Conflicts of Belief Functions: Continuity and Frame Resizement

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Abstract. Plausibility and pignistic conflict of belief functions are briefly recalled in this study. These measures of conflict are based on two different probability transformations of belief functions, normalised plausibility of singletons Pl_C and Smets' pignistic probability $BetP$.

Continuity properties and relationship of these conflict measures to extension and refinement of a frame of discernment are investigated here. A new continuous improvement of both the measures which is preserved by a frame extension is introduced. A relation of the new conflict measures to refinement of a frame of discernment is also discussed. Finally a comparison between the new measure and the two original measures as well as W. Liu's degree of conflict cf is presented.

Keywords: Belief functions, Dempster-Shafer theory, uncertainty, plausibility conflict, pignistic conflict, degree of conflict, continuity, extension of a frame of discernment, refinement of a frame of discernment.

1 Introduction

When combining belief functions (BFs) by the conjunctive rules of combination, conflicts often appear (which are assigned to $\emptyset$ by non-normalised conjunctive rule $\circledcirc$ or normalised by Dempster's rule of combination $\oplus$). Combination of conflicting BFs and interpretation of conflicts are often questionable in real applications. Thus a series of papers were published on alternative combination rules, conflicting belief functions, e.g. [2,4,13,15,16,22], and measures of conflicts, e.g. [12,17,18].

A new interpretation of conflicts of BFs was introduced in [6]. Important distinction of conflicts between BFs due to internal conflict of a single BF, and due to the difference between BFs was introduced there. The most elaborated perspective of the three approaches initiated in [6] — plausibility conflict of BFs — was analysed in [9] and improved in [10]. An alternative pignistic conflict based on Smets' pignistic probability $BetP$ was introduced there as well.

The presented study investigates plausibility and pignistic conflicts from the point of view of continuity and resizement of a frame of discernment: extension

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and refinement of the frame. New improvements of both the measures of conflicts between BFs with respect to these properties are presented.

Similarly to [6,9,10] we use W. Liu's assumption that conflict between BFs appears when the BFs strongly support mutually non-compatible hypotheses [16], and also the assumptions from [6] that there is no conflict between BFs when the BFs (strongly) support same or compatible hypotheses. Moreover, starting from Section 4, we assume continuity of conflict measures defined in Section 3; starting from Section 5, we assume keeping of conflictness/non-conflictness when extending a frame of discernment; and further, starting from Section 6, we assume also keeping of conflictness/non-conflictness when refining the frame. Section 7 compares and summarizes the presented results, several ideas for a future research are stated in Section 8.

2 State of the Art

We assume classic definitions of basic notions from theory of belief functions [19] on finite frames of discernment $\Omega_n = \{\omega_1, \omega_2, ..., \omega_n\}$. Due to a limited space we do not repeat all the notions used in [6,9,10], but only the important of those, which were introduced there.

A *basic belief assignment (bba)* $m : \mathcal{P}(\Omega) \longrightarrow [0,1]$, $\sum_{A\subseteq\Omega} m(A) = 1$, its values are called *basic belief masses (bbms)*; a *belief function (BF)* $Bel : \mathcal{P}(\Omega) \longrightarrow [0,1]$, $Bel(A) = \sum_{\emptyset\neq X\subseteq A} m(X)$. A *plausibility function* $Pl(A) = \sum_{\emptyset\neq A\cap X} m(X)$. There is a unique correspondence between m and the corresponding Bel and Pl; thus we often speak about m as a belief function. A *focal element* is a subset X of the frame of discernment such that $m(X) > 0$. *Normalised plausibility of singletons* corresponding to Bel: $Pl\_P(\omega) = \frac{Pl(\{\omega\})}{\sum_{\omega'\in\Omega} Pl(\{\omega'\})}$ [3,5] (this is normalised contour function); *pignistic probability*: $BetP(\omega) = \sum_{\omega\in X\subseteq\Omega} \frac{1}{|X|}\frac{m(X)}{1-m(\emptyset)}$ [20].

We say that $\omega \in \Omega$ is *supported* or *preferred by a belief function Bel* defined on Ω_n when $Pl\_P(\omega) > \frac{1}{n}$, ω is *opposed by Bel* if $Pl\_P(\omega) < \frac{1}{n}$. Analogously for $BetP(\omega)$ if Smets pignistic probability is used. U_n is a BF on Ω_n such that $m(\omega) = \frac{1}{n}$ for all $\omega \in \Omega_n$.

Conflict between BFs is distinguished from internal conflict in [6,9,10], where internal conflict of a BF is included inside the individual BF. Total conflict of two BFs Bel_1, Bel_2, which is equal to sum of all conflicting belief mases: $m_{\bigcirc\!\!\!\cap}(\emptyset) = \sum_{X\cap Y=\emptyset} m_1(X)m_2(Y)$, includes internal conflicts of individual BFs Bel_1, Bel_2 and a conflict between them. Thus two definitions were introduced in [6]; we are interested in conflict between belief BFs in this study.

Definition 1. *The* internal plausibility conflict *Pl-IntC* of BF Bel *is defined as*

$$Pl\text{-}IntC(Bel) = 1 - max_{\omega\in\Omega} Pl(\{\omega\}),$$

where Pl is the plausibility corresponding to Bel.

Definition 2. *Let Bel_1, Bel_2 be two belief functions on Ω_n given by bbms m_1 and m_2 which have normalised plausibility of singletons Pl_P_1 and Pl_P_2. The*

conflicting set $\Omega_{PlC}(Bel_1, Bel_2)$ *is defined to be the set of* elements $\omega \in \Omega_n$ with conflicting Pl_P masses *it is conditionally extended with union of sets max Pl_P_i value elements under condition that they are disjoint. Formally we have* $\Omega_{PlC}(Bel_1, Bel_2) = \Omega_{PlC_0}(Bel_1, Bel_2) \cup \Omega_{smPlC}(Bel_1, Bel_2)$, *where* $\Omega_{PlC_0}(Bel_1, Bel_2) = \{\omega \in \Omega_n \mid (Pl\_P_1(\omega) - \frac{1}{n})(Pl\_P_2(\omega) - \frac{1}{n}) < 0\}$, $\Omega_{smPlC}(Bel_1, Bel_2) = \{\omega \in \Omega_n \mid \omega \in \{max_{\omega\in\Omega_n} Pl\_P_1(\omega)\} \cup \{max_{\omega\in\Omega_n} Pl\_P_2(\omega)\}$ & $\{max_{\omega\in\Omega_n} Pl\_P_1(\omega)\} \cap \{max_{\omega\in\Omega_n} Pl\_P_2(\omega)\} \neq \emptyset\}$.
Plausibility conflict between BFs Bel_1 *and* Bel_2 *is then defined by the formula*

$$Pl\text{-}C(Bel_1, Bel_2) = min(\, Pl\text{-}C_0(Bel_1, Bel_2), (m_1 \circledcirc m_2)(\emptyset)\,),$$

where[1]

$$Pl\text{-}C_0(Bel_1, Bel_2) = \sum_{\omega \in \Omega_{PlC}(Bel_1, Bel_2)} \frac{1}{2} |Pl\_P(Bel_1)(\omega) - Pl\_P(Bel_2)(\omega)|.$$

There are two reasons for minimising with $(m_1 \circledcirc m_2)(\emptyset)$ (briefly with $m_{\circledcirc}(\emptyset)$ if Bel_1 and Bel_2 are clear from a context): at first the original from [6], see Example 1, where two obviously non-conflicting BFs have non-empty conflicting set and positive $Pl\text{-}C_0$, whereas $m_{\circledcirc}(\emptyset) = 0$; the second is that $m_{\circledcirc}(\emptyset)$ was found to be an upper bound for conflict between BFs [10].

Example 1. Let us suppose two categorical BFs on Ω_6 given by $m_1(\{\omega_1\}) = 1$ and $m_2(\{\omega_1, \omega_2, \omega_3, \omega_4\}) = 1$, thus $Pl\_P_1 = (1, 0, 0, 0, 0, 0)$ and $Pl\_P_2 = (0.25, 0.25, 0.25, 0.25, 0, 0)$ thus $\Omega_{PlC} = \{\omega_2, \omega_3, \omega_4\}$ as $Pl\_P_1(\omega_i) = 0 < \frac{1}{6}$ and $Pl\_P_2(\omega_i) = 0.25 > \frac{1}{6}$ for $i = 1, 2, 3, 4$ (other elements are non-conflicting), hence $Pl\text{-}C_0 = 0.375$ and this should be minimised with $m_{\circledcirc}(\emptyset) = 0$.

Four variants of $\Omega_{PlC}(Bel_1, Bel_2)$ are defined and analysed in [10]: Ω_{smPlC}, $\Omega_{spPlC} = \Omega_{smPlC}(Bel_1, Bel_2) \cup \Omega_{PlC_0}(Bel_1, Bel_2)$ (as above), Ω_{cpPlC} which includes ω with different order of $Pl\_P_i(\omega)$ values, and $\Omega_{cbPlC} = \Omega_{cpPlC}(Bel_1, Bel_2) \cup \Omega_{PlC_0}(Bel_1, Bel_2)$. I.e., Ω_{PlC} is constructed using either max Pl_P_i values, ordering Pl_P_i values, support/opposition of elements of the frame of discernment (+ max Pl_P_i values), or combination of these options; for detail see [10]. (All of these variants coincide on Ω_2).

Example 2. Four variants of conflicting sets. Let us suppose Bel_1, Bel_2 on Ω_5.

X :	$\{\omega_1\}$	$\{\omega_2\}$	$\{\omega_3\}$	$\{\omega_4\}$	$\{\omega_5\}$	$\{\omega_1,\omega_2\}$	$\{\omega_2,\omega_4\}$	$\{\omega_1 - \omega_3\}$	$\{\omega_1 .. \omega_4\}$	Ω_5
$m_1(X)$:	0.225	0.195	0.19	0.19	0.01	0.02	0.01	0.03	0.11	0.02
$m_2(X)$:	0.110	0.410	0.16	0.00	0.01	0.03	0.04	0.02	0.09	0.05

We obtain $Pl\_P_1 = (0.27, 0.25, 0.24.0.22, 0.02)$, $Pl\_P_2 = (0.20, 0.40, 0.24.0.12, 0.04)$, and $\Omega_{smPlC} = \{\omega_1, \omega_2\}$, $\Omega_{PlC_0} = \{\omega_4\}$, $\Omega_{spPlC} = \{\omega_1, \omega_2, \omega_4\}$, $\Omega_{cpPlC} = \{\omega_1, \omega_2, \omega_3\}$, and $\Omega_{cbPlC} = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, hence Bel_1 and Bel_2 are considered to be mutually conflicting by all variants of *Pl-C*, but values of conflict are different in this case: $smPl\text{-}C(Bel_1, Bel_2) = 0.11$, $spPl\text{-}C = 0.16$, $cpPl\text{-}C = 0.11$, $cbPl\text{-}C = 0.16$. (ω_3 has different order of Pl_P_i values thus it is included in both Ω_{cpPlC} and Ω_{cbPlC} but both its Pl_P_i values are the same: $Pl\_P_1(\omega_3) = Pl\_P_2(\omega_3) = 0.24$, thus $smPl\text{-}C = cpPl\text{-}C$ and $spPl\text{-}C = cbPl\text{-}C$ in this special case.

[1] $Pl\text{-}C_0$ is not a separate measure of conflict in general; it is just a component of *Pl-C*.

Definition 3. *Let Bel_1, Bel_2 be two belief functions on Ω_n given by bbms m_1 and m_2 which have pignistic probabilities $BetP_1$ and $BetP_2$. The* pignistic conflicting set $\Omega_{BetC}(Bel_1, Bel_2)$ *is defined analogously to plausibility conflicting set $\Omega_{PlC}(Bel_1, Bel_2)$, having analogously four variants Ω_{smBetC}, Ω_{spBetC}, Ω_{cpPlC} and Ω_{cbPlC}, see [10].*

Pignistic conflict between BFs Bel_1 *and* Bel_2 *is then defined by the formula*

$$Bet\text{-}C(Bel_1, Bel_2) \ = min(\, Bet\text{-}C_0(Bel_1, Bel_2), (m_1 \circledcirc m_2)(\emptyset)\,),$$

where[2]

$$Bet\text{-}C_0(Bel_1, Bel_2) = \sum_{\omega \in \Omega_{BetC}(Bel_1, Bel_2)} \frac{1}{2}\, |BetP_1(\omega) - BetP_2(\omega)|.$$

Quantitative aspect of conflict — conflictness/non-conflictness is classified by emptyness/non-emptyness of related conflicting set by both *Pl-C* and *Bet-C*. Quantitative conflict is then computed only for mutually conflicting BFs.

Whereas qualitative values are computed for any pair of BFs by Liu's two-component degree of conflict $cf = (difBetP_{m_i}^{m_j}, m_{\cap}(\emptyset))$ [10,16], where $difBetP_{m_i}^{m_j} = max_{A \subseteq \Omega}(|BetP_{m_i}(A) - BetP_{m_j}(A)|)$, $m_{\cap}(\emptyset) = (m_1 \circledcirc m_2)(\emptyset)$. Qualitative question of conflictness/non-conflictness is not addressed there, in fact; and 'high conflictness' / 'not high conflictness' is determined from the qualitative values using empirically/heuristically given threshold of conflict tolerance ε.

Unfortunately, jumps of *Pl-C* and *Bet-C* values were observed, see the following examples. Such a jump in conflict values is counter-intuitive, moreover neither $m_{\cap}(\emptyset)$ nor the other component $difBetP_{m_i}^{m_j}$ of Liu's degree of conflict cf have similar jumps. Hence, we are interested in how to remove the jumps from conflict measures in this study, i.e., how to modify measures of conflict *Pl-C* and *Bet-C* to be continuous, or jump-free.

Example 3. Let us suppose two BFs on Ω_2: $Bel_1 = (m_1(\{\omega_1\}), m_1(\{\omega_2\}) = (0.8, 0.1)$ $(m_1(\{\omega_1, \omega_2\}) = 1 - m_1(\{\omega_1\}) - m_1(\{\omega_2\}) = 0.1)$, $Bel_2 = (0.3, 0.3)$, thus we obtain $Pl\_P_1 = (0.888, 0.111)$, $Pl\_P_2 = (0.5, 0.5)$, hence these two BFs are non-conflicting. Let us suppose a very small change of Bel_2, thus we expect zero conflict again or a very small conflict value corresponding to the very small change. Let $Bel_2' = (0.3, 0.31)$, thus $Pl\_P_2' = (0.4964, 0.5036)$, hence $\Omega_{PlC}(Bel_1, Bel_2') = \Omega_2$ and $Pl\text{-}C(Bel_1, Bel_2') = 0.3925$, which is significantly higher than the slight changes made on $m_2(\{\omega_2\})$ and $m_2(\{\omega_1, \omega_2\})$ by 0.01. Analogously, we have $BetP_1 = (0.85, 0.15)$, $BetP_2 = (0.5, 0.5)$, and $BetP_2' = (0.495, 0.505)$, which leads to $BetP\text{-}C(Bel_1, Bel_2) = 0$, $BetP\text{-}C(Bel_1, Bel_2') = 0.355$.

Let us suppose a free BF $Bel_1 = (a_1, b_1)$ and a fixed BF $Bel_2 = (a_2, b_2)$ on Ω_2, such that $Pl\_P_2 = (u, 1-u)$ where $u \geq \frac{1}{2}$. We can show how the value $Pl\text{-}C(Bel_1, Bel_2)$ depends on the value $Pl\_P_1(\omega_1)$ (i.e., $Pl\_P_1(\omega_1) = \frac{1-b_1}{2-a_1-b_1}$, as $Pl\_P_1(\omega_2) = 1 - Pl\_P_1(\omega_1)$ and u is fixed). A jump is obvious at $Pl\_P_1(\omega_1) = \frac{1}{2}$, see Fig. 1. For another example of jumps of conflict see Example 4.

[2] *Bet-C*$_0$ is again not a separate measure of conflict in general; it is just a component of *Bet-C*.

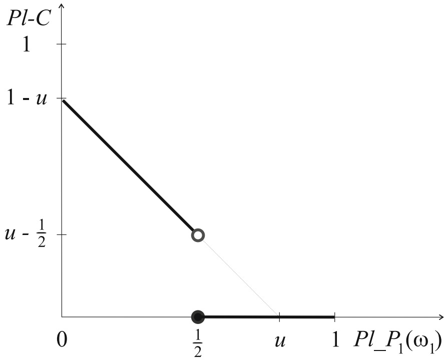

Fig. 1. Jump of *Pl-C*

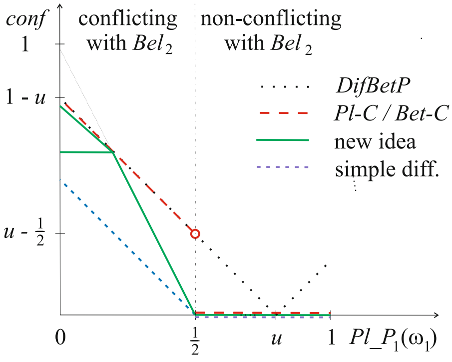

Fig. 2. A comparison of approaches

Example 4. Let us suppose BFs on Ω_4 given by the following table now:

X :	$\{\omega_1\}$	$\{\omega_2\}$	$\{\omega_3\}$	$\{\omega_4\}$	$\{\omega_1,\omega_2\}$	$\{\omega_4,\omega_3\}$	Ω_4
$m_1(X)$:	0.15	0.15	0.15	0.15	0.01		0.39
$m_2(X)$:	0.15	0.15	0.15	0.15		0.01	0.39
$m_3(X)$:	0.20	0.15	0.10	0.05	0.15	0.05	0.30
$m_4(X)$:	0.90	0.03	0.02		0.01	0.01	0.03

$Pl\_C_1 = (0.2523, 0.2523, 0.2477, 0.2477)$, $Pl\_C_2 = (0.2477, 0.2477, 0.2523, 0.2523)$, $Pl\_C_3 = (0.3095, 0.2857, 0.2143, 0.1905)$, $Pl\_C_4 = (0.8468, 0.0631, 0.0541, 0.0360)$, $Pl\text{-}C(Bel_1, Bel_3) = 0$, $Pl\text{-}C(Bel_2, Bel_3) = 0.0998$, which is about ten times larger than the changes on $\{\omega_1, \omega_2\}$ and $\{\omega_3, \omega_4\}$ by 0.01.
$smPl\text{-}C(Bel_1, Bel_4) = cpPl\text{-}C(Bel_1, Bel_4) = 0$, $spPl\text{-}C(Bel_1, Bel_4) = cbPl\text{-}C(Bel_1, Bel_4) = 0.2027$, $smPl\text{-}C(Bel_2, Bel_4) = 0.5068$, $cpPl\text{-}C(Bel_2, Bel_4) = 0.5991$, $spPl\text{-}C(Bel_2, Bel_4) = 0.5068$, $cbPl\text{-}C(Bel_2, Bel_4) = 0.5991$, all the changes on conflict values are significantly greater than the difference between m_1 values and m_2 values (i.e., 0.01).

3 Continuity of Measures of Conflict between Belief Functions

Let us define continuity of a measure of conflict using the conventional $\epsilon - \delta$ way. That is, we first define a δ-surrounding for any BF, and we then use δ-surrounding to define continuity of conflict measures.

Formally, we have the following definitions.

Definition 4. *We say that a belief function Bel' is in δ surrounding of a belief function Bel (briefly $Bel' \in \delta(Bel)$) if $|m(X) - m'(X)| \leq \delta$.*

Definition 5. *We say that a measure of conflict of belief functions $conf$ is continuous if for any $\varepsilon > 0$ and any BFs Bel_1, Bel_2, there exits a δ surrounding of Bel_1, such that for any $Bel' \in \delta(Bel_1)$, $|conf(Bel', Bel_2) - conf(Bel_1, Bel_2)| \leq \varepsilon$.*

Lemma 1. *(i)* $(m_1 \circledcirc m_2)(\emptyset)$ *is continuous measure of conflict.*
(ii) $min(conf(Bel_1, Bel_2), (m_1 \circledcirc m_2)(\emptyset))$ *is continuous for any continuous measure of conflict* $conf$ *and any pair of BFs* Bel_1, Bel_2 *given by* m_1 *and* m_2.
(iii) $min(difBet_{Bel_1}^{Bel_2}, (m_1 \circledcirc m_2)(\emptyset))$ *is a continuous measure of conflict of BFs.*

Proof. Proofs are verifications of the statements, for detail see [11].

4 Continuous Improvement of Plausibility and Pignistic Conflicts

There is a non-conflicting area around any BF (a half of the belief triangle in the case of BFs on Ω_2; there is a possibility of different variants of such areas using different conflicting sets Ω_{PlC} (or Ω_{BetC}) for BFs on Ω_n, $n > 2$). The idea is that conflict is zero on the border of conflicting area and it should continually increase without any jump behind the border. In the case of Ω_2, we compute a difference of Pl_P (or $BetP$) from U_2; for obtaining continuity we use minimal difference. Its value should be doubled to obtain normalised conflict, i.e. to obtain conflict between $(1, 0)$ and $(0, 1)$ equal to 1; see green line in Fig. 2, for simple (non doubled) difference see blue line. This is equal to the sum of minimal differences over Ω_{PlC} (it is $\emptyset$ or entire Ω_2 in the case of Ω_2). Thus we obtain the following modification of $Pl\text{-}C_0$:

$$Pl\text{-}C_1(Bel_1, Bel_2) = Pl\text{-}C_0(Bel_1, Bel_2) = 0,$$

if $(Bel_1(\{\omega_1\}) - Bel_1(\{\omega_2\}))(Bel_2(\{\omega_1\}) - Bel_2(\{\omega_2\})) \geq 0$.

$$Pl\text{-}C_1(Bel_1, Bel_2) = 2\, min\ (\ |Pl\_P(Bel_1)(\omega_1) - \frac{1}{2}|,\ |Pl\_P(Bel_2)(\omega_1) - \frac{1}{2}|\ )$$
$$= \sum_{i=1,2} min\ (\ |Pl\_P(Bel_1)(\omega_i) - \frac{1}{2}|,\ |Pl\_P(Bel_2)(\omega_i) - \frac{1}{2}|\ );$$

as $Pl\_P(Bel_i)(\omega_2) = 1 - Pl\_P(Bel_i)(\omega_1)$ for $i = 1, 2$.

The situation is more complicated on $\Omega_n = \{\omega_1, \omega_2, ...\omega_n\}$: We have to distinguish two parts of $\Omega_{PlC}(Bel_1, Bel_2)$: $\Omega_{PlC}^{opp}(Bel_1, Bel_2)$ ... ω's which are opposed by Bel_1 and Bel_2 (i.e. where $(Bel_1(\omega) - \frac{1}{n})(Bel_2(\omega) - \frac{1}{n}) < 0$) and $\Omega_{PlC}^{ord}(Bel_1, Bel_2) = \Omega_{PlC}(Bel_1, Bel_2) \setminus \Omega_{PlC}^{opp}(Bel_1, Bel_2)$ ... ω's from corresponding Ω_{cpPlC} and Ω_{cbPlC} which have different order of $Pl\_P(Bel_i)(\omega)$ values, but they are not opposed by Bel_1 and Bel_2, thus we have to handle them separately.

Let us assume that all ω's from $\Omega_{PlC}(Bel_1, Bel_2)$ are opposed by Bel_1 and Bel_2, i.e. $\Omega_{PlC}^{ord}(Bel_1, Bel_2) = \emptyset$, now. We can compute $Pl\text{-}C_1(Bel_1, Bel_2)$ 'per elements' directly in the same way as it is computed on Ω_2:

$$Pl\text{-}C_1(Bel_1, Bel_2) = \sum_{\omega \in \Omega_{PlC}(Bel_1, Bel_2)} min\ (\,|Pl\_P(Bel_1)(\omega) - \frac{1}{n}|,\ |Pl\_P(Bel_2)(\omega) - \frac{1}{n}|\,).$$

Let us look at the following example of belief functions Bel_1, ..., Bel_4 on $\Omega_{10} = \{\omega_1, \omega_2, ..., \omega_{10}\}$ such that, $Pl\_P_1(\omega_1) = 1$, $Pl\_P_2(\omega_{10}) = 1$. In this case,

we obtain $\Omega_{PlC}(Bel_1, Bel_2) = \Omega^{opp}_{PlC}(Bel_1, Bel_2) = \{\omega_1, \omega_{10}\}$, $Pl\text{-}C_1(Bel_1, Bel_2)$ $= \sum_{\omega\in\Omega_{PlC}} min\ (|Pl\_P_1(\omega) - \frac{1}{10}|, |Pl\_P_2(\omega) - \frac{1}{10}|) = \frac{1}{10} + \frac{1}{10} = \frac{2}{10}$.
$Pl\_P_3(\omega_1) = Pl\_P_3(\omega_2) = \frac{1}{2}$, $Pl\_P_4(\omega_9) = Pl\_P_4(\omega_{10}) = \frac{1}{2}$, $\Omega_{PlC}(Bel_3, Bel_4)$ $= \Omega^{opp}_{PlC}(Bel_3, Bel_4) = \{\omega_1, \omega_2, \omega_9, \omega_{10}\}$, thus $Pl\text{-}C_1(Bel_3, Bel_4) = 4 \cdot \frac{1}{10} = \frac{4}{10}$.
The conflict between two different categorical singletons $Pl\text{-}C_1(Bel_1, Bel_2)$ should be maximal/greatest (as different elements (disjoint hypotheses) are fully (categorically) supported). More precisely, it should be equal to 1 for normalised conflict. Moreover conflict $Pl\text{-}C_1(Bel_1, Bel_2)$ should be the same or greater than conflict $Pl\text{-}C_1(Bel_3,\ Bel_4)$, definitely not a half of it.

Considering the above example, we have to proportionalise comparison of $Pl\_P(Bel_j)(\omega_i)$ with $\frac{1}{n}$; i.e., to multiply $|Pl\_P(Bel_j)(\omega_i) - \frac{1}{n}|$ by appropriate coefficient(s):
- a coefficient $\frac{n}{2}$ determined by the size of frame of discernment;
  this factor is equal to $\frac{2}{2} = 1$ for $n = 2$;
- a coefficient $\frac{1}{2}(Pl\_P(Bel_1)(\omega_i) + Pl\_P(Bel_2)(\omega_i))$,
  i.e., by the relative size of sum of relative plausibilities of corresponding ω_i.

Thus we obtain:

$$Pl\text{-}C_{11}(Bel_1, Bel_2) = \sum_{\omega\in\Omega_{PlC}(Bel_1,Bel_2)} \frac{n}{2}\frac{Pl\_P_1(\omega) + Pl\_P_2(\omega)}{2}\ min_{i=1,2}\ (|Pl\_P_i(\omega) - \frac{1}{n}|).$$

For proving of continuity of $Pl\text{-}C_{11}$ we will use the following technical lemma, for proofs see [11].

Lemma 2. *(i) For any BFs Bel and Bel' on Ω_n such that $Bel' \in \delta(Bel)$ for $\delta = \frac{\varepsilon}{2^{n-1}}$ it holds that $|Pl\_P(\omega) - Pl\_P'(\omega)| \le \varepsilon$ for any $\omega \in \Omega_n$.*
(ii) For any BFs Bel_1, Bel_2 and Bel' on Ω_n such that $Bel' \in \delta(Bel_1)$ for $\delta \le min_{\omega_i\in\Omega_n}|Pl\_P_1(\omega_i) - \frac{1}{n}|$ it holds that $\Omega_{PlC}(Bel', Bel_2) = \Omega_{PlC}(Bel_1, Bel_2)$.

Analogously to the original version $Pl\text{-}C_0$ we need to minimize $m_{\bigcirc\!\!\!\cap}(\emptyset)$ (also for $Pl\text{-}C_{11}$), see BFs and Ω_{PlC} from Example 1 again. From the previous proof and Lemma 1 we obtain also continuity of $min(Pl\text{-}C_{11}, m_{\bigcirc\!\!\!\cap}(\emptyset))$.

For $\omega \in \Omega^{ord}_{PlC}$ we cannot use min of $(Pl\_P(\omega) - \frac{1}{n})$ as both the BFs are in accordance with respect to ω, and both of them support (or oppose) ω (thus min may be relatively high for BFs with same or similar $Pl\_P(\omega)$ and, on the other hand, it is very small for $Pl\_P_1(\omega)$ close to $\frac{1}{n}$ and $Pl\_P_2(\omega)$ close to 0 or 1.) Hence we have to use difference of differences, i.e., we have $||Pl\_P_1(\omega) - \frac{1}{n}| - |Pl\_P_2(\omega) - \frac{1}{n}|| = |Pl\_P_1(\omega) - Pl\_P_2(\omega)|$ as it is in $Pl\text{-}C$ (see [10]).

Thus we obtain the following formula:

$$Pl\text{-}C_{12}(Bel_1, Bel_2) = \sum_{\omega\in\Omega^{opp}_{PlC}(Bel_1,Bel_2)} \frac{n}{2}\frac{Pl\_P_1(\omega) + Pl\_P_2(\omega)}{2}\ min_{i=1,2}(|Pl\_P_i(\omega) - \frac{1}{n}|)$$

$$+ \sum_{\omega\in\Omega^{ord}_{PlC}(Bel_1,Bel_2)} (\,|Pl\_P_1(\omega) - Pl\_P_2(\omega)|\,).$$

A difference of Pl_P values is continuous, thus continuity is not lost upgrading $Pl\text{-}C_{11}$ to $Pl\text{-}C_{12}$.

The situation is analogous for $Bet\text{-}C$. For proving of continuity of $Bet\text{-}C_{12}$, where $BetP$'s are used instead of Pl_P's; we use the analogy of Lemma 2:

Lemma 3. *(i) For any BFs Bel and Bel′ on Ω_n such that $Bel' \in \delta(Bel)$ for $\delta = \frac{\varepsilon}{2^{n-1}}$ it holds that $|BetP(\omega) - BetP'(\omega)| \leq \varepsilon$ for any $\omega \in \Omega_n$.*
(ii) For any BFs Bel_1, Bel_2 and Bel' on Ω_n such that $Bel' \in \delta(Bel_1)$ for $\delta \leq min_{\omega_i \in \Omega_n} |BetP_1(\omega_i) - \frac{1}{n}|$ it holds that $\Omega_{BetC}(Bel', Bel_2) = \Omega_{BetC}(Bel_1, Bel_2)$.

For proofs see [11].

5 Extension of a Frame of Discernment

We have to note a relationship of $Pl\text{-}C_{12}(Bel_1, Bel_2)$ to resizement of a frame of discernment. An extension of a frame of discernment is to add one or more elements into the frame of discernment but keeping the BFs not changed. More precisely, let us suppose a frame $\Omega_m = \{\omega_1, \omega_2, ..., \omega_m\}$ and BFs Bel_i's given by bbms m_i's. Let us extend the frame with $\{\omega_{m+1}, ..., \omega_{m+k}\}$ for $m \geq 2$, $k \geq 1$. Let Bel_i''s be given by m_i''s such that $m_i'(X) = m_i(X)$ for $X \subseteq \Omega_m$, $m_i'(X) = 0$ for $X \cap \{\omega_{m+1}, ..., \omega_{m+k}\} \neq \emptyset$. Thus we have $Pl\_P_i'(\omega) = Pl\_P_i(\omega)$ for $\omega \in \Omega_m$ and $Pl\_P_i'(\omega) = 0$ for $\omega \in \Omega_{m+k} \setminus \Omega_m$. Comparing $Pl\_P_i'(\omega)$ with $\frac{1}{m+k} < \frac{1}{m}$ some ω's may be opposed by one of both Bel_i's but supported by Bel_i''s. If such ω is opposed by just one of Bel_i's and supported by both Bel_i''s, or it is opposed by both Bel_i's and supported just by one of Bel_i''s, there may be $\Omega'^{opp}_{PlC} \subsetneq \Omega^{opp}_{PlC}$ or $\Omega'^{opp}_{PlC} \supsetneq \Omega^{opp}_{PlC}$ (or $\Omega'^{opp}_{PlC} = \Omega^{opp}_{PlC}$ of course). Hence conflict may be increased or decreased with greater or less conflicting set on the extended frame of discernment. $\Omega'^{ord}_{PlC} = \Omega^{ord}_{PlC}$ as $Pl\_P_i'(\omega) = Pl\_P_i(\omega)$ for $\omega \in \Omega_m$ and $Pl\_P_i'(\omega) = 0$ out of Ω_m. See the following example:

Let $\Omega_m = \Omega_3$, $\Omega_{m+k} = \Omega_4$ and Bel_i's are given by m_i's as follows:

X :	$\{\omega_1\}$	$\{\omega_2\}$	$\{\omega_3\}$	$\{\omega_1,\omega_2\}$	$\{\omega_1,\omega_3\}$	$\{\omega_2,\omega_3\}$	$\{\omega_1,\omega_2,\omega_3\}$
$m_1(X)$:	0.60		0.10	0.1			0.2
$m_2(X)$:	0.20	0.05	0.35			0.3	0.1
$m_3(X)$:	0.45	0.15	0.10	0.1			0.2
$m_4(X)$:		0.25	0.40			0.2	0.15

Thus we obtain the following normalised plausibilities:
$Pl\_P_1 = (0.6, 0.2, 0.2)$ ... $Pl\_P_1' = (0.6, 0.2, 0.2, 0.0)$,
$Pl\_P_2 = (0.2, 0.3, 0.5)$ ... $Pl\_P_2' = (0.2, 0.3, 0.5, 0.0)$.
Only ω_1 is supported on both the frames by Bel_1 and Bel_1'; ω_2, ω_3 (and ω_4) are opposed. Only ω_3 is supported by Bel_2 on Ω_3, but ω_2 and ω_3 are supported by Bel_2' on Ω_4, thus we obtain different conflicting sets on the frames, namely, $\Omega_{PlC}(Bel_1, Bel_2) = \{\omega_1, \omega_3\}$ on Ω_3, whereas $\Omega_{PlC}(Bel_1', Bel_2') = \{\omega_1, \omega_2, \omega_3\}$ on Ω_4, as $\frac{1}{4} < Pl\_P_2(\omega_2) = 0.3 = Pl\_P_2'(\omega_2) < \frac{1}{3}$ and $Pl\_P_1(\omega_2) = 0.2 < \frac{1}{4} < \frac{1}{3}$. Hence conflicting set was increased with the extension of the frame.

Analogously $\Omega_{PlC}(Bel_3, Bel_4) = \{\omega_1, \omega_2, \omega_3\} \supsetneq \Omega_{PlC}(Bel_3', Bel_4') = \{\omega_1, \omega_3\}$.

Behaviour of conflict $Pl\text{-}C_{12}$ may be even more different on the original and extended frames in some cases. Let us look at the following example:

X :	$\{\omega_1\}$	$\{\omega_2\}$	$\{\omega_3\}$	$\{\omega_1,\omega_2\}$	$\{\omega_1,\omega_3\}$	$\{\omega_2,\omega_3\}$	$\{\omega_1,\omega_2,\omega_3\}$
$m_5(X)$:	0.15	0.05	0.15	0.2		0.1	0.35
$m_6(X)$:	0.20			0.6			0.2
$m_7(X)$:	0.50	0.05	0.05		0.3		0.1
$m_8(X)$:	0.20		0.20		0.2		0.4

Thus we obtain the following normalised plausibilities:
$Pl\_P_5 = (0.35, 0.35, 0.3)$ ... $Pl\_P'_5 = (0.35, 0.35, 0.3, 0.0)$,
$Pl\_P_6 = (0.5, 0.4, 0.1)$ ... $Pl\_P'_6 = (0.5, 0.4, 0.1, 0.0)$.
ω_1 and ω_2 are supported by both Bel_5 and Bel_6 on both the frames, ω_3 is opposed by both BFs on Ω_3 but only by Bel'_6 on Ω_4 whereas supported by Bel'_5, thus conflicting set $\{\omega_3\}$ appears when extending the frame. Hence the extension makes conflict between two originally $Pl\text{-}C_{12}$ non-conflicting BFs Bel_5 and Bel_6. Analogously $\Omega_{PlC}(Bel_7, Bel_8) = \{\omega_3\}$ becomes empty when extending the frame.

A very simple example is $Pl\text{-}C_{12}$ from BFs $(0.4, 0.6)$ and $(0.65, 0.35)$ on Ω_2 which becomes non-conflicting by any extension of the frame.

The above problem of $Pl\text{-}C_{12}$ with a change of conflicting sets when extending the frame of discernment is related only to Ω^{opp}_{PlC} not to Ω^{ord}_{PlC} as orderings of $Pl\_P(\omega_i)$ values are the same on the original Ω_m and on corresponding extended Ω_{m+k}. Thus the problem is related only to *spPl-C*, *cbPl-C*, not to *smPl-C*, *cpPl-C*. Hence we have obtained a new argument for using the latter versions of *Pl-C*.

We have either to accept a strange behaviour of $Pl\text{-}C_{12}$ when extending the frame of discernment, or to change the definition of supporting/opposing elements by BFs to be independent of extension of the frame of discernment or to concentrate ourselves to *smPl-C*, *cpPl-C* versions, as it is in the following.

Unfortunately, we are again at the beginning of the continuity problems. As *smPl-C*$((0.5, 0.5), (0.9, 0.1)$ is zero for BFs on Ω_2 as ω_1 has maximal Pl_P for both of them, but *smPl-C*$((0.49, 0.51), (0.9, 0.1) = \frac{1}{2}(0.41 + 0.41) = 0.41$. Moreover we have the same results and same problem regardless computing $\frac{1}{2}(|Pl\_P_1(\omega_1) - Pl\_P_2(\omega_1)| + |Pl\_P_1(\omega_2) - Pl\_P_2(\omega_2)|)$ or $\frac{1}{2}(|Pl\_P_1(\omega_1) - Pl\_P_1(\omega_2)| + |Pl\_P_2(\omega_1) - Pl\_P_2(\omega_2)|)$ on Ω_2; thus regardless whether we use differences of Pl_P_i per elements or differences between max and max but one value of the same Pl_P per BFs.

Nevertheless, we can apply the 'min idea' from $Pl\text{-}C_1$ to differences of the max Pl_P's values from max but one values of Pl_P's (Pl_P_1 and Pl_P_2) instead of differences of these values from $\frac{1}{n}$. Thus we obtain

$$Pl\text{-}C_{13}(Bel_1, Bel_2) = min\,(\,|Pl\_P_1(\omega_1) - Pl\_P_1(\omega_2)|,\ |Pl\_P_2(\omega_1) - Pl\_P_2(\omega_2)|\,)$$

for Bel_1, Bel_2 on Ω_2 such that $\Omega_{smPlC}(Bel_1, Bel_2) = \Omega_2$ on Ω_2; and generally

$$Pl\text{-}C_{13}(Bel_1, Bel_2) = min\,(\,|Pl\_P_1(\omega_i) - Pl\_P_1(\omega_j)|,\ |Pl\_P_2(\omega_k) - Pl\_P_2(\omega_l)|\,)$$

for Bel_1, Bel_2 on general finite frame Ω_n, where max $Pl\_P(\omega)$ values appear for ω_i and ω_k, $i \neq k$, and $Pl\_P_1(\omega_j)$, $Pl\_P_2(\omega_l)$ are max but one values of Pl_P's;

$$Pl\text{-}C_{13}(Bel_1, Bel_2) = 0$$

if sets of max values of Pl_P_1 and Pl_P_2 are not disjoint.

A proof of continuity. Let suppose a pair of BFs Bel_1, Bel_2 and a BFs Bel' in δ surrounding of Bel_1, such that $|m_1(X) - m'(X)| \leq \delta = \frac{\varepsilon}{2^n}$, thus $|Pl_1(X) - Pl'(X)| \leq 2^{n-1}\frac{\varepsilon}{2^n} = \frac{\varepsilon}{2}$ for any $X \subseteq \Omega$ and sequently also $|Pl\_P_1(\omega) - Pl\_P'(\omega)| \leq \frac{\varepsilon}{2}$ for any $\omega \in \Omega$, hence we obtain $|Pl\_P_1(\omega_i) - Pl\_P_1(\omega_j)| - |Pl\_P'(\omega_i) - Pl\_P'(\omega_j)| \leq |Pl\_P_1(\omega_i) - Pl\_P'(\omega_i)| + |Pl\_P_1(\omega_j) - Pl\_P'(\omega_j)| \leq \varepsilon$ and sequently $|Pl\text{-}C_{13}(Bel_1, Bel_2) - Pl\text{-}C_{13}(Bel', Bel_2)| = |min(|Pl\_P_1(\omega_i) - Pl\_P_1(\omega_j)|, |Pl\_P_2(\omega_k) - Pl\_P_2(\omega_l)|) - min(|Pl\_P'(\omega_i) - Pl\_P'(\omega_j)|, |Pl\_P_2(\omega_k) - Pl\_P_2(\omega_l)|)| \leq \varepsilon$ (for detail see [11]). Hence $Pl\text{-}C_{13}$ is continuous.

Values of m, Bel, Pl and of Pl_P are kept with an extension of the frame of discernment, thus also conflictness/non-conflictness and the size of $Pl\text{-}C_{13}$ are kept with a frame extension. Thus using $Pl\text{-}C_{13}$ instead of $Pl\text{-}C_0$ we obtain a continuous improvement $min(Pl\text{-}C_{13}, m_{\cap}(\emptyset))$ of $Pl\text{-}C$ which is preserved when extending the frame of discernment.

$Pl\text{-}C_{13}$ is a modification or analogy of $smPl\text{-}C$ in fact: if max but one value of Pl_P_1 appears for element(s) which has/have the max value of Pl_P_2 and vice versa then $Pl\text{-}C_{13}$ coincides with *sm* version of $Pl\text{-}C$ in some cases, but not in general. Thus, it seems neither easy nor useful useful to try to define a similar continuous improvement which is a modification of $cpPl\text{-}C$.

The above problems of Ω^{opp}_{PlC} and $Pl\text{-}C_{12}$ are in the same way relevant also to Ω^{opp}_{BetC} (not to Ω^{ord}_{BetC}) and $Bet\text{-}C_{12}$. Thus completely analogously to $Pl\text{-}C_{13}$, just using $BetP$'s instead of Pl_P's we can define $Bet\text{-}C_{13}$. Having Lemma 3, we can use also the above proof of continuity substituting Pl_P's with $BetP$'s. Values of $BetP$ are also kept with an extension of the frame of discernment, thus conflictness/non-conflictness and the size of $Bet\text{-}C_{13}$ are kept with a frame extension as well. Thus using $Bet\text{-}C_{13}$ instead of $Bet\text{-}C_0$ we obtain a continuous improvement $min(Bet\text{-}C_{13}, m_{\cap}(\emptyset))$ of $Bet\text{-}C$ which is preserved when extending the frame of discernment.

As in the case of $Pl\text{-}C_{13}$, $Bet\text{-}C_{13}$ is a modification of *sm* version conflict measure and it does not seems to be useful to try to define similar modification of $cpBet\text{-}C$.

6 Refinement of a Frame of Discernment

There is a completely different case of resizement of a frame of discernment, or the refinement of a frame. In this case, there are no new elements added but some of the original is/are split into one or more new one(s), thus bbm(s) of the split singleton(s) is/are transferred to the corresponding resulting set(s) and bbms of sets containing split element(s) are transferred to the corresponding larger sets. *Pl_P* and *BetP* have different behaviour in this case, hence *Pl-C* and *Bet-C* as well.

We can easily show that using neither *Bet-*C_{12} or *Bet-*C_{13} conflictness or non-conflictness of a pair of BFs is kept when refining the corresponding frame of discernment. It is enough to show the simple examples of BFs on Ω_2 and its refinement to $\{\omega_{11}; \omega_{12}; \omega_2\}$. Let us suppose a non-conflicting pair $(\mathbf{0.6}, 0.4)$ and $(\mathbf{0.8}, 0.2)$ where both the BFs support ω_1 and oppose ω_2. Refining the frame, we obtain $m'_1(\{\omega_{11}, \omega_{12}\}) = 0.6$, $m'_2(\{\omega_{11}, \omega_{12}\}) = 0.8$ and $BetP'_1 = (0.3, 0.3, \mathbf{0.4})$, $BetP'_2 = (\mathbf{0.4}, \mathbf{0.4}, 0.2)$, where ω_{11}, ω_{12} are supported by Bel'_2 but opposed by Bel'_1 and ω_2 is supported by Bel'_1 but opposed by Bel'_2. Thus *Bet-*C_{12} conflict has appeared when refining the frame. ω_2 has max $BetP'_1$ value, but max $BetP'_2$ value appears at ω_{11} and ω_{12}, hence also *Bet-*C_{13} conflict has appeared.

Let us further suppose Bel_3 given by $(0.2, \mathbf{0.8})$ on Ω_2. Refining the frame we obtain $m'_3(\{\omega_{11}, \omega_{12}\}) = 0.2$ and $BetP'_3 = (0.1, 0.1, \mathbf{0.8})$. Thus ω_{11}, ω_{12} are opposed by Bel'_3 and ω_2 is supported by Bel'_3, as by Bel'_1; moreover max $BetP'_3$ value appears at ω_2 as in the case of $BetP'_1$. Hence two conflicting BFs Bel_1 and Bel_3 became both *Bet-*C_{12} and *Bet-*C_{13} non-conflicting when the frame was refined.

Note that we can use the same examples to show the same property for *Bet-C* and *Bet-*C_{11}.

On the other hand the *Pl-*C_{13} conflictness/non-conflicteness is preserved by refinement of the frame (see Corollary 2; for proof of the lemma see [11]):

Lemma 4. *Ordering of Pl_P values is not changed with a refinement of a frame of discernment.*[3]

Corollary 1. *The sets of elements with the maximal (minimal) value of Pl_P are the same (up to refinement) for a belief functions Bel and Bel′ on an extended frame of discernment.*

Corollary 2. *Measure of conflict Pl-*C_{13} *keeps conflictness/non-conflictness of a pair of belief functions when the frame of discernment is refined.*

The situation is more complicated for *Pl-C*, *Pl-*C_{11} and *Pl-*C_{12}: Orderings of the *Pl_P* values (and max/min values) are kept when refining a frame; thus also *sm* and *cp* versions of conflictness/non-conflictness. But there is possibility of change of support/opposition of other elements; thus change of $sp\Omega_{PlC}$ and $cb\Omega_{PlC}$ and also of *sp* and *cb* versions of *Pl-C*, *Pl-*C_{11} and *Pl-*C_{12} conflictness/ non-conflictness.

7 A Comparison of the Presented Measures of Conflict

Comparing the series of *Pl-C* using *Pl-*C_0, *Pl-*C_{11}, *Pl-*C_{12}, *Pl-*C_{13} (and analogously *Bet-C* using *Bet-*C_0, *Bet-*C_{11}, *Bet-*C_{12}, *Bet-*C_{13}) we see step-wise improvement from *Pl-*C_0 to *Pl-*C_{13} (and from *Bet-*C_0 to *Bet-*C_{13}) from the point of view of the investigated properties, see Table 1.

[3] Unfortunately, after completion of this text, we have realized, that Lemma 4 and its corollaries hold true only under a special condition; for correction see [11].

Table 1. A comparison of properties of conflict measures and their components

property → measure ↓	Cont.	Conf/NonC distinguish.	SmallVal	BigVal	Extens. equal.	Extension Conf/NonC	Refinement Conf/NonC	Refin. equal.
$\emptyset$ (Diff *Pl_P*)	+	-	-	+/-	=	=	+/- / N.A.	≠
cf	+	-	-	+/-	=	=	C↔N / N.A.	≠
Pl-C	-	+	-	+/-	≠	C↔N	(*)	≠
Bet-C	-	+	-	+/-	≠	C↔N	C↔N	≠
Pl-C_{11}	+	+	+/-	+/-	≠	C↔N	(*)	≠
Bet-C_{11}	+	+	+/-	+/-	≠	C↔N	C↔N	≠
Pl-C_{12}	+	+	+/-	+/-	≠	C↔N	(*)	≠
Bet-C_{12}	+	+	+/-	+/-	≠	C↔N	C↔N	≠
Pl-C_{13}	+	+	+/-	+/-	=	=	+	≠
Bet-C_{13}	+	+	+/-	+/-	=	=	C↔N	≠

Explanation:

+	property is satisfied,
-	property is not satisfied / values are not acceptable,
+/-	we can accept the values as an approximation of values of conflict,
C↔N	**c**onflicting pair of BFs may become a **n**on-conflicting (and vice versa) when resizing the frame of discernment,
(*)	elements with maximal preprerence / opposition are the same, nevertheless the property is not satisfied in general (*cp* and *cb* conflicts).

Further we have to note that:
"$\emptyset$ (Diff *Pl_P*)" is a *Pl_P* version of *cf* (is has not been mentioned anywhere, it is here just for a comparison of the properties);
each of *Pl-C*, *Bet-C*, *Pl-*C_{12}, *Bet-*C_{12} have four variants (*sm*, *sp*, *cp*, *cb* according to 4 variants of conflicting sets Ω_{PlC} or Ω_{BetC});
Pl-C_{11}, *Bet-*C_{11} suppose $\Omega_{PlC}^{ord} = \emptyset$, $\Omega_{BetC}^{ord} = \emptyset$, thus there are two variants of each of them (*sm* and *sp*);
Pl-C_{13}, *Bet-*C_{13} classify conflictness/non-conflictness according to max and max but one values of Pl_P_i, $BetP_i$ (there is the only variant analogous to *sm* but not the same as 2–4 elements play their role here).

The conflict measures using *Pl-*C_{13} and *Bet-*C_{13} are also simpler in comparison with previous both theoretically and from the computational point of view. Only a modified version of simple conflicting set is used there, hence there are not four variants (*sm*, *sp*, *cp*, *cb*) and thus the computation is also simpler or equal in comparison with the previous measures.

In the case of the series of the measures based on *BetP*, *Bet-C* using *Bet-*C_{13} is also improvement of Liu's degree of conflict *cf* from the point of view all these properties, whereas original version using *Bet-*C_0 was improvement only from the point of view better and clearer distinguishing of conflictness/non-conflictness and using $m_{\bigcirc\!\!\!\!\cap}$ as upper bound, on the other hand continuity and robustness with respect to an extension of a frame of discernment was lost.

When comparing *Pl-C* with *Bet-C* we have obtained a new argument in favour of *Pl-C*, that is its keeping of conflictness/non-conflictness when a frame of discernment is refined. The original arguments mentioned already in [10] are better interpretation of *Pl-C* and its compatibility with Dempster's rule based on commutativity of Pl_P with Dempster's rule [3,5]. It is also strengthened by keeping zero/non-zero values by $difPl\_P_{m_i}^{m_j}$ (a Pl_P version of $difBetP_{m_i}^{m_j}$ when a frame of discernment is refined, see value "+/- / N.A." in "$\emptyset$" row of Table 1.

8 Open Problems and Ideas for a Future Research

Investigating and improving measures of conflicts of BFs we have met the following open problems:

- Pl-C_{13} does not use conflicting sets, there is no problem with BFs from Example 1, thus there is a question whether it holds Pl-$C_{13}(Bel_1, Bel_2) \leq (m_1 \circledcirc m_2)(\emptyset)$ or not.
- Analogously whether it holds Bet-$C_{13}(Bel_1, Bel_2) \leq (m_1 \circledcirc m_2)(\emptyset)$ or not.
- To look for an alternative support/opposition of ω by a BF not depending from resizement of a frame of discernment.
- Investigation of an idea to use Pl_P or $BetP$ for classification of conflictness/ non-conflictness only, and look for an appropriate distance of BFs (not transformed to probabilities) to use it for determination of conflict of BFs which were already classified as conflicting, i.e., which are in some positive conflict. (This partial "step back" may be either useful or a dead end procedure).

9 Conclusion

A series of gradual improvements of two measures of conflict between belief functions, plausibility conflict Pl-C and pignistic conflict Bet-C, are presented in this theoretical contribution. The measures are improved from the point of view of their continuity and robustness with respect to resizing of a frame of discernment: its extension and refinement. Bet-C is now a real improvement of Liu's degree of conflict cf.

Higher robustness of Pl-C with respect to frame refinement is a new argument in favour of the measure based on normalised plausibility of singletons against the measure based on Smets' pignistic probability.

Improved conflict measures both increase our general understanding of the nature of belief functions and can be applied in better combination of conflicting belief functions in numerous applications of the real world.

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