On the Expressiveness of Metric Temporal Logic over Bounded Timed Words*

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Abstract. It is known that Metric Temporal Logic (MTL) is strictly less expressive than the Monadic First-Order Logic of Order and Metric ($\mathsf{FO}[<, +1]$) in the pointwise semantics over bounded time domains (i.e., timed words of bounded duration) [15]. In this paper, we present an extension of MTL which has the same expressive power as $\mathsf{FO}[<, +1]$ in both the pointwise and continuous semantics over bounded time domains.

1 Introduction

One of the most prominent specification formalisms used in verification is *Linear Temporal Logic* (LTL), which is typically interpreted over the non-negative integers or reals. A celebrated result of Kamp [9] states that, in either case, LTL has precisely the same expressive power as the *Monadic First-Order Logic of Order* ($\mathsf{FO}[<]$). These logics, however, are inadequate to express specifications for systems whose correct behaviour depends on quantitative timing requirements. Over the last three decades, much work has therefore gone into lifting classical verification formalisms and results to the real-time setting. *Metric Temporal Logic* (MTL), which extends LTL by constraining the temporal operators by time intervals, was introduced by Koymans [10] in 1990 and has emerged as a central real-time specification formalism.

MTL enjoys two main semantics, depending intuitively on whether atomic formulas are interpreted as *state predicates* or as (instantaneous) *events*. In the former, the system is assumed to be under observation at every instant in time, leading to a 'continuous' semantics based on *flows* or *signals*, whereas in the latter, observations of the system are taken to be (finite or infinite) sequences of timestamped snapshots, leading to a 'pointwise' semantics based on *timed words*. Timed words are the leading interpretation, for example, for systems modelled as timed automata [1]. In both cases, the time domain is usually taken to be the non-negative real numbers. Both semantics have been extensively studied; see, e.g., [12] for a historical account.

Alongside these developments, researchers proposed the *Monadic First-Order Logic of Order and Metric* ($\mathsf{FO}[<, +1]$) as a natural quantitative extension of

* More extensive technical details as well as all proofs can be found in the full version of this paper [5].

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$\mathsf{FO}[<]$. Unfortunately, Hirshfeld and Rabinovich [4] showed that no 'finitary' extension of MTL—and *a fortiori* MTL itself—could have the same expressive power as $\mathsf{FO}[<,+1]$ over the reals.[1] Still, in the continuous semantics, MTL can be made expressively complete for $\mathsf{FO}[<,+1]$ by extending the logic with an infinite family of '*counting modalities*' [7] or considering only *bounded* time domains [11,13]. Nonetheless, and rather surprisingly, MTL with counting modalities remains strictly less expressive than $\mathsf{FO}[<,+1]$ over bounded time domains in the pointwise semantics, i.e., over timed words of bounded duration, as we will see in Section 3.

The main result of this paper is to show that MTL, equipped with both the forwards and backwards temporal modalities 'generalised Until' ($\mathfrak{U}$) and 'generalised Since' ($\mathfrak{S}$), has precisely the same expressive power as $\mathsf{FO}[<,+1]$ over bounded time domains in the pointwise semantics (and also, trivially, in the continuous semantics). This extended version of Metric Temporal Logic, written $\mathsf{MTL}[\mathfrak{U},\mathfrak{S}]$, therefore yields a definitive real-time analogue of Kamp's theorem over bounded domains.

It is worth noting that $\mathsf{MTL}[\mathfrak{U},\mathfrak{S}]$ satisfiability and model checking (against timed automata) are decidable over bounded time domains, thanks to the decidability of $\mathsf{FO}[<,+1]$ over such domains as established in [11,13]. Unfortunately, $\mathsf{FO}[<,+1]$ has non-elementary complexity, whereas the time-bounded satisfiability and model-checking problems for MTL are EXPSPACE-complete [11,13]. However, it can easily be seen by inspecting the relevant constructions that the complexity bounds for MTL carry over to our new logic $\mathsf{MTL}[\mathfrak{U},\mathfrak{S}]$.

2 Preliminaries

2.1 Timed Words

A *time sequence* $\tau = \tau_1\tau_2\dots\tau_n$ is a non-empty finite sequence over non-negative reals (called *timestamps*) that satisfies the requirements below (we denote the length of τ by $|\tau|$):

- *Initialisation*[2]: $\tau_1 = 0$
- *Strict monotonicity*: For all $i, 1 \leq i < |\tau|$, we have $\tau_i < \tau_{i+1}$.

A ***timed word*** over finite alphabet Σ is a pair $\rho = (\sigma, \tau)$, where $\sigma = \sigma_1\sigma_2\dots\sigma_n$ is a non-empty finite word over Σ and τ is a time sequence of the same length. We refer to each (σ_i, τ_i) as an *event*. In this sense, a timed word

[1] Hirshfeld and Rabinovich's result was only stated and proved for the continuous semantics, but we believe that their approach would also carry through for the pointwise semantics. In any case, using different techniques Prabhakar and D'Souza [15] and Pandya and Shah [14] independently showed that MTL is strictly weaker than $\mathsf{FO}[<,+1]$ in the pointwise semantics.

[2] This requirement is natural in the present context as all the logics we consider in this paper are *translation invariant*: two timed words are indistinguishable by formulas (of these logics) if they only differ by a fixed delay.

can be regarded as a sequence of events. We denote by $|\rho|$ the number of events in ρ. A *position* in ρ is a number i such that $1 \le i \le |\rho|$. The *duration* of ρ is defined as $\tau_{|\rho|}$. A $\mathbb{T}$-timed word is a timed word all of whose timestamps are in $\mathbb{T}$, where $\mathbb{T}$ is either $[0, N)$, for some $N \in \mathbb{N}$, or $\mathbb{R}_{\ge 0}$.

Note that we are focussing on finite timed words. Our results carry over to the case of (Zeno) infinite timed words as well, with some modifications.

2.2 Metric Logics

We first define a metric predicate logic $\mathsf{FO}[<, +1]$ and its pointwise interpretation. This logic will serve as a 'yardstick' of expressiveness. In the sequel, we write $\Sigma_{\mathbf{P}} = 2^{\mathbf{P}}$ for a set of monadic predicates $\mathbf{P}$.

Definition 1. *Given a set of monadic predicates* $\mathbf{P}$*, the set of* $\mathsf{FO}[<, +1]$ *formulas is generated by the grammar*

$$\vartheta ::= P(x) \mid x < x' \mid d(x, x') \sim c \mid \mathbf{true} \mid \vartheta_1 \land \vartheta_2 \mid \neg\vartheta \mid \exists x\, \vartheta\,,$$

where $P \in \mathbf{P}$*,* x, x' *are variables,* $\sim \in \{=, \neq, <, >, \le, \ge\}$ *and* $c \in \mathbb{N}$*.*[3]

With each $\mathbb{T}$-timed word $\rho = (\sigma, \tau)$ over $\Sigma_{\mathbf{P}}$ we associate a structure M_ρ. Its universe U_ρ is the finite subset $\{\tau_i \mid 1 \le i \le |\rho|\}$ of $\mathbb{T}$. The order relation $<$ and monadic predicates in $\mathbf{P}$ are interpreted in the expected way. For example, $P(\tau_i)$ holds in M_ρ iff $P \in \sigma_i$. The binary *distance predicate* $d(x, x') \sim c$ holds iff $|x - x'| \sim c$. The satisfaction relation is defined inductively as usual. We write $M_\rho, t_1, \ldots, t_n \models \vartheta(x_1, \ldots, x_n)$ (or $\rho, t_1, \ldots, t_n \models \vartheta(x_1, \ldots, x_n)$) if $t_1, \ldots, t_n \in U_\rho$ and $\vartheta(t_1, \ldots, t_n)$ holds in M_ρ. We say that $\mathsf{FO}[<, +1]$ formulas $\vartheta_1(x)$ and $\vartheta_2(x)$ are *equivalent* over $\mathbb{T}$-timed words if for all $\mathbb{T}$-timed words ρ and $t \in U_\rho$,

$$\rho, t \models \vartheta_1(x) \iff \rho, t \models \vartheta_2(x)\,.$$

Formulas of metric temporal logics are built from monadic predicates using Boolean connectives and ***modalities***. A k-ary modality is defined by an $\mathsf{FO}[<, +1]$ formula $\varphi(x, X_1, \ldots, X_k)$ with a single free first-order variable x and k free monadic predicates $X_1, \ldots, X_k$. For example, the MTL modality $\mathsf{U}_{(0,5)}$ is defined by

$$\mathsf{U}_{(0,5)}(x, X_1, X_2) = \exists x' \Big(x < x' \land d(x, x') < 5 \land X_2(x') \land \forall x'' \left(x < x'' \land x'' < x' \implies X_1(x'')\right)\Big)\,.$$

The MTL formula $\varphi_1 \, \mathsf{U}_{(0,5)} \, \varphi_2$ (using infix notation) is obtained by substituting MTL formulas φ_1, φ_2 for X_1, X_2, respectively.

[3] Note that whilst we still refer to the logic as $\mathsf{FO}[<, +1]$, we adopt here an equivalent definition using a binary distance predicate $d(x, x')$ (as in [16]) in place of the usual $+1$ function symbol.

Definition 2. *Given a set of monadic predicates* $\mathbf{P}$*, the set of* MTL *formulas is generated by the grammar*

$$\varphi ::= P \mid \mathbf{true} \mid \varphi_1 \wedge \varphi_2 \mid \neg\varphi \mid \varphi_1 \mathsf{U}_I \varphi_2 \mid \varphi_1 \mathsf{S}_I \varphi_2 \,,$$

where $P \in \mathbf{P}$ *and* $I \subseteq (0, \infty)$ *is an interval with endpoints in* $\mathbb{N} \cup \{\infty\}$*.*

The (future-only) fragment $\mathsf{MTL}_{\mathsf{fut}}$ is obtained by banning subformulas of the form $\varphi_1 \mathsf{S}_I \varphi_2$. If I is not present as a subscript to a given modality then it is assumed to be $(0, \infty)$. We sometimes use pseudo-arithmetic expressions to denote intervals, e.g., '≥ 1' denotes $[1, \infty)$ and '$= 1$' denotes the singleton $\{1\}$. We also employ the usual syntactic sugar, e.g., $\mathbf{false} \equiv \neg\mathbf{true}$, $\mathsf{F}_I\varphi \equiv \mathbf{true}\, \mathsf{U}_I \varphi$, $\overleftarrow{\mathsf{F}}_I\varphi \equiv \mathbf{true}\, \mathsf{S}_I \varphi$, $\mathsf{G}_I\varphi \equiv \neg\mathsf{F}_I\neg\varphi$ and $\mathsf{X}_I\varphi \equiv \mathbf{false}\, \mathsf{U}_I \varphi$, etc. For the sake of completeness, we give a traditional inductive definition of the satisfaction relation of MTL below.

Definition 3. *The satisfaction relation* $(\rho, i) \models \varphi$ *for an* MTL *formula* φ*, a timed word* $\rho = (\sigma, \tau)$ *and a position* i *in* ρ *is defined as follows:*

- $(\rho, i) \models P$ *iff* $P(\tau_i)$ *holds in* M_ρ
- $(\rho, i) \models \mathbf{true}$
- $(\rho, i) \models \varphi_1 \wedge \varphi_2$ *iff* $(\rho, i) \models \varphi_1$ *and* $(\rho, i) \models \varphi_2$
- $(\rho, i) \models \neg\varphi$ *iff* $(\rho, i) \not\models \varphi$
- $(\rho, i) \models \varphi_1 \mathsf{U}_I \varphi_2$ *iff there exists* $j, i < j \leq |\rho|$ *such that* $(\rho, j) \models \varphi_2$, $\tau_j - \tau_i \in I$, *and* $(\rho, k) \models \varphi_1$ *for all* k *with* $i < k < j$
- $(\rho, i) \models \varphi_1 \mathsf{S}_I \varphi_2$ *iff there exists* j, $1 \leq j < i$ *such that* $(\rho, j) \models \varphi_2$, $\tau_i - \tau_j \in I$ *and* $(\rho, k) \models \varphi_1$ *for all* k *with* $j < k < i$.

Note that we adopt strict versions of temporal modalities, e.g., φ_2 holds at i does not imply that $\varphi_1 \mathsf{U}\, \varphi_2$ holds at i. We write $\rho \models \varphi$ if $(\rho, 1) \models \varphi$.

2.3 Relative Expressiveness

Let L, L' be two metric logics. We say that L' is ***expressively complete for*** L over $\mathbb{T}$-timed words if for any formula $\vartheta(x) \in L$, there is an equivalent formula $\varphi(x) \in L'$ over $\mathbb{T}$-timed words.

3 Expressiveness

In this section, we present a sequence of successively more expressive extensions of $\mathsf{MTL}_{\mathsf{fut}}$ over bounded timed words. Along the way we highlight the key features that give rise to the differences in expressiveness. The necessity of a 'new' extension (such as the one in the next section) is justified by the fact that no known extension can lead to expressive completeness.

3.1 Definability of Time 0

Recall that $\mathsf{MTL}_{\mathsf{fut}}$ and $\mathsf{FO}[<, +1]$ have the same expressiveness over continuous domains of the form $[0, N)$ [11,13], a result that fails over $[0, N)$-timed words. To account for this difference between the two semantics, observe that a distinctive feature of the continuous interpretation of $\mathsf{MTL}_{\mathsf{fut}}$ is exploited in [11,13]: in any $[0, N)$-flow, the formula $\mathsf{F}_{=(N-1)}\mathbf{true}$ holds in $[0, 1)$ and nowhere else. One can make use of conjunctions of similar formulas to determine which unit interval the current instant is in. Unfortunately, this trick does not work for $\mathsf{MTL}_{\mathsf{fut}}$ in the pointwise semantics. However, it can be achieved in MTL by using past modalities. Let

$$\varphi_{i,i+1} = \overleftarrow{\mathsf{F}}_{[i,i+1)}(\neg\overleftarrow{\mathsf{F}}\mathbf{true})$$

and $\Phi_{unit} = \{\varphi_{i,i+1} \mid i \in \mathbb{N}\}$. It is clear that $\varphi_{i,i+1}$ holds only in $[i, i+1)$ and nowhere else. Denote by $\mathsf{MTL}_{\mathsf{fut}}[\Phi_{unit}]$ the extension of $\mathsf{MTL}_{\mathsf{fut}}$ obtained by allowing these formulas as subformulas. This very restrictive use of past modalities strictly increases the expressiveness of $\mathsf{MTL}_{\mathsf{fut}}$. Indeed, our main result depends crucially on the use of these formulas.

Proposition 1. *$\mathsf{MTL}_{\mathsf{fut}}[\Phi_{unit}]$ is strictly more expressive than $\mathsf{MTL}_{\mathsf{fut}}$ over $[0, N)$-timed words.*

3.2 Past Modalities

The following proposition says that the conservative extension in the last subsection is not sufficient for obtaining expressive completeness: non-trivial nesting of future modalities and past modalities provides more expressiveness.

Proposition 2. *MTL is strictly more expressive than $\mathsf{MTL}_{\mathsf{fut}}[\Phi_{unit}]$ over $[0, N)$-timed words.*

3.3 Counting Modalities

The modality $C_n(x, X)$ asserts that X holds at least at n points in the open interval $(x, x+1)$. The modalities C_n for $n \geq 2$ are called *counting modalities*. It is well-known that these modalities are inexpressible in MTL over $\mathbb{R}_{\geq 0}$-flows [3]. For this reason, they (or variants thereof) are often used to separate the expressiveness of various metric logics (cf., e.g., [2,14,15]). For example, the $\mathsf{FO}[<, +1]$ formula

$$\vartheta_{pqr}(x) = \exists y \Big(x < y \land P(y) \land \exists y' \big(y < y' \land d(y, y') > 1 \land d(y, y') < 2 \land Q(y') \\ \land \exists y'' \left(y' < y'' \land d(y, y'') > 1 \land d(y, y'') < 2 \land R(y'') \right) \big) \Big)$$

has no equivalent in MTL over $\mathbb{R}_{\geq 0}$-timed words [14]. Indeed, it was shown recently that in the continuous semantics, MTL with counting modalities and their past versions (which we denote by $\mathsf{MTL}[\{C_n, \overleftarrow{C}_n\}_{n=2}^{\infty}]$) is expressively complete for $\mathsf{FO}[<, +1]$ [7]. However, counting modalities add no expressiveness to MTL in the time-bounded setting. To see this, observe that the following formula is equivalent to ϑ_{pqr} over $[0, N)$-timed words (we make use of formulas in Φ_{unit} defined in Section 3.1)

$$\mathsf{F}\Bigg(\bigvee_{i\in[0,N-1]} \Big(P \wedge \varphi_{i,i+1} \wedge \Big(\mathsf{F}_{>1}\big(Q \wedge \mathsf{F}(R \wedge \varphi_{i+1,i+2})\big) \vee \mathsf{F}_{<2}\big(R \wedge \varphi_{i+2,i+3} \wedge \overleftarrow{\mathsf{F}}(Q \wedge \varphi_{i+2,i+3})\big) \vee \big(\mathsf{F}_{>1}(Q \wedge \varphi_{i+1,i+2}) \wedge \mathsf{F}_{<2}(R \wedge \varphi_{i+2,i+3})\big)\Big)\Big)\Bigg) .$$

The same idea can be generalised to handle counting modalities and their past counterparts.

Proposition 3. MTL *is expressively complete for* $\mathsf{MTL}[\{C_n, \overleftarrow{C}_n\}_{n=2}^{\infty}]$ *over* $[0, N)$*-timed words.*

3.4 Non-Local Properties: One Reference Point

Proposition 3 shows that part of the expressiveness hierarchy over $\mathbb{R}_{\geq 0}$-timed words collapses in the time-bounded setting. Nonetheless, MTL is still not expressive enough to capture all of $\mathsf{FO}[<, +1]$. Recall that another feature of the continuous interpretation of $\mathsf{MTL}_{\mathsf{fut}}$ used in the proof in [11, 13] is that $\mathsf{F}_{=k}\varphi$ holds at t *iff* φ holds at $t + k$. Suppose that we want to specify the following property over $\mathbf{P} = \{P, Q\}$ at the current time t_1 for some integer constant $a > 0$:

- There is an event at time $t_2 > t_1 + a$ where Q holds
- P holds at all events in $(t_1 + a, t_2)$.

In the continuous semantics, by introducing a special monadic predicate P_ϵ that holds at all 'no-event' points in the flow, the property can easily be expressed as

$$\varphi_{cont1} = \mathsf{F}_{=a}\big((P \vee P_\epsilon) \,\mathsf{U}\, Q\big) .$$

See Figure 1 for an illustration. Filled boxes denote events at which $\neg P \wedge Q$ holds whereas hollow boxes denote events at which $P \wedge \neg Q$ holds. The formula φ_{cont1} holds at t_1 in the continuous semantics.

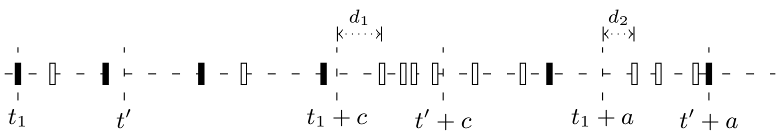

Fig. 1. φ_{cont1} holds at t_1 in the continuous semantics

In essence, when the current time is t_1, the continuous interpretation of MTL allows one to speak of properties 'around' t_1+a regardless of whether there is an event at t_1+a. The same is not readily possible with the pointwise interpretation of MTL if there is no event at t_1+a. To handle this issue within the pointwise semantic framework, we introduce a relatively simple family of modalities $\mathcal{B}^{\rightarrow}_I$ ('Beginning') and their past versions $\mathcal{B}^{\leftarrow}_I$. They can be used to specify the *first* events in given intervals. For example, the following modality asserts that X holds at the first event in (a,b):

$$\mathcal{B}^{\rightarrow}_{(a,b)}(x,X) = \exists x' \Big(x < x' \wedge d(x,x') > a \wedge d(x,x') < b \wedge X(x') \\ \wedge \nexists x'' \left(x < x'' \wedge x'' < x' \wedge d(x,x'') > a\right)\Big) .$$

Now the property above can be defined as $\mathcal{B}^{\rightarrow}_{(a,\infty)}\big(Q \vee (P \,\mathsf{U}\, Q)\big)$. We refer to the extension of MTL with $\mathcal{B}^{\rightarrow}_I, \mathcal{B}^{\leftarrow}_I$ as $\mathsf{MTL}[\mathcal{B}^{\leftrightarrows}]$.

The following proposition states that this extension is indeed non-trivial.

Proposition 4. $\mathsf{MTL}[\mathcal{B}^{\leftrightarrows}]$ *is strictly more expressive than* MTL *over* $[0,N)$-*timed words.*

3.5 Non-local Properties: Two Reference Points

Adding modalities $\mathcal{B}^{\rightarrow}_I, \mathcal{B}^{\leftarrow}_I$ to MTL allows one to specify properties with respect to a distant time point even when there is no event at that point. However, the following proposition shows that this is still not enough for expressive completeness.

Proposition 5. $\mathsf{FO}[<,+1]$ *is strictly more expressive than* $\mathsf{MTL}[\mathcal{B}^{\leftrightarrows}]$ *over* $[0,N)$-*timed words.*

Proof. This is similar to a proof in [15, Section 7]. Given $m \in \mathbb{N}$, we construct two models as follows. Let

$$\mathcal{G}_m = (\emptyset,0)(\emptyset,\frac{0.5}{2m+3})(\emptyset,\frac{1.5}{2m+3})\dots(\emptyset,1-\frac{0.5}{2m+3}) \\ (\emptyset,1+\frac{0.5}{2m+2})(\emptyset,1+\frac{1.5}{2m+2})\dots\dots(\emptyset,2-\frac{0.5}{2m+2}) .$$

$\mathcal{H}_m$ is constructed as $\mathcal{G}_m$ except that the event at time $\frac{m+1.5}{2m+3}$ is missing.

Figure 2 illustrates the models for the case $m=2$ where hollow boxes represent events at which no monadic predicate holds. It can be proved that no $\mathsf{MTL}[\mathcal{B}^{\leftrightarrows}]$ formula of modal depth $\leq m$ distinguishes $\mathcal{G}_m$ and $\mathcal{H}_m$ while the $\mathsf{FO}[<,+1]$ formula

$$\exists x \Bigg(\nexists y\,(y < x) \wedge \exists x' \Big(d(x,x') > 1 \wedge d(x,x') < 2 \\ \wedge \exists x'' \big(x' < x'' \wedge \nexists y'\,(x' < y' \wedge y' < x'') \\ \wedge \nexists y''\,\left(d(x',y'') < 1 \wedge d(x'',y'') > 1\right)\big)\Big)\Bigg)$$

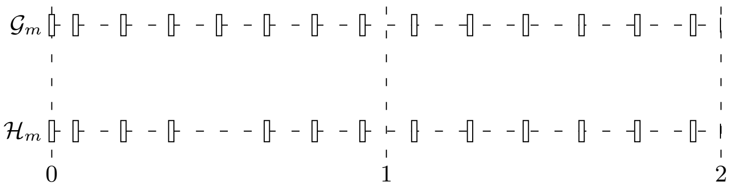

Fig. 2. Models $\mathcal{G}_m$ and $\mathcal{H}_m$ for $m = 2$

distinguishes $\mathcal{G}_m$ and $\mathcal{H}_m$ for any $m \in \mathbb{N}$. □

One way to understand this phenomenon is to consider the arity of MTL operators. Let the current time be t_1. Suppose that we want to specify the following property ($a > c > 0$):

- There is an event at $t_2 > t_1 + a$ where Q holds
- P holds at all events in $\big(t_1 + c, t_1 + c + (t_2 - t_1 - a)\big)$.

In the continuous semantics one can simply write

$$\varphi_{cont2} = \big(\mathsf{F}_{=c}(P \vee P_\epsilon)\big) \,\mathsf{U}\, (\mathsf{F}_{=a}Q)\,.$$

Observe how this formula (effectively) talks about properties around two points: $t_1 + c$ and $t_1 + a$. In the same vein, the following formula distinguishes $\mathcal{G}_m$ and $\mathcal{H}_m$ in the continuous semantics:

$$\varphi_{cont3} = \mathsf{F}_{(1,2)}\big(\neg P_\epsilon \wedge (\overleftarrow{\mathsf{F}}_{=1}P_\epsilon) \,\mathsf{U}\, (\neg P_\epsilon)\big)\,.$$

In the next section, we propose new modalities that add this ability to MTL in the pointwise semantics. We show later that this ability is exactly the missing piece of expressiveness.

4 New Modalities

4.1 Generalised 'Until' and 'Since'

We introduce a family of modalities which can be understood as generalisations of the usual 'Until' and 'Since' modalities. Let $I \subseteq (0, \infty)$ be an interval with endpoints in $\mathbb{N} \cup \{\infty\}$ and $c \in \mathbb{N}$. The formula $\varphi_1 \,\mathfrak{U}^c_I\, \varphi_2$ (using infix notation), when imposed at t_1, asserts that

- There is an event at t_2 where φ_2 holds and $t_2 - t_1 \in I$
- φ_1 holds at all events in $\Big(c, c + \Big(t_2 - \big(t_1 + \inf(I)\big)\Big)\Big)$.

Formally, for $I = (a, b)$ and $a \geq c \geq 0$, we define the generalised 'Until' modality $\mathfrak{U}^c_{(a,b)}$ by the following $\mathsf{FO}[<, +1]$ formula:

$$\begin{aligned}\mathfrak{U}^c_{(a,b)}(x, X_1, X_2) = \exists x' \Big(x < x' \wedge d(x, x') > a \wedge d(x, x') < b \wedge X_2(x') \\ \wedge \forall x'' \big(x < x'' \wedge d(x, x'') > c \wedge x'' < x' \\ \wedge d(x', x'') > (a - c) \implies X_1(x'')\big)\Big) .\end{aligned}$$

Symmetrically, we define the generalised 'Since' modality $\mathfrak{S}^c_{(a,b)}$ for $I = (a, b)$ and $a \geq c \geq 0$:

$$\begin{aligned}\mathfrak{S}^c_{(a,b)}(x, X_1, X_2) = \exists x' \Big(x' < x \wedge d(x, x') > a \wedge d(x, x') < b \wedge X_2(x') \\ \wedge \forall x'' \big(x'' < x \wedge d(x, x'') > c \wedge x' < x'' \\ \wedge d(x', x'') > (a - c) \implies X_1(x'')\big)\Big) .\end{aligned}$$

The modalities for I being a half-open interval or a closed interval can be defined similarly. We will refer to the logic obtained by adding these modalities to MTL as $\mathsf{MTL}[\mathfrak{U}, \mathfrak{S}]$. Note that the usual 'Until' and 'Since' modalities can be written in terms of generalised modalities. For instance,

$$\varphi_1 \, \mathsf{U}_{(a,b)} \, \varphi_2 = \varphi_1 \, \mathfrak{U}^a_{(a,b)} \, \varphi_2 \wedge \neg \Big(\mathbf{true} \, \mathfrak{U}^0_{(0,a]} \, (\neg \varphi_1)\Big) .$$

4.2 More Liberal Bounds

In the definition of modalities $\mathfrak{U}^c_I$ and $\mathfrak{S}^c_I$ in the last subsection, we stressed that $I \subseteq (0, \infty)$ and $\inf(I) \geq c \geq 0$. This is because more liberal usage of bounds are indeed merely syntactic sugar. For instance, one may define

$$\begin{aligned}\mathfrak{U}^{10}_{(2,5)}(x, X_1, X_2) = \exists x' \Big(x < x' \wedge d(x, x') > 2 \wedge d(x, x') < 5 \wedge X_2(x') \\ \wedge \forall x'' \big(x < x'' \wedge d(x, x'') > 10 \\ \wedge d(x', x'') < 8 \implies X_1(x'')\big)\Big) ,\end{aligned}$$

but this is indeed equivalent to

$$\mathsf{F}_{(2,5)} \varphi_2 \wedge \neg \Big((\neg \varphi_2) \, \mathfrak{U}^2_{(10,13)} \, \big(\neg \varphi_1 \wedge \neg (\overleftarrow{\mathsf{F}}_{=8} \varphi_2)\big)\Big)$$

over $[0, N)$-timed words. In fact, we can even use $c \in \mathbb{Z}$ and $I \subseteq (-\infty, \infty)$ for free. For example, over $[0, N)$-timed words, $\varphi_1 \, \mathfrak{U}^{-7}_{(5,10)} \, \varphi_2$ is equivalent to

$$\mathsf{F}_{(5,10)} \big(\varphi_2 \wedge (\varphi_1 \mathfrak{S}^{12}_{(5,10)} \mathbf{true})\big) \wedge \Big((\mathbf{false} \, \mathfrak{U}^0_{(5,10)} \varphi_2) \vee \Big(\varphi' \, \mathsf{U} \big((\mathbf{false} \, \mathfrak{U}^0_{(5,10)} \varphi_2) \wedge \varphi'\big)\Big)\Big)$$

where $\varphi' = \varphi_1 \, \mathfrak{S}^7_{(0,5)} \, \big(\mathbf{true} \wedge \neg (\overleftarrow{\mathsf{F}}_{=7} \neg \varphi_1)\big)$. Other cases can be handled with similar ideas.

We can now give an $\mathsf{MTL}[\mathfrak{U},\mathfrak{S}]$ formula that distinguishes, in the pointwise semantics, the models $\mathcal{G}_m$ and $\mathcal{H}_m$ in Section 3.5:

$$\mathsf{F}_{(1,2)}\big(\mathbf{true} \wedge (\mathbf{false}\,\mathfrak{U}^{-1}_{(0,\infty)}\,\mathbf{true})\big)\,.$$

Compare this with the formula φ_{cont3} defined in Section 3.5, which distinguishes $\mathcal{G}_m$ and $\mathcal{H}_m$ in the continuous semantics.

5 The Translation

We give a translation from an arbitrary $\mathsf{FO}[<,+1]$ formula with one free variable into an equivalent $\mathsf{MTL}[\mathfrak{U},\mathfrak{S}]$ formula (over $[0,N)$-timed words). Our proof strategy is similar to that in [11]: we eliminate the metric by introducing extra predicates, convert to LTL, and then replace the new predicates by their equivalent $\mathsf{MTL}[\mathfrak{U},\mathfrak{S}]$ formulas.

5.1 Eliminating the Metric

We introduce fresh monadic predicates $\overline{\mathbf{P}} = \{P_i \mid P \in \mathbf{P}, 0 \le i \le N-1\}$ as in [11] and, additionally, $\overline{\mathbf{Q}} = \{Q_i \mid 0 \le i \le N-1\}$. Intuitively, $P_i(x)$ holds (for $x \in [0,1)$) iff $P \in \mathbf{P}$ holds at time $i+x$ in the corresponding $[0,N)$-timed word, and $Q_i(x)$ holds iff there is an event at time $i+x$ in the corresponding $[0,N)$-timed word, regardless of whether any $P \in \mathbf{P}$ holds there. Let $\varphi_{event} = \forall x \left(\bigvee_{i\in[0,N-1]} Q_i(x)\right) \wedge \forall x \left(\bigwedge_{i\in[0,N-1]} \left(P_i(x) \implies Q_i(x)\right)\right)$ and $\varphi_{init} = \exists x \left(\nexists x' \,(x' < x) \wedge Q_0(x)\right)$. There is an obvious 'stacking' bijection (indicated by overlining) between $[0,N)$-timed words over $\Sigma_{\mathbf{P}}$ and $[0,1)$-timed words over $\Sigma_{\overline{\mathbf{P}}\cup\overline{\mathbf{Q}}}$ satisfying $\varphi_{event} \wedge \varphi_{init}$.

Let $\vartheta(x)$ be an $\mathsf{FO}[<,+1]$ formula with one free variable and in which each quantifier uses a fresh new variable. Without loss of generality, we assume that $\vartheta(x)$ contains only existential quantifiers (this can be achieved by syntactic rewriting). Replace the formula by

$$\big(Q_0(x) \wedge \vartheta[x/x]\big) \vee \big(Q_1(x) \wedge \vartheta[x+1/x]\big) \vee \ldots \vee \big(Q_{N-1}(x) \wedge \vartheta[x+(N-1)/x]\big)$$

where $\vartheta[e/x]$ denotes the formula obtained by substituting all free occurrences of x in ϑ by (an expression) e. Then, similarly, recursively replace every subformula $\exists x'\,\theta$ by

$$\exists x' \left(\big(Q_0(x') \wedge \theta[x'/x']\big) \vee \ldots \vee \big(Q_{N-1}(x') \wedge \theta[x'+(N-1)/x']\big)\right).$$

Note that we do not actually have the $+1$ function in our structures; it only serves as annotation here and will be removed later, e.g., $x'+k$ means that $Q_k(x')$ holds. We then carry out the following syntactic substitutions:

- For each inequality of the form $x_1 + k_1 < x_2 + k_2$, replace it with
  - $x_1 < x_2$ if $k_1 = k_2$

  - **true** if $k_1 < k_2$
  - $\neg$**true** if $k_1 > k_2$
- For each distance formula, e.g., $d(x_1 + k_1, x_2 + k_2) \leq 2$, replace it with
  - **true** if $|k_1 - k_2| \leq 1$
  - $(\neg(x_1 < x_2) \wedge \neg(x_2 < x_1)) \vee (x_2 < x_1)$ if $k_2 - k_1 = 2$
  - $(\neg(x_1 < x_2) \wedge \neg(x_2 < x_1)) \vee (x_1 < x_2)$ if $k_1 - k_2 = 2$
  - $\neg$**true** if $|k_1 - k_2| > 2$
- Replace terms of the form $P(x_1 + k)$ with $P_k(x_1)$.

This gives a non-metric first-order formula $\overline{\vartheta}(x)$ over $\overline{\mathbf{P}} \cup \overline{\mathbf{Q}}$. Denote by $frac(t)$ the fractional part of a non-negative real t. It is not hard to see that for each $[0, N)$-timed word $\rho = (\sigma, \tau)$ over $\Sigma_{\mathbf{P}}$ and its stacked counterpart $\overline{\rho}$, the following holds:

- $\rho, t \models \vartheta(x)$ implies $\overline{\rho}, \overline{t} \models \overline{\vartheta}(x)$ where $\overline{t} = frac(t)$
- $\overline{\rho}, \overline{t} \models \overline{\vartheta}(x)$ implies there exists $t \in U_\rho$ with $frac(t) = \overline{t}$ s.t. $\rho, t \models \vartheta(x)$.

Moreover, if $\rho, t \models \vartheta(x)$, then the integral part of t indicates which clause in $\overline{\vartheta}(x)$ is satisfied when x is substituted with $\overline{t} = frac(t)$, and vice versa.

By Kamp's theorem [9], $\overline{\vartheta}(x)$ is equivalent to an $\mathsf{LTL}[\mathsf{U}, \mathsf{S}]$ formula $\overline{\varphi}$ of the following form:

$$(Q_0 \wedge \overline{\varphi}_0) \vee (Q_1 \wedge \overline{\varphi}_1) \vee \ldots \vee (Q_{N-1} \wedge \overline{\varphi}_{N-1}) .$$

5.2 From Non-Metric to Metric

We now construct an $\mathsf{MTL}[\mathfrak{U}, \mathfrak{S}]$ formula that is equivalent to $\vartheta(x)$ over $[0, N)$-timed words. Note that we make heavy use of the formulas in Φ_{unit} defined in Section 3.1.

Proposition 6. *Let $\overline{\psi}$ be a subformula of $\overline{\varphi}_i$ for some $i \in [0, N-1]$. There is an $\mathsf{MTL}[\mathfrak{U}, \mathfrak{S}]$ formula ψ such that for any $[0, N)$-timed word ρ, $t \in \rho$ and $frac(t) = \overline{t} \in \overline{\rho}$, we have*

$$\overline{\rho}, \overline{t} \models \overline{\psi} \iff \rho, t \models \psi .$$

Proof. The $\mathsf{MTL}[\mathfrak{U}, \mathfrak{S}]$ formula ψ is constructed inductively as follows:

- *Base step.* Consider the following cases:
  - $\overline{\psi} = P_j$: Let

$$\psi = (\varphi_{0,1} \wedge \mathsf{F}_{=j} P) \vee \ldots \vee (\varphi_{j,j+1} \wedge P) \vee \ldots \vee (\varphi_{N-1,N} \wedge \overleftarrow{\mathsf{F}}_{=((N-1)-j)} P) .$$

  - $\overline{\psi} = Q_j$: Similarly we let

$$\psi = (\varphi_{0,1} \wedge \mathsf{F}_{=j} \mathbf{true}) \vee \ldots \vee (\varphi_{j,j+1} \wedge \mathbf{true}) \vee \ldots \vee (\varphi_{N-1,N} \wedge \overleftarrow{\mathsf{F}}_{=((N-1)-j)} \mathbf{true}) .$$

– *Induction step.* The case for boolean operations is trivial and hence omitted.
  - $\overline{\psi} = \overline{\psi}_1 \,\mathsf{U}\, \overline{\psi}_2$: By IH we have ψ_1 and ψ_2. Let

$$\psi^{j,k,l} = \psi_1 \,\mathfrak{U}^k_{(j,j+1)}\, (\psi_2 \wedge \varphi_{l,l+1})\,.$$

    The desired formula is

$$\psi = \bigvee_{i\in[0,N-1]} \left(\varphi_{i,i+1} \wedge \bigvee_{\substack{j\in[-i,\ldots,(N-1)-i] \\ l=i+j}} \left(\bigwedge_{k\in[-i,\ldots,(N-1)-i]} \psi^{j,k,l} \right) \right).$$

  - $\overline{\psi} = \overline{\psi}_1 \,\mathsf{S}\, \overline{\psi}_2$: This is symmetric to the case for $\overline{\psi}_1 \,\mathsf{U}\, \overline{\psi}_2$.

The claim holds by a straightforward induction on the structure of $\overline{\psi}$ and ψ. ☐

Construct corresponding formulas φ_i for each $\overline{\varphi}_i$ using the proposition above. Substitute them into $\overline{\varphi}$ and replace all Q_i by $\varphi_{i,i+1}$ to obtain our final formula φ. We claim that it is equivalent to $\vartheta(x)$ over $[0, N)$-timed words.

Proposition 7. *For any $[0, N)$-timed words ρ and $t \in U_\rho$, we have*

$$\rho, t \models \varphi(x) \iff \rho, t \models \vartheta(x)\,.$$

Proof. Follows directly from Section 5.1 and Proposition 6. ☐

We are now ready to state our main result.

Theorem 1. $\mathsf{MTL}[\mathfrak{U}, \mathfrak{S}]$ *is expressively complete for* $\mathsf{FO}[<, +1]$ *over* $[0, N)$*-timed words.*

6 Conclusion

Our main result is that over bounded timed words, MTL extended with our new modalities 'generalised until' and 'generalised since' is expressively complete for $\mathsf{FO}[<, +1]$. Along the way we obtain a strict hierarchy of metric temporal logics, based on their expressiveness over bounded timed words:

$$\mathsf{MTL_{fut}} \subsetneq \mathsf{MTL_{fut}}[\Phi_{unit}] \subsetneq \mathsf{MTL} \subsetneq \mathsf{MTL}[\mathcal{B}^{\leftrightarrows}] \subsetneq \mathsf{MTL}[\mathfrak{U}, \mathfrak{S}] = \mathsf{FO}[<, +1].$$

We are currently working on adapting the result to the case of $\mathbb{R}_{\geq 0}$-timed words. This might require a separation theorem (in the style of [8]) that works in the pointwise semantics [6].

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