KBE}$ is a kind of generalized '*many*' or '*most*' quantifier, whose origins go back to the work of J. Burgess and A. Herzig, but its model-theoretic incarnation ('weak filters') has been introduced by K. Schlechta and V. Jauregui. We work with *complete weak filters* ('*weak ultrafilters*') as we are interested in situations where an estimation can be always reached. The axiomatization of $\mathbf{KBE}$ comprises 'bridge' axioms which reflect the intuitive relationship of '*estimation*' to '*knowledge*' and '*belief*', several introspective properties are shown to hold and it comes out that *believing* φ can be equivalently defined in $\mathbf{KBE}$ as '*estimating that* φ *is known*', an interesting fact and an indication of the intuitive correctness of the introduced *estimation* operator. The model theory of $\mathbf{KBE}$ comprises a class of frames combining relational Kripke frames with Scott-Montague semantics, in which neighborhoods are collections of 'large' sets of possible worlds. Soundness and completeness is mentioned and a tableaux proof procedure is sketched.

1 Introduction

The various logics of *Knowledge* and *Belief* have found very important applications in *Knowledge Representation*, *Distributed Computing*, *Security* and *Cryptography*, *Game Theory* and *Economics*. On the other hand, it is natural to ask *whether knowledge and belief suffice* to guide the decision-making process of an agent acting in a complex environment. Given the fact that an agent typically reasons in terms of incomplete information, it is natural to consider that its epistemic state is incomplete; the same for its belief set. In the absence of knowledge, the agent can proceed to an estimation on the truth (or falsity) of a certain fact, in view of the available evidence and in several situations where a decision *has* to be taken at any rate, the estimation should necessarily be accomplished. The **interaction** of **Knowledge**, **Belief** and **Estimation** crops all over, implicitly or explicitly.

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In our logic, **estimation** is intended to capture the intuition that the agent can estimate that φ is true (in the sense that she 'bets' on its truth rather than its falsity) in the case φ is true in '*many*' alternative situations to the one the agent is situated in. The axiomatization of $\mathbf{KBE}$ comprises four '*bridge*' axioms that pin down the relationship of estimation to knowledge and belief, as suggested by our intuition on what '*estimation*' actually means. We insist on estimation being consistent and complete, if it is to be really useful in the decision-making process. We prove some 'introspective' properties of estimation and **it turns out that, in $\mathbf{KBE}$, belief can be equivalently defined in terms of estimation and knowledge**: believing that φ is true amounts exactly to the agent estimating that she knows φ. This is clearly close to our intuition on '*estimation*' as a weak version of belief, which traditionally comes in many facets and many variants. Regarding the model theory of $\mathbf{KBE}$, we work with a class of frames which combine a subclass of $\mathbf{S4.2}$ relational frames (those with a *final* cluster), endowed with Scott-Montague semantics. Due to space limitations, proofs of the results and full presentation of the tableaux proof procedure, is left for the full report [20].

The logic $\mathbf{KBE}$ resembles the approach of J. Burgess in [4], where a 'probably' operator is added to $\mathbf{S5}$. In [15], Andreas Herzig employs the same operator, providing an axiomatization which is very close to the 'most' modality underlying our estimation operator. However, our approach is the first to combine such a generalized quantifier with a normal modal system, providing a full completeness proof both for the Hilbert-style axiomatization and the tableaux proof procedure introduced; see [20].

2 Background Material

We assume that the reader is well acquainted with the notation and the terminology of Modal Logic; we refer to [16,7,3] for Modal Logic and to [11] for a tour in epistemic logic (see also [24,2,26]). In particular, we assume that the reader is readily aware of the epistemic interpretation of the widely used modal axioms. We will work with $\mathbf{S4.2}$, which is the normal modal logic $\mathbf{KT4G}$, assuming Lenzen's approach [23,22] in which belief can be defined through knowledge: see ('abbreviation') $\mathbf{DB}$ in Section 3.1 and [20] for details. It is well-known that $\mathbf{S4.2}$ is determined by the class of reflexive, transitive and directed relational frames; in [18] (and independently in [21]) it is proved that $\mathbf{S4.2}$ is also determined by the subclass of frames which possess a non-empty *final* (*terminal*) *cluster*, intuitively, a non-empty universally-related set of worlds 'seen' by every world in the frame.

The interpretation of estimation as a '*many*' ('*most*') quantifier requires a model-theoretic interpretation of this notion. In classical Model Theory, it is the notion of 'filter' (*non-empty* collection of sets, *upwards closed* and *closed under intersection*) that captures the 'large' subsets of the universe. For various reasons this notion is not entirely appropriate for our purposes and we work with the '*weak filters*' (non-empty collections of pairwise-disjoint sets, upwards closed) introduced in [25,17]. Actually, we work with '*weak ultrafilters*' introduced in [1], requiring further that either a set or its complement (but not both) is a '*large*' set. The reader is referred to [19], in which it is shown that the notion of '*weak ultrafilter*' is non-trivial and that every 'consistent' weak filter can be extended to a weak ultrafilter. It is worth noting that similar notions of generalized quantification have been introduced earlier by W. Carnielli et al. in [5,6].

Definition 1. *Consider sets $W \neq \varnothing$, $Z \subseteq W$, $F \subseteq \mathcal{P}(W)$ and the following properties:*

$$
\begin{array}{ll}
(\mathbf{wf1}) & W \in F \\
(\mathbf{wf2}) & (\forall X \in F)(\forall Y \subseteq W)(X \subseteq Y \Longrightarrow Y \in F) \\
(\mathbf{wf3}) & (\forall X \subseteq W)(X \in F \Longrightarrow W \setminus X \notin F) \\
(\mathbf{wuf}) & (\forall X \subseteq W)(X \notin F \Longrightarrow W \setminus X \in F) \\
(\mathbf{inZ}) & (\forall A, D \in F)(A \cap D \cap Z \neq \varnothing)
\end{array}
$$

If (wf1) *to* (wf3) *hold for F, then it is called a* ***weak filter over*** *W [25,17]. If* (wuf) *holds for weak filter F, then it is called a* ***weak ultrafilter over*** *W. If* (inZ) *holds for a weak filter F then it is called* ***weak filter over*** *W* ***with intersections in*** *Z.*

3 The Logic KBE

The logic $\mathbf{S4.2}$ has been advocated by W. Lenzen as the 'correct' logic of knowledge, as it contains practically every one of the 'plausible' principles governing knowledge, belief and their interaction. In the full report [20] we discuss in detail the epistemic importance of $\mathbf{S4.2}$ and the work of W. Lenzen and R. Stalnaker. We proceed to enrich $\mathbf{S4.2}$ with *estimation*; see [20] for the rationale of the axioms.

3.1 Axiomatization of KBE

We consider the propositional bimodal language $\mathcal{L}_{KBE}$ with the propositional variables $\Phi = \{p_0, p_1, \ldots\}$, the *falsum* $\bot$, the implication connective $\supset$ and the modal operators K and E. The intended interpretation is that $\mathsf{K}\varphi$ is read as '*the agent* ***knows*** *φ*', $\mathsf{B}\varphi$ (it is an abbreviation) is read as '*the agent* ***believes*** *φ*', $\mathsf{E}\varphi$ is read as '*the agent* ***estimates*** *that φ is true*'. We proceed now to list the axioms of $\mathbf{KBE}$, including the abbreviation for belief.

Abbreviation

$\mathbf{DB}$. $\mathsf{B}\varphi \equiv \neg\mathsf{K}\neg\mathsf{K}\varphi$ *Belief definition.*

Axioms

$\mathbf{K}$. $\mathsf{K}\varphi \wedge \mathsf{K}(\varphi \supset \psi) \supset \mathsf{K}\psi$
Knowledge is closed under logical consequence.

$\mathbf{T}$. $\mathsf{K}\varphi \supset \varphi$
Only true things are known.

4. $\mathsf{K}\varphi \supset \mathsf{K}\mathsf{K}\varphi$
Positive introspection, with respect to knowledge.

$\mathbf{CB}$. $\mathsf{B}\varphi \supset \neg\mathsf{B}\neg\varphi$
Belief is consistent.

$\mathbf{BE}$. $\mathsf{B}\varphi \supset \mathsf{E}\varphi$

Beliefs are estimations.

CCE. $\mathsf{E}\varphi \equiv \neg\mathsf{E}\neg\varphi$
Estimation is consistent and complete.

EK. $\mathsf{E}\varphi \wedge \mathsf{K}(\varphi \supset \psi) \supset \mathsf{E}\psi$
Estimation can be inferred, only through knowledge.

PIE. $\mathsf{E}\varphi \supset \mathsf{KE}\varphi$
Introspection with respect to estimation.

Definition 2. $\mathbf{KBE}$ *is the propositional bimodal logic axiomatized by* $\mathbf{K}$*,* $\mathbf{T}$*,* $\mathbf{4}$*,* $\mathbf{CB}$*,* $\mathbf{BE}$*,* $\mathbf{CCE}$*,* $\mathbf{EK}$*,* $\mathbf{PIE}$ *and closed under rule* $\mathbf{RN}_{\mathsf{K}}$. $\frac{\varphi}{\mathsf{K}\varphi}$

The next result, whose proof consists of formal $\mathbf{KBE}$ derivations (see [20]) clarifies some properties of the logic, of 'introspective' nature.

Proposition 1.

i. *Positive 'Introspection' wrt estimation is valid in all three epistemic 'degrees':*
$\mathsf{E}\varphi \supset \mathsf{KE}\varphi,\ \mathsf{E}\varphi \supset \mathsf{BE}\varphi,\ \mathsf{E}\varphi \supset \mathsf{EE}\varphi \in \mathbf{KBE}$

ii. *So is the negative 'Introspection' wrt estimation:*
$\neg\mathsf{E}\varphi \supset \mathsf{K}\neg\mathsf{E}\varphi,\ \neg\mathsf{E}\varphi \supset \mathsf{B}\neg\mathsf{E}\varphi,\ \neg\mathsf{E}\varphi \supset \mathsf{E}\neg\mathsf{E}\varphi \in \mathbf{KBE}$

iii. *Non-estimation implies introspection wrt ignorance and 'lack of certainty':*
$\neg\mathsf{E}\varphi \supset \mathsf{K}\neg\mathsf{K}\varphi,\ \neg\mathsf{E}\varphi \supset \mathsf{B}\neg\mathsf{K}\varphi,\ \neg\mathsf{E}\varphi \supset \mathsf{E}\neg\mathsf{K}\varphi \in \mathbf{KBE}$
$\neg\mathsf{E}\varphi \supset \mathsf{K}\neg\mathsf{B}\varphi,\ \neg\mathsf{E}\varphi \supset \mathsf{B}\neg\mathsf{B}\varphi,\ \neg\mathsf{E}\varphi \supset \mathsf{E}\neg\mathsf{B}\varphi \in \mathbf{KBE}$

iv. $\mathsf{KE}\varphi \equiv \mathsf{E}\varphi \in \mathbf{KBE}$

v. $\mathsf{EK}\varphi \equiv \mathsf{B}\varphi \in \mathbf{KBE}$

Remark 1. Note that, by the last item of Prop. 1 above, belief can be equivalently defined as '*estimation that the agent knows*'. Defining knowledge in terms of belief and vice versa, is a very interesting topic in epistemic logic (see [14]). In that respect, it is interesting that belief can be equivalently defined in an $\mathbf{S4.2}$ framework, in a rather intuitive way, through an 'estimation' operator. In the same fashion, item (iv) says that knowledge about estimation amounts exactly to estimation itself.

3.2 The Possible-Worlds Models of $\mathbf{KBE}$

In this section, we define the frames and models of $\mathbf{KBE}$. These structures properly mix an interesting subclass of $\mathbf{S4.2}$-frames (the reflexive, transitive, with a final cluster FC) with Scott-Montague semantics [7, *neighborhood semantics*], in which each neighborhood is a *complete collection of large sets* on the epistemic alternatives of the world at hand - a weak ultrafilter. In the following definition, the properties $(\mathbf{cce})$ and $(\mathbf{ek})$ are essentially $(\mathbf{wf2})$, $(\mathbf{wf3})$ and $(\mathbf{wuf})$ of Definition 1 of weak ultrafilters, properly stated, as we define **weak ultrafilters** on $\mathcal{R}(w)$.

Definition 3. *Consider the triple $\mathfrak{F} = \langle W, \mathcal{R}, \mathcal{N} \rangle$, where W is a non-empty set, $\mathcal{R} \subseteq W \times W$, $\mathcal{N} : W \to \mathcal{P}(\mathcal{P}(W))$ and*

- *$\mathcal{R}$ is reflexive, transitive and has a nonempty* final cluster $FC = \{v \in W \mid (\forall w \in W)\ w\mathcal{R}v\}$.
- *$\mathcal{N}$ is such, that* $\forall w \in W$

$$(\mathbf{nr}) \quad \mathcal{N}(w) \subseteq \mathcal{P}(\mathcal{R}(w))$$

$$(\mathbf{be}) \quad FC \in \mathcal{N}(w)$$

$$(\mathbf{pie}) \quad \forall X \subseteq \mathcal{R}(w)\ \forall u \in W\ \big(X \in \mathcal{N}(w)\ \&\ w\mathcal{R}u \Longrightarrow X \cap \mathcal{R}(u) \in \mathcal{N}(u)\big)$$

$$(\mathbf{cce}) \quad \forall X \subseteq \mathcal{R}(w)\ \big(X \in \mathcal{N}(w) \Longleftrightarrow \mathcal{R}(w) \setminus X \notin \mathcal{N}(w)\big)$$

$$(\mathbf{ek}) \quad \forall X, Y \subseteq \mathcal{R}(w)\ \big(X \in \mathcal{N}(w)\ \&\ Y \supseteq X \Longrightarrow Y \in \mathcal{N}(w)\big)$$

$\mathfrak{F}$ is called a **kbe-frame**. $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ is called a **kbe-model**, if it is based on a kbe-frame and $V : \Phi \to \mathcal{P}(W)$ is a valuation. It is not hard to show that the class of kbe-frames is nonempty [20]. Given a model $\mathfrak{M} = \langle W, \mathcal{R}, \mathcal{N}, V \rangle$ for the language $\mathcal{L}_{KBE}$, the valuation $V : \Phi \to \mathcal{P}(W)$ can be extended to all formulae of $\mathcal{L}_{KBE}$ in a straightforward way. In [20] the following result is proved.

Theorem 1. (Soundness & Completeness) $\mathbf{KBE}$ *is sound and strongly complete w.r.t. the class of all kbe-frames.*

Using Theorem 1 we can show that various 'introspective' principles are not $\mathbf{KBE}$-axioms. Having in mind that 'estimation' is conceived as a weak form of a belief-like attitude, the fact that $\mathsf{E}\varphi \supset \mathsf{B}\varphi$ is not a $\mathbf{KBE}$-theorem is consistent with our intuition. Among the formulae of Fact 2.(iii), $\mathsf{E}\varphi \supset \mathsf{EK}\varphi$ deserves a comment. The fact that it is not a theorem of $\mathbf{KBE}$ is welcomed; otherwise, given that $\mathsf{EK}\varphi \supset \mathsf{B}\varphi \in \mathbf{KBE}$ (Prop. 1.(v)) and $\mathbf{BE}$, estimation would collapse to belief ($\mathsf{E}\varphi \equiv \mathsf{B}\varphi$ would be a theorem of $\mathbf{KBE}$) and this would immediately invalidate our attempt to define estimation as a weak form of belief. In the same fashion, it is really good news that $\mathsf{EK}\varphi \supset \mathsf{K}\varphi$ is not a $\mathbf{KBE}$-theorem. This formula introduces a strong form of the '*infallibility argument*' or else the '*paradox of the perfect believer*' (see [11]): in view of axiom $\mathbf{T}$, it finally requires that something is true whenever our agent estimates that she knows it.

Fact 2. i. $\mathsf{E}\varphi \supset \mathsf{B}\varphi \notin \mathbf{KBE}$

ii. $\neg\mathsf{B}\neg\varphi \supset \mathsf{B}\varphi \notin \mathbf{KBE}$

iii. $\mathsf{E}\varphi \supset \mathsf{KK}\varphi,\ \mathsf{E}\varphi \supset \mathsf{KB}\varphi \notin \mathbf{KBE}$
$\mathsf{E}\varphi \supset \mathsf{BK}\varphi,\ \mathsf{E}\varphi \supset \mathsf{BB}\varphi \notin \mathbf{KBE}$
$\mathsf{E}\varphi \supset \mathsf{EK}\varphi,\ \mathsf{E}\varphi \supset \mathsf{EB}\varphi \notin \mathbf{KBE}$

iv. $\mathsf{EK}\varphi \supset \mathsf{K}\varphi \notin \mathbf{KBE}$

v. $\mathsf{E}\varphi \wedge \mathsf{E}(\varphi \supset \psi) \supset \mathsf{E}\psi \notin \mathbf{KBE}$

4 Tableaux for $\mathbf{KBE}$

In this section we sketch a tableau system for $\mathbf{KBE}$ using *prefixed formulas*. A reminder on terminology is in order: a *prefix* is a finite sequence of natural numbers, separated by periods. A *prefixed formula* is an expression of the form $\sigma\ \varphi$, where σ is a prefix and φ is a formula. A tableau branch is *closed* if it contains both $\sigma\ \varphi$ and $\sigma\ \neg\varphi$ for some prefix σ and formula φ. A tableau is *closed* if all of its branches are closed. A tableau or branch is *open* if it is not closed. The terminology and most of the techniques we use, draw from [8,9].

The intention for the prefixes is that they name worlds in a model, and the world named by $\sigma.n$ is accessible from the world named by σ. The worlds of a kbe-model either belong to its final cluster or not, so we will be using two kinds of prefixed formulas; of the form $0.n, n \in \mathbb{N}$ to represent the first, and of the form $1.\sigma$ to represent the latter. Prefixes of the form $0.n$ do not allow tracking of some accessibility relation, but are sufficient for the final cluster, exactly because it is a cluster, i.e. relation $\mathcal{R}$ is universal in it.

4.1 Tableaux Rules

Before presenting the rules themselves we need a notion of *accessibility* between prefixes, proper for kbe-models. For the alphabet of our tableaux, we assume $\mathsf{B}\varphi, \langle\mathsf{K}\rangle\,\varphi, \langle\mathsf{E}\rangle\,\varphi, \varphi \supset \psi, \varphi \equiv \psi$ are abbreviations for $\neg\mathsf{K}\neg\mathsf{K}\varphi, \neg\mathsf{K}\neg\varphi, \neg\mathsf{E}\neg\varphi, \neg\varphi \vee \psi, (\varphi \supset \psi) \wedge (\psi \supset \varphi)$ respectively, thus no corresponding rules have to be specified.

Definition 4. *A prefix σ' is* accessible *from a prefix σ if and only if σ is an initial segment of σ' (proper or otherwise), or σ' is of the form $0.n, n \in \mathbb{N}$.*

Definition 5. *A* ***kbe-tableau*** *for a formula φ is a tableau that starts with the prefixed formulas $1\ \neg\varphi$ and $0.1\ \top$ and is extended using any of the rules below.*

A few words on the rules that follow: $\mathbf{KBE}$ is normal with respect to K, and our rules for K (and $\neg\mathsf{K}$) state what they should, regarding semantics. What makes the rules appropriate for an $\mathbf{S4.2}$*-frame,* is that we introduce at least a prefix for the final cluster with $0.1\top$, and reflexivity, transitivity and that the final cluster is both 'final' and a cluster, is integrated into our notion of prefix *accessibility.* Regarding [$\mathbf{CCE}$-rule] and [$\mathbf{PIE}$-rule], as their name suggests, they exist to tend to axioms $\mathbf{CCE}$ and $\mathbf{PIE}$. Axiom $\mathbf{CCE}$ is in fact an equivalence, but we decide to transform all $\langle\mathsf{E}\rangle$ into E and have no use for the other direction. Axiom $\mathbf{PIE}$ is also not exactly what our rule implies, but for the sake of shortening proofs, one can observe the only applicable rule to a formula $\mathsf{KE}\varphi$ is [$\mathsf{K}\nu$-rule]; we do it outright. Finally, regarding modality E, $\mathbf{KBE}$ is monotonic with respect to it. The proper rule, is that for any pair $\langle\mathsf{E}\rangle\,\varphi$, $\mathsf{E}\psi$ there is a world such that φ, ψ hold (see [9] regarding the Logic $\mathbf{U}$, and specifically Chapter 6.13 for a tableau for $\mathbf{U}$). In our case, $\langle\mathsf{E}\rangle$ has turned into E, and not just any world will do, but one accessible with respect to $\mathcal{R}$; [$\mathbf{E}$-rule] is created accordingly. Also note that φ can be the same as ψ.

For prefixes σ of the form $1.\sigma'$:

[Double negation rule] $\dfrac{\sigma\ \neg\neg\varphi}{\sigma\ \varphi}$

[Conjunctive rules] $\dfrac{\sigma\ \varphi \wedge \psi}{\begin{array}{c}\sigma\ \varphi\\ \sigma\ \psi\end{array}}$ $\quad \dfrac{\sigma\ \neg(\varphi \vee \psi)}{\begin{array}{c}\sigma\ \neg\varphi\\ \sigma\ \neg\psi\end{array}}$

[Disjunctive rules] $\dfrac{\sigma\ \varphi \vee \psi}{\sigma\ \varphi \mid \sigma\ \psi}$ $\quad \dfrac{\sigma\ \neg(\varphi \wedge \psi)}{\sigma\ \neg\varphi \mid \sigma\ \neg\psi}$

[Kν-rule] $\dfrac{\sigma\ \mathsf{K}\varphi}{\sigma'\ \varphi}$ for all σ' accessible from σ and already existing on the branch.

[Kπ-rule] $\dfrac{\sigma\ \neg\mathsf{K}\varphi}{\sigma.n\ \neg\varphi}$ for any prefix $\sigma.n$ new to the branch.

[**CCE**-rule] $\dfrac{\sigma\ \neg\mathsf{E}\varphi}{\sigma\ \mathsf{E}\neg\varphi}$

[**PIE**-rule] $\dfrac{\sigma\ \mathsf{E}\varphi}{\sigma'\ \mathsf{E}\varphi}$ for all σ' accessible from σ and already existing on the branch.

[E-rule] $\dfrac{\begin{array}{c}\sigma\ \mathsf{E}\varphi\\ \sigma\ \mathsf{E}\psi\end{array}}{\begin{array}{c}\sigma.n\ \varphi\\ \sigma.n\ \psi\end{array}}$ for any prefix $\sigma.n$ new to the branch.

For prefixes $0.n$ we have, in essence, the same rules, but the exact notation for rules introducing a new world is:

[Kπ-rule] $\dfrac{0.n\ \neg\mathsf{K}\varphi}{0.m\ \neg\varphi}$ for any prefix $0.m$ new to the branch.

[E-rule] $\dfrac{\begin{array}{c}0.n\ \mathsf{E}\varphi\\ 0.n\ \mathsf{E}\psi\end{array}}{\begin{array}{c}0.m\ \varphi\\ 0.m\ \psi\end{array}}$ for any prefix $0.m$ new to the branch.

Definition 6. *A closed kbe-tableau for a formula φ is a* ***kbe-tableau proof*** *for φ.*

Let us see an example from Prop. 1.

$$\mathsf{E}\mathsf{K}p \equiv \neg\mathsf{K}\neg\mathsf{K}p$$

1 $\neg((\neg\mathsf{EK}p \vee \neg\mathsf{K}\neg\mathsf{K}p) \wedge (\neg\neg\mathsf{K}\neg\mathsf{K}p \vee \mathsf{EK}p))$ 1.	
0.1 $\top$ 2.	
1 $\neg(\neg\mathsf{EK}p \vee \neg\mathsf{K}\neg\mathsf{K}p)$ 3.	1 $\neg(\neg\neg\mathsf{K}\neg\mathsf{K}p \vee \mathsf{EK}p)$ 10.
1 $\neg\neg\mathsf{EK}p$ 4.	1 $\neg\neg\neg\mathsf{K}\neg\mathsf{K}p$ 11.
1 $\neg\neg\mathsf{K}\neg\mathsf{K}p$ 5.	1 $\neg\mathsf{EK}p$ 12.
1 $\mathsf{EK}p$ 6.	1 $\neg\mathsf{K}\neg\mathsf{K}p$ 13.
1 $\mathsf{K}\neg\mathsf{K}p$ 7.	1 $\mathsf{E}\neg\mathsf{K}p$ 14.
1.1 $\mathsf{K}p$ 8.	1.1 $\neg\neg\mathsf{K}p$ 15.
1.1 $\neg\mathsf{K}p$ 9.	1.1 $\mathsf{K}p$ 16.
	0.1 $\mathsf{E}\neg\mathsf{K}p$ 17.
	0.2 $\neg\mathsf{K}p$ 18.
	0.3 $\neg p$ 19.
	0.3 p 20.

Item 1 is the negation of the formula we want to prove expressed in the tableaux language and item 2 is standard. Items 3 and 10 are from 1 by [Disjunctive Rule]. Items 4 and 5 are from 3 by a [Conjunctive Rule]. Items 6 and 7 are from 4 and 5 respectively by [Double negation rule]. Item 8 is from 6 by [E-rule]. Item 9 is from 7 by [Kν-rule]. Item 11 and 12 are from 10 by a [Conjunctive rule]. Item 13 is from 11 by [Double negation rule]. Item 14 is from 12 by [**CCE**-rule]. Item 15 is from 13 by [Kπ-rule]. Item 16 is from 15 by [Double negation rule]. Item 17 is from 14 by [**PIE**-rule]. Item 18 is from 17 by [E-rule]. Item 19 is from 18 by [Kπ-rule] and item 20 is from 16 by [Kν-rule]. In the full report [20] we develop a systematic tableaux-based procedure and prove finite model property and decidability of **KBE**.

5 Conclusions

To the best of our knowledge (belief and estimation) our work is the first to provide a modal treatment of (qualitative) estimation, with respect to its interaction with knowledge and belief. The analysis of **KBE** is in line with the tradition of possible-worlds analysis in epistemic logic and sheds light on the nature of belief as '*estimation that φ is known*'. There exist similar approaches, involving the notion of *certainty*. The relation *of knowledge, belief* and *certainty* has been investigated by Halpern [13], Lenzen [22] and other authors. *Certainty* is also called '*robust belief*' by some authors, as opposed to '*strong belief*'; belief is a delicate interesting notion with a lot of useful variants.

As far as future research is concerned, we believe that the most important question is the identification of the computational properties of **KBE**. Moreover, it seems very challenging to try to embed a similar modal 'estimation' operator in first-order modal epistemic logic. This is bound to raise several technical and philosophical issues, but it seems a very promising and interesting problem.

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